

# Catmull-Clark subdivision algorithm

Michael Spitzer - vo3xel@gmail.com

February 2019

## 1 Introduction

This document describes the application of the Catmull-Clark algorithm on a cube. All steps are available as GeoGebra files.

## 2 Catmull-Clark algorithm

This document is based on the algorithm described in [1]. The algorithm consists of the following steps:

1. Add face point for each face (average of all original points on face).
2. Add an edge point for each edge (average of the two neighboring face points and its two original endpoints).
3. For each face point, add an edge for every edge of the face, connecting the face point to each edge point for the face.
4. For each original point  $P$ , take the average  $F$  of all  $n$  (recently created) face points for faces touching  $P$ , and take the average  $R$  of all  $n$  edge midpoints for (original) edges touching  $P$ , where each edge midpoint is the average of its two endpoint vertices (not to be confused with new "edge points" above). Move each original point to the point:

$$P' = \frac{F + 2R + (n - 3)P}{n}$$

5. Connect each new vertex point to the new edge points of all original edges incident on the original vertex.

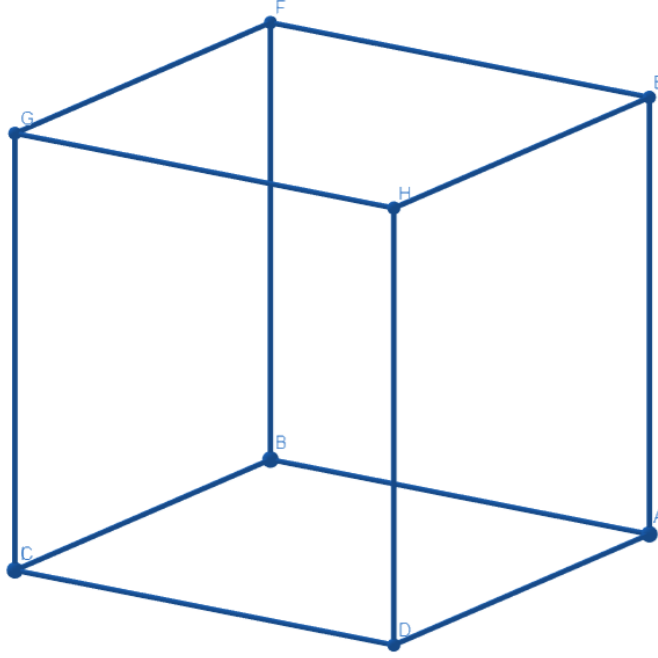
## 3 Cube example

This section describes the application of the algorithm on a simple cube. Figure 1 shows the cube. The cube has the following vertices:

$$A = \begin{pmatrix} -1 & -1 & 0 \end{pmatrix} \quad E = \begin{pmatrix} -1 & -1 & 2 \end{pmatrix}$$

$$\begin{aligned}
B &= \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} & F &= \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \\
C &= \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} & G &= \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} \\
D &= \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} & H &= \begin{pmatrix} -1 & 1 & 2 \end{pmatrix}
\end{aligned}$$

Figure 1: Example cube with length = 2. (GeoGebra link: <https://ggbm.at/dfz4phk9>)



### 3.1 Add face point for each face

The first step is to compute the face points. The face points are calculated with the following equation:

$$f_x = \frac{1}{4} \cdot (P_1 + P_2 + P_3 + P_4)$$

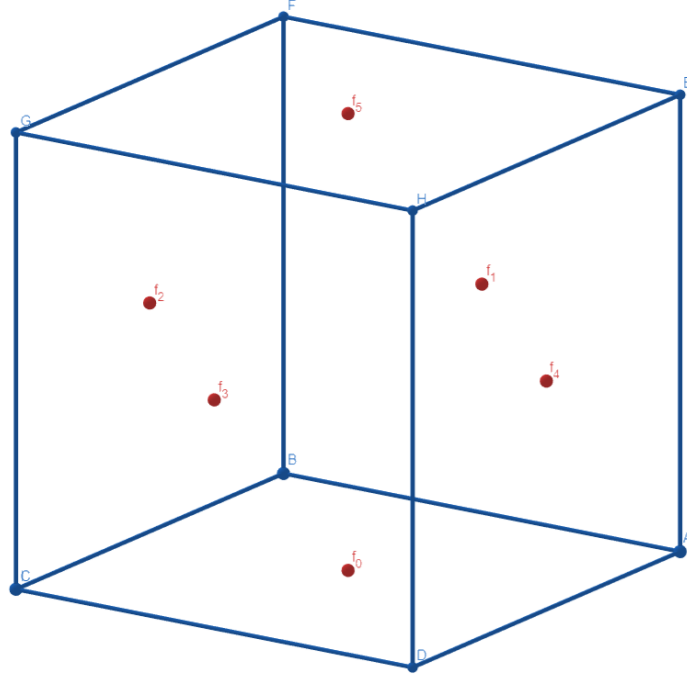
All four points must be on the same face.

$$\begin{aligned}
f_0 = f_{ABCD} &= \frac{1}{4} \cdot (A+B+C+D) = \frac{1}{4} \cdot \left[ \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
f_1 = f_{AEFB} &= \frac{1}{4} \cdot (A+E+F+B) = \frac{1}{4} \cdot \left[ \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
f_2 = f_{BFGC} &= \frac{1}{4} \cdot (B+F+G+C) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
f_3 = f_{CDHG} &= \frac{1}{4} \cdot (C+D+H+G) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
f_4 = f_{DAEH} &= \frac{1}{4} \cdot (D+A+E+H) = \frac{1}{4} \cdot \left[ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\
f_5 = f_{EFGH} &= \frac{1}{4} \cdot (E+F+G+H) = \frac{1}{4} \cdot \left[ \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}
\end{aligned}$$

The computed face points are shown in figure 2.

Figure 2: Example cube with computed face points. (GeoGebra link: <https://ggbm.at/tmfeuwrn>)



### 3.2 Add an edge point for each edge

The edge point is calculated as average of the two neighboring face points and its two original endpoints.

$$e_{P_x, P_y} = \frac{1}{4} \cdot (f_1^{P_x, P_y} + f_2^{P_x, P_y} + P_x + P_y)$$

The edge point is computed for every line.

$$e_{AB} = \frac{1}{4} \cdot (f_0 + f_1 + A + B) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ -\frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$$

$$e_{BC} = \frac{1}{4} \cdot (f_0 + f_2 + B + C) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} \frac{3}{4} \\ 0 \\ \frac{1}{4} \end{pmatrix}$$

$$e_{CD} = \frac{1}{4} \cdot (f_0 + f_3 + C + D) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$$

$$e_{DA} = \frac{1}{4} \cdot (f_0 + f_4 + D + A) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} -\frac{3}{4} \\ 0 \\ \frac{1}{4} \end{pmatrix}$$

$$e_{AE} = \frac{1}{4} \cdot (f_1 + f_4 + A + E) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} -\frac{3}{4} \\ -\frac{3}{4} \\ 1 \end{pmatrix}$$

$$e_{BF} = \frac{1}{4} \cdot (f_1 + f_2 + B + F) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ 1 \end{pmatrix}$$

$$e_{CG} = \frac{1}{4} \cdot (f_2 + f_3 + C + G) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ 1 \end{pmatrix}$$

$$e_{DH} = \frac{1}{4} \cdot (f_3 + f_4 + D + H) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} -\frac{3}{4} \\ \frac{3}{4} \\ 1 \end{pmatrix}$$

$$e_{EF} = \frac{1}{4} \cdot (f_1 + f_5 + E + F) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ -\frac{3}{4} \\ \frac{7}{4} \end{pmatrix}$$

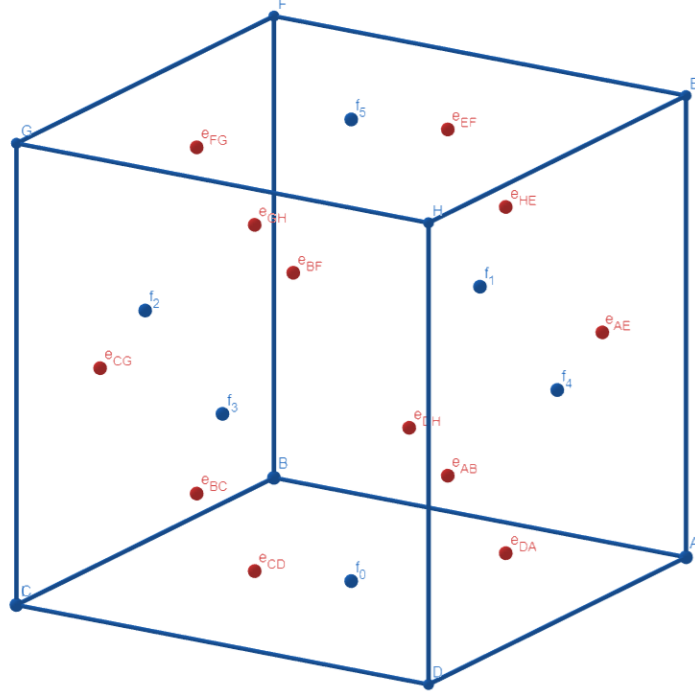
$$e_{FG} = \frac{1}{4} \cdot (f_2 + f_5 + F + G) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} \frac{3}{4} \\ 0 \\ \frac{7}{4} \end{pmatrix}$$

$$e_{GH} = \frac{1}{4} \cdot (f_3 + f_5 + G + H) = \frac{1}{4} \cdot \left[ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \frac{3}{4} \\ \frac{7}{4} \end{pmatrix}$$

$$e_{HE} = \frac{1}{4} \cdot (f_4 + f_5 + H + E) = \frac{1}{4} \cdot \left[ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} -\frac{3}{4} \\ 0 \\ \frac{7}{4} \end{pmatrix}$$

The computed edge points are shown in figure 3.

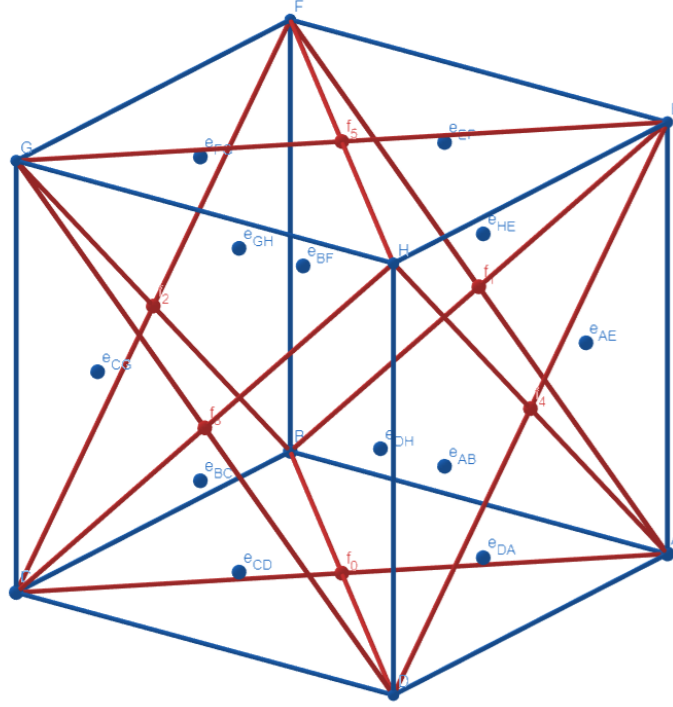
Figure 3: Example cube with computed edge points. (GeoGebra link: <https://ggbm.at/kmwfec2m>)



### 3.3 Connect face points and face vertices with edges

Figure 4 shows the face points connected with face vertices.

Figure 4: Example cube with computed edge points. (GeoGebra link: <https://ggbm.at/ungtevr>)



### 3.4 Move each original point

Move each original point to the point:

$$P' = \frac{F + 2R + (n - 3)P}{n}$$

Since the shape is a cube (n=3) the equation can be simplified to:

$$P' = \frac{F + 2R}{3}$$

(F ... average of all face points of faces touching P, R ... average of all edge mid points of edges touching P, n ... number of faces touching the point, P ... original point)

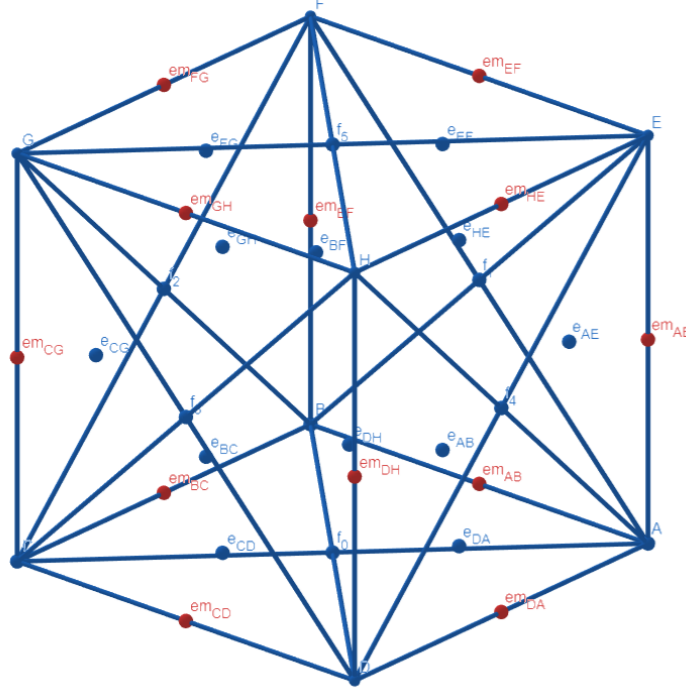
#### 3.4.1 Compute edge mid points

$$em_{P_x, P_y} = \frac{1}{2} \cdot (P_x + P_y)$$

$$\begin{aligned}
em_{AB} &= \frac{1}{2} \cdot (A + B) = \frac{1}{2} \cdot \left[ \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \\
em_{BC} &= \frac{1}{2} \cdot (B + C) = \frac{1}{2} \cdot \left[ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
em_{CD} &= \frac{1}{2} \cdot (C + D) = \frac{1}{2} \cdot \left[ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
em_{DA} &= \frac{1}{2} \cdot (D + A) = \frac{1}{2} \cdot \left[ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \\
em_{AE} &= \frac{1}{2} \cdot (A + E) = \frac{1}{2} \cdot \left[ \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \\
em_{BF} &= \frac{1}{2} \cdot (B + F) = \frac{1}{2} \cdot \left[ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\
em_{CG} &= \frac{1}{2} \cdot (C + G) = \frac{1}{2} \cdot \left[ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
em_{DH} &= \frac{1}{2} \cdot (D + H) = \frac{1}{2} \cdot \left[ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\
em_{EF} &= \frac{1}{2} \cdot (E + F) = \frac{1}{2} \cdot \left[ \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \\
em_{FG} &= \frac{1}{2} \cdot (F + G) = \frac{1}{2} \cdot \left[ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \\
em_{GH} &= \frac{1}{2} \cdot (G + H) = \frac{1}{2} \cdot \left[ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\
em_{HE} &= \frac{1}{2} \cdot (H + E) = \frac{1}{2} \cdot \left[ \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}
\end{aligned}$$

Figure 5 shows the computed edge points.

Figure 5: Example cube with computed edge mid points. (GeoGebra link: <https://ggbm.at/ng8zqgpr>)



### 3.4.2 Move original vertices

$$P' = \frac{F + 2R}{3}$$

$$A' = \frac{\frac{1}{3} \cdot [f_0 + f_1 + f_4] + \frac{2}{3} \cdot [em_{AB} + em_{DA} + em_{AE}]}{3}$$

$$A' = \frac{\frac{1}{3} \cdot \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right] + \frac{2}{3} \cdot \left[ \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right]}{3} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{5}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$B' = \frac{\frac{1}{3} \cdot [f_0 + f_1 + f_2] + \frac{2}{3} \cdot [em_{AB} + em_{BC} + em_{BF}]}{3}$$

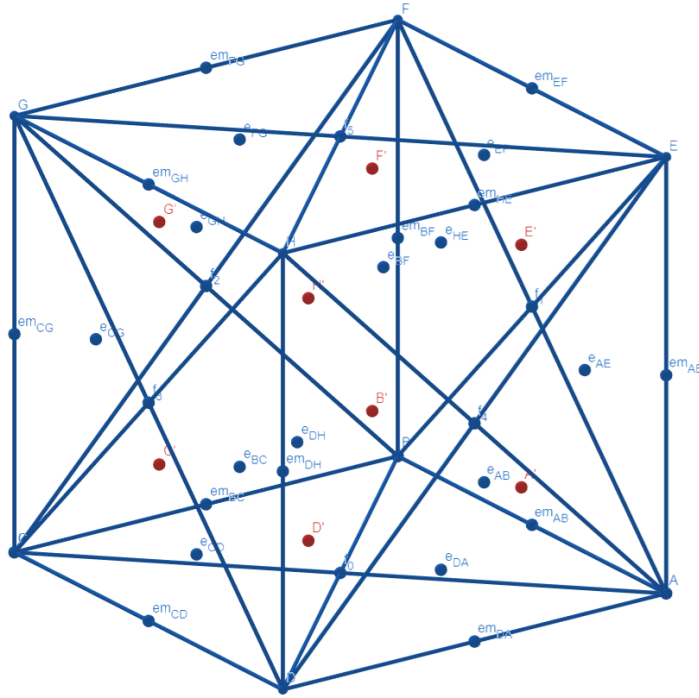
$$B' = \frac{\frac{1}{3} \cdot \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right] + \frac{2}{3} \cdot \left[ \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right]}{3} = \begin{pmatrix} \frac{5}{9} \\ -\frac{5}{9} \\ \frac{4}{9} \end{pmatrix}$$



$$\begin{aligned}
C' &= \frac{\frac{1}{3} \cdot [f_0 + f_2 + f_3] + \frac{2}{3} \cdot [em_{BC} + em_{CD} + em_{CG}]}{3} \\
C' &= \frac{\frac{1}{3} \cdot \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right] + \frac{2}{3} \cdot \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]}{3} = \begin{pmatrix} \frac{5}{9} \\ \frac{5}{9} \\ \frac{14}{9} \end{pmatrix} \\
D' &= \frac{\frac{1}{3} \cdot [f_0 + f_3 + f_4] + \frac{2}{3} \cdot [em_{CD} + em_{DA} + em_{DH}]}{3} \\
D' &= \frac{\frac{1}{3} \cdot \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right] + \frac{2}{3} \cdot \left[ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right]}{3} = \begin{pmatrix} -\frac{5}{9} \\ \frac{5}{9} \\ \frac{4}{9} \end{pmatrix} \\
E' &= \frac{\frac{1}{3} \cdot [f_1 + f_4 + f_5] + \frac{2}{3} \cdot [em_{AE} + em_{EF} + em_{HE}]}{3} \\
E' &= \frac{\frac{1}{3} \cdot \left[ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right] + \frac{2}{3} \cdot \left[ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right]}{3} = \begin{pmatrix} -\frac{5}{9} \\ -\frac{5}{9} \\ \frac{14}{9} \end{pmatrix} \\
F' &= \frac{\frac{1}{3} \cdot [f_1 + f_2 + f_5] + \frac{2}{3} \cdot [em_{BF} + em_{FG} + em_{EF}]}{3} \\
F' &= \frac{\frac{1}{3} \cdot \left[ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right] + \frac{2}{3} \cdot \left[ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right]}{3} = \begin{pmatrix} \frac{5}{9} \\ -\frac{5}{9} \\ \frac{14}{9} \end{pmatrix} \\
G' &= \frac{\frac{1}{3} \cdot [f_2 + f_3 + f_5] + \frac{2}{3} \cdot [em_{CG} + em_{FG} + em_{GH}]}{3} \\
G' &= \frac{\frac{1}{3} \cdot \left[ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right] + \frac{2}{3} \cdot \left[ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right]}{3} = \begin{pmatrix} \frac{5}{9} \\ \frac{5}{9} \\ \frac{14}{9} \end{pmatrix} \\
H' &= \frac{\frac{1}{3} \cdot [f_3 + f_4 + f_5] + \frac{2}{3} \cdot [em_{DH} + em_{GH} + em_{HE}]}{3} \\
H' &= \frac{\frac{1}{3} \cdot \left[ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right] + \frac{2}{3} \cdot \left[ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right]}{3} = \begin{pmatrix} -\frac{5}{9} \\ \frac{5}{9} \\ \frac{14}{9} \end{pmatrix}
\end{aligned}$$

Figure 6 shows the moved vertices.

Figure 6: Example cube with moved vertices. (GeoGebra link: <https://ggbm.at/s9s9jbta>)

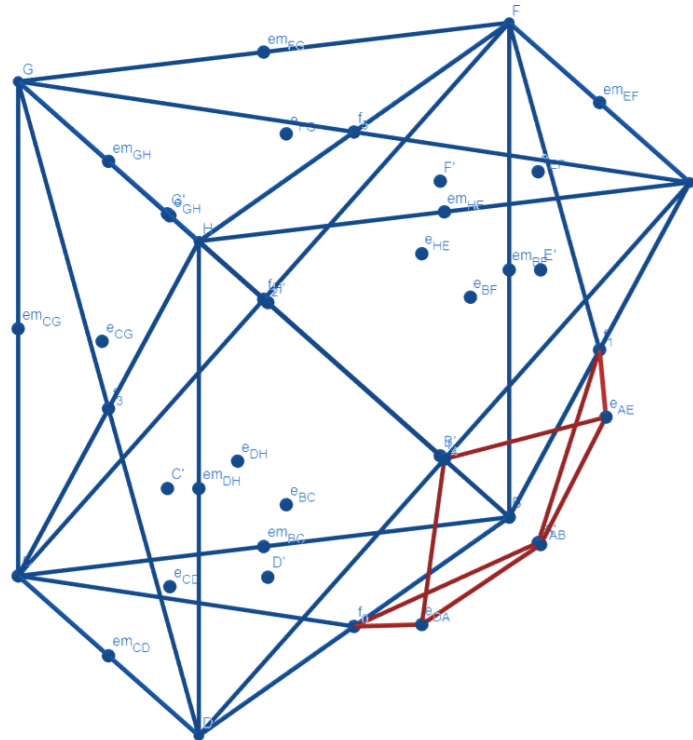


### 3.5 Connect each new vertex point

Connect each new vertex point to the new edge points of all original edges incident on the original vertex.

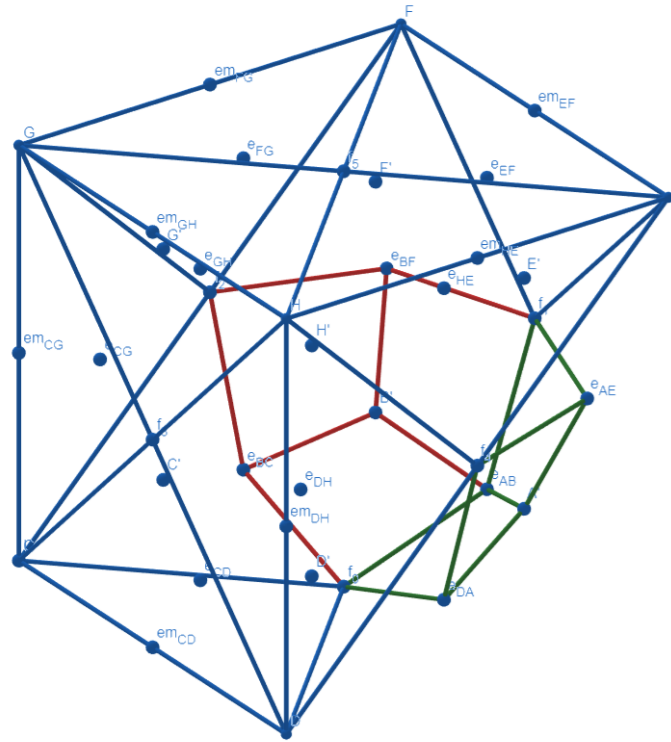
### 3.5.1 $\mathbf{A} \rightarrow A'$

Figure 7: A' (GeoGebra link: <https://ggbm.at/u7rp4pe8>)



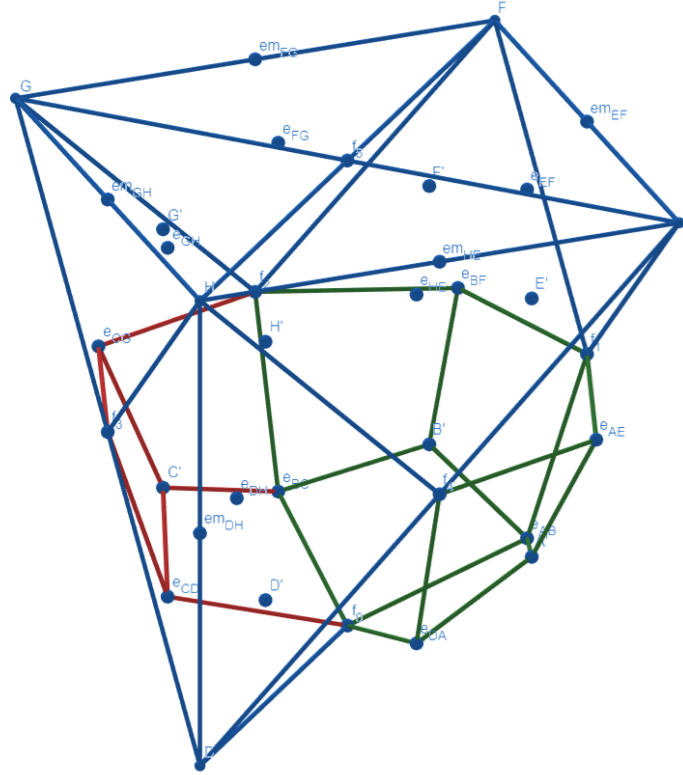
### 3.5.2 $B \rightarrow B'$

Figure 8:  $B'$  (GeoGebra link: <https://ggbm.at/se5hwy89>)



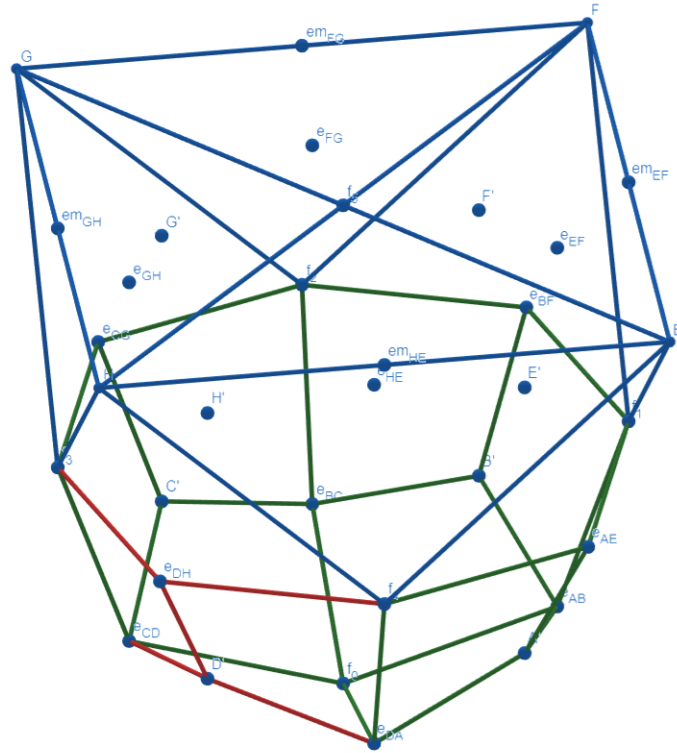
### 3.5.3 $C \rightarrow C'$

Figure 9:  $C'$  (GeoGebra link: <https://ggbm.at/vwx3rxrm>)



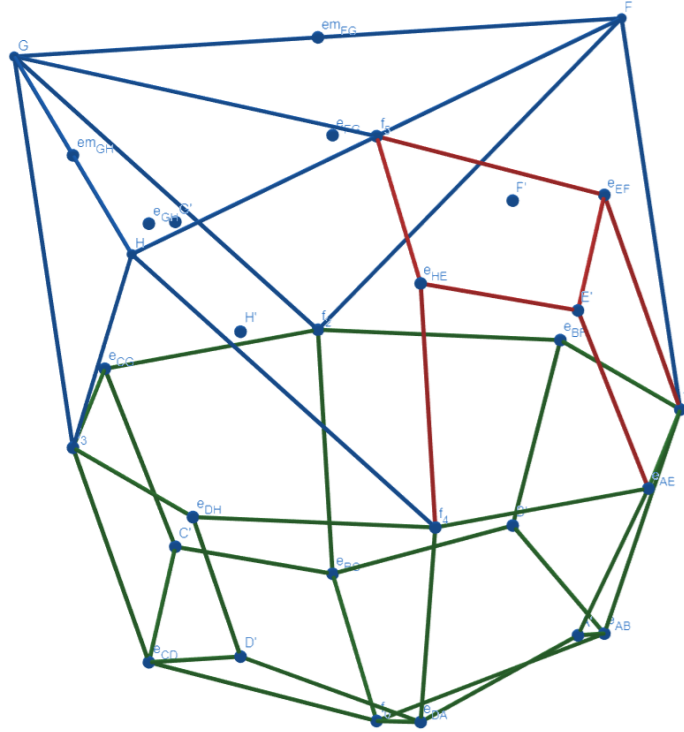
### 3.5.4 $D \rightarrow D'$

Figure 10:  $D'$  (GeoGebra link: <https://ggbm.at/u4hnhdtc>)



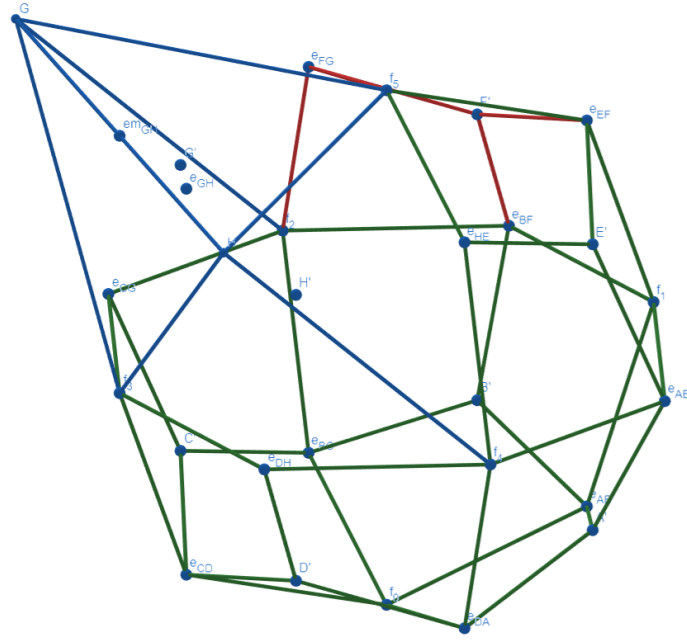
### 3.5.5 $E \rightarrow E'$

Figure 11:  $E'$  (GeoGebra link: <https://ggbm.at/kwqrtdzs>)



### 3.5.6 $F \rightarrow F'$

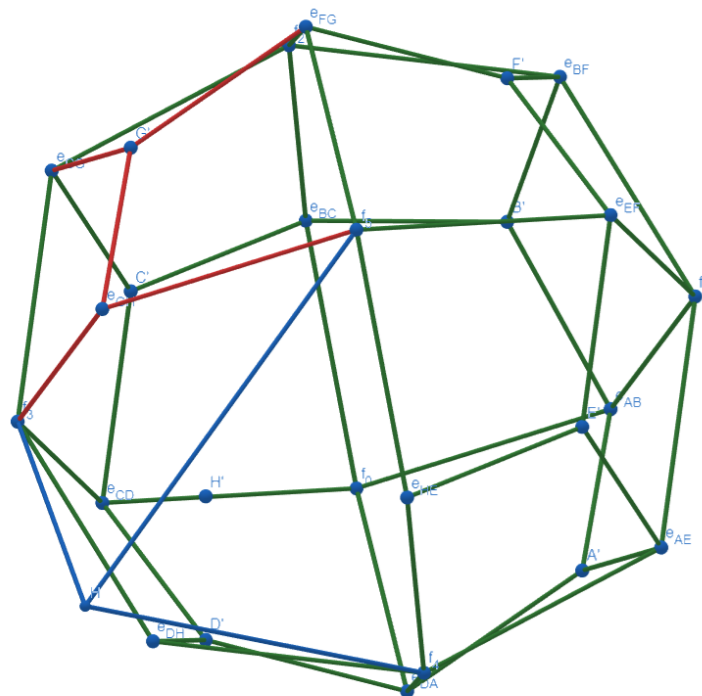
Figure 12:  $F'$  (GeoGebra link: <https://ggbm.at/gqnfxyjk>)





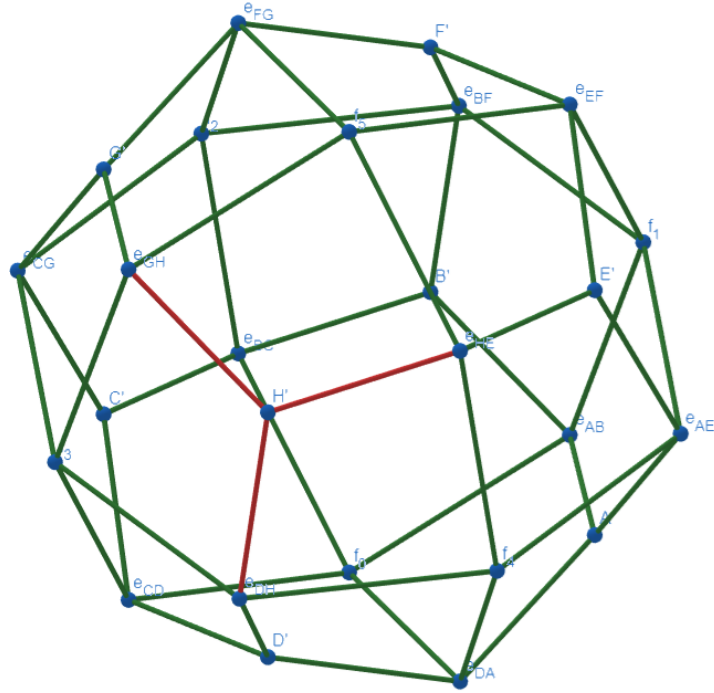
### 3.5.7 $G \rightarrow G'$

Figure 13:  $G'$  (GeoGebra link: <https://ggbm.at/exzncjsu>)



### 3.5.8 $H \rightarrow H'$

Figure 14:  $H'$  (GeoGebra link: <https://ggbm.at/nr3s6n4y>)



### 3.6 Finished sub division

Figure 15: Finished Catmull-Clark sub division (GeoGebra link: <https://ggbm.at/a4dng2bt>)

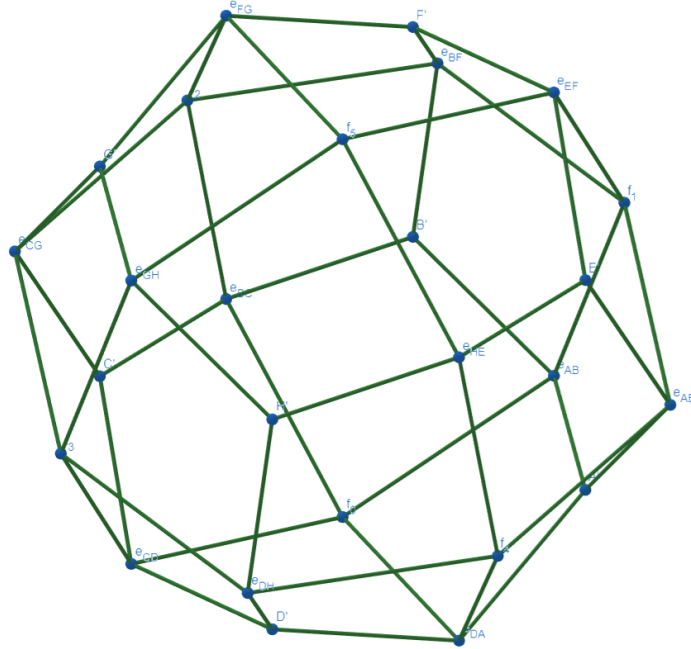
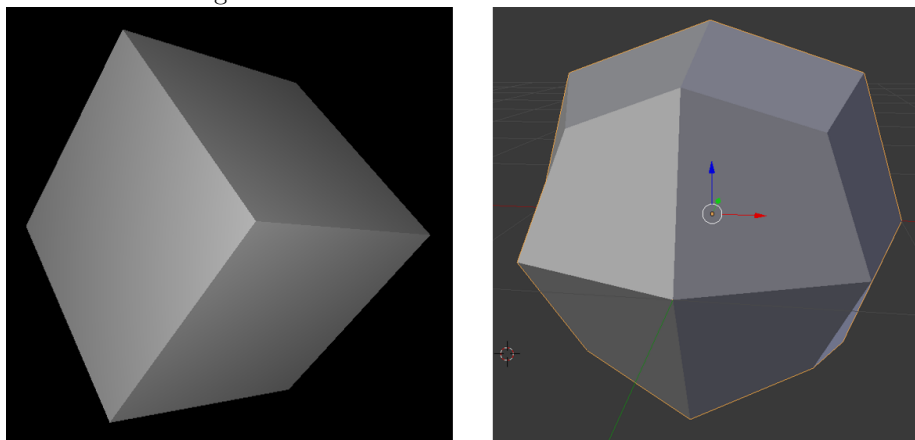


Figure 16: Finished Catmull-Clark sub division



## 4 Verification

This section compares the manually calculated points with the computed values from JavaScript algorithm found on GitHub (<https://github.com/Erkaman/gl-catmull-clark>). Table 1 shows the vertex results. The results are the same but other vertex order, this is caused by internal structure of the algorithm.

## References

- [1] Edwin Catmull and James Clark. Recursively generated b-spline surfaces on arbitrary topological meshes. *Computer-aided design*, 10(6):350–355, 1978.

Table 1: Left: manual calculation, right: algorithm results

vertex	x	y	z
$f_0$	0	0	0
$f_1$	0	-1	1
$f_2$	1	0	1
$f_3$	0	1	1
$f_4$	-1	0	1
$f_5$	0	0	2
$e_{AB}$	0	$-\frac{3}{4}$	$\frac{1}{4}$
$e_{BC}$	$\frac{3}{4}$	0	$\frac{1}{4}$
$e_{CD}$	0	$\frac{3}{4}$	$\frac{1}{4}$
$e_{DA}$	$-\frac{3}{4}$	0	$\frac{1}{4}$
$e_{AE}$	$-\frac{3}{4}$	$-\frac{3}{4}$	1
$e_{BF}$	$\frac{3}{4}$	$-\frac{3}{4}$	1
$e_{CG}$	$\frac{3}{4}$	$\frac{3}{4}$	1
$e_{DH}$	$-\frac{3}{4}$	$\frac{3}{4}$	1
$e_{EF}$	0	$-\frac{3}{4}$	$\frac{7}{4}$
$e_{FG}$	$\frac{3}{4}$	0	$\frac{7}{4}$
$e_{GH}$	0	$\frac{3}{4}$	$\frac{7}{4}$
$e_{HE}$	$-\frac{3}{4}$	0	$\frac{7}{4}$
$A'$	$-\frac{5}{9}$	$-\frac{5}{9}$	$\frac{4}{9}$
$B'$	$\frac{5}{9}$	$-\frac{5}{9}$	$\frac{4}{9}$
$C'$	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{4}{9}$
$D'$	$-\frac{5}{9}$	$\frac{5}{9}$	$\frac{4}{9}$
$E'$	$-\frac{5}{9}$	$-\frac{5}{9}$	$\frac{14}{9}$
$F'$	$\frac{5}{9}$	$-\frac{5}{9}$	$\frac{14}{9}$
$G'$	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{14}{9}$
$H'$	$-\frac{5}{9}$	$\frac{5}{9}$	$\frac{14}{9}$

vertex	x	y	z
$D'$	$-\frac{5}{9}$	$\frac{5}{9}$	$\frac{4}{9}$
$e_{CD}$	0	$\frac{3}{4}$	$\frac{1}{4}$
$f_0$	0	0	0
$e_{DA}$	$-\frac{3}{4}$	0	$\frac{1}{4}$
$C'$	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{4}{9}$
$e_{BC}$	$\frac{3}{4}$	0	$\frac{1}{4}$
$B'$	$\frac{5}{9}$	$-\frac{5}{9}$	$\frac{4}{9}$
$e_{AB}$	0	$-\frac{3}{4}$	$\frac{1}{4}$
$A'$	$-\frac{5}{9}$	$-\frac{5}{9}$	$\frac{4}{9}$
$e_{BF}$	$\frac{3}{4}$	$-\frac{3}{4}$	1
$f_1$	0	-1	1
$F'$	$\frac{5}{9}$	$-\frac{5}{9}$	$\frac{14}{9}$
$e_{EF}$	0	$-\frac{3}{4}$	$\frac{7}{4}$
$E'$	$-\frac{5}{9}$	$-\frac{5}{9}$	$\frac{14}{9}$
$e_{AE}$	$-\frac{3}{4}$	$-\frac{3}{4}$	1
$e_{CG}$	$\frac{3}{4}$	$\frac{3}{4}$	1
$f_2$	1	0	1
$G'$	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{14}{9}$
$e_{FG}$	$\frac{3}{4}$	0	$\frac{7}{4}$
$e_{DH}$	$-\frac{3}{4}$	$\frac{3}{4}$	1
$f_3$	0	1	1
$H'$	$-\frac{5}{9}$	$\frac{5}{9}$	$\frac{14}{9}$
$e_{GH}$	0	$\frac{3}{4}$	$\frac{7}{4}$
$f_4$	-1	0	1
$e_{HE}$	$-\frac{3}{4}$	0	$\frac{7}{4}$
$f_5$	0	0	2