1. Să se arate că $\frac{\mathbb{R}[X]}{(X^2-1)} \simeq \mathbb{R}\times\mathbb{R}$, dar $\frac{\mathbb{Z}[X]}{(X^2-1)} \not\simeq \mathbb{Z}\times\mathbb{Z}$.

În $\mathbb{R}[X]$:

$$X^2-1=(X-1)(X+1)$$

$$I=(X-1),\ J=(X+1)$$

$$IJ=\big((X-1)(X+1)\big)=(X^2-1)$$

I și J sunt comaximale, adică

$$I+J=\mathbb{R}[X]$$

$$\underbrace{-\tfrac{1}{2}(X-1)}_{\in I}+\underbrace{\tfrac{1}{2}(X+1)}_{\in J}=1$$

$$1\in I+J \Rightarrow I+J=\mathbb{R}[X]$$

Lema chineză $\Rightarrow I\cap J=IJ=(X^2-1)$

$$\frac{\mathbb{R}[X]}{(X^2-1)}\simeq\frac{\mathbb{R}[X]}{I\cap J}\simeq\frac{\mathbb{R}[X]}{I}\times\frac{\mathbb{R}[X]}{J}\simeq\underbrace{\frac{\mathbb{R}[X]}{(X-1)}}_{\simeq\,\mathbb{R}}\times\underbrace{\frac{\mathbb{R}[X]}{(X+1)}}_{\simeq\,\mathbb{R}}$$

$A[X]$, A inel comutativ
$f,g\in A[X]$, $g\neq 0$
coef. dominant din g este în $U(A)$ $\Bigg|\Rightarrow$

$\Rightarrow (\exists!)\ c, r \in A[X]$ cu $\begin{cases} f = c\cdot g + r \\ \deg r \leq \deg g \end{cases}$

$$\begin{array}{l} a_m X^m + \dots \\ a_m X^m + \dots \\ \hline \end{array} \qquad \begin{array}{l|l} b_m X^m + \dots & \\ \hline b_m^{-1} a_m X^{m-m} & \end{array}$$

$\deg g = m$

$$\frac{A[X]}{(g)} = \{ \hat{f} \mid f \in A[X] \} \qquad \hat{f_1} = \hat{f_2} \Leftrightarrow f_1 - f_2 \in (g)$$

$$\Leftrightarrow g \mid f_1 - f_2$$

$$\left(\frac{R}{I} = \{ \hat{r} \mid r \in R \} \quad \hat{r} = \hat{s} \Leftrightarrow r - s \in I \right)$$

$$\hat{g} = \hat{0} \Leftrightarrow g \in (g)$$

$$\hat{f} = \widehat{cg + r} = \hat{c}\cancel{\hat{g}} + \hat{r}, \quad \hat{g} = 0$$

$$\frac{A[X]}{(g)} = \{ \hat{r} \mid r \in A[X],\ \deg r < m \}$$

Dacă r_1, r_2 au grad $< m$:

$$\hat{r_1} = \hat{r_2} \Leftrightarrow g \mid r_1 - r_2 \Leftrightarrow \underbrace{r_1 - r_2}_{\text{grad} < m} = \underbrace{g \cdot h}_{\text{grad } g + \text{grad } h}$$

$$\Downarrow$$

$$r_1 = r_2$$

Deci $\frac{A[X]}{(g)} = \{\widehat{a_0+a_1X+\ldots+a_{m-1}X^{m-1}} \mid a_0,\ldots,a_{m-1}\in A\}$

$$\frac{\mathbb{Z}[X]}{(X^2-1)} = \{\widehat{a+bX} \mid a,b\in\mathbb{Z}\}$$

$$\widehat{a+bX} = \widehat{c+dX} \iff \begin{cases} a=c \\ b=d \end{cases}$$

$$\widehat{a+bX} + \widehat{c+dX} = \widehat{(a+c)+(b+d)X}$$

$$\widehat{a+bX} \cdot \widehat{c+dX} = \widehat{ac+(ad+bc)\cdot X+bdX^2}$$

$$= \widehat{bd(X^2-1)+bd+ac+(ad+bc)X}$$

$$= \widehat{bd+ac+(ad+bc)X}$$

$\varphi:\mathbb{Z}\times\mathbb{Z}\to\frac{\mathbb{Z}[X]}{(X^2-1)}$

$\varphi((a,b)) = \widehat{a+bX}$

funcție bijectivă

Câte elemente u în $\frac{\mathbb{Z}[X]}{(X^2-1)}$ cu $u^2=u$ există?

$\mathbb{Z}\times\mathbb{Z}$ $\quad (a,b)^2=(a^2,b^2) = (a,b)$ $\quad (0,0),(0,1),(1,0),(1,1)$

Fie $\widehat{a+bX}=u$. Atunci

$$(\widehat{a+bX})^2 = \widehat{b^2+a^2+2abX} = \widehat{a+bX}$$

$\Downarrow$

$$\begin{cases} b^2+a^2=a \\ 2ab=b \end{cases} \Rightarrow \underset{\overset{\parallel}{0}}{b}(2a-1)=0 \quad \text{sau } a=\tfrac{1}{2}\notin\mathbb{Z}$$

Deci $b=0 \Rightarrow a^2=a \Rightarrow a\in\{0,1\}$

$x=x^2$ are $\hat{0}$ și $\hat{1}$ ca soluții

$x^2=1 \qquad \mathbb{Z}\times\mathbb{Z} \qquad (1,1)$

$$(a,b)^2=(1,1) \Leftrightarrow (a^2,b^2)=(1,1)$$

$$a,b\in\{1,-1\}$$

$\Rightarrow$ ec. $x^2=1$ are tot 4 soluții.

în $\frac{\mathbb{Z}[X]}{X^2-1}$: $(\widehat{a+bX})^2=\widehat{b^2+a^2+2abX}=\hat{1} \Rightarrow \begin{cases} b^2+a^2=1 \\ 2ab=0 \end{cases}$

$a=0,\ b=\pm1 \qquad \hat{X}, -\hat{X}, \hat{1}, -\hat{1}$

$a=\pm1,\ b=0$

2. $\frac{\mathbb{Z}_2[X]}{(X^2+X+\hat{1})}$ corp cu 4 elemente

$\Downarrow$

$\{\widehat{\alpha+\beta x} \mid \alpha,\beta\in\mathbb{Z}_2\}$

	$\hat 0$	$\hat 1$	$\hat x$	$\widehat{1+x}$
$\hat 0$	$\hat 0$	$\hat 0$	$\hat 0$	$\hat 0$
$\hat 1$	$\hat 0$	$\hat 1$	$\hat x$	$\widehat{1+x}$
$\hat x$	$\hat 0$	$\hat x$	$\widehat{x+\bar 1}$	$\hat 1$
$\widehat{1+x}$	$\hat 0$	$\widehat{1+x}$	$\hat 1$	$\hat x$

$$\hat x \cdot \hat x = \widehat{x^2} = \widehat{\cancel{x^2+x+1} + x + 1}$$

$$\hat x \cdot \widehat{x+\bar 1} = \widehat{x^2+x} = \widehat{\cancel{x^2+x}+\bar 1+\bar 1}$$ funcție bijectivă

$$\widehat{\bar 1+x} \cdot \widehat{\bar 1+x} = \widehat{\bar 1+x^2} = \bar 1 + \widehat{\cancel{x+x^2}+x}$$

$$\frac{\mathbb{Z}_2[x]}{(x^2+x)}$$

$I=(x)$, $J=(x+\hat 1)$

$x + (x+\hat 1) = \hat 1$ $\Rightarrow$ I, J comaximale

Lema Chineză $\Rightarrow$ $\dfrac{\mathbb{Z}_2[X]}{(X^2+X)} \simeq \dfrac{\mathbb{Z}_2[X]}{(X)} \times \dfrac{\mathbb{Z}_2[X]}{(X+\hat 1)} = \mathbb{Z}_2 \times \mathbb{Z}_2$, nu este corp pt. că

$(\hat 1, \hat 0)\cdot(\hat 0, \hat 1) = (\hat 0, \hat 0)$

$\dfrac{\mathbb{Z}_2[X]}{(X^2+\hat{1})}$. Arătați că are 4 elemente, nu este corp, nu este $\simeq \mathbb{Z}_2\times\mathbb{Z}_2$

$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$

$= \{\widehat{a+bX} \mid a, b \in \mathbb{Z}_2\}$

$= \{\hat{0}, \hat{1}, \hat{X}, \widehat{\bar{1}+X}\}$

$\hat{X}^2 = \hat{X^2} = \cancel{\widehat{X^2+\bar{1}}} + \hat{\bar{1}}$

$\widehat{\bar{1}+X}^2 = \widehat{\bar{1}+X^2} = \hat{0}$

$u = \widehat{\bar{1}+X}$ $\quad u^2 = 0$

$\in R$ $\quad u \neq 0$

$u \neq 0 \Rightarrow u$ inv.

$u^2 = 0 \overset{u^{-1}}{\Longrightarrow} u = 0$ Deci nu este corp.

În $\mathbb{Z}_2\times\mathbb{Z}_2$ ec. $u^2=0$ are doar o soluție: $(\bar{0},\bar{0})$

$(\alpha,\beta)^2 = (\alpha^2, \beta^2) = (\hat{0}, \hat{0})$

$\alpha = \beta = \hat{0}$

În R ecuația $u^2=0$ are 2 soluții: $\hat{0}$, $\widehat{\bar{1}+X}$

sau

ec. $u^2 = 1$.

În $\mathbb{Z}_2\times\mathbb{Z}_2$:

o soluție $(\hat{1},\hat{1})$

În $\mathbb{R}$:

$\hat{1}$, $\hat{X}$

două soluții

6 / Probleme-tip

$K = \{ a + b\sqrt{3} \mid a, b \in \mathbb{Q} \}$

(1) · K subcorp al lui $\mathbb{R}$

(2) · morfismele de corpuri $f: K \to K$

$$\left.\begin{array}{l} 1 \in K \\ x, y \in K \Rightarrow x - y,\ xy \in K \\ x \in K \Rightarrow x^{-1} \in K \end{array}\right\} \text{pentru a arăta că } K \text{ este subcorp}$$

$x = a + b\sqrt{3}$

$y = c + d\sqrt{3}$

$a, b, c, d \in \mathbb{Q}$

$$x - y = (\underbrace{a - c}_{\in \mathbb{Q}}) + (\underbrace{b - d}_{\in \mathbb{Q}})\sqrt{3} \in K$$

$$xy = (\underbrace{ac + 3bd}_{\in \mathbb{Q}}) + (\underbrace{ad + bc}_{\in \mathbb{Q}})\sqrt{3} \in K$$

$0 \neq x = a + b\sqrt{3} \in K$

$a + b\sqrt{3} = 0 \Rightarrow b\sqrt{3} = -a$

Dacă $b \neq 0 \Rightarrow \sqrt{3} = -\frac{a}{b} \in \mathbb{Q}$ contradicție

Deci $b = 0 \Rightarrow a = 0$

$$x^{-1} = \frac{1}{a+b\sqrt{3}} = \frac{a-b\sqrt{3}}{(a-b\sqrt{3})(a+b\sqrt{3})}$$

$$= \frac{a-b\sqrt{3}}{a^2-3b^2} = \frac{a}{a^2-3b^2} - \frac{b}{a^2-3b^2}\sqrt{3}$$

$\in K$

De ce $\sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$?

Presupunem prin absurd că $\sqrt{3} \in \mathbb{Q}$.

$\sqrt{3} = \frac{a}{b}$ cu $a, b \in \mathbb{N}^*$ și $(a, b) = 1$

$\Downarrow$

$$3 = \frac{a^2}{b^2} \Rightarrow a^2 = 3b^2 \Rightarrow 3 \mid a^2 \Rightarrow 3 \mid a$$

$a = 3x$

$\Downarrow$

$9x^2 = 3b^2$

$\Downarrow$

$3x^2 = b^2$

$\Downarrow$

$3 \mid b^2 \Rightarrow 3 \mid b$ Contradicție!

$3 \mid (a, b)$

$a^2 = 3b^2$

$a = 3^{\alpha} \dots$

$b = 3^{\beta} \dots$

$3^{2\alpha} \dots = \underbrace{3 \cdot 3^{2\beta}}_{= 3^{2\beta+1}} \dots$

Morfismele de corpuri?

$f : K \to K$

$f(x+y) = f(x) + f(y)$ $(\forall)\ x, y \in K$

$f(xy) = f(x) f(y)$

$f(1) = 1$

$\Rightarrow f(x_1 + \dots + x_n) = f(x_1) + \dots + f(x_n)$

$x_1 = \dots = x_n = 1$

$\Downarrow$

$f(n) = n f(1) = n$

$(\forall)\, n \in \mathbb{N}^*$

$f(0) = 0$

$f(-n) = -f(n) =$

$= -n$

$f(x) = x, (\forall)\ x \in \mathbb{Z}$

$$f\Big(\underbrace{\frac{m}{n} + \dots + \frac{m}{n}}_{n \text{ ori}}\Big) = \underbrace{f\Big(\frac{m}{n}\Big) + \dots + f\Big(\frac{m}{n}\Big)}_{n \text{ ori}}$$

$\parallel$

$f(m) = n f\left(\frac{m}{n}\right)$

$\parallel$

m

$\Downarrow$

$f\left(\frac{m}{n}\right) = \frac{m}{n}$

$f(x) = x, (\forall)\ x \in \mathbb{Q}$

$$f(a+b\sqrt{3}) = f(a) + f(b\sqrt{3})$$

$$\underbrace{f(a)}_{a} + \underbrace{f(b)}_{b} f(\sqrt{3}) = a + b f(\sqrt{3})$$

$$\underbrace{f(\sqrt{3}\cdot\sqrt{3})}_{f(3) = 3} = f(\sqrt{3})\cdot f(\sqrt{3}) = (f(\sqrt{3}))^2$$

$$\Rightarrow f(\sqrt{3}) \in \{\sqrt{3}, -\sqrt{3}\}$$

Cazul 1. $f(\sqrt{3}) = \sqrt{3} \Rightarrow$

$$\Rightarrow f(a+b\sqrt{3}) = a+b\sqrt{3}$$

$$(\forall)\, a, b$$

$$\Downarrow$$

$$f = Id_K$$

Cazul 2. $f(\sqrt{3}) = -\sqrt{3}$

$$\Downarrow$$

$$f(a+b\sqrt{3}) = a-b\sqrt{3}$$

$$f(c+d\sqrt{3}) = c-d\sqrt{3}$$

$$f((a+b\sqrt{3})+(c+d\sqrt{3})) = f((a+c)+(b+d)\sqrt{3})$$

$$= (a+c)-(b+d)\sqrt{3}$$

$$= f(a+b\sqrt{3}) + f(c+d\sqrt{3})$$

$$f\big((a+b\sqrt{3})\cdot(c+d\sqrt{3})\big) \overset{?}{=} f(a+b\sqrt{3})\cdot f(c+d\sqrt{3})$$

$$f\big((a+b\sqrt{3})\cdot(c+d\sqrt{3})\big) = f\big((ac+3bd)+(ad+bc)\sqrt{3}\big) = (ac+3bd)-(ad+bc)\sqrt{3}$$

$$f(a+b\sqrt{3})\cdot f(c+d\sqrt{3}) = (a-b\sqrt{3})\cdot(c-d\sqrt{3}) = (ac+3bd)-(ad+bc)\sqrt{3}$$