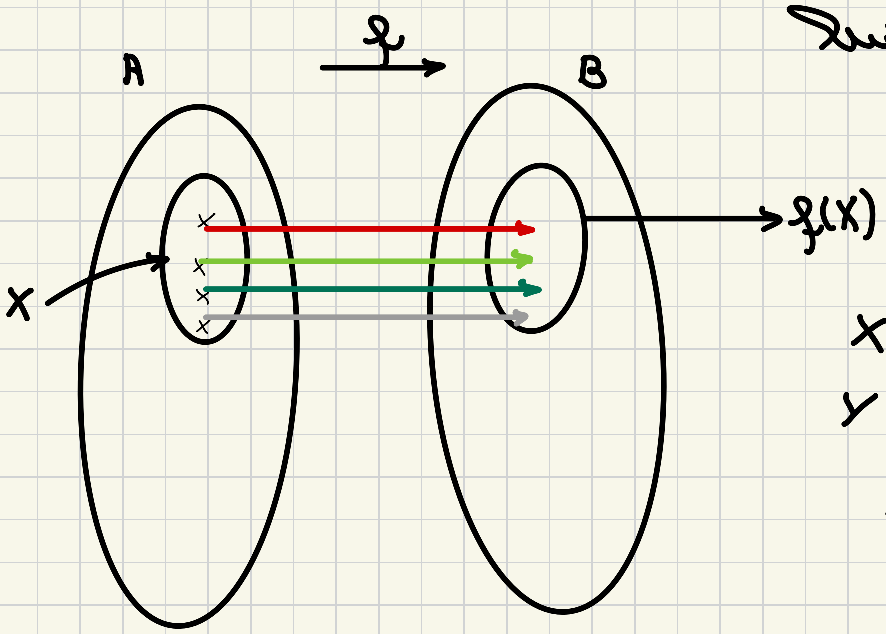
1. $f: A \to B$

$X \subset A$

$f(X) \overset{def.}{=\!=} \{f(x) \mid x \in X\}$ imaginea lui X prin f $(\subset B)$

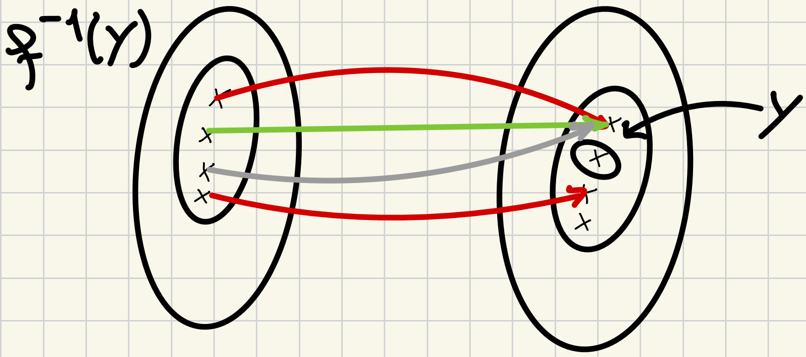
$X = A$ $\quad f(A) = \operatorname{Im}(f)$ imaginea lui f

$Y \subset B$ $\quad f^{-1}(Y) \overset{def.}{=\!=} \{x \in A \mid f(x) \in Y\}$ preimaginea (imaginea inversată) $\subset A$ lui Y prin f



$X \neq \emptyset \Rightarrow f(X) \neq \emptyset$

$Y \neq \emptyset$ se poate ca $f^{-1}(Y) = \emptyset$



$f: A \to B$ funcție

(i) $X_1 \subset X_2 \subset A \Rightarrow f(X_1) \subset f(X_2)$

Fie $y \in f(X_1) \Rightarrow (\exists)\, x \in X_1$ cu $y = f(x)$

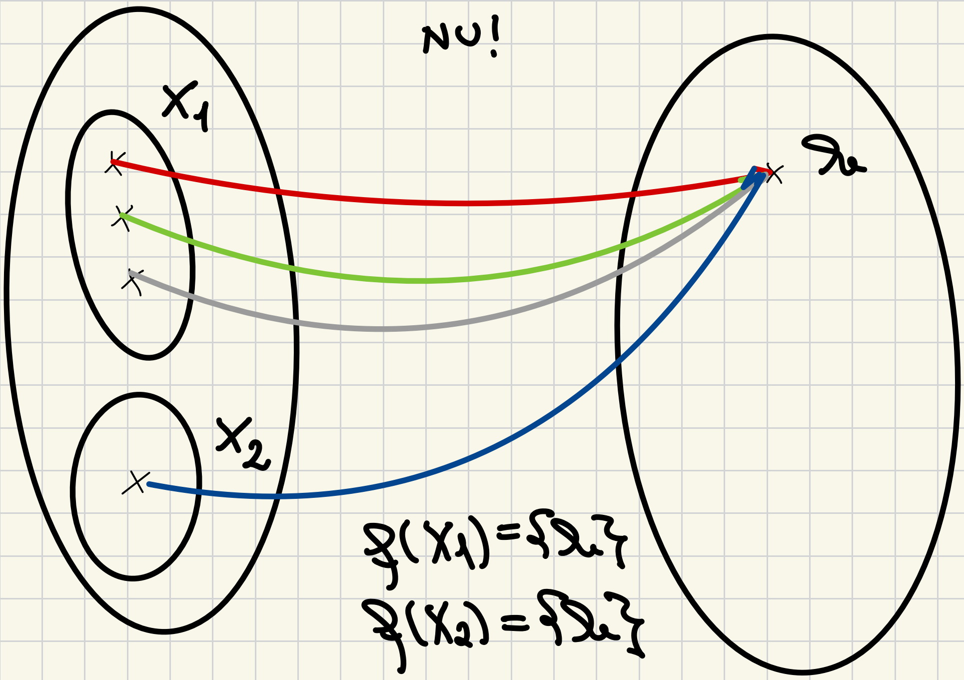
$X_1 \subset X_2 \Rightarrow x \in X_2 \Rightarrow$

$\Rightarrow y = f(x) \in f(X_2)$

(ii) $X_1, X_2 \subset A$

$f(X_1) \subset f(X_2) \overset{?}{\Rightarrow} X_1 \subset X_2$

NU!



$f(X_1) = \{x\}$
$f(X_2) = \{x\}$

(ii) $Y_1 \subset Y_2 \subset B \Rightarrow f^{-1}(Y_1) \subset f^{-1}(Y_2)$

Fie $x \in f^{-1}(Y_1) \Rightarrow \left.\begin{array}{l} f(x) \in Y_1 \\ Y_1 \subset Y_2 \end{array}\right| \Rightarrow f(x) \in Y_2$

$f(x) \in Y_2 \Rightarrow x \in f^{-1}(Y_2)$

(iii) $X_1, X_2 \subset A \Rightarrow f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$

„$\subset$": Fie $y \in f(X_1 \cup X_2) \Rightarrow (\exists)\ x \in X_1 \cup X_2$ cu $y = f(x)$

$$\left.\begin{array}{l} x \in X_1 \cup X_2 \Rightarrow x \in X_1 \text{ sau } x \in X_2 \\ \qquad\qquad\qquad\quad \Downarrow \qquad\qquad\qquad \Downarrow \\ \qquad\qquad y = f(x) \in f(X_1) \quad y = f(x) \in f(X_2) \end{array}\right\} \Rightarrow$$

$\Rightarrow y \in f(X_1) \cup f(X_2)$

„$\supset$": Fie $y \in f(X_1) \cup f(X_2) \Rightarrow y \in f(X_1)$ sau $y \in f(X_2)$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$(\exists)\, x \in X_1$ cu $y = f(x)$ $\quad$ $(\exists)\, x \in X_2$ cu $y = f(x)$

$\Downarrow \qquad \Leftarrow$

$(\exists)\, x \in X_1 \cup X_2$ cu $y = f(x)$

$\Downarrow$

$\underline{y \in f(X_1 \cup X_2)}$

sau

$$\left.\begin{array}{l} X_1 \subset X_1 \cup X_2 \Rightarrow f(X_1) \subset f(X_1 \cup X_2) \\ X_2 \subset X_1 \cup X_2 \Rightarrow f(X_2) \subset f(X_1 \cup X_2) \end{array}\right\} \Rightarrow f(X_1) \cup f(X_2) \subset f(X_1 \cup X_2)$$

(iii) $Y_1, Y_2 \subset B \Rightarrow f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$

Fie $x \in f^{-1}(Y_1 \cup Y_2) \Rightarrow f(x) \in Y_1 \cup Y_2 \Rightarrow$

$$\left.\begin{array}{l} \Rightarrow f(x) \in Y_1 \Rightarrow x \in f^{-1}(Y_1) \\ \quad\text{SAU} \qquad\qquad\qquad \text{SAU} \\ f(x) \in Y_2 \Rightarrow x \in f^{-1}(Y_2) \end{array}\right\} \Rightarrow x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$$

$$\left.\begin{array}{l} Y_1 \subset Y_1 \cup Y_2 \Rightarrow f^{-1}(Y_1) \subset f^{-1}(Y_1 \cup Y_2) \\ Y_2 \subset Y_1 \cup Y_2 \Rightarrow f^{-1}(Y_2) \subset f^{-1}(Y_1 \cup Y_2) \end{array}\right\} \Rightarrow$$

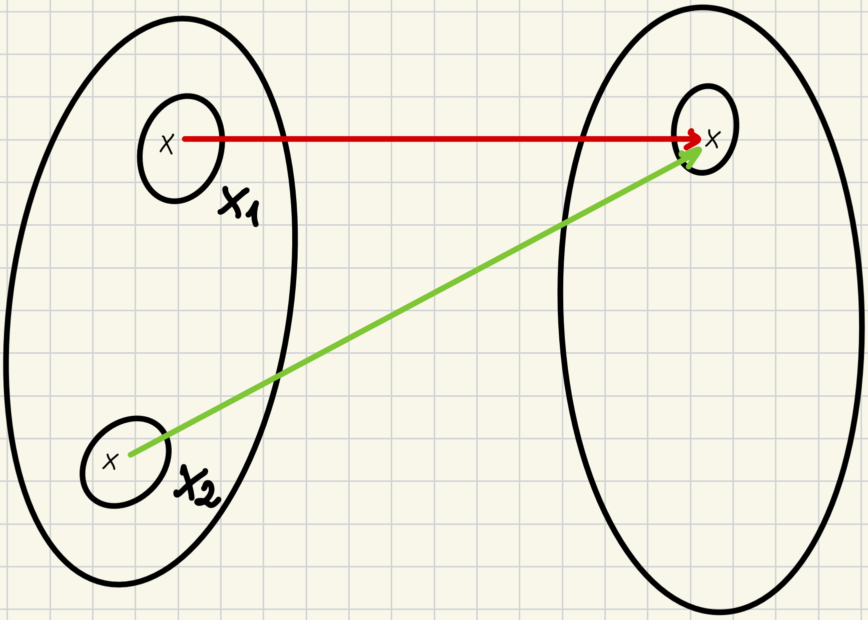
$\Rightarrow f^{-1}(Y_1) \cup f^{-1}(Y_2) \subset f^{-1}(Y_1 \cup Y_2)$

(iv) $X_1, X_2 \subset A \Rightarrow f(X_1 \cap X_2) \subset f(X_1) \cap f(X_2)$

$$\left.\begin{array}{l} X_1 \cap X_2 \subset X_1 \overset{i)}{\Rightarrow} f(X_1 \cap X_2) \subset f(X_1) \\ X_1 \cap X_2 \subset X_2 \Rightarrow f(X_1 \cap X_2) \subset f(X_2) \end{array}\right| \Rightarrow f(X_1 \cap X_2) \subset f(X_1) \cap f(X_2)$$

Este egalitate?

NU!



Arătați că, dacă f este injectivă, avem $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$.

„$\supset$":

Fie $y \in f(X_1) \cap f(X_2) \Rightarrow y \in f(X_1)$ și $y \in f(X_2)$

$$\left.\begin{array}{l} \Rightarrow (\exists)\ x_1 \in X_1 \text{ cu } y = f(x_1) \text{ și} \\ (\exists)\ x_2 \in X_2 \text{ cu } y = f(x_2) \end{array}\right\} \xRightarrow{f \text{ inj.}} x_1 = x_2 \in X_1 \cap X_2$$

$$\Rightarrow y = f(x_1) \in f(X_1 \cap X_2)$$

(vi) $Y_1, Y_2 \subset B \Rightarrow f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$

„$\subset$": $\left.\begin{array}{l} Y_1 \cap Y_2 \subset Y_1 \Rightarrow f^{-1}(Y_1 \cap Y_2) \subset f^{-1}(Y_1) \\ Y_1 \cap Y_2 \subset Y_2 \Rightarrow f^{-1}(Y_1 \cap Y_2) \subset f^{-1}(Y_2) \end{array}\right\} \Rightarrow$

$f^{-1}(Y_1 \cap Y_2) \subset f^{-1}(Y_1) \cap f^{-1}(Y_2)$

Fie $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$

$\Rightarrow x \in f^{-1}(Y_1)$ și $x \in f^{-1}(Y_2)$

$\Rightarrow f(x) \in Y_1$ și $f(x) \in Y_2$

$\Rightarrow f(x) \in Y_1 \cap Y_2 \Rightarrow x \in f^{-1}(Y_1 \cap Y_2)$

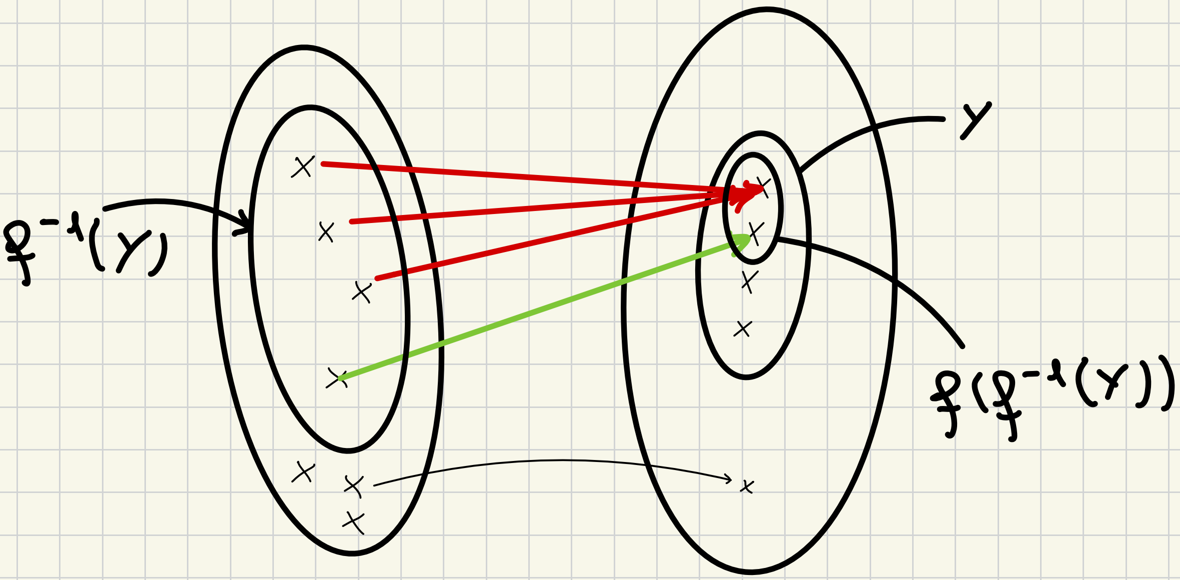
(vii) $Y \subset B \Rightarrow f(f^{-1}(Y)) \subset Y$

În general, $\neq$.

Dacă f este surjectivă, avem $=$.

Fie $u \in f(f^{-1}(Y))$.

$\left.\begin{array}{l} \Rightarrow (\exists)\, x \in f^{-1}(Y) \text{ a.î. } u = f(x) \\ \text{Dar } x \in f^{-1}(Y) \Rightarrow f(x) \in Y \end{array}\right\} \Rightarrow u \in Y$



Fie f surjectivă.

Fie $y \in Y$.

$(\exists)\, x \in A$ cu $y = f(x) \Rightarrow f(x) \in Y \Rightarrow$

$\Rightarrow x \in f^{-1}(Y)$

Atunci $y = f(x) \in f(f^{-1}(Y))$

(viii) $X \subset A \Rightarrow X \subset f^{-1}(f(X))$

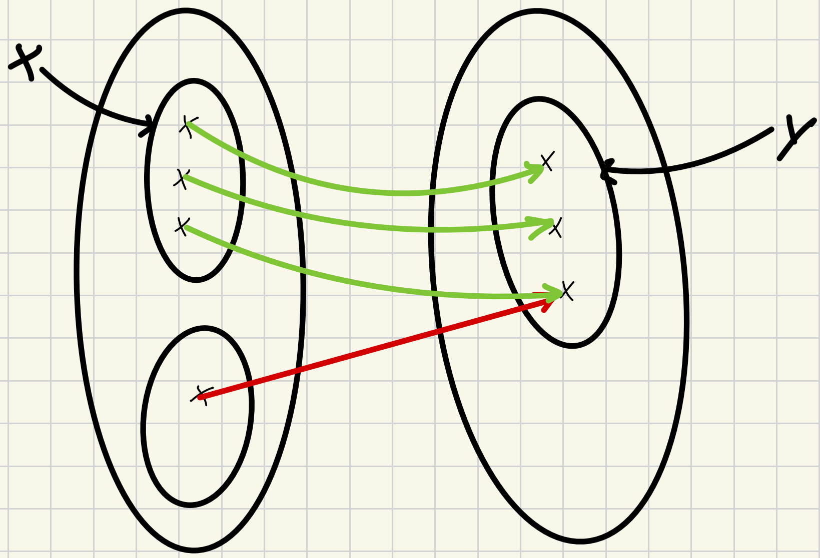
În general, $\neq$.

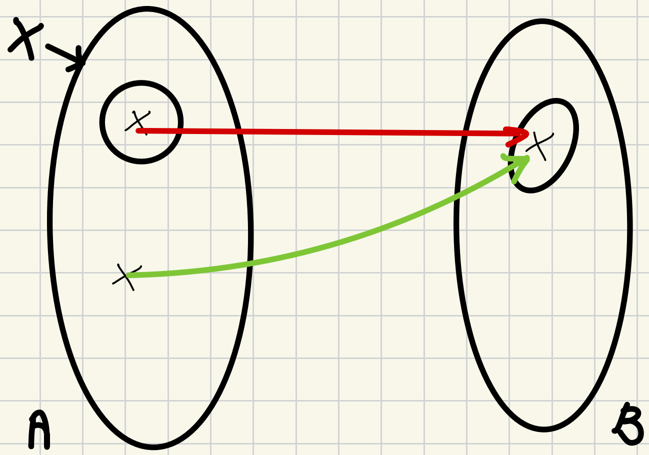
Dacă f injectivă, avem $=$.

Fie $x \in X$.

Atunci $f(x) \in f(X)$.

$\Rightarrow x \in f^{-1}(f(X))$





$$f^{-1}(\underbrace{f(X)}_{B}) = A \underset{\neq}{\supset} X$$

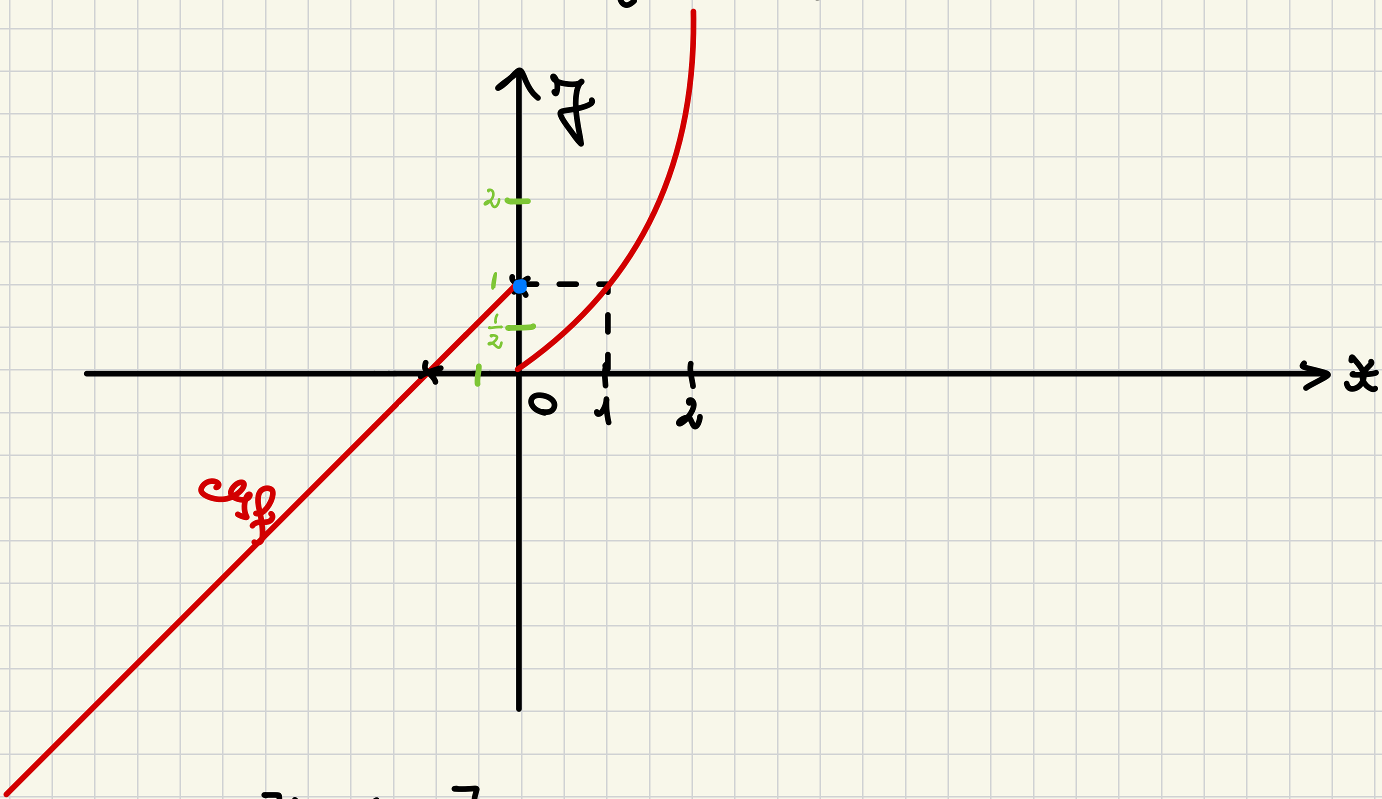
Presupunem că f este injectivă.

$u \in f^{-1}(f(X)) \Rightarrow f(u) \in f(X) \Rightarrow (\exists)\, x \in X$ a.î. $f(u) = f(x)$

Cum f este injectivă $\Rightarrow u = x \in X$

$\Rightarrow u \in X$

2. Fie $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} x+1, & x\le 0 \\ x^2, & x>0 \end{cases}$.



$f([0,1])=(0,1]$

$f([0,\frac{1}{2}])=(0,\frac{1}{4}]\cup\{1\}$

$\operatorname{Im}(f)=\mathbb{R}$

$f^{-1}([0,+\infty))=[-1,+\infty)$

$f^{-1}((0,1))=(-1,1)\setminus\{0\}$

$f^{-1}([\frac{1}{2},2])=[-\frac{1}{2},0]\cup[\frac{\sqrt{2}}{2},\sqrt{2}]$

3. $\mathbb{R}, \sim$

$x \sim y \Leftrightarrow x - y \in \mathbb{Z}$

$\sim$ rel. de echivalență

$\widehat{\frac{2025}{2}} = ?$

1) Reflexivitate:

$a \sim a \Leftrightarrow a - a \in \mathbb{Z}$

$0 \in \mathbb{Z}$ (A)

2) Simetrie:

$a \sim b \overset{?}{\Rightarrow} b \sim a$

$\Downarrow$

$a - b \in \mathbb{Z} \Rightarrow -(a-b) \in \mathbb{Z} \Rightarrow b - a \in \mathbb{Z} \Rightarrow b \sim a$

3) Tranzitivitate:

$a \sim b, b \sim c \overset{?}{\Rightarrow} a \sim c$

$$\left.\begin{array}{l} a - b \in \mathbb{Z} \\ b - c \in \mathbb{Z} \end{array}\right\} \Rightarrow a - \cancel{b} + \cancel{b} - c \in \mathbb{Z}$$

$a - c \in \mathbb{Z} \Rightarrow a \sim c$

$\Rightarrow$ Relație de echivalență

$\widehat{\frac{2025}{2}} = \left\{ a \in \mathbb{R} \mid a \sim \frac{2025}{2} \right\}$

$a \sim \frac{2025}{2} \Leftrightarrow a - \frac{2025}{2} \in \mathbb{Z}$

$a - 1012 - \frac{1}{2} \in \mathbb{Z} \Leftrightarrow a - \frac{1}{2} \in \mathbb{Z}, a - \frac{1}{2} = m, m \in \mathbb{Z} \Rightarrow$

$\Rightarrow a = m + \frac{1}{2}, m \in \mathbb{Z}$

Deci $\frac{2025}{2} = m + \frac{1}{2}, m \in \mathbb{Z}$