1. $S_4 = \{ \sigma \mid \sigma : \{1,2,3,4\} \to \{1,2,3,4\}$ bijectivă $\}$

$|S_4| = 4! = 24$

- Cine este $\{ o(\sigma) \mid \sigma \in S_4 \}$?

Pentru fiecare k, determinați câte elemente $\sigma \in S_4$ au ordin k.

$\sigma \in S_4 \Rightarrow \sigma =$ produs de cicluri disjuncte

- Dacă $\sigma = e \Rightarrow o(\sigma) = 1$
- Dacă σ este un ciclu $\Rightarrow$

a) $\sigma = (i\ j)$, $i \neq j \Rightarrow o(\sigma) = 2$

b) $\sigma = (i\ j\ k)$, $i \neq j \neq k \Rightarrow o(\sigma) = 3$

c) $\sigma = (i\ j\ k\ l)$, $i \neq j \neq k \neq l$

$\Rightarrow o(\sigma) = 4$

a) $C_4^2 = 6$

b) $\frac{A_4^3}{3} = 8$

$((i\ j\ k) = (j\ k\ i) = (k\ i\ j))$

$\neq$

$(i\ k\ j)$

$4 \cdot 2 = 8$

c) $\frac{4!}{4} = 3! = 6$

$(i\ j\ k\ l) = (j\ k\ l\ i) = (k\ l\ i\ j) = (l\ i\ j\ k$

$(\ 1\ \ *\ \ *\ \ *\ )$

$3\ \ 2$

- Dacă σ este produs de două cicluri disjuncte?

$(i\ \ j)(k\ \ l)$, $i \neq j \neq k \neq l$

τ_1, τ_2 cicluri distincte $\Rightarrow \tau_1\tau_2 = \tau_2\tau_1$

$$(\tau_1\tau_2) = \tau_1\tau_2\tau_1\tau_2 = \tau_1^2\tau_2^2$$

$\Rightarrow o(\sigma) = 2$

$(1\ \ 2)(3\ \ 4)$, $(1\ \ 3)(2\ \ 4)$, $(1\ \ 4)(2\ \ 3)$

$\Rightarrow$ ordinele posibile sunt $\{1, 2, 3, 4\}$.

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 9 & 8 & 6 \end{array}$$

- Să se determine toate $\sigma \in S_4$ pentru care $\sigma^2 = (1\ \ 2)(3\ \ 4)$

$o(\sigma^2) = 2 \Rightarrow \boxed{o(\sigma) = 4}$

$(\sigma^2)^2 = e \Rightarrow \sigma^4 = e \Rightarrow o(\sigma) \in \{\cancel{1}, 2, 4\}$

$\sigma = (1\ \ a\ \ b\ \ c)$

$$(1\ \ 2)(3\ \ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$\sigma^2 = (1\ \ a\ \ b\ \ c)(1\ \ a\ \ b\ \ c) =$

$$\begin{pmatrix} 1 & a & b & c \\ b & c & 1 & a \end{pmatrix} = (1\ \ b)(a\ \ c)$$

$\sigma^2 = (1\ \ 2)(3\ \ 4) \Leftrightarrow b = 2$ şi $\{a, c\} = \{3, 4\}$

Pentru $a=3$ şi $c=4$:

$\sigma=(1\ 3\ 2\ 4)$

Pentru $a=4$ şi $c=3$:

$\sigma=(1\ 4\ 2\ 3)$

• Să se rezolve ecuaţia $\tau^2=(1\ 2)$ în S_4.

$$\Rightarrow \underbrace{\varepsilon(\tau^2)}_{\substack{\parallel \\ \varepsilon(\tau)\varepsilon(\tau) \\ \parallel \\ 1}} = \underbrace{\varepsilon((1\ 2))}_{\substack{\parallel \\ -1}}$$

Ecuaţia nu are soluţii!

2. În S_n, $\sigma=\tau_1\dots\tau_r$ $\quad \underset{\text{lung. } m_1}{\overset{\text{orbitele } O_1}{\tau_1}},\dots,\underset{m_r}{\overset{O_r}{\tau_r}}$ cicluri disjuncte

$\Downarrow$

$o(\sigma)=[m_1,\dots,m_r]$ $\qquad \tau=(i_1\ \dots\ i_m)$

orbite $\{i_1,\dots,i_m\}$

$O_i\cap O_j=\emptyset$, $(\forall)\ i\neq j$

$\underline{x\in O_k}$ $\qquad \tau_i(x)=x$, $(\forall)\ i\neq k$

$(\tau_1\dots\tau_r)(x)=\tau_k(x)$

$(\tau_1\dots\tau_r)^2(x)=\tau_k^2(x)$

- - - - - - - - - - - - - - - - - - -

$(\tau_1\dots\tau_r)^{m_k}(x)=\dots(x)$

Pentru $x \in O_k$, $h \in \mathbb{N}^*$,

$\sigma^h(x) = x \iff m_k \mid h$

Dacă $x \notin O_1 \cup \ldots \cup O_r \Rightarrow \sigma(x) = x$

$\sigma^h = e \iff (\forall)\, x,\ \sigma^h(x) = x \iff (\forall)\, k$

$$\underbrace{(\forall)\, x \in O_k \quad \sigma^h(x) = x}_{\Updownarrow \; m_k \mid h} \iff \begin{cases} m_1 \mid h \\ \vdots \\ m_r \mid h \end{cases} \iff$$

$\iff [m_1, \ldots, m_r] \mid h$

$\Rightarrow o(\sigma) = [m_1, \ldots, m_r]$

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$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 6 & 9 & 11 & 10 & 2 & 5 & 8 & 12 & 1 & 4 & 3 \end{pmatrix} \in S_{12}$$

a) Să se scrie σ ca produs de cicluri disjuncte

b) Să se scrie σ ca produs de transpoziții

c) $o(\sigma) = ?$ $\quad \varepsilon(\sigma) = ?$

d) $\sigma^{1000} = ?$

a) $\sigma = (1\ 7\ \overset{-1}{5}\ 10)(2\ \overset{-1}{6})(3\ 9\ 12)(4\ \overset{-1}{11})$

b) $(i_1\ i_2\ \dots\ i_m) = (i_1\ i_2)(i_2\ i_3)\dots(i_{m-1}\ i_m)$

$$\cancel{=}$$

$(i_1\ i_m)(i_1\ i_{m-1})\dots(i_1\ i_2)$

$\sigma = (1\ 7)(7\ 5)(5\ 10)(2\ 6)(3\ 9)(9\ 12)(4\ 11)$

c) $\varepsilon(\sigma) = (-1)^7 = -1$

$\|$

$(-1)^{\mathrm{Inv}(\sigma)}$

$\mathrm{o}(\sigma) = [4, 2, 3] = 12$

d) $\sigma^{1000} = (\sigma^{12})^{83} \cdot \sigma^4 = \sigma^4$

$\sigma^4 = \underset{\substack{\| \\ e}}{(1\ 7\ 5\ 10)^4} \cdot \underset{\substack{\| \\ e}}{(2\ 6)^4} (3\ 9\ 12)^4 \underset{\substack{\| \\ e}}{(4\ 11)^4} =$

$= (3\ 9\ 12)$

Fie $1 \le k \le m$

Fie $k = 1$ $\quad \sigma(i_1) = i_2$

$\tau(i_1) = i_2$

Fie $k = 2$ $\quad \sigma(i_2) = i_3$

$\tau(i_2) = i_3$

Fie $1 \le k < m$

$\sigma(i_k) = i_{k+1}$
$\tau(i_k) = i_{k+1}$

Fie $k = m$

$\sigma(i_m) = i_1$
$\tau(i_m) = i_1$

Fie $i \neq i_1 \dots i_m$

$\sigma(i) = i$
$\tau(i) = i$

$\Rightarrow \sigma = \tau$

4. $f: R \to S$ morfism de inele

$f(x+y) = f(x) + f(y)$ $\quad (\forall)\ x, y \in R$
$f(xy) = f(x) \cdot f(y)$
$f(1) = 1$

Determinați toate morfismele de inele:

a) $f: \mathbb{Z} \to \mathbb{Z}$
b) $f: \mathbb{Q} \to \mathbb{Q}$
c) $f: \mathbb{R} \to \mathbb{R}$

Dacă $f: (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ morfism de grupuri

$f(1) = m \in \mathbb{Z}$
$\Downarrow$

$f(a) = am$, $(\forall)\, a \in \mathbb{Z}$

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f morfism de inele $\Rightarrow f(1) = 1 \Rightarrow f(a) = a$, $(\forall)\, a \in \mathbb{Z}$

$\Downarrow$

$f = Id$

Pentru $\mathbb{Q}$:

$f : (\mathbb{Q}, +) \to (\mathbb{Q}, +)$ morfism de grupuri

$f(1) = q \in \mathbb{Q}$

$\Downarrow$

$(\forall)\, m \in \mathbb{Z}$ $\quad f(m) = mq$

$\Downarrow$

$(\forall)\, \frac{m}{n} \in \mathbb{Q}$

$$\underbrace{f\left(\frac{m}{n} + \ldots + \frac{m}{n}\right)}_{n \text{ ori}} = \underbrace{f\left(\frac{m}{n}\right) + \ldots + f\left(\frac{m}{n}\right)}_{n}$$

$\Downarrow$

$$f(m) = \boxed{mq = n f\left(\frac{m}{n}\right)}$$

$\Downarrow$

$$f\left(\frac{m}{n}\right) = q\frac{m}{n}$$

$q \in \mathbb{Q}$

$f : \mathbb{Q} \to \mathbb{Q}$

$f(x) = qx$

f morfism de inele $\Rightarrow q = 1 \Rightarrow \boxed{f = Id}$

Pentru $\mathbb{R}$:

$f: \mathbb{R} \to \mathbb{R}$ morfism de inele

$f(1) = 1$

$\Downarrow$ calcule

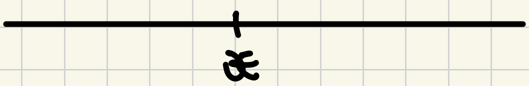
$f(x) = x, \ (\forall)\, x \in \mathbb{Q}$

$x, y \in \mathbb{R}, \ x < y$

$\Rightarrow f(y) - f(x) = f(y - x) = f(\sqrt{y-x} \cdot \sqrt{y-x})$

$= f(\sqrt{y-x}) \cdot f(\sqrt{y-x}) \geq 0$

$\Rightarrow f(x) \leq f(y)$

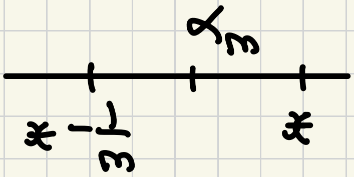


$(\exists)\ (\alpha_n)_n \subset \mathbb{Q}$

$\alpha_n < x$

$\Rightarrow \lim \alpha_n = x$

Pentru fiecare $n \in \mathbb{N}$ iau un



$\alpha_n \in \mathbb{Q} \cap (x - \frac{1}{n}, x)$

$(\alpha_n)_n \subset \mathbb{Q}$

$x - \frac{1}{n} < \alpha_n < x \quad (\forall)\, n$

$\searrow$ crit $\swarrow$

x

$(\exists)\ (\beta_n)_n \subset \mathbb{Q}$

$x < \beta_n$

$\lim \beta_n = x$

$\alpha_n < x < \beta_n$

$\Downarrow f$

$$\underbrace{f(\alpha_n)}_{\alpha_n} \le f(x) \le \underbrace{f(\beta_n)}_{\beta_n}$$

$$\alpha_n \to x \leftarrow \beta_n$$

$\Rightarrow f(x) = x$