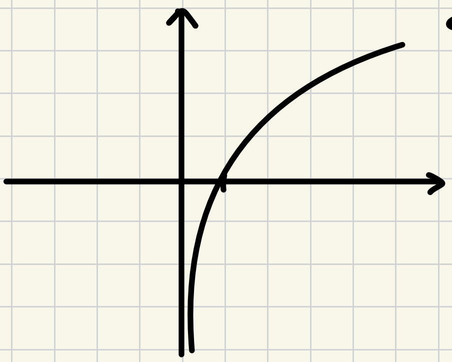
1. $\mathbb{R}$

$(0,1) \nsim \mathbb{N}$

$\mathbb{R}$ numărabilă $\Rightarrow (0,1)$ numărabilă, Contradicție!

Să se construiască (explicit) o funcție bijectivă $f:(0,1)\to\mathbb{R}$.



$\ln : (0,+\infty) \to \mathbb{R}$

bij.: $\mathbb{R} \sim (0,+\infty)$

$\ln_{|(0,1)} : (0,1) \to (-\infty, 0)$

$(0,1) \sim (-\infty,0)$

$$(0,1) \xrightarrow[\sim]{u} (-\infty,0) \xrightarrow[\sim]{p} (0,+\infty) \xrightarrow[\sim]{v} \mathbb{R}$$

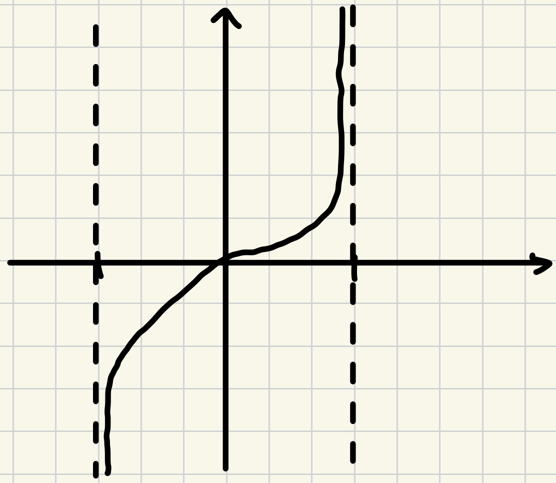
$p(x) = -x \quad v(x) = \ln x$

$u(x) = \ln x$

Deci $f:(0,1)\to\mathbb{R}$

$f = v \circ p \circ u$

$$\begin{aligned} f(x) &= (v\circ p\circ u)(x) \\ &= (v\circ p)(\ln x) \\ &= v(-\ln x) \\ &= \ln(-\ln x) \end{aligned}$$



$h(x) = tg\, x$

$h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

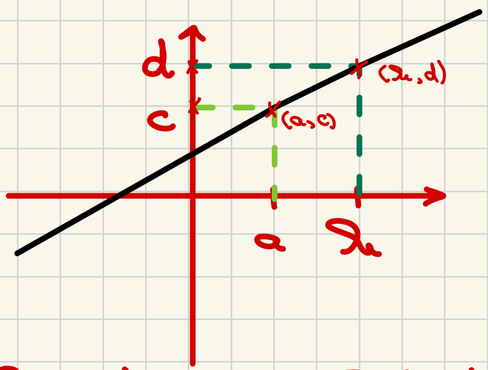
bij.

$(0,1) \xrightarrow{g} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow[2]{h} \mathbb{R}$

pt. $a=0, b=1, c=-\frac{\pi}{2}, d=\frac{\pi}{2}$

$a<b, \ c<d$

$[a,b] \qquad [c,d]$



$m = c - \frac{d-c}{b-a} \cdot a$

$m = \frac{cb - \cancel{ca} - da + \cancel{ca}}{b-a}$

Caut $m, n \in \mathbb{R}$ $(m \neq 0)$

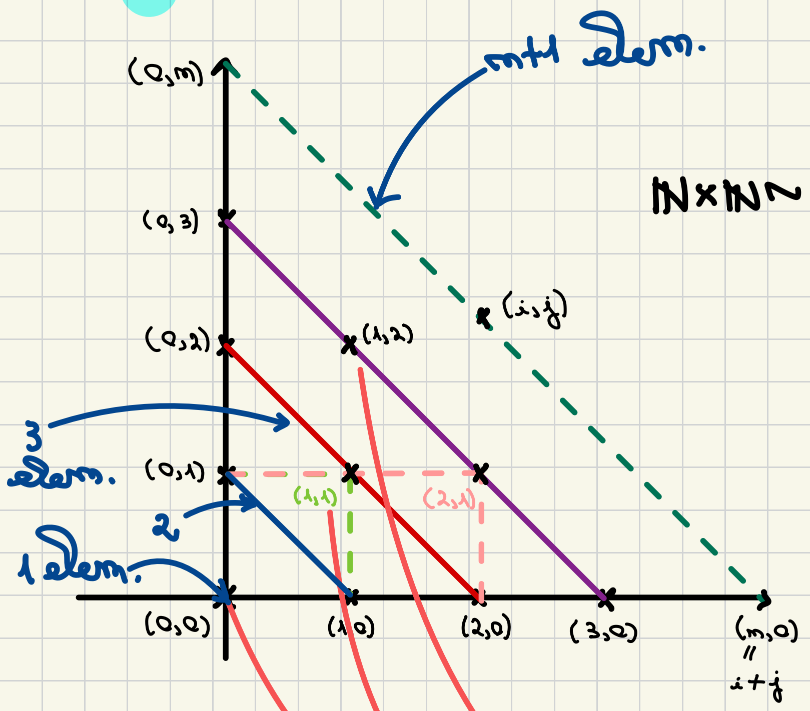
$\begin{cases} ma + n = c \\ mb + n = d \end{cases} \Leftrightarrow \quad m = \frac{d-c}{b-a}; \quad n = \frac{cb-da}{b-a}$

Dacă există $g : [a,b] \to [c,d]$, $g(x) = mx + n$ bijectivă

$g : (a,b) \to (c,d)$

bij. $\quad g(x) = \dfrac{d-c}{b-a} \cdot x + \dfrac{cd-da}{b-a}$

2.



$\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

$(0,0),\ \underbrace{(0,1),(1,0)},\ \underbrace{(0,2),(1,1),(2,0)},\ \dots$

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$$f((i,j)) = \frac{(i+j)(i+j+1)}{2} + i$$

Arătați că f este bijectivă.

$m \in \mathbb{N}$

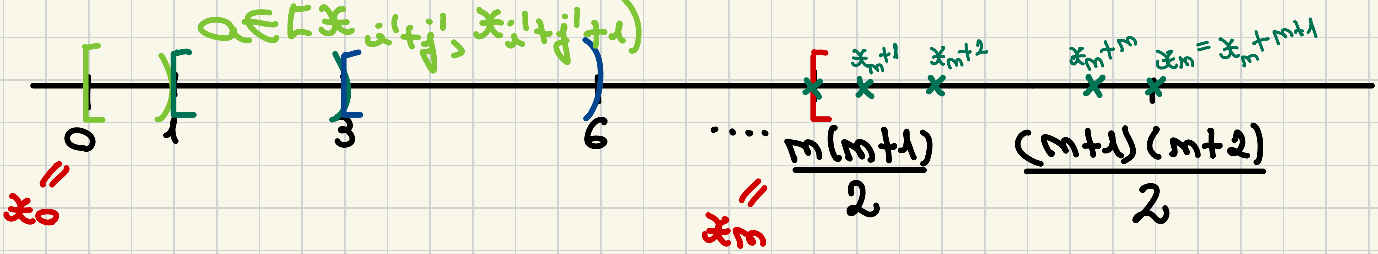
$\dfrac{m(m+1)}{2}$

$m=0 \to 0$

$m=1 \to 1$

$m=2 \to 3$

$m=3 \to 6$



$$\underbrace{\frac{(m+1)(m+2)}{2}}_{x_{m+1}-x_m} - \frac{m(m+1)}{2} = \frac{(m+1)(m+2-m)}{2} = m+1$$

$$0 = x_0 < x_1 < x_2 < \dots \ (\subset \mathbb{N})$$

$$x_{m+1} - x_m = m+1$$

Fie $a \in \mathbb{N}$. Arăt că:

$(\exists!)\ (i,j) \in \mathbb{N} \times \mathbb{N}$ cu $a = f((i,j))$

$$\mathbb{N} = \underbrace{\left([x_0, x_1) \cap \mathbb{N}\right)}_{=\{0\}} \cup \underbrace{\left([x_1, x_2) \cap \mathbb{N}\right)}_{=\{1,2\}} \cup \dots$$

reuniune disjunctă

$\{3,4,5\}$

$(\exists!)\ m$ cu $x_m \le a < x_{m+1} = x_m + m+1$

$$x_m = \frac{m(m+1)}{2}, \qquad x_{m+1} = \frac{m(m+1)}{2} + m+1$$

$$a = \frac{m(m+1)}{2} + \left[a - \frac{m(m+1)}{2}\right]$$

$$\begin{cases} i+j = m \\ i = a - \dfrac{m(m+1)}{2} \end{cases}$$

Clar $\underbrace{a - \frac{m(m+1)}{2}}_{} \ge 0$

i

$j = m - i \qquad i \le m \,?$

$a - \frac{m(m+1)}{2} \le m \iff a \le \frac{m(m+1)}{2} + m$, adevărat

$\Downarrow$ $i, j \in \mathbb{N}$

$a = f((i,j))$

Dacă $a = f((i',j'))$, $i', j' \in \mathbb{N}$

$$a = \frac{(i'+j')(i'+j'+1)}{2} + i' \le \frac{(i'+j')(i'+j'+1)}{2} + i' + j'$$

$$< \underbrace{\frac{(i'+j')(i'+j'+1)}{2} + i' + j' + 1}_{= x_{i'+j'+1}}$$

$$a \ge \underbrace{\frac{(i'+j')(i'+j'+1)}{2}}_{x_{i'+j'}}$$

$a \in [x_{i'+j'}, x_{i'+j'+1}) \Rightarrow i'+j' = m\,(= i+j)$

$$\boxed{\frac{(i+j)(i+j+1)}{2}} + i = a = \frac{(i'+j')(i'+j'+1)}{2} + i'$$

$\frac{m(m+1)}{2} \qquad \frac{m(m+1)}{2}$

$\Downarrow$

$i = i' \xRightarrow{i+j = i'+j'} j = j'$

Abordare geometrică:

Al câtelea element este (i,j)? (unde $i,j \in \mathbb{N}$)

(i,j) se găsește pe segmentul care unește $(0,m)$ cu $(m,0)$, unde $m=i+j$

Al câtelea este?

$$\underbrace{1+2+\dots+m}+i+1$$

$$= \frac{m(m+1)}{2}+i+1$$

$$= \frac{(i+j)(i+j+1)}{2}+i+1$$

$$\begin{array}{c|ccccccc}
\boxed{\mathbb{N}\times\mathbb{N}} & (0,0) & (0,1) & (1,0) & \dots\dots & (i,j) & \dots\dots \\
\downarrow & & & & & & \\
\mathbb{N}^* & 1 & 2 & 3 & & \frac{(i+j)(i+j+1)}{2}+i+1 & \dots\dots \\
\downarrow {\scriptstyle *-1} & \downarrow & \downarrow & \downarrow & & \downarrow & \\
\mathbb{N} & 0 & 1 & 2 & & \frac{(i+j)(i+j+1)}{2}+i &
\end{array}$$

3. **A, B numărabile $\Rightarrow A\cup B$ este numărabilă!**

Pentru orice $m \in \mathbb{N}^*$:

$A_1, \dots, A_m$ numărabile $\Rightarrow A_1 \cup \dots \cup A_m$ numărabilă

(inducție)

$A_0, A_1, A_2, \dots, A_m, \dots$ mulțimi numărabile $\Rightarrow \bigcup_{i\geq 0} A_i$ numărabilă

$A_0 = \{a_{00}, a_{01}, a_{02}, \dots\}$

$A_1 = \{a_{10}, a_{11}, a_{12}, \dots\}$

$A_i = \{a_{i0}, a_{i1}, \dots, a_{ij}, \dots\}$

$A_m = \{a_{m0}, a_{m1}, a_{m2}, \dots\}$

$f : \mathbb{N}\times\mathbb{N} \to \bigcup_{i\geq 0} A_i$

$f((i,j)) = a_{ij}$

f surjectivă

$(\exists)\ g : \underbrace{\bigcup_{i\geq 0} A_i}_{\text{infinită}} \to \underbrace{\mathbb{N}\times\mathbb{N}}_{\text{numărabilă}}$ injectivă $\Rightarrow \bigcup_{i\geq 0} A_i$ numărabilă

4. Pe $\mathbb{R}^*$ definim $x \rho y \overset{def.}{\Longleftrightarrow} xy > 0 \Rightarrow \rho$ relație de echivalență

Dem.:

1) Reflexivitate:

$x \rho x \Leftrightarrow x \cdot x > 0 \Leftrightarrow x^2 > 0$ (A)

2) Simetrie:

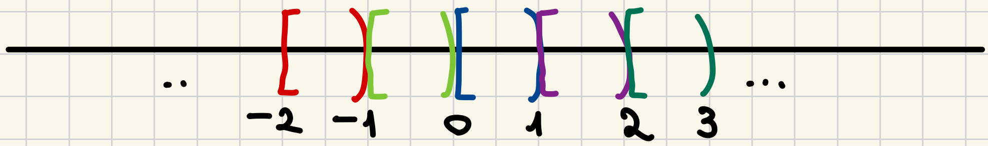
$$\begin{array}{ccc} x \rho y & \Rightarrow & y \rho x \\ \Downarrow & & \Uparrow \\ xy > 0 & \Rightarrow & yx > 0 \end{array}$$

3) Tranzitivitate:

$x \rho y,\ y \rho z \overset{?}{\Rightarrow} x \rho z$

$$\begin{array}{l} xy > 0 \\ yz > 0 \\ \hline \left.\begin{array}{l} xy^2z > 0 \\ y^2 > 0 \end{array}\right\} \Rightarrow xz > 0 \Rightarrow x \rho z \end{array}$$

5. Pe $\mathbb{R}$ definim $x \sim y \overset{def.}{\Longleftrightarrow} [x] = [y] \Rightarrow \sim$ relație de echivalență; $\widehat{\sqrt{2}} = ?$



Dem.:

1) **Reflexivitate:**

$x \sim x \Leftrightarrow [x] = [x]$ (A)

2) **Simetrie:**

$x \sim y \Leftrightarrow [x] = [y] \Rightarrow [y] = [x] \Rightarrow y \sim x$ (A)

3) **Tranzitivitate:**

$x \sim y,\ y \sim z \Rightarrow x \sim z$

$$\left.\begin{array}{l} x \sim y \Leftrightarrow [x] = [y] \\ y \sim z \Leftrightarrow [y] = [z] \end{array}\right\} \Rightarrow [x] = [y] = [z] \Rightarrow [x] = [z] \Rightarrow x \sim z$$

$\widehat{\sqrt{2}} = \{x \in \mathbb{R} \mid \sqrt{2} \sim x\}$

$= \{x \in \mathbb{R} \mid [\sqrt{2}] = [x]\} = \{x \in \mathbb{R} \mid [x] = 1\} = [1, 2)$

$[\sqrt{2}] = 1$