1. $(A,\cdot)$ monoid

$U(A)=\{a\in A \mid a \text{ inversabil}\}$ $\to$ $(\exists)\, b\in A$

$ab=ba=1$

b unic determinat

$b \overset{\text{not.}}{=} a^{-1}$

Arătați că $U(A)$ este parte stabilă în raport cu $\cdot$ și $(U(A),\cdot)$ grup.

$\downarrow$

grupul elementelor inversabile din A

U(A) parte stabilă

$(\forall)\, a,b \in U(A)$, $ab\in U(A) \Leftarrow (\exists)\,(ab)^{-1}$

$\Downarrow$

$(\exists)\, a^{-1}, b^{-1} \in U(A)$

$$\left.\begin{array}{l} ab\cdot b^{-1}a^{-1}=1 \\ b^{-1}a^{-1}ab=1 \end{array}\right\} \Rightarrow ab \text{ inversabil}, \quad (ab)^{-1}=b^{-1}a^{-1}$$

$U(A)\neq\emptyset$ $(1\in U(A))$

Din asociativitatea lui $(A,\cdot) \Rightarrow (U(A),\cdot)$ este asociativă.

$1\in A \Rightarrow 1\in U(A)$ (element neutru)

$a\in U(A) \Rightarrow a$ inversabil

$aa^{-1}=a^{-1}\cdot a=1$, $a^{-1}\in U(A)$

Deci $(U(A),\cdot)$ grup.

$U(\mathbb{Z},\cdot) = \{1,-1\}$

Ψ

$a \quad (\exists)\, b \in \mathbb{Z} \quad ab = 1$

$\Downarrow$

$a \mid 1 \Rightarrow a \in \{-1;1\}$

$U((\mathbb{N},\cdot)) = \{1\}$

$U((\mathbb{Q},\cdot))$ monoid

$\parallel$

$\mathbb{Q}^*$

2. X mulțime, $X \neq 0$

$(\text{Func}(X,X),\circ)$ monoid

$U(\underbrace{\qquad\qquad}_{f}) \quad (\exists)\, g: X \to X \quad \left.\begin{array}{l} g\circ f = 1_X \\ f\circ g = 1_X \end{array}\right) \Leftrightarrow$

$\Leftrightarrow f$ inversabilă $\Leftrightarrow f$ bijectivă

$U(\text{Func}(X,X)) = S(X)$

$(\text{Func}(X,X),\circ)$ comutativ $\Leftrightarrow |X| = 1$

$(S(X),\circ)$ comutativ $\Leftrightarrow |X| \in \{1,2\}$

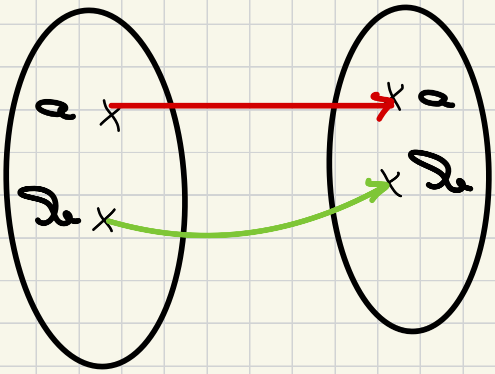
"$\Leftarrow$":

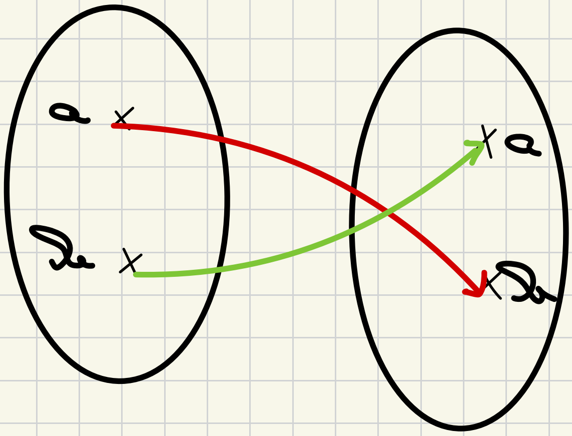
Pentru $|X| = 1$ avem $S(X) = \{1_X\}$

$1_X \circ 1_X = 1_X \circ 1_X$

Pentru $|X| = 2$ avem:

$X = \{a, b\}$





$S(X) = \{1_X, f\}$

$1_X \circ g = g \circ 1_X, \ (\forall)\, g \in S(X)$

$g = g$

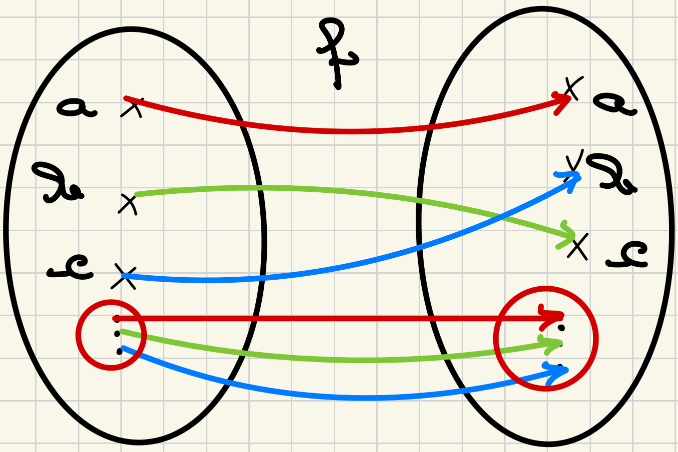
$f \circ f = f \circ f$ (A)

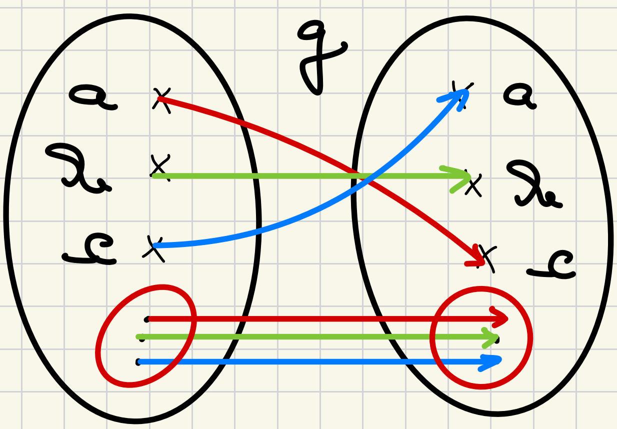
	1_X	f
1_X	1_X	f
f	f	1_X

"$\Rightarrow$":

Arăt că, dacă X are cel puțin 3 elemente, $S(X)$ nu este comutativ.

$(\exists)\ f, g : X \to X$ bijective cu $g \circ f \neq f \circ g$





$$\left.\begin{array}{l} f\circ g(a) = f(g(a)) = f(c) = b \\ g\circ f(a) = g(f(a)) = g(a) = c \end{array}\right\} \Rightarrow f\circ g \neq g\circ f$$

3. $(G,\cdot)$ grup

$a, b \in G$

$ab = 1 \overset{\cdot a^{-1}}{\Longrightarrow} b = a^{-1}$

Dacă $(A,\cdot)$ monoid, $a\in A$ și există $b\in A$ cu $ab=1$, rezultă că a este inversabil?

Dacă $(A,\cdot)$ este monoid finit și $a\in A$ este inversabil la dreapta (adică $(\exists)\, b\in A$ cu $ab=1$), atunci a este inversabil.

$ab=1$

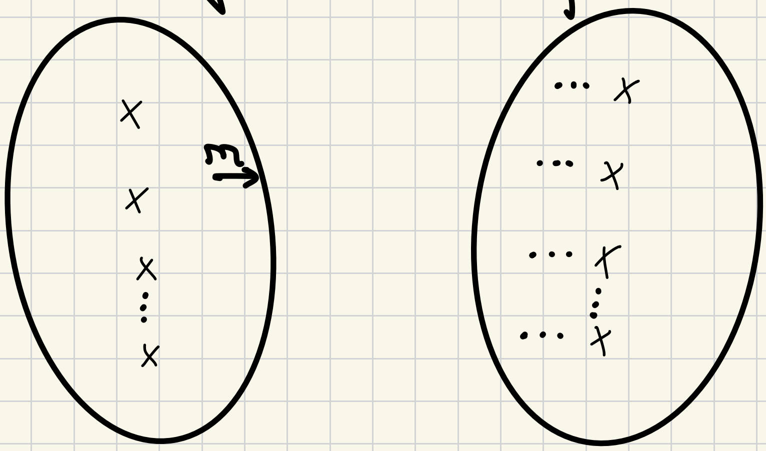
$\varphi: A\to A,\ \varphi(x)=xa$

φ injectivă

$\varphi(x)=\varphi(y) \Rightarrow xa=ya \overset{\cdot b}{\Longrightarrow} x\underbrace{ab}_{=1}=y\underbrace{ab}_{=1}$

$\Rightarrow x=y$

φ surjectivă $\quad$ A finită



φ surjectivă $\Rightarrow (\exists)\ x \in A$

$\varphi(x) = 1$

$\Downarrow$

$xa = 1$

$/ \cdot x$

$$\underbrace{c \underbrace{a x}_{1}}_{c} = x$$

$(\mathrm{Func}(X, X), \circ)$

Caut f

$f : X \to X$

inversabilă la dreapta

$(\exists)\ g : X \to X$ cu $f \circ g = 1_X$

Dar f să nu fie bijectivă

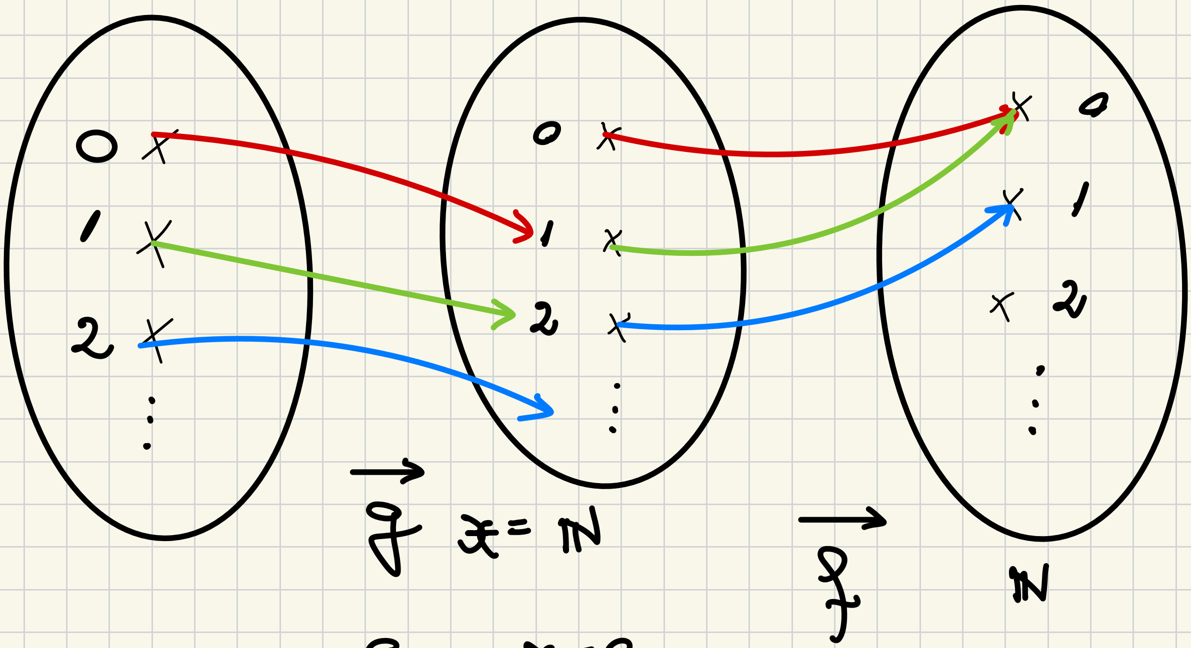
$f : A \to B$ este surjectivă

$\Updownarrow$

$(\exists)\ g : B \to A$ cu $f \circ g = 1_B$

X infinită

f surjectivă, neinjectivă



$$f(x) = \begin{cases} 0, & x = 0 \\ x-1, & x \geq 1 \end{cases}$$

$$g(x) = x+1$$

$$\boxed{f \circ g = 1_{\mathbb{N}}}$$

$$g \circ f \neq 1_{\mathbb{N}}$$

(altfel f bijectivă)

4. G, H grupuri, $f: G \to H$ morfism de grupuri

Dacă $f(xy) = f(x) \cdot f(y)$, $(\forall)\, x, y \in G$

1) $G \xrightarrow{f} H \xrightarrow{g} K$

f, g morfisme $\Rightarrow g \circ f$ morfism

$$(g \circ f)(xy) = g(f(xy)) = g(f(x) \cdot f(y)) =$$
$$= g(f(x)) \cdot g(f(y)) = (g \circ f)(x) \cdot (g \circ f)(y)$$

2) $f: G \to H$ morfism bijectiv de grupuri $\Rightarrow$

$\Rightarrow f^{-1}: H \to G$ morfism

$(\forall)\ x, y \in H \Rightarrow f^{-1}(xy) = f^{-1}(x) \cdot f^{-1}(y)$

$$f(f^{-1}(xy)) \stackrel{?}{=} f(\underbrace{f^{-1}(x) \cdot f^{-1}(y)})$$

$$\begin{array}{ccc} \| & & \| \\ xy & = & f(f^{-1}(x))\, f(f^{-1}(y)) = xy \end{array}$$

$\Leftrightarrow xy = xy$

5. Să se determine morfismele de grupuri:

(i) $f: (\mathbb{Z}, +) \to (\mathbb{Z}, +)$, $f(x+y) = f(x) + f(y)$, $(\forall)\ x, y \in \mathbb{Z}$

$\boxed{f(0) = 0}$

$a \in \mathbb{Z}$

$f_a(x) = ax, (\forall)\ x$ verifică relația

$f(-x) = -f(x), (\forall)\ x \in \mathbb{Z}$

Notăm $f(1) = a \in \mathbb{Z}$

$f(2) = f(1) + f(1) = 2a$

$f(3) = f(2) + f(1) = 3a$

$\vdots$

? $\underline{f(n) = na}, (\forall)\ n \in \mathbb{N}^*$

$f(n+1) = f(n) + f(1)$

$= ma + a = (m+1)a$

Dacă $x \in \mathbb{Z}$, $x < 0$:

$$f(x) = -f(\underbrace{-x}_{\in \mathbb{N}}) = -(-x)a = xa$$

$$\boxed{f(x) = xa, \ (\forall)\, x \in \mathbb{Z}}$$

Care sunt izomorfismele de grupuri $f: \mathbb{Z} \to \mathbb{Z}$?

$a = 0$ nu este inj.

$a \neq 0$ f_a este injectivă

$a = 1$ $f_a = f_1 = Id_{\mathbb{Z}}$ bij.

$a = -1$ $f_{-1}(x) = -x$ bij.

$a \in \mathbb{Z}^*$, f_a surj. $\Rightarrow (\exists)\, x \in \mathbb{Z}$ cu $\underbrace{f_a(x)}_{ax} = 1$

$\Rightarrow a \mid 1 \Rightarrow a \in \{-1; 1\}$

(ii) $f: (\mathbb{Z}, +) \to (\mathbb{Q}, +)$

$a \in \mathbb{Q}$

$f_a(x) = ax$, $f_a: \mathbb{Z} \to \mathbb{Q}$

există un izomorfism $\mathbb{Z} \to \mathbb{Q}$?

Pentru $a \neq 0$, f_a nu este surjectivă.

$f_a(x) = ?$

$ax = ?$

$ax = \frac{a}{2}$

Nu există $x \in \mathbb{Z}$ a.î. $ax = \frac{a}{2}$

$(x = \frac{1}{2} \notin \mathbb{Z})$

(iii) $f : (\mathbb{Q}, +) \to (\mathbb{Q}, +)$

Fie $f(1) = a \in \mathbb{Q}$ Analog...

$\Downarrow$

$f(n) = na, (\forall) n \in \mathbb{Z}$

$m \in \mathbb{Z}, n \in \mathbb{N}^*$

$$f\Big(\underbrace{\frac{m}{n} + \frac{m}{n} + \dots + \frac{m}{n}}_{n \text{ ori}}\Big) = \underbrace{f\left(\frac{m}{n}\right) + f\left(\frac{m}{n}\right) + \dots + f\left(\frac{m}{n}\right)}_{n \text{ ori}}$$

$$\underbrace{f(m)}_{= ma} = n\, f\left(\frac{m}{n}\right)$$

$\Downarrow$

$f\left(\frac{m}{n}\right) = \frac{m}{n} a$

$(\forall)\ a \in \mathbb{Q}$, $f_a : \mathbb{Q} \to \mathbb{Q}$

$f_a(x) = ax$

$\downarrow$

morfism

Orice morfism este $= f_a$

f_a izomorfism $(\forall)\, a \neq 0$ st. ca

$(\forall)\, y \in \mathbb{Q}\ (\exists)\, x \in \mathbb{Q}$

$f_a(x) = y \Leftrightarrow a x = y,\ x = \frac{y}{a}$

(iv) $f: \mathbb{Q} \to \mathbb{Z}$

Dacă f este morfism,

fie $f(1) = a$, atunci $f(n) = na,\ (\forall)\, n \in \mathbb{Z}$

şi $f(x) = xa,\ (\forall)\, x \in \mathbb{Q}$

$(\exists)\, a \in \mathbb{Q},\ f(x) = xa,\ (\forall)\, x \in \mathbb{Q}$

$\boxed{\underline{a=0}\quad f: \mathbb{Q} \to \mathbb{Z},\ f(x) = 0 \text{ morfism}}$

$\underline{a \neq 0}$ $f\left(\frac{1}{2a}\right) = \frac{1}{2a} \cdot a = \frac{1}{2}$