1. Să se determine $A=\{a \mid a\in\mathbb{Z} \text{ și } \frac{2a+1}{a+2}\in\mathbb{Z}\}$.

$$a\in A \Leftrightarrow \frac{2a+1}{a+2}\in\mathbb{Z} \Leftrightarrow a+2 \mid 2a+1$$

Dar $a+2 \mid 2(a+2)$ $(-)$

$$\Downarrow$$

$$a+2\mid 3 \Rightarrow a+2\in D_3 \Rightarrow$$

$$\Rightarrow a+2\in\{-3,-1,1,3\} \mid -2$$

$$a\in\{-5,-3,-1,1\} \Rightarrow A\subset\{-5,-3,-1,1\}$$

Pentru $a=-5$, avem $\frac{2(-5)+1}{-5+2}=\frac{-9}{-3}=3\in\mathbb{Z}$

$a=-3$, avem $\frac{2\cdot(-3)+1}{-3+2}=\frac{-5}{-1}=5\in\mathbb{Z}$

$a=-1$, avem $\frac{2(-1)+1}{-1+2}=\frac{-1}{1}=-1\in\mathbb{Z}$

$a=1$, avem $\frac{2\cdot 1+1}{1+2}=1\in\mathbb{Z}$

Așadar, $A=\{-5,-3,-1,1\}$

Sau $\frac{2a+1}{a+2}=\frac{2a+4-3}{a+2}=2-\frac{3}{a+2}$

Pt. $a\in\mathbb{Z}$ $\frac{2a+1}{a+2}\in\mathbb{Z} \Leftrightarrow \frac{3}{a+2}\in\mathbb{Z} \Leftrightarrow$

$$\Leftrightarrow a+2\in\{-3,-1,1,3\} \Leftrightarrow a\in\{-5,-3,-1,1\}$$

2. Arătați că, dacă A, B, C mulțimi, atunci
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

„$\subset$": $x \in A \cap (B \cup C) \Rightarrow x \in A$ și $x \in B \cup C$

$x \in B \cup C \Rightarrow x \in B$ sau $x \in C$

Dacă $x \in B$: Cum $x \in A \Rightarrow x \in A \cap B \Rightarrow x \in (A \cap B) \cup (A \cap C)$

Dacă $x \in C$: Cum $x \in A \Rightarrow x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C)$

„$\supset$": Fie $x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in A \cap B$ sau $x \in A \cap C$

Dacă $x \in A \cap B \Rightarrow x \in A$ și $x \in B \Rightarrow x \in B \cup C$

Cum $x \in A \Rightarrow x \in A \cap (B \cup C)$

Dacă $x \in A \cap C \Rightarrow x \in A$ și $x \in C \Rightarrow x \in B \cup C$

Cum $x \in A \Rightarrow x \in A \cap (B \cup C)$

Sau

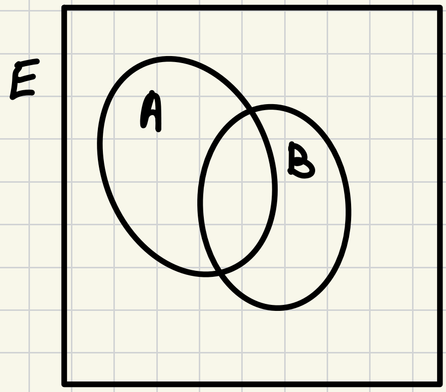
A	B	C	$B \cup C$	$A \cap B$	$A \cap C$	$A \cap (B \cup C)$	$(A \cap B) \cup (A \cap C)$
0	0	0	0	0	0	0 $\longleftrightarrow$	0
0	1	0					
0	0	1					
0	1	1					
1	0	0					
1	1	0	1	1	0	1 $\longleftrightarrow$	1
1	0	1					
1	1	1					

3. Legile lui De Morgan

$A, B \subset E$, atunci

$$C_E(A \cup B) = (C_E A) \cap (C_E B)$$

$$C_E(A \cap B) = (C_E A) \cup (C_E B)$$



$$C_E(A \cup B) = (C_E A) \cap (C_E B)$$

„$\subset$": Fie $x \in C_E(A \cup B)$.

$\Rightarrow x \in E \setminus (A \cup B) \Rightarrow x \in E$ și $x \notin A \cup B \Rightarrow x \notin A$ și $x \notin B$ și $x \in E \Rightarrow$

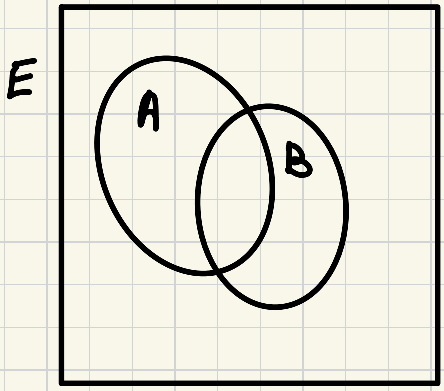
$$\left.\begin{array}{l} \Rightarrow x \in E \setminus A = (C_E A) \\ \Rightarrow x \in E \setminus B = (C_E B) \end{array}\right\} \Rightarrow x \in (C_E A) \cap (C_E B)$$

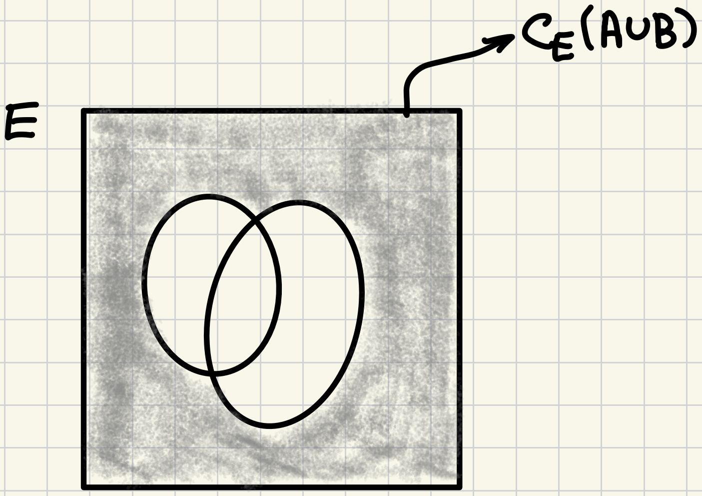
„$\supset$": Fie $x \in (C_E A) \cap (C_E B) \Rightarrow x \in (E \setminus A) \cap (E \setminus B) \Rightarrow$

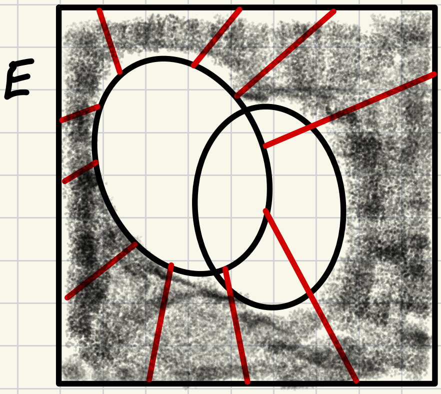
$\Rightarrow x \in E \setminus A$ și $x \in E \setminus B$.

$$\Rightarrow \left.\begin{array}{l} x \in E \\ x \notin A \\ x \notin B \end{array}\right\} \Rightarrow \begin{array}{l} x \in E \\ x \notin A \cup B \end{array} \Rightarrow x \in E \setminus (A \cup B) \Rightarrow x \in C_E(A \cup B)$$

[illegible]







A	B	$A \cup B$	$C_E A$	$C_E B$	$C_E(A \cup B)$	$(C_E A) \cap (C_E B)$
0	0	0	1	1	1	1
1	0	1	0	1	0	0
0	1	1	1	0	0	0
1	1	1	0	0	0	0

4. Câte numere naturale între 1 și 1000 există cu proprietatea că se divid cu cel puțin unul dintre 3, 5 și 7?

A_3, A_5, A_7

$$|A_3 \cup A_5 \cup A_7| = |A_3| + |A_5| + |A_7| - \underbrace{|A_3 \cap A_5|}_{A_{15}} - \underbrace{|A_3 \cap A_7|}_{A_{21}} - \underbrace{|A_5 \cap A_7|}_{A_{35}}$$

$$+ \underbrace{|A_3 \cap A_5 \cap A_7|}_{A_{105}} = E$$

$|A_3|: \ 3 \leq 3k \leq 999 \Rightarrow 1 \leq k \leq 333 \Rightarrow k = 333$

$|A_5|: \ 5 \leq 5k \leq 1000 \Rightarrow k = 200$

$|A_7|: \ 7 \leq 7k \leq 1000 \Rightarrow 1 \leq k \leq 142 \Rightarrow k = 142$

$|A_{15}|: \ 15 \leq 15k \leq 1000 \Rightarrow 1 \leq k \leq 66 \Rightarrow k = 66$

$|A_{21}|: \ 21 \leq 21k \leq 1000 \Rightarrow 1 \leq k \leq 47 \Rightarrow k = 47$

$|A_{35}|: \ 35 \leq 35k \leq 1000 \Rightarrow 1 \leq k \leq 28 \Rightarrow k = 28$

$|A_{105}|: \ 105 \leq 105k \leq 1000 \Rightarrow 1 \leq k \leq 9 \Rightarrow k = 9$

$E = 333 + 200 + 142 - 66 - 47 - 28 + 9 = 543$

5. $m \in \mathbb{N}, m \geq 2$, $A_1, \dots, A_m$ mulțimi finite

Atunci $|A_1 \cup \dots \cup A_m| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i_1 < i_2 \leq m} |A_{i_1} \cap A_{i_2}| +$

$$+ \sum_{1 \leq i_1 < i_2 < i_3 \leq m} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots +$$

$$+ (-1)^{m-1} |A_1 \cap \dots \cap A_m|$$

Demonstrăm prin inducție matematică că formula are loc pt. orice $m \in \mathbb{N}, m \geq 2$ și orice mulțimi finite $A_1, \dots, A_m$.

$$P_m: |A_1 \cup \dots \cup A_m| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i_1 < i_2 \leq m} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq m} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{m-1} |A_1 \cap \dots \cap A_m|, m \geq 2$$

Etapa I. Verificare:

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ adev.

Etapa II. Demonstrație:

Presupunem P_m adev. și demonstrăm P_{m+1} adev., $m \geq 2$.

$$P_{m+1}: |A_1 \cup \dots \cup A_{m+1}| = |(A_1 \cup \dots \cup A_m) \cup A_{m+1}| =$$

$$= |A_1 \cup \dots \cup A_m| + |A_{m+1}| - |(A_1 \cup \dots \cup A_m) \cap A_{m+1}|$$

$$\left((A_1 \cup \dots \cup A_m) \cap A_{m+1} = |\overbrace{(A_1 \cap A_{m+1})}^{X_1} \cup \underbrace{(A_2 \cap A_{m+1})}_{X_2} \cup \dots \cup \underbrace{(A_m \cap A_{m+1})}_{X_m}| \; (**) \right)$$

$$= |A_{m+1}| + \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i_1 < i_2 \leq m} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq m} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{m-1} |A_1 \cap \dots \cap A_m| - (**) = \mathrm{I}$$

$$(**) = \sum_{1 \le i \le m} |X_i| - \sum_{1 \le i_1 < i_2 \le m} |X_{i_1} \cap X_{i_2}| +$$

$$+ \sum_{1 \le i_1 < i_2 < i_3 \le m} |X_{i_1} \cap X_{i_2} \cap X_{i_3}| - \ldots +$$

$$+ (-1)^{m-1} |X_1 \cap \ldots \cap X_m|$$

$$X_{i_1} \cap X_{i_2} = A_{i_1} \cap A_{m+1} \cap A_{i_2} \cap A_{m+1}$$
$$= A_{i_1} \cap A_{i_2} \cap A_{m+1}$$

$$= \sum_{i=1,m} |A_i \cap A_{m+1}| - \sum_{1 \le i_1 < i_2 \le m} |A_{i_1} \cap A_{i_2} \cap A_{m+1}| +$$

$$\sum_{1 \le i_1 < i_2 < i_3 \le m} |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{m+1}| \ldots + (-1)^{m-1} |A_1 \cap \ldots \cap A_{m+1}|$$

$$\{(j_1, j_2) \mid 1 \le j_1 < j_2 \le m+1\} = \{(i_1, i_2) \mid 1 \le i_1 < i_2 \le m\} \cup$$
$$\cup \{(i, m+1) \mid 1 \le i \le m\}$$

$$\{(j_1, j_2, j_3) \mid 1 \le j_1 < j_2 < j_3 \le m+1\} =$$
$$= \{(i_1, i_2, i_3) \mid 1 \le i_1 < i_2 < i_3 \le m\} \cup \{(i_1, i_2, m+1) \mid 1 \le i_1 < i_2 \le m\}$$

$$\mathrm{I} = \sum_{i=1,m+1} |A_i| - \sum_{1 \le j_1 < j_2 \le m+1} |A_{j_1} \cap A_{j_2}| + \sum_{1 \le j_1 < j_2 < m+1} |A_{j_1} \cap A_{j_2} \cap A_{j_m}| -$$

$$\ldots + (-1)^m |A_1 \cap A_2 \ldots \cap A_{m+1}|$$

$\Rightarrow P_m$ adevărat