Norbert Wiener's Elevator Problem: Pi Day 2025*

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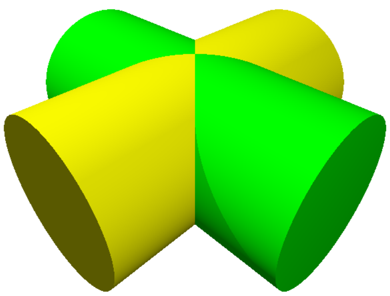

Figure 1: Overview: intersecting cylinders.

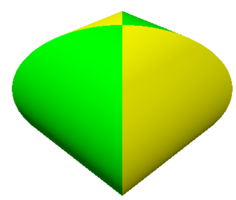

Figure 2: Boundary of the volume removed by the drill.

1 A Norbert Wiener Story

There are lots of Norbert Wiener stories [2]. In one of them, a colleague set a question for Wiener as the two of them waited for an elevator; Wiener announced the solution as the elevator reached the fourth floor. The problem:

> You have a dowel, two inches in diameter. Use a two-inch diameter bit to drill a hole through the dowel, perpendicular to the axis and straight through the center. What is the volume of the wood removed from the dowel?

See Figure 1 for an illustration: call the yellow cylinder the dowel and the green cylinder the drill bit.[1]

To illustrate the symmetry, see Figure 2, showing the intersection, with the rest of the two cylinders removed. Note also the apparent color swap: the left side, where the yellow cylinder was in Figure 1, is now green. Why?

The colors did not change. The green bits in Figure 2 are part of the surface of the green cylinder that were covered up by the yellow cylinder in Figure 1. *All* of the yellow and green surfaces in Figure 2 belong the surface of one of the two original cylinders. That fact will be useful in computing the volume.

[1] Vice versa would work as well, since the problem is symmetric.

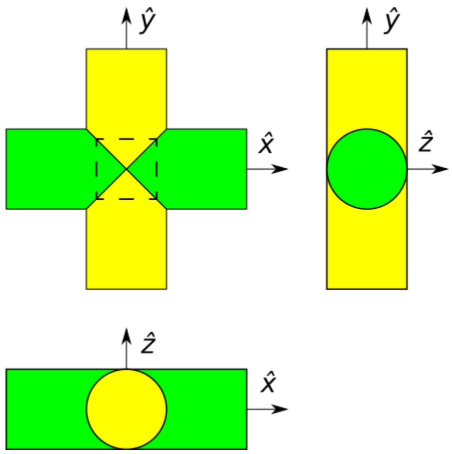

Figure 3: Three plane projections of intersecting cylinders. Upper left: looking down the $\hat{z}$-axis; upper right: looking down the $\hat{x}$-axis; lower left: looking along the $\hat{y}$-axis. The dashed square on the upper left projection is explained in the text.

2 Geometry

As is clear from Figure 2, the drilled-out volume is not one of the easy standard shapes, e,g, a sphere. So, what *is* it?

Let's build it from its components. First, what is a cylinder? Let's define it as the set of points (*locus*, if you want the jargon) at a constant perpendicular distance, r, from a line in 3-D space. So for a cylinder centered on, say, the $\hat{x}$ axis, then the cylinder satisfies

$$y^2 + z^2 = r^2. \tag{1}$$

For any given value of x, this defines a circle; over all values of x it's a cylinder, with circular cross-section. This becomes clearer in orthogonal projections of our intersecting cylinders onto the xy, yz, and xz planes, shown in Figure 3.

What happens when we run a dowel through a planer? That is, geometrically, we take the intersection of a cylinder with a plane running parallel to the axis of the cylinder? Then the result looks like Figure 4, which illustrates a useful property of cylinders. Consider the boundary between the (blue) plane and (yellow) cylinder. I claim that this is a straight line. Proof: the equation of the plane is $z = k$, where $0 < k < r$ is some constant. So to find an equation for the intersection of the plane and the cylinder, substitute for z in Equation 2:

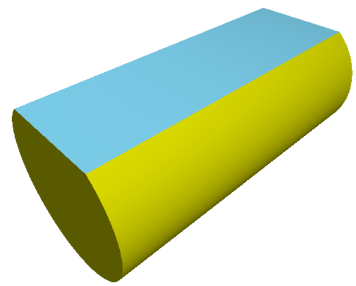

Figure 4: A cylinder with part shaved off by a plane parallel to the axis of the cylinder.

$$\begin{aligned} x^2 + k^2 &= r^2 \\ x^2 &= r^2 - k^2 \\ x &= \pm\sqrt{r^2 - k^2}, \end{aligned}$$

that is, along the intersection of plane and cylinder, x takes on two possible constant values, straddling the $y = 0$ plane, while z is a constant and y varies however it wants. This is a line! The intersection is a line, parallel to the $\hat{y}$ axis, which is also the axis of the cylinder.

The same reasoning applies to a green cylinder centered on the $\hat{x}$ axis. Or any other color, for that matter.

What about the intersection of two perpendicular cylinders with equal radii, as in Figure 1? Let's assume that the green cylinder is centered on the $\hat{x}$ axis and the yellow cylinder is centered on the $\hat{y}$ axis. Then their respective equations are:

$$\begin{aligned} x^2 &= r^2 - z^2 \\ y^2 &= r^2 - z^2, \end{aligned}$$

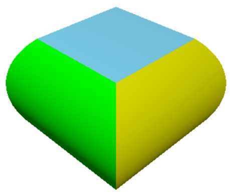

Figure 5: The blue patch, corresponding to the dashed box in Figure 3, is a square.

or:

$$\begin{aligned} x(z) &= \pm\sqrt{r^2 - z^2} \quad (2) \\ y(z) &= \pm\sqrt{r^2 - z^2}. \quad (3) \end{aligned}$$

Notice that $|x(z)| = |y(z)|$ for any given value of z. That implies that the corners of the plane figure form a square centered on $x = y = 0$.

Consider, in Figure 5, the intersection, as in Figure 2, but with everything above $z = 0.75$ planed off. The blue shape is a square, because we've established that, first, the edges are line segments; and second, the vertices are the corners of a square. It's a square!

That is the unit we'll use to compute the volume of our shape.

3 Computing the Volume

We're going to compute the volume V of the octant of the intersection with all nonnegative coordinates. Because the intersection is symmetrical, the volume of the whole is $8V$.

We'll integrate a stack of squares, each in a plane normal to the $\hat{z}$ axis, starting with $z = 0$ and proceeding to $z = r$.

One corner of each square is on the $\hat{z}$ axis, at $x = y = 0$. The other corner is at the positive-$\hat{x}, \hat{y}$ intersection of the surfaces of the two cylinders. For now we'll leave those coordinates as a function of z:

$$V = \int_{z=0}^{r} x(z)y(z)dz.$$

We already have expressions for $x(z)$ and $y(z)$ from Equations 2 and 3. Substituting into the equation for V:

$$\begin{aligned} V &= \int_{z=0}^{r} \sqrt{r^2 - z^2}\sqrt{r^2 - z^2}dz \\ &= \int_{z=0}^{r} r^2 - z^2 dz \\ &= (zr^2 - z^3/3)\Big|_{z=0}^{r} \\ &= (r^3 - r^3/3) - (0r^2 - 0^3/3) \\ &= 2r^3/3. \end{aligned}$$

As described above, V covers one octant of the intersection, and $r = 1$, so the total intersection volume T is:

$$\begin{aligned} T &= 8V \\ &= 8(2r^3/3) \\ &= 16(1^3)/3 \\ &= 16/3. \end{aligned}$$

There's no π to be seen. Surprise!

4 Coda

Somehow the answer is always in the back of the book, if you look hard enough [1].

References

[1] Wikipedia contributors. Steinmetz solid, 2025. Online; accessed 24-February-2025.

[2] Larry Hardesty. The original absent-minded professor. *MIT Technology Review*, 2021.