Max-flow, min-cut for MRF inference

Max-flow, min-cut

> Theres's a difference between knowing the path and walking the path -- {Morpheus}

(s,t)-cut: is a partition of the vertices into disjoint subsets S and T- meaning $S \cup T = V$
The **minimum cut problem** is to compute an (s,t)-cut whose capacity is small as possible. Intuitively, the minimum cut is the cheapest way to disrupt all from from s to t.

Maxflow-Mincut Theorem:

> In every flow network with source s and target t, the value of the maximum (s,t)-flow is equal to the capacity of the minimum (s,t)-cut.

Application of Graph Cuts into MRFs

Remember with MRF, we want to do inference:

$$z_1, \ldots, z_n = argmax_{z_1,\ldots,z_n} \frac{1}{Z} e^{-\sum_i E(x_i,z_i) - \sum_{i,j} E(z_i,z_j)}$$

where we have unary(hidden&observed) and pairwise(hidden&hidden) energy terms. We want to maximize the potential function $e^{-\sum}$
This is equivalent to minimizing the sum of energy functions:

$$z_1, \ldots, z_n = argmin_{z_1,\ldots,z_n} \sum_i E(x_i, z_i) + \sum_{i,j} E(z_i, z_j)$$

Graph's minimum cut provides a solution to the maximum likelihood task

Constraints on Graph Construction

- binary labels
- maximum clique size of 2
- submodularity condition for pairwise energy terms:

  $$E(0,0) + E(1,1) \leq E(0,1) + E(1,0)$$

  α**-expansion** algorithm extends the method to **non-binary labels**. It is locally optimal, but can be globally optimal with some guaranteed margin.

Homomorphism

Every energy term can be transformed into a graph snippet (subgraph). This is possible because the energy minimization via min cut is homomorphic under graph composition:

- if G_1 encodes $min(E_1)$ and G_2 encodes $min(E_2)$, then $G_1 \cup G_2$ encodes $min(E_1 + E_2)$
  .
  Without homomorphism property, we cannot easily apply graph cuts into energy minimization in MRFs.

Construction

1. Start with neighborhood relationship of hidden variables z_i
2. Add a source node s with label 0
3. Add a sink node t with label 1
4. Connect the nodes with chosen edges and edge weights
5. Calculate minimum cut between s and t
6. The solution is to assign label 0 to all nodes that are connected to s, and label 1 to all nodes that are connected to t.
   Homomorphism
   It allows us to write every energy term as a graph snippet. If G_1 encodes $min(E_1)$ and G_2 encodes $min(E_2)$, then $G_1 G_2$ encodes $min(E_1 + E_2)$

Focus on Energy function for calculating submodularity condition. Be careful about sign.

Gibbs sampling makes the assumption that a joint distribution p(z_1, ..., z_N) can be approximated by updating one variable at a time when it is conditioned on the other variables, as you write: update z_1 via p(z_1 | z_2, ..., z_N), and so on. In MRF inference we are also doing exactly this: we update one node by using the labels of the other nodes, which is just a practical realization of the conditional probability.