Combinatorial methods in algebraic geometry

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Abstract

In this thesis, we study how to use combinatorics to solve problems in algebraic geometry. We succeed in our goal in many different areas. Firstly, we obtain results about normality and Gorenstein property of the varieties associated to phylogenetic group-based models. In particular, we prove a conjecture of Michałek about the normality of the 3-Kimura model and extend results of Buczynska and Wisniewski about the Gorenstein property from the group $\mathbb{Z}_2$ to groups $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. We also obtain a full classification of graphical matroids whose associated varieties satisfy the Gorenstein property. Moreover, we classify tangential varieties to Segre-Veronese varieties which are Gorenstein or Cohen-Macaulay. Finally, we find formulas for computing the intersection products in the space of complete quadrics. We find the connection between these numbers, the maximum-likelihood degree of general linear concentration models, and the degree of semidefinite programming. These results allow us to prove Nie-Ranestadt-Sturmfels conjecture about the degree of semidefinite programming and also Sturmfels-Uhler conjecture about the polynomiality of maximum-likelihood degree.

Keywords: toric variety, lattice polytope, normality, Gorenstein property, phylogenetic group-based models, 3-Kimura model, graphical matroid, Segre-Veronese variety, tangential variety, intersection theory, complete quadrics, linear concentration model, ML-degree, degree of semidefinite programming

Abstrakt

In dieser Arbeit untersuchen wir, wie Kombinatorik zur Lösung von Problemen der algebraischen Geometrie eingesetzt werden kann. Unser Ziel erreichen wir in vielen Bereichen. Zunächst erhalten wir Ergebnisse zur Normalität und Gorenstein-Eigenschaft der Varietäten, die mit phylogenetischen gruppenbasierten Modellen assoziiert sind. Insbesondere beweisen wir eine Vermutung von Michałek über die Normalität des 3-Kimura-Modells und erweitern die Ergebnisse von Buczynska und Wisniewski über die Gorenstein-Eigenschaft von der Gruppe $\mathbb{Z}_2$ auf die Gruppen $\mathbb{Z}_3$ und $\mathbb{Z}_2 \times \mathbb{Z}_2$. Wir erhalten auch eine vollständige Klassifikation grafischer Matroide, deren zugehörige Varietäten die Gorenstein-Eigenschaft erfüllen. Darüber hinaus klassifizieren wir tangentiale Varietäten in Segre-Veronese-Varietäten, die Gorenstein oder Cohen-Macaulay sind. Schließlich finden wir Formeln zur Berechnung der Schnittprodukte im Raum vollständiger Quadriken. Wir finden den Zusammenhang zwischen diesen Zahlen, dem Maximum-Likelihood-Grad allgemeiner linearer Konzentrationsmodelle und dem Grad der semidefiniten Programmierung. Diese Ergebnisse erlauben es uns, die Nie-Ranestadt-Sturmfels-Vermutung über den Grad der semidefiniten Programmierung und auch die Sturmfels-Uhler-Vermutung über die Polynomialität des Maximum-Likelihood-Grades zu beweisen.

Schlüsselwörter: torische Varietät, Gitterpolytop, Normalität, Gorenstein-Eigenschaft, phylogenetische gruppenbasierte Modelle, 3-Kimura-Modell, grafisches Matroid, Segre-Veronese-Varietät, tangentiale Varietät, Schnittpunkttheorie, vollständige Quadrik, lineares Konzentrationsmodell, ML-Grad, Grad der semidefiniten Programmierung

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Contents

Introduction

Algebraic geometry is about studying algebraic varieties. There are many techniques and theorems used in algebraic geometry. In this thesis, we are interested in the proofs which use combinatorial methods. Usually, the process is that using some theory one is able to transform the problem from algebraic geometry to combinatorial or near-combinatorial problem, which is often much easier to solve. In this thesis, we will show that this is not a rare situation. Using this technique we obtain several strong new results, including proofs of conjectures by Michałek about the normality of the 3-Kimura model [68], Sturmfels and Uhler about the polynomiality of maximum-likelihood degree [98], and Ranestad and Sturmfels about the degree of semidefinite programming [77].

Toric varieties

One of the most typical and important examples of this technique are toric varieties [18]. The toric varieties form a class of algebraic varieties that can be described by combinatorial objects. Moreover, there are many ways how to do it. Affine toric variety can be described by several lattice points, affine semigroup, or cone in the lattice. Projective toric variety is described by the lattice polytope and in general, abstract toric varieties can be described by the fan. Since all information about the toric variety is encoded in the combinatorial data, in theory, proving anything about any toric variety should be possible by using just combinatorics. However, this often leads to surprisingly difficult combinatorial problems.

The best thing about toric varieties is that they are not rare. To cite Sturmfels and Michałek [73] "the world is toric". In other words, toric varieties appear in many places in various areas of mathematics.

In this thesis we will give a brief introduction into the toric varieties in Chapter 1. Then we will study specific toric varieties coming from three different areas:

- **Algebraic Phylogenetics**
- **Matroids**
- **Tangential varieties to Segre-Veronese varieties**

Now we briefly introduce these three areas.

Algebraic phylogenetics

Phylogenetics is a science that models evolution and describes mutations in this process [33]. This topic reveals many connections to several branches of mathematics such as algebraic geometry [30], [13], [66], and combinatorics [67], [24].

One of the central objects in phylogenetics is the *tree model*. In general, a statistical model is a parametric family of probability distributions. The tree model is based on rooted tree and finite set B and gives us probability distribution on B^l where l is the number of leaves of the tree. The parameters are the distribution on the root and $|B| \times |B|$ transition matrices along the edges of the tree. Phylogenetic models often reflect symmetries among the elements of B. These are usually encoded by the action of a finite group G on B. This is the case also for the model we will focus on in this thesis, the group-based model. It is a tree model with a group G acting transitively and freely on the set B and parameters are invariant with respect to the group action.

Since everything is finite, a distribution allowed by a tree model may be represented as a vector $(p_1, \ldots, p_n)$ where p_i's are nonnegative and sum to one. Thus a tree model may be regarded as a map from the parameter space to the n-dimensional vector space.

In algebraic phylogenetics, we are interested in the geometric locus of all probability distributions allowed by a given model. Precisely, the Zariski closure of this locus is an algebraic variety and one is interested in its geometric and algebraic properties [30, 101].

We consider the algebraic variety $X(T, G)$ associated to a tree T, describing the evolution of species, and an abelian group G, distinguishing a model of evolution in Chapter 2, which is based on my articles "Normality of the Kimura 3-parameter model" [113] and "Gorenstein property for phylogenetical trees" [23]. The latter is joint work with Rodica Dinu. The construction of the variety $X(T, G)$ is described in [30]. For an arbitrary model, this variety is not necessarily toric. However, by [31], [103] and [97], group-based models allow a monomial parametrization, thus this variety is toric. We are interested in investigating the geometric properties of these algebraic varieties, following the path initiated by Sturmfels and Sullivant [97]. The construction of the defining polytope of the toric variety corresponding to a group-based model depends on the tree and the group. There is an algorithm for finding the polytope, which is presented in [66] and implemented in [25]. The biological motivation for considering variety $X(T, G)$ can be consulted in [84] and [30].

The main focus will be on the case of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, which is the so-called 3-parameter Kimura model [51]. It is the most biologically meaningful group-based model since in this case, the basic elements correspond to the nucleotides of DNA: adenine (A), cytosine (C), guanine (G), and thymine (T). The action of the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is actually the Watson-Crick complementarity. Furthermore, this model is very interesting from a mathematical perspective. Recent results may be found e.g. in [15], [14], [16], [52], [68], [75].

We will study the varieties by studying associated polytopes and we will obtain many new fascinating results. Two main results of Chapter 2 concentrate on normality and Gorenstein property:

Theorem 1. *The polytopes $P_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$ and $P_{\mathbb{Z}_3, m}$ associated to the m-claw tree and the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3$ are normal for every positive integer m.*

Theorem 2. *Let $P_{G,m}$ be a polytope associated to the m-claw tree ($m \geq 3$) and the* *group $G \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. Then the polytope $P_{G,m}$ is Gorenstein if and only if $(G, m) \in \{(\mathbb{Z}_2, 3), (\mathbb{Z}_2, 4), (\mathbb{Z}_3, 3), (\mathbb{Z}_2 \times \mathbb{Z}_2, 3)\}$. The Gorenstein indices in these cases are 4,2,3,4 respectively.*

The first theorem confirms the conjecture of Michałek about the normality of the 3-Kimura model [68]. The second theorem extends the result of Buczynska and Wisniewski about the Gorenstein property for the group $\mathbb{Z}_2$ [13].

Graphical matroids

The toric variety of a matroid is defined as the toric variety associated to the base polytope of the matroid. White [114] proved that the toric variety of a matroid is projectively normal. Consequently, by the celebrated result of Hochster, it satisfies Cohen–Macaulay property. Additionally, White [115] provided a conjectural description of generators of the toric ideal of a matroid – the ideal defining the matroid variety. In particular, he conjectured that the toric ideal of a matroid is generated by quadrics. White's conjecture in full generality remains open since its formulation in 1980. Blasiak [6] confirmed it for graphical matroids. For general matroids, it was proved 'up to saturation' [57], see also [56] for further improvements. The toric variety of a representable matroid (in particular of a graphic matroid) has a nice geometric description – by a result of Gelfand, Goresky, MacPherson and Serganova [36] it is isomorphic to the torus orbit closure in a Grassmannian and any such orbit arises this way.

In Chapter 3, which is based on my article "Gorenstein graphic matroids" [44] which is joint work with T. Hibi, M. Lason, K. Matsuda, and M. Michałek, we provide a complete classification of graphical matroids whose toric variety satisfies the Gorenstein property. This study was initiated by Herzog and Hibi [39] who observed that the Gorenstein property partitions matroids in an interesting way. As they doubt at full classification (among all matroids), we focus on graphic matroids – one of the basic classes motivating the concept of a matroid. However, there is a recent result by Lason and Michałek [58], where the authors use our result to generalize it to a general matroid, completely answering the question by Hibi and Herzog. We believe that our study, apart from algebraic meaning, is interesting for its combinatorial sake as well.

Tangential varieties to Segre-Veronese varieties

Let $\mathbb{P}^N$ be the projective space over the complex ground field $\mathbb{C}$ and $X \subset \mathbb{P}^N$ a projective variety. We define the *tangential variety* $\tau(X)$ of X as the union of all the tangent lines to X. Likewise the *(second) secant variety* $\sigma_2(X)$ is defined as the union of all the secant lines to X together with all the points lying on $\tau(X)$.

In Chapter 4, which is based on my article "Cohen-Macaulay and Gorenstein tangential varieties of the Segre-Veronese Varieties" [48] which is joint work with M. Azeem Khadam, we will consider the case of X being Segre-Veronese variety. That means X is the embedding of $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_k}$ into $\mathbb{P}^N$ given by the very ample line bundle $\mathcal{O}(a_1, \ldots, a_k)$.

Investigating tangential and secant varieties is a part of classical algebraic geometry which was studied, among others, by Terracini and were brought into a modern light by F.L. Zak [116]. One of the most basic questions concerns the dimension, which can in most cases be calculated by Terracini Lemma (see [106] for the original statement and [1, 19] for modern versions). Another basic question is to provide a complete list of generators of the ideal of the tangential and secant varieties. In [87] Raicu solved this problem for secant varieties and in [79] Oeding and Raicu obtained the analogous results for tangential varieties. In both of the cases, their methods were based on representation theory.

Furthermore, the secant variety $\sigma_2(X)$ is known to be normal [112, Theorem 2.2] and Cohen-Macaulay (which followed from [49, Proposition 3.5]). On the other hand, the tangential variety $\tau(X)$ is not always normal [72, Example 2.22]. This is one of the determining reasons that investigating Cohen-Macaulay and Gorenstein properties of $\tau(X)$ have been found challenging and it had remained an open problem to classify those tangential varieties of the Segre-Veronese varieties which are Cohen-Macaulay or Gorenstein. In [49, Theorem 4.4] M. Azeem Khadam, Mateusz Michałek, and Piotr Zwiernik classified those secant varieties of the Segre-Veronese varieties which are Gorenstein. In Chapter 4 we will turn our attention to tangential varieties and obtain a full classification of those tangential varieties of Segre-Veronese varieties which are Gorenstein or Cohen-Macaulay:

Theorem 3. *The tangential variety of the Segre-Veronese variety is smooth if and only if*

(S1) $k = 2$, $\mathbf{a} = (1,1)$, $b_1 = 1$, *or*

(S2) $k = 1$, $a = 1$ *or (*$a = 2$ *and* $b = 1$*).*

If the tangential variety of the Segre-Veronese variety is not smooth, then it is Cohen–Macaulay if and only if one of the following holds

(CM1) $k \geq 3$, $\mathbf{a} = (1,\dots,1)$,

(CM2) $k = 2$, $\mathbf{a} = (2,2)$, $\mathbf{b} = (1,1)$,

(CM3) $k = 2$, $\mathbf{a} = (1,2)$, $\mathbf{b} = (1,b_2)$ *for all* $b_2 \geq 1$,

(CM4) $k = 2$, $\mathbf{a} = (1,1)$, $b_i > 1$ *for all* $i = 1,2$,

(CM5) $k = 1$, $a \geq 3$, $b = 1$,

(CM6) $k = 1$, $a = 2$, $b > 1$.

If the tangent variety of the Segre-Veronese variety is not smooth, then it is Gorenstein if and only if one of the following holds

(G1) $k = 3$, $\mathbf{a} = (1,1,1)$, $\mathbf{b} = (1,1,1)$,

(G2) $k = 2$, $\mathbf{a} = (1,2)$, $\mathbf{b} = (1,1)$,

(G3) $k = 2$, $\mathbf{a} = (1,1)$, $b_1 = b_2$, $b_1 > 1$.

(G4) $k = 1$, $a \geq 3$, $b = 1$,

(G5) $k = 1$, $a = 2$, b *is even.*

In a special case, the tangential variety $\tau(X)$ and the secant variety $\sigma_2(X)$ coincide with that of the locus of matrices of rank at most two. That is, its classification of Cohen-Macaulay or Gorenstein was classically known (see Remark 4.2.11 for references and discussion).

A part of the research on the geometry of tangential varieties has also been motivated by applications. In [78], Oeding pointed out applications of the tangential variety of an n-factor Segre where the equations allow one to answer the question of membership for the following sets: the set of tensors with border rank 2 and rank $k \leq n$ (the secant variety is stratified by such tensors [3]), a special Context-Specific Independence model, and a certain type of inverse eigenvalue problem.

The investigation of properties like normal, Cohen-Macaulay, or Gorenstein for the tangential (and secant) varieties remains a difficult problem. The techniques from algebraic statistics combined with toric geometry recently made it possible to study them (see [99] for a seminal paper). Our main result is based on methods from algebraic statistics, in particular cumulants, and toric geometry. The main idea behind cumulants is to treat points of the variety as (formal) probability distributions and apply methods from algebraic statistics [117, 118]. Cumulants have already been applied successfully on several occasions [49, 61, 71, 72, 99].

A change of coordinates, inspired by cumulants, leads to new structures on secant and tangential varieties. In our setting, cumulant methods turn the tangential variety of the Segre-Veronese variety locally into a toric variety, although, in general, the tangential variety is not a toric variety. Since Cohen-Macaulay and Gorenstein are local properties and our main object is locally a toric variety, then we apply methods from toric geometry. *However*, it is not as easy as it seems to be, since in our case the tangential variety is *not* normal hence in this case using toric geometry is highly nontrivial. To deal with this problem we apply a criterion of Cohen-Macaulay and Gorenstein developed by Hoa and Trung [108, Theorem 4.1], which is equally applicable to non-normal toric varieties.

Complete quadrics

Another area of algebraic geometry where we may use combinatorial methods is enumerative geometry. There we answer questions such as: "How many lines in the spaces pass through four general lines?" Since, in essence, to answer such a question, one has to compute the number of objects with a certain property, it resembles a combinatorial problem. Of course, the problem is not purely combinatorial by its nature, yet the use of combinatorial methods has been found convenient

For instance, consider the intersection theory on the Grassmannian $G(n, r)$. Its Chow group is generated by Schur polynomials S_λ, for all partitions λ which Young diagrams fit inside an $r \times (n - r)$ rectangle, see e.g [29, Chapter 4]. There are many combinatorial rules, e.g. Pieri's rule, which tells us how to multiply Schur polynomials corresponding to Young diagrams. In this thesis, we will use this theory to study intersection theory on more complicated space - on the space of complete quadrics. The special case, the space of complete conics, is the space that allows us to come up with the famous number 3264 as the number of conics in the plane that are tangent to five general conics. This number also appears in the title of the book by Eisenbud and

Harris on intersection theory [29].

The intersection theory in the space of complete quadrics will allow us to answer the following question: "How many quadrics in $\mathbb{P}^n$ are passing through d given points and are tangent to $\binom{n+1}{2} - d - 1$ given hyperplanes?"

In Chapter 5, which is based on my article "Complete quadrics: Schubert calculus for Gaussian models and semidefinite programming" [62] which is joint work with L. Manivel, M. Michałek, L. Monin, and T. Seynnaeve, we provide recurrence formulas that will allow us to find the answer. While the question is interesting by itself we will not stop there. We provide the connection between these numbers, the degree of semidefinite programming, and the maximum likelihood degree of the linear concentration model. This will allow us to prove the following conjecture by Sturmfels, Uhler [98, p. 611] about the polynomiality of ML-degree:

Theorem 4. *For any fixed positive integer d, the ML-degree of general d-dimensional subspace of $\mathbb{C}^n$ is a polynomial in n.*

Moreover, we also prove Nie-Ranestad-Sturmfels conjecture [77, Conjecture 21] about the degree of semidefinite programming:

Theorem 5. *Let m, n, s be positive integers. Then*

$$\delta(m,n,n-s) = \sum_{\sum I \leq m-s} (-1)^{m-s-\sum I} \psi_I b_I(n) \binom{m-1}{m-s-\sum I}$$

where the sum goes trough all sets of nonnegative integers of cardinality s.

The definitions of coefficients $\delta_{m,n,n-s}$, ψ_I, b_I can be found in Sections 5.2, 5.5.

This thesis is organised as follows: In Chapter 1 we give a small introduction to toric varieties and show our point of view on them.

In Chapter 2 we deal with the toric varieties arising from phylogenetics, in Chapter 3 we look at the varieties associated to graphical matroids and in Chapter 4 we study tangential varieties to Segre-Veronese varieties. Each of the chapters may be read separately. The chapters are based on the articles [113], [23],[48],[44] which are joint works with Rodica Dinu, Mateusz Michałek, Michal Lason, Kazunori Matsuda, Takayuki Hibi and M. Azeem Khadam.

In Chapter 5 we study the intersection theory on the space of complete quadrics. The chapter may be read separately and is based on the article [62] which is joint work with Laurent Manivel, Mateusz Michałek, Leonin Monin, and Tim Seynnaeve.

Chapter 1

Introduction to toric varieties

Toric varieties are the most typical example of geometrical objects which can be described by the combinatorial data. This is the reason why in the big part of this thesis we study the varieties which are toric. Moreover, they appear in different branches of mathematics. In Chapter 2 we study toric varieties arising from phylogenetics, in Chapter 3 we study the toric varieties associated to matroids. In Chapter 4 we study the toric varieties which cover tangential varieties to Segre-Veronese varieties and therefore allow us to study their local properties.

In this chapter, we recall basic definitions of toric varieties and their properties which we will study in the following chapters, such as normality and Gorenstein property. More details can be found in the literature, see [18], [34].

We note that the definition of a toric variety is not unified in the literature. Fulton in his book [34] requires toric variety to be normal. We will use the definition from [18] where the toric varieties do not have to be normal. This will be essential in Chapter 4 where we deal with a lot of non-normal toric varieties.

We will start with the abstract definition of the toric variety [18, Definition 3.1.1]:

Definition 1.0.1. A toric variety is an irreducible variety that contains a complex torus $T = (\mathbb{C} \setminus \{0\})^n$ as the dense subset and the natural action of the torus T on itself can be extended to an algebraic action on the whole variety X.

From the previous definition, it is not so clear that the toric varieties are very combinatorial objects. In the thesis we will work only with affine and projective toric varieties, thus we give an alternative equivalent definition of these, see [73, Chapter 8], [18, Theorem 1.1.17, Definition 2.1.1]:

Definition 1.0.2. Let $a_1, \dots, a_k \in \mathbb{N}^n$. The affine toric variety is the closure of the image of the monomial map $\varphi : (\mathbb{C} \setminus \{0\})^n \to \mathbb{C}^k$, $x = (x_1, \dots, x_n) \mapsto (x^{a_1}, \dots, x^{a_k})$.

Projective toric variety is the closure of the image of the monomial map $\varphi : (\mathbb{C} \setminus \{0\})^n \to \mathbb{P}^{k-1}$, $x = (x_1, \dots, x_n) \mapsto (x^{a_1} : \dots : x^{a_k})$.

From this definition it is clear that affine or projective toric variety is determined by the set of lattice points $\mathcal{A} = \{a_1, \dots, a_k\}$. Thus, we should be able to find all information about the toric variety just by studying the set $\mathcal{A}$.

In the case of affine toric varieties, we can also consider the affine semigroup $S_{\mathcal{A}} \subset \mathbb{Z}^n$ generated by the points in the set $\mathcal{A}$. On the other hand, given affine semigroup S, we define the toric variety given by semigroup S as the affine toric variety determined by any generating set of S. It can be shown that you always get isomorphic toric variety regardless of the choice of the generating set. Moreover, the associated toric variety is equal to $\mathrm{Spec}(\mathbb{C}[S])$. More details about the equivalence of these constructions can be found in [18, Chapter 1]. We will work with affine toric varieties defined by affine semigroups in Chapter 4.

In the case of projective varieties, it is more natural to consider the lattice polytope P which is the convex hull of the points in the set $\mathcal{A}$. Conversely, from the polytope P we get the set $\mathcal{A}$ as the set of all lattice points in P. Notice that shifting the polytope in the lattice does not change the associated toric variety. For more details, see [18, Chapter 2]. We will study projective toric varieties defined by lattice polytopes in Chapter 2 and Chapter 3.

For the projective variety X given by the lattice polytope $P \subset \mathbb{R}^n$, another important variety is the affine cone over X. We can obtain the affine cone over X as the affine toric variety associated to the set $(P \cap \mathbb{Z}^n) \times \{1\}$.

Remark 1.0.3. *Given a set of points $\mathcal{A}$, affine semigroup S or a polytope P, we initially assume they lie in the lattice $\mathbb{Z}^n$. However, for the purpose of toric geometry, it is better to consider the lattice L generated by the set $\mathcal{A}$, S or $P \cap \mathbb{Z}^n$, respectively since it will not change the associated toric variety. Of course, any lattice is isomorphic to $\mathbb{Z}^k$ for suitable k. However, sometimes there will be a natural choice of coordinates where the lattice L will be a proper sublattice of $\mathbb{Z}^n$. We will see that in Chapters 2 and 4. Thus, in the next section, we will define the properties of polytopes or affine semigroups with respect to the lattice L.*

1.1 Properties of toric varieties

In this section, we recall some properties of algebraic varieties which we will be interested in in this thesis. In particular, we recall normality and the Gorenstein property. All of these properties can be defined for arbitrary algebraic varieties. However, in the case of toric varieties, it is much easier to check if a variety has the desired property than in the case of a general algebraic variety. All of these properties will be studied in the following chapters for different types of toric varieties.

1.1.1 Normality

We start by recalling the definition.

Definition 1.1.1. Algebraic variety X is *normal* if at every point p the local ring is integrally closed.

In general, the normality of algebraic variety is an important property, since many theorems in algebraic geometry require normality. However, for toric varieties the normality is essential. As we mentioned before, there are two schools regarding the definition of a toric variety. Fulton in his book [34] requires the toric variety to be normal. We will not work with this definition and allow also non-normal toric varieties, which we will see in Chapter 4. However, normal toric varieties have many interesting properties. For example, normal toric varieties admit a nice combinatorial description in terms of a fan [18, Chapter 3].

We mention another strong result about normal toric varieties which is the following theorem by Hochster [47], see also [18, Theorem 9.2.9]:

Theorem 1.1.2. *Normal toric variety X is Cohen-Macaulay.*

Thus, for normal toric varieties, the question about Cohen-Macaulayness is trivial. However, in Chapter 4 we deal with non-normal toric varieties and we will study their Cohen-Macaulay property.

There is a quite easy criterium how to check whether an affine toric variety given by a semigroup is normal, see [18, Theorem 1.3.5]:

Theorem 1.1.3. *Affine toric variety X given by the semigroup S which generates lattice L, is normal if and only if $S = \mathrm{Cone}(S) \cap L$, in other words, there are no holes in S.*

Another important notion is projective normality. Projective toric variety X is called projectively normal if the affine cone over X is normal. Projective normality implies normality, but the converse is not true.

As we will discuss in this thesis mainly projective varieties determined by lattice polytopes, we also consider the normality of polytopes. Normality is an important property of lattice polytopes by itself, see [9], [96], [41], [18]. We recall a definition of a normal polytope:

Definition 1.1.4. A lattice polytope P whose integral points generate lattice L is called *normal* if every point in $kP \cap L$ can be expressed as a sum of k points from $kP \cap L$.

There is a connection between the normality of a polytope P and normality of affine cone over the associated projective toric variety, see [18, Lemma 2.2.14]:

Lemma 1.1.5. *A lattice polytope P whose integral points generate lattice L is normal if and only if the points from $(P \cap L) \times \{1\}$ generate the semigroup $\mathrm{Cone}(P \times \{1\}) \cap (L \times \mathbb{Z})$.*

From the previous lemma, it follows that the normality of polytope is equivalent to the projective normality of the associated projective toric variety. The normality of polytope also implies that the normal fan of the polytope describes the associated toric variety. The normality and normality of polytopes play an important role when one discusses the Gorenstein property as we will see in the next section.

1.1.2 Gorenstein property

The Gorenstein property of a ring was introduced by Grothendieck. It reflects many symmetries of the cohomological properties of the ring. It is also a condition that implies that the singularities of the spectrum of the ring are not 'too bad'. In particular, regular rings and complete intersections are Gorenstein, and furthermore, Gorenstein rings are always Cohen–Macaulay.

We said that the (toric) variety is Gorenstein if all of their local rings are Gorenstein.

We do not present a formal definition of the Gorenstein property of the ring. Instead, we provide several equivalent conditions when a normal affine toric variety X defined by affine semigroup S is Gorenstein, see [4, 42]:

Theorem 1.1.6. *Let $S \subset \mathbb{Z}^n$ be an affine semigroup with the property $S = \operatorname{Cone}(S) \cap L$ and let $X = \operatorname{Spec} \mathbb{C}[S]$ be the associated normal affine toric variety. Then the following conditions are equivalent*

1. *X is Gorenstein variety,*

2. *$\mathbb{C}[S]$ is a Gorenstein algebra,*

3. *canonical divisor of X is Cartier,*

4. *there is a lattice element $v \in S$, which has lattice distance one from all facets of $\operatorname{Cone}(S)$.*

The Gorenstein property connects to reflexive polytopes. Firstly, we recall the definition:

Definition 1.1.7. Lattice polytope P is *reflexive*, if 0 is its only interior point and it has lattice distance one to all facets of P.

Equivalently, P is reflexive if its dual polytope is also a lattice polytope. By virtue of the work of Batyrev [4] reflexive polytopes play an important role in mirror symmetry and are very intensively studied, e.g. [18, Section 8.3] and references therein. Furthermore, relations of the Gorenstein property to combinatorics are also an important research topic [11, 81, 43, 45, 46, 93].

Consider a normal lattice polytope P whose integral points generate lattice L. When we apply Theorem 1.1.6 to semigroup S_P generated by $(P \cap L) \times \{1\}$ we see that $\mathbb{C}[P] := \mathbb{C}[S_P]$ is Gorenstein if there is a lattice point in $v \in kP \cap L$ such that v has lattice distance one from all facets of kP, i.e. $kP - v$ is reflexive.

We will use this fact to define the Gorenstein property for polytopes, see also [40, Section 12.5].

Definition 1.1.8. For a positive integer k, a normal lattice polytope P whose integral points generate lattice L is *Gorenstein of index k* if there exists a lattice point $v \in kP$ such that for every facet F of P, the point v has lattice distance one from kF. A normal lattice polytope P is *Gorenstein* if P is Gorenstein of index k for some positive integer k.

In other words, P is Gorenstein of index k if kP is shifted reflexive polytope. The consequence of Theorem 1.1.6 is that the cone over projective variety X defined by a normal lattice polytope P is Gorenstein if and only if the polytope P is Gorenstein. We sum up this observation and mention another equivalent characterization of Gorenstein property for polytopes in the following proposition [4, 42]:

Proposition 1.1.9. *Let P be a normal lattice polytope whose integral points generate lattice L. Let X be the projectively normal projective toric variety defined by P. Then the following conditions are equivalent:*

- *Affine cone over X is Gorenstein variety,*

- *P is Gorenstein polytope,*

- *the numerator of the Hilbert series of $\mathbb{C}[P]$ is palindromic.*

Furthermore, for a Gorenstein polytope P the associated projective toric variety is the so-called Gorenstein Fano variety, [18, Section 8.3]. We will be proving the Gorenstein property for polytopes associated to toric varieties in Chapters 2 and 3.

In the case of non-normal toric varieties, the situation is much more difficult, but we will be dealing with it in Chapter 4.

Chapter 2

Algebraic phylogenetics

This chapter is based on my articles "Normality of the Kimura 3-parameter model" [113] and "Gorenstein property for phylogenetical trees" [23]. The latter is joint work with Rodica Dinu.

In this chapter, we will study the toric varieties associated to the group-based phylogenetical models by studying their associated polytopes. We will focus on the properties mentioned in Chapter 1.

To give a reader a better view, we start by giving a brief description of phylogenetic group-based models in Section 2.1, next in Section 2.2 we give the explicit vertex description of their associated polytopes.

An important tool in the proof of many properties is the facet description of the polytope. In general, it is a difficult task to get such a description from the vertex description of the polytope. For the 3-Kimura model such a description was provided in [65]. For the group $\mathbb{Z}_3$ we give the facet description of the defining polytope of $X(T, \mathbb{Z}_3)$, where T is the m-claw tree, in Theorem 2.3.3.

Our next step is to look at the normality since not all group-based models give rise to normal toric varieties and there is no known complete classification of the models for which the associated toric varieties are normal. In Section 2.4, Proposition 2.4.1, we show that for an abelian group G such that $|G| = 2k, k \geq 3$, and the tripod T, the projective algebraic variety $X(T, G)$ representing the model is not projectively normal, extending the result of [26].

So far normality was confirmed only in the simplest case: $G = \mathbb{Z}_2$. We extend this result to groups $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ and show that the polytopes associated to any tree and one of these groups is always normal in Theorem 2.4.3. The result for 3-Kimura model also confirms a conjecture of Michałek [67, Conjecture 9.5], [69, Conjecture 12.1].

In Section 2.5, we investigate the toric fiber product of two Gorenstein polytopes of the same Gorenstein index and we prove that Gorenstein property is under specific conditions preserved by taking fiber products which will help us in the Section 2.6.

In a seminal paper of Buczyńska and Wiśniewski [13], in Theorem 2.15, the authors showed that $X(T, \mathbb{Z}_2)$, the variety associated to any trivalent tree and the group $\mathbb{Z}_2$, is Gorenstein Fano variety of index 4. In Section 2.6, we extend this result also to groups $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.1 Phylogenetic trees

In this section, we briefly describe phylogenetic group based-models. We start by describing a tree model. We fix a rooted tree T which corresponds to the evolution and a finite alphabet B. To every vertex of T we assign a copy of B and we associate to the root r a probability distribution π. To every edge e we associate a transition matrix M_e of type $|B| \times |B|$ which rows sum up to 1. These matrices encode the probability of mutating one element of B to another by the process of evolution. This data gives us a probability distribution on leaves which we can encode as a point in the vector space. This defines a map from the parameter space to the space of probability distribution on the leaves. We are interested in a locus of all probability distribution allowed by our model which is the image of the map described above. Zariski closure of this set is an affine algebraic variety associated to the model. We will work with the corresponding projective variety which is a projectivization of this affine variety.

In the case of the group-based model there is a group G acting transitively and freely on the set B. The distribution π is uniform and there is a natural action of G on the matrices M_e. All matrices must be G-invariant with respect to that action. As it was already mentioned in the introduction, the most meaningful group-based model is the 3-Kimura model, which is a model for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, the elements of the set B correspond to four nucleobases of DNA.

Example 2.1.1. *In the case of 3-Kimura model, i.e $\mathbb{Z}_2 \times \mathbb{Z}_2$ all transition matrices are in the following form:*

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{pmatrix}$$

The variety $X(T, G)$ associated to the tree T with the set of edges E and the set of leaves L is the closure of the image of the map $\phi : \mathbb{R}^{|G||E|} \to \mathbb{R}^{|G||L|}$ which sends choice of matrices M_e to the probability distribution on the leaves of T.

This map is quite complicated. However, we can simplify the map by making the linear change of coordinates known as the Discrete Fourier Transform, where we pass to the basis of characters of the group G. Precisely, in all vector spaces generated by the elements of B associated to the vertices of T, we pass from coordinates x_g to coordinates y_χ:

$$y_\chi = \sum g \in G \chi(g) x_g,$$

for all characters $\chi : G \to \mathbb{C}$.

Example 2.1.2. *In the case of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, is the change of coordinates as follows.*

$$y_0 = x_0 + x_1 + x_2 + x_3,\ y_1 = x_0 + x_1 - x_2 - x_3,$$

$$y_2 = x_0 - x_1 + x_2 - x_3,\ y_3 = x_0 - x_1 - x_2 + x_3$$

After this base change, the matrices M_e become diagonal, the map ϕ becomes a monomial map and therefore its image is a toric variety. For more details, see [97]. As we mentioned in chapter 1, toric varieties are nice, in particular, they have a combinatorial description by the polytope associated to it.

In this chapter, we study the properties of the variety $X_{G,T}$ by studying associated polytopes $P_{G,T}$. The description of these polytopes is known and it will be given in the next section.

2.2 The polytopes for group-based models

Before we describe the polytopes $P_{G,T}$ we start by the key fact which is the following result due to Sullivant [100, Theorem 3.10]:

Theorem 2.2.1. *The polytope associated to any tree, $P_{T,G}$, can be expressed as the fiber product of polytopes associated to the m-claw trees $P_{m,G}$ for every inner vertex of T. In particular, for any trivalent tree T, the polytope $P_{T,G}$ can be expressed as the fiber product of polytopes associated to the tripod, $P_{3,G}$.*

This theorem allows us to study a lot of properties of the polytopes $P_{G,T}$ simply by restricting to polytopes $P_{m,G}$ associated to the m-claw trees. Thus, we give the explicit description of the associated polytopes only for m-claw trees since we will be our main object of interest in the rest of the chapter.

Let $P_{G,m} \subseteq \mathbb{R}^{m|G|}$ be the polytope associated to the m-claw tree and the group G. We label the coordinates of a point $x \in \mathbb{R}^{m|G|}$ by x_g^j where $1 \leq j \leq m$ and $g \in G$, i.e. g corresponds to group element and j to the edge of the tree. We will often work with the lattice points using their G-presentations:

Definition 2.2.2. We say that *the G-presentation* of a point $x \in \mathbb{Z}_{\geq 0}^{n|G|}$ is an n-tuple $(G_1, \dots, G_n)$ of multisets of elements of G such that the element $g \in G$ appears exactly x_g^j times in the multiset G_j. We may identify the n-tuple $(g_1, \dots, g_n) \in G^n$ with the n-tuple of multisets $(\{g_1\}, \dots, \{g_n\})$.

In this language adding lattice points corresponds to taking the union of multisets. The vertex description of the polytope $P_{G,m}$ is known; see [97], [13], [66]. We recall this description and we formulate in the language of G-presentations the vertex description of $P_{G,m}$.

Theorem 2.2.3. *The vertices of the polytope $P_{G,m}$ associated to the m-claw tree and the finite abelian group G are exactly the points $x(g_1, \dots, g_m)$ with $g_1 + \dots + g_m = 0$.*

Let $L_{G,m}$ be the lattice generated by vertices of $P_{G,m}$. Then

$$L_{G,m} = \{x \in \mathbb{Z}^{m|G|} : \sum_{g,j} x_g^j \cdot g = 0, \forall\ 1 \leq j, j' \leq m, \sum_g x_g^j = \sum_g x_g^{j'}\}.$$

where the first sum is taken in the group G.

Alternatively, we can characterize G-presentations of points in $L_{G,m} \cap \mathbb{Z}_{\geq 0}^{m|G|}$ as follows: Every multiset has the same size and sum of all elements in multisets is 0.

Example 2.2.4. *Let $G = \mathbb{Z}_2$, $m = 3$. Then*

$$\begin{aligned} P_{\mathbb{Z}_2,3} &= \mathrm{Conv}(\{x(0,0,0), x(1,1,0), x(1,0,1), x(0,1,1)\}) \\ &= \mathrm{Conv}(\{(1,0,1,0,1,0), (0,1,0,1,1,0), (0,1,1,0,0,1), (1,0,0,1,0,1)\}). \end{aligned}$$

Then the lattice generated by the vertices of the polytope $P_{\mathbb{Z}_2,3}$ is

$$L_{\mathbb{Z}_2,3} = \{x \in \mathbb{Z}^6 : 2 \mid x_2 + x_4 + x_6,\ x_1 + x_2 = x_3 + x_4 = x_5 + x_6\}.$$

Polytopes $P_{G,m}$ have a lot of symmetries that can be described by group actions on $\mathbb{R}^{m|G|}$:

- Action of $\mathbb{S}_m$:

  For $\sigma \in \mathbb{S}_m$ and $x \in \mathbb{R}^{m|G|}$ we define $\sigma(x)_g^{\sigma(j)} = x_g^j$. Intuitively, we only permute $|G|$-tuples of coordinates by the upper index.

- Action of $H_m := \{(g_1, \ldots, g_m) \in G^m : g_1 + g_2 + \cdots + g_m = 0\}$

  For $h = (g_1, \ldots, g_m) \in H_m$ and $x \in \mathbb{R}^{m|G|}$ we define $(hx)_g^j = x_{(g+g_j)}^j$. Intuitively, if we look at G-presentation of a point in $\mathbb{Z}^{m|G|}$ we add g_j to all elements in G_j.

- Action of $\mathrm{Aut}(G)$: For $\varphi \in \mathrm{Aut}(G)$ and $x \in \mathbb{R}^{m|G|}$ we define $\varphi(x)_{\varphi(g)}^j = x_g^j$. Again, if we consider G-presentation of x this is application of the automorphism φ to the elements in multisets.

All of these actions only permute coordinates in $\mathbb{R}^{m|G|}$ and therefore are automorphisms of $\mathbb{R}^{m|G|}$ as a vector space. It can be easily verified that they map vertices of $P_{G,m}$ to vertices of $P_{G,m}$ and therefore preserve $P_{G,m}$. It follows that these actions restricted to $L_{G,m}$ are automorphisms of this lattice. The symmetries will be a useful tool to prove statements about polytopes $P_{G,m}$.

2.3 Facet description for the polytope associated to $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$

There are two basic ways to describe a polytope. One can either say what the vertices of the polytope are, or one can specify the facets of the polytope. Both can be useful since sometimes it is easier to work with the vertex description and sometimes with the facet description. However, passing from the vertex description of the polytope to its facet description is a difficult problem in general.

As we have seen, the vertex description for polytopes representing group-based models is already known. On the contrary, the facet description is known only in a few cases, namely, in the case of $G = \mathbb{Z}_2$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, see [65].

Theorem 2.3.1. *The polytope $P_{\mathbb{Z}_2 \times \mathbb{Z}_2,m}$ is defined by the following inequalities:*

- $x_g^j \geq 0$ *for all* $g \in \mathbb{Z}_2 \times \mathbb{Z}_2, 1 \leq j \leq m$,
- $x_0^j + x_\alpha^j + x_\beta^j + x_\gamma^j = 1,$ *for all* $1 \leq j \leq m$,
- *For all $A \subseteq \{1, 2, \ldots, m\}$ with $|A|$ being an odd number:*

$$\sum_{j \in A} (x_0^j + x_\alpha^j) + \sum_{j \notin A} (x_\beta^j + x_\gamma^j) \geq 1,$$

$$\sum_{j \in A} (x_0^j + x_\beta^j) + \sum_{j \notin A} (x_\alpha^j + x_\gamma^j) \geq 1,$$

$$\sum_{j \in A} (x_0^j + x_\gamma^j) + \sum_{j \notin A} (x_\alpha^j + x_\beta^j) \geq 1.$$

In the next result, we are presenting the facet description of the polytope $P_{\mathbb{Z}_3,m}$. For this part, in this section, we consider the projection of $P_{\mathbb{Z}_3,m}$ on the $2m$ coordinates which correspond to non-zero elements of $\mathbb{Z}_3$. It is easy to see that this projection is isomorphic with the original polytope.

Lemma 2.3.2. *The vertices of $P_{\mathbb{Z}_3,m}$ which are connected with 0 by an edge are only those with at most 3 non-zero coordinates. More precisely, these are the following:*

- $v(j_1, j_2)$ *such that* $v(j_1, j_2)_1^{j_1} = 1, v(j_1, j_2)_2^{j_2} = 1$ *and all other coordinates are equal to 0, where* $1 \le j_1, j_2 \le m, j_1 \neq j_2$.
- $v(i; j_1, j_2, j_3)$ *such that* $v(i; j_1, j_2, j_3)_i^{j_1} = v(i; j_1, j_2, j_3)_i^{j_2} = v(i; j_1, j_2, j_3)_i^{j_3} = 1$ *and all other coordinates are equal to 0, where* $1 \le j_1 < j_2 < j_3 \le m$, $i \in \{1, 2\}$.

Proof. Consider any other vertex v of $P_{\mathbb{Z}_3,m}$. If we can write $v = w_1 + w_2$ for some vertices w_1, w_2 then there can not be an edge between 0 and v. Clearly, there must either exist two indices j_1, j_2 such that $v_1^{j_1} = 1, v_2^{j_2} = 1$ or three indices j_1, j_2, j_3 and i such that $v_i^{j_1} = v_i^{j_2} = v_i^{j_3} = 1$. In the first case we can write $v = v(j_1, j_2) + w$ and in the second $v = v(i; j_1, j_2, j_3) + w$ for some vertex w. Thus, there are no other vertices connected by an edge with 0.

It is not difficult to show that the vertices from the statement are connected by an edge with 0 but we do not need this fact so we leave the proof for the reader. $\square$

Theorem 2.3.3. *The facet description of the polytope $P_{\mathbb{Z}_3,m}$ is given by:*

$$x_i^j \ge 0, \text{ for } 1 \le i \le 2, 1 \le j \le m,$$

$$x_1^j + x_2^j \le 1, \text{ for } 1 \le j \le m,$$

$$\langle u_{a_1}, x^1 \rangle + \langle u_{a_2}, x^2 \rangle + \cdots + \langle u_{a_m}, x^m \rangle \ge 2 - a_1 - a_2 - \cdots - a_m,$$

$$\langle w_{a_1}, x^1 \rangle + \langle w_{a_2}, x^2 \rangle + \cdots + \langle w_{a_m}, x^m \rangle \ge 2 - a_1 - a_2 - \cdots - a_m,$$

for all $(a_1, a_2, \ldots, a_m) \in \{0, 1, 2\}^m$ *such that* $a_1 + a_2 + \cdots + a_m \equiv 2 \pmod 3$, *where* $u_0 = (1, 2), u_1 = (1, -1), u_2 = (-2, -1), w_0 = (2, 1), w_1 = (-1, 1), w_2 = (-1, -2)$ *and* $x^j = (x_1^j, x_2^j)$.

Proof. Since by acting with suitable $h \in H_m$ we can map any vertex of $PP_{\mathbb{Z}_3,m}$ to 0 it is enough to describe all facets containing 0.

Firstly, we show that the inequality $x_1^1 \ge 0$ gives us a facet of $P_{\mathbb{Z}_3,m}$ containing 0. This inequality holds for every vertex of P_m and therefore for every point in $P_{\mathbb{Z}_3,m}$. On the other hand, for vertices $0, v(2,3), v(3,2)$, $v(j,1)$ for $2 \le j \le m$ and $v(2;1,2,j)$ for $3 \le j \le m$ equality $x_1^1 = 0$ holds. These $2m$ vertices span a $2m-1$-dimensional affine subspace. We conclude that $x_1^1 \ge 0$ gives us a facet of $P_{\mathbb{Z}_3,m}$. Analogously, we can prove that $x_i^j \ge 0$ defines a facet for all $i \in \{1, 2\}, 1 \le j \le m$.

Consider the facet F containing 0 given by the inequality

$$A_1 x_1^1 + B_1 x_2^1 + A_2 x_1^2 + B_2 x_2^2 + \cdots + A_m x_1^m + B_m x_2^m \ge 0$$

for $A_j, B_j \in \mathbb{R}$. Since $P_{\mathbb{Z}_3,m}$ is a lattice polytope we may assume that all A_j, B_j are integers.

Suppose that all $A_j, B_j \geq 0$. If only one of them is non-zero we obtain the facet $x_i^j \geq 0$. If two or more coefficients are non-zero the inequality defines the intersection of such facets and therefore it is not a facet. Thus, there are no other facets for which all $A_j, B_j \geq 0$.

We may assume that at least one coefficient is negative. Without loss of generality $A_m < 0$. If not, we can just act with suitable $\sigma \in \mathbb{S}_m$ and $\varphi \in \mathrm{Aut}(\mathbb{Z}_3)$ to get to this case. We claim that there exists a vertex $v \in F$ with $v_1^m = 1$. Indeed, if there was no such vertex it would imply that F is a subset of the facet of $P_{\mathbb{Z}_3,m}$ given by $x_1^m \geq 0$, which is impossible.

By definition of $P_{\mathbb{Z}_3,m}$, to the vertex v of $P_{\mathbb{Z}_3,m}$ corresponds one element $h_v \in H_m$. If we act with $h_v \in H_m$ on the facet F we get a facet of $P_{\mathbb{Z}_3,m}$ containing 0 since it maps v to 0. Last two coefficients defining $h_v(F)$ are $(-B_m, A_m - B_m)$. If $B_m > 0$, then these coefficients for $h_v(F)$ are both non-positive. Thus, we may also assume that $B_m \leq 0$. Without loss of generality we may assume $A_m \leq B_m$, by acting with suitable $\varphi \in \mathrm{Aut}(\mathbb{Z}_3)$.

Since vertices $v(m, j)$ and $v(j, m)$ satisfy the inequality we get $B_j + A_m \geq 0$ and $A_j + B_m \geq 0$. In particular, $A_j \geq 0, B_j > 0$ for all $1 \leq j \leq m-1$. Moreover, from vertices $v(1; j_1, j_2, m)$ we get that at most one coefficient A_j can be equal to 0.

Since F is a facet containing 0, there must exist $2m-1$ linearly independent non-zero vertices connected to 0 by an edge lying on F. It is easy to check that the only possible candidates for that are vertices $v(j, m)$ and $v(i; j_1, j_2, m)$ by Lemma 2.3.2.

That means that $2m-1$ linearly independent equations from the following must hold:

$$A_j = -B_m, \tag{$*_j$}$$

$$A_{j_1} + A_{j_2} = -A_m, \tag{$*_{j_1,j_2}$}$$

$$B_j = -A_m, \tag{$\diamond_j$}$$

$$B_{j_1} + B_{j_2} = -B_m, \tag{$\diamond_{j_1,j_2}$}$$

for $1 \leq j, j_1, j_2 \leq m-1$.

Since in equations $(*_j)$ and $(*_{j_1,j_2})$ appear only $m+1$ coefficients we can choose at most m linearly independent from them. The same is true for $(\diamond_j)$ and $(\diamond_{j_1,j_2})$. So we have to choose m equations from $(*)$ equations and $m-1$ from $(\diamond)$ or the other way around.

Vertices $v(m, j)$ satisfy the inequality of the facet F and therefore $B_j \geq -A_m$ which leads to $B_{j_1} + B_{j_2} \geq -2A_m > -A_m \geq -B_m$. Thus, none of the equalities $(\diamond_{j_1,j_2})$ can hold. We conclude that all $m-1$ equalities $(\diamond_j)$ hold and $B_j = -A_m$ for all $1 \leq j \leq m-1$.

Suppose that $A_m \neq 2B_m$. From the $(*)$ equalities we need to choose m linearly independent and therefore at least one from $(*)_j$ since others do not contain the coefficient B_m. Without loss of generality we choose $(*)_1, (*)_2, \ldots, (*)_k$, $1 \leq k \leq m-1$. Thus $A_1 = A_2 = \cdots = A_k = -B_m$. The equalities $(*_{j_1,j_2})$ for $j_1, j_2 \leq k$ can not hold since $A_m \neq 2B_m$. From the equalities $(*_{j_1,j_2})$ for $j_1, j_2 > k$ we can choose at most $m-k-1$ linearly independent so we must choose at least one with $j_1 \leq k, j_2 > k$. For the fixed $j_2 > k$ all equalities $(*_{j_1,j_2})$ are the same. Hence if we choose more than one of them we need to choose different j_2. Without loss of generality we choose $(*_{1,j_2})$ for all $k+1 \leq j_2 \leq l$, where $k < l \leq m-1$. We are left with the equalities $(*_{j_1,j_2})$ for

$j_1, j_2 > l$. However, we can choose at most $m-k-l-1$ linearly independent which means we have together only at most $k+l+m-k-l-1 = m-1$ equalities which is a contradiction. Therefore, we obtain that $A_m = 2B_m$.

In this case all equalities $(*)$ have common solution $A_1 = A_2 = \cdots = A_{m-1} = -B_m = -A_m/2$ so it does not matter which of them we choose.

Thus, we have only one solution. After dividing by A_1 we obtain the facet given by

$$x_1^1 + 2x_2^1 + x_1^2 + 2x_2^2 + \cdots + x_1^{m-1} + 2x_2^{m-1} - 2x_1^m - x_2^m \geq 0.$$

All the other facets of $P_{\mathbb{Z}_3,m}$ can be obtained by acting with the groups $\mathbb{S}_m$, H_m and $\mathrm{Aut}(\mathbb{Z}_3)$. It is easy to check that we get exactly the facets from the statement of our theorem. □

2.4 Normality

It was explained in Chapter 1 that normality is an important property of lattice polytopes. It will be also important in the following sections where we will study the Gorenstein property since we will do it only for normal polytopes. Thus, we would like to know which polytopes associated to the phylogenetic tree models are normal.

Unfortunately, for many models, we do not know whether the associated polytopes are normal. We give a summary of known results and then we prove new results. From Michałek's result [66, Lemma 5.1] and Theorem 2.2.1 it follows that, for a given group, if one wants to check the normality for the algebraic variety for this group and any tree, it is enough to verify normality for claw trees. In particular, in order to check the normality for any trivalent tree, it is enough to check the normality for the chosen group and the tripod.

Buczyńska and Wiśniewski [13] proved that the toric variety associated to any trivalent tree and the group $\mathbb{Z}_2$ is projectively normal. In section 2.7 we will extend this result and prove normality for groups $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

However, there are examples of group-based models whose associated toric variety is not normal. By [26, Computation 4.1], the polytope associated to the tripod and any $G \in \{\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2\}$ gives non-normal algebraic varieties representing the model. The next result shows that, in fact, we always obtain non-normal models for the tripod and an abelian group G such that $|G| = 2k, k \geq 3$.

Proposition 2.4.1. *Let G be an abelian group with even cardinality and at least 6 elements. Then the polytope $P_{G,3}$ associated to the tripod and the group G is not normal. Thus, the projective algebraic variety representing the model is not projectively normal.*

Proof. Suppose that $G \neq \mathbb{Z}_2^n$. We prove that there exist elements $g, h \in G$ such that $2g = 0$ and $2h \neq 0, g$.

Suppose for contradiction that such elements do not exist. Since G has an even number of elements we can find an element g of order 2. By assumption, for every $h \in G$ we have $2h = 0$ or $2h = g$. Therefore, every element of G has order 2 or 4. This implies $G = \mathbb{Z}_2^k \times \mathbb{Z}_4^l$ with $k \geq 0, l \geq 1$. However, for the group $\mathbb{Z}_2 \times \mathbb{Z}_4$ we are able to find such g, h, e.g. $g = (1,0), h = (0,1)$. All other groups in this form (except $\mathbb{Z}_4$ which does not have 6 elements) contain $\mathbb{Z}_2 \times \mathbb{Z}_4$ as a subgroup. In this case, we may pick the same elements from the subgroup.

We pick such g, h and consider the point

$$p = x(\{0, g, h, g+h\}, \{0, g, h, g+h\}, \{0, g, -2h, g-2h\}).$$

We notice that

$$2p = x(0,0,0) + x(g,g,0) + x(g,0,g) + x(0,g,g) + x(h,h,-2h)+$$

$$+x(g+h, g+h, -2h) + x(g+h, h, g-2h) + x(h, g+h, g-2h).$$

Since each summand is a vertex of $P_{G,3}$ we obtain that $p \in 4P$. Moreover, $p \in L_{G,3}$ since the sum of all elements in G-presentation of p is 0. We show that p cannot be written as a sum of 4 vertices of $P_{G,3}$.

Expressing p in such a way is equivalent to splitting elements from G-presentation into triples with sum 0 and consisting of one element from each multiset. It is easy to verify that this is not possible. Thus, $P_{G,3}$ is not normal.

For the group $G = \mathbb{Z}_2^3$ we conclude in an analogous way by considering the point

$$p = x(\{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\}, \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\},$$

$$\{(0,0,0),(0,0,1),(1,1,0),(1,1,1)\}).$$

Similarly as in the previous case, we can check that $p \in 4P_{G,3}$ but it cannot be written as the sum of 4 vertices of $P_{G,3}$. The groups $G = \mathbb{Z}_2^n$ contain $\mathbb{Z}_2^3$ as a subgroup and, by [26, Proposition 4.2], we obtain the non-normality for the polytope associated to the tripod and the group $G = \mathbb{Z}_2^n$. $\square$

Remark 2.4.2. *Non-normality for the tripod gives non-normality for any non-trivial tree (not a path). Indeed, the polytope for any tree always contains a face that is isomorphic to the polytope for the tripod (and the same group).*

Using the facet description of the polytopes $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ and $P_{\mathbb{Z}_3,m}$ we are able to prove our main result about normality:

Theorem 2.4.3. *The polytopes $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ and $P_{\mathbb{Z}_3,m}$ are normal for every positive integer m.*

Since the proof is quite long and technical we postpone it to the last section of this chapter, Section 2.7, which is dedicated to the proof of Theorem 2.4.3.

2.5 The toric fiber product

We start this section by defining *toric fiber products.* This concept was introduced by Sullivant, in [100]. Given a positive integer m, we denote $[m] = \{1, \dots, m\}$. Let r be a positive integer and let s and t be two vectors of positive integers in $\mathbb{Z}^r_{>0}$. We consider the following multigraded polynomial rings

$$\mathbb{K}[x] = \mathbb{K}[x^i_j : i \in [r], j \in [s_i]] \text{ and } \mathbb{K}[y] = \mathbb{K}[y^i_k : i \in [r], k \in [t_i]],$$

with the same multigrading

$$\deg(x_j^i) = \deg(y_k^i) = \mathbf{a}^i \in \mathbb{Z}^d.$$

We assume that there exists a vector $w \in \mathbb{Q}^d$ such that $\langle w, \mathbf{a}^i \rangle = 1$, for any i. Denote $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^d\}$. If $I \subseteq \mathbb{K}[x]$ and $J \subseteq \mathbb{K}[y]$ are homogeneous ideals, then the quotients rings $R = \mathbb{K}[x]/I$ and $S = \mathbb{K}[y]/J$ are also multigraded rings. Let

$$\mathbb{K}[z] = \mathbb{K}[z_{jk}^i : i \in [r], j \in [s_i], k \in [t_i]],$$

and consider the ring homomorphism

$$\phi_{I,J} : \mathbb{K}[z] \to R \otimes_{\mathbb{K}} S; \quad z_{jk}^i \longmapsto x_j^i \otimes y_k^i.$$

Definition 2.5.1. The toric fiber product of I and J is

$$I \times_{\mathcal{A}} J = \mathrm{Ker}(\phi_{I,J}).$$

Let $P \subset \mathbb{R}^d$ be an integral convex polytope and $A_P \in \mathbb{Z}^{(d+1)\times n}$ its configuration matrix given by the column vectors $(\alpha_1, 1)^t, \ldots, (\alpha_n, 1)^t$ with $\{\alpha_1, \ldots, \alpha_n\} = P \cap \mathbb{Z}^d$. Let $S = K[x_1, \ldots, x_n]$ the polynomial ring in n indeterminates over the field K and let $T = K[t_1^{\pm}, \ldots, t_d^{\pm}, s]$ be the Laurent polynomial ring in $(d+1)$ indeterminates over K. The toric ideal associated to P, denoted by I_P is the toric ideal associated to the configuration matrix A_P, i.e. I_P is the ideal of S which is the kernel of the ring homomorphism $\pi : S \to T$ with $x_i \mapsto t^{\alpha_i} s$ for $1 \leq i \leq n$. For more details, see [41].

Because of this association, we will work with toric fiber products of polytopes.

Lemma 2.5.2. *Let Q_1 and Q_2 be two polytopes. Let $\pi_i : Q_i \to \mathbb{R}^n, (i = 1, 2)$ be projections such that $\pi_1(Q_1) = \pi_2(Q_2) = \Delta_n$, where Δ_n is the standard simplex. Then all facets of the toric fiber product $Q_1 \times_{\Delta_n} Q_2$ are of the form $F_1 \times_{\Delta_n} Q_2$ or $Q_1 \times_{\Delta_n} F_2$ where F_i is a facet of Q_i.*

Proof. Let d_1, d_2 be the dimensions of Q_1, Q_2, respectively and $d = d_1 + d_2 - n$ be the dimension of $Q = Q_1 \times_{\Delta_n} Q_2$. For $(x_i)_{i\in[d]} = x \in \mathbb{R}^d$ and subset $S \subseteq [d]$ we denote by x_S the projection of x to $\mathbb{R}^S$. Let $A, B \subseteq [d]$ be sets with $|A| = d_1, |B| = d_2, |A \cap B| = n$ such that $Q_1 \subseteq \mathbb{R}^A$, $Q_2 \subseteq \mathbb{R}^B$, $Q \subseteq \mathbb{R}^{A\cup B}$ and $\Delta_n \subseteq \mathbb{R}^{A\cap B}$.

For a face F of Q, we say that F is *good* if F is of the form $F_1 \times_{\Delta_n} Q_2$ or $Q_1 \times_{\Delta_n} F_2$ where F_i is a face of Q_i.

Let F be a face of Q given by $u \in \mathbb{Z}^d$, i.e. $\langle x, u \rangle \geq a$ for any $x \in Q$ and $F = \{x \in Q : \langle x, u \rangle = a\}$.

We prove that if F is not good, then there exists a face $\widetilde{F} \supseteq F$ such that $\widetilde{F}$ is good. This implies that there is no facet of Q which is not good.

For this we will find $0 \neq v \in \mathbb{Z}^d$ such that, for any vertex $x \in Q$, $\langle x, v \rangle \geq a$ and $\langle x, u \rangle = a$ implies $\langle x, v \rangle = a$ and $v_{B\setminus A} = 0$. Then the face given by v and a satisfies all conditions.

Denote

$$m_k = \min_{x \in Q, x_{A\cap B} = e_k} \langle x_B, u_B \rangle.$$

It exists because we are taking minimum in the compact set. Take v such that

$$v_A = u_A + \sum_{k \in A\cap B} (m_k - u_k) e_k, \;\; v_{B\setminus A} = 0.$$

Then for any vertex $x \in Q$ such that $x_{A\cap B} = e_k$ we have

$$\langle x, v\rangle = \langle x_A, v_A\rangle = \langle x_A, u_A\rangle + m_k - u_k \geq a,$$

since for all x with $x_{A\cap B} = e_k$ we have $\langle x, u\rangle = \langle x_A, u_A\rangle + \langle x_B, u_B\rangle - u_k \geq a$. The equality statement follows in the same way. Since F was not in the form from the statement we must have $u_{A\setminus B} \neq 0$ which means that $0 \neq v$ and therefore this is the v which we were looking for.

Now let F be a facet of Q. We know that F is be good. Without loss of generality $F = F_1 \times_{\Delta_n} Q_2$. But if F_1 is not a facet of Q_1 then there exists a facet $\widetilde{F_1} \supsetneq F_1$. Then $F_1 \times_{\Delta_n} Q_2 \subsetneq \widetilde{F_1} \times_{\Delta_n} Q_2$ which means that F is not a facet of Q, which is a contradiction. □

We give now the main result of this section.

Theorem 2.5.3. *Let Q_1, Q_2 be two Gorenstein polytopes with the same Gorenstein index k. Let $\pi_i : Q_i \to \mathbb{R}^n, (i = 1, 2)$ be projections such that $\pi_1(Q_1) = \pi_2(Q_2) = \Delta_n$, where Δ_n is the standard simplex. Let p_1, p_2 be the unique interior lattice points of kQ_1 and kQ_2 respectively. Assume that $\pi_1(p_1) = \pi_2(p_2)$. Then the toric fiber product $Q = Q_1 \times_{\Delta_n} Q_2$ is also Gorenstein of index k. Moreover, the unique interior point p of kQ satisfies $\varphi_1(p) = p_1, \varphi_2(p) = p_2$ where $\varphi_i : Q \to Q_i$ are toric fiber product maps.*

Proof. We show that in the dilated polytope kQ, the point p has a distance 1 to all facets. Let F be a facet of Q. By Lemma 2.5.2, without loss of generality, $F = F_1 \times_{\Delta_n} Q_2$. It is clear that the point p has the distance to F the same as the distance of the point p_1 to F_1, which is 1, by our assumption on Q_1 being Gorenstein. □

Corollary 2.5.4. *Let Q_1, Q_2 be two normal lattice polytopes. The product polytope $Q_1 \times Q_2$ is Gorenstein of index k if and only if both Q_1 and Q_2 are Gorenstein of index k.*

Proof. The "if" implication is a special case of Theorem 2.5.3, where $n = 0$ and Δ_n is a point. For the other implication, note that $F \times Q_2$ is the facet of the product polytope $Q_1 \times Q_2$ for any facet F of Q_1. Suppose that $Q_1 \times Q_2$ is Gorenstein of index k and (p_1, p_2) is the unique interior lattice point of $k(Q_1 \times Q_2)$. Then the point $p_1 \in kQ_1$ is the lattice point with lattice distance one from all facets of kQ_1. Analogously, p_2 proves the Gorenstein property for Q_2. □

2.6 Gorenstein property for claw trees and small groups

In this section, we prove our main result, Gorenstein property for polytopes associated to group $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and any trivalent tree.

Theorem 2.6.1. *Let $P_{G,m}$ be a polytope associated to the m-claw tree $(m \geq 3)$ and the group $G \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. Then the polytope $P_{G,m}$ is Gorenstein if and only if $(G, m) \in \{(\mathbb{Z}_2, 3), (\mathbb{Z}_2, 4), (\mathbb{Z}_3, 3), (\mathbb{Z}_2 \times \mathbb{Z}_2, 3)\}$. The Gorenstein indices in these cases are 4,2,3,4 respectively.*

Proof. In these cases, the polytopes are normal by Theorem 2.4.3. Therefore, our polytope is Gorenstein if and only if there exists some k such that $kP_{G,m}$ has a unique interior lattice point whose lattice distance from all facets is 1, see Subsection 1.1.2.

Suppose that there exists a unique interior lattice point $\omega_{G,m} \in kP_{G,m}$. We claim that all coordinates of $\omega_{G,m}$ must be equal. If not we can act with a suitable $h \in H_{G,m}$ to obtain a different interior lattice point. Therefore, in the case of $G \in \{\mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$ or m even, the only candidate for $\omega_{G,m}$ is $(1, 1, \dots, 1)$ since it belongs to $L_{G,m}$. In the case $G = \mathbb{Z}_2$ and m odd, $(1, 1, \dots, 1)$ is not a lattice point in the lattice $L_{G,m}$, but the point $(2, 2, \dots, 2)$ is. Thus, in all cases we can denote by $\omega_{G,m}$ the only possible candidate. We have the inequalities for $P_{G,m}$ from [65] and Theorem 2.3.3. Now we compute the lattice distances from $\omega_{G,m}$ to the facets of $P_{G,m}$ and conclude our desired result. We analyze in detail the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is the 3-Kimura model, and the other cases are similar.

Let us denote the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, \alpha, \beta, \gamma\}$.The inequalities describing the k-dilation of the polytope are

$$x_g^j \geq 0, \forall g \in \mathbb{Z}_2 \times \mathbb{Z}_2, 1 \leq j \leq m, \qquad (\spadesuit_1)$$

For all $A \subseteq \{1, 2, \dots, m\}$ of odd cardinality:

$$\begin{aligned} &\sum_{j \in A}(x_0^j + x_\alpha^j) + \sum_{j \notin A}(x_\beta^j + x_\gamma^j) \geq k, \\ &\sum_{j \in A}(x_0^j + x_\beta^j) + \sum_{j \notin A}(x_\alpha^j + x_\gamma^j) \geq k, \qquad (\spadesuit_2) \\ &\sum_{j \in A}(x_0^j + x_\gamma^j) + \sum_{j \notin A}(x_\alpha^j + x_\beta^j) \geq k. \end{aligned}$$

For the last three inequalities we know that the difference of left and right side is always even if $x \in L_{G,m}$ [113, Lemma 2]. This means that for lattice point x its distance from the facet is the difference of both sides divided by 2. In other words this facet is given by the point $y \in L_{G,m}^\vee$ where $y_0^j = y_\alpha^j = 1/2, y_\beta^j = y_\gamma^j = 0$ for $j \in A$ and $y_0^j = y_\alpha^j = 0, y_\beta^j = y_\gamma^j = 1/2$ for $j \notin A$.

If we plug $\omega_{G,m} = (1, 1, \dots, 1)$ in inequality $(\spadesuit_1)$ and we obtain that the distance is always one. If we plug it in one of the inequalities of $(\spadesuit_2)$ on the left side we obtain:

$$\sum_{j \in A}((\omega_{G,m})_0^j + (\omega_{G,m})_\alpha^j) + \sum_{j \notin A}((\omega_{G,m})_\beta^j + (\omega_{G,m})_\gamma^j) = 2|A| + 2(m - |A|) = 2m.$$

Since $\omega_{G,m} \in 4P_{G,m}$ the lattice distance of $\omega_{G,m}$ to this facet is $(2m - 4)/2$. This is 1 only for $m = 3$. Therefore, for $m = 3$ our polytope is Gorenstein and otherwise not. □

Corollary 2.6.2. *Let $\mathcal{T}$ be a trivalent tree. Then the polytope associated to $\mathcal{T}$ and the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is Gorenstein of index* 4 *and the polytope associated to $\mathcal{T}$ and the group $\mathbb{Z}_3$ is Gorenstein of index* 3.

Proof. By Theorem 2.5.3 and Theorem 2.6.1. □

2.7 Normality of the polytopes associated to $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$

Our goal is to prove that polytopes $P_{\mathbb{Z}_3,m}$ and $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ are normal for every positive integer m. The proofs for both groups are similar and use the same techniques. We start with definitions that will be needed and some general steps which work for both groups. Then we finish with the most technical parts for both groups separately. Firstly, we describe briefly the idea of the proof.

It is easy to check by computer, that $P_{G,m}$ for $G = \mathbb{Z}_3$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, and $m \leq 3$ are normal. Hence, in this chapter we consider only $m \geq 4$.

We prove normality by induction. We consider a point $x \in kP_{G,m} \cap L_{G,m}$ and show that it can be can be written in the form $x = y + v$ where $y \in (k-1)P_{G,m-1}$ and v is a vertex of $P_{G,m}$. This, of course, means that also $y \in L_{G,m}$ since all vertices of $P_{G,m}$ belong to $L_{G,m}$ and this implies that $P_{G,m}$ is normal.

In the section 2.2 we described the symmetries of the polytope $P_{G,m}$. This allows us to consider only points x which are close to the vertex $v(0)$. Firstly, we solve the special cases, when $k = 2$ of $k = 3$ or x lies on a specific face. After that we consider the rest of the cases and we find for each x a vertex v such that $x - v \in kP_{G,m}$. We prove that by checking all inequalities from the facet description of $P_{G,m}$ which we have from section 2.3.

We will need the facet description for the dilated polytope $kP_{G,m}$. It is straightforward to derive it from the facet description of $P_{G,m}$ given in Section 2.3. However, in the case of $G = \mathbb{Z}_3$, we need to pass back to $3m$ coordinates. It follows from Theorem 2.3.3 that the facet description of $kP_{\mathbb{Z}_3,m}$ is given by the following equalities and inequalities:

$$x_i^j \geq 0, \text{ for } 0 \leq i \leq 2, 1 \leq j \leq m,$$

$$x_0^j + x_1^j + x_2^j = k, \text{ for } 1 \leq j \leq m,$$

$$\langle u_{a_1}, x^1 \rangle + \langle u_{a_2}, x^2 \rangle + \cdots + \langle u_{a_m}, x^m \rangle \geq 2k,$$

$$\langle w_{a_1}, x^1 \rangle + \langle w_{a_2}, x^2 \rangle + \cdots + \langle w_{a_m}, x^m \rangle \geq 2k,$$

for all $(a_1, a_2, \ldots, a_m) \in \{0,1,2\}^m$ such that $a_1 + a_2 + \cdots + a_m \equiv 2 \pmod 3$, where $u_0 = (0,1,2), u_1 = (1,2,0), u_2 = (2,0,1), w_0 = (0,2,1), w_1 = (1,0,2), u_2 = (2,1,0)$ and $x^j = (x_0^j, x_1^j, x_2^j)$.

For the m-tuple $A = (a_1, \ldots, a_m) \in \{0,1,2\}^m$ we denote the facets of kP_m corresponding to A and first and second inequality by $F_1(k, A)$ and $F_2(k, A)$, respectively. For the point x we denote left sides of the inequalities by $S_1(x, A)$ and $S_2(x, A)$. That means $x \in F_i(k, A) \Leftrightarrow S_i(x, A) = 2k$.

In the case of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, we denote the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, \alpha, \beta, \gamma\}$. From Theorem 2.3.1 the polytope $kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ defined by the following inequalities:

- $x_g^j \geq 0$ for all $g \in G, 1 \leq j \leq m$,
- $x_0^j + x_\alpha^j + x_\beta^j + x_\gamma^j = k, \text{ for all } 1 \leq j \leq m,$

- For all $A \subseteq \{1, 2, \dots, m\}$ with $|A|$ being an odd number:

$$\sum_{j\in A}(x_0^j + x_\alpha^j) + \sum_{j\notin A}(x_\beta^j + x_\gamma^j) \geq k,$$

$$\sum_{j\in A}(x_0^j + x_\beta^j) + \sum_{j\notin A}(x_\alpha^j + x_\gamma^j) \geq k,$$

$$\sum_{j\in A}(x_0^j + x_\gamma^j) + \sum_{j\notin A}(x_\alpha^j + x_\beta^j) \geq k$$

Analogously, we denote the left sides of the last three inequalities by $S_\alpha(x, A)$, $S_\beta(x, A)$, $S_\gamma(x, A)$ respectively. Each inequality gives us a facet of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$. We define

$$F_g(A) = \{x \in P_{\mathbb{Z}_2\times\mathbb{Z}_2,m} : S_g(x, A) = 1\}.$$

We want to prove that every point $x \in kP_{G,m} \cap L_{G,m}$ decomposes to a sum of lattice points from $P_{G,m}$. It is sufficient to prove it for an image of x under any of group actions described as symmetries of $P_{G,m}$.

We define a linear ordering on multisets of r real numbers with sum k, for $r = 3, 4$. For two multisets $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ where $a_1 \geq \dots \geq a_r$ and $b_1 \geq \dots \geq b_r$ we say

$$\{a_1, \dots, a_r\} \succ \{b_1, \dots, b_r\} \Leftrightarrow \exists i : (a_i > b_i) \wedge \forall j < i : (a_j = b_j).$$

Consider $x \in kP_{G,m} \cap L_{G,m}$. If we order multisets $\{x_g^j; g \in G\}$ we can ensure that multiset for $j = m$ is the smallest one in this ordering by acting with suitable permutation from $\mathbb{S}_m$.

If we denote by g_j the most frequent element in j-th multiset from G-presentation of x we can act by $(g_1, g_2, \dots, g_{m-1}, g_1 + \dots + g_{m-1}) \in H_{G,m}$ to obtain a point x for which $x_0^j \geq x_g^j$ for all $1 \leq j < m, g \in G$.

This means that for a point $x \in kP_{G,m} \cap L_{G,m}$, without loss of generality, we may assume the following two facts:

$$\forall j \in \{1, 2, \dots, m-1\} : \ \{x_g^j; g \in G\} \succeq \{x_g^m; g \in G\}. \tag{2.1}$$

$$\forall j \in \{1, 2, \dots, m-1\} : \ x_0^j = \max\{x_g^j; g \in G\}. \tag{2.2}$$

Definition 2.7.1. Let $x \in kP_{G,m} \cap L_{G,m}$. A vertex v of $P_{G,m}$ is called *x-good* if all coordinates of the point $x - v$ are non-negative.

If we add some assumptions on point $x \in kP_{G,m}$ we will need to subtract some of the specific vertices of $P_{G,m}$. Thus, we define these subsets of vertices:

Definition 2.7.2. Let $v(0)$ be the vertex of $P_{G,m}$ corresponding to the m-tuple $(0, \dots, 0)$.

For $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ let $v(g)_{j,j'}$ be the vertex of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ corresponding to the m-tuple which has on j-th and j'-th place g and all other places 0.

Let $V_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ be the following set of vertices of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$:

$$V_{\mathbb{Z}_2\times\mathbb{Z}_2,m} = \{v(0)\} \cup \{v(g)_{j,m} | 1 \leq j \leq m-1, g \in \{\alpha, \beta, \gamma\}\}.$$

For $G = \mathbb{Z}_3$ we denote by $V_{\mathbb{Z}_3,m} = \{v(0), v(j, m), v(m, j) | 1 \leq j \leq m-1\}$, where by $v(j_1, j_2)$ we mean the vertices from Lemma 2.3.2, but in R^{3m}.

Lemma 2.7.3. *a) Let $x \in L_{\mathbb{Z}_3,m} \cap kP_{\mathbb{Z}_3,m}$, $i \in \{1,2\}$ and $A = (a_1, \dots, a_m) \in \{0,1,2\}^m$. Then*

$$S_i(x, A) \equiv 2k \pmod 3.$$

b) Let $x \in L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ be a point such that $x_0^j + x_\alpha^j + x_\beta^j + x_\gamma^j = k$ for all $1 \le j \le m$. Let $g \in \{\alpha, \beta, \gamma\}$ and $A \subseteq \{1, 2, \dots, n\}$ be a set of odd cardinality. Then

$$S_g(x, A) \equiv k \pmod 2.$$

Proof. We start with part a). We consider only the case $i = 1$, the other case is analogous.

Since $x \in L_{\mathbb{Z}_3,m}$ we get

$$\sum_{j=1}^{m}(x_j^1 + 2x_2^j) \equiv 0 \pmod 3.$$

Moreover, it is easy to verify that

$$\langle u_i, x^j\rangle \equiv x_1^j + 2x_2^j + i(x_0^j + x_1^j + x_2^j) \equiv x_1^j + 2x_2^j + ik \pmod 3.$$

Therefore, we obtain that

$$S_i(x, A) = \sum_{j=1}^{m}\langle u_{a_j}, x^j\rangle \equiv \sum_{j=1}^{m} x_1^j + 2x_2^j + k\sum_{j=1}^{m} a_j \equiv 2k \pmod 3.$$

In a similar fashion we prove part b). To make everything clear we denote the elements of $\mathbb{Z}_2$ by $\mathbf{0}$ and $\mathbf{1}$. Again, we consider only the case $g = \alpha$, other cases are analogous.

Consider the homomorphism

$$\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$$

$$0, \alpha \mapsto \mathbf{0},\ \beta, \gamma \mapsto \mathbf{1}.$$

For $x \in L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ we get

$$\mathbf{0} = \varphi(0) = \varphi\left(\sum_{\substack{g\in\mathbb{Z}_2\times\mathbb{Z}_2\\ 1\le j\le n}}(x_g^j g)\right) = \sum_{\substack{g\in\mathbb{Z}_2\times\mathbb{Z}_2\\ 1\le j\le n}}(x_g^j\varphi(g)) = \sum_{j=1}^{n}(x_\beta^j + x_\gamma^j)\cdot\mathbf{1}.$$

Therefore $\sum_{j=1}^{n}(x_\beta^j + x_\gamma^j)$ must be even. This implies

$$\sum_{j\in A}(x_0^j + x_\alpha^j) + \sum_{j\notin A}(x_\beta^j + x_\gamma^j) = \sum_{j=1}^{n}(x_\beta^j + x_\gamma^j) + \sum_{j\in A}(x_0^j + x_\alpha^j - x_\beta^j - x_\gamma^j) =$$

$$= \sum_{j=1}^{n}(x_\beta^j + x_\gamma^j) - \sum_{j\in A}(x_0^j + x_\alpha^j + x_\beta^j + x_\gamma^j) + 2\sum_{j\in A}(x_0^j + x_\alpha^j) =$$

$$\sum_{j=1}^{n}(x_\beta^j + x_\gamma^j) - k|A| + 2\sum_{j\in A}(x_0^j + x_\alpha^j) \equiv k|A| \equiv k \pmod 2.$$

□

The previous lemma allows us to weaken the inequalities for the facets $F_g(A)$ by one in the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ or by two in the case $G = \mathbb{Z}_2$. Thus, to prove that $S_g(x, A) \geq k$ it is enough to prove that $S_g(x, A) \geq k - 1$. Despite the fact, the equality, of course, can not hold it will come very handy in the proof where we will be making estimates for $S_g(x, A)$.

The following result implies the fact that it is sufficient to consider only such points x for which the following condition holds:

$$\forall j \in \{1, 2, \dots, m\}, g \in G : x_0^j < k. \tag{2.3}$$

Geometrically, points on specific facets can be easily decomposed from the induction hypothesis, thus it is sufficient to consider only the points inside the polytope.

Lemma 2.7.4. *Suppose that for every positive integers k, m and every $x \in kP_{G,m} \cap L_{G,m}$ such that $x_0^j < k$ for all j we can write x as a sum of k vertices of $P_{G,m}$. Then $P_{G,m}$ is normal, for every positive integer m.*

Proof. Proof by induction on m. $P_{G,1}$, $P_{G,2}$ and $P_{G,3}$ are normal.

Suppose that $P_{G,m-1}$ is normal. We prove that also P_{m-1} is normal. Consider a point $x \in kP_{G,m} \cap L_{G,m}$. If $x_g^j < k$ for all g, j then x decomposes by assumption. Therefore we may assume that $x_g^j = k$ for some g, j. By acting with suitable permutation we can assume that $j = m$ and then by acting with $(g, 0, \dots, 0, -g)$ we obtain $g = 0$.

Consider now the projection $\pi : \mathbb{R}^{|G|m} \to \mathbb{R}^{|G|(m-1)}$ on the first $|G|(m-1)$ coordinates. Since $x \in kP_{G,m}$ there exist positive real numbers $\lambda_1, \dots, \lambda_s$ with $\lambda_1 + \dots + \lambda_s = k$ such that $\lambda_1 v_1 + \dots + \lambda_s v_s = x$, where $v_1, \dots, v_s$ are some vertices of $P_{G,m}$. But $(v_i)_0^m \leq 1$ and $x_0^m = k$ implies $(v_i)_0^m = 1$ for all i.

Consequently, $\pi(v_i)$ is a vertex of $P_{G,m-1}$ and $\pi(x) \in kP_{G,m-1}$. By induction hypothesis $\pi(x)$ decomposes to $\pi(x) = u_1 + \dots + u_k$, where u_i are vertices of $P_{G,m-1}$.

Now we simply put $u_i' \in \pi^{-1}(u_i)$ such that $(u_i')_0^m = 1$ and $(u_i')_g^m = 0$ for $g \neq 0$. Obviously all u_i' are vertices of $P_{G,m}$ and we have $x = u_1' + \dots + u_m'$. □

2.7.1 The proof for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$

In this subsection, we finish the proof of normality for the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. We start with a very simple lemma which will be used later to prove the inequalities.

Lemma 2.7.5. *Let $x \in kP_{\mathbb{Z}_2 \times \mathbb{Z}_2, m} \cap L_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$ and $j \in \{1, 2, \dots, n\}$. Suppose that $x_0^j \geq x_\alpha^j, x_\beta^j, x_\gamma^j$ and let $g \in \{\alpha, \beta, \gamma\}$. Then $x_0^j + x_g^j \geq \lceil k/3 \rceil$ and the equality holds if and only if $x_g^j = 0$ and $x_h^j = k/3$ for $h \neq g$.*

Proof.

$$3(x_0^j + x_g^j) \geq 3x_0^j + x_g^j \geq x_0^j + x_\alpha^j + x_\beta^j + x_\gamma^j = k.$$

We divide by 3 and realise that x_g^j are integers to obtain wanted inequality. The part about the equality is obvious. □

In the following three lemmas we consider specific choices of points x. In these specific cases, it is quite easy to prove that we can decompose the point x as the sum of the vertices of $P_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$. However, it is necessary to consider these cases separately, since they are in some sense too small and the general proof does not work for them.

Lemma 2.7.6. *Let $x \in 2P_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$. Then x can be written in the form $x = v + v'$ where v, v' are vertices of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.*

Proof. We assume, without loss of generality, (1), (2) and (3). Thus $x_0^j > 0$ for $1 \le j \le m-1$. If also $x_0^m > 0$ then $x = v(0) + (x - v(0))$. Further, $x - v(0)$ must be a vertex of P_m since it has non-negative coordinates and sum of elements in G-presentation of $x - v(0)$ is 0 since it is 0 for both x and v_0. If $x_0^n = 0$ then by acting with suitable $\varphi \in \mathrm{Aut}(G)$ we have $x_\alpha^n = x_\beta^m = 1$ since $x_g^m < 2$ for all g by condition (3).

Since $S_\gamma(x, \{m\}) \ge 2$ at least one of the numbers x_g^j for $g = \alpha, \beta$; $1 \le j \le m-1$ is greater than 0. Then $x = v(g)_{j,m} + (x - v(g)_{j,m})$ for such g, j. By the same arguments as above $(x - v(g)_{j,m})$ must be a vertex of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$. □

Lemma 2.7.7. *Let $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ be such that there are at least three multisets $\{k/3, k/3, k/3, 0\}$ in G-presentation of x. Then $x = y + v$, where v is a vertex of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ and $y \in (k-1)P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.*

Proof. By acting with suitable permutation from $\mathbb{S}_m$ we may assume that these three multisets are the first three. Then by acting with suitable $(g_1, g_2, \ldots, g_n) \in H_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ we may assume $x_0^j = x_\alpha^j = x_\beta^j = k/3$ for $j = 1, 2, 3$. We describe the G-presentation of v (which is a n-tuple $(g_1, g_2, \ldots, g_n)$ of elements from $\mathbb{Z}_2 \times \mathbb{Z}_2$). We pick the last $m-2$ elements arbitrarily, the only condition is that g_j belongs to the j-th multiset from G-presentation of x. Then we pick g_1 and g_2 such that sum of this m-tuple is 0 and g_1, g_2, g_3 are not all equal. Since g_1 and g_2 can be any from $0, \alpha, \beta$, it is possible.

Now we need to check that $x - v = y \in (k-1)P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$. We only need to check the inequalities for sets A. However, if we try to compute $S_g(y, A)$ we always get at least $k-2$ already on the first three coordinates. Therefore, due to Lemma 2.7.3, the inequalities hold. □

From now, we may assume that $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ satisfies the following condition since the other case is covered by the previous lemma.

$$\text{At most two multisets from } G\text{-presentation of } x \text{ are } \{k/3, k/3, k/3, 0\}. \tag{2.4}$$

Lemma 2.7.8. *Let $x \in 3P_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ satisfy (2), (4). Let $A \subseteq \{1, 2, \ldots, m\}$ be a set with $|A| \ge 5$. Then $S_g(x - v, A) \ge 2$ for any $g \in \{\alpha, \beta, \gamma\}$ and any x-good vertex v of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.*

Proof. Let $B \subseteq \{1, 2, \ldots, m\}$ be the set of those indices for which multisets in G-presentation of x are equal to $\{1, 1, 1, 0\}$. This together with condition (2) yields $x_0^j \ge 2$ for $j \notin B$. Condition (4) implies $|B| \le 2$. It follows that

$$S(x - v, A) \ge \sum_{j \in A \setminus B} (x_0^j - 1) \ge |A \setminus B| \ge 2.$$

□

Idea of the proof

Now we prove for all positive integers k, m that every point $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_n$ can be written in the form $x = y + v$ where $y \in (k-1)P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ and v is a vertex of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.

This, of course, means that also $y \in L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ since all vertices of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ belong to $L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ and this implies that $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ is normal.

Consider a point $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$. It is sufficient to consider $k \geq 3$ because the case $k = 2$ is solved by Lemma 2.7.6. Without loss of generality, from now we will suppose that x satisfies (1), (2), (3) and (4). To conclude we need to pick an x-good vertex v and then check that $y = x - v$ belongs to $(k-1)P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$. We prove this by checking all inequalities from facet characterization of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ for every set A with odd cardinality.

Regarding the vertex v, we show we can always use some vertex $v \in V_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ as in Definition 3.

Big sets A

Proposition 2.7.9. *Let $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}, k \geq 3$ satisfy $(1)-(4)$ and let $A \subseteq \{1, 2, \ldots, m\}$ be a set with odd cardinality.*

a) If $|A| \geq 5$ then $S_g(x - v, A) \geq k - 1$ for any $g \in \{\alpha, \beta, \gamma\}$ and any x-good vertex v of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.

b) If $|A| = 3, m \notin A$ and x satisfies (4) then $S_g(x-v, A) \geq k-1$ for any $g \in \{\alpha, \beta, \gamma\}$ and any x-good vertex v of $P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.

Proof. Let $y = x - v$. Clearly, it is sufficient to prove the inequality for $g = \alpha$. We begin with part a):

$$\begin{aligned}\sum_{j\in A}(y_0^j + y_\alpha^j) + \sum_{j\notin A}(y_\beta^j + y_\gamma^j) &\geq \sum_{j\in A\setminus\{m\}}(y_0^j + y_\alpha^j)\\ &\geq \sum_{j\in A\setminus\{m\}}(x_0^j + x_\alpha^j - 1)\\ &\geq \sum_{j\in A\setminus\{m\}}(\lceil k/3\rceil - 1)\\ &\geq 4\lceil k/3\rceil - 4 \geq k-2.\end{aligned}$$

The last inequality holds for $k \geq 4$. Case $k = 3$ is covered in Lemma 2.7.8. We also used Lemma 2.7.5 and $|A \setminus \{m\}| \geq 4$. Inequality $S_g(y, A) \geq k - 2$ together with Lemma 2.7.3 implies $S_g(y, A) \geq k - 1$.

Proof of part b) is similar:

$$\begin{aligned}\sum_{j\in A}(y_0^j + y_\alpha^j) + \sum_{j\notin A}(y_\beta^j + y_\gamma^j) &\geq \sum_{j\in A}(y_0^j + y_\alpha^j)\\ &\geq \sum_{j\in A}(x_0^j + x_\alpha^j - 1)\\ &\geq 3\lceil k/3\rceil - 3 \geq k-3,\end{aligned}$$

where we again used Lemma 2.7.5. Lemma 2.7.3 implies that $S_g(y, A) \neq k - 2$. Therefore, the only bad case is when we have equality. This is possible only if we have equality everywhere, in particular $x_0^j = x_\beta^j = x_\gamma^j = k/3$ for all $j \in A$. But this means that x does not satisfy (4) which is a contradiction. □

Therefore it is sufficient to check inequalities for $|A| = 1$ and $|A| = 3$ such that $m \in A$.

Small sets A

Since $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ we have the inequalities $S_g(x,A) \geq k$ for any g and any set A with odd cardinality. For big sets A discussed in Section 5.2 we have not used them. However, we use them for smaller sets. Our first step is to observe how does $S_g(x,A)$ change when we subtract some vertex $v \in V_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ from x.

Lemma 2.7.10. *Let $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$, $v \in V_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$, $g \in \{\alpha,\beta,\gamma\}$ and $|A| = 1$ or $|A| = 3$ with $m \in A$. Then*

$$S_g(x - v, A) = S_g(x, A) - 3 \text{ or } S_g(x - v, A) = S_g(x, A) - 1.$$

Moreover, for $|A| = 1$ we have $S_g(x - v, A) = S_g(x, A) - 1$ if and only if one of the following conditions holds:

- $v = v(0)$
- $v = v(g)_{j,m}$ *for any* $1 \leq j \leq n - 1$
- $v = v(g')_{j,m}$ *for* $g' \neq g$ *and* $A = \{j\}$ *or* $|A| = \{m\}$

Also $S_g(x - v, \{m\}) \geq k - 1$.

Proof. For the first part, one checks how many summands in $S_g(x,A)$ will decrease by 1 when we subtract v. The last part is clear consequence since $S_g(x,\{m\}) \geq k$ for $x \in kP_m$. □

Now we consider the following:

Proposition 2.7.11. *Let $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ satisfy conditions $(1)-(4)$. Suppose that 0 is also the most frequent element in the m-th multiset from G-presentation of x. Then $x - v(0) \in (k-1)P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.*

Proof. Obviously, every multiset from G-presentation of x contains 0 so $x - v(0)$ has non-negative coordinates and therefore $v(0)$ is x-good. Inequalities for sets with $|A| \geq 3$ hold for $x - v$ by Proposition 2.7.9 since for sets with $|A| = 3$ and $n \in A$ we can use same arguments. Inequalities for $|A| = 1$ hold by Lemma 2.7.10 since we are subtracting $v(0)$. It follows that $x - v(0) \in (k-1)P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$. □

The previous proposition implies that we can assume that for $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ satisfying $(1),(2),(3)$ also the following condition holds:

$$\begin{array}{l}\text{There exists no } h \in H_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \text{ such that the following conditions holds:}\\ 0 \text{ is the most frequent element in all multisests from } G\text{-presentation of } hx.\end{array} \tag{2.5}$$

Proposition 2.7.12. *Let $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ satisfy $(1) - (5)$. Then:*

a) *x does not belong to any facet $kF_g(A)$ for $|A| = 3, m \in A$, i.e. $S_g(x,A) > k$ for all such A and $g = \alpha,\beta,\gamma$.*

b) *$S_g(x - v, A) \geq k - 1$ for all $v \in V_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$, $g \in \{\alpha,\beta,\gamma\}$ and $|A| = 3, n \in A$.*

Proof. We prove part a) by contradiction: Suppose that we have an equality for $A = \{1, 2, m\}$ and $g = \alpha$. We may get to this situation by acting with suitable $\sigma \in \mathbb{S}_n$ and $\varphi \in \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. We compute $S_\alpha(x, A)$:

$$S_\alpha(x, A) = \sum_{j \in A}(x_0^j + x_\alpha^j) + \sum_{j \notin A}(x_\beta^j + x_\gamma^j) \geq x_0^1 + x_0^2 + x_0^m + x_\alpha^m =$$
$$= x_0^1 + x_0^2 + k - (x_\beta^n + x_\gamma^m) \geq x_0^1 + x_0^2 + k - 2\min\{x_0^1, x_0^2\} \geq k.$$

An equality holds only if there is equality in all inequalities. In particular, it means that $x_0^1 = x_0^2 = x_\beta^n = x_\gamma^m$ and $x_\alpha^1 = 0$. But from ordering of multisets, we get that also some $x_{g_0}^1 = x_0^1$ for $g_0 = \beta$ or $g_0 = \gamma$. By acting with $(g_0, 0, \ldots, 0, g_0) \in H_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$ we get to the situation where 0 is the most frequent also in m-th multiset and still is also most frequent on the first one. This is a contradiction with condition (5).

We continue with proof of part b). Part a) together with Lemma 2.7.3 implies that $S_g(x, A) \geq k + 2$. Consequently, Lemma 2.7.10 implies $S_g(x - v, A) \geq k - 1$ for any $v \in V_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$. □

Proposition 2.7.13. *Let $x \in kP_{\mathbb{Z}_2 \times \mathbb{Z}_2, m} \cap L_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$ satisfy $(1) - (5)$ and $x_0^m > 0$. Then $x - v(0) \in (k-1)P_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$.*

Proof. Clearly, $v(0)$ is x-good. Inequalities for $|A| \geq 3$ hold by Propositions 2.7.9 and 2.7.12. For $|A| = 1$ we have $S_g(x, A) \geq k$, then by Lemma 2.7.10 we get $S_g(x - v(0), A) \geq k - 1$. Since all inequalities hold $x - v(0) \in (k-1)P_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$. □

Therefore we are left only with the case $x_0^m = 0$.

Special case $x_0^m = 0$

In this case we will subtract a vertex $v(g)_{j,m}$ for a special choice of g and j. Propositions 2.7.9 and 2.7.12 and Lemma 2.7.10 imply that it is enough to check inequalities for $|A| = 1, A \neq \{m\}$. We distinguish two cases depending on whether x lies or does not lie on a facet $kF_g(A)$ for such A.

Proposition 2.7.14. *Let $x \in kP_{\mathbb{Z}_2 \times \mathbb{Z}_2, m} \cap L_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$ satisfy $(1) - (5)$, $x_0^m = 0$ and x does not belong to any facet $kF_g(A)$ for $|A| = 1, A \neq \{m\}$. Then there exists a vertex $v \in V_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$ such that $x - v \in (k-1)P_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$.*

Proof. For any $v \in V_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$ Lemma 2.7.10 implies that for any A with $|A| = 1, A \neq \{m\}$ we have $S_g(x - v, A) \geq S_g(x, A) - 3 \geq k - 1$. We used Lemma 2.7.3 to deduce inequality $S_g(x, A) \geq k + 2$. Therefore, inequalities for every set A hold for any x-good vertex $v \in V_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$, since bigger sets are taken care of by Propositions 2.7.9 and 2.7.12. Consequently, it is sufficient to pick any x-good vertex $v \in V_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$.

At least two of the numbers x_g^m for $g \in \{\alpha, \beta, \gamma\}$ must be non-zero by condition (3) and the fact that $x_0^m = 0$. Without loss of generality, let those two coordinates be x_α^m and x_β^m.

Since $S_\gamma(x, \{m\}) \geq k$ and $x_0^m + x_\gamma^m < k$, at least one of the numbers x_α^j and x_β^j for $1 \leq j \leq m - 1$ must be non-zero. Let it be $x_{g_0}^j$. For $v = v(g_0)_{j,m}$ all coordinates of $x - v$ are non-negative since condition (2) implies $x_0^j > 0$ for $1 \leq j \leq m - 1$. This means we have found x-good vertex $v \in V_{\mathbb{Z}_2 \times \mathbb{Z}_2, m}$ and the proposition is proved. □

If x belongs to a facet we prove that it belongs to only one facet and that we can as well subtract a vertex $v \in V_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$:

Proposition 2.7.15. *Let $x \in kP_{\mathbb{Z}_2\times\mathbb{Z}_2,m} \cap L_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ satisfy $(1)-(4)$, $x_0^m = 0$ and x belongs to some facet $kF_g(A)$ for $|A| = 1, A \neq \{m\}$. Then*

a) x belongs to only one such facet.

b) There exists a vertex $v \in V_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$ such that $x - v \in (k-1)P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.

Proof. By acting with suitable permutation from $\mathbb{S}_m$ and $\varphi \in \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ we can get to situation where $x \in kF_\alpha(\{1\})$. We have

$$k = S_\alpha(x, \{1\}) \geq x_0^1 + x_\beta^m + x_\gamma^m = x_0^1 + k - x_\alpha^m \geq k.$$

To get an equality, there must be an equality in all inequalities, specifically $x_\alpha^1 = x_\beta^j = x_\gamma^j = 0$ for all $2 \leq j \leq m$ and $x_0^1 = x_\alpha^n = \max_{g\in\mathbb{Z}_2\times\mathbb{Z}_2}\{x_g^m\}$.

Assumption that x belongs to a facet give us strong conditions. It is easy to see that x cannot belong to some other facet $kF_\alpha(\{j\})$ for $j < m$ because it would imply $x_\beta^1 = x_\gamma^1 = 0$. But this is a contradiction with condition (3). Also x cannot belong to some $kF_\beta(\{j\})$ for $1 \leq j < m$ because it would imply $x_\alpha^i = x_\beta^i = x_\gamma^i$ for $i \neq 1, j, n$ which is again a contradiction with (3). Same arguments hold for $kF_\gamma(\{j\})$. This proves part a).

For part b), by the same arguments as in the proof of Proposition 2.7.14 for any x-good vertex $v \in V_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$, $g \in \{\alpha, \beta, \gamma\}$ and set A we have $S_g(x - v, A) \geq k - 1$, except the case when $g = \alpha$ and $A = \{1\}$.

Since $k \leq S_\alpha(x, \{m\}) = x_\beta^1 + x_\gamma^1 + x_\alpha^m$ at least one of the numbers x_β^1, x_γ^1 must be greater than 0. Also one of the numbers x_β^m and x_γ^m is greater than zero by condition (3).

If both numbers x_g^m and x_g^1 are greater than zero for $g = \beta$ or $g = \gamma$ then the vertex $v = v(g)_{1,m}$ is x-good. By Lemma 2.7.10 also $S_\alpha((x - v), \{1\}) \geq k - 1$ and therefore $x - v \in (k-1)P_{\mathbb{Z}_2\times\mathbb{Z}_2,m}$.

Suppose the opposite, i.e. $x_\beta^1 = 0$ and $x_\gamma^m = 0$ (we can get to this case by acting with $\varphi \in \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$). Then $S_\gamma(x, \{m\}) \geq k$ implies that at least one of the numbers x_α^j for $2 \leq j \leq m-1$ is greater than 0. Then we can subtract $v = v(\alpha)_{j,m}$ for such j. Again $x - v$ has non-negative coordinates and by Lemma 2.7.10 $S_\alpha((x - v), \{1\}) \geq k - 1$. □

proof of Theorem 2.4.3 for $\mathbb{Z}_2 \times \mathbb{Z}_2$. Consider point $x \in kP_n \cap L_n$ for some positive integer k. If $k = 2$ then x decomposes due to Lemma 2.7.6. To prove normality of P_n it is sufficient for $k \geq 3$ to prove that there exists a vertex v of P_n such that $x - v \in (k-1)P_n$. Also it is sufficient to consider only points x which satisfy $(1)-(3)$. The existence of such v is implied by Lemma 2.7.7 and Propositions 2.7.11, 2.7.13, 2.7.14 and 2.7.15. □

2.7.2 The proof for the group $\mathbb{Z}_3$

In this section, we finish the proof for the group $\mathbb{Z}_3$. It will be quite similar to the previous section - just the inequalities and coordinate are little but different.

We prove that for every point $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}$ there exists a vertex v such that $x - v \in (k-1)P_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}$. It is sufficient to consider the points which satisfy $(1)-(3)$ and we will show that we can use a vertex $v \in V_{\mathbb{Z}_3,m}$. The normality of $P_{\mathbb{Z}_3,m}$ will follow by induction. Since $x - v \in L_{\mathbb{Z}_3,m}$ it remains to prove that $x - v \in (k-1)P_{\mathbb{Z}_3,m}$. We will show this by checking inequalities from the facet description of $(k-1)P_{\mathbb{Z}_3,m}$.

We examine all inequalities. We split them into groups depending on the sum $a_1 + \cdots + a_m$.

Proposition 2.7.16. *Let $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}, k \geq 3$ satisfy $(1) - (3)$ and $A = (a_1, \ldots, a_m) \in \{0,1,2\}^m$.*

a) If $a_1 + \cdots + a_m \geq 8$, then $S_i(x - v, A) \geq 2(k-1)$ for any $i \in \{1,2\}$ and any x-good vertex v of $P_{\mathbb{Z}_3,m}$.

b) If $a_1 + \cdots + a_m = 5, a_m < 2$, then $S_i(x - v, A) \geq 2(k-1)$ for any $i \in \{1,2\}$ and any x-good vertex v of $P_{\mathbb{Z}_3,m}$.

c) If $a_1 + \cdots + a_m = 5, a_m = 2$, then $S_2(x - v, A) \geq 2(k-1)$ for any x-good vertex v of $P_{\mathbb{Z}_3,m}$.

Proof. Let $y = x - v$. It is sufficient to prove parts a) and b) when $i = 1$, the other case is analogous.

We first notice the following two inequalities which hold for all $1 \leq j \leq m-1$:

$$\langle u_2, y^j \rangle = 2y_0^j + y_2^j = 2x_0^j + x_2^j - 2v_0^j - v_2^j \geq x_0^j + x_1^j + x_2^j - 2 = k - 2,$$

$$\langle u_1, y^j \rangle = y_0^j + 2y_1^j = x_0^j + 2x_1^j - v_0^j - 2v_1^j \geq \frac{1}{2}(x_0^j + x_1^j + x_2^j) - 1 + \frac{3}{2}x_1^j - v_1^j \geq \frac{k}{2} - 1.$$

This yields $\langle u_i, y^j \rangle \geq i(k/2 - 1)$. Further, in all cases we have $a_1 + \cdots + a_{m-1} \geq 4$. Now we can prove the desired inequality:

$$S_1(x - v, A) \geq ka_1/2 + \cdots + ka_{m-1}/2 \geq 4(k/2 - 1) = 2k - 4.$$

We know that $S_i(x - v, A) \equiv 2k - 2 \pmod 3$, which implies $S_i(x - v, A) \geq 2k - 2$.

For the part c): from the fact $x_1^m \geq x_2^m$ we get an analogous inequality

$$\langle u_2, y^m \rangle \geq 2y_0^j + y_1^j = 2x_0^j + x_1^j - 2v_0^j - v_1^j \geq \frac{1}{2}(x_0^j + x_1^j + x_2^j) - 1 + \frac{3}{2}x_0^j - v_0^j \geq \frac{k}{2} - 1.$$

Together with the fact that $a_1 + \cdots + a_{m-1} = 3$ we obtain

$$S_2(x - v, A) \geq ka_1/2 + \cdots + ka_{m-1}/2 + \frac{k}{2} - 1 \geq 3(\frac{k}{2} - 1) + \frac{k}{2} - 1 = 2k - 4,$$

Now we conclude our desired result in the same way as in the parts a) and b). $\square$

For the cases which are not covered by the Proposition 2.7.16 we use induction. We know from induction hypothesis that $S_i(x, A) \geq 2k$, for all n-tuples $A = (a_1, \ldots, a_m)$. We observe how $S_i(x, A) \geq 2k$ changes when we subtract the vertex v from x.

Lemma 2.7.17. *Let $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m},\ v \in V_{\mathbb{Z}_3,m},\ A = (a_1, \ldots, a_m) \in \{0,1,2\}^m$. Denote $D_i(A) = S_i(x, A) - S_i(x - v, A)$.*

- *If* $a_1 + \cdots + a_m = 5$, $a_m = 2$, *then* $D_1(A) \in \{2, 5\}$.
- *If* $a_1 + \cdots + a_m = 2$, *then* $D_1(A) \in \{2, 5\}$. *Moreover, if* $v = v(0)$ *or* $a_m = 2$ *then* $D_1(A) = 2$. *If* $a_m = 1, a_j = 1$ *and* $v = v(m, j)$, *then also* $D_1(A) = 2$.
- *If* $a_1 + \cdots + a_m = 2$ *then* $D_2(A) \in \{2, 5\}$. *Moreover, if* $v = v(0)$ *or* $a_m = 2$ *then* $D_2(A) = 2$. *If* $a_m = 1$ *and* $v = v(m, j)$ *for some* j, *then also* $D_2(A) = 2$.

Proof. We check for every possible v and A how many summands in $S_i(x, A)$ will decrease when we subtract v. We see that it will be always decreased by 2 or 5. □

Remark 2.7.18. *We could list all cases when* $D_i(A) = 2$, *but since it would be a long list, we just list those which will be important later in the proof.*

Now we consider the following special case:

Proposition 2.7.19. *Let* $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}$ *satisfying conditions* $(1) - (3)$. *Suppose that* 0 *is also the most frequent element in the* m*-th multiset from* G*-presentation of* x. *Then* $x - v(0) \in (k-1)P_{\mathbb{Z}_3,m}$.

Proof. Obviously, every multiset from G-presentation of x contains 0, so $v(0)$ is x-good. Inequalities for n-tuples with $a_1 + \cdots + a_m \geq 5$ hold for $x - v$ by Proposition 2.7.16, because when 0 is also most frequent in the m-th multiset there is no special case when $a_1 + \cdots + a_m = 5$. Inequalities for $a_1 + \cdots + a_m = 2$ hold by Lemma 2.7.17, since we are subtracting vertex $v(0)$. It follows that $x - v(0) \in (k-1)P_{\mathbb{Z}_3,m}$. □

This result implies that we can assume that for $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}$ satisfying $(1), (2), (3)$ also holds the following condition:

$$\begin{aligned}&\text{There exists no } h \in H_m \text{ such that the following conditions holds:}\\ &0 \text{ is the most frequent element in all multisests from } G\text{-presentation of } hx.\end{aligned} \tag{2.6}$$

We show that in our case the point x cannot lie on many facets.

Proposition 2.7.20. *Let* $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}$ *satisfy* $(1) - (4)$, $A = (a_1, \ldots, a_m) \in \{0, 1, 2\}^m$ *with* $a_1 + \cdots + a_m = 5$ *and* $a_m = 2$. *Then:*

a) x *does not belong to facet* $F_1(A)$, *i.e.* $S_1(x, A) > 2k$, *for all such* A *and* x.

b) $S_1(x - v, A) \geq 2(k-1)$, *for all* $v \in V_{\mathbb{Z}_3,m}$.

Proof. We prove part a) by contradiction: Suppose that we have an equality for some A. Since $a_m = 2$ there are two cases regarding $a_1, \ldots, a_{m-1}$: either three of them are equal to 1 or one of them is equal to 1 and other to 2. By acting with suitable $\sigma \in S_m$ we get to the case where either $a_1 = a_2 = a_3 = 1$ or $a_1 = 1, a_2 = 2$. Consider the first case. Then

$$\begin{aligned} S_1(x, A) &\geq x_0^1 + 2x_1^1 + x_0^2 + 2x_1^2 + x_0^3 + 2x_1^3 + 2x_0^m + x_2^m \\ &\geq \frac{1}{2}(x_0^1 + x_1^1 + x_2^1) + \frac{1}{2}(x_0^2 + x_1^2 + x_2^2) + (x_0^3 - x_1^m) + (x_0^m + x_1^m + x_2^m) \\ &\geq \frac{k}{2} + \frac{k}{2} + 0 + k = 2k. \end{aligned}$$

We use the ordering of multisets (condition (1)) for $(x_0^3 - x_1^m) \geq 0$. To get an equality we must have equality everywhere. In particular, we get $x_0^1 = x_2^1$, $x_1^1 = x_1^3 = x_0^m = 0$ and $x_0^3 = x_1^m$. From condition (2) we obtain that $\{x_0^1, x_1^1, x_2^1\} = \{1/2, 1/2, 0\} \succeq \{x_0^m, x_1^m, x_2^m\} = \{0, x_1^m, x_2^m\}$, which means $x_1^m = x_2^m = 1/2$. By acting with $(2, 0, 0, \ldots, 0, 1)$ we get to the situation where 0 is the most frequent element in all multisets, which is a contradiction with (4).

The case $a_1 = 1, a_2 = 2$ is analogous:

$$\begin{aligned} S_1(x, A) &\geq x_0^1 + 2x_1^1 + 2x_0^2 + x_2^2 + 2x_0^m + x_2^m \\ &\geq \frac{1}{2}(x_0^1 + x_1^1 + x_2^1) + \frac{1}{2}(x_0^2 + x_1^2 + x_2^2) + (x_0^2 - x_1^m) + (x_0^m + x_1^m + x_2^m) \\ &\geq \frac{k}{2} + \frac{k}{2} + 0 + k = 2k. \end{aligned}$$

Similarly, we get that $x_1^m = x_2^m = x_0^1 = x_2^1 = 1/2$ and we can act with $(2, 0, 0, \ldots, 0, 1)$ to obtain the same situation, where 0 is the most frequent element in all multisets, contradicting (4).

We continue with the proof of part b). Part a) together with Lemma 2.7.3 implies that $S_i(x, A) \geq k + 2$. Consequently, Lemma 2.7.17 implies $S_1(x - v, A) \geq 2(k - 1)$, for any $v \in V_{\mathbb{Z}_{3,m}}$. □

Proposition 2.7.21. *Let $x \in kP_{\mathbb{Z}_{3,m}} \cap L_{\mathbb{Z}_{3,m}}$ satisfying $(1) - (4)$ and $x_0^m > 0$. Then $x - v(0) \in (k-1)P_{\mathbb{Z}_{3,m}}$.*

Proof. Clearly, $v(0)$ is x-good. Inequalities for A with $a_1 + \cdots + a_m \geq 5$ hold by Propositions 2.7.16 and 2.7.20. For $a_1 + \cdots + a_m = 2$ we have $S_i(x, A) \geq 2k$, then by Lemma 2.7.17 we get $S_i(x - v(0), A) \geq 2(k - 1)$. Therefore, all inequalities hold and $x - v(0) \in (k - 1)P_{\mathbb{Z}_{3,m}}$. □

This means we are left with the special case $x_0^m = 0$. In this case, we cannot subtract $v(0)$ since it is not an x-good vertex. We will use some other vertex from $V_{\mathbb{Z}_32,m}$. We now have to also check facets for m-tuples A with $a_1 + \cdots + a_m = 2$. Again, we show that our assumptions imply that x doe not lie on some of these facets:

Proposition 2.7.22. *Let $x \in kP_{\mathbb{Z}_{3,m}} \cap L_{\mathbb{Z}_{3,m}}$ satisfy $(1) - (4)$, $x_0^m = 0$ and $A = (a_1, \ldots, a_m) \in \{0, 1, 2\}^m$ with $a_1 + \cdots + a_m = 2$ and $a_m = 0$. Then x does not belong to the facet $F_i(A)$.*

Proof. We have two options: either two of the numbers a_j are 1 or one of the numbers is 2. We can assume without loss of generality $a_1 = a_2 = 1$ in the first case and $a_1 = 2$ in the second. Also, without loss of generality, we can assume $i = 1$.

We start with the first case and compute $S_1(x, A)$:

$$\begin{aligned} S_1(x, A) &\geq x_0^1 + 2x_1^1 + x_0^1 + 2x_1^1 + x_1^m + 2x_2^m \\ &\geq \frac{1}{2}(x_0^1 + x_1^1 + x_2^1) + \frac{1}{2}(x_0^2 + x_1^2 + x_2^2) + (x_1^m + x_2^m) \geq \frac{k}{2} + \frac{k}{2} + k = 2k. \end{aligned}$$

To get an equality we must have $x_2^m = 0$ and $x_1^m = k$. Condition (1) and (2) then gives us $x_0^1 = k$ which is contradiction with the condition (3).

The case $a_1 = 2$ is similar:

$$\begin{aligned} S_1(x, A) &\geq 2x_0^1 + x_1^1 + x_1^m + 2x_2^m \\ &\geq (x_0^2 + x_1^2 + x_2^2) + (x_1^m + x_2^m) \geq k + k = 2k. \end{aligned}$$

Again, we must have $x_2^m = 0$ and $x_1^m = k$ which is impossible for the same reason. □

For the point x satisfying $(1) - (4)$ we can also assume that $x_1^m \geq x_2^m$. We can achieve that by acting with suitable $\varphi \in \mathrm{Aut}(\mathbb{Z}_3)$. Now we solve the case when x does not lie on any facet $F_1(A)$ with $a_1 + \cdots + a_m = 2, a_m = 1$ by adding one assumption:

Proposition 2.7.23. *Let $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}$ satisfy $(1) - (4)$, $x_0^m = 0, x_1^m \geq x_2^m$ and x does not belong to any facet $F_1(A)$ with $a_1 + \cdots + a_m = 2$, $a_m = 1$. Moreover, suppose that there exists an index $j < m$ such that $x_2^j > 0$. Then $x - v(m, j) \in (k-1)P_{\mathbb{Z}_3,m}$.*

Proof. Clearly, the vertex $v(m, j)$ is x-good. Then Propositions 2.7.16 and 2.7.20 and Lemma 2.7.17 imply that the inequalities for m-tuple A with $a_1 + \ldots a_m \geq 5$ hold. For $a_1 + \cdots + a_m = 2$, Lemma 2.7.17 and Lemma 2.7.22 implies $S_i(x - v, A) \geq S_i(x, A) - 5 \geq 2(k-1)$, where $S_i(x, A) \geq 2k + 3$ follows from Lemma 2.7.3.

Therefore, inequalities for every m-tuple A hold, which proves our result. □

If x belongs to facet $F_1(A)$ with $a_1 + \cdots + a_m = 2, a_m = 1$ we prove that it belongs only to one such facet and that we can as well subtract some vertex $v(m, j)$:

Proposition 2.7.24. *Let $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}$ satisfy $(1) - (4)$, $x_0^m = 0$, $x_1^m \geq x_2^m$ and x belongs to some facet $F_1(A)$ for $a_1 + \cdots + a_m = 2$, $a_m = 1$. Moreover, suppose that there exists an index $j < m$ such that $x_2^j > 0$. Then*

a) x belongs to only one such facet.

b) There exists a vertex $v \in V_{\mathbb{Z}_3,m}$ such that $x - v \in (k-1)P_{\mathbb{Z}_3,m}$.

Proof. To prove part a) suppose that x belongs to two such facets. By acting with suitable permutation from $\mathbb{S}_m$ we can get to the situation where these facets are $F_1((1, 0, \ldots, 0, 1))$ and $F_1(0, 1, 0, \ldots, 0, 1)$.

Then we compute the following sum:

$$S_1(x, (1, 0, \ldots, 0, 1)) + S_1(x, (0, 1, 0, \ldots, 0, 1))$$

$$\begin{aligned} &\geq (x_0^1 + 3x_1^1 + 2x_2^1) + (x_0^2 + 3x_1^2 + 2x_2^2) + (4x_1^m) \\ &\geq k + k + 2(x_1^m + x_2^m) = 4k. \end{aligned}$$

We know that there must be equality and therefore $x_1^1 = x_2^1 = 0$ which is a contradiction with condition (3).

For part b) we again suppose without loss of generality that $x \in F_1((1, 0, \ldots, 0, 1)$. We show by contradiction that $x_2^1 > 0$. Suppose that $x_2^1 = 0$. Then

$$\begin{aligned} S_1(x, (1, 0, \ldots, 0, 1)) &\geq (x_0^1 + 2x_1^1) + 2x_1^m \\ &\geq (x_0^1 + x_1^1) + x_1^1 + (x_1^m + x_2^m) \geq k + 0 + k = 2k \end{aligned}$$

To get an equality we must have $x_1^1 = 0$ and $x_0^1 = k$ which is contradiction with condition (3). This implies that vertex $v(m,1)$ is x-good. We claim that $x - v(m,1)$ satisfy all inequalities and hence belongs to $(k-1)P_{\mathbb{Z}_3,m}$.

Propositions 2.7.16 and 2.7.20 and Lemma 2.7.17 imply that the inequalities for m-tuple A with $a_1 + \dots a_m \geq 5$ hold. Lemma 2.7.17 together with Lemma 2.7.22 and part a) gives the inequalities for $a_1 + \dots a_m = 2$.

□

We are left only with one special case. That is the case when x does not belong to facet $F_1(A)$ with $a_1 + \cdots + a_m = 2, a_m = 1$ and $x_2^j = 0$ for all $1 \leq j \leq m-1$.

Proposition 2.7.25. *Let $x \in kP_n \cap L_n$ satisfy $(1)-(4)$, $x_0^m = 0, x_1^m \geq x_2^m$ and x does not belong to any $F_1(A)$ for $a_1 + \cdots + a_m = 2$, $a_m = 1$. Moreover, suppose that $x_2^j = 0$ for all $1 \leq j \leq m-1$. Then $x - v(1,m) \in (k-1)P_{\mathbb{Z}_3,m}$.*

Proof. We notice that x_1^1 and x_2^m are non-zero due to (3). This means that the vertex $v(1,m)$ is x-good. We need to check the inequalities. As in the previous cases everything is covered by Propositions 2.7.16 and 2.7.20 and Lemmas 2.7.17 and 2.7.22 except the case of facets $F_2(A)$ with $a_1 + \cdots + a_m = 2, a_m = 1$. If we can prove that x does not lie on any such facet then we can conclude by Lemma 2.7.17.

Suppose by contradiction that (without loss of generality) $x \in F_2((1,0,\dots,0,1))$. Then

$$\begin{aligned} k = S_2(x,(1,0,\dots,0,1)) &= x_0^1 + 2(x_2^1 + \cdots + x_1^{m-1}) + 2x_2^m \\ &\geq (x_1^1 + \cdots + x_1^{m-1} + x_2^m) \\ &= S_1(x,(0,0,\dots,0,2) \geq k. \end{aligned}$$

To get the equality we must have $x_2^m = 0$ but we have already shown that it is not the case.

□

Now we can sum up and prove the normality also for the group $\mathbb{Z}_3$.

Proof of Theorem 2.4.3 for the group $\mathbb{Z}_3$. Consider a point $x \in kP_{\mathbb{Z}_3,m} \cap L_{\mathbb{Z}_3,m}$ for some positive integer k. To prove normality of $P_{\mathbb{Z}_3,m}$ it is sufficient for $k \geq 2$ to prove that there exists a vertex v of $P_{\mathbb{Z}_3,m}$ such that $x - v \in (k-1)P_{\mathbb{Z}_3,m}$. Also it is sufficient to consider only points x which satisfy $(1)-(3)$. Then the existence of such v is implied by Propositions 2.7.19, 2.7.21, 2.7.23, 2.7.24 and 2.7.25. □

Chapter 3

Gorenstein graphic matroids

This chapter is based on my article "Gorenstein graphic matroids" [44] which is joint work with T. Hibi, M. Lason, K. Matsuda, and M. Michałek.

3.1 Matroid polytopes and graphic matroids

Let M be a matroid on a ground set E with the set of bases $\mathfrak{B}$ and the set of independent sets $\mathfrak{I}$. We refer the reader to [83] for a comprehensive monograph on matroids.

Definition 3.1.1. The *base polytope* of a matroid M, denoted by $B(M)$, is the convex hull of all bases indicator vectors in $\mathbb{Z}^{|E|}$, that is $e_B := \sum_{b \in B} e_b$ for $B \in \mathfrak{B}$.
The *independence polytope* of a matroid M, denoted by $P(M)$, is the convex hull of all independent sets indicator vectors in $\mathbb{Z}^{|E|}$, that is $e_I := \sum_{i \in I} e_i$ for $I \in \mathfrak{I}$.

Notice that $B(M)$ is contained in a hypersimplex, and $B(M)$ is a facet of $P(M)$. Moreover, every edge of a matroid base polytope $B(M)$ is a parallel translate of $e_i - e_j$ for some $i, j \in E$ [36]. In other words, the edges of $B(M)$ correspond exactly to the pairs of bases that satisfy the symmetric exchange property (see [55] for more exchange properties). As already mentioned both polytopes $B(M)$ and $P(M)$ are normal. [114].

Definition 3.1.2. Let $G = (V, E)$ be a finite undirected (multi-)graph. The *graphic matroid* corresponding to G, denoted by $M(G)$, is a matroid on the set E whose independent sets are the forests in G. A set $B \subset E$ is a basis of $M(G)$ if and only if B is an inclusion maximal forest of G.

Example 3.1.3. *Consider the* 3*-cycle* C_3*. The base polytope of the graphic matroid of* C_3*, that is* $B(M(C_3))$*, is a simplex spanned by three vertices* $(1,1,0)$*,* $(1,0,1)$*,* $(0,1,1)$*. Thus the algebra* $\mathbb{C}[B(M(C_3))]$ *is a polynomial ring, in particular, it is Gorenstein.*

Recall that a matroid is called *connected* if it is not a direct sum of two matroids. In particular, graphic matroid $M(G)$ is connected if and only if G is 2-connected. Moreover, the base polytope (and the independence polytope) of a matroid is the product of base polytopes (correspondingly independence polytopes) of its connected components. Hence, the base polytope (and the independence polytope) of $M(G)$ is the product of base polytopes (correspondingly independence polytopes) of graphic matroids corresponding to 2-connected components of G.

By the above remark and Corollary 2.5.4, the polytope $B(M(G))$ (resp. $P(M(G))$) is Gorenstein of index δ if and only if for every 2-connected component H of G the polytope $B(M(H))$ (resp. $P(M(H))$) is Gorenstein of index k. Therefore in the proceeding sections we restrict to 2-connected graphs.

If e is a loop in a matroid M, then the polytopes $P(M)$ and $P(M \setminus \{e\})$ (also $B(M)$ and $B(M \setminus \{e\})$) are lattice isomorphic. Thus, $P(M)$ is Gorenstein if and only if $P(M/\{e\})$ is. Since adding or subtracting loops does not affect the property of being Gorenstein, we restrict to loopless matroids.

We provide a complete graph-theoretic classification of graphs G for which the semigroup algebra of lattice points in the base polytope of the graphic matroid of G, that is $\mathbb{C}[B(M(G))]$, is a Gorenstein algebra. Theorems 3.3.1, 3.4.9 and 3.5.5 add up to the following result.

Theorem 3.1.4. *Let G be a finite simple undirected graph. The following conditions are equivalent:*

1. $\mathbb{C}[B(M(G))]$ *is a Gorenstein algebra,*
2. *there exists a positive integer δ such that every 2-connected component of G can be obtained:*
   - *if $\delta > 2$ using constructions from Propositions 3.4.1 and 3.4.2 from a δ-cycle,*
   - *if $\delta = 2$ using construction from Proposition 3.5.2 from the clique K_4.*

The same type of result we obtain for the algebra associated to the independence polytope $\mathbb{C}[P(M(G))]$ corresponding to a multigraph G. Theorems 3.2.2, 3.2.4 add up to the following.

Theorem 3.1.5. *Let G be a finite undirected multigraph. The following conditions are equivalent:*

1. $\mathbb{C}[P(M(G))]$ *is a Gorenstein algebra,*
2. *there exists an integer $\delta \geq 2$ such that every 2-connected component of G is a $(\delta-1)$-blow up of a graph that is δ-chordal (any cycle without a chord has exactly $\delta+1$ elements) and has no K_4 minor,*
3. *there exists an integer $\delta \geq 2$ such that every 2-connected component of G is a $(\delta-1)$-blow up of a graph that can be constructed from the clique K_2, by at each step adding a new $(\delta+1)$-cycle to an edge of the preceding graph.*

The case of the independence polytope is way much easier compared to the base polytope. We study independence polytopes in Section 3.2, the remaining Sections 3.3, 3.4, and 3.5 treat base polytopes.

3.2 Characterization of Gorenstein polytopes $P(M(G))$

In this section, we characterize multigraphs G for which $P(M(G))$ is Gorenstein. As we will see such multigraphs are related to graphs, by the blow-up construction.

Definition 3.2.1. The m-blow up of a graph is a multigraph on the same set of vertices, in which every edge of the graph is replaced by m parallel edges.

The following theorem is a corollary of a result of Herzog and Hibi [39] for the case of graphic matroids. In the end of the section we provide Examples 3.2.5, 3.2.6, 3.2.7 to illustrate this result. For a subset S of vertices of a graph G, we denote $E(G)$ the set of all edges with the endpoints in S.

Theorem 3.2.2. *Let G be a multigraph. The polytope $P(M(G))$ is Gorenstein if and only if there exists an integer $\delta \geq 2$ such that G is a $(\delta - 1)$-blow up of a graph $H = (V, E)$ satisfying condition $(\clubsuit)_\delta$:*

$$(\delta - 1)|E(S)| + 1 = \delta(|S| - 1),$$

for every set $S \subset V$ inducing a 2-connected subgraph.

In order to prove this theorem, we recall a result of Herzog and Hibi.

Theorem 3.2.3 (Theorem 7.3 [39])**.** *The independence polytope $P(M)$ of a loopless matroid M is Gorenstein if and only if there exists a constant $\alpha \in \mathbb{Z}_{\geq 2}$, such that*

$$\operatorname{rank}(A) = \frac{1}{\alpha}(|A| + 1),$$

for every indecomposable flat A, i.e. a flat that is connected.

Proof of Theorem 3.2.2. Suppose a set of edges A forms an indecomposable flat in a graphic matroid $M(G)$. Let S be the set of all endpoints of edges from A. As the restriction to the flat is a connected matroid, the pair (S, A) is a 2-connected multigraph. In particular, it is connected. Since A is a flat, it becomes clear that any edge in G between vertices of S is in A. The converse is clear, that is if S induces a 2-connected subgraph of G, then the set A of all edges between vertices S is an indecomposable flat. Identifying an indecomposable flat A with a set of vertices S we obtain: $\alpha(|S| - 1) = |E(S)| + 1$.

Assume G is Gorenstein. Applying the above equality to any two vertices that are connected by an edge in G we see that there are exactly $\alpha - 1$ parallel edges between them. Hence, G is indeed an $(\alpha - 1)$-blow-up of a graph H. Taking $\alpha = \delta$ we see that H satisfies the desired equalities, as induced subgraphs of H are $(\alpha - 1)$-blow-ups of induced subgraphs of G. The opposite implication follows the same way. □

Theorem 3.2.4. *Let H be a 2-connected graph, and let $\delta \geq 2$. The following conditions are equivalent:*

1. *H satisfies $(\clubsuit)_\delta$,*
2. *H is δ-chordal (any cycle without a chord has exactly $\delta + 1$ elements) and has no K_4 minor,*
3. *H can be constructed from the clique K_2, by at each step adding a new $(\delta+1)$-cycle to an edge of the preceding graph.*

Proof. (1) $\Rightarrow$ (2) Consider a chordless cycle C. It is an indecomposable flat in H. We have $(\delta-1)|C|+1=\delta(|C|-1)$. This implies $|C|=\delta+1$.

We now prove by induction on the number of vertices of H that it is K_4-minor free. When H is a single edge it is trivial. Let H' be inclusion maximal proper 2-connected induced subgraph. Clearly, H' satisfies $(\clubsuit)_\delta$, hence by induction H' is K_4-minor free. There exists an edge $e_1 \in E(H)\setminus E(H')$, that has one vertex in H'. As H is 2-connected we may pick a shortest path $p=(e_1,\dots,e_q)$ in H, that finishes also with a vertex in H'. As H' union p is 2-connected induced subgraph of H, larger than H', it cannot be proper. Thus, H is H' union p. Applying $(\clubsuit)_\delta$ to H' and H we obtain $q=\delta$. As H is δ-chordal we see that it is obtained from H' by attaching path p to an edge. In particular, since H' is K_4-minor free, so is H.

(2) $\Rightarrow$ (3) We prove by induction on the number of edges that if H is 2-connected, δ-chordal, and has no K_4 minor, then either $H=K_2$ or H contains an attached $(\delta+1)$-cycle. The graph H has no K_4 minor, so it has treewidth at most 2. Thus, H has a vertex of degree at most 2. If $H\neq K_2$, then since H is 2-connected it has a vertex of degree 2. Let e,f be edges incident to such a vertex. Consider a 2-connected component of $H\setminus\{e,f\}$. It satisfies inductive assumptions, so it contains an attached $(\delta+1)$-cycle to an edge. This is also an attached $(\delta+1)$-cycle to an edge in H, as otherwise minor K_4 would appear.

(3) $\Rightarrow$ (1) When H is constructed from H' by attaching a $(\delta+1)$-cycle C to an edge e, there are two types of indecomposable flats, that is 2-connected induced subgraphs: 2-connected induced subgraphs of H', and 2-connected induced subgraphs of H' containing e together with the whole cycle C. In both cases equalities $(\clubsuit)_\delta$ hold. $\square$

Example 3.2.5. *Let G be the 3-cycle C_3. Then G satisfies $(\clubsuit)_2$. Hence $P(M(G))$ is Gorenstein of index 2 by Theorem 3.2.2. The h-vector of $\mathbb{C}[P(M(G))]$ is $(1,3,1)$.*

Example 3.2.6. *Let G be a graph on $\{1,2,3,4\}$ with the set of edges $E=\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\}\}$. Then G satisfies the condition (3) for $\delta=2$ in Theorem 3.2.4. Hence $P(M(G))$ is Gorenstein of index 2 by Theorem 3.2.2. The h-vector of $\mathbb{C}[P(M(G))]$ is $(1,18,43,18,1)$.*

Example 3.2.7. *Let G be a multigraph on $\{1,2,3,4\}$ with 8 edges $e_1=e_2=\{1,2\}$, $e_3=e_4=\{2,3\}, e_5=e_6=\{3,4\}, e_7=e_8=\{1,4\}$. Then G is the 2-blow up of the cycle C_4 which is 3-chordal and K_4-minor free. Hence $P(M(G))$ is Gorenstein of index 3 by Theorem 3.2.2 and Theorem 3.2.4. The h-vector of $\mathbb{C}[P(M(G))]$ is $(1,56,450,865,450,56,1)$.*

3.3 Graphical translation of Gorenstein property for polytope $B(M(G))$

Theorem 3.3.1. *Fix a positive integer δ. Let $G=(V,E)$ be a 2-connected graph. The polytope $B(M(G))$ is Gorenstein of index δ if and only if G possess the weight function $w: E\to\{1,\delta-1\}$ defined by*

$$w(e)=\begin{cases} 1 & \text{if } G\setminus e \text{ is 2-connected,}\\ \delta-1 & \text{if } G/e \text{ is 2-connected,}\end{cases}$$

and the following equalities $(\spadesuit)_\delta$ are satisfied:

1. $w(E) = \delta(|V| - 1)$, *where* $w(E) = \sum_{e \in E} w(e)$,

2. $w(E(S)) + 1 = \delta(|S| - 1)$ *for every* good flat $S \subset V$, *i.e. a set such that both: restriction of* G *to* S *and contraction of* $E(S)$ *in* G *are* 2*-connected.*

In the end of the section we provide Examples 3.3.5, 3.3.6, 3.3.7 to illustrate this theorem. Before we start the proof, we recall the general matroid base polytope facets description.

Lemma 3.3.2 ([50, 32])**.** *Let* M *be a connected matroid on the ground set* E *with the rank function* r*. Then, the base polytope* $B(M)$ *is full dimensional in an affine sublattice of* $\mathbb{Z}^E$ *given by* $\sum_{e \in E} x_e = r$*, and all facets of* $B(M)$ *are of one of the following two types:*

1. $x_e \geq 0$*, if* $M \setminus \{e\}$ *is connected,*

2. $\sum_{e \in F} x_e \leq \frac{r(F)}{r(E)} \sum_{e \in E} x_e$*, where* $F \subsetneq E$ *is a* good flat *– a flat such that both: restriction of* M *to* F *and contraction of* F *in* M *are connected.*

We extract from Lemma 3.3.2 the facet presentation of the polytope $B(M(G))$.

Corollary 3.3.3. *Let* $G = (V, E)$ *be a* 2*-connected graph. The polytope* $B(M(G))$ *has two types of facets:*

1. $x_e \geq 0$*, if* $G \setminus e$ *is* 2*-connected,*

2. $\sum_{e \in E(S)} x_e \leq \frac{|S|-1}{|V|-1} \sum_{e \in E} x_e$*, where* $S \subsetneq V$ *is a* good flat *– a set such that* $E(S)$ *is a flat and both: restriction of* G *to* S *and contraction of* $E(S)$ *in* G *are* 2*-connected.*

Proof. Inequalities (1) correspond directly to those from Lemma 3.3.2.

For (2) we need to show a correspondence between good flats in a graphic matroid and good flats in a graph. Suppose that a set of edges E' forms a good flat in a graphic matroid. Let S be the set of all endpoints of edges from E'. As the restriction to the flat is a connected matroid, the pair (S, E') is a 2-connected graph. In particular, it is connected. Since E' is a flat, any edge of G between vertices of S is in E'. Thus, restriction of G to S and contraction of $E(S)$ in G are 2-connected – S is a good flat in G. The converse is clear. □

Now, we proceed to the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. Weight function w is supposed to correspond to lattice point v from Definition 1.1.8. Namely, if weight function w exists, then, by Corollary 3.3.3, properties of w together with equalities $(\spadesuit)_\delta$ guarantee that $v \in \delta P$. Further, we claim that both inequalities in Corollary 3.3.3 provide *reduced* equations of the facets. The equalities $(\spadesuit)_\delta$ imply that for every facet of P, its reduced equation $h(x) = c$ satisfies $h(v) = c + 1$. Therefore, the polytope $B(M(G))$ is Gorenstein of index δ.

Conversely, suppose the polytope $B(M(G))$ is Gorenstein of index δ. If $G \setminus e$ is 2-connected, then by Corollary 3.3.3 $x_e \geq 0$ is a facet, and hence $v_e = 1$. Otherwise, there is no cycle in which e is a chord, so contraction of e does not influence cycles (cycle C becomes a cycle C/e), so G/e is 2-connected. Then, e is a good flat, and (by Corollary 3.3.3) $x_e \leq \frac{|S|-1}{|V|-1} \sum_{e \in E} x_e$ is a facet, and hence $1 + v_e = \frac{1}{|V|-1} \sum_{e \in E} v_e = \frac{1}{|V|-1}(|V| - 1)\delta$, so $v_e = \delta - 1$. Hence, weight function $w := v$ exists, and (by Corollary 3.3.3) properties of v imply equalities $(\spadesuit)_\delta$. □

Theorem 3.3.4. *Fix a positive integer δ. Let $G = (V, E)$ be a 2-connected graph. The polytope $B(M(G))$ is Gorenstein of index δ if and only if the equality $(\heartsuit)_\delta$ is satisfied:*

$$w(E(S)) + k(S) = \delta(|S| - 1) \text{ for every set } S \subseteq V \text{ such that } G_{|S} \text{ is 2-connected,}$$

where $w : E \to \{1, \delta - 1\}$ is the weight function defined in Theorem 3.3.1, and $k(S)$ is the number of 2-connected components in $G/E(S)$ (notice that $k(V) = 0$).

Proof. The implication $\Leftarrow$ is straightforward as $S = V$ gives the first equality of $(\spadesuit)_\delta$ and for any good flat S we obtain the second equality of $(\spadesuit)_\delta$ with $k(S) = 1$.

For the implication $\Rightarrow$ let $C_1, \dots, C_k$ be 2-connected components in $G/E(S)$ (where $k = k(S)$). Sets $V \setminus (C_i \setminus S)$ are good flats in G, thus

$$w(E(V \setminus (C_i \setminus S))) + 1 = \delta(|V \setminus (C_i \setminus S)| - 1).$$

Summing up

$$(k - 1) \cdot w(E) + w(E(S)) + k = \delta((k - 1)|V| + |S| - (k - 1) - 1),$$

$$w(E(S)) + k = \delta(|S| - 1).$$

□

Example 3.3.5. *Let G be the complete graph K_4 and put $w(e) = 1$ for any edge e of K_4. Then G satisfies $(\spadesuit)_2$, hence $B(M(G))$ is Gorenstein of index 2 by Theorem 3.3.1. The h-vector of $\mathbb{C}[B(M(G))]$ is $(1, 10, 20, 10, 1)$.*

Example 3.3.6. *Let G be a graph on $\{1, 2, 3, 4\}$ with 5 edges $e_1 = \{1, 2\}, e_2 = \{2, 3\}, e_3 = \{3, 4\}, e_4 = \{1, 4\}$ and $e_5 = \{1, 3\}$. Put $w(e_1) = w(e_2) = w(e_3) = w(e_4) = 2$ and $w(e_5) = 1$. Then G satisfies $(\spadesuit)_3$, hence $B(M(G))$ is Gorenstein of index 3 by Theorem 3.3.1. The h-vector of $\mathbb{C}[B(M(G))]$ is $(1, 3, 1)$.*

Example 3.3.7. *Let G be a graph on $\{1, 2, 3, 4, 5\}$ with 6 edges $e_1 = \{1, 2\}, e_2 = \{2, 3\}, e_3 = \{3, 4\}, e_4 = \{4, 5\}, e_5 = \{1, 5\}$ and $e_6 = \{1, 3\}$. Assume for contradiction that $B(M(G))$ is Gorenstein of index δ. Then $w(e_1) = w(e_2) = w(e_3) = w(e_4) = w(e_5) = \delta - 1$, $w(e_6) = 1$ and $5\delta - 4 = 4\delta$ by Theorem 3.3.1. Hence $\delta = 4$. Let $S = \{1, 3, 4, 5\}$. Then S is a good flat but $w(E(S)) + 1 = 11 \neq 4(|S| - 1)$, a contradiction. Thus $B(M(G))$ is not Gorenstein. Indeed, h-vector of $\mathbb{C}[B(M(G))]$ is $(1, 5, 3)$, not symmetric.*

3.4 Characterization of Gorenstein polytopes $B(M(G))$ of index δ for $\delta > 2$

Let $\delta > 2$ be fixed. The following two propositions show how to construct new Gorenstein graphs from those already known.

Proposition 3.4.1. *Suppose $G_1, \dots, G_{\delta-1}$ are 2-connected graphs satisfying equalities $(\spadesuit)_\delta$ from Theorem 3.3.1. Let $e_1, \dots, e_{\delta-1}$ be edges of the corresponding graphs with weights equal to $\delta - 1$. Then, the* glueing *of $G_1, \dots, G_{\delta-1}$* along *$e_1, \dots, e_{\delta-1}$, that is a graph G which is a disjoint union of $G_1, \dots, G_{\delta-1}$ with edges $e_1, \dots, e_{\delta-1}$ unified to a distinguished edge e, satisfies equalities $(\spadesuit)_\delta$ (and the weight of e is equal to 1).*

Proof. The first equality $(\spadesuit)_\delta$ is easy to check.

For the second, let S be a good flat in G. If S is disjoint from the endpoints of e, then since $G_{|S}$ is connected, S must be contained in some G_j. Further, $G_{|S} = G_{j|S}$ is 2-connected. If the contraction of S in G_j would lead to a separating vertex, it would also be the case in G. Hence, S is a good flat in G_j and satisfies the second equality $(\spadesuit)_\delta$.

The same argument works if S contains just one endpoint v_1 of e. Indeed, if S was not contained in some G_j, then v_1 would be a cut vertex of $G_{|S}$.

In the remaining case e is an edge of $G_{|S}$. In particular, contraction of S also contracts e. If there were two parts G_{j_1} and G_{j_2} not fully contained in S, then contraction of S would be a separating vertex. Thus, we may assume S contains all vertices of $G_1, \dots, G_{\delta-2}$. Contraction of S contracts also these parts, and hence $S \cap V(G_{\delta-1})$ must be a good flat F of $G_{\delta-1}$. We obtain:

$$
\begin{aligned}
&w(E(S)) + 1 = \\
&= \sum_{i=1}^{\delta-2}(w_i(E(G_i)) - w_i(e)) + (w_{\delta-1}(F) - w_{\delta-1}(e)) + w(e) + 1 = \\
&= \sum_{i=1}^{\delta-2} w_i(E(G_i)) + w_{\delta-1}(F) - (\delta-1)^2 + 1 + 1 = \\
&= \delta\sum_{i=1}^{\delta-2}(|V(G_i)| - 1) + \delta(|F| - 1) - (\delta-1)^2 + 1 = \\
&= \delta(|S| - 1) + \delta + 2\delta(\delta-2) - \delta(\delta-1) - (\delta-1)^2 + 1 = \\
&= \delta(|S| - 1)
\end{aligned}
$$

□

Proposition 3.4.2. *Suppose G is a 2-connected graph satisfying equalities $(\spadesuit)_\delta$ from Theorem 3.3.1. Let e be an edge with weight equal to 1. Then, the $(\delta-1)$*-subdivision of *e, that is a graph G' equal to G with e replaced by a path $e_1, \dots, e_{\delta-1}$, satisfies equalities $(\spadesuit)_\delta$ (and the weight of each e_i is equal to $\delta - 1$).*

Proof. It is straightforward to check the first equality in $(\spadesuit)_\delta$.

For the second, let S be a good flat in G'. If S does not contain the endpoints of e then, as $G'_{|S}$ is connected, it cannot contain any of the vertices on the added path. In such a case, S may be identified with a good flat of G and the second equality in $(\spadesuit)_\delta$ follows.

The same argument applies when S contains precisely one endpoint of e. Indeed, then it cannot contain any of the other vertices on the added path, as $G'_{|S}$ is 2-connected.

Suppose now that both endpoints of e belong to S. By 2-connectivity of $G'_{|S}$ we have only the following two cases.

- S contains all vertices on the added path. By forgetting the added vertices we have an induced good flat S' in G. We have:

$$w(E(S)) + 1 = w(E(S')) + (\delta-1)^2 = \delta(|S'| - 1) - 1 + (\delta-1)^2 = \delta(|S| - 1).$$

- S contains no inner vertices on the added path. Since $G/E(S)$ is 2-connected, S must contain all other vertices. In particular $|S| = |V(G)|$. We have:

$$w(E(S)) + 1 = w(E(G)) = \delta(|S| - 1).$$

□

Before we present our main Theorem 3.4.9 characterizing graphs for which $B(M(G))$ is Gorenstein we prove six technical lemmas.

Lemma 3.4.3. *Suppose a 2-connected graph G satisfying equalities $(\spadesuit)_\delta$ has an edge $e = uv$ of weight 1. Then, G is the glueing of graphs $G_1, \ldots, G_{\delta-1}$ satisfying equalities $(\spadesuit)_\delta$ along edges $e_1, \ldots, e_{\delta-1}$, where G_i is a subgraph of G induced on the union of $\{u, v\}$ and a connected component of $G \setminus \{u, v\}$.*

Proof. Let $G_1, \ldots, G_k$ be the subgraphs of G inducing the 2-connected components after the contraction of e. Note that subgraphs G_i are 2-connected, and the number of 2-connected components after contraction of G_i equals $k(G_i) = k - 1$.

By equalities $(\heartsuit)_\delta$ from Theorem 3.3.4 we know that:

$$w(E(G_i)) + k - 1 = \delta(|V(G_i)| - 1).$$

Summing up we obtain:

$$\sum_{i=1}^{k} w(E(G_i)) + k(k-1) = \delta(\sum_{i=1}^{k} |V(G_i)| - k).$$

Knowing that:

$$\sum_{i=1}^{k} w(E(G_i)) = w(E(G)) + k - 1,$$
$$\sum_{i=1}^{k} |V(G_i)|) = |V(G)| + 2(k-1),$$
$$w(E(G)) = \delta(|V(G)| - 1)$$

we get that $k = \delta - 1$.

It remains to show that each G_i satisfies equalities $(\spadesuit)_\delta$ (note that now $w_i(e) = \delta - 1$). The first one follows from

$$w(E(G_i)) + \delta - 2 = w_i(E(G_i)) = \delta(|V(G_i)| - 1).$$

For the second equality suppose S is a good flat in G_i. If S contains at most one endpoint of e, then S is also a good flat in G and the second equality of $(\spadesuit)_\delta$ follows.

Suppose contrary, that is $u, v \in S$. We extend S to a good flat S' in G by adding all vertices not in G_i. We have:

$$w_i(E(S)) + 1 = w(E(S')) + 1 - (\sum_{j: j \neq i} w_j(E(G_j))) + \delta(\delta - 2).$$

Applying equalities $(\spadesuit)_\delta$ this gives:

$$w_i(E(S)) + 1 = \delta(|S'| - 1) - (\sum_{j: j \neq i} \delta(|V(G_j)| - 1)) + \delta(\delta - 2),$$
$$w_i(E(S)) + 1 = \delta(|S| - 1).$$

□

By *s-ear* in a graph G we mean a path of length s whose inner vertices have degree 2.

Lemma 3.4.4. *Let G be a 2-connected graph satisfying equalities $(\spadesuit)_\delta$ in which the weight of every edge equals $\delta - 1$. Let $P = (v_0, \ldots, v_s)$ be an s-ear in G. Then, the number of 2-connected components of $G \setminus \{v_1, \ldots, v_{s-1}\}$ equals $\delta - s$.*

Proof. Let $C_1, \ldots, C_k$ be the 2-connected components of $G \setminus \{v_1, \ldots, v_{s-1}\}$. As adding the path P makes the graph 2-connected, there exists an ordering of components C_i for which:

- if C_i and C_j share a vertex, then $i = j \pm 1$,
- C_i has a common vertex with C_{i+1} and this vertex is unique,
- $v_0 \in C_1$ and $v_s \in C_k$.

As each C_i is a good flat we have

$$(\delta - 1)|E(C_i)| + 1 = \delta(|V(C_i)| - 1).$$

Summing up we get:

$$(\delta - 1)\sum_{i=1}^{k} |E(C_i)| + k = \delta(\sum_{i=1}^{k} |V(C_i)| - k).$$

Substituting $|E| - s = \sum_{i=1}^{k} |E(C_i)|$ and $|V| + k - 1 - (s - 1) = \sum_{i=1}^{k} |V(C_i)|$ we obtain:

$$(\delta - 1)(|E| - s) + k = \delta(|V| - 1 - (s - 1)).$$

By the first equality in $(\spadesuit)_\delta$ we know that $(\delta - 1)|E| = \delta(|V| - 1)$, which gives us:

$$-s(\delta - 1) + k = \delta(-s + 1).$$

This is precisely the assertion of the lemma. □

Lemma 3.4.5. *Let G be a 2-connected graph satisfying equalities $(\spadesuit)_\delta$ in which the weight of every edge equals $\delta - 1$. Then, $(\delta - 1)$-ears are complements of inclusion maximal 2-connected induced proper subgraphs.*

Proof. Clearly, by Lemma 3.4.4 every such path must be a complement of a 2-connected induced subgraph. Of course, it is not possible to extend such a subgraph to a larger proper 2-connected subgraph.

For the opposite inclusion assume B is an inclusion maximal 2-connected induced proper subgraph. Consider a shortest path $P = (v_0, v_1, \ldots, v_s)$ in G, such that $v_0, v_s \in V(B)$ and $v_i \notin V(B)$ for $i = 1, \ldots, s - 1$. Such a path exists as B is proper (we may find v_1) and G is 2-connected.

By maximality of B, we must have $V(G) = V(B) \cup \{v_1, \ldots, v_{s-1}\}$. P is an s-ear, since if v_i was not of degree 2 for some $i = 1, \ldots, s - 1$ we would be able to construct a shorter path.

Since B is 2-connected, by Lemma 3.4.4, length of P equals $s = \delta - 1$. □

Lemma 3.4.6. *Suppose a 2-connected graph G satisfying equalities $(\spadesuit)_\delta$ with all edges of weight $\delta - 1$ contains an $(\delta-1)$-ear P whose contraction is not 2-connected. Then, the graph obtained from G by replacing P by a single edge also satisfies $(\spadesuit)_\delta$.*

Proof. Firstly, notice that δ-cycle does not contain an $(\delta-1)$-ear whose contraction is not 2-connected.

Secondly, if G is not a δ-cycle, then endpoints of a $(\delta-1)$-ear are not joined by an edge. Indeed, if they were joined by an edge e, the contraction of e would be not 2-connected, implying $w(e) = 1$, which contradicts the assumption.

Let G' be the graph G with P replaced by a single edge e. By the above remarks, G' is again a simple graph. Consider a good flat S' in G'. By assumption e itself is not a good flat, and $w_{G'}(e) = 1$. As G and G' have the same topology, it follows that the only nontrivial case to consider is when both endpoints of e are in S'.

Let S be the subset of vertices of G that coincides with S' on G', but also contains all inner vertices of the path P. As $G'/E(S') = G/E(S)$ and the second is 2-connected, it follows that S is a good flat. Thus:

$$w_G(E(S)) + 1 = \delta(|S| - 1),$$

$$w_{G'}(E(S')) + (\delta-1)^2 - 1 + 1 = \delta(|S'| + \delta - 2 - 1),$$

$$w_{G'}(E(S')) + 1 = \delta(|S'| - 1).$$

□

Lemma 3.4.7. *There is no graph $G(V, E)$ satisfying the following conditions:*

- *G is 2-connected,*
- *deletion of every edge is 2-connected,*
- *contraction of every edge is 2-connected (thus, every edge is a good flat),*
- *$|E| = \delta(|V| - 1)$,*
- *$|E(S)| + 1 = \delta(|S| - 1)$ for every good flat $S \subset V$ distinct from an edge.*

Proof. Suppose such graphs exist and let G be one of them with a minimum size of $V(G)$.

Notice that not every vertex in G is of degree $\delta + 1$. Indeed, in such a case the average degree would be:

$$\frac{2|E|}{|V|} = \frac{2\delta(|V|-1)}{|V|} = \delta + 1.$$

This gives $|V|(\delta - 1) = 2\delta$. As $\delta - 1$ and δ are coprime, we would have $\delta = 3$ and $|V| = 3$, which is not possible.

Let v be a vertex in G of degree distinct from $\delta + 1$. Let T be an inclusion minimal set of edges incident to v such that the graph $(V, E \setminus T)$ is not 2-connected. Denote its 2-connected components by $C_1, \ldots, C_k$, for $k \geq 2$.

Let H be the block graph of this decomposition – a graph on all 2-connected components and all cut vertices, with edges between a cut vertex and a component to which it belongs. Clearly, H is acyclic. For $e \in T$, the graph $(V, (E \setminus T) \cup e)$ is 2-connected, hence $(V, E \setminus T)$ is connected, so H is a tree. Moreover, since for every $e \in T$ the graph $(V, (E \setminus T) \cup e)$ is 2-connected, the above structure has to be the following (renumbering indices of C_i if necessary):

- H is a path: C_i has one common vertex with C_{i+1} for $i = 1, \dots, k-1$,
- $v \in C_1$,
- every $e \in T$ joins v with C_k.

Notice that no component C_i is a single edge. Indeed, if $k > 2$ and C_i was an edge, its deletion would be not 2-connected, contradicting the assumption. If $k = 2$ and C_2 was an edge, G would have a vertex of degree 2, contradicting the assumption. Finally, if $k = 2$ and C_1 was an edge, then from equalities $|E(C_2)| + 1 = \delta(|V(C_2)| - 1)$ (as C_2 is a good flat) and $|E| = \delta(|V| - 1)$ we would get that v is of degree $\delta + 1$, which is not the case.

Thus, every component C_i is a good flat distinct from an edge, and we have the equation

$$|E(C_i)| + 1 = \delta(|V(C_i)| - 1).$$

Summing up we obtain

$$|E(G)| - |T| + k = \delta(|V(G)| + (k-1) - k).$$

As $|E(G)| = \delta(|V(G)| - 1)$ we have $|T| = k$. Further, $C_k \cup \{v\}$ is also a good flat distinct from an edge, hence

$$|E(C_k)| + |T| + 1 = \delta(|V(C_k)| + 1 - 1).$$

Thus $|T| = k = \delta$.

Let G' be the graph $G(C_\delta \cup \{v\})$ together with an edge e' between vertices v and $w = V(C_{\delta-1}) \cap V(C_\delta)$. In other words, G' is equal to G with components $C_1, \dots, C_{\delta-1}$ replaced by an edge e'. Notice that G' is again a simple graph, i.e. in G there was no edge between v and w. Indeed, if there was such an edge e, then since C_δ is larger than e, the contraction of e would be not 2-connected.

Now, G' satisfies all conditions of the lemma:

- G' is 2-connected, as $G(C_\delta \cup \{v\})$ is,
- deletion of every edge is 2-connected, as deletion of every edge in G is 2-connected, and $G(C_\delta \cup \{v\})$ is 2-connected (for deletion of e'),
- contraction of every edge is 2-connected, as in G is, and $G(C_\delta)$ is 2-connected (for contraction of e'),
- $|E(G')| = \delta(|V(G')| - 1)$, as
$|E(G')| = |E(C_\delta)| + 1 + \delta = \delta(|V(C_\delta)| - 1) + \delta = \delta(|V(G')| - 1)$,

- $|E(S')| + 1 = \delta(|S'| - 1)$ for every good flat S' in G' distinct from an edge. Indeed, every good flat in G' that does not contain e' identifies with a good flat in G and the equality holds. If S' contains e' then S' together with all vertices from $C_1, \ldots, C_{\delta-1}$ forms a good flat S in G, and we have:

$$|E(S')| + 1 = |E(S)| + 1 - \sum_{i=1}^{\delta-1} |E(C_i)| - (\delta - 1) + \delta =$$

$$= \delta(|V(S)| - 1 - \sum_{i=1}^{\delta-1} (|V(C_i)| - 1) + 1) = \delta(|V(S')| - 1).$$

Of course, G' is smaller than G. Contradiction. $\square$

Lemma 3.4.8. *Let G be a 2-connected graph satisfying equalities $(\spadesuit)_\delta$ in which the weight of every edge equals $\delta - 1$. Suppose G is not a δ-cycle. Then, contraction of every $(\delta - 1)$-ear is not 2-connected.*

Proof. We proceed by induction on the size of $V(G)$.

First, we notice that in order to prove the assertion for every $(\delta - 1)$-ear, it is enough to show that contraction of some $(\delta - 1)$-ear P is not 2-connected. Indeed, if this is the case, then by Lemma 3.4.6 we can replace P by a single edge (which has weight 1) receiving a graph G' satisfying equalities $(\spadesuit)_\delta$. By Lemma 3.4.3 graph G' is the glueing of graphs $G_1, \ldots, G_{\delta-1}$ satisfying equalities $(\spadesuit)_\delta$ along copies of the edge e. Since $(\delta-1)$-ears in G are disjoint, every $(\delta-1)$-ear in G different from P, is a $(\delta-1)$-ear in some G_i. By induction, either its contraction in G_i is not 2-connected, and therefore its contraction in G is not 2-connected, or G_i is a δ-cycle and then contraction of that $(\delta - 1)$-ear in G is not 2-connected as contraction of P is not 2-connected.

Consider the case when there exists a maximal ear P in G of length l distinct from $\delta - 1$. By Lemma 3.4.4 the complement of P in G has $\delta - l$ 2-connected components $C_1, \ldots, C_{\delta-l}$. In particular, the number of 2-connected components is at least 2. Let G' be the graph G in which we replace components $C_2, \ldots, C_{\delta-l}$ by single edges. Clearly, G' is smaller than G. It is straightforward to check that G' is a 2-connected graph satisfying equalities $(\spadesuit)_\delta$ in which the weight of every edge equals $\delta - 1$. Moreover, G' is not a δ-cycle. By Lemma 3.4.5 graph G' has many $(\delta - 1)$-ears. Let P' be a $(\delta - 1)$-ear contained in C_1. By inductive assumption of the lemma, the contraction of P' is not 2-connected in G'. Thus, it is not 2-connected in G. By the initial remark, G satisfies the assertion of the lemma.

The remaining case is when all maximal ears in G are of length $\delta - 1$. In particular, every edge is a part of a single $(\delta - 1)$-ear. If contraction of some of $(\delta - 1)$-ear is not 2-connected we are done by the initial remark. Otherwise, graph G satisfies the following conditions:

- G is 2-connected,
- deletion of every $(\delta - 1)$-ear is 2-connected (by Lemma 3.4.4),
- contraction of every $(\delta - 1)$-ear is 2-connected (by assumption of the case),
- $(\delta - 1)|E| = \delta(|V| - 1)$,

- $(\delta-1)|E(S)|+1=\delta(|S|-1)$ for every good flat $S \subset V$.

Let G' be a graph obtained from G by replacing every $(\delta-1)$-ear by a single edge. Notice that every good flat S' in G' distinct from an edge is achieved from a good flat S in G by replacing its $(\delta-1)$-ears to single edges. Graph G' satisfies assumptions of Lemma 3.4.7, and therefore by Lemma 3.4.7 this case is empty. □

Theorem 3.4.9. *Let G be a 2-connected graph and let $\delta > 2$. The following conditions are equivalent:*

1. *G satisfies equalities $(\spadesuit)_\delta$ from Theorem 3.3.1,*

2. *G can be obtained using constructions described in 3.4.1, 3.4.2 from a δ-cycle.*

We illustrate this theorem in Examples 3.4.10, 3.4.11 presented after the proof.

Proof. It is easy to check that δ-cycle satisfies equalities $(\spadesuit)_\delta$. Implication (2) $\Rightarrow$ (1) follows from Propositions 3.4.1 and 3.4.2.

We next prove the implication (1) $\Rightarrow$ (2). The proof is by induction on the size of $V(G)$.

First, suppose that G contains an edge with a weight 1. In this case we conclude by Lemma 3.4.3, applying construction inverse to the one from Proposition 3.4.1. Hence, from now on we may assume that all edges of G have weight $\delta-1$.

Lemma 3.4.5 allows us to find many $(\delta-1)$-ears in G. Lemma 3.4.6 tells us that G is constructed from smaller graphs applying operation from Proposition 3.4.2, provided there is a $(\delta-1)$-ear whose contraction is not 2-connected. Finally, Lemma 3.4.8 guarantees existence of such $(\delta-1)$-ears.

□

Example 3.4.10. *Let G be a graph on $\{1,2,\ldots,8\}$ with 10 edges $\{1,2\}$, $\{2,3\}$, $\{3,4\},\{4,5\},\{5,6\},\{1,6\},\{1,4\},\{1,7\},\{7,8\}$ and $\{4,8\}$. Then G can be obtained using constructions described in Proposition 3.4.1 from three C_4. Precisely, G is the glueing of three cycles C_4 along an edge. Hence $B(M(G))$ is Gorenstein of index 4 by virtue of Theorem 3.4.9. The h-vector of $\mathbb{C}[B(M(G))]$ is $(1,44,315,586,315,44,1)$.*

Example 3.4.11. *Let G be a graph on $\{1,2,3,4,5\}$ with 6 edges $\{1,2\},\{2,3\}$, $\{3,4\}$, $\{1,4\}$, $\{1,5\}$ and $\{3,5\}$. Then G can be obtained by the 2-subdivision of e_5 on the graph which appears in Example 3.3.6. Hence $B(M(G))$ is Gorenstein of index 3 by virtue of Proposition 3.4.2. The h-vector of $\mathbb{C}[B(M(G))]$ is $(1,6,6,1)$.*

3.5 Characterization of Gorenstein polytopes $B(M(G))$ of index δ for $\delta = 2$

We begin with the following simplification of equalities $(\spadesuit)_2$ from Theorem 3.3.1.

Corollary 3.5.1. *Let $G=(V,E)$ be a 2-connected graph. The polytope $B(M(G))$ is Gorenstein of index 2 if and only if the following equalities $(\spadesuit)_2$ are satisfied:*

1. *$|E| = 2(|V|-1)$, and*

2. $|E(S)| = 2|S| - 3$ *for every good flat* $S \subsetneq V$*.*

Proof. Suppose $B(M(G))$ is Gorenstein of index 2. Notice that since G is 2-connected, for every edge e one of the graphs $G \setminus e, G/e$ is 2-connected. Thus, in Theorem 3.3.1 we must have $w(e) = 1$ for every edge e. Further, substituting $\delta = 2$ we obtain equalities from the assertion.

For the opposite implication, define $w(e) := 1$. Now G satisfies all conditions from Theorem 3.3.1 and $B(M(G))$ is Gorenstein of index 2. $\square$

Proposition 3.5.2. *Suppose* G_1, G_2 *are* 2*-connected graphs satisfying equalities* $(\spadesuit)_2$ *from Theorem 3.3.1. The* collision of G_1, G_2 on the corresponding edges e_1, e_2*, that is the graph obtained by glueing of* G_1, G_2 *along edges* e_1, e_2 *and removing these edges, satisfies equalities* $(\spadesuit)_2$*.*

Proof. The first equality $(\spadesuit)_2$ (using Corollary 3.5.1) is easy to check.

For the second, let S be a good flat in G – the collision of G_1 and G_2 on the corresponding edges e_1, e_2. Denote by v_1, v_2 vertices in G that come from both G_1 and G_2.

First suppose that at most one vertex, v_1 or v_2, belongs to S. Then S must be contained either in G_1 or G_2, as otherwise it would not be 2-connected. In this case, without loss of generality, $S \subset G_1$ and we claim that S is a good flat in G_1. Indeed, it is clearly 2-connected, and contraction of S in G_1 cannot lead to a separating vertex, as it would also in the contraction of S in G. Hence S is a good flat in G_1 and the second equality $(\spadesuit)_2$ holds.

Now suppose that $v_1, v_2 \in S$. Then S would be separating, unless S contains G_1 or G_2. Thus without loss of generality we may assume $G_1 \subset S$. Further, $S \cap V(G_2)$ is a good flat in G_2. Thus $|S| = |V(G_1)| - 2 + |S \cap V(G_2)|$, and $|E_G(S)| =$

$$|E(G_1)| + |E_{G_2}(S \cap V(G_2))| - 2 = 2(|V(G_1)| - 1) + 2|S \cap V(G_2)| - 3 - 2 = 2|S| - 3.$$

$\square$

The following lemma characterizes 3-connected Gorenstein graphs of index 2.

Lemma 3.5.3. *Suppose* G *is a* 3*-connected graph satisfying* $(\spadesuit)_2$*. Then,* $G = K_4$*.*

Proof. For a vertex $v \in V$ the graph $G \setminus \{v\}$ is 2-connected, so $V \setminus \{v\}$ is a good flat for every $v \in V$. Therefore, the number of edges in $G \setminus \{v\}$ equals $2|V| - 5$. On the other hand, the number of edges in $G \setminus \{v\}$ is $|E| - \deg(v) = 2|V| - 2 - \deg(v)$, which implies $\deg(v) = 3$ for every vertex $v \in V$. This yields to

$$2|V| - 2 = |E| = \frac{3}{2}|V|,$$

which implies $|V| = 4$, and $G = K_4$. $\square$

Lemma 3.5.4. *Let* G *be a* 2*-connected graph satisfying* $(\spadesuit)_2$*. Suppose that* $\{v_1, v_2\}$ *is separating set and that* $H_1, \ldots, H_n$ *are connected components of* $G \setminus \{v_1, v_2\}$*. Then the following holds:*

1. $n = 2$ *(after removing* v_1, v_2 *there are exactly two connected components),* *2. there is no edge between v_1 and v_2,*

3. $G \setminus H_i$ is 2-connected for every i.

Proof. We prove (1) by contradiction. Suppose that $n \geq 3$.

Suppose that $G \setminus H_i$ has a separating vertex v. However, v is not separating vertex in G, since G is 2-connected. Therefore, v separates vertices v_1 and v_2. This is impossible since they are connected by a component H_j such that $j \neq i$, $v \notin H_j$.

This means that $G \setminus H_i$ is 2-connected. Consequently, $G \setminus H_i$ is a good flat, since G is 2-connected and H_i is (by definition) connected (implying $G/(G \setminus H_i)$ is 2-connected).

We sum the number of edges in $G \setminus H_i$. Every edge in G is counted exactly $n-1$ times except of the edge between v_1 and v_2 which is counted n times, if it exists. Therefore:

$$(n-1)|E| \leq \sum_{i=1}^{n} |E(G \setminus H_i)| = \sum_{i=1}^{n} (2|V(G \setminus H_i)| - 3) =$$

$$= \sum_{i=1}^{n} (2|V| - 2|H_i| - 3) = 2(n-1)|V| + 4 - 3n.$$

We get an inequality

$$2(n-1)|V| - 2(n-1) \leq 2(n-1)|V| + 4 - 3n,$$

$$n \leq 2,$$

which is a contradiction. Thus, $n = 2$ and we proved (1).

For (2) observe that if sets $G \setminus H_i$ are 2-connected (which is the statement of part (3)) we can do the same calculation and conclude that v_1 and v_2 are not joint by an edge. Thus, it remains to prove (3).

For (3) notice first, that if there is an edge between v_1 and v_2 (that is if the statement of (2) is false), then (3) is clear since G is 2-connected. Otherwise, denote by $C_1^1, \ldots, C_k^1$ 2-connected components of $G \setminus H_1$ and by $C_1^2, \ldots, C_l^2$ 2-connected components of $G \setminus H_2$. If $G \setminus H_1$ is not 2-connected, then vertices v_1 and v_2 cannot lie in the same component C_i^1 because that would mean that also G is not 2-connected. Thus, every C_j^i is a good flat. The fact that the block graph of 2-connected components is a tree (in our case in fact a path) implies

$$\sum_{j=1}^{k} |V(C_j^1)| = |V(G \setminus H_1)| + k - 1.$$

We may again compute the edges assuming (2) is true:

$$2|V| - 2 = |E| = \sum_{j=1}^{k} |E(C_j^1)| + \sum_{j=1}^{l} |E(C_j^2)| =$$

$$= \sum_{j=1}^{k} (2|V(C_j^1)| - 3) + \sum_{j=1}^{l} (2|V(C_j^2)| - 3) =$$

$$= 2\sum_{j=1}^{k} |V(C_j^1)| - 3k + 2\sum_{j=1}^{l} |V(C_j^2)| - 3l =$$

$$= 2|V(G \setminus H_1)| + 2k - 2 - 3k + 2|V(G \setminus H_2)| + 2l - 2 - 3l = 2|V| - k - l.$$

We conclude $k + l = 2$ which gives $k = l = 1$ and that $G \setminus H_i$ are 2-connected. $\square$

Theorem 3.5.5. *Let G be a 2-connected graph. The following conditions are equivalent:*

1. *G satisfies $(\spadesuit)_2$ from Theorem 3.3.1,*
2. *G can be obtained using construction described in Proposition 3.5.2 from the clique K_4.*

Proof. Since K_4 satisfies $(\spadesuit)_2$ implication (2) $\Rightarrow$ (1) is a consequence of Proposition 3.5.2.

We now prove the implication (1) $\Rightarrow$ (2). The proof is by induction on the number of vertices of a graph. Let G be a 2-connected graph satisfying $(\spadesuit)_2$. Suppose that the statement is true for all graphs with a smaller number of vertices than G.

If G is 3-connected, then $G = K_4$ by Lemma 3.5.3. Otherwise, G is not 3-connected and has two separating vertices v_1 and v_2 which by Lemma 3.5.4 are not joined by an edge. Using the notation from Lemma 3.5.4 define G_i as the graph induced by $V \setminus H_i$ with an additional edge between v_1 and v_2. It follows that the graph G is the collision of graphs G_1 and G_2 on the edge $\{v_1, v_2\}$.

Now, it is sufficient to show that graphs G_1, G_2 are 2-connected and satisfy conditions $(\spadesuit)_2$, since then by induction hypothesis both can be obtained by construction from Proposition 3.5.2 and so can G. Cases for G_1 and for G_2 are analogous. Lemma 3.5.4 gives that G_1 is 2-connected. Since $V \setminus H_1$ is a good flat

$$|E(G_1)| = 1 + 2|V(G_1)| - 3 = 2|V(G_1)| - 2,$$

so G_1 has the right number of edges. It remains to check that every good flat has the right number of edges. Let $S \subset V \setminus H_1$ be a good flat in G_1. There are 2 cases:

- $\{v_1, v_2\} \not\subset S$. Then S is also a good flat in the graph G because the component H_1 is joined with both v_1 and v_2. Consequently, the number of edges in the induced graph $G_1|_S$ is $2|S| - 3$.
- $v_1, v_2 \in S$. Then the graph induced by $S \cup H_1$ is still 2-connected because both $G|_S$ and $G \setminus H_2$ are 2-connected. It follows that $S \cup H_1$ is a good flat in G and the number of edges in the induced subgraph of G is $2(|S| + |H_1|) - 3$. Clearly, $|E_G(S \cup H_1)| = |E(G \setminus H_2)| + |E_{G_1}(S)| - 1$ which yields to $|E_{G_1}(S)| = 2|S| - 3$.

$\square$

Chapter 4

Gorenstein tangential varieties

This chapter is based on my article "Cohen-Macaulay and Gorenstein tangential varieties of the Segre-Veronese Varieties" [48] which is joint work with M. Azeem Khadam.

The purpose of this chapter is to present the complete classification of those tangential varieties of the Segre-Veronese varieties which are Cohen-Macaulay or Gorenstein. Here we state the main theorem of this chapter (see Theorem 4.2.9 for the proof). To this end, fix $k \in \mathbb{N}$ a positive integer and $\mathbf{a}, \mathbf{b} \in \mathbb{N}^k$ where $\mathbf{a} = (a_1, \ldots, a_k), \mathbf{b} = (b_1, \ldots, b_k)$ such that a_i, b_i are positive integers. Throughout this chapter we assume that X is the corresponding Segre-Veronese variety, i.e. the embedding of $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_k}$ into $\mathbb{P}^N$ given by the very ample line bundle $\mathcal{O}(a_1, \ldots, a_k)$.

Theorem 4.0.1. *The tangential variety of the Segre-Veronese variety is smooth if and only if*

(S1) $k = 2$, $\mathbf{a} = (1, 1)$, $b_1 = 1$, *or*

(S2) $k = 1$, $a = 1$ *or* ($a = 2$ *and* $b = 1$).

If the tangential variety of the Segre-Veronese variety is not smooth, then it is Cohen–Macaulay if and only if one of the following holds

(CM1) $k \geq 3$, $\mathbf{a} = (1, \ldots, 1)$,

(CM2) $k = 2$, $\mathbf{a} = (2, 2)$, $\mathbf{b} = (1, 1)$,

(CM3) $k = 2$, $\mathbf{a} = (1, 2)$, $\mathbf{b} = (1, b_2)$ *for all* $b_2 \geq 1$,

(CM4) $k = 2$, $\mathbf{a} = (1, 1)$, $b_i > 1$ *for all* $i = 1, 2$,

(CM5) $k = 1$, $a \geq 3$, $b = 1$,

(CM6) $k = 1$, $a = 2$, $b > 1$.

If the tangent variety of the Segre-Veronese variety is not smooth, then it is Gorenstein if and only if one of the following holds

(G1) $k = 3$, $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (1, 1, 1)$,

(G2) $k = 2$, $\mathbf{a} = (1, 2)$, $\mathbf{b} = (1, 1)$,

(G3) $k = 2$, $\mathbf{a} = (1, 1)$, $b_1 = b_2$, $b_1 > 1$.

(G4) $k = 1$, $a \geq 3$, $b = 1$,

(G5) $k = 1$, $a = 2$, *b is even.*

Note that the tangential variety is normal for the cases (S1 - S2). Other cases when the tangential variety is normal fall inside (CM1 - CM6). Precisely, the tangential variety of the Segre-Veronese variety is normal only in the following cases:

(N1) The Segre case, that is, $k \geq 1$, $\mathbf{a} = (1, \ldots, 1)$ (cf. [71, Proposition 8.5]),

(N2) The Veronese case, that is, $k = 1$, $a = 2$, b is arbitrary (see Remark 4.2.11).

This means we have *non* normal tangential varieties of the Segre-Veronese varieties which are Cohen-Macaulay but not Gorenstein or which are Gorenstein (and hence Cohen-Macaulay as well) or which are not Cohen-Macaulay (hence not Gorenstein). Likewise, we have normal tangential varieties of the Segre-Veronese varieties which are not Gorenstein but this fact was already known (for Segre case (N1) by [71, Theorem 8.9] and for Veronese case (N2) see Remark 4.2.11 for several references).

Our approach is to use the change of coordinates inspired by cumulants [117, 118]. Although the tangential variety $\tau(X)$ is not a toric variety, the change of coordinates will turn it locally into a toric variety. This will be done in Proposition 4.1.6. A similar approach was used in the case of secant varieties in [49]. However, in our case, we have to overcome another obstacle since the variety $\tau(X)$ is not normal. Note that, according to our best knowledge, cumulant methods have so far applied only to normal varieties (or to decide when a variety is normal) and then to use the normal toric machinery to investigate above-mentioned properties as in Chapter 1. To deal with not-normal toric varieties we use a criterion of Cohen-Macaulay and Gorenstein property developed by Hoa and Trung [108, Theorem 4.1].

This chapter is organized as follows. The first half of Section 4.1 is mainly based on [49] where we recall the background results needed later. In particular, we define toric varieties and simplicial embeddings. For more general references for this and others (Cohen-Macaulay, Gorenstein rings, etc), we recommend [10, 18, 73, 76, 96]. In the second half, we present and elucidate [108, Theorem 4.1] by using a few examples. In section 4.2, we study the toric geometry of our varieties and present the main theorem.

4.1 Background results

Our approach is to use methods from [49] where the authors mainly deal with the secant varieties of the Segre-Veronese varieties. In this section, we recall a few definitions and results which we need to prove the main results of this chapter. More details and examples can be found in [49]. Let $\mathbb{N}$ denote the set of nonnegative integers.

Definition 4.1.1. (a) (Toric variety) Let $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{t} = (t_1, \ldots, t_n)$, and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_N\}$ be a fixed subset of $\mathbb{N}^n$. The set $\mathcal{C}$ defines a map $e_{\mathcal{C}}$ from $\mathbb{C}^n$ to $\mathbb{C}^N$ where $x_i = \mathbf{t}^{\mathbf{c}_i} := t_1^{c_{i1}} \cdots t_n^{c_{in}}$ for $1 \leq i \leq N$. The closure of the image of this map $V_{\mathcal{C}} := \overline{e_{\mathcal{C}}(\mathbb{C}^n)}$ is called an *affine toric variety.* We note that this differs from the classical definition of a toric variety, where, in addition, the variety needs to be normal; see [96, Chapter 13] for discussion.

(b) (Simplicial complex) A *simplicial complex* Δ on the vertex set $\{1,\dots,n\}$ is a collection of subsets, called *simplices*, closed under taking subsets, that is, if $\sigma \in \Delta$ and $\tau \subset \sigma$ then $\tau \in \Delta$. A simplex $\sigma \in \Delta$ of cardinality $|\sigma| = i+1$ has *dimension* $\dim(\sigma) = i$. In this chapter we allow vertices to have repeated labels. In this case $\{1,\dots,n\}$ always refers to the labelling set of Δ rather than its vertex set; see [49, Example 2.1] for an example.

By the standard construction a simplicial complex defines an affine toric variety. Let Δ be a simplicial complex with vertices labelled by variables $\mathbf{t} = (t_1,\dots,t_n)$ (with possible repetitions). Suppose Δ contains N distinct simplices. Then Δ induces an embedding $e_\Delta : \mathbb{C}^n \to \mathbb{C}^N$, where the coordinates of the codomain are indexed by $\sigma \in \Delta$, by

$$\mathbf{t} \mapsto \mathbf{x} = (x_\sigma)_{\sigma\in\Delta}, \qquad x_\sigma = \prod_{i\in\sigma} t_i.$$

By convention, the monomial corresponding to the empty set is $x_\emptyset = 1$. If two simplices have exactly the same labels, as multisets, we may identify them. We define the variety $V_\Delta := \overline{e_\Delta(\mathbb{C}^n)}$. Denote by $\Delta_{\geq 2}$ the set of simplices in Δ of dimension at least one. The toric variety T_Δ associated with the embedding e_Δ is the affine toric variety in $\mathbb{C}^{N-n-1}$ obtained as the closure of the projection of V_Δ to the coordinates x_σ with $\sigma \in \Delta_{\geq 2}$.

We define the *tangential variety* $\tau(Y)$ of a variety Y as the union of all tangent lines to Y. The following is one of the main results from [49] which plays the central role in our investigations.

Theorem 4.1.2. *The tangential variety of V_Δ is isomorphic to the product of $\mathbb{C}^n$ and the variety T_Δ.*

Proof. See [49, Theorem 2.9]. □

Remark 4.1.3. *The proof of the above Theorem 4.1.2 uses coordinate changes from $\mathbb{C}^N$ to $\mathbb{C}^N$ called simplicial cumulants [49, Definition 2.5], which were generalization of secant cumulants [71]. Interested reader can find more detail about cumulant coordinates in [117, 118].*

Next we present an example of a simplicial complex whose associated toric variety is the (affine) Segre–Veronese variety — our main object of investigations.

Example 4.1.4. *Fix $k \in \mathbb{N}$ a positive integer and $\mathbf{a}, \mathbf{b} \in \mathbb{N}^k$ where $\mathbf{a} = (a_1,\dots,a_k)$, $\mathbf{b} = (b_1,\dots,b_k)$ such that a_i, b_i are positive integers. Consider the vertex set $V = V_1 \sqcup \cdots \sqcup V_k$, where each V_i has a_ib_i vertices such that for each $j = 1,\dots,b_i$, exactly a_i vertices get labelled $t_{i,j}$. We denote Δ_{SV} the simplicial complex with vertex set V. A subset σ of V forms a simplex of Δ_{SV} if and only if $|\sigma \cap V_i| \leq a_i$ for all $i = 1,\dots,k$.*

If $\sigma \in \Delta_{\mathrm{SV}}$ then $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_k$, where each σ_i is a multiset of labels $t_{i,j}$ with $|\sigma_i| \leq a_i$. Let $n = b_1 + \cdots + b_k$ and N be the number of simplices in Δ_{SV}. The toric embedding $e_{\Delta_{\mathrm{SV}}} : \mathbb{C}^n \to \mathbb{C}^N$ is given by

$$x_\sigma \quad = \quad \prod_{i=1}^{k} \prod_{j\in\sigma_i} t_{i,j} \qquad \text{for all} \quad \sigma \in \Delta_{\mathrm{SV}}.$$

The corresponding projective variety is obtained by introducing additional variables $t_{i,0}$ for $i = 1, \dots, k$ (the coordinates of each $\mathbb{P}^{b_i}$ are $(t_{i,0}, \dots, t_{i,b_i})$) and considering now a homogeneous parameterization $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_k} \to \mathbb{P}^N$

$$x_\sigma \;=\; \prod_{i=1}^{k} t_{i,0}^{a_i - |\sigma_i|} \prod_{j \in \sigma_i} t_{i,j} \qquad \text{for all} \quad \sigma \in \Delta_{\mathrm{SV}}.$$

The image of this map is the *Segre–Veronese variety*

$$X \;:=\; v_{a_1}(\mathbb{P}^{b_1}) \times \cdots \times v_{a_k}(\mathbb{P}^{b_k}),$$

which is the embedding of the product $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_k}$ given by the very ample line bundle $\mathcal{O}(a_1, \dots, a_k)$. **From this point onward Δ_{SV} will always be denoted by Δ.**

Remark 4.1.5. *(a) The original affine variety V_Δ is isomorphic to the open subset of the Segre–Veronese variety obtained by setting $t_{i,0} \neq 0$ for all $i = 1, \dots, k$. This amounts to assuming $x_\emptyset \neq 0$. The Segre–Veronese variety can be covered by such varieties obtained by assuming that exactly one variable $t_{i,j}$ for each $i = 1, \dots, k$ is necessarily nonzero, or in other words, that a given coordinate x_σ is nonzero.*

(b) Consider the open subset of $\tau(X)$ given by $\tau(X) \cap \{x_\emptyset \neq 0\}$. On this subset $\tau(X)$ is isomorphic to the tangential variety of the affine variety V_Δ. By Theorem 4.1.2 this (affine) tangential variety is isomorphic to the product of $\mathbb{C}^n$ and the variety T_Δ associated to the simplicial complex Δ. This means that $\tau(X)$ can be covered by toric varieties, which is our main motivation to study the variety T_Δ.

Above Remark 4.1.5(b) gives us the following proposition:

Proposition 4.1.6. *The tangential variety $\tau(X)$ is covered by toric varieties isomorphic to a product of an affine space of dimension $n = \sum_{i=1}^{k} b_i$ and the toric variety T_Δ.*

Next we describe the semigroup $S_\Delta \subseteq \mathbb{N}^n$ associated to the toric variety T_Δ, that is, when $T_\Delta = \operatorname{Spec} \mathbb{C}[S_\Delta]$. By following [49] S_Δ is generated by those lattice points $(x_{i,j}) \in \mathbb{N}^n$ which satisfy the inequalities

(1) $\sum_j x_{i,j} \leq a_i$ for all $i = 1, \dots, k$, and

(2) $\sum_{i,j} x_{i,j} \geq 2$.

In the next section, we study when the tangential variety $\tau(X)$ of the Segre–Veronese variety X is Cohen–Macaulay or Gorenstein. Here comes the main difference between secant varieties which were studied in [49] and the tangential varieties we study in this chapter. In the case of secant varieties, the associated toric variety T_Δ is always normal. Normal toric are Cohen-Macaulay, and there is quite easy criteria to check, whether they are Gorenstein 1.1.6, which was used in the case of secant varieties.

However, in our case, the variety T_Δ is not always normal, thus we can not use this criteria. Instead, we employ [108, Theorem 4.1], which we recall for readers' convenience adapted to $T_\Delta = \operatorname{Spec} \mathbb{C}[S_\Delta]$. First, we need to introduce a few notations (which will be clarified below by using a list of examples, see Example 4.1.9).

Notation 4.1.7.
- *Let G_Δ denote the additive group in $\mathbb{Z}^n$ generated by S_Δ and put $r = \operatorname{rank}_{\mathbb{Z}} G_\Delta$.*
- *Let C_Δ denote the convex rational polyhedral cone spanned by S_Δ in $\mathbb{Q}^n_{\geq 0}$. Hence $\dim_{\mathbb{Q}} C_\Delta = r$.*
- *Let $\mathcal{F}$ be the set of all facets of C_Δ.*
- *For any facet $F \in \mathcal{F}$ let*
$$S_F = \{x = (x_{i,j}) \in G_\Delta \mid x + y \in S_\Delta \text{ for some } y = (y_{i,j}) \in S_\Delta \cap F\}$$
and $S'_\Delta = \bigcap_{F \in \mathcal{F}} S_F$. Note that $S_\Delta \subseteq S_F$ for every $F \in \mathcal{F}$.
- *In our case, we will have only two kinds of facets so we denote them by*
$$F_{i,j} := \{x = (x_{i,j}) \mid x_{i,j} = 0\} \quad \text{and} \quad F_i := \{x = (x_{i,j}) \ : \ \sum_j x_{i,j} = \sum_{l \neq i} \sum_j x_{l,j}\},$$
cf. Lemma 4.2.4. We denote $S_{i,j} := S_{F_{i,j}}$ and $S_i := S_{F_i}$.
- *For a subset J of $\mathcal{F}$, we set $G_J = \bigcap_{F \notin J} S_F \setminus \bigcup_{F' \in J} S_{F'}$. In particular,*
$$G_{\mathcal{F}} = G_\Delta \setminus \bigcup_{F' \in \mathcal{F}} S_{F'}.$$
- *For a subset J of $\mathcal{F}$, let π_J be the simplicial complex of nonempty subsets I of J with the property $\bigcap_{F \in I}(S_\Delta \cap F) \neq \{0\}$.*

We recall that a simplicial complex Δ is called *acyclic* if the reduced homology group $\tilde{H}_q(\Delta; \mathbb{C})$ vanishes for all $q \geq 0$ (see [76] for basics on reduced homology groups). Moreover, let $n = b_1 + \cdots + b_k$ and $\mathcal{I} = \{(i,j) : 1 \leq i \leq k, 1 \leq j \leq b_i\}$, and note that $|\mathcal{I}| = n$. The canonical unit vectors of $\mathbb{R}^n$ are denoted by $e_{i,j}$ where $(i,j) \in \mathcal{I}$ and $x = (x_{i,j}) \in \mathbb{R}^n$. We follow the convention that the elements of $\mathcal{I}$ are ordered lexicographically. Furthermore, from this point onward, without loss of generality, we assume that $a_1 \leq a_2 \leq \ldots \leq a_k$.

Theorem 4.1.8. *([108, Theorem 4.1]) $\mathbb{C}[S_\Delta]$ is a Cohen–Macaulay (resp. Gorenstein) ring if and only if the following conditions are satisfied:*

(i) $S'_\Delta = S_\Delta$ (resp. $G_{\mathcal{F}} = x - S_\Delta$ for some $x \in G_\Delta$), and

(ii) for every nonempty proper subset J of $\mathcal{F}$, $G_J = \emptyset$ or π_J is acyclic.

Let us look at a few examples.

Example 4.1.9. *(1) Let $k = 2$, $\mathbf{a} = (2,2)$, $\mathbf{b} = (1,1)$. Then*
$$S_\Delta = \mathbb{N}^2 \setminus \left(\{x \in \mathbb{N}^2 : 2 \nmid x_{1,1}, x_{2,1} = 0\} \cup \{x \in \mathbb{N}^2 : x_{1,1} = 0, 2 \nmid x_{2,1}\}\right),$$
$G_\Delta = \mathbb{Z}^2$, see Lemma 4.2.1 for general description of G_Δ, and the cone C_Δ has two facets $F_{1,1}$ and $F_{2,1}$, see Lemma 4.2.4 for a more general facet description of our cone. Also,
$$S_{1,1} = \{x \in \mathbb{Z}^2 : x_{1,1} > 0\} \cup \{x \in \mathbb{Z}^2 : x_{1,1} = 0, 2 \mid x_{2,1}\}$$

and

$$S_{2,1} = \{x \in \mathbb{Z}^2 : x_{2,1} > 0\} \cup \{x \in \mathbb{Z}^2 : 2 \mid x_{1,1}, x_{2,1} = 0\},$$

and hence $S'_\Delta = S_{1,1} \cap S_{2,1} = S_\Delta$. *Moreover,*

$$G_{\mathcal{F}} = \{x \in \mathbb{Z}^2 : x_{1,1} < 0, x_{2,1} < 0\} \cup \{(-1-2n, 0) : n \in \mathbb{N}\} \cup \{(0, -1-2m) : m \in \mathbb{N}\}$$

and there is no point $x \in \mathbb{Z}^2$ *such that* $G_{\mathcal{F}} = x - S_\Delta$, *see Figure 4.1 below. Finally, for* I *equal to* $\{F_{1,1}\}$ *or* $\{F_{2,1}\}$, π_I *is a point and hence acyclic. Therefore* $\mathbb{C}[S_\Delta]$ *is Cohen-Macaulay but neither Gorenstein nor normal. This means that the tangential variety* $\tau(v_2(\mathbb{P}^1) \times v_2(\mathbb{P}^1))$ *is Cohen-Macaulay but neither Gorenstein nor normal.*

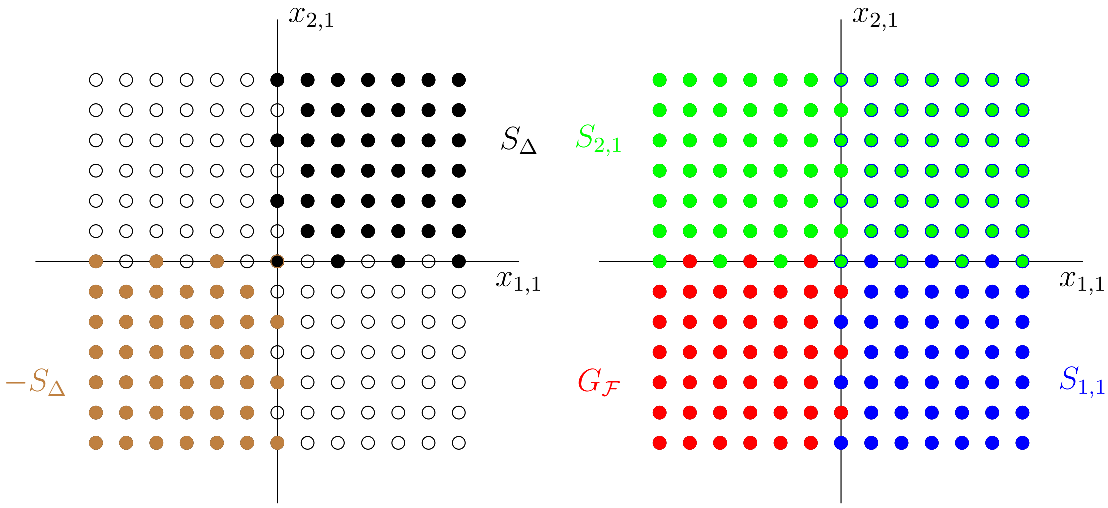

Figure 4.1: Example 4.1.9(1), $\mathbf{a} = (2,2)$, $\mathbf{b} = (1,1)$

(2) Let $k = 2$, $\mathbf{a} = (2,2)$, $\mathbf{b} = (1,2)$. *Then* C_Δ *has three facets* $F_{1,1}, F_{2,1}$ *and* $F_{2,2}$. *Also,* $G_\Delta = \mathbb{Z}^3$, $S_{1,1} = \{x \in \mathbb{Z}^3 : x_{1,1} > 0\} \cup \{x \in \mathbb{Z}^3 : x_{1,1} = 0, 2 \mid x_{2,1} + x_{2,2}\}$,

$$S_{2,1} = \{x \in \mathbb{Z}^3 : x_{2,1} \geq 0\} \qquad \textit{and} \qquad S_{2,2} = \{x \in \mathbb{Z}^3 : x_{2,2} \geq 0\}.$$

Therefore the point $e_{1,1} \in S'_\Delta \setminus S_\Delta$ *and hence* $\mathbb{C}[S_\Delta]$ *is not Cohen-Macaulay. This means that the tangential variety* $\tau(v_2(\mathbb{P}^1) \times v_2(\mathbb{P}^2))$ *is not Cohen-Macaulay and hence neither Gorenstein nor normal. We in fact can generalize this example to any* $b_2 \geq 2$, *cf. Lemma 4.2.6 (2).*

(3) Let $k = 2$, $\mathbf{a} = (1,2)$, $\mathbf{b} = (1,1)$. *Then*

$$S_\Delta = \{x \in \mathbb{N}^2 : x_{1,1} \leq x_{2,1}\} \setminus \{x \in \mathbb{N}^2 : x_{1,1} = 0, 2 \nmid x_{2,1}\},$$

$G_\Delta = \mathbb{Z}^2$, *and* C_Δ *has two facets* $F_{1,1}$ *and* F_1. *Also,*

$$S_{1,1} = \{x \in \mathbb{Z}^2 : x_{1,1} > 0\} \cup \{x \in \mathbb{Z}^2 : x_{1,1} = 0, 2 \mid x_{2,1}\} \quad \textit{and} \quad S_1 = \{x \in \mathbb{Z}^2 : x_{1,1} \leq x_{2,1}\},$$

and hence $S'_\Delta = S_{1,1} \cap S_1 = S_\Delta$. *Moreover,*

$$G_{\mathcal{F}} = \{x \in \mathbb{Z}^2 : x_{1,1} < 0, x_{2,1} < 0, x_{1,1} > x_{2,1}\} \cup \{(0, -1-2n) : n \in \mathbb{N}\}$$

and hence $G_{\mathcal{F}} = (0,-1) - S_\Delta$, *see Figure 4.2 below. Finally, for* I *equal to* $\{F_{1,1}\}$ *or* $\{F_1\}$, π_I *is a point and hence acyclic. Therefore* $\mathbb{C}[S_\Delta]$ *is Gorenstein (so is Cohen-Macaulay). Note that* $\mathbb{C}[S_\Delta]$ *is not normal. This means that the tangential variety* $\tau(v_1(\mathbb{P}^1) \times v_2(\mathbb{P}^1))$ *is Gorenstein but not normal.*

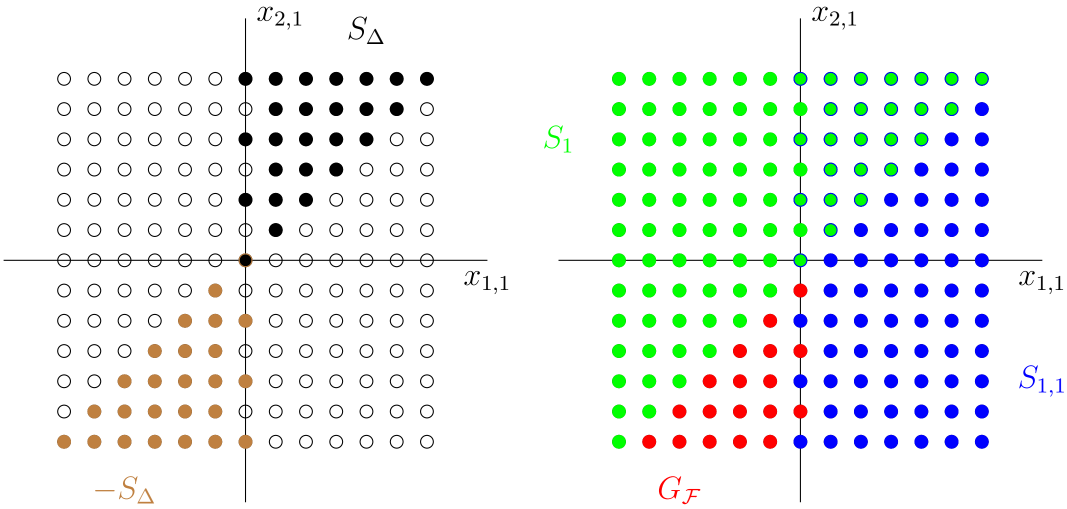

Figure 4.2: Example 4.1.9(3), $\mathbf{a} = (1, 2)$, $\mathbf{b} = (1, 1)$

(4) Let $k = 2$, $\mathbf{a} = (1, 2)$, $\mathbf{b} = (1, 2)$. Then C_Δ has four facets $F_{1,1}, F_{2,1}, F_{2,2}$ and F_1. Also

$$S_{1,1} = \{x \in \mathbb{Z}^3 : x_{1,1} > 0\} \cup \{(0, x_{2,1}, x_{2,2}) : 2 \mid x_{2,1} + x_{2,2}\}, \quad S_{2,1} = \{x \in \mathbb{Z}^3 : x_{2,1} \geq 0\},$$

$$S_{2,2} = \{x \in \mathbb{Z}^3 : x_{2,2} \geq 0\} \quad \text{and} \quad S_1 = \{x \in \mathbb{Z}^3 : x_{1,1} \leq x_{2,1} + x_{2,2}\}.$$

We first claim that S_Δ satisfies condition (ii) of the above theorem. Indeed if J is a singleton subset of $\mathcal{F}$, then clearly π_J is acyclic. When J has two elements, we need to consider all the cases separately:

- *If $J = \{F_{1,1}, F_{2,1}\}$, then $\pi_J = \{\emptyset, \{F_{1,1}\}, \{F_{2,1}\}, \{F_{1,1}, F_{2,1}\}\}$ which is a simplex and hence acyclic. Note that $G_J = S_{2,2} \cap S_1 \setminus S_{1,1} \cup S_{2,1} \neq \emptyset$, since it for example contains $(-1, -1, 5)$.*

- *If $J = \{F_{1,1}, F_{2,2}\}$, then $\pi_J = \{\emptyset, \{F_{1,1}\}, \{F_{2,2}\}, \{F_{1,1}, F_{2,2}\}\}$ which is acyclic. Note that $G_J = S_{2,1} \cap S_1 \setminus S_{1,1} \cup S_{2,2} \neq \emptyset$, since it for example contains $(-1, 5, -1)$.*

- *If $J = \{F_{1,1}, F_1\}$, then $\pi_J = \{\emptyset, \{F_{1,1}\}, \{F_1\}\}$ which is* not *acyclic, since $\tilde{H}_0(\pi_J; \mathbb{C}) \cong \mathbb{C}$. But in this case $G_J = S_{2,1} \cap S_{2,2} \setminus S_{1,1} \cup S_1 = \emptyset$.*

- *If $J = \{F_{2,1}, F_{2,2}\}$, then $\pi_J = \{\emptyset, \{F_{2,1}\}, \{F_{2,2}\}\}$ which is* not *acyclic, since $\tilde{H}_0(\pi_J; \mathbb{C}) \cong \mathbb{C}$. But in this case $G_J = S_{1,1} \cap S_1 \setminus S_{2,1} \cup S_{2,2} = \emptyset$.*

- *If $J = \{F_{2,1}, F_1\}$, then $\pi_J = \{\emptyset, \{F_{2,1}\}, \{F_1\}, \{F_{2,1}, F_1\}\}$ which is acyclic. Note that $G_J = S_{1,1} \cap S_{2,2} \setminus S_{2,1} \cup S_1 \neq \emptyset$, since it for example contains $(1, -1, 1)$.*

- *If $J = \{F_{2,2}, F_1\}$, then $\pi_J = \{\emptyset, \{F_{2,2}\}, \{F_1\}, \{F_{2,2}, F_1\}\}$ which is acyclic. Note that $G_J = S_{1,1} \cap S_{2,1} \setminus S_{2,2} \cup S_1 \neq \emptyset$, since it for example contains $(1, 1, -1)$.*

  When J has three elements, we need to consider all the cases separately:

- *If $J = \{F_{1,1}, F_{2,1}, F_{2,2}\}$, then $\pi_J = \{\emptyset, \{F_{1,1}\}, \{F_{2,1}\}, \{F_{2,2}\}, \{F_{1,1}, F_{2,1}\}, \{F_{1,1}, F_{2,2}\}\}$ which is acyclic. Note that $G_J = S_1 \setminus S_{1,1} \cup S_{2,1} \cup S_{2,2} \neq \emptyset$, since it for example contains $(-2, -1, -1)$.*

- *If* $J = \{F_{1,1}, F_{2,1}, F_1\}$*, then* $\pi_J = \{\emptyset, \{F_{1,1}\}, \{F_{2,1}\}, \{F_1\}, \{F_{1,1}, F_{2,1}\}, \{F_{2,1}, F_1\}\}$ *which is acyclic. Note that* $G_J = S_{2,2} \setminus S_{1,1} \cup S_{2,1} \cup S_1 \neq \emptyset$*, since it for example contains* $(-1, -4, 1)$*.*

- *If* $J = \{F_{1,1}, F_{2,2}, F_1\}$*, then* $\pi_J = \{\emptyset, \{F_{1,1}\}, \{F_{2,2}\}, \{F_1\}, \{F_{1,1}, F_{2,2}\}, \{F_{2,2}, F_1\}\}$ *which is acyclic. Note that* $G_J = S_{2,1} \setminus S_{1,1} \cup S_{2,2} \cup S_1 \neq \emptyset$*, since it for example contains* $(-1, 1, -4)$*.*

- *If* $J = \{F_{2,1}, F_{2,2}, F_1\}$*, then* $\pi_J = \{\emptyset, \{F_{2,1}\}, \{F_{2,2}\}, \{F_1\}, \{F_{2,1}, F_1\}, \{F_{2,2}, F_1\}\}$ *which is acyclic. Note that* $G_J = S_{1,1} \setminus S_{2,1} \cup S_{2,2} \cup S_1 \neq \emptyset$*, since it for example contains* $(1, -1, -1)$*.*

Moreover, it is easy to check that $S'_\Delta = S_\Delta$*, but there does not exist any* $x \in \mathbb{Z}^3$ *such that* $G_\mathcal{F} = x - S_\Delta$*. Therefore* $\mathbb{C}[S_\Delta]$ *is Cohen-Macaulay, but neither Gorenstein nor normal. This means that the tangential variety* $\tau(v_1(\mathbb{P}^1) \times v_2(\mathbb{P}^2))$ *is Cohen-Macaulay but neither Gorenstein nor normal (the same statement is true for any* $b_2 \geq 2$*, see Theorem 4.2.9; see also Lemma 4.2.7).*

(5) Let $k = 1$*,* $a = 3$*,* $b = 1$ *(same argument applies to any* $a \geq 3$*). Then* $S_\Delta = \mathbb{N} \setminus \{1\}$*,* $G_\Delta = \mathbb{Z}$ *and* C_Δ *has only one facet* $F_{1,1}$*. Also,* $S'_\Delta = S_{1,1} = S_\Delta$*,*

$$G_{\{F_{1,1}\}} = \{-1 - n : n \in \mathbb{N}\} \cup \{1\} = 1 - S_\Delta$$

and condition (ii) of the previous Theorem 4.1.8 trivially holds. Therefore $\mathbb{C}[S_\Delta]$ *is Gorenstein (so is Cohen-Macaulay). Note that* $\mathbb{C}[S_\Delta]$ *is not normal. This means that the tangential variety* $\tau(v_3(\mathbb{P}^1))$ *is Gorenstein but not normal.*

(6) Let $k = 1$*,* $a = 2$*,* $b = 2$*. Then*

$$S_\Delta = \{x \in \mathbb{N}^2 : 2 \mid x_{1,1} + x_{1,2}\}, \quad G_\Delta = \{x \in \mathbb{Z}^2 : 2 \mid x_{1,1} + x_{1,2}\}$$

and C_Δ *has two facets* $F_{1,1}, F_{1,2}$*. Also,*

$$S_{1,1} = \{x \in G_\Delta : x_{1,1} \geq 0\} \quad \text{and} \quad S_{1,2} = \{x \in G_\Delta : x_{1,2} \geq 0\},$$

and hence $S'_\Delta = S_{1,1} \cap S_{1,2} = S_\Delta$*. Moreover,* $G_\mathcal{F} = \{x \in G_\Delta : x_{1,1} < 0, x_{1,2} < 0\}$ *and hence* $G_\mathcal{F} = (-1, -1) - S_\Delta$*. Finally, for* I *equal to* $\{F_{1,1}\}$ *or* $\{F_{1,2}\}$*,* π_I *is a point and hence acyclic. Therefore* $\mathbb{C}[S_\Delta]$ *is Gorenstein (see Theorem 4.2.9(G5) for a generalization). Note that* $\mathbb{C}[S_\Delta]$ *is also normal. This means that the tangential variety* $\tau(v_2(\mathbb{P}^2))$ *is Gorenstein as well as normal.*

(7) Let $k = 1$*,* $a = 2$*,* $b = 3$*. Then*

$$S_\Delta = \{x \in \mathbb{N}^3 : 2 \mid x_{1,1} + x_{1,2} + x_{1,3}\}, \quad G_\Delta = \{x \in \mathbb{Z}^3 : 2 \mid x_{1,1} + x_{1,2} + x_{1,3}\}$$

and C_Δ *has three facets* $F_{1,1}, F_{1,2}$ *and* $F_{1,3}$*. Also,*

$$S_{1,j} = \{x \in G_\Delta : x_{1,j} \geq 0\} \quad \text{for} \quad j = 1, 2, 3,$$

and hence $S'_\Delta = S_{1,1} \cap S_{1,2} \cap S_{1,3} = S_\Delta$*. Moreover,*

$$G_\mathcal{F} = \{x \in G_\Delta : x_{1,1} < 0, x_{1,2} < 0, x_{1,3} < 0\}$$

which is not equal to $x - S_\Delta$*, since the only possibility for* x *is* $(-1, -1, -1)$ *which is not an element of* G_Δ*. Finally, for any subset* $I = \{F, F'\}$ *of* $\mathcal{F}$*, we have* $\pi_I = \{\emptyset, \{F\}, \{F'\}, \{F, F'\}\}$ *which is a simplex and hence acyclic. Therefore* $\mathbb{C}[S_\Delta]$ *is Cohen-Macaulay, but not Gorenstein (see Theorem 4.2.9(CM6) and (G5) for a generalization). Note that* $\mathbb{C}[S_\Delta]$ *is also normal. This means that the tangential variety* $\tau(v_2(\mathbb{P}^3))$ *is Cohen-Macaulay and normal but not Gorenstein.*

4.2 Cohen–Macaulay and Gorenstein tangential varieties

In this section, we study when a tangential variety $\tau(X)$ of the Segre–Veronese variety X is Cohen–Macaulay or Gorenstein. Firstly, we present the description of G_Δ when it is equal to the whole $\mathbb{Z}^n$.

Lemma 4.2.1. *We have $G_\Delta = \mathbb{Z}^n$, that is the tangential variety is of expected dimension $2n$, unless:*

(i) $k = 2, \mathbf{a} = (1, 1)$, *when* $G_\Delta = \{x \in \mathbb{Z}^n \ : \ \sum x_{1,j} = \sum x_{2,j}\}$,

(ii) $k = 1$, $a = 2$, *when* $G_\Delta = \{x \in \mathbb{Z}^n \ : \ 2 \mid \sum_j x_{1,j}\}$, *or*

(iii) $k = 1$, $a = 1$, *when* $G_\Delta = \{0\}$.

Proof. We separately consider cases $k \geq 3$, $k = 2$, and $k = 1$.

Case I. $k \geq 3$
Consider the set of vectors

1. $e_{1,1} + e_{i,j}$ for all $(i, j) \in \mathcal{I}$ with $i \neq 1$,
2. $e_{1,j} + e_{2,1}$ for all $1 < j \leq b_1$,
3. $e_{2,1} + e_{3,1}$, and
4. $e_{1,1} + e_{2,1} + e_{3,1}$

that lie in S_Δ. Now combining (3) and (4) we get $e_{1,1} \in G_\Delta$, and hence by using (1), we obtain $e_{i,j} \in G_\Delta$ for all $(i, j) \in \mathcal{I}$ with $i \neq 1$. Finally use (2) to get $e_{1,j} \in G_\Delta$ for all $1 < j \leq b_1$, showing $G_\Delta = \mathbb{Z}^n$.

Case II. $k = 2$
Consider the set of vectors

1. $e_{1,1} + e_{2,j}$ for all $1 \leq j \leq b_2$,
2. $e_{1,j} + 2e_{2,1}$ for all $1 \leq j \leq b_1$, and
3. $2e_{2,1}$

that lie in S_Δ if $a_2 \geq 2$. Now combining (2) and (3) we get $e_{1,j} \in G_\Delta$ for all $1 \leq j \leq b_1$, and hence by using (1), we obtain $e_{2,j} \in G_\Delta$ for all $1 \leq j \leq b_2$, showing $G_\Delta = \mathbb{Z}^n$. If $a_2 = 1$ then the vectors in (1) are the only generators of S_Δ, so it is easy to check that $G_\Delta = \{x \in \mathbb{Z}^n \ : \ \sum x_{1,j} = \sum x_{2,j}\} \neq \mathbb{Z}^n$.

Case III. $k = 1$
Consider the set of vectors

1. $2e_{1,j}$ for all $1 \leq j \leq b$, and
2. $3e_{1,j}$ for all $1 \leq j \leq b$

that lie in S_Δ if $a \geq 3$. Combining (1) and (2), we get $e_{1,j} \in G_\Delta$ for all $1 \leq j \leq b$, showing $G_\Delta = \mathbb{Z}^n$. If $a = 2$ then all generators are of the form $e_{1,j_1} + e_{1,j_2}$, i.e. with the sum of the coordinates equal to two. It can be easily seen that they generate

$$G_\Delta = \{x \in \mathbb{Z}^n \ : \ 2 \mid \sum_j x_{1,j}\}.$$

Finally, the case when $a = 1$ is trivial. $\square$

Remark 4.2.2. *The dimension of the tangential variety of the Segre-Veronese variety was already known. It, in fact, can be derived by using Terracini Lemma [106]. Also, from Lemma 4.2.1(ii) it can be derived that the tangential variety is of expected dimension $2n$ in that case as well, although the group G_Δ is not equal to $\mathbb{Z}^n$ (instead it is isomorphic to $\mathbb{Z}^n$).*

We now have the description of the cone C_Δ.

Lemma 4.2.3. *The cone C_Δ is defined by the following set of inequalities:*

1. *$x_{i,j} \geq 0$ for all $(i,j) \in \mathcal{I}$, and*
2. *$\sum_j x_{i,j} \leq \sum_{l \neq i} \sum_j x_{l,j}$ for all i such that $a_i = 1$.*

Proof. Let C be the cone defined by the above inequalities $(1)-(2)$. It is easy to check that all generators of S_Δ lie in C so $C_\Delta \subseteq C$. To prove the other inclusion, consider a point $x \in C \cap \mathbb{Z}^n$. It is sufficient to show that $2x \in C_\Delta$. We, in fact, show a more general statement: any point $y = (y_{i,j}) \in C \cap \mathbb{Z}^n$ with even sum of coordinates can be written as a sum of generators $(x_{i,j})$ of S_Δ with $\sum_{i,j} x_{i,j} = 2$. We only need to prove this statement since it implies the lemma.

We denote $\sum_{i,j} y_{i,j} = 2m$ and prove the statement by induction on m. For $m = 0$ it is true. Consider the case $m > 0$. If there exists an index i_0 such that all non-zero coordinates of y are in the form (i_0, j) then inequality (2) for i_0 implies $a_{i_0} \neq 1$. Therefore $e_{i_0,j_1} + e_{i_0,j_2}$ are generators of S_Δ for all $1 \leq j_1, j_2 \leq b_{i_0}$. We can easily write y as a sum of m such generators.

Otherwise, we look at the inequalities (2) for the point y in which an equality holds, i.e. the point y lies on the corresponding face of C. Since the inequalities (2) is equivalent with

$$\sum_j y_{i,j} \leq \frac{1}{2} \sum_{i,j} y_{i,j} = m,$$

therefore we can have at most two indices i for which the equality holds. Moreover, for every index i for which the equality holds there exists a pair (i,j) such that $y_{i,j} > 0$ since $\sum_j y_{i,j} = m > 0$. So we pick two pairs $(i_1, j_1), (i_2, j_2)$ with $i_1 \neq i_2$ such that $y_{i_1,j_1}, y_{i_2,j_2} > 0$, and for every index i for which there is an equality in (2) we have $i \in \{i_1, i_2\}$. This is clearly possible since there are at most two such indices i. The point $p = e_{i_1,j_1} + e_{i_2,j_2}$ is a generator of S_Δ and we claim that $z = y - p \in C$. We show that by checking all inequalities $(1) - (2)$.

The inequalities (1) obviously hold for the point z, hence we have to check the inequalities (2) for every $1 \leq i \leq k$. Note that the inequalities (2) for point z are equivalent with

$$\sum_j z_{i,j} \leq \frac{1}{2} \sum_{i,j} z_{i,j} = m - 1.$$

We distinguish two cases.

Case I. $\sum_j y_{i,j} < m$.
In this case we have

$$\sum_j z_{i,j} \leq \sum_j y_{i,j} \leq m-1,$$

and therefore the inequalities hold.

Case II. $\sum_j y_{i,j} = m$.
By our definition of point p we have

$$\sum_j z_{i,j} = \sum_j y_{i,j} - \sum_j p_{i,j} = m-1.$$

We can conclude that $z \in C$ and $\sum_{i,j} z_{i,j} = 2m-2$. By the induction hypothesis, we can write $z = y - p$ as a sum of generators which shows that y can be written as a sum of generators of S_Δ as well. This completes the proof. $\square$

In order to employ Theorem 4.1.8, we require the description of the facets of the cone C_Δ. To this end, let us first recall the following from Notations 4.1.7:

(a) $F_{i,j} = \{(x_{i,j}) \ : \ x_{i,j} = 0\}$ for $(i,j) \in \mathcal{I}$, and

(b) $F_i = \{(x_{i,j}) \ : \ \sum_j x_{i,j} = \sum_{l \neq i} \sum_j x_{l,j}\}$ for $1 \leq i \leq k$.

We have the facet description of the cone C_Δ:

Lemma 4.2.4. *1. $F_{i,j}$ defines a facet of C_Δ, unless:*

(i) $k=3$, $a_1 = a_2 = 1$, $a_3 \geq 2$ and $b_3 = 1$, when $F_{3,1}$ is not a facet,

(ii) $k=3$, $a_1 = a_2 = a_3 = 1$ and $b_i = 1$ for some i, when $F_{i,1}$ is not a facet,

(iii) $k=2$, $a_1 = 1, a_2 \geq 2$, $b_2 = 1$, when $F_{2,1}$ is not a facet,

(iv) $k=2$, $a_1 = a_2 = 1$, when $F_{i,j}$ is not a facet for every $(i,j) \in \mathcal{I}$, or

(v) $k=1$, $a=1$, when $C_\Delta = \{0\}$.

2. F_i defines a facet of C_Δ for all i such that $a_i = 1$, unless:

(i) $k=2$, $a_1 = a_2 = 1$, when $F_1 = F_2 = C_\Delta$, or

(ii) $k=1$, $a=1$, when $C_\Delta = \{0\}$.

Proof. If $b_{i_0} \geq 2$, we show that $F_{i_0,1}$ forms a facet (the same proof applies to every $F_{i_0,j}$). We consider another semigroup $S_{\Delta'}$ which corresponds to $(\mathbf{a}', \mathbf{b}')$, where $\mathbf{a}' = \mathbf{a}$, $b'_{i_0} = b_{i_0} - 1$ and $b'_i = b_i$ for all $i \neq i_0$. Moreover, we get coordinates of $S_{\Delta'}$ from those of S_Δ by skipping the coordinate $x_{i_0,1}$. Now there is a trivial bijection between the points of S_Δ which satisfy $x_{i_0,1} = 0$ and the points of $S_{\Delta'}$. Therefore, $F_{i_0,1}$ defines a facet if and only if $S_{\Delta'}$ is full-dimensional, i.e. when $k = 1, a \neq 1$, or $k = 2, \mathbf{a}' \neq (1,1)$, or $k \geq 3$, showing a part of $(iv)-(v)$ of the statement (1). If $b_{i_0} = 1$ and $k \geq 2$ (the case $k=1$ is obvious), we use the same argument as above for $k' = k-1$, $\mathbf{a}' = (a_1, \ldots, \widehat{a_{i_0}}, \ldots, a_k)$ and $\mathbf{b}' = (b_1, \ldots, \widehat{b_{i_0}}, \ldots, b_k)$, where $\widehat{\cdot}$ means we skip the corresponding coordinate. It gives us $(i) - (iii)$ and remaining part of $(iv) - (v)$ of the statement (1). This finishes the proof of (1).

To show that F_1 also forms a facet when $a_1 = 1$ (the same proof applies to every F_i), we need to find $n-1$ linearly independent points lying on it. We take the following points

1. $e_{1,1} + e_{i,j}$ for all $(i,j) \in \mathcal{I}$ with $i > 1$, and
2. $e_{1,j} + e_{2,1}$ for all $1 < j \leq b_1$.

This also shows (i) of the statement (2). Part (ii) is obvious and so this finishes the proof of (2). $\square$

The following result tells us about holes inside S_Δ.

Lemma 4.2.5. *(i) If $a_i > 2$, then the point $e_{i,j}$, for all $1 \leq j \leq b_i$, belongs to $(C_\Delta \cap \mathbb{Z}^n) \setminus S_\Delta$, and*

(ii) if $a_i = 2$, then the points $e_{i,j}$, for all $1 \leq j \leq b_i$, and $\sum_j c_j e_{i,j}$ with $(c_j) \in \mathbb{N}^{b_i}$ such that $2 \nmid \sum_j c_j$ belong to $(C_\Delta \cap \mathbb{Z}^n) \setminus S_\Delta$, unless $k = 1$, $a = 2$.
In other words, we have holes inside S_Δ.

Proof. All of the points listed clearly belong to $(C_\Delta \cap \mathbb{Z}^n) \setminus S_\Delta$. $\square$

In the following lemma, we study the condition (i) of Theorem 4.1.8.

Lemma 4.2.6. *1. In the case $a_k \geq 3$ we have $S_\Delta \neq S'_\Delta$, unless $k = 1$, $b = 1$.*

2. In the case $a_k = 2$ we have $S_\Delta \neq S'_\Delta$, unless:

(i) $k = 2$, $\mathbf{a} = (2,2)$, $\mathbf{b} = (1,1)$,

(ii) $k = 2$, $\mathbf{a} = (1,2)$, $b_1 = 1$, or

(iii) $k = 1$.

Proof. We show that the point $e_{k,1} \in S'_\Delta \setminus S_\Delta$. Clearly $e_{k,1} \notin S_\Delta$, cf. Lemma 4.2.5. It remains to show that for every facet F_i and $F_{i,j}$ of C_Δ we have $e_{k,1} \in S_i$ resp. $e_{k,1} \in S_{i,j}$, which implies that $e_{k,1} \in S'_\Delta$.

For $a_k \geq 2$ and for any i such that $a_i = 1$ we have $e_{k,1} + (e_{i,1} + e_{k,1}) \in S_\Delta$ and $e_{i,1} + e_{k,1} \in S_\Delta \cap F_i$, therefore $e_{k,1} \in S_i$. To show $e_{k,1} \in S_{i,j}$ we need to consider the cases $a_k > 2$ and $a_k = 2$ separately.

Case I. $a_k > 2$

For any $(i,j) \neq (k,1)$ we have $e_{k,1} + 2e_{k,1} \in S_\Delta$ and $2e_{k,1} \in S_\Delta \cap F_{i,j}$ which implies $e_{k,1} \in S_{i,j}$. For $S_{k,1}$, again we want to find a point $x \in S_\Delta \cap F_{k,1}$ such that $e_{k,1} + x \in S_\Delta$. We look at several cases:

- If $b_k \geq 2$ we can take $x = 2e_{k,2}$.
- If $a_i \geq 2$ for some $i \neq k$ we can take $x = 2e_{i,1}$.
- If $a_1 = a_2 = 1$ we can take $x = e_{1,1} + e_{2,1}$.

We are left with the cases $k = 1, b = 1$ and $k = 2, \mathbf{a} = (1, a_2), b_2 = 1$. However, in our statement we do not consider the first case and in the second case $F_{k,1}$ is not a facet by Lemma 4.2.4, so we are done. This concludes Case I.

Case II. $a_k = 2$

For the facet F_{i_0,j_0} with $i_0 \neq k$ we consider any pair $(i, j) \neq (i_0, j_0)$ with $i \neq k$. Note that such a pair does exist, unless $k = 2$ and $b_1 = 1$. Then we have $e_{k,1}+(e_{k,1}+e_{i,j}) \in S_\Delta$ with $e_{k,1} + e_{i,j} \in S_\Delta \cap F_{i_0,j_0}$, which implies that $e_{k,1} \in S_{i_0,j_0}$. For the facet $F_{k,j}$ we again need to find a point $x \in S_\Delta \cap F_{k,j}$ such that $x + e_{k,1} \in S_\Delta$. We consider several cases:

- If $a_i \geq 2$ for some $i \neq k$, then $x = 2e_{i,1}$.
- If $a_1 = a_2 = 1$, then $x = e_{1,1} + e_{2,1}$.
- If $\mathbf{a} = (1, 2)$ and $b_2 \geq 2$, then $x = e_{1,1} + e_{2,j_2}$ for some $j_2 \neq j$.

To sum up we always can find such x, unless $k = 1, a = 2$ or $k = 2, \mathbf{a} = (1, 2), b_2 = 1$. The first case is excluded in the statement and in the second case $F_{2,1}$ is not a facet by Lemma 4.2.4 so we covered all cases. This concludes Case II and hence completes the proof. □

In the following lemma, we study a special case of $k = 2$.

Lemma 4.2.7. *If $k = 2$, $\mathbf{a} = (1, 2)$, $b_1 = 1, b_2 \geq 2$ then:*

(i) $S_\Delta = (C_\Delta \cap \mathbb{Z}^n) \setminus \{x \in C_\Delta \cap \mathbb{Z}^n \ : \ x_{1,1} = 0, 2 \nmid \sum x_{2,j}\}$,

(ii) $S_\Delta = S'_\Delta$,

(iii) for any proper subset J of $\mathcal{F}$, π_J is acyclic, unless $J = \{F_1, F_{1,1}\}$ or $J = \mathcal{F} \setminus \{F_1, F_{1,1}\}$, and

(iv) for $J = \{F_1, F_{1,1}\}$ or $J = \mathcal{F} \setminus \{F_1, F_{1,1}\}$, we have $G_J = \emptyset$.

Proof. The points $e_{2,j_1} + e_{2,j_2}$ are the only generators of S_Δ that lying on the facet $F_{1,1}$, therefore on this facet, we have only the points with an even sum of coordinates. Furthermore, from the proof of Lemma 4.2.3, we know that all points with an even sum of coordinates in $C_\Delta \cap \mathbb{Z}^n$ also lie in S_Δ. To prove (i) it remains to show that any point $x \in C_\Delta \cap \mathbb{Z}^n$ with odd sum of coordinates and $x_{1,1} > 0$ is in S_Δ.

It is easy to check that $x - e_{1,1} \in C_\Delta$ and therefore x can be written as the sum of generators of S_Δ with the sum of coordinates equal to two. Since $x_{1,1} \leq \sum_j x_{2,j}$, at least one of these generators must be in the form $e_{2,j_1} + e_{2,j_2}$. So we simply replace this generator by $e_{1,1} + e_{2,j_1} + e_{2,j_2}$ to write x as the sum of generators of S_Δ.

To prove (ii) we notice that we have $S_\Delta \subseteq S'_\Delta \subseteq C_\Delta \cap G_\Delta$. This, in fact, holds for any affine semigroup S. Thus, it is sufficient to show that for any point x with $x_{1,1} = 0$ and $2 \nmid \sum x_{2,j}$ we have $x \notin S'_\Delta$. However, this is easy since for any such point x we have $x \notin S_{1,1}$.

For part (iii) note that by Lemma 4.2.4 we have $\mathcal{F} = \{F_{1,1}, F_1\} \cup \{F_{2,j} : 1 \leq j \leq b_2\}$. We claim that $\bigcap_{F \in I}(S_\Delta \cap F) = \{0\}$ if and only if $\{F_1, F_{1,1}\} \subseteq I$ or $\mathcal{F} \setminus \{F_1, F_{1,1}\} \subseteq I$. It is easy to see that the only point from S_Δ which lies on both F_1 and $F_{1,1}$ is the origin, and the same is true for a point lying on all $F_{2,j}$. To prove the "only if" part it is sufficient to consider maximal subset I which does not contain two forbidden sets:

- For $I = \mathcal{F} \setminus \{F_{1,1}, F_{2,j_0}\}$ we have $e_{1,1} + e_{2,j_0} \in \bigcap_{F \in I}(S_\Delta \cap F)$,
- for $I = \mathcal{F} \setminus \{F_1, F_{2,j_0}\}$, we have $2e_{2,j_0} \in \bigcap_{F \in I}(S_\Delta \cap F)$.

From this statement it follows that for $J = \{F_1, F_{1,1}\}$ or $J = \mathcal{F} \setminus \{F_1, F_{1,1}\}$ π_J is not acyclic, because we have $\tilde{H}_0(\pi_J; \mathbb{C}) \cong \mathbb{C}$ or $\tilde{H}_{b_2-2}(\pi_J; \mathbb{C}) \cong \mathbb{C}$ respectively. Moreover, it is straightforward to check that for any other set J the complex π_J is either a simplex, a union of two simplices with a common facet, or a simplex without a facet, which are all acyclic.

For the part (iv), we first consider the case $J = \{F_1, F_{1,1}\}$. Suppose on contrary that $x \in G_J$. Then the condition $x \in S_{2,j}$ for every $1 \leq j \leq b_2$ implies that $x_{2,j} \geq 0$ for every $1 \leq j \leq b_2$. If now $x_{1,1} > 0$ or $x_{1,1} = 0$ and $2 \mid \sum_j x_{2,j}$, then $x \in S_{1,1}$ which is not possible by the definition of G_Δ. So $x_{1,1} < 0$ or $x_{1,1} = 0$ and $2 \nmid \sum_j x_{2,j}$. As $x \notin S_1$, therefore $x_{1,1} > \sum_j x_{2,j}$ and hence $x_{2,j} < 0$ for some j, which is a contradiction. Now we consider the case $J = \mathcal{F} \setminus \{F_1, F_{1,1}\}$. Again we suppose on contrary that $x \in G_J$. Therefore $x_{2,j} < 0$ for all $1 \leq j \leq b_2$. On the other hand, from $x \in S_{1,1}$ we get that $x_{1,1} \geq 0$, and hence $x_{1,1} - \sum_j x_{2,j} > 0$. The last inequality implies that $x \notin S_1$ which is a contradiction. This completes the proof.

□

The following lemma is about the non smoothness of T_Δ.

Lemma 4.2.8. *(i) If $G_\Delta = \mathbb{Z}^n$, then T_Δ is not smooth.*

(ii) If $k = 2$, $\mathbf{a} = (1, 1)$, $b_i > 1$ for all $i = 1, 2$, then T_Δ is not smooth.

(iii) If $k = 1$, $a = 2$ and $b > 1$, then T_Δ is not smooth.

Proof. We know that an affine toric variety is smooth if and only if it is of the form $\operatorname{Spec} \mathbb{C}[C \cap G]$, where C is the cone in the lattice G, and the rays of the cone C form a basis of the lattice G [18, Theorem 1.3.12.].

For (i), in our setting, if $G_\Delta = \mathbb{Z}^n$ we have that T_Δ is smooth if the ray generators of C_Δ form a basis and $C_\Delta \cap \mathbb{Z}^n = S_\Delta$, which is not possible unless $a_i = 1$ for all i, cf. Lemma 4.2.5. On the other hand, in this case all generators of S_Δ and therefore all ray generators of C_Δ are of the form $e_{i_1,j_1} + e_{i_2,j_2}$, where $i_1 \neq i_2$. These generators lie in the sublattice of all points with even sum of coordinates and therefore can not form a basis of $\mathbb{Z}^n$. This finishes the proof of (i).

For (ii), note that all points of the form $e_{1,j_1} + e_{2,j_2}$ are generators of S_Δ and G_Δ as well as are ray generators of C_Δ, so $C_\Delta \cap G_\Delta = S_\Delta$ and $\dim G_\Delta = b_1 + b_2 - 1$. On the other hand, there are $b_1 b_2$ points of form $e_{1,j_1} + e_{2,j_2}$, which satisfy $b_1 b_2 > b_1 + b_2 - 1$, unless b_1 or b_2 equals 1. Therefore T_Δ can be smooth only if $b_1 = 1$ or $b_2 = 1$. This finishes the proof of (ii).

For (iii), note that $S_\Delta = \{x \in \mathbb{N}^b : 2 \mid \sum_j x_{1,j}\}$, $G_\Delta = \{x \in \mathbb{Z}^b : 2 \mid \sum_j x_{1,j}\}$ and hence $C_\Delta \cap G_\Delta = S_\Delta$. Further $2e_{1,j}$ are ray generators of C_Δ which are a basis of G_Δ only if $b = 1$. This finishes the proof of (iii). □

Finally, we have the main result of this chapter.

Theorem 4.2.9. *The tangential variety of the Segre-Veronese variety is smooth if and only if*

(S1) $k = 2$, $\mathbf{a} = (1, 1)$, $b_1 = 1$, *or*

(S2) $k = 1$, $a = 1$ *or* ($a = 2$ *and* $b = 1$).

If the tangential variety of the Segre-Veronese variety is not smooth, then it is Cohen–Macaulay if and only if one of the following holds

(CM1) $k \geq 3$, $\mathbf{a} = (1, \ldots, 1)$,

(CM2) $k = 2$, $\mathbf{a} = (2, 2)$, $\mathbf{b} = (1, 1)$,

(CM3) $k = 2$, $\mathbf{a} = (1, 2)$, $\mathbf{b} = (1, b_2)$ *for all* $b_2 \geq 1$,

(CM4) $k = 2$, $\mathbf{a} = (1, 1)$, $b_i > 1$ *for all* $i = 1, 2$,

(CM5) $k = 1$, $a \geq 3$, $b = 1$,

(CM6) $k = 1$, $a = 2$, $b > 1$.

If the tangent variety of the Segre-Veronese variety is not smooth, then it is Gorenstein if and only if one of the following holds

(G1) $k = 3$, $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (1, 1, 1)$,

(G2) $k = 2$, $\mathbf{a} = (1, 2)$, $\mathbf{b} = (1, 1)$,

(G3) $k = 2$, $\mathbf{a} = (1, 1)$, $b_1 = b_2$, $b_1 > 1$.

(G4) $k = 1$, $a \geq 3$, $b = 1$,

(G5) $k = 1$, $a = 2$, b *is even.*

Proof. By applying Lemma 4.2.6 the following are the only possible candidates for $\mathbb{C}[S_\Delta]$ (equivalently for $\tau(X)$) to be Cohen-Macaulay and to be Gorenstein:

(i) $a_i = 1$ for all $1 \leq i \leq k$,

(ii) $k = 2$, $\mathbf{a} = (2, 2)$, $\mathbf{b} = (1, 1)$,

(iii) $k = 2$, $\mathbf{a} = (1, 2)$, $b_1 = 1$,

(iv) $k = 1$, $a \geq 3$, $b = 1$,

(v) $k = 1$, $a = 2$.

The candidates (ii, iv) can be resolved by using Example 4.1.9(1) and Example 4.1.9(5) respectively, showing (CM2), (CM5), and (G4). For $k = 1, a = 1$, we have $S_\Delta = \{0\}$, which shows a part of (S2).

Now we consider candidate (iii). The case $b_2 = 1$ is resolved by Example 4.1.9(3), showing (G2) and a part of (CM3). Assume $b_2 \geq 2$. By Lemma 4.2.7 both conditions from Theorem 4.1.8 are satisfied, showing (CM3). Suppose that T_Δ is also Gorenstein. Then there exists $x_0 \in \mathbb{Z}^n$ such that $G_\mathcal{F} = x_0 - S_\Delta$. Since in S_Δ the point 0 has the minimal possible value in every coordinate of all points in S_Δ, x_0 must be a point in $G_\mathcal{F}$ which has maximal value in every coordinate. Thus, $x_0 = (0, -1, -1, \ldots, -1)$ since clearly for all points $x \in \mathbb{Z}^n$, $x_{2,j} \geq 0$ implies $x \in S_{2,j}$ and $x_{1,1} > 0$ implies $x \in S_{1,1}$. Note that the $(1,1)$-coordinate must be positive since there are holes in S_Δ. However, we also have $y = (-1, \ldots, -1) \in G_{\mathcal{F}}$ and $x_0 - y = e_{1,1} \notin S_\Delta$ which is a contradiction.

We now consider candidate (v). Note that in this case $S_\Delta = \{x \in \mathbb{N}^b : 2 \mid \sum_j x_{1,j}\}$, $G_\Delta = \{x \in \mathbb{Z}^b : 2 \mid \sum_j x_{1,j}\}$ and $F_{1,j}$ is a facet of C_Δ for all $1 \leq j \leq b$. If $b = 1$, then $S_\Delta \cong \mathbb{N}$ which gives part of (S2), hence completes (S2), cf. Lemma 4.2.8(iii). If $b > 1$, then $S_{1,j} = \{x \in G_\Delta : x_{1,j} \geq 0\}$ for any $1 \leq j \leq b$ and hence $S'_\Delta = \cap_j S_{1,j} = S_\Delta$, and $G_{\mathcal{F}} = \{x \in G_\Delta : x_{1,j} < 0 \text{ for all } 1 \leq j \leq b\}$. Now an element $x \in G_\Delta$ satisfying $G_{\mathcal{F}} = x - S_\Delta$ must be equal to $(-1, \ldots, -1)$, which is only possible when k is even. Moreover, for any proper subset I of $\mathcal{F}$, π_I is a simplex over I which is well-known to be acyclic. This shows (CM6) and (G5).

Finally, we consider the candidate (i). First assume that $k \geq 3$. In this case we have $S_\Delta = C_\Delta \cap G_\Delta$ so T_Δ is normal and therefore Cohen-Macaulay, showing (CM1). Suppose it is also Gorenstein. That is, there exists some $x_0 \in G_\Delta$ such that $G_{\mathcal{F}} = x_0 - S_\Delta$. Since in S_Δ there is a unique point with the smallest sum of coordinates, therefore there is a unique point in $G_{\mathcal{F}}$ with the largest sum of coordinates. On the other hand, we have

$$G_{\mathcal{F}} = \{x \in \mathbb{Z}^n | -x \in C_\Delta \setminus (\cup_i F_i) \cup (\cup_{i,j} F_{i,j})\},$$

so this implies that there exists a unique lattice point in the interior of C_Δ with the smallest sum of coordinates. However, there is no point in S_Δ with the sum of coordinates one, and all point with the sum of coordinates two are of the form $e_{i_1,j_1} + e_{i_2,j_2}$ with $i_1 \neq i_2$ which lie on F_{i_1}.

Further, by checking all inequalities we can see that the point $e_{1,1}+e_{2,1}+e_{3,1}$ is in the interior of C_Δ so it must be the unique lattice point with the sum of coordinates equal to three. However, if $k \geq 4$ then also the point $e_{1,1} + e_{2,1} + e_{4,1}$ is there and if $b_1 \geq 2$ also the point $e_{1,2} + e_{2,1} + e_{3,1}$ is the interior lattice point in C_Δ. This implies $k = 3$ and $\mathbf{b} = (1,1,1)$. In this case, the point $(-1,-1,-1)$ has lattice distance one from all facets of C_Δ and therefore $G_{\mathcal{F}} = (-1,-1,-1) - S_\Delta$. Condition (ii) from Theorem 4.1.8 must be satisfied since we know that T_Δ is Cohen-Macaulay. This shows (G1).

Now we consider the case $\mathbf{a} = (1,1)$. We have $G_\Delta = \{x \in \mathbb{Z}^n; \sum x_{1,j} = \sum x_{2,j}\} \neq \mathbb{Z}^n$ and $S_\Delta = C_\Delta \cap G_\Delta$, so again T_Δ is normal, showing (CM4). If $b_1 = 1$, we have $S_\Delta = \{x \in \mathbb{Z}^n | x_{1,1} = \sum_j x_{2,j}\}$ and therefore $S_\Delta \cong \mathbb{N}^{b_2}$, which together with Lemma 4.2.8(ii) shows (S1).

Assume that $b_1 > 1$ and suppose that T_Δ is Gorenstein. As in the previous cases, this implies that there is a unique interior lattice point in C_Δ with the smallest sum of coordinates. The only candidate is the point $x = (1,1,\ldots,1)$. From $x \in C_\Delta$ we have $b_1 = \sum_j x_{1,j} = \sum x_{2,j} = b_2$. On the other hand, it is easy to check that in this case $G_{\mathcal{F}} = x - S_\Delta$. Moreover, condition (ii) from Theorem 4.1.8 holds since T_Δ is Cohen-Macaulay, showing (G3).

□

We conclude the chapter with the following two remarks.

Remark 4.2.10. *The cases (S1) and (S2) are precisely those cases where the tangential variety $\tau(X)$ is the whole projective space $\mathbb{P}^N$.*

Remark 4.2.11. *For the case when* Δ *is a graph, that is when the dimension of each of its simplices is at most one, then the tangential variety of* V_Δ *coincides with the secant variety of* V_Δ*. In our setting, this occurs in the following two cases:*

1. $k = 1$*,* $a = 2$ *(Veronese case), and*

2. $k = 2$*,* $\mathbf{a} = (1, 1)$ *(Segre Case).*

Therefore, in both of these cases, Cohen-Macaulay tangential varieties were classically known (as they are normal). For the case (2), the tangential variety coincides with the locus of $(b_1 + 1) \times (b_2 + 1)$ *matrices of rank at most two. That is, it is a determinantal variety. In this setting, (G3) of the above classification is classically known as the case of square matrices. It was, after partial attempts by Eagon [28] and Goto [37], proved by Svanes (see [102, Theorem 5.5.6]); see [12, Chapter 8] for more discussion. Also,* $(G3)$ *were proved in [71] (see also [49] for (CM4), (CM6), (G3) and (G5)). Moreover,* $(G5)$ *was also considered by Hibi and Ohsugi in [80, 82].*

Chapter 5

Complete quadrics

This chapter is based on my article "Complete quadrics: Schubert calculus for Gaussian models and semidefinite programming" [62] which is joint work with L. Manivel, M. Michałek, L. Monin, and T. Seynnaeve.

5.1 Motivation

Maximum likelihood degree and quadrics

Although this chapter is mainly about enumerative geometry and symmetric functions, the main motivations come from algebraic statistics and multivariate Gaussian models. These are generalizations of the well-known Gaussian distribution to higher dimensions. In the one-dimensional case, in order to determine a Gaussian distribution on $\mathbb{R}$, one needs to specify its mean $\mu \in \mathbb{R}$ and its variance $\sigma \in \mathbb{R}_{>0}$. In the n-dimensional case, the mean is a vector $\mu \in \mathbb{R}^n$, and the second parameter is a positive-definite $n \times n$ covariance matrix Σ. The corresponding Gaussian distribution on $\mathbb{R}^n$ is given by

$$f_{\mu,\Sigma}(x) := \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)},$$

where ^{T} denotes the transpose. Equivalently to determining it by μ and Σ, one may represent the distribution by μ and the *concentration matrix* $K := \Sigma^{-1}$, which is also positive definite. Our primary interest lies in *linear concentration models*, i.e. statistical models which assume that K belongs to a fixed d-dimensional space $\mathcal{L}$ of $n \times n$ symmetric matrices. These were introduced by Anderson half a century ago [2]. In particular, this means that Σ should belong to the set $\mathcal{L}^{-1}$ of inverses of matrices from $\mathcal{L}$.

In statistics, typically one gathers data as sample vectors $x_1, \dots, x_s \in \mathbb{R}^n$. This allows to estimate the mean μ as the mean of the x_i's. Furthermore, each x_i provides a matrix $\Sigma_i := (x_i - \mu)(x_i - \mu)^T$. Next, one considers the *sample covariance matrix* S, that is the mean of the Σ_i's. Note that in most situations, it is not true that $S \in \mathcal{L}^{-1}$. The aim is then to find Σ that best explains the observations. From the point of view of statistics, it is natural to maximize the likelihood function

$$f_{\mu,\Sigma}(x_1) \cdots f_{\mu,\Sigma}(x_s),$$

that is, to find a positive definite matrix $\Sigma \in \mathcal{L}^{-1}$ for which the above value is maximal. Classical theorems in statistics assert that the solution to this optimization problem is essentially geometric [8, Theorem 3.6, Theorem 5.5], [74, Theorem 4.4]. Namely, under mild genericity assumptions, the optimal Σ is the unique positive definite matrix in $\mathcal{L}^{-1}$ that maps to the same point as S under projection from $\mathcal{L}^{\perp}$.

This is one of the main reasons why the complex variety that is the Zariski closure of $\mathcal{L}^{-1}$ (which abusing notation we also denote by $\mathcal{L}^{-1}$) and the rational map π defined as the projection from $\mathcal{L}^{\perp}$ are intensively studied in algebraic statistics. Note that for generic $\mathcal{L}$, and after projectivization, π becomes a finite map. The following is the central definition of the chapter.

Definition 5.1.1 (ML-degree)**.** The *ML-degree* of a linear concentration model represented by a space $\mathcal{L}$ is the degree of the projection from the space $\mathcal{L}^{\perp}$ restricted to the variety $\mathcal{L}^{-1}$.

The ML-degree is the basic measure of the complexity of the model. When $\mathcal{L}$ is a generic space, the ML-degree only depends on the size n of the symmetric matrices and on the (affine) dimension d of $\mathcal{L}$. By a theorem of Teissier [104, 105] (cf. [70, Corollary 2.6]) or Sturmfels and Uhler [98, Theorem 1], the ML-degree equals the degree of the variety $\mathcal{L}^{-1}$. Following Sturmfels and Uhler [98] we denote it by $\phi(n,d)$. We refer algebraists interested in statistics to [27] for more information about the subject.

Definition 5.1.2. For $n \in \mathbb{Z}_{>0}$ and $1 \leq d \leq \binom{n+1}{2}$, we define $\phi(n,d)$ to be the degree of the variety $\mathcal{L}^{-1}$, where $\mathcal{L}$ is a general d-dimensional linear subspace of $S^2\mathbb{C}^n$.

Thus, our main result concerns a very basic algebro-geometric object: the degree of the variety obtained by inverting all symmetric matrices in a general linear space. In Section 5.4, we confirm the following conjecture of Sturmfels and Uhler [98, p. 611], [70, Conjecture 2.8]. This theorem is actually a corollary of a similar polynomiality result for the *algebraic degree of semidefinite programming*, which will be discussed in the next section.

Theorem 5.1.3. *For any fixed positive integer d, the ML-degree $\phi(n,d)$ is polynomial in n.*

Astonishingly, it appears that the numbers $\phi(n,d)$ were studied for the last 150 years! In 1879 Schubert presented his fundamental results on quadrics satisfying various tangency conditions [89]. His contributions shaped the field of enumerative geometry, inspiring many mathematicians for centuries to come. A nondegenerate quadric being given, the set of its tangent hyperplanes (its projective dual, in modern language) is nothing else than the inverse quadric. This implies that $\phi(n,d)$ is also the solution to the following enumerative problem:

> *What is the number of nondegenerate quadrics in n variables, passing through $\binom{n+1}{2} - d$ general points and tangent to $d-1$ general hyperplanes?*

In modern language, such problems can be solved by performing computations in the cohomology ring of the *variety of complete quadrics.* This is now a classical topic with many beautiful results [90, 92, 109, 110, 21, 22, 53, 20, 107, 63]. In particular, the cohomology ring has been described by generators and relations, and algorithms have been devised that allow computing any given intersection number. But this only applies for n fixed. Algebraic statistics suggested changing the perspective and fixing d instead of n. This explains, in a way, why the polynomiality property of $\phi(n,d)$ is only proved now.

Semidefinite programming and projective duality

The second domain of mathematics that inspired our research is semidefinite programming (SDP), a very important and effective subject in optimization theory. The goal is to study linear optimization problems over spectrahedra. This subject is a direct generalization of linear programming, that is optimization of linear functions over polyhedra. For a short introduction to the topic, we refer to [73, Chapter 12].

The coordinates of the optimal solution for an SDP problem, defined over rational numbers, are algebraic numbers. Their algebraic degree is governed by *the algebraic degree of semidefinite programming*. For more information, we refer to the fundamental article [77]. To stress the importance of this degree let us just quote this paper:

> *"The algebraic degree of semidefinite programming addresses the computational complexity at a fundamental level. To solve the semidefinite programming exactly essentially reduces to solve a class of univariate polynomial equations whose degrees are the algebraic degree."*

Let us provide a precise definition of the algebraic degree of SDP, in the language of algebraic geometry, without referring to optimization. (However, the fact that this definition is correct is actually a nontrivial result [77, Theorem 13].)

Definition 5.1.4. For $0 < m < \binom{n+1}{2}$ and $0 < r < n$, let $\mathcal{L} \subset S^2\mathbb{C}^n$ be a general linear space of symmetric matrices, of (affine) dimension $m+1$, and let $SD_m^{r,n} \subset \mathbb{P}(\mathcal{L})$ denote the projectivization of the cone of matrices of rank at most r in $\mathcal{L}$. The *algebraic degree of semidefinite programming* $\delta(m,n,r)$ is the degree of the projective dual $(SD_m^{r,n})^*$ of $\mathrm{S}D_m^{r,n}$ if this dual is a hypersurface, and zero otherwise.

Projective duality is a very classical topic, to which a huge literature has been devoted. Computing the degree of a dual variety is well-known to be very hard, especially when the variety in question is singular, which is often the case for our $SD_m^{r,n}$. Nevertheless, Ranestad and Graf von Bothmer [38] suggested to use conormal varieties, and managed to obtain an algebraic expression for $\delta(m,n,r)$ in terms of what we call the *Lascoux coefficients*. These are integer coefficients that govern the Segre classes of the symmetric square of a given vector bundle; algebraically, they are defined by the formal identity

$$\prod_{1\leq i\leq j\leq s} \frac{1}{1-(x_i+x_j)} = \sum_I \psi_I s_{\lambda(I)}(x_1,\ldots,x_s),$$

where the sum is over the increasing sets $I = (i_1 < i_2 < \cdots < i_s)$ of nonnegative integers, $\lambda(I) = (i_s - s + 1, \ldots, i_2 - 1, i_1)$ is the associated partition, and $s_{\lambda(I)}(x_1,\ldots,x_s)$ the corresponding Schur function in the variables $x_1,\ldots,x_s$. These coefficients were introduced and studied in [53, 85], whose influence on our work cannot be underestimated. Graf von Bothmer and Ranestad found a formula for $\delta(m,n,r)$ in terms of the

Lascoux coefficients (see 5.3.7). Diving into the combinatorics of those coefficients, in Section 5.4 we prove the following polynomiality result:

Theorem 5.1.5. *For any fixed $m, s > 0$, the function $\delta(m, n, n-s)$ is a polynomial in n.*

Moreover, in Section 5.5 we confirm [77, Conjecture 21], providing another explicit formula for $\delta(m, n, r)$, and another proof of the above theorem.

Theorem 5.1.6. *(NRS, Conjecture 21) Let m, n, s be positive integers. Then*

$$\delta(m, n, n-s) = \sum_{\sum I \leq m-s} (-1)^{m-s-\sum I} \psi_I b_I(n) \binom{m-1}{m-s-\sum I}$$

where the sum goes trough all sets of nonnegative integers of cardinality s.

In this formula, $\Sigma I = i_1 + \cdots + i_s$, and $b_I(n)$ is a polynomial function of n defined in Section 5.5. Actually, $b_I(n)$ is obtained by evaluating a Q-Schur polynomial on n identical variables; by the work of Stembridge [94], it counts certain shifted tableaux of shape determined by I, numbered by integers not greater than n. This also proves the polynomiality of the ML-degree, since elementary relations in the cohomology ring of the variety of complete quadrics imply the fundamental identity (see 5.3.6):

$$\phi(n, d) = \frac{1}{n} \sum_{\binom{s+1}{2} \leq d} s\delta(d, n, n-s).$$

Our approach also applies to linear spaces of general square matrices or of skew-symmetric matrices and allows us to obtain closed formulas for the dual degrees of their determinantal loci.

5.2 Notation and preliminaries

5.2.1 Partitions, Schur polynomials

Notation 5.2.1. *For a set of nonnegative integers $I = \{i_1, \dots, i_r\}$, we assume $i_1 < \cdots < i_{r-1} < i_r$ and we define the corresponding partition*

$$\lambda(I) := (i_r - (r-1), i_{r-1} - (r-2), \dots, i_2 - 1, i_1).$$

Analogously, for a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, which means $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ (zeroes are allowed), we define the corresponding set

$$I(\lambda) := \{\lambda_r, \lambda_{r-1} + 1, \dots, \lambda_2 + r - 2, \lambda_1 + r - 1\}.$$

The length of a partition λ, (i.e. the number of nonzero entries) will be denoted by $\mathrm{length}(\lambda)$. *By $|\lambda|$ we denote the size of the partition $\sum_{i=1}^{r} \lambda_r$. By $\tilde{\lambda}$ we denote the partition conjugate to λ, e.g. $\widetilde{(3,1)} = (2,1,1)$.*

We will abbreviate $\{0, \dots, n-1\}$ to $[n]$.

Let $\sum I := i_1 + \cdots + i_r$ denote the sum of elements of I and $\#I = r$ its cardinality. For two sets $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_r\}$ we say that $I \leq J$ if $i_k \leq j_k$ for all $1 \leq k \leq r$.

Notation 5.2.2. *For a partition λ we denote by s_λ the corresponding Schur polynomial.*

Definition 5.2.3. Let I, J be two sets of nonnegative integers of cardinality r. We define the numbers $s_{I,J}$ to be the unique integers which satisfy the polynomial equation

$$s_{\lambda(I)}(x_1+1,\dots,x_r+1) = \sum_{J\leq I} s_{I,J}s_{\lambda(J)}(x_1,\dots,x_r).$$

Note that since $s_{\lambda(I)}$ is a homogeneous polynomial, we also have the identity

$$s_{\lambda(I)}(x_1-1,\dots,x_r-1) = \sum_{J\leq I} (-1)^{\sum I-\sum J} s_{I,J}s_{\lambda(J)}(x_1,\dots,x_r).$$

As a consequence, the triangular matrices $(s_{I,J})_{I,J}$ and $((-1)^{\sum I-\sum J}s_{I,J})_{I,J}$ are inverses of each other.

Lemma 5.2.4. *Let $I = \{i_1,\dots,i_r\}$ and $J = \{j_1,\dots,j_r\}$ be two sets of nonnegative integers. Let $M_{I,J} = (m_{kl})$ be the $r\times r$ matrix with $m_{kl} = \binom{i_k}{j_l}$. Then*

a) $s_{I,J} = \det(M_{I,J})$

b) $s_{(d)}(x_1+1,\dots,x_r+1) = \sum_{i=0}^{d}\binom{d+r-1}{d-i}s_{(i)}(x_1,\dots,x_r)$

Proof. Part a) is proved in [60, Section I.3, example 10]. In particular, it implies

$$s_{[r+d],[r+i]} = \binom{d+r-1}{r+i-1} = \binom{d+r-1}{d-i}.$$

From this, the equation in part b) becomes the defining equation for $s_{I,J}$. □

5.2.2 Lascoux coefficients

Definition 5.2.5. We define the *Lascoux coefficients* ψ_I by the following formula:

$$s_{(d)}(\{x_i+x_j \mid 1\leq i\leq j\leq r\}) = \sum_{\substack{\lambda(I)\,\vdash\, d\\ \#I=r}} \psi_I s_{\lambda(I)}(x_1,\dots,x_r),$$

Here $s_{(d)}$ is the complete symmetric polynomial of degree d, in the $\binom{r+1}{2}$ variables x_i+x_j. Hence, the coefficients ψ_I appear in the expansion in the Schur basis, of the complete symmetric polynomial evaluated at sums of variables.

Equivalently, the Lascoux coefficients appear in the expansion of the d-th Segre class of the second symmetric power of any rank r vector bundle in terms of its Schur classes. In particular, for the universal bundle $\mathcal{U}$ over the Grassmannian $G(r,n)$ for $n\geq r+d$,

$$Seg_d(S^2\mathcal{U}) = \sum_{\substack{\lambda(I)\,\vdash\, d\\ \#I=r}} \psi_I\sigma_{\lambda(I)},$$

where σ_λ denote the Schubert classes in the Chow ring of the Grassmannian. (For $r\leq n<r+d$ the identity is still true, but some of the Schubert classes $\sigma_{\lambda(I)}$ will be zero.)

Example 5.2.6. *Let us consider $r = 2$ and $n = 4$, i.e. the Grassmannian $G(2, 4)$. The rank two universal vector bundle $\mathcal{U}$ has two Chern roots x_1, x_2. Recall that the cohomology ring of $G(2, 4)$ is six-dimensional with basis corresponding to Young diagrams contained in the 2×2 square. We have formal equalities:*

$$x_1 + x_2 = -\ydiagram{1}, \quad x_1 \cdot x_2 = \ydiagram{1,1}.$$

The Chern roots of $S^2\mathcal{U}$ are $2x_1, x_1 + x_2, 2x_2$. Computing the elementary symmetric polynomials in those we obtain the three respective Chern classes:

$$-3\ydiagram{1}, \quad 2\ydiagram{2} + 6\ydiagram{1,1}, \quad -4\ydiagram{2,1}.$$

By inverting the Chern polynomial we obtain the Segre classes:

$$3\ydiagram{1}, \quad \mathbf{7}\ydiagram{2} + 3\ydiagram{1,1}, \quad 10\ydiagram{2,1}, \quad 10\ydiagram{2,2}.$$

Their coefficients are the Lascoux coefficients, namely:

$$\psi_{0,2} = 3, \quad \psi_{0,3} = \mathbf{7}, \quad \psi_{1,2} = 3, \quad \psi_{1,3} = 10, \quad \psi_{2,3} = 10.$$

We use boldface and emphasis above and below to indicate the same numbers. We may also compute them by expanding complete symmetric polynomials, where now x_1, x_2 are simply formal variables.

$$s_{(2)}(2x_1, x_1 + x_2, 2x_2) = 7x_1^2 + 7x_2^2 + 10x_1x_2 =$$

$$= 7(x_1^2 + x_1x_2 + x_2^2) + 3x_1x_2 = \mathbf{7}s_{(2,0)}(x_1, x_2) + 3s_{(1,1)}(x_1, x_2).$$

We note that Lascoux coefficients appear in many publications with different notations. In particular one needs to be careful with the shift: $\psi_{\{j_1,\dots,j_r\}}$ as defined above equals $\psi_{\{j_1+1,\dots,j_r+1\}}$ in [38]. On the other hand our notation is consistent with [54, 77].

The lemma below gives a closed formula for the Lascoux coefficients, in terms of Pfaffians.

Lemma 5.2.7 ([85, Prop. 7.12])**.** *Let $I = \{i_1, \dots, i_r\}$ be a set of nonnegative integers. If $r = 1, 2$ then ψ_I is given by $\psi_{\{i\}} = 2^i$ and $\psi_{\{i,j\}} = \sum_{k=i+1}^{j} \binom{i+j}{k}$ respectively. For $r > 2$, ψ_I can be computed as*

$$\psi_I = \mathrm{Pf}(\psi_{\{i_k,i_l\}})_{0<k<l\leq n} \text{ for even } r,$$

$$\psi_I = \mathrm{Pf}(\psi_{\{i_k,i_l\}})_{0\leq k<l\leq n} \text{ for odd } r,$$

where $\psi_{\{i_0,i_k\}} := \psi_{\{i_k\}}$.

5.2.3 SDP-degree and ML-degree

Recall the Definitions 5.1.2 and 5.1.4 of the ML-degree $\phi(n,d)$ and the SDP-degree $\delta(m,n,r)$.

Remark 5.2.8. *For our polynomiality results in Section 5.4, it will be useful to extend the definitions of ϕ and δ as follows:*

- *For $d > \binom{n+1}{2}$, we put $\phi(n,d) = 0$.*
- *For $m = 0$ and $r < n$, we define $\delta(0,n,r) = 0$.*
- *For $m \geq \binom{n+1}{2}$ or $s \geq n$, we put $\delta(m,n,n-s) = 0$, with one exception: in the case $m = \binom{n+1}{2}$ and $s = n$, we define $\delta(\binom{n+1}{2}, n, 0) = 1$. See also Remark 5.3.9.*

Now $\phi(n,d)$ is defined for all $n,d > 0$, and $\delta(m,n,n-s)$ is defined for all $m,n,s > 0$.

For later reference, we recall the description of $\delta(m,n,r)$ in terms of the bidegrees of a conormal variety:

Theorem 5.2.9 ([77, Theorem 10])**.** *Let $Z_r \subseteq \mathbb{P}(S^2V)\times\mathbb{P}(S^2V^*)$ be the conormal variety to the variety $SD^r \subseteq \mathbb{P}(S^2V)$ of matrices of rank at most r. Explicitely, Z_r consisit of pairs of symmetric matrices (X,Y), up to scalars, with* $\operatorname{rank} X \leq r$, $\operatorname{rank} Y \leq n-r$, *and $X \cdot Y = 0$. Then the multidegree of Z_r is given by*

$$[Z_r] = \sum_{m=0}^{\binom{n+1}{2}} \delta(m,n,r) H_1^m H_{n-1}^{\binom{n+1}{2}-m}.$$

Here H_1 and H_{n-1} denote the pull-backs of the hyperplane classes from $\mathbb{P}(S^2V)$ and $\mathbb{P}(S^2V^*)$.

Remark 5.2.10. *From this description, we immediately get the following duality relation (see also [77, Proposition 9])*

$$\delta(m,n,n-s) = \delta(\binom{n+1}{2} - m, n, s).$$

5.3 ML-degrees via complete quadrics

5.3.1 Spaces of complete quadrics

Let V be a n-dimensional vector space over $\mathbb{C}$. The space of complete quadrics $CQ(V)$ is a particular compactification of the space of smooth quadrics in $\mathbb{P}(V)$, or equivalently, of the space of invertible symmetric matrices (up to scalar) $\mathbb{P}(S^2(V))^\circ \subset \mathbb{P}(S^2(V))$. The space of complete quadrics $CQ(V)$ has several equivalent descriptions, below we will describe some of them. For more information we refer the reader to [53, 107, 63].

For $A \in S^2(V)$ let $\bigwedge^k A \in S^2(\bigwedge^k V)$ be the corresponding form on $\bigwedge^k V$. If we view A as a symmetric matrix, then $\bigwedge^k A$ is the matrix of $k \times k$ minors of A. In particular, $\bigwedge^{n-1} A$ is the inverse of A up to scaling (by the determinant).

Definition 5.3.1. The space of complete quadrics $CQ(V)$ is the closure of $\phi(\mathbb{P}(S^2(V))^\circ)$, where

$$\phi : \mathbb{P}(S^2(V))^\circ \to \mathbb{P}(S^2(V)) \times \mathbb{P}\left(S^2\left(V \wedge V\right)\right) \times \ldots \times \mathbb{P}\left(S^2\left(\bigwedge^{n-1} V\right)\right)$$

is given by

$$[A] \mapsto \left([A], [\bigwedge^2 A], \ldots, [\bigwedge^{n-1} A]\right).$$

To simplify the notation we will also denote $CQ(V)$ by CQ_n.

The natural projection to the j-th factor induces a regular map

$$\pi_j : CQ(V) \to \mathbb{P}\left(S^2\left(\bigwedge^j V\right)\right).$$

In particular, the map $\pi_1 : CQ(V) \to \mathbb{P}(S^2(V))$, which is an isomorphism on $\phi(\mathbb{P}(S^2(V))^\circ)$, can be described as a sequence of blow-downs. This provides the second description of the space of complete quadrics.

Definition 5.3.2. The space of complete quadrics $CQ(V)$ is the successive blow-up of $\mathbb{P}(S^2(V))$:

$$CQ(V) = Bl_{\widetilde{D}^{n-1}} Bl_{\widetilde{D}^{n-2}} \ldots Bl_{D^1} \mathbb{P}(S^2(V)),$$

where $\widetilde{D^i}$ is the proper transform of the space of symmetric matrices of rank at most i under the previous blow-ups.

Theorem 5.3.3 ([110, Theorem 6.3])**.** *Definitions 5.3.1 and 5.3.2 of the space of complete quadrics are equivalent.*

The space of complete quadrics admits other descriptions, we would like to mention two of them. The first one describes the space of complete quadrics as an equivariant compactification of the space of invertible symmetric matrices, which is a spherical homogeneous space:

$$\mathbb{P}(S^2(V))^\circ \simeq \mathrm{SL}_n / N(\mathrm{SO}_n),$$

where $N(\mathrm{SO}_n)$ is the normalizer of SO_n. The second description realises the space of complete quadrics as a subvariety of the Kontsevich moduli space of stable maps to the Lagrangian Grassmannian, see for details [107, 70].

The space of complete quadrics has two natural series of special classes of divisors. The first series $S_1, \ldots, S_{n-1}$ is given by the classes of the (strict transforms) of the exceptional divisors $E_1, \ldots, E_{n-1}$ of the successive blow-ups in Definition 5.3.2 (which are precisely the SL_n-invariant prime divisors on $CQ(V)$). The second series $L_1, \ldots, L_{n-1}$ is obtained by pulling back the hyperplane classes by the projections $\pi_1, \ldots, \pi_{n-1}$.

Proposition 5.3.4. *The classes $L_1, \ldots, L_{n-1}$ form a basis of* $\mathrm{Pic}(CQ(V))$*, in which the classes $S_1, \ldots, S_{n-1}$ are given by the relations*

$$S_i = -L_{i-1} + 2L_i - L_{i+1},$$

with $L_0 = L_n := 0$.

Proof. These relations were already known to Schubert [88]. For a modern treatment, see for example [63, Proposition 3.6 and Theorem 3.13]. $\square$

The inverse relations are given by the $(n-1) \times (n-1)$ matrix M given by:

$$M_{i,j} = \min(i,j) - \frac{ij}{n},$$

in particular, we have:

$$\begin{aligned} nL_1 &= (n-1)S_1 + (n-2)S_2 + \ldots + S_{n-1}, \\ nL_{n-1} &= S_1 + 2S_2 + \ldots + (n-1)S_{n-1}. \end{aligned} \tag{5.1}$$

5.3.2 Intersection theory

We are ready to relate the computation of the *ML*-degree and of the algebraic degree of semidefinite programming to the intersection theory of $CQ(V)$.

Proposition 5.3.5. *The ML-degree and the algebraic degree of semidefinite programming can be computed as the following intersection numbers on $CQ(V)$:*

$$\phi(n,d) = \int_{CQ_n} L_1^{\binom{n+1}{2}-d} L_{n-1}^{d-1},$$

$$\delta(m,n,r) = \int_{E_r} L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1} = \int_{CQ_n} S_r L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1}.$$

Proof. Since the morphisms π_1 and π_{n-1} resolve the inversion map $\mathbb{P}(S^2V) \dashrightarrow \mathbb{P}(S^2V^*)$, we can compute the degree of $\mathcal{L}^{-1}$, for $\mathcal{L} \subseteq \mathbb{P}(S^2V^*)$ a general $d-1$-dimensional linear subspace, as $\pi_1^*(H_1^{\binom{n+1}{2}-d})\pi_{n-1}^*(H_{n-1}^{d-1})$, where H_1 and H_{n-1} are hyperplane classes in $\mathbb{P}(S^2V)$ and $\mathbb{P}(S^2V^*)$ respectively. Indeed the factor $H_1^{\binom{n+1}{2}-d}$ imposes $\binom{n+1}{2}-d$ linear conditions to matrices in $\mathbb{P}(S^2V)$, defining a linear subspace $\mathbb{P}(\mathcal{L})$ of projective dimension $d-1$, while the factor H_{n-1}^{d-1} imposes $d-1$ linear conditions on $\mathbb{P}(\mathcal{L}^{-1})$, hence computing the degree $\phi(n,d)$.

For $\delta(m,n,r)$, the main observation is that E_r is birationally isomorphic to the conormal variety Z_r. Indeed, the inductive properties of the spaces of complete quadrics imply that E_r can be described as a space of relative complete quadrics; more precisely

$$E_r = CQ_r(\mathcal{U}) \times_{G(r,n)} CQ_{n-r}(\mathcal{Q}^*).$$

The equality above may be derived from the quotient construction of the variety of complete quadrics, cf. [70], [107]. In particular E_r is birationally isomorphic to $Y_r = \mathbb{P}(S^2\mathcal{U}) \times_{G(r,n)} \mathbb{P}(S^2\mathcal{Q}^*)$. As observed for example in [38], this is also a smooth model for the conormal variety Z_r. As the divisors L_i are base point free, by Theorem 5.2.9

$$\delta(m,n,r) = \int_{Z_r} L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1} = \int_{Y_r} L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1}$$

can be computed on E_r, and this implies our claim. $\square$

Since S_r projects in $\mathbb{P}(S^2(V))$ to the locus of matrices of rank at most r, we must have $S_r L_1^{\binom{n+1}{2}-m-1} = 0$ when m is smaller that the codimension of this locus. Similarly $S_r L_{n-1}^{m-1} = 0$ when m is big enough. One can deduce that the following *Pataki inequalities* are necessary and sufficient conditions for $\delta(m,n,r)$ to be nonzero [77, Proposition 5 and Theorem 7]:

$$\binom{n-r+1}{2} \leq m \leq \binom{n+1}{2} - \binom{r+1}{2}. \tag{5.2}$$

We can then use (5.1) to write the ML-degree in terms of the SDP-degree:

Corollary 5.3.6. *For any $n, d > 0$, the following fundamental relation does hold:*

$$\phi(n,d) = \frac{1}{n} \sum_{1 \leq \binom{s+1}{2} \leq d} s\delta(d,n,n-s).$$

Proof. First, for $d > \binom{n+1}{2}$, we have both sides equal to 0, and for $d = \binom{n+1}{2}$ both sides equal 1 (see Remark 5.2.8), so the relation holds. For $1 \leq d < \binom{n+1}{2}$, we can write

$$\begin{aligned}
\phi(n,d) &= \int_{CQ_n} L_1^{\binom{n+1}{2}-d} L_{n-1}^{d-1} \\
&= \frac{1}{n} \int_{CQ_n} L_1^{\binom{n+1}{2}-d-1} L_{n-1}^{d-1} \sum_{s=1}^{n-1} sS_{n-s} \\
&= \frac{1}{n} \sum_{s=1}^{n-1} s\delta(d,n,n-s) = \frac{1}{n} \sum_{1 \leq \binom{s+1}{2} \leq d} s\delta(d,n,n-s).
\end{aligned} \tag{5.3}$$

The last equality follows from the Pataki inequalities, since $\delta(d,n,n-s) = 0$ whenever $\binom{s+1}{2} > d$. ☐

All our computations can now be reduced to the intersection theory of the Grassmannians.

Theorem 5.3.7 ([38, Theorem 1.1])**.** *For $0 < m < \binom{n+1}{2}$ and $0 < r < n$,*

$$\delta(m,n,r) = \sum_{\substack{I \subset [n] \\ \#I = n-r \\ \sum I = m-n+r}} \psi_I \psi_{[n] \setminus I}$$

Idea of proof. As already mentioned, $\delta(m,n,r)$ can be computed as an intersection number on Y_r, which is a fiber bundle over the Grassmannian $G(r,n)$. By push-forward, one obtains

$$\delta(m,n,r) = \int_{G(r,n)} \operatorname{Seg}_{\left(\binom{n+1}{2}-m-\binom{r+1}{2}\right)}(S^2\mathcal{U}) \operatorname{Seg}_{\left(m-\binom{n-r+1}{2}\right)}(S^2\mathcal{Q}^*).$$

We then obtain the theorem by expanding these Segre classes and using the fundamental duality properties of Schubert classes. ☐

Remark 5.3.8. *Recall that our definition of ψ_I is shifted w.r.t. [38], which explains why our formula looks slightly different.*

Remark 5.3.9. *One can easily verify that the above formula is still true for the extended definition of δ from 5.2.8. The only nontrivial case is $\delta\left(\binom{n+1}{2}, n, 0\right) = \psi_{[n]}\psi_{[n]\setminus[n]} = 1$.*

5.3.3 Representation theory

In this subsection, we will establish a formula that expresses the ML-degree as a linear combination of dimensions of irreducible representations of SL_n. Our construction is based on the following folklore lemma.

Lemma 5.3.10. *Let X be a smooth complete N-dimensional algebraic variety, and D_1, D_2 two divisors on X. Then the following identity holds:*

$$\int_X D_1^i D_2^{N-i} = \chi\left((1-\mathcal{O}(-D_1))^i(1-\mathcal{O}(-D_2))^{N-i}\right),$$

where χ denotes the holomorphic Euler characteristic.

Proof. By the additivity and multiplicativity properties of the Chern character we have

$$\mathrm{ch}(1-\mathcal{O}(-D_i)) = \sum_{k\geq 1}(-1)^{k+1}\frac{D_i^k}{k!},$$

and therefore

$$\mathrm{ch}\Big((1-\mathcal{O}(-D_1))^i(1-\mathcal{O}(-D_2))^{N-i}\Big) = \left(\sum_{k\geq 1}(-1)^{k+1}\frac{D_1^k}{k!}\right)^i\left(\sum_{k\geq 1}(-1)^{k+1}\frac{D_2^k}{k!}\right)^{N-i} = \int_X D_1^i D_2^{N-i}.$$

Finally, by the Riemann-Roch theorem, we get

$$\chi\left((1-\mathcal{O}(-D_1))^i(1-\mathcal{O}(-D_2))^{N-i}\right) = \int_X D_1^i D_2^{N-i}\,\mathrm{td}(X) = \int_X D_1^i D_2^{N-i}.$$

□

We are going to apply Lemma 5.3.10 to the computation of the ML-degree

$$\phi(n,d) = \int_{CQ_n} L_1^{N+1-d} L_{n-1}^{d-1},$$

where $N = \binom{n+1}{2} - 1$ denotes the dimension of the variety of complete quadrics CQ_n. We will denote by Λ the character lattice of SL_n, and for a fundamental $\lambda \in \Lambda$, we will denote by V_λ the corresponding irreducible representation of SL_n. We will also denote by $\alpha_1, \ldots, \alpha_{n-1}$ and $\omega_1, \ldots, \omega_{n-1}$ the simple roots and fundamental weights of SL_{n-1}, respectively.

For a character $\lambda \in \Lambda$, let us define a SL_n-representation W_λ by

$$W_\lambda := \bigoplus_{\nu\in\Lambda} V_{2\nu}^*,$$

where the sum is taken over dominant weights of the form $\nu = \lambda - \sum_{i=1}^{n-1} k_i\alpha_i$, with $k_i \in \mathbb{Z}_{\geq 0}$. In particular, $W_\lambda = 0$ if λ cannot be represented as a sum $\nu + \sum_{i=1}^{n-1} k_i\alpha_i$, with ν a dominant weight and $k_i \in \mathbb{Z}_{\geq 0}$. For $\lambda = (i-1)\omega_1 + (j-1)\omega_{n-1} - \sum_{l=1}^{n-1}\omega_l$, we will denote the representation W_λ by $W_{i,j}^n$.

Theorem 5.3.11. *With the notation as above, the following identity holds:*

$$\phi(n,d) = 1 + \sum_{\substack{0\leq i\leq N-d+1\\ 0\leq j\leq d-1\\ i+j>0}} (-1)^{i+j+N}\binom{N+1-d}{i}\binom{d-1}{j}\dim(W_{i,j}^n).$$

Proof. By Lemma 5.3.10 we have:

$$\phi(n,d) = \chi((1-\mathcal{L}_1^{-1})^{N+1-d}(1-\mathcal{L}_{n-1}^{-1})^{d-1}) = \sum_{\substack{0\le i\le N+1-d\\ 0\le j\le d-1}} (-1)^{i+j}\binom{N+1-d}{i}\binom{d-1}{j}\chi(\mathcal{L}_1^{-i}\otimes\mathcal{L}_{n-1}^{-j}).$$

Now, both $\mathcal{L}_1$ and $\mathcal{L}_{n-1}$ are globally generated, and since they are pull-backs of ample line bundles by birational morphisms their Iitaka dimensions $\kappa(\mathcal{L}_1) = \kappa(\mathcal{L}_{n-1}) = N$. By [7, Theorem 2.2], the cohomology of their negative powers must therefore vanish in degree lower than the dimension, so that for $i \geq 0, j \geq 0$ and $i + j > 0$

$$\chi(\mathcal{L}_1^{-i}\otimes\mathcal{L}_{n-1}^{-j}) = (-1)^N h^N(CQ_n, \mathcal{L}_1^{-i}\otimes\mathcal{L}_{n-1}^{-j}) = (-1)^N h^0(CQ_n, K_{CQ_n}\otimes\mathcal{L}_1^{i}\otimes\mathcal{L}_{n-1}^{j}).$$

The canonical divisor K_{CQ_n} of the space of complete quadrics is given by ([59, Corollary 3])

$$K_{CQ_n} = -L_1 - L_{n-1} - \sum_{l=1}^{n-1} L_l.$$

In particular, CQ_n is Fano [63] and $\chi(CQ_n, \mathcal{O}) = 1$. Finally, the spaces of global sections of line bundles on complete quadrics have been computed, as SL_n-representations, by De Concini and Procesi. It is direct corollary of [21, Theorem 8.3] that

$$H^0(CQ_n, \mathcal{L}_1^{a_1}\otimes\ldots\otimes\mathcal{L}_{n-1}^{a_{n-1}}) = W_{a_1\omega_1+\ldots+a_{n-1}\omega_{n-1}}.$$

In particular, $H^0(CQ_n, K_{CQ_n}\otimes\mathcal{L}_1^{i}\otimes\mathcal{L}_{n-1}^{j}) = W^n_{ij}$, so our result follows. □

Example 5.3.12. *Let us compute $\phi(3,1), \phi(3,2)$ and $\phi(3,3)$ with the help of Theorem 5.3.11. First we notice that:*

$$W^3_{i,0} = W^3_{0,i} = 0 \textit{ for } i \leq 5, \quad W^3_{i,1} = W^3_{1,i} = 0 \textit{ for } i \leq 3.$$

Therefore we get:

$$\phi(3,1) = 1 + \sum_{1\le i\le 5}(-1)^{i+1}\binom{5}{i}\dim\left(W^3_{i,0}\right) = 1;$$

$$\phi(3,2) = 1 + \sum_{\substack{0\le i\le 4\\ 0\le j\le 1\\ i+j>0}}(-1)^{i+j+1}\binom{4}{i}\dim\left(W^3_{i,j}\right) = 1 + \dim(W^3_{4,1}) = 1 + \dim V_0 = 2;$$

$$\phi(3,3) = 1 + \sum_{\substack{0\le i\le 3\\ 0\le j\le 2\\ i+j>0}}(-1)^{i+j+1}\binom{3}{i}\binom{2}{j}\dim\left(W^3_{i,j}\right) = 1 - 3\cdot\dim(W^3_{2,2}) + \dim(W^3_{3,2}),$$

where $W^3_{2,2} = V_0$ and $W^3_{3,2} = V^*_{2\omega_1}$. By Weyl's dimension formula:

$$\dim V_{i\omega_1+j\omega_2} = \frac{(i+1)(j+1)(i+j+2)}{2},$$

we get $\phi(3,3) = 1 - 3\cdot 1 + 6 = 4$.

Remark 5.3.13. *Our representation-theoretic approach gives a closed formula for the ML-degree. However, already for $n = 4$, the computation analogous to Example 5.3.12 is quite involved. One reason why this computation is more complicated than other formulas is that we obtain the answer to our intersection problem as a virtual representation, not only its dimension. One could say that this gives more information than we ask for.*

5.4 Polynomiality of the ML-degree

In this section and the next one, we present three proofs of the following polynomiality result for the algebraic degree of semidefinite programming:

Theorem 5.4.1. *For any fixed $m, s > 0$, the function $\delta(m, n, n - s)$ is a polynomial in n. Moreover this polynomial vanishes at $n = 0$.*

As an immediate corollary, we obtain one of the main results of this chapter: the polynomiality of the ML-degree for linear concentration models. This property was first conjectured by Sturmfels and Uhler [98] and confirmed in small, special cases in [17, 95, 70].

Theorem 5.4.2. *For any fixed $d > 0$, the function $\phi(n, d)$ is a polynomial for $n > 0$.*

Proof. For all $n, d > 0$, by Corollary 5.3.6, we have:

$$\phi(n, d) = \frac{1}{n} \sum_{1 \leq \binom{s+1}{2} \leq d} s\delta(d, n, n - s). \tag{5.4}$$

By Theorem 5.4.1 every term in the right hand side of (5.4) is a polynomial divisible by n, hence the theorem follows. □

Each of our proofs of Theorem 5.4.1 has its advantages. The first one is quite elementary, being based on algebraic recursive formulas, which also have a geometric meaning. It provides very efficient methods for explicit computations. The second proof is more technical, however, it allows us to derive the leading coefficients of the polynomials we study. The last one is simply a corollary of the conjecture of Nie, Ranestad, and Sturmfels that we prove in Section 5.5.

Our first two proofs of Theorem 5.4.1 are based on the following theorem.

Theorem 5.4.3. *Let $I = \{i_1, \ldots, i_r\}$ be a set of strictly increasing nonnegative integers. For $n \geq 0$ the function:*

$$LP_I(n) := \begin{cases} \psi_{[n]\setminus I} & \text{if } I \subseteq [n], \\ 0 & \text{otherwise.} \end{cases}$$

is a polynomial.

Before we prove Theorem 5.4.3 let us note that it immediately implies Theorem 5.4.1. Indeed, by Theorem 5.3.7, we have

$$\delta(m, n, n - s) = \sum_{\substack{I \subset [n] \\ \#I = s \\ \sum I = m - s}} \psi_I \psi_{[n]\setminus I} = \sum_{\substack{\#I = s \\ \sum I = m - s}} \psi_I LP_I(n)$$

By Theorem 5.4.3, each of the summands is a polynomial in n that vanishes for $n = 0$. Thus $\delta(m, n, n-s)$ is also a polynomial in n, which proves Theorem 5.4.1, and hence Theorem 5.4.2.

In the remainder of this section, we will present two proofs of Theorem 5.4.3. But let us first give a few examples.

Example 5.4.4. *By induction, one can check the following formulas for LP_I, when I has cardinality one or two:*

$$LP_{(i)}(n) = \binom{n}{j+1}, \qquad LP_{(0,j)}(n) = j\binom{n+1}{j+2},$$

and more generally, for $i < j$,

$$LP_{(i,j)}(n) = \frac{(j-i)[n+1]_{j+2}}{(i+1)!(j+1)!(i+j+2)!}\sum_{d=0}^{i}(-1)^d a_{i,d}(i+j+1-d)![n]_{i-d},$$

where $a_{i,d} = \prod_{k=0}^{d-1}(i-k)(i-k+1)$ and $[n]_d = n(n-1)\cdots(n-d+1)$.

5.4.1 First proof

The following recursive relations are central for our first proof.

Lemma 5.4.5. *1. For $j_1 > 0$ we have:*

$$\psi_{\{j_1,\ldots,j_r\}} = (r+1)\psi_{\{0,j_1,\ldots,j_r\}} - 2\sum_{\ell=1}^{r}\psi_{\{0,j_1,\ldots,j_\ell-1,\ldots,j_r\}}, \tag{5.5}$$

where the summation is over all ℓ for which $j_\ell - 1 > j_{\ell-1}$ and we set $j_0 := 0$.

2. For $j_1 = 0$ we have:

$$\psi_{\{j_1,j_2,\ldots,j_r\}} = \sum_{j_\ell \le j'_\ell < j_{\ell+1}} \psi_{\{j'_1,\ldots,j'_{r-1}\}}. \tag{5.6}$$

Proof. The first formula is: [85, p. 446], [54, (A.15.7)] and [86, p. 163-166].

To prove the second formula, recall that $s_{(d)}$ is the complete homogeneous symmetric polynomial of degree d, and that we have:

$$s_{(d)}(\{x_i + x_j \mid 1 \le i \le j \le r\}) = \sum_{\substack{\lambda(I)\,\vdash\, d\\ \#I\,=\,r}} \psi_I s_{\lambda(I)}(x_1,\ldots,x_r).$$

Substituting $x_r = 0$ we obtain:

$$\sum_{i=0}^{d} s_{(i)}(\{x_i + x_j \mid 1 \le i \le j \le r-1\}) s_{(d-i)}(x_1,\ldots,x_{r-1}) =$$

$$s_{(d)}(\{x_i + x_j \mid 1 \le i \le j \le r-1\}, x_1,\ldots,x_{r-1}) = \sum_{\substack{\lambda(I)\,\vdash\, d\\ \mathrm{length}(\lambda(I))\,\le\, r-1}} \psi_I s_{\lambda(I)}(x_1,\ldots,x_{r-1}).$$

We note that length$(\lambda(I)) \leq r-1$ if and only if $0 \in I$. On the other hand we may apply Pieri's rule to

$$\sum_{i=0}^{d} s_{(i)}(\{x_i + x_j \mid 1 \leq i \leq j \leq r-1\}) s_{(d-i)}(x_1, \dots, x_{r-1}) =$$

$$\sum_{i=0}^{d} \left(\sum_{\substack{\lambda(I) \vdash i \\ \#I = r-1}} \psi_I s_{\lambda(I)}(x_1, \dots, x_{r-1}) \right) s_{(d-i)}(x_1, \dots, x_{r-1}).$$

Comparing the coefficients of Schur polynomials in both expressions gives the formula. $\square$

First proof of Theorem 5.4.3. We proceed by induction first on $\#I$, then on $\sum I := \sum_{i_j \in I} i_j$. The base case is $I = \emptyset$, when $\psi_{\{0,\dots,n-1\}} = 1$.

For the induction step, fix I, and assume the theorem has been proven for all I' with $\#I' < \#I$, and for all I' with $\#I' = \#I$ and $\sum I' < \sum I$. We consider two cases:

Case 1. $i_1 = 0$. We claim that for every $n \geq 0$,

$$LP_I(n) = (n-r+1) LP_{I \setminus \{0\}}(n) - 2 \sum_{\ell : i_{\ell+1} > i_\ell + 1} LP_{I \setminus \{0, i_\ell\} \sqcup \{i_\ell + 1\}}(n),$$

where for summation we formally assume $i_{r+1} = +\infty$. Indeed: if $n \leq i_r$ then both sides are 0, and if $n > i_r$ then the equation is precisely Lemma 5.4.5 (1).

Case 2. $i_1 > 0$. We claim that for every $n \geq 0$,

$$LP_I(n) - LP_I(n-1) = \sum_J LP_J(n-1),$$

where the sum is over all $J \neq I$ of the form $\{i_1 - \epsilon_1, \dots, i_r - \epsilon_r\}$ with $\epsilon_\ell \in \{0, 1\}$. Again, if $n \leq i_r$ then both sides are 0, and if $n > i_r$ then the equation is precisely Lemma 5.4.5 (2).

In both cases, it follows that LP_I is a polynomial. $\square$

5.4.2 Second proof

Our second proof is based on an explicit interpretation of ψ_I as a sum of minors in the Pascal triangle. We denote by E the Pascal triangle matrix, i.e. $E_{ij} = \binom{i}{j}$. We will always consider only finite submatrices of E so despite the fact that it is an infinite matrix there will be no computations with infinite matrices.

Notation 5.4.6. *For an $n \times n$ matrix A and sets $I, J \subset [n]$ we denote by $A_{I,J}$ the $\#I \times \#J$ matrix which is obtained from A by taking rows indexed by I and columns indexed by J. Here we index rows and columns from 0. In the case $I = J$ we write simply $A_{I,I} = A_I$.*

For sets $K, C \subset [n]$ with $\#K = \#C$ we denote $V(K, C)$ the Vandermonde matrix with entries $V(K, C)_{ij} = k_{i+1}^{c_{j+1}}$. We also set $V(K) := V(K, [\#K])$, i.e. $V(K)_{ij} = k_{i+1}^{j}$.

For two sets $A, B \subset [n]$ we denote by $\varepsilon^{A,B}$ the sign of the permutation of $A \cup B$ determined by A, B if they are disjoint. If they are not, we define $\varepsilon^{A,B} = 0$.

We begin with a characterization of ψ_I as a sum of the minors of the matrix E which follows from [54, Proposition 2.8].

Proposition 5.4.7. *The following equality holds:*

$$\psi_I = \sum_{J \leq I} \det(E_{I,J}).$$

In what follows we will need the following lemma that may be easily proved by induction.

Lemma 5.4.8. *Let a, b be nonnegative integers.*

a) If $a > b$ then $\sum_{i=0}^{a} (-1)^i \binom{a}{i} i^b = 0$.

b) If $a = b$ then $\sum_{i=0}^{a} (-1)^{a-i} \binom{a}{i} i^b = a!$.

To compute special minors of the matrix E we use the following lemma.

Lemma 5.4.9. *Let $I = \{i_1, \dots, i_r\} \subset [n]$ be a set of nonnegative integers. Then*

$$\det E_{[n]\setminus[r],[n]\setminus I} = \frac{\prod_{1 \leq j < k \leq n-r} (i_k - i_j)}{(r-1)!(r-2)! \dots 2!1!} = \frac{\det(V(I))}{(r-1)!(r-2)! \dots 2!1!}$$

Proof. We fix r and proceed by induction on n. The case $i_r < n-1$ is trivial. In the case $i_r = n-1$ we express the determinant via Laplace expansion on the n-th row, use the induction hypothesis and Lemma 5.4.8 to conclude. $\square$

Now we are able to present our second proof of Theorem 5.4.3.

Second proof of Theorem 5.4.3. Let $\#I = r$ and $m := i_r + 1$. First, assume $n \geq m$. We use the formula from Proposition 5.4.7. We express the determinants $E_{[n]\setminus I,J}$ using the Laplace expansion along the first $m-r$ rows, we choose the columns indexed by set L. For the rest we use the Lemma 5.4.9. To simplify notation we let $K := [n] \setminus J$.

$$\begin{aligned}
\psi_{[n]\setminus I} &= \sum_{J \leq [n]\setminus I} \det(E_{[n]\setminus I,J}) \\
&= \sum_{J \leq [n]\setminus I} \sum_{\substack{L \subseteq J \\ \#L = m-r}} \varepsilon^{L,J\setminus L} \det(E_{[m]\setminus I,L}) \det(E_{[n]\setminus[m],J\setminus L}) \\
&= \sum_{\substack{\#L=m-r \\ L \leq [m]\setminus I}} \det(E_{[m]\setminus I,L}) \sum_{\substack{\#K=r \\ K\cap L=\emptyset \\ K \subset [n]}} \varepsilon^{L,[n]\setminus(K\cup L)} \det(E_{[n]\setminus[m],[n]\setminus(K\cup L)}) \\
&= \sum_{\substack{\#L=m-r \\ L \leq [m]\setminus I}} \det(E_{[m]\setminus I,L}) \sum_{\substack{\#K=r \\ K \subset [n]}} \varepsilon^{L,K} \varepsilon^{L,[n]\setminus L} \frac{\det(V(L \cup K))}{(m-1)!(m-2)! \dots 2!1!} \\
&= \sum_{\substack{\#L=m-r \\ L \leq [m]\setminus I}} \frac{\varepsilon^{L,[n]\setminus L} \det(E_{[m]\setminus I,L})}{(m-1)!(m-2)! \dots 2!1!} \sum_{\substack{\#K=r \\ K \subset [n]}} \det(V^*(L \cup K))
\end{aligned}$$

where $V^*(L \cup K)$ is the matrix $V(L \cup K)$ where we first put the rows indexed by L. Note that we may drop the assumption $L \leq [m] \setminus I$, since otherwise $\det(E_{[m]\setminus I, L}) = 0$. Similarly, we can extend our sum and drop the condition $L \cap K = \emptyset$ since we add only zero terms. If we fix L and denote the elements of K by $k_1 < \cdots < k_r$, then $\det(V^*(L \cup K))$ is clearly a polynomial in $k_1, \ldots, k_r$. Then

$$\sum_{\substack{\#K=r\\K\subset[n]}} \det(V^*(L \cup K)) = \sum_{0 \leq k_1 < \cdots < k_r < n} \det(V^*(L \cup K))$$

is a polynomial in n for the fixed L. Moreover, the sum through L does not depend on n and therefore also $\psi_{[n]\setminus I}$ is a polynomial in n. Our computations are correct only for $n \geq m$. However, the last expression makes sense and is a polynomial for all $n \geq 0$. Clearly, it is equal 0 for $n < m$. This proves the theorem. □

With this approach, we can even compute the leading coefficient of LP_I. For this, we will need two technical lemmas. The proof of the first one is straightforward, e.g. by induction.

Lemma 5.4.10. *Let $a_1, \ldots, a_r$ be nonnegative integers. Then*

$$\sum_{0 \leq k_1 < \cdots < k_r < n} k_1^{a_1} k_2^{a_2} \ldots k_r^{a_r}$$

is a polynomial in n of degree $\sum_{i=1}^r a_i + r$, with leading coefficient equal to

$$\frac{1}{(a_1+1)(a_1+a_2+2)\ldots(a_1+\cdots+a_r+r)}.$$

Lemma 5.4.11. *The following identity of rational functions in r variables holds:*

$$\sum_{\sigma \in S_r} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)}+x_{\sigma(2)})\ldots(x_{\sigma(1)}+\cdots+x_{\sigma(r)})} = \frac{\prod_{i>j}(x_i - x_j)}{\prod_i x_i \prod_{i>j}(x_i+x_j)}$$

Proof. We proceed by induction on r. It is easy to check that for $r = 1, 2$ the statement holds. For $r > 2$, we split the sum depending on $\sigma(r)$ and apply the induction hypothesis to the partial sums:

$$\begin{aligned}
&\sum_{\sigma \in S_r} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)}+x_{\sigma(2)})\ldots(x_{\sigma(1)}+\cdots+x_{\sigma(r)})} = \\
&= \frac{1}{x_1+\cdots+x_r} \sum_{k=1}^r \sum_{\substack{\sigma\in S_r\\ \sigma(r)=k}} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)}+x_{\sigma(2)})\ldots(x_{\sigma(1)}+\cdots+x_{\sigma(r-1)})} \\
&= \frac{1}{x_1+\cdots+x_r} \sum_{k=1}^r (-1)^{r-k} \frac{\prod_{i>j;i,j\neq k}(x_i - x_j)}{\prod_{i\neq k} x_i \prod_{i>j;i,j,\neq k}(x_i+x_j)} \\
&= \frac{1}{(x_1+\cdots+x_r)\prod_i x_i \prod_{i>j}(x_i+x_j)} \sum_{k=1}^r (-1)^{r-k} x_k \prod_{i>j;i,j\neq k}(x_i - x_j) \prod_{i \neq k}(x_i + x_k) \\
&= \frac{1}{(x_1+\cdots+x_r)\prod_i x_i \prod_{i>j}(x_i+x_j)} Q(x_1, \ldots, x_r),
\end{aligned}$$

where Q is a homogeneous polynomial of degree $\binom{r}{2}+1$. Moreover, Q is skew-symmetric, that if we exchange values of x_i and x_j we just change the sign. Therefore

$$Q(x_1,\dots,x_r)=\prod_{i>j}(x_i-x_j)R(x_1,\dots,x_r)$$

for R a symmetric polynomial of degree one. This implies that R is a multiple of $x_1+\dots+x_r$. Finally, it is easy to check that the coefficient of $x_r^r x_{r-1}^{r-2} x_{r-2}^{r-3}\dots x_2$ in Q is 1. Therefore $R=x_1+\dots+x_r$ and the proof is complete. □

Theorem 5.4.12. *The polynomial LP_I has degree $\sum I+\#I$. Its leading coefficient is equal to*

$$\frac{\prod_{j>k}(i_j-i_k)}{(i_1+1)!\dots(i_r+1)!\prod_{j>k}(i_j+i_k+2)}$$

Proof. Let us denote by $DS(k)$ the sign diagonal matrix $k\times k$, i.e. the diagonal matrix which diagonal entries are $1,-1,1,\dots,(-1)^{k-1}$. We continue with the calculation from the second proof of Theorem 5.4.3. We use Laplace expansion of Vandermonde by first $m-r$ rows. We get

$$\begin{aligned}
&\sum_{\substack{\#L=m-r\\ L\leq[m]\setminus I}} \varepsilon^{L,[m]\setminus L}\det(E_{[m]\setminus I,L})\sum_{\substack{\#K=r\\ K\subset[n]}}\det(V^*(L\cup K)) =\\
&=\sum_{\substack{\#L=m-r\\ L\leq[m]\setminus I}} \varepsilon^{L,[m]\setminus L}\det(E_{[m]\setminus I,L})\sum_{\substack{\#C=m-r\\ C\subset[m]}}\varepsilon^{C,[m]\setminus C}\det(V(L,C))\sum_{\substack{\#K=r\\ K\subset[n]}}\det(V(K,[m]\setminus C))\\
&=\sum_{\substack{\#C=m-r\\ C\subset[m]}}\varepsilon^{C,[m]\setminus C}\sum_{\substack{\#L=m-r\\ L\leq[m]\setminus I}} \varepsilon^{L,[m]\setminus L}\det(E_{[m]\setminus I,L})\det(V(L,C))\sum_{\substack{\#K=r\\ K\subset[n]}}\det(V(K,[m]\setminus C))\\
&=\sum_{\substack{\#C=m-r\\ C\subset[m]}}\varepsilon^{C,[m]\setminus C}\det\Big(DS(m)E_{[m]\setminus I,[m]}V([m],C)\Big)\sum_{\substack{\#K=r\\ K\subset[n]}}\det(V(K,[m]\setminus C)).
\end{aligned}$$

Consider the matrix $A:=\Big(DS(m)E_{[m]\setminus I,[m]}V([m],C)\Big)$. Let $[m]\setminus I=\{b_1,\dots,b_{m-r}\}$, $C=\{c_1,\dots,c_{m-r}\}$, where, as always, we assume that the elements of these sets are ordered increasingly. Notice that $c_{m-r}<b_{m-r}$ implies that the last row of the matrix A is 0 by Lemma 5.4.8 and so is $\det(A)$. In general, if $c_i<b_i$, then $A_{[m-r]\setminus[i-1],[i]}=0$ and we also get $\det A=0$.

The necessary condition for $\det A\neq 0$ is $c_i\geq b_i$ for all $1\leq i\leq m-r$. Therefore, we will sum only through such sets C. In the border case when $C=[m]\setminus I$ we get that the matrix A is upper triangular and by Lemma 5.4.8 we have $\varepsilon^{C,[m]\setminus C}\det A=(b_1)!\dots(b_{m-r})!$.

The sum $\sum_{\substack{\#K=r\\ K\subset[n]}}\det(V(K,[m]\setminus C))$ is clearly a polynomial in n of degree at most $\sum([m]\setminus C)+r=\binom{m}{2}+r-\sum C$. Since we are summing only trough C with $\sum C\geq\sum([m]\setminus I)$ we immediately get that the degree of the polynomial P is at most $\sum I+r$. Moreover, the only summand which contributes to the term of degree $\sum I+r$ is the one with $C=[m]\setminus I$. We finish the proof of the theorem by computing this summand. In this case, we get the polynomial

$$\widetilde{LP}_I(n):=\sum_{\substack{\#K=r\\ K\subset[n]}}\det(V(K,[m]\setminus C))=\sum_{\sigma\in S_r}\sum_{0\leq k_1<\dots<k_r<n}(-1)^\sigma k_1^{i_{\sigma(1)}}\dots k_r^{i_{\sigma(r)}}.$$

By Lemma 5.4.10 the leading coefficient of $\widetilde{LP}_I$ is

$$\sum_{\sigma\in\mathbb{S}_r}(-1)^\sigma\frac{1}{(i_{\sigma(1)}+1)(i_{\sigma(1)}+i_{\sigma(2)}+2)\dots(i_{\sigma(1)}+\dots+i_{\sigma(r)}+r)}$$

Now we apply Lemma 5.4.11 for $x_j = i_j+1$ to conclude that the leading coefficient of $\widetilde{LP}_I$ is

$$\frac{\prod_{j>k}(i_j-i_k)}{\prod_j(i_j+1)\prod_{j>k}(i_j+i_k+2)}$$

which is obviously non-zero. This shows that the degree of the polynomial LP_I is $\sum I+r$ and its leading coeffient is

$$\frac{1}{(m-1)!(m-2)!\dots1!}\cdot(b_1!)\dots(b_{m-r})!\cdot\frac{\prod_{j>k}(i_j-i_k)}{\prod_j(i_j+1)\prod_{j>k}(i_j+i_k+2)}=$$
$$=\frac{\prod_{j>k}(i_j-i_k)}{(i_1)!\dots(i_r)!\prod_j(i_j+1)\prod_{j>k}(i_j+i_k+2)}.$$

□

Corollary 5.4.13. *The polynomial $\delta(m,n,n-s)$ from Theorem 5.4.1 has degree m, and the polynomial $\phi(n,d)$ from Theorem 5.4.2 has degree $d-1$.*

5.5 The Nie-Ranestad-Sturmfels conjecture

In this section, we present a proof of the formula for the degree of semidefinite programming which was conjectured by Nie, Ranestad, and Sturmfels [77]. The formula was known so far only for special values of the parameters. To state it we introduce the following coefficients.

Definition 5.5.1 (Coefficients b_I)**.** Let I be a set of s nonnegative integers. We define $b_I(n)$ by the following formula:

$$b_I(n)=Q_{I+\mathbf{1}_s}(\underbrace{1/2,\dots,1/2}_{n\text{ times}}),$$

where $I+\mathbf{1}_s$ is the set obtained from I by adding one to each of its elements. The function $Q_{I+\mathbf{1}_s}$ is the Schur Q-function [60, Section III.8] and its argument $1/2$ appears n times.

These coefficients may be computed recursively as described in [77, Section 6]. We note that in this reference the authors use the convention that I is a subset of the set $\{1,\dots,n\}$ while in this thesis $I\subset[n]=\{0,\dots,n-1\}$. This results in the difference in the notation for the coefficient b_I exchanging I and $I+\mathbf{1}_s$.

The main theorem of this section, confirming the Nie-Ranestad-Sturmfels conjecture, is the following.

Theorem 5.5.2. *([77], Conjecture 21) Let m,n,s be positive integers. Then*

$$\delta(m,n,n-s)=\sum_{\sum I\le m-s}(-1)^{m-s-\sum I}\psi_I b_I(n)\binom{m-1}{m-s-\sum I},$$

where the sum goes through all sets of nonnegative integers of cardinality s.

As we already mentioned, Theorem 5.4.1 is an immediate corollary of Theorem 5.5.2, since the coefficients $b_I(n)$ are known to be polynomials. Hence, as soon as we have proven Theorem 5.5.2, we have a third proof of Theorem 5.4.1.

Remark 5.5.3. *We note that if the Pataki inequality* (5.2) $m \geq \binom{s+1}{2}$ *is not satisfied, then both sides of the equality above are trivially zero.*

For the rest of the section, we fix the numbers m, n, s as in the statement of the theorem. Theorem 5.5.2 presents a relation between numbers $b_I(n)$ and ψ_I, our proof of which will be algebraic, with the coefficients $s_{I,J}$ from Definition 5.2.3 playing a prominent role. The following lemma describes the relations between $b_I(n)$ and $s_{I,J}$:

Lemma 5.5.4. *Let I be a set of s nonnegative integers. Then*

$$b_I(n) = \sum_{J \leq I} \left(\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} LP_J(n),$$

$$LP_I(n) = \sum_{J \leq I} \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} b_J(n).$$

These two identities are equivalent, by the discussion following Definition 5.2.3. We present two proofs: one based on simple algebra, and one on more sophisticated methods from algebraic geometry.

For the first proof, let us recall two statements from linear algebra which will allow us to prove Pfaffian formulas also for the set complements.

Lemma 5.5.5. *(Jacobi's Theorem.) Let A be an $n \times n$ matrix, and A^C its cofactor matrix. Then*

$$\det(A_{[n]\setminus I,[n]\setminus J}) = \det(A^C_{I,J}) \det(A)^{\#I-1}$$

for all sets $I, J \subset [n]$ with $\#I = \#J$.

Corollary 5.5.6. *The cofactor matrix A^C of an $n \times n$ skew-symmetric matrix A is given by*

$$A^C_{ij} = \operatorname{Pf}(A_{[n]\setminus\{i,j\}}) \operatorname{Pf}(A).$$

Lemma 5.5.7. *Let $I = \{i_1, \ldots, i_r\}$ be a set of nonnegative integers. Then*

$$\psi_{[n]\setminus I} = \operatorname{Pf}(\psi_{[n]\setminus\{i_k,i_l\}})_{0<k<l\leq r} \text{ for even } \#I,$$

$$\psi_{[n]\setminus I} = \operatorname{Pf}(\psi_{[n]\setminus\{i_k,i_l\}})_{0\leq k<l\leq r} \text{ for odd } \#I,$$

where $\psi_{[n]\setminus\{i_0,i_k\}} := \psi_{[n]\setminus\{i_k\}}$.

Proof. Let us consider the case where both n and $\#I$ are even. Consider the skew-symmetric matrix A such that $A_{k,l} = \psi_{\{k,l\}}$ for $0 \leq k < l < n$. Then using Lemmas 5.5.5 and 5.2.7 we get

$$\psi_{[n]\setminus I} = \operatorname{Pf}(A_{[n]\setminus I}) = \operatorname{Pf}(A^C_I) \operatorname{Pf}(A)^{\#I-1} = \operatorname{Pf}(A^C_I),$$

since $\det(A) = \psi_{\{0,1,\ldots,n-1\}} = 1$. Moreover, by Corollary 5.5.6, the entries of the cofactor matrix A^C are $\operatorname{Pf}(A_{[n]\setminus\{k,l\}}) \operatorname{Pf}(A) = \psi_{[n]\setminus\{k,l\}}$ which proves the lemma in this case.

The proof in the other cases is similar. The only difference is that we consider a different matrix A. If n is odd we take $A = (\psi_{\{k,l\}})_{-1\leq k<l<n}$ and if n is even and $\#I$ is odd we take $A = (\psi_{\{k,l\}})_{-2\leq k<l<n}$. We interpret $\psi_{\{-1,k\}}$ and $\psi_{\{-2,k\}}$ as $\psi_{\{k\}}$ and we put $\psi_{\{-1,-2\}} = 1$. Then we conclude in the same way. $\square$

Corollary 5.5.8.

$$\#I\psi_{[n]\setminus I} = \begin{cases} 2\sum_{1\le k<l\le r}(-1)^{k+l+1}\psi_{[n]\setminus\{i_k,i_l\}}\psi_{[n]\setminus(I\setminus\{i_k,i_l\})} & \textit{if } \#I \textit{ is even} \\ 2\sum_{0\le k<l\le r}(-1)^{k+l+1}\psi_{[n]\setminus\{i_k,i_l\}}\psi_{[n]\setminus(I\setminus\{i_k,i_l\})} & \textit{if } \#I \textit{ is odd.} \end{cases}$$

where $\psi_{[n]\setminus\{i_0,i_k\}} := \psi_{[n]\setminus\{i_k\}}$.

Proof. For every skew-symmetric $r\times r$ matrix A (with r even) and every $k = 1,\dots,r$, we have the following recursive formula for the Pfaffian:

$$\mathrm{Pf}(A) = \sum_{l=1}^{k-1}(-1)^{k+l}a_{k,l}\,\mathrm{Pf}(A_{\hat{k}\hat{l}}) - \sum_{l=k+1}^{r}(-1)^{k+l}a_{k,l}\,\mathrm{Pf}(A_{\hat{k}\hat{l}}),$$

where $A_{\hat{k}\hat{l}}$ is the submatrix obtained by removing the k-th and l-th rows and columns. Summing over all k gives the desired equality. □

Remark 5.5.9. *If we define* $\psi_{[n]\setminus I} = 0$ *for* $I = \{i_1,\dots,i_r\}$ *a multiset/partition with at least one repeated entry, the recursion from Corollary 5.5.8 still holds.*

Remark 5.5.10. *Corollary 5.5.8 can be seen as a recursive relation between the polynomials* $LP_I(n)$ *from Theorem 5.4.3. In particular, we can obtain in this way one more proof of Theorem 5.4.3.*

Proof of Lemma 5.5.4. We will use induction on the length of I, which we will denote by s. The base of induction, i.e. the cases $s = 1, 2$ are left for the reader.

We proceed with the general case $s > 2$. We will assume that s is even; the odd case is analogous. Since $b_I = \mathrm{Pf}(b_{i_p,i_q})_{1\le p<q\le s}$, we have (as in Corollary 5.5.8) the following recursive relations between the b_I's:

$$sb_I = 2\sum_{1\le p<q\le n}(-1)^{p+q+1}b_{\{i_p,i_q\}}b_{I\setminus\{i_p,i_q\}}.$$

In order to use induction, we need to show that

$$s\sum_{J\le I}2^{\sum J}s_{I,J}\psi_{[n]\setminus J} =$$

$$2\sum_{1\le p<q\le n}(-1)^{p+q+1}\left(\sum_{J\le\{i_p,i_q\}}2^{\sum J}s_{\{i_p,i_q\},J}\psi_{[n]\setminus J}\right)\left(\sum_{J\le I\setminus\{i_p,i_q\}}2^{\sum J}s_{I\setminus\{i_p,i_q\},J}\psi_{[n]\setminus J}\right).$$

This follows immediately from the following claim:

Claim 5.5.11. *For every* $J \le I$, *where* J *can have repeated elements,*

$$s_{I,J}\psi_{[n]\setminus J} =$$

$$\frac{2}{s}\sum_{1\le p<q\le n}(-1)^{p+q+1}\left(\sum_{1\le s<t<n}s_{\{i_p,i_q\},\{j_s,j_t\}}\psi_{[n]\setminus\{j_s,j_t\}}s_{I\setminus\{i_p,i_q\},J\setminus\{j_s,j_t\}}\psi_{[n]\setminus(J\setminus\{j_s,j_t\})}\right).$$

Indeed, using Laplace expansion, for any t, u we can write:

$$s_{I,J} = \sum_{p<q}(-1)^{p+q+t+u}s_{\{i_p,i_q\},\{j_t,j_u\}}s_{I\setminus\{i_p,i_q\},J\setminus\{j_t,j_u\}}.$$

Hence, the right-hand side can be rewritten as

$$2s_{I,J} \sum_{1 \le t < u < n} (-1)^{t+u+1} \psi_{[n]\setminus\{j_t,j_u\}} \psi_{[n]\setminus(J\setminus\{j_t,j_u\})}.$$

It remains to show that

$$s\psi_{[n]\setminus J} = 2 \sum_{1 \le t < u < n} (-1)^{t+u+1} \psi_{[n]\setminus\{j_t,j_u\}} \psi_{[n]\setminus(J\setminus\{j_t,j_u\})}.$$

But this is precisely Corollary 5.5.8, and this concludes the first proof of the formula. □

Lemma 5.5.12. *Let J be a set of non-negative integers of length s with $\sum J \le m - s$. Then*

$$\sum_{\substack{I \ge J \\ \sum I \le m-s}} \psi_I \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} \binom{m-1}{m-s-\sum I} = \begin{cases} 0 & \text{if } \sum J < m-s \\ \psi_J & \text{if } \sum J = m-s \end{cases}$$

Proof. We prove the lemma at the same time for all the J's by multiplying the above equation by the Schur polynomial $s_{\lambda(J)}(x_1, \ldots, x_s)$ and summing up. Since Schur polynomials form a basis of the space of symmetric polynomials, the statement of the lemma is equivalent to the following polynomial identity:

$$\sum_{\sum J \le m-s} \sum_{\substack{I \ge J \\ \sum I \le m-s}} \psi_I \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} \binom{m-1}{m-s-\sum I} s_{\lambda(J)}(x_1, \ldots, x_s) =$$

$$= \sum_{\sum J = m-s} \psi_J s_{\lambda(J)}(x_1, \ldots, x_s).$$

By 5.2.5, the right hand side is equal to $s_{(m-s-\binom{s}{2})}(x_i + x_j | 1 \le i \le j \le s)$. For the left hand side we can use Definition 5.2.3 of the coefficients $s_{I,J}$:

$$\sum_{\sum J\leq m-s}\ \sum_{\substack{I\geq J\\ \sum I\leq m-s}} \psi_I \left(-\frac{1}{2}\right)^{\sum I-\sum J} s_{I,J}\binom{m-1}{m-s-\sum I} s_{\lambda(J)}(x_1,\ldots,x_s) =$$

$$\sum_{\sum I\leq m-s} \psi_I \binom{m-1}{m-s-\sum I} \sum_{J\leq I} \left(-\frac{1}{2}\right)^{\sum I-\sum J} s_{I,J} s_{\lambda(J)}(x_1,\ldots,x_s) =$$

$$\sum_{\sum I\leq m-s} \psi_I \binom{m-1}{m-s-\sum I} s_{\lambda(I)}(x_1-1/2,\ldots,x_s-1/2) =$$

$$\sum_{i=\binom{s}{2}}^{m-s}\ \sum_{\sum I=i} \binom{m-1}{m-s-i} \psi_I s_{\lambda(I)}(x_1-1/2,\ldots,x_s-1/2) =$$

$$\sum_{i=\binom{s}{2}}^{m-s} \binom{m-1}{m-s-i} \sum_{\sum I=i} \psi_I s_{\lambda(I)}(x_1-1/2,\ldots,x_s-1/2) =$$

$$\sum_{i=\binom{s}{2}}^{m-s} \binom{m-1}{m-s-i} s_{(i-\binom{s}{2})}(x_i+x_j-1|1\leq i\leq j\leq s) =$$

$$s_{(m-s-\binom{s}{2})}(x_i+x_j|1\leq i\leq j\leq s).$$

In the last equality we applied Lemma 5.2.4 to the variables x_i+x_j-1. $\square$

Now we are able to present the proof of Theorem 5.5.2:

Proof of Theorem 5.5.2. We replace $b_I(n)$ by the expression from Lemma 5.5.4, change the order of summation and use Lemma 5.5.12 in the last step:

$$\sum_{\sum I\leq m-s} (-1)^{m-s-\sum I}\psi_I b_I(n)\binom{m-1}{m-s-\sum I} =$$

$$= \sum_{\sum I\leq m-s}\ \sum_{J\leq I} s_{I,J}\psi_{[n]\setminus J}\left(\frac{1}{2}\right)^{\sum I-\sum J} (-1)^{m-s-\sum I}\psi_I\binom{m-1}{m-s-\sum I}$$

$$= \sum_{\sum J\leq m-s} (-1)^{m-s-\sum J}\psi_{[n]\setminus J} \sum_{\substack{I\geq J\\ \sum I\leq m-s}} s_{I,J}\left(-\frac{1}{2}\right)^{\sum I-\sum J} \psi_I\binom{m-1}{m-s-\sum I}$$

$$= \sum_{\sum J=m-s} (-1)^{m-s-\sum J}\psi_{[n]\setminus J}\psi_J = \delta(m,n,n-s).$$

$\square$

5.6 General square matrices

The results from the previous sections have natural analogues if we replace the space of symmetric matrices ("type C") with the space of skew-symmetric matrices ("type D"), or with the space of general matrices ("type A"). This section will be devoted to the latter case and the next section to the former one.

5.6.1 Codegrees of smooth determinantal loci

Let M_n denote the space of complex matrices of size n, and $D^{n-r,n} \subset \mathbb{P}(M_n)$ the locus of matrices of rank at most $n-r$. Denote by $D_m^{n-r,n}$ its intersection with a general m-dimensional projective space. Its dimension is $d = m - r^2$ when this is non negative, otherwise it is empty. The analogues of Pataki's inequalities are given by:

Proposition 5.6.1. *The dual variety of $D_m^{n-r,n}$ is a hypersurface if and only if*

$$r^2 \leq m \leq n^2 - (n-r)^2.$$

As was done in [77] for symmetric matrices, the degree of this dual variety can be computed by classical means when $D_m^{n-r,n}$ is smooth, which is equivalent to $r^2 \leq m \leq r^2 + 2r$. The class formula gives, in terms of topological Euler characteristics,

$$\deg(D_m^{n-r,n})^* = (-1)^d\Big(\chi(D_m^{n-r,n}) - 2\chi(D_{m-1}^{n-r,n}) + \chi(D_{m-2}^{n-r,n})\Big).$$

Euler characteristics of smooth degeneracy loci have been computed by Pragacz [85]. For $\varphi : F \to E$ a morphism of vector bundles of ranks f, e over a variety X, the formula given in [35, page 57] is

$$\chi(D_r(\varphi)) = \int_X P_r(E,F)c(X),$$

where $c(X)$ denotes the total Chern class of X, while $P_r(E,F)$ is a universal polynomial in the Chern classes of E and F. Explicitely,

$$P_r(E,F) = \sum_{\lambda,\mu} (-1)^{|\lambda|+|\mu|} D_{\lambda,\mu}^{n-r,m-r} s_{(m-r)^{n-r}+\lambda,\tilde{\mu}}(E-F),$$

where the sum is over partitions λ and μ of length $n-r$ and $m-r$ respectively, and $\tilde{\mu}$ is the dual partition of μ. Moreover the coefficients denoted $D_{\lambda,\mu}^{n-r,m-r}$ in [35] encode the Segre classes of a tensor product of vector bundles. (We will rather use in the sequel the notations of [54], see Definition 5.6.7.)

We want to apply this formula to $D_m^{n-r,n}$, which we consider as the degeneracy locus $D_{n-r}(\varphi)$ of the tautological morphism $\varphi : F = \mathcal{O}(-1)^{\oplus n} \longrightarrow \mathcal{O}^{\oplus n}$ over $X = \mathbb{P}^m$. Since $c(\mathbb{P}^m) - 2hc(\mathbb{P}^{m-1}) + h^2 c(\mathbb{P}^{m-2}) = (1+h)^{m-1}$, where h denotes the hyperplane class, we get the formula

$$\deg(D_m^{n-r,n})^* = \sum_{\lambda,\mu} (-1)^{|\lambda|+|\mu|} \binom{m-1}{r^2+|\lambda|+|\mu|} D_{\lambda,\mu}^{r,r} s_{(r)^r+\lambda,\tilde{\mu}}(\underbrace{1,\ldots,1}_{n \text{ times}}),$$

the sum being taken over partitions λ and μ of length r. Note that the dependence on n for r and m fixed is only in the last term, more precisely in the number of one's on which the Schur functions are evaluated. This dependence is well known to be polynomial in n; very explicitely, for any partition ν,

$$s_\nu(\underbrace{1,\ldots,1}_{n \text{ times}}) = \dim S_\nu \mathbb{C}^n = c_\nu(n)/h(\nu),$$

where c_ν is the content polynomial and $h(\nu)$ is the product of the hook lengths of ν [60]. A priori this formula is only valid in the range $r^2 \leq m \leq r^2 + 2r$, when $D_m^{n-r,n}$ is smooth. That it should be true in general would be an analogue of the NRS conjecture in type A. We will prove below that this statement is correct.

We introduce the following notations, similar to those we used for symmetric matrices.

Definition 5.6.2. We define $\delta_A(m,n,r)$ to be the degree of the variety $(D_m^{r,n})^*$ if it is a hypersurface, and zero otherwise. Here $D_m^{r,n}$ is the variety of $n\times n$ matrices of rank at most r, intersected with a general (projective) m dimensional subspace. Equivalently, if we let $Z_r \subset \mathbb{P}(V^*\otimes V)\times\mathbb{P}(V^*\otimes V)$ be the variety of pairs of matrices (X,Y), up to scalars, with $X\cdot Y = Y\cdot X = 0$, $\operatorname{rank} X\le r$, $\operatorname{rank} Y \le n-r$, then the multidegree of Z_r is equal to

$$[Z_r] = \sum_m \delta_A(m,n,r) H_1^{n^2-m} H_{n-1}^m,$$

where H_1 and H_{n-1} denote the pull-backs of the hyperplane classes from $\mathbb{P}(V^*\otimes V)$ and $\mathbb{P}(V^*\otimes V)$, respectively.

Definition 5.6.3. The number $\phi_A(n,d)$ is the degree of the variety $\mathcal{L}^{-1}$, where $\mathcal{L}\subseteq \mathbb{P}(M_n)$ is a general linear subspace of dimension $d-1$.

5.6.2 Complete collineations

The correct space to work with is the *space of complete collineations* [91, 109, 111, 53, 107, 63]. It can actually be defined for rectangular matrices, but for sake of simplicity we will restrict ourselves to square matrices.

Definition 5.6.4. Let V and W be two vector spaces of equal dimension n. The space $\mathbb{P}(V^*\otimes W)$ represents linear maps from V to W; the open subset of rank n linear maps is denoted by $\mathbb{P}(V^*\otimes W)^\circ$. Then the space of complete collineations $CC(V,W)$ is defined as the closure of the image of the map

$$\phi : \mathbb{P}(V^*\otimes W)^\circ \to \mathbb{P}(V^*\otimes W)\times \mathbb{P}\left(\bigwedge^2 V^*\otimes \bigwedge^2 W\right)\times\ldots\times\mathbb{P}\left(\bigwedge^{n-1} V^*\otimes\bigwedge^{n-1} W\right),$$

given by

$$[A]\mapsto([A],[\wedge^2 A],\ldots,[\wedge^{n-1}A]).$$

As before, in coordinates, this map sends a matrix to its minors of various sizes.

As in the symmetric case, the space of complete collineations can be constructed by blowing-up $\mathbb{P}(M_n)$ along the subvariety of rank one matrices, then the strict transform of the subvariety of matrices of rank at most two, and so on. As such, it admits a first series $S_1,\ldots,S_{n-1}$ of special classes of divisors: the classes of (the strict transforms of) the exceptional divisors $E_1,\ldots,E_{n-1}$ of these successive blow-ups. A second natural series $L_1,\ldots,L_{n-1}$ of classes of divisors can be obtained by pulling back the hyperplane classes under the projections $\pi_i : CC(V,W)\to\mathbb{P}\left(\wedge^i V^*\otimes\wedge^i W\right)$.

The analogue of Proposition 5.3.4 holds:

Proposition 5.6.5. *$L_1,\ldots,L_{n-1}$ form a basis of* $\operatorname{Pic}(CC(V,W))$*, in which the S_i's are given by the formulas*

$$S_i = -L_{i-1}+2L_i-L_{i+1},$$

with the convention that $L_0 = L_n := 0$.

Proof. Follows from [63, Proposition 3.6, Theorem 3.13]. □

Proposition 5.6.6. *The numbers ϕ_A and δ_A can be computed as intersection products of the variety of complete collineations:*

$$\phi_A(n,d) = \int_{CC_n} L_1^{n^2-d} L_{n-1}^{d-1},$$

$$\delta_A(m,n,r) = \int_{CC_n} S_r L_1^{n^2-m-1} L_{n-1}^{m-1} = \int_{E_r} L_1^{n^2-m-1} L_{n-1}^{m-1}.$$

This implies the analogue of eq. (5.3):

$$\phi_A(n,d) = \frac{1}{n}\sum_{r=1}^{n-1} r\delta_A(d,n,n-r). \tag{5.7}$$

Definition 5.6.7. We define type A Lascoux coefficients $d_{I,J}$ as follows. For $X = (x_1,\ldots,x_k)$ and $Y = (y_1,\ldots,y_l)$ two sets of indeterminates, we denote by $X+Y$ the set of indeterminates $x_i + y_j$, $1 \le i \le k$, $1 \le j \le l$. Then the $d_{I,J}$'s are defined by the formal identity

$$s_{(d)}(X+Y) = \sum_{\substack{\#I=k,\#J=l\\ |\lambda(I)|+|\lambda(J)|=d}} d_{I,J} s_{\lambda(I)}(X) s_{\lambda(J)}(Y).$$

Equivalently, for the product of the universal bundles $\mathcal{U}_1 \otimes \mathcal{U}_2$ over a product of Grassmannians $G(k,m)\times G(l,n)$:

$$Seg_d(\mathcal{U}_1\otimes\mathcal{U}_2) = \sum_{\substack{\#I=k,\#J=l\\ |\lambda(I)|+|\lambda(J)|=d}} d_{I,J}\sigma^1_{\lambda(I)}\sigma^2_{\lambda(J)}$$

Analogously to Theorem 5.3.7, we have the following formula for δ_A:

Theorem 5.6.8.

$$\delta_A(m,n,r) = \sum_{\substack{I,J\subset[n]\\ \#I=\#J=n-r\\ \sum I+\sum J=m-n+r}} d_{I,J} d_{[n]\setminus I,[n]\setminus J}$$

5.6.3 Induction relations and polynomiality

We denote by $D(t)$ the infinite matrix with entries $D(t)_{ij} = \binom{t+i+j}{i}$. This matrix gives us a formula for $d_{I,J}$ [54, Proposition 2.8].

Proposition 5.6.9. *Let $I = \{i_1,\ldots,i_r\}$, $J = \{j_1,\ldots,j_s\}$ be two sets of nonnegative integers with $r \le s$. Then*

$$d_{I,J} = \begin{cases} \det D(s-r)_{I,\{j_{s-r+1}-(s-r),\ldots,j_s-(s-r)\}} & \text{if } j_i = i-1 \text{ for all } 1\le i\le s-r\\ 0 & \text{otherwise .}\end{cases}$$

In particular, if $\#I = \#J$ then $d_{I,J} = \det D(0)_{I,J}$.

Lemma 5.6.10. *1. Let $I = \{i_1, \ldots, i_s\}$, $J = \{j_1, \ldots, j_s\}$ with $i_1, j_1 > 1$. Write $I_0 = \{0\} \cup I$ and $J_0 = \{0\} \cup J$. Then*

$$d_{I,J} = (s+1)d_{I_0,J_0} - \sum_{p=1}^{s} d_{I_0\setminus\{i_p\}\cup\{i_p-1\},J_0} - \sum_{q=1}^{s} d_{I_0,J_0\setminus\{j_q\}\cup\{j_q-1\}}.$$

(Here, if $I_0 \setminus \{i_p\} \cup \{i_p - 1\}$ is a multiset, then $d_{I_0\setminus\{i_p\}\cup\{i_p-1\},J_0} = 0$.)

2. For $i_1 = 0$ or $j_1 = 0$ we have:

$$d_{\{i_1,\ldots,i_s\},\{j_1,j_2,\ldots,j_s\}} = \sum_{\substack{i_\ell \le i'_\ell < i_{\ell+1} \\ j_\ell \le j'_\ell < j_{\ell+1}}} d_{\{i'_1,\ldots,i'_{s-1}\},\{j'_1,\ldots,j'_{s-1}\}}.$$

Proof. 1. We expand the determinant $\det D(0)_{I_0,J_0}$ in each row, and sum up:

$$\begin{aligned}
(s+1)d_{I_0,J_0} = &\sum_{p,q=0}^{s} (-1)^{p+q}\binom{i_p+j_q}{i_p} d_{I_0\setminus\{i_p\},J_0\setminus\{j_q\}} \\
=&d_{I,J} + \sum_{p=1}^{s}(-1)^p d_{I_0\setminus\{i_p\},J} + \sum_{q=1}^{s}(-1)^q d_{I,J_0\setminus\{j_q\}} \\
&+ \sum_{p,q=1}^{s}(-1)^{p+q}\left(\binom{i_p+j_q-1}{i_p} + \binom{i_p+j_q-1}{i_p-1}\right) d_{I_0\setminus\{i_p\},J_0\setminus\{j_q\}} \\
=&d_{I,J} + \sum_{p=1}^{s}\sum_{q=0}^{s}(-1)^{p+q}\binom{i_p+j_q-1}{i_p-1} d_{I_0\setminus\{i_p\},J_0\setminus\{j_q\}} \\
&+ \sum_{q=1}^{s}\sum_{p=0}^{s}(-1)^{p+q}\binom{i_p+j_q-1}{i_p} d_{I_0\setminus\{i_p\},J_0\setminus\{j_q\}} \\
=&d_{I,J} + \sum_{p=1}^{s} d_{I_0\setminus\{i_p\}\cup\{i_p-1\},J_0} + \sum_{q=1}^{s} d_{I_0,J_0\setminus\{j_q\}\cup\{j_q-1\}}.
\end{aligned}$$

2. The proof of the second formula is similar to the proof of formula 5.6 in Lemma 5.4.5. We only consider the case $i_1 = 0$ and in $s_{(d)}(\{x_i + y_j \mid 1 \le i, j \le s\})$ we subsitute $x_s = 0$. This yields

$$d_{\{i_1,\ldots,i_s\},\{j_1,j_2,\ldots,j_s\}} = \sum_{j_{\ell-1} < j'_\ell \le j_\ell} d_{\{i_2-1,\ldots,i_s-1\},\{j'_1,\ldots,j'_s\}}.$$

Then by Proposition 5.6.9 all summands with $j'_1 > 0$ are zero. This allows to substitute $y_s = 0$ in $s_{(d)}(\{x_i + y_j \mid 1 \le i \le s-1, 1 \le j \le s\})$ and conclude the lemma analogously to formula 5.6 in Lemma 5.4.5.

□

Theorem 5.6.11. *Let $I = \{i_1, \ldots, i_r\}$, $J = \{j_1, \ldots, j_r\}$ be two sets of strictly increasing nonnegative integers. The function defined for $n \ge 0$ by*

$$LP^A_{I,J}(n) := \begin{cases} d_{[n]\setminus I,[n]\setminus J} & \text{if } I, J \subset [n], \\ 0 & \text{otherwise,} \end{cases}$$

is polynomial in n.

Proof. From Lemma 5.6.10 it follows that

$$LP^A_{I,J}(n) = (n-r+1)LP^A_{I\setminus\{0\},J\setminus\{0\}}(n) \\ - \sum_{\ell:i_{\ell+1}>i_\ell+1} LP^A_{I\setminus\{0,i_\ell\}\sqcup\{i_\ell+1\},J\setminus\{0\}}(n) - \sum_{\ell:j_{\ell+1}>j_\ell+1} LP^A_{I\setminus\{0\},J\setminus\{0,j_\ell\}\sqcup\{j_\ell+1\}}(n)$$

if $i_0 = j_0 = 0$, and otherwise

$$LP^A_{I,J}(n) = \sum_{I',J'} LP^A_{I',J'}(n-1),$$

where the sum is over all pairs (I', J') of the form $(\{i_1-\epsilon_1, \dots, i_r-\epsilon_r\}, \{j_1-\mu_1, \dots, j_r-\mu_r\})$ with $\epsilon_\ell, \mu_\ell \in \{0,1\}$. As in the first proof of Theorem 5.4.3, it follows by induction that $LP^A_{I,J}$ is polynomial. □

Theorem 5.6.12. *For every fixed m, s, the function $\delta_A(m,n,n-s)$ is a polynomial in n.*

Proof. Follows from Theorems 5.6.8 and 5.6.11. □

Theorem 5.6.13. *For any fixed d, the function $\phi_A(n,d)$ is a polynomial for $n > 0$.*

Proof. Follows from eq. (5.7) and Theorem 5.6.12. □

5.7 Skew-symmetric matrices

5.7.1 Codegrees of smooth skew-symmetric determinantal loci

Let A_n denote the space of skew-symmetric complex matrices of size n, and $AD^{n-r,n} \subset \mathbb{P}(A_n)$ the locus of matrices of rank at most $n-r$, where $n-r$ is always supposed to be even. Denote by $AD^{n-r,n}_m$ its intersection with a general m-dimensional projective space. Its dimension is $d = m - \binom{r}{2}$ when this is non negative, otherwise it is empty. The analogues of Pataki's inequalities are given by:

Proposition 5.7.1. *The dual variety of $AD^{n-r,n}_m$ is a hypersurface if and only if*

$$\binom{r}{2} \leq m \leq \binom{n}{2} - \binom{n-r}{2}.$$

As in the previous case, the degree of this dual variety can be computed by classical means when $AD^{n-r,n}_m$ is smooth, which is equivalent to $\binom{r}{2} \leq m \leq \binom{r}{2} + 2r$. The class formula gives, in terms of topological Euler characteristics,

$$\deg(AD^{n-r,n}_m)^* = (-1)^d\Big(\chi(AD^{n-r,n}_m) - 2\chi(AD^{n-r,n}_{m-1}) + \chi(AD^{n-r,n}_{m-2}\Big).$$

Euler characteristics of smooth skew-symmetric degeneracy loci have also been computed by Pragacz [85]. For E a vector bundle of rank e over a variety X, and $\varphi : E^* \to E$ a skew-symmetric morphism, the formula given in [35, page 64] is

$$\chi(D_s(\varphi)) = \int_X P_s(E)c(X),$$

where $c(X)$ denotes the total Chern class of X, while $P_s(E)$ is a universal polynomial in the Chern classes of E. Explicitely,

$$P_s(E) = \sum_{\ell(\lambda)\leq n-s} (-1)^{|\lambda|}[\lambda+\rho(n-s-1)]P_{\lambda+\rho(n-s-1)}(E),$$

where the coefficients $[\lambda+\rho(n-s-1)]$ are those appearing in the Segre class of the skew-symmetric square of a vector bundle of rank $n-s$. These coefficients were denoted α_I in [54], that we will rather follow, where I is a set of $r=n-s$ nonnegative integers.

Let us apply this formula to $AD_m^{n-r,n}$, which we consider formally as the degeneracy locus $D_s(\varphi)$ of the tautological skew-symmetric morphism $\phi : F = \mathcal{O}(-\frac{1}{2})^{\oplus n} \longrightarrow \mathcal{O}(\frac{1}{2})^{\oplus n}$ over $X=\mathbb{P}^m$. Since $c(\mathbb{P}^m) - 2hc(\mathbb{P}^{m-1}) + h^2c(\mathbb{P}^{m-2}) = (1+h)^{m-1}$, where h denotes the hyperplane class, we get the formula

$$\deg(AD_m^{n-r,n})^* = \sum_I \binom{m-1}{m-\sum I}\alpha_I P_I(\underbrace{1,\dots,1}_{n \text{ times}}),$$

where the sum goes over the sets I of r nonnegative integers. Once again the dependence on n for r fixed is only in the last term, more precisely in the number of one's on which the P-Schur functions are evaluated. We have already seen that this dependence is well known to be polynomial in n.

A priori this formula is only valid in the range $\binom{r}{2} \leq m \leq \binom{r}{2}+2r$, when $AD_m^{n-r,n}$ is smooth. That it should be true in general would be an analogue of the NRS conjecture in type D. We will prove below that this statement is correct. Our notations for the dual degrees will be as follows

Definition 5.7.2. Define $\delta_D(m,n,r)$ to be the degree of the variety $(AD_m^{2r,2n})^*$ if it is a hypersurface, and zero otherwise. Here $AD_m^{2r,2n}$ is the variety of rank at most $2r$ skew-symmetric $2n\times 2n$ matrices, intersected with a general (projective) m dimensional subspace. Equivalently, if we let $Z_r \subset \mathbb{P}(\wedge^2 V^*)\times\mathbb{P}(\wedge^2 V)$ be the variety of pairs of matrices (X,Y), up to scalars, with $X\cdot Y=0$, $\operatorname{rank} X \leq 2r$, $\operatorname{rank} Y \leq n-2r$. Then the multidegree of Z_r is equal to

$$[Z_r] = \sum_m \delta_D(m,n,r) H_1^{\binom{n}{2}-m} H_{n-1}^m,$$

where H_1 and H_{n-1} are the pullbacks of the hyperplane classes from $\mathbb{P}(\wedge^2 V^*)$ and $\mathbb{P}(\wedge^2 V)$.

5.7.2 Complete skew-symmetric forms

A well-known particularity of skew-symmetric forms is that the cases of odd and even sizes are quite different. In particular, the following definition only makes sense in the even case.

Definition 5.7.3. The number $\phi_D(n,d)$ is the degree of the variety $\mathcal{L}^{-1}$, where $\mathcal{L} \subseteq \mathbb{P}(A_{2n})$ is a general linear subspace of dimension $d-1$.

In this section, we will be only working with skew-symmetric matrices of even size $2n\times 2n$. The relevant space to deal with is then the *space of complete skew-forms*.

Just as with complete quadrics, there are many ways of constructing this space. Here we give just two, referring the reader to the literature [5, 107, 64] for other equivalent definitions.

Definition 5.7.4. Let V be a $2n$-dimensional vector space. The space of complete skew-forms $CS(V)$ is defined as the closure of $\phi(\mathbb{P}(\bigwedge^2(V))^\circ)$, where

$$\phi : \mathbb{P}\left(\bigwedge^2 V\right)^\circ \to \mathbb{P}\left(\bigwedge^2 V\right) \times \mathbb{P}\left(\bigwedge^4 V\right) \times \ldots \times \mathbb{P}\left(\bigwedge^{2n-2} V\right),$$

given by

$$[A] \mapsto ([A], [\wedge^2 A], \ldots, [\wedge^{n-1} A]).$$

We note that here $\wedge^i A$ is viewed as an element of $\bigwedge^{2i} V$, see also [5, Section 3]. In coordinates, the map $\bigwedge^2 V \to \bigwedge^{2i} V$ sends the entries of a skew-symmetric matrix to the Pfaffians of its principal $2i \times 2i$ submatrices.

For simplicity we will also use the notation $CS_{2n} = CS(\mathbb{C}^{2n})$.

As in the symmetric case, the space of complete skew-forms can be constructed by blowing up $\mathbb{P}(A_{2n})$ along the subvariety of rank two matrices, then the strict transform of the subvariety of matrices of rank at most four, and so on. As such, it admits a first series $S_1, \ldots, S_{n-1}$ of special classes of divisors: the classes of (the strict transforms of) the exceptional divisors $E_1, \ldots, E_{n-1}$ of these successive blow-ups. A second natural series $L_1, \ldots, L_{n-1}$ of classes of divisors can be obtained by pulling back the hyperplane classes under the projections $\pi_i : CS(V) \to \mathbb{P}\left(\bigwedge^{2i} V\right)$.

The analogue of Proposition 5.3.4 holds:

Proposition 5.7.5. *The classes $L_1, \ldots, L_{n-1}$ for a basis* $\mathrm{Pic}(CS(V))$*, in which the S_i's are given by*

$$S_i = -L_{i-1} + 2L_i - L_{i+1},$$

with $L_0 = L_n := 0$.

Proof. Follows from [64, Proposition 3.6, Theorem 3.9]. □

As with symmetric matrices, the numbers ϕ_D and δ_D can be expressed as intersection products in the Chow ring of CS_{2n}:

Proposition 5.7.6.

$$\phi_D(n, d) = \int_{CS_{2n}} L_1^{\binom{2n}{2}-d} L_{n-1}^{d-1}$$

$$\delta_D(m, n, r) = \int_{CS_{2n}} S_r L_1^{\binom{2n}{2}-m-1} L_{n-1}^{m-1} = \int_{E_r} L_1^{\binom{2n}{2}-m-1} L_{n-1}^{m-1}.$$

Proof. Analogous to the proof of Proposition 5.3.5. □

From the two propositions above and the Pataki inequalities, we deduce that

$$\phi_D(n, d) = \frac{1}{n} \sum_{\binom{r}{2} \le d} r \delta_D(d, n, n - r), \tag{5.8}$$

the analogue of eq. (5.3).

Definition 5.7.7. We define type D Lascoux coefficients α_I as follows. For $X = (x_1, \ldots, x_k)$ a set of indeterminates, we denote by $\lambda(X)$ the set of indeterminates x_i+x_j, $1 \leq i < j \leq k$. Then the α_I's are defined by the formal identity

$$s_{(d)}(\lambda(X)) = \sum_{\substack{\#I=k,\\ |\lambda(I)|=d}} \alpha_I s_{\lambda(I)}(X).$$

Equivalently, for the universal bundle $\mathcal{U}$ over a Grassmannian $G(k,m)$,

$$Seg_d\left(\bigwedge^2 \mathcal{U}\right) = \sum_{\substack{\#I=k\\ |\lambda(I)|=d}} \alpha_I \sigma_{\lambda(I)}.$$

For more about these coefficients, see [54, Proposition A.16].

Theorem 5.7.8.

$$\delta_D(m,n,r) = \sum_{\substack{I\subset[2n]\\ \#I=2n-2r\\ \sum I=m}} \alpha_I \alpha_{[2n]\setminus I}$$

Proof. Analogous to the proof of Theorem 5.3.7. $\square$

5.7.3 Induction relations and polynomiality

We will now prove the polynomiality (or more precisely, quasipolynomiality) of $\alpha_{[k]\setminus I}$. The following recursive relations will be central to our proof:

Lemma 5.7.9. *1. For $j_1 > 0$ we have:*

$$\alpha_{\{j_1,\ldots,j_s\}} = \begin{cases} \alpha_{\{0,j_1,\ldots,j_s\}} & \textit{if } s \textit{ is even} \\ 0 & \textit{if } s \textit{ is odd} \end{cases} \tag{5.9}$$

2. For $j_1 = 0$ we have:

$$\alpha_{\{j_1,j_2,\ldots,j_s\}} = \sum_{j_\ell \leq j'_\ell < j_{\ell+1}} \alpha_{\{j'_1,\ldots,j'_{s-1}\}}. \tag{5.10}$$

Proof. First formula is [85, p. 446], [54, (A.16.3)] and [86, p. 163-166]. The proof of the second formula is analogous to the proof of eq. (5.6) in Lemma 5.4.5. $\square$

Theorem 5.7.10. *Let $I = \{i_1, \ldots, i_s\}$ be a set of strictly increasing nonnegative integers. For $k \geq 0$ the function:*

$$LP_I^D(k) := \begin{cases} \alpha_{[k]\setminus I} & \textit{if } I \subset [k], \\ 0 & \textit{otherwise.} \end{cases}$$

is a quasi-polynomial in k with period 2, i.e. for both even k and odd k it is a polynomial.

Proof. We proceed as in the first proof of Theorem 5.4.3 by induction on $\#I$ and then on $\sum I$ using relations from Lemma 5.7.9. The difference is that in the case $i_0 = 0$ we have

$$LP_I^D(n) = \begin{cases} LP_{I\setminus 0}^D(n) & \text{if } n - \#I \text{ is even} \\ 0 & \text{if } n - \#I \text{ is odd} \end{cases}$$

which is clearly by induction hypothesis a quasipolynomial in n with period 2. The rest is analogous as in the proof of Theorem 5.4.3. $\square$

From Theorem 5.7.8 and Theorem 5.7.10 we deduce the polynomiality of δ_D:

Theorem 5.7.11. *For every fixed m, s, the function $\delta_D(m, n, n - s)$ is a polynomial in n.*

Using eq. (5.8),we also get the polynomiality of ϕ_D:

Theorem 5.7.12. *For any fixed d, the function $\phi_D(n, d)$ is a polynomial for $n > 0$.*

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