The Comparative Study of Finite Difference Method and Monte Carlo Method for Pricing European Option


Sunday Fadugba
Ekiti State University, Ado Ekiti


Teniola Babalola
Ekiti State University/Afe Babalola University

Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.2, No.4, 2012

The Comparative Study of Finite Difference Method and Monte Carlo Method for Pricing European Option

Sunday Fadugba[1] Chukwuma Nwozo[2] Teniola Babalola[3]

1. Department of Mathematical and Physical Sciences, Afe Babalola University, Ado Ekiti, Nigeria.

2. Department of Mathematics, University of Ibadan, Nigeria.

3. Department of Mathematical and Physical Sciences, Afe Babalola University, Ado Ekiti, Nigeria.

* Email of the corresponding author: emmasfad2006@yahoo.com.

Abstract

Numerical methods form an important part of options pricing and especially in cases where there is no closed form analytic formula. We discuss two of the primary numerical methods that are currently used by financial professionals for determining the price of an options namely Monte Carlo method and finite difference method. Then we compare the convergence of the two methods to the analytic Black-Scholes price of European option. Monte Carlo method is good for pricing exotic options while Crank Nicolson finite difference method is unconditionally stable, more accurate and converges faster than Monte Carlo method when pricing standard options.

Keywords: Option, European option, Asian option, Monte Carlo Method, Finite difference method.

1. Introduction

Black-Scholes published their seminar work on options valuation (Black and Scholes 1973) in which they described a mathematical frame work for finding the fair price of a European option by the use of a non-arbitrage argument to describe a partial differential equation which governs the evolution of the option price with respect to the time to expiry and the price of the underlying asset. Numerical techniques are needed for pricing options in cases where analytic solutions are either unavailable or easily computable (Hull 2003). Now, we present an overview of two popular numerical methods available in the context of Black-Scholes (Merton 1973) for vanilla and path dependent options valuation which are finite difference method for pricing derivative governed by solving the underlying partial difference equation which was considered by (Brennan and Schwarz 1998) and Monte Carlo approach introduced by (Boyle 1997) is used for pricing European option and path dependent options. The sufficient conditions for dynamic stability and convergence to equilibrium of the growth rate of the function of stock shares were given by (Osu 2011) and binomial model for pricing options based on risk-neutral valuation was derived by (Cox *et al*. 2003). These procedures provide much of the infrastructure in which many contributions to the field over the past three decades have been centered.

2. Numerical Methods for Pricing European Option

This section presents two numerical methods for pricing options namely:

- Monte Carlo method.
- Finite difference method.

2.1 Monte Carlo Method.

(Boyle 1997) was the first researcher to introduce Monte Carlo method into finance. Monte Carlo method is a numerical method that is useful in many situations when no closed form solution is available. This method is good for pricing both vanilla and path dependent options and uses the risk valuation result.

The expected payoff in a risk neutral world is calculated using a sampling procedure. The main procedures are followed when using this method:

- Simulate a path of the underlying asset under the risk neutral condition within the desired time horizon.
- Discount the payoff corresponding to the path at the risk free interest rate.
- Repeat the procedure for a high number of simulated sample paths.
- Average the discounted cash flows over sample paths to obtain option's value.

A Monte Carlo method followed the geometric Brownian motion for stock price

$$dS = \mu S dt + \sigma S dW(t) \tag{1}$$

Where $dW(t)$ a Brownian motion or Wiener process and S is the stock price. If ΔS is the increase in the stock price in the next small interval of time Δt then,

$$\frac{\Delta S}{S} = \mu S dt + \sigma z \sqrt{\Delta t} \tag{2}$$

Where z is normally distributed with mean zero and variance one, σ is the volatility of the stock price and μ is the expected return in a risk neutral world, (2) is expressed as

$$S(t+\Delta t) - S(t) = \mu S(t)\Delta t + \sigma S(t) z\sqrt{\Delta t} \tag{3}$$

It is more accurate to estimate InS than S , we transform the asset price process using Ito's lemma.

$$d(InS) = (\mu - \frac{\sigma^2}{2})dt + \sigma dW(t)$$

So that,

$$InS(t+\Delta t) - InS(t) = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma z\sqrt{\Delta t}$$

Or

$$S(t+\Delta t) = S(t)\exp[(\mu - \sigma^2/2) + \sigma z\sqrt{\Delta t}] \tag{4}$$

This method is particularly relevant when the financial derivatives payoff depends on the path followed by the underlying asset during the life of the option (Weston *et* al. 2005).

The fair price for pricing option at maturity date is given by

$$S_T^j = S\exp[(\mu - \sigma^2/2)T + \sigma z\sqrt{T}] \tag{5}$$

Where $j = 1,2,\ldots,M$ and M denotes the number of trials. The estimated European call option value is

$$C = \frac{1}{M}\sum_{J=1}^{M}\exp(-rT)\max[S_T^j - S_t, 0] \tag{6}$$

Similarly, for a European put option, we have

$$C = \frac{1}{M}\sum_{J=1}^{M}\exp(-rT)\max[S_t - S_T^j, 0]$$

(7)

Where S_t is the strike price determined by either arithmetic or geometric mean.

2.2 Finite Difference Method.

(Brennan and Schwartz 1978) first applied the finite difference method to price option for which closed form solutions are unavailable and considered the valuation of an American option on stock which pays discrete dividends. The finite difference method attempts to solve the Black-Scholes partial differential equation by approximating the differential equation over the area of integration by system of algebraic equations (Tveito and Winther 1998).

The most common finite difference methods for solving the partial differential equations are:

- Explicit scheme.
- Implicit scheme.
- Crank Nicolson scheme.

These schemes are closely related but differ in stability, accuracy and execution speed, but we shall only consider Crank Nicolson scheme. In the formulation of a partial differential equation problem, there are three components to be considered.

- The partial differential equation.
- The region of space time on which the partial differential is required to be satisfied.
- The ancillary boundary and initial conditions to be met.

2.3 Discretization of the Black-Scholes Equation

The finite difference method consists of discretizing the partial differential equation and the boundary conditions using a forward or a backward difference approximation. The Black-Scholes partial differential equation is given by

$$f_t(t,S_t) + rS_t f_{S_t} + \frac{\sigma^2 S^2{}_t}{2} f_{S_t S_t} = rf(t,S_t) \qquad (8)$$

We discretize (8) with respect to time and to the underlying price of the asset. Divide the (t, S_t) plane into a sufficiently dense grid or mesh and approximate the infinitesimal steps Δ_t and Δ_{S_t} by some small fixed finite steps. Further, define an array of $N+1$ equally spaced grid points $t_0,...,t_N$ to discretize the time derivative with $t_{n+1} - t_n = \frac{T}{N} = \Delta_t$. Using the same procedures, we obtain for the underlying price of the asset as follows:

$S_{M+1} - S_M = \frac{S_{\max}}{M} = \Delta_{S_t}$. This gives us a rectangular region on the (t, S_t) plane with sides $(0, S_{\max})$ and $(0, T)$. The grid coordinates (n, m) enables us to compute the solution at discrete points. We will denote the value of the derivative at time step t_n when the underlying asset has value S_m as

$$f_{m,n} = f(n\Delta t, m\Delta S) = f(t_n, S_m) = f(t, S_t)$$

(9)

Where n and m are the numbers of discrete increments in the time to maturity and stock price respectively.

2.4 Crank Nicolson Finite Difference Equation

In finite difference method, we replace the partial derivative occurring in the partial differential equation by approximations based on Taylor series expansions of function near the points of interest (Travella and Randall 2000). Expanding $f(t, \Delta S + S)$ and $f(t, S - \Delta S)$ in Taylor series we have the forward and backward difference respectively with $f(t, S)$ represented in the grid by $f_{n,m}$ (Ames 1997)

$$f_{S_t} \approx \frac{f_{n,m+1} - f_{n,m}}{\Delta S_t} \qquad (10)$$

$$f_{S_t} \approx \frac{f_{n,m} - f_{n,m-1}}{\Delta S_t} \qquad (11)$$

Also the first order partial derivative results in the central difference given by

$$f_{S_t} \approx \frac{f_{n,m+1} - f_{n,m-1}}{2\Delta S_t} \tag{12}$$

And the second order partial derivative gives symmetric central difference approximation of the form

$$f_{S_t S_t} \approx \frac{f_{n,m+1} - 2f_{n,m} - f_{n,m-1}}{\Delta S_t^{\ 2}} \tag{13}$$

Similarly, we obtained forward difference approximation for the maturity time given by

$$f_t \approx \frac{f_{n+1,m} - f_{n,m}}{\Delta S_t} \tag{14}$$

Substituting equations (12), (13) and (14) into (8), we have

$$\rho_{1m} f_{n,m-1} + \rho_{2m} f_{n,m} + \rho_{3m} f_{n,m+1} = f_{n+1,m}$$

(15)

Where

$\rho_{1m} = \frac{1}{2} rm\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t$, $\rho_{2m} = 1 + r\Delta t + \sigma^2 m^2 \Delta t$, $\rho_{3m} = -\frac{1}{2} rm\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t$,

(15) is called a finite difference equation which gives equation that we use to approximate the solution of $f(t, S)$ (Boyle *et al* 1997). Similarly, we obtained for the Crank Nicolson finite difference method which is the average of the explicit and implicit schemes given by.

$$v_{1m} f_{n,m-1} + v_{2m} f_{n,m} + v_{3m} f_{n,m+1} = \varphi_{1m} f_{n+1,m+1} + \varphi_{2m} f_{n+1,m} + \varphi_{3m} f_{n+1,m+1}$$

(16)

For $n = 0,1,\ldots, N-1$ and $m = 0,1,\ldots, M-1$. Then the parameters

$$v_{1m} = \frac{rm\Delta t}{4} - \frac{\sigma^2\ m^2\Delta t}{4},\ v_{2m} = 1 + \frac{r\Delta t}{2} + \frac{\sigma^2\ m^2\Delta t}{2}, v_{3m} = -\frac{rm\Delta t}{4} - \frac{\sigma^2\ m^2\Delta t}{4}$$

$$\varphi_{1m} = -\frac{rm\Delta t}{4} + \frac{\sigma^2\ m^2\Delta t}{4}, \varphi_{21m} = 1 - \frac{r\Delta t}{2} - \frac{\sigma^2\ m^2\Delta t}{2},\ \varphi_{3m} = \frac{rm\Delta t}{4} + \frac{\sigma^2\ m^2\Delta t}{4}.$$

2.5 Stability Analysis

The two fundamental sources of error are the truncation error in the stock price and time discretization. The importance of truncation error is that the numerical schemes solve a problem that is not exactly the same as the problem we are trying to solve (Smith 1985). The three fundamental factors that characterize a numerical scheme are consistency, stability and convergence. These three factors are linked by Lax equivalence theorem which state as follows:

- Lax Equivalence Theorem (Merton 1973)

  Given a well posed linear initial value problem and a consistent finite difference method, stability is the necessary and sufficient condition for convergence.

Generally, a problem is said to be well posed if the following holds:

- A solution to the problem exists.
- The solution is unique when it exists.
- The solution depends continuously on the problem data.

3. Numerical Example

We consider the performance of the two methods against the analytic Black-Scholes price for a European put with the following parameters:

$$K = 50, r = 0.05, \sigma = 0.25, T = 3.0 .$$

The results obtained are shown in the Table below:

4. Discussion of Results

The Table 1 below shows the variation of the option price with the underlying price S. The results demonstrate that the two schemes perform well, are consistent and agree with the Black-Scholes value. However, finite difference method is the most accurate and converges faster than Monte Carlo method when pricing European option.

5. Conclusion

In general, each of the two numerical methods has its advantages and disadvantages of use: finite difference method converges faster and more accurate, they are fairly robust and good for pricing vanilla option. They can also require sophisticated algorithms for solving large sparse linear systems of equations and are relatively difficult to code.

Moreover, Monte Carlo method works very well for pricing both European and exotic options, it is flexible in handling varying and even high dimensional financial problems, hence despite its significant progress, early exercise is problematic.

Finally, we conclude that Crank Nicolson method is unconditionally stable, more accurate and converges faster than Monte Carlo method when pricing European option.

References

Ames, W. (1997). Numerical Methods for Partial Differential Equation. Academic Press, New York.

Black, F. and Scholes M., (1973). The Pricing of Option and Corporate Liabilities. Journal of Political Economy, Vol. 81(3), 637-654.

Boyle, P. (1977). Options: A Monte Carlo Approach. Journal of Financial Economics, Vol. 4(3), 323-338.

Boyle, P., Broadie M. and Glasserman P. (1997). Monte Carlo Methods for Security Pricing, Journal of Economic Dynamics and Control, Vol. (8-9), 1267-1231.

Brennan, M. and Schwartz, E. (1978). Finite Difference Method and Jump Processes arising in the Pricing of Contingent Claim. Journal of Financial and Quantitative Analysis, Vol. 5(4), 461-474.

Cox, J., Ross, S. and Rubinstein M. (1979). Option Pricing: A Simplified Approach. Journal of Financial Economics, Vol. 7, 229-263.

Hull, J. (2003). Options, Futures and other Derivatives. Pearson Education Inc. Fifth Edition, Prentice Hall, New Jersey.

Merton, R. C. (1973). Theory of Rational Option Pricing. Bell Journal of Economics and Management Science, Vol. 4(1), 141-183.

Osu, B. O. (2010). A Stochastic Model of the variation of the capital market price. International Journal of Trade Economics and Finance, Vol. 1(3), 2010-23X.

Smith, G.D. (1985). Numerical Solution of Partial Differential Equations. Finite Difference Methods. Clarendon Press, Third Edition, Oxford.

Travella, D. and Randall, C. (2000). Pricing Financial Instruments. The Finite Difference Method. John Wiley and Sons, New York.

Tveito, A. and Winther, R. (1998). Introduction to Partial Differential Equation Instruments: The Finite Difference Method. John Wiley and Son, New York.

Weston, J., Copeland T. and Shastri, K. (2005). Financial Theory and Corporate Policy. Fourth Edition, New York, Pearson Addison Wesley.

Table 1. A Comparison with the Black-Scholes Price for a European Put Option.

Stock price S.	**Black-Scholes Analytic Price**	**Crank Nicolson Method**	**Monte Carlo Method**
10	33.0363	33.0362	33.0345
15	28.0619	28.0616	28.0595
20	23.2276	23.2271	23.2291
25	18.7361	18.7350	18.7339
30	14.7739	14.7734	14.7748
35	11.4384	11.4390	11.4402
40	8.7338	8.7334	8.7374
45	6.6021	6.6019	6.6014
50	4.9564	4.9563	4.9559
55	3.7046	3.7042	3.7076
60	2.7621	2.7613	2.7602
65	2.0574	2.7613	2.7602
70	1.5328	1.5326	1.5324
75	1.1430	1.1427	1.1407
80	0.8538	0.8537	0.8543
85	0.6392	0.6391	0.6405
90	0.4797	0.4795	0.4790
95	0.2501	0.2490	0.2487
100	0.2319	0.2315	0.2318