Numerical Analysis for Finance: What is ADI?

Financial Risk Management MSc & Financial Mathematics MSc

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1. The problem

1.1. The two dimensional heat equation.

ADI stands for alternate direction implicit. It is a method used for two dimensional parabolic PDEs such as the two dimensional heat equation

$$f_t = f_{xx} + f_{yy}.$$

For the sake of concreteness we will consider this equation in the cube

$$\left\{(x, y, t) \in \mathbb{R}^3 | 0 \leq x, y, t \leq 1\right\},$$

together with initial conditions

$$f(x, y, 0) = 1,$$

and boundary conditions

$$\begin{aligned} f(0, y, t) &= f(1, y, t) = 0, \\ f(x, 0, t) &= f(x, 1, t) = 0. \end{aligned}$$

As in the one dimensional case we have various ways in which we can approximate the derivatives that lead to methods with different properties.

1.2. Set-up for numerical resolution.

In each of these cases space and time are discretised with a grid of N_x, N_y and M steps and sizes $\Delta x = 1/N_x$, $\Delta y = 1/N_y$ and $\Delta t = 1/M$. As is usual we note $f_{i_x i_y j}$ the approximation to $f(i_x \Delta x, i_y \Delta, j \Delta t)$.

For the purposes of book-keeping note there are $N_x + 1$ x values and $N_y + 1$ y values. At two of these levels the function is known by the boundary conditions. So at each time step we have $(N_x-1)\times(N_y-1)$ unknown values. There is a total of $M+1$ time values. At one of these the function is given by the initial condition. So there are $(N_x-1)\times(N_y-1)\times M$ unknown values of the function. As usual these will be calculated using a recurrence that derives values at time step $j\Delta t$ from values at the previous time step $(j-1)\Delta t$.

2. The explicit method

In this case we just approximate the x and y derivatives by their standard central differences and the time derivative by a forward difference:

$$\frac{f_{i_x,i_y,j+1}-f_{i_x,i_y,j}}{\Delta t}=\frac{f_{i_x+1,i_y,j}-2f_{i_x,i_y,j}+f_{i_x-1,i_y,j}}{\Delta x^2}+\frac{f_{i_x,i_y+1,j}-2f_{i_x,i_y,j}+f_{i_x,i_y-1,j}}{\Delta y^2}.$$

As in the one dimensional case this leads to an recurrence scheme in which $f_{i_x,i_y,j+1}$ can be expressed trivially as a linear combination of estimates of the function at the previous time step $j\Delta t$:

$$\begin{aligned} f_{i_x,i_y,j+1} &= \frac{\Delta t}{\Delta x^2}f_{i_x+1,i_y,j}+\left(1-2\frac{\Delta t}{\Delta x^2}\right)f_{i_x,i_y,j}+\frac{\Delta t}{\Delta x^2}f_{i_x-1,i_y,j}+ \\ &+ \frac{\Delta t}{\Delta y^2}f_{i_x,i_y+1,j}+\left(1-2\frac{\Delta t}{\Delta y^2}\right)f_{i_x,i_y,j}+\frac{\Delta t}{\Delta y^2}f_{i_x,i_y-1,j}. \end{aligned}$$

The downside of the explicit method is that unless the quantities $\frac{\Delta t}{\Delta y^2}$ and $\frac{\Delta t}{\Delta y^2}$ are small enough the method will be unstable. This in practice means that M needs to so large that the performance of the calculation is compromised.

As in the one dimensional case implicit methods solve this stability problem.

3. The implicit method

For the implicit method we proceed as above but use backward time differences to approximate the time derivative

$$\frac{f_{i_x,i_y,j}-f_{i_x,i_y,j-1}}{\Delta t}=\frac{f_{i_x+1,i_y,j}-2f_{i_x,i_y,j}+f_{i_x-1,i_y,j}}{\Delta x^2}+\frac{f_{i_x,i_y+1,j}-2f_{i_x,i_y,j}+f_{i_x,i_y+1,j}}{\Delta y^2}$$

as in the one dimensional case this leads to a linear system linking the unknown values $f_{i_x,i_y,j}$ to the values in the previous time step $f_{i_x,i_y,j-1}$. As in the one dimensional case this relationship is not trivial (explicit) and we need to resolve a linear system to find the $(N_x-1)\times(N_y-1)$ unknowns.

Unfortunately in dimension 2 this system is no longer tridiagonal (note that in order to write the actual system matrix we need to order the $(N_x - 1)(N_y - 1)$ unknowns $f_{i_x,i_y,j}$ for a fixed j which can be done in more than one way). It is still a sparse system but the simple Thomas algorithms for solving tridiagonal matrices is no longer applicable.

On the positive side, the implicit method is stable.

The *raison d'être* of the ADI method is to provide a scheme that is stable and that can be reduced to the resolution of tridiagonal systems.

4. Compact notation

4.1. Finite difference operators

Given a step size Δ we use the standard notation δ^2 for the difference operator on sequences of numbers that takes $(a_i)_i$ to

$$\delta^2(a_i)_i = \left(\frac{a_{i-1} - 2a_i + a_{i-1}}{\Delta^2}\right)_i.$$

We extend the notation to our set-up as follows:

$$\begin{aligned}
\delta_x^2 f_{i_x,i_y,j} &= \frac{f_{i_x+1,i_y,j} - 2f_{i_x,i_y,j} + f_{i_x-1,i_y,j}}{\Delta x^2}, \\
\delta_y^2 f_{i_x,i_y,j} &= \frac{f_{i_x,i_y+1,j} - 2f_{i_x,i_y,j} + f_{i_x,i_y-1,j}}{\Delta y^2};
\end{aligned}$$

then the explicit scheme can be abbreviated as

$$\frac{f_{i_x,i_y,j+1} - f_{i_x,i_y,j}}{\Delta t} = \delta_x^2 f_{i_x,i_y,j} + \delta_y^2 f_{i_x,i_y,j},$$

and the implicit as

$$\frac{f_{i_x,i_y,j} - f_{i_x,i_y,j-1}}{\Delta t} = \delta_x^2 f_{i_x,i_y,j} + \delta_y^2 f_{i_x,i_y,j}.$$

Note this notation is not fully mathematically rigorous. To be clear $\delta_x^2 f_{i_x,i_y,j}$ means: the operator above δ^2 with $\Delta = \Delta x$ is applied to the sequence $\left(f_{i,i_y,j}\right)_i$ (with fixed i_y and j) and we select the i_xth element of the resulting sequence.

4.2. Change of time indices.

So far we have based both implicit and explicit methods on the second derivatives evaluated at time $j\Delta t$. As a result the explicit method involves time values $(j+1)\Delta t$ and $j\Delta t$ whereas the implicit method involves $j\Delta t$ and $(j-1)\Delta t$.

In order to compare these methods more effectively we will shift the indices so that in both cases we are considering time steps $(j+1)\Delta t$ and $j\Delta t$:

- Explicit
$$\frac{f_{i_x,i_y,j+1} - f_{i_x,i_y,j}}{\Delta t} = \delta_x^2 f_{i_x,i_y,j} + \delta_y^2 f_{i_x,i_y,j}.$$
- Implicit
$$\frac{f_{i_x,i_y,j+1} - f_{i_x,i_y,j}}{\Delta t} = \delta_x^2 f_{i_x,i_y,j+1} + \delta_y^2 f_{i_x,i_y,j+1}.$$

Which is a forward time difference equated to a second derivative evaluated at either $j\Delta t$ or $(j+1)\Delta t$. [1]

Other methods can also be written compactly using this notation; for example Crank-Nicolson in one dimension becomes:

$$\frac{f_{i_x,j+1} - f_{i_x,j}}{\Delta t} = \frac{1}{2}\left(\delta_x^2 f_{i_x,j+1} + \delta_x^2 f_{i_x,j}\right).$$

which can be seen as an average of implicit at explicit.

In this notation the difference between the different methods reduces to the choice of a time at which to estimate second derivative.

We can further abbreviate notations by using the identity operator Id:

- Explicit $f_{i_x,i_y,j+1} = \left(Id + \Delta t\delta_x^2 + \Delta t\delta_y^2\right) f_{i_x,i_y,j}$.
- Implicit $\left(Id + \Delta t\delta_x^2 + \Delta t\delta_y^2\right) f_{i_x,i_y,j+1} = f_{i_x,i_y,j}$.
- Crank-Nicolson $\left(Id - \frac{1}{2}\Delta t\delta_x^2\right) f_{i_x+1,j} = \left(Id + \frac{1}{2}\Delta t\delta_x^2\right) f_{i_x,j}$.

In each of these cases the RHS are the known variables at time $j\Delta t$ and the LHS holds the unknown variables. The one dimensional implicit and CN schemes require inverting the matrix $Id - \frac{1}{2}\Delta t\delta_x^2$ which is tridiagonal.

[1] Note however that we are not changing the equations, we are just changing the indices equation we display as generic. The methods are identical to the previously described.

5. Alternate Implicit Directions

ADI originally refers to the scheme first described in [2]. That scheme is now often called the Peaceman-Rachford scheme as ADI has also come to refer to other methods based on similar ideas, namely splitting time. For that reason general ADI methods are also referred to as time splitting schemes or fractional step methods.

5.1. The Peaceman-Rachford scheme

In the original ADI method we divide each time step $[j\Delta t, (j+1)\Delta t]$ in half and introduce a new temporary intermediate unknown $f_{i_x,i_y,j+1/2}$.

The key idea is based in the previous subsection and consists of approximating the differential equation using the explicit scheme in y and the implicit scheme in x, which is defined to mean:

$$\frac{f_{i_x,i_y,j+1/2} - f_{i_x,i_y,j}}{\Delta t/2} = \delta_x^2 f_{i_x,i_y,j+1} + \delta_y^2 f_{i_x,i_y,j}.$$

This is a set of $(N_x - 1) \times (N_y - 1)$ linear equations. However the each of the set of $N_x - 1$ equations for a fixed i_y are independent of the other sets. And each of these can be arranged as a tridiagonal system. So in summary:

- The system of equations for the unknowns $f_{1,1,j+1/2}$, $f_{2,1,j+1/2}$, $\ldots$, $f_{N_x-1,1,j+1/2}$ is tridiagonal. It can be solved using the Thomas algorithm.
- The system of equations for the unknowns $f_{1,2,j+1/2}$, $f_{2,2,j+1/2}$, $\ldots$, $f_{N_x-1,2,j+1/2}$ is tridiagonal. It can be solved using the Thomas algorithm.
- $\ldots$
- The system of equations for the unknowns $f_{1,N_y-1,j+1/2}$, $f_{2,N_y-1,j+1/2}$, $\ldots$, $f_{N_x-1,N_y-1,j+1/2}$ is tridiagonal. It can be solved using the Thomas algorithm.

To calculate the unknowns $f_{i_x,i_y,j+1}$ we use a similar scheme but where we change the roles of δ_x^2 and δ_y^2. That is:

$$\frac{f_{i_x,i_y,j} - f_{i_x,i_y,j+1/2}}{\Delta t/2} = \delta_x^2 f_{i_x,i_y,j} + \delta_y^2 f_{i_x,i_y,j+1}.$$

Again this leads to a set of tridiagonal systems that can be solved by application of the Thomas algorithm.

The remarkable result is that the ADI method is always stable. This way we achieve what the implicit method does in one dimension: a stable method with elementary linear algebra.

5.2. Operator notation.

The Peaceman-Rachford method can be written in operator notation as

$$\begin{aligned}\left(Id - \frac{1}{2}\Delta t \delta_x^2\right) f_{i_x,i_y,j+1/2} &= \left(Id + \frac{1}{2}\Delta t \delta_y^2\right) f_{i_x,i_y,j} \\ \left(Id - \frac{1}{2}\Delta t \delta_y^2\right) f_{i_x,i_y,j+1} &= \left(Id + \frac{1}{2}\Delta t \delta_x^2\right) f_{i_x,i_y,j+1/2}\end{aligned}$$

or more compactly as

$$\left(Id + \frac{1}{2}\Delta t \delta_y^2\right)^{-1} \left(Id - \frac{1}{2}\Delta t \delta_y^2\right) f_{i_x,i_y,j+1} = \left(Id - \frac{1}{2}\Delta t \delta_x^2\right)^{-1} \left(Id + \frac{1}{2}\Delta t \delta_y^2\right) f_{i_x,i_y,j}$$

Which can be approximated to second order

$$\left(Id - \frac{1}{2}\Delta t \delta_y^2 + \frac{1}{4}\Delta t^2 \delta_y^4 + \dots\right) \left(Id - \frac{1}{2}\Delta t \delta_y^2\right) f_{i_x,i_y,j+1} = \left(Id + \frac{1}{2}\Delta t \delta_x^2 + \frac{1}{4}\Delta t^2 \delta_x^4 + \dots\right) \left(Id + \frac{1}{2}\Delta t \delta_y^2\right.$$

equal to

$$\left(Id - \frac{1}{2}\Delta t \delta_x^2 - \frac{1}{2}\Delta t \delta_y^2 + \frac{1}{4}\Delta t^2 \delta_y^4 + \dots\right) f_{i_x,i_y,j+1} = \left(Id + \frac{1}{2}\Delta t \delta_x^2 + \frac{1}{2}\Delta t \delta_y^2 + \frac{1}{4}\Delta t^2 \delta_x^4 + \dots\right) f_{i_x,i_y,j}$$

which is approximately the Crank Nicolson two dimensional iteration (not solvable with tridiagonal systems)

$$\left(Id - \frac{1}{2}\Delta t \delta_x^2 - \frac{1}{2}\Delta t \delta_y^2\right) f_{i_x,i_y,j+1} = \left(Id + \frac{1}{2}\Delta t \delta_x^2 + \frac{1}{2}\Delta t \delta_y^2\right) f_{i_x,i_y,j}.$$

References

[1] James M. McDonough, *Lectures on computational numerical analysis of partial differential equations*, University of Kentucky, 2008.

[2] Donald W. Peaceman and Henry H. Rachford Jr., *The numerical solution of parabolic and elliptic differential equations*, Journal of the Society for Industrial and Applied Mathematics **3** (1955), no. 1, 28–41.

