MASM006 FINANCIAL MATHEMATICS

(7) THE BLACK-SCHOLES EQUATION

Now that we have discussed the mathematical tools of stochastic calculus, we are ready to return to our financial applications.

Recall that we had a discrete model for the price of an asset such as a share price: we argued that the *return* on the shares over each small timestep of length δt (rather than the actual change in the share price) should behave like one step of a random walk with drift. Thus if S_n represents the share price at time $n\ \delta t$, then

$$\frac{S_{n+1} - S_n}{S_n} = \mu\ \delta t + \sigma\sqrt{\delta t} Y_n, \tag{1}$$

where μ is the expected rate of return (per unit time), σ is the volatility of the return, and the Y_n are independent random variables, taking the values ± 1 each with probability $\frac{1}{2}$. Notice that this is driven by a sequence of independent random variables $\sqrt{\delta t} Y_n$ of expectation 0 and variance δt. Using the Central Limit Theorem, we saw that for large n, the logarithm of the share price $S(T) = S_n$ at time $T = n\ \delta t$ is (approximately) normally distributed:

$$\log\left(\frac{S(T)}{S(0)}\right) \cong \mathcal{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right) T, \sigma^2\right).$$

We will now consider a continuous-time version of the model of (1), describing a continuous-time stochastic process $(S_t)_{t\geq 0}$. To form this model, we simply replace the $\sqrt{\delta t} Y_n$ by increments of a standard Brownian motion $(W_t)_{t\geq 0}$ and let $\delta t \to 0$. This gives the stochastic differential equation

$$\mathrm{d}S_t = S_t \mu \mathrm{d}t + S_t \sigma \mathrm{d}W_t. \tag{2}$$

Using Itô's Lemma, it can be verified (exercise!) that this has solution

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{3}$$

The continuous-time stochastic process satisfying (3) (or (2)) is called *geometric Brownian motion.*

We will need a version of Itô's Lemma for a process $X_t = f(t, S_t)$ which is described in terms of the geometric Brownian motion $(S_t)_{t\geq 0}$, rather than directly in terms of the underlying Brownian motion $(W_t)_{t\geq 0}$ itself.

Itô's Lemma for Geometric Brownian Motion

If $X_t = f(t, S_t)$, where $(S_t)_{t\geq 0}$ satisfies (2), then, arguing as in the heuristic derivation of Itô's Lemma itself, we have

$$\mathrm{d}X_t = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial S_t}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(\mathrm{d}S_t)^2.$$

Here we have retained the term in $(\mathrm{d}S_t)^2$, but ignored the other terms of second or higher order. Substituting for $\mathrm{d}S_t$ from (2), and using the "multiplication table"

	$\mathrm{d}t$	$\mathrm{d}W_t$
$\mathrm{d}t$	0	0
$\mathrm{d}W_t$	0	$\mathrm{d}t$

we have

$$\begin{aligned} \mathrm{d}X_t &= \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial S_t}(S_t\mu\mathrm{d}t + S_t\sigma\mathrm{d}W_t) + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(S_t\mu\mathrm{d}t + S_t\sigma\mathrm{d}W_t)^2 \\ &= \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial S_t}(S_t\mu\mathrm{d}t + S_t\sigma\mathrm{d}W_t) + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}S_t^2\sigma^2\mathrm{d}t, \end{aligned}$$

which simplifies to

$$\mathrm{d}X_t = \left(\frac{\partial f}{\partial t} + \mu S_t\frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S_t^2}\right)\mathrm{d}t + \sigma S_t\frac{\partial f}{\partial S_t}\mathrm{d}W_t. \tag{4}$$

Derivation of the Black-Scholes Partial Differential Equation

Now consider a financial derivative whose payoff $V(T, S_T)$ at the expiry time T depends only on the value S_T of the underlying asset at expiry. The price of the underlying asset is assumed to follow the geometric Brownian motion (2). The derivative could, for example, be a European call or put option, but not an American option, whose value at time T depends also on whether the option has been exercised early. We assume that the value of the derivative at a time $t < T$ is a smooth function $V(t, S)$ of the time t and the asset price S. Our goal is to determine this function V.

Following the intuition we gained from the discrete-time Binomial Model, we will try to hedge a portfolio involving the derivative, so that the value of the portfolio is protected from the effects of random movements in the asset price.

Consider the portfolio obtained by buying one unit of the derivative, and selling short a units of the underlying asset (where the quantity a is to be determined). The value at time t of such a portfolio is $\Pi_t = V(t, S_t) - aS_t$. Over an infinitesimal time interval $\mathrm{d}t$, the change in the portfolio value is $\mathrm{d}\Pi_t = \mathrm{d}V_t - a\mathrm{d}S_t$. From (4) we have

$$\mathrm{d}V_t = \left(\frac{\partial V}{\partial t} + \mu S_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2}\right)\mathrm{d}t + \sigma S_t\frac{\partial V}{\partial S_t}\mathrm{d}W_t.$$

Using this and (2), we obtain

$$\mathrm{d}\Pi_t = \left(\frac{\partial V}{\partial t} + \mu S_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} - a\mu S_t\right)\mathrm{d}t + \left(\sigma S_t\frac{\partial V}{\partial S_t} - a\sigma S_t\right)\mathrm{d}W_t.$$

We wish to choose a to eliminate the randomness in $\mathrm{d}\Pi_t$. This means that $\mathrm{d}\Pi_t$ must depend only on $\mathrm{d}t$, and not on $\mathrm{d}W_t$, so we must take

$$a = \frac{\partial V}{\partial S_t}.$$

Thus we must continually hedge our portfolio to ensure that the quantity of the underlying asset held at any time t is $-\frac{\partial V}{\partial S_t}$. We then have

$$\mathrm{d}\Pi_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \right) \mathrm{d}t.$$

We have now constructed a (continually rehedged) portfolio which is riskfree. In the absence of arbitrage opportunities, this riskfree investment must give the same return over the interval $\mathrm{d}t$ as a riskfree cash bond. Writing r for the riskfree interest rate, as usual, we therefore have

$$\mathrm{d}\Pi_t = r\Pi_t \mathrm{d}t,$$

that is

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \right) \mathrm{d}t = r \left(V_t - S_t \frac{\partial V}{\partial S_t} \right) \mathrm{d}t.$$

Dropping the subscripts t, we can rewrite this as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{5}$$

The partial differential equation (5) is the famous **Black-Scholes equation**. We pause to make some remarks:

(i) The Black-Scholes equation is a deterministic (rather than stochastic) differential equation: by hedging the portfolio, we have eliminated the randomness.

(ii) The continuously hedged portfolio

$$\Pi_t = V(t, S_t) - \frac{\partial V}{\partial S_t} S_t$$

is riskfree and self-financing. The time-varying quantity $\frac{\partial V}{\partial S_t}$ therefore has a natural financial interpretation: it is the short position we must hold in the underlying asset at time t in order to balance the randomness in the behaviour of the derivative. (Alternatively, it is the amount of the underlying asset we should hold at time t in a portfolio which replicates the derivative.) This quantity is called the *Delta* of the derivative:

$$\Delta = \frac{\partial V}{\partial S_t}.$$

(iii) The expected rate of return μ on the underlying asset does not appear in the Black-Scholes equation (although the volatility σ and the riskfree interest rate r do): the two terms in $\mathrm{d}\Pi_t$ involving μ cancel out once we have made the correct choice of a. In particular, we are free to replace μ by r, so we can put ourselves in the risk-neutral situation (as we did in the Binomial Method).

(iv) As with any other differential equation, we need to impose suitable boundary conditions in order to obtain a specific solution to the Black-Scholes equation. This is where the difference between call and put options (or any other derivatives) arises, and where the exercise price E and the expiry time T enter the picture. Let us write $C(t, S)$ (respectively $P(t, S)$) for the value of a European call option (respectively, put option).

Since the Black-Scholes equation involves only the first derivative with respect to time, but the first and second derivatives with respect to the asset value S, we will need one boundary condition with respect to time (namely the value at the expiry time $t = T$) and two boundary conditions with respect to S (coming from the behaviour of the option at $S = 0$ and as $S \to \infty$).

For the call option, the expiry time condition is $C(T, S) = \max(S - E, 0)$ for all S. If the underlying asset becomes worthless, then it will remain worthless, so the option will also be worthless. Thus $C(t, 0) = 0$ for all t. On the other hand, if S becomes very large, then the option will almost certainly be exercised, and the exercise price is negligible compared to S. Thus the option will have essentially the same value as the underlying asset itself. The three boundary conditions for a European call option are therefore:

$$\begin{aligned} C(T, S) &= \max(S - E, 0) \quad \text{for all } S; \\ C(t, 0) &= 0 \quad \text{for all } t; \\ C(t, S) &\sim S \quad \text{as } S \to \infty, \text{ for fixed } t. \end{aligned}$$

The corresponding boundary conditions for a European put option can be deduced, for instance from the Put-Call Parity Formula:

$$\begin{aligned} P(T, S) &= \max(E - S, 0) \quad \text{for all } S; \\ P(t, 0) &= Ee^{-r(T-t)} \quad \text{for all } t; \\ P(t, S) &\to 0 \quad \text{as } S \to \infty, \text{ for fixed } t. \end{aligned}$$

Transformation of the Black-Scholes Equation

Problems involving partial differential equations frequently arise in the form of initial value problems, where the boundary conditions specify the state of some system at time $t = 0$, and we wish to find the behaviour of the system for $t > 0$. In contrast, the Black-Scholes equation (together with the boundary conditions we have just obtained) gives as a final value problem: we know the value $V(T, S)$ at the expiry time T and we wish to determine $V(t, S)$ for $0 \leq t < T$. We can transform our problem to the more familiar case of an initial value problem, and at the same time greatly simplify the Black-Scholes equation, by an appropriately chosen change of variables. For the moment, we concentrate on the PDE itself, without worrying about the precise form of the boundary conditions, but we suppose that we have in mind a fixed expiry time T at which V is known, and a fixed exercise price E.

We begin by replacing the two independent variables t, S by new independent variables τ, x defined by

$$t = T - \frac{2\tau}{\sigma^2}, \qquad S = Ee^x.$$

We also replace the dependent variable V by a new variable v where

$$V(t,S) = Ev(\tau, x).$$

We have scaled the variable prices S and V by the fixed price E, enabling us to work with the dimensionless quantities x and v, and we have "reversed time" so that we now have an initial value problem, where v is known at $\tau = 0$ and we are interested in the behaviour of v for $\tau > 0$. The exponential function in the equation defining x will eliminate the multipliers S and S^2 which occur in the Black-Scholes equation.

Since $x = \log(S/E)$ and $\tau = \frac{1}{2}\sigma^2(T-t)$, we have

$$\frac{\partial x}{\partial S} = \frac{1}{(S/E)} \times \frac{1}{E} = \frac{1}{S}, \qquad \frac{\partial x}{\partial t} = 0.$$

$$\frac{\partial \tau}{\partial t} = -\frac{1}{2}\sigma^2, \qquad \frac{\partial \tau}{\partial S} = 0.$$

Thus we have

$$\frac{\partial V}{\partial t} = E\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t}\right) = -\frac{1}{2}\sigma^2 E\frac{\partial v}{\partial \tau}$$

and

$$\frac{\partial V}{\partial S} = E\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial S}\right) = \frac{E}{S}\frac{\partial v}{\partial x}.$$

Differentiating again with respect to S, we obtain

$$\begin{aligned}
\frac{\partial^2 V}{\partial S^2} &= E\frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right) \\
&= E\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S}\frac{\partial^2 v}{\partial x^2}\frac{\partial x}{\partial S} + \frac{1}{S}\frac{\partial^2 v}{\partial \tau \partial x}\frac{\partial \tau}{\partial S}\right) \\
&= E\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2} + 0\right).
\end{aligned}$$

The Black-Scholes PDE (5) then becomes

$$-\frac{1}{2}\sigma^2 E\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 E\left(-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\right) + rS\frac{E}{S}\frac{\partial v}{\partial x} - rEv = 0.$$

Setting

$$k = \frac{2r}{\sigma^2},$$

this simplifies to

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv. \tag{6}$$

We have now managed to transform the Black-Scholes equation into a considerably simpler PDE, in which there is only one parameter k (in place of the two parameters r, σ), and in which the various derivatives occur with constant coefficients (instead of with the multipliers S and S^2 occurring in the original Black-Scholes equation). To simplify even further, we try writing

$$v(\tau, x) = e^{\alpha x+\beta\tau} u(\tau, x),$$

where u is the new dependent variable, and α, β are constants which we will choose later. Differentiating, we find

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= e^{\alpha x+\beta\tau}\left(\beta u + \frac{\partial u}{\partial \tau}\right); \\
\frac{\partial v}{\partial x} &= e^{\alpha x+\beta\tau}\left(\alpha u + \frac{\partial u}{\partial x}\right); \\
\frac{\partial^2 v}{\partial x^2} &= e^{\alpha x+\beta\tau}\left(\alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right).
\end{aligned}$$

Substituting into (6), and cancelling $e^{\alpha x+\beta\tau}$ throughout, we obtain

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku,$$

which simplifies to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (k-1+2\alpha)\frac{\partial u}{\partial x} + (\alpha^2 + \alpha k - \alpha - k - \beta)u.$$

We now choose α and β to eliminate the last two terms: take

$$\alpha = \frac{1-k}{2},$$

$$\beta = \alpha^2 + \alpha k - \alpha - k = \left(\frac{1-k}{2}\right)\left(\frac{1+k}{2}\right) - \left(\frac{1+k}{2}\right) = -\frac{1}{4}(1+k)^2.$$

The final result of this long sequence of transformations is that the Black-Scholes equation has been reduced to the much simpler PDE

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \tag{7}$$

Boundary Conditions

We now translate the boundary conditions for a European call or put option into conditions on the function u in (7). Solving (7) subject to these boundary conditions, and then reversing our transformations to express everything in terms of the original financial variables, we will obtain the explicit Black-Scholes formulae for the price of these options. As usual, the options have expiry time T and exercise price E. Recall that

$$V(t, S) = Ev(\tau, x) = Ee^{\alpha x+\beta\tau}u(\tau, x), \qquad t = T - \frac{2\tau}{\sigma^2}, \qquad S = Ee^x.$$

The price V of a call option satisfies

$$\begin{array}{rcl} V(T,S) &=& \max(S-E,0) \quad \text{for all } S; \\ V(t,0) &=& 0 \quad \text{for all } t; \\ V(t,S) &\sim& S \quad \text{as } S \to \infty, \text{ for fixed } t. \end{array}$$

Now $\max(S-E,0) = E\max(e^x-1,0)$, and as $\alpha = (1-k)/2$, the first condition becomes

$$u(0,x) = e^{-\alpha x}\max(e^x-1,0) = \max(e^{(k+1)x/2} - e^{(k-1)x/2}, 0). \tag{8}$$

The other two conditions become

$$u(\tau,x) \to 0 \text{ as } x \to -\infty, \qquad u(\tau,x) \sim e^{(k+1)x/2} \text{ as } x \to \infty. \tag{9}$$

Similarly, the boundary conditions for a European put option,

$$\begin{array}{rcl} V(T,S) &=& \max(E-S,0) \quad \text{for all } S; \\ V(t,0) &=& Ee^{-r(T-t)} \quad \text{for all } t; \\ V(t,S) &\to& 0 \quad \text{as } S \to \infty, \text{ for fixed } t; \end{array}$$

become

$$u(0,x) = \max(e^{(k-1)x/2} - e^{(k+1)x/2}, 0), \tag{10}$$

$$u(\tau,x) \to 0 \text{ as } x \to \infty, \qquad u(\tau,x) \sim e^{-(k+\beta)\tau} = e^{-(k-1)^2/4} \text{ as } x \to -\infty. \tag{11}$$

The Heat Equation

Equation (7) is a much-studied partial differential equation called the Heat Diffusion Equation, or simply the Heat Equation. The techniques developed by mathematicians over nearly two centuries to handle this equation can now be applied to investigate solutions of the Black-Scholes equation.

For the sake of motivation, we now briefly describe the physical meaning of (7). Consider an infinite metal rod. The points on the rod are specified by a coordinate x with $-\infty < x < \infty$. Let $u(\tau,x)$ denote the temperature at the point x at time τ, and suppose that we have some initial temperature distribution $u(0,x)$. As time progresses, heat will dissipate from warmer regions to cooler regions, so that the temperature distribution gradually evens out. The 2-variable function $u(\tau,x)$ describes this dissipation process. We would expect the flow of heat to be governed by the temperature gradient $\partial u/\partial x$: if the temperature gradient is constant near the point x, then as much heat flows into x (from the side which is hotter) as flows out from x (to the side which is cooler), so the temperature at x does not change. More generally, we would expect the *rate of change* (with respect to time) of the temperature at x to be proportional to the difference in the temperature gradients at points close to, but on opposite sides of, x. Thus, with appropriate choice of units, a physically plausible model for the evolution of the temperature distribution is the PDE

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

In addition to the initial temperature distribution $u(0, x)$, we also assume that no heat enters or leaves the rod, so that the total amount of heat in the rod is constant (i.e. does not vary with time). With appropriate scaling, we may assume this constant is 1. We therefore impose the condition

$$\int_{-\infty}^{\infty} u(\tau, x)\mathrm{d}x = 1. \qquad (12)$$

Fundamental Solution of the Heat Equation

We could at this point simply state the Black-Scholes formulae for European call and put options, and then verify that they do indeed satisfy the Black-Scholes PDE and the appropriate boundary conditions. Instead of this, however, we will see how solutions of the Heat Equation, and hence of the Black-Scholes equation, can be built up systematically. We begin by trying to solve the Heat Equation with a very special boundary condition: we consider the (physically impossible) situation where, at time 0, all the heat is concentrated at the point $x = 0$. Our boundary condition is then

$$u(0, x) = \delta(x), \qquad (13)$$

where δ is the Dirac delta function, defined by the requirements

$$\begin{aligned} \delta(x) &= 0 \text{ if } x \neq 0, \\ \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x &= 1. \end{aligned}$$

(We remark in passing that the Dirac delta function is not in fact a function in the strict mathematical sense of the word, since it is not determined simply by giving its value for each x. Instead, it is a *generalised function*, or *distribution*, which is determined by the effect of integrating it over the whole real line, after multiplication by a suitable *test function*.)

We will look for a solution of the Heat Equation (7) under the boundary conditions (12) and (13). The trick is to look for symmetry properties of the Heat Equation under scaling transformations. If we can find such a symmetry, we can use it reduce the number of independent variables from two to one, replacing the Partial Differential Equation by an Ordinary Differential Equation. Since the Heat Equation involves a second derivative with respect to x, but only a first derivative with respect to τ, we guess that τ should be scaled "twice as much" as x. More precisely, for a constant $\lambda \neq 0$, let

$$\tau^* = \lambda^2 \tau, \qquad x^* = \lambda x.$$

Then

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{\partial u}{\partial \tau^*}\frac{\partial \tau^*}{\partial \tau} = \lambda^2 \frac{\partial u}{\partial \tau^*}\\
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x*}\frac{\partial x^*}{\partial x} = \lambda \frac{\partial u}{\partial x^*}\\
\frac{\partial^2 u}{\partial x^2} &= \lambda^2 \frac{\partial^2 u}{\partial x^{*2}}.
\end{aligned}$$

Hence

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \Leftrightarrow \frac{\partial u}{\partial \tau^*} = \frac{\partial^2 u}{\partial x^{*2}}.$$

This suggests that we should try to find a solution of the Heat Equation depending only on the single independent variable

$$\xi = \frac{x}{\sqrt{\tau}} = \frac{x^*}{\sqrt{\tau^*}}.$$

First try: Write

$$u(\tau, x) = U(\xi). \tag{14}$$

As $\mathrm{d}x = \sqrt{\tau}\,\mathrm{d}\xi$, the heat conservation condition (12) becomes

$$1 = \int_{-\infty}^{\infty} u(\tau, x)\mathrm{d}x = \sqrt{\tau}\int_{-\infty}^{\infty} U(\xi)\mathrm{d}\xi.$$

This is no good: the right-hand side depends on τ, so we do not get a boundary condition on U which depends only on the new variable ξ. Thus (14) is not compatible with (12).

Second try: To get round this difficulty, we try instead

$$u(\tau, x) = \frac{1}{\sqrt{\tau}}U(\xi), \tag{15}$$

so that (12) now becomes

$$1 = \int_{-\infty}^{\infty} u(\tau, x)\mathrm{d}x = \sqrt{\tau}\int_{-\infty}^{\infty} (\sqrt{\tau})^{-1}U(\xi)\mathrm{d}\xi = \int_{-\infty}^{\infty} U(\xi)\mathrm{d}\xi. \tag{16}$$

This time the heat conservation condition does make sense as a boundary condition for U. We must translate the Heat Equation into an ordinary differential equation for U. Writing U' for $\mathrm{d}U/\mathrm{d}\xi$, we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{\sqrt{\tau}}U'\frac{\partial \xi}{\partial x} = \frac{1}{\tau}U';\\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{\tau}U''\frac{\partial \xi}{\partial x} = \tau^{-\frac{3}{2}}U'';\\
\frac{\partial u}{\partial \tau} &= -\frac{1}{2}\tau^{-\frac{3}{2}}U + \tau^{-\frac{1}{2}}U'\frac{\partial \xi}{\partial \tau}\\
&= -\frac{1}{2}\tau^{-\frac{3}{2}}U + \tau^{-\frac{1}{2}}\left(-\frac{1}{2}x\tau^{-\frac{3}{2}}\right)U'\\
&= -\frac{1}{2}\tau^{-\frac{3}{2}}(U + \xi U')\\
&= -\frac{1}{2}\tau^{-\frac{3}{2}}(\xi U)'
\end{aligned}$$

The Heat Equation then becomes

$$U'' = -\frac{1}{2}(\xi U)'.$$

Integrating once, we get

$$U' = -\frac{1}{2}\xi U + A$$

for some constant A. Multiplying by the integrating factor $e^{\xi^2/4}$, we have

$$(Ue^{\xi^2/4})' = \left(U' + \frac{1}{2}\xi U\right) e^{\xi^2/4} = Ae^{\xi^2/4},$$

so that

$$e^{\xi^2/4}U(\xi) = A\int_2^{\xi} e^{s^2/4}\mathrm{d}s + B,$$

or

$$U(\xi) = Ae^{-\xi^2/4}\int_2^{\xi} e^{s^2/4}\mathrm{d}s + Be^{-\xi^2/4},$$

for some constants A and B. (We can choose the lower limit on the integral arbitrarily: 2 turns out to be convenient for the calculation below). Now (16) cannot hold unless $A = 0$. Indeed, making the substitution $y = s^2/4$ and integrating by parts, we have

$$\int_2^{\xi} e^{s^2/4}\mathrm{d}s = \int_1^{\xi^2/4} \frac{e^y}{\sqrt{y}}\,\mathrm{d}y = \left[\frac{e^y}{\sqrt{y}}\right]_1^{\xi^2/4} + \frac{1}{2}\int_2^{\xi^2/4} y^{-3/2}e^y\mathrm{d}y.$$

Thus

$$e^{-\xi^2/4}\int_2^{\xi} e^{s^2/4}\mathrm{d}s = \frac{2}{\xi} - e^{1-\xi^2/4} + 2e^{-\xi^2/4}\int_2^{\xi} s^{-2}e^{s^2/4}\mathrm{d}s,$$

so that for large ξ we have

$$e^{-\xi^2/4}\int_2^{\xi} e^{s^2/4}\mathrm{d}s > \frac{1}{\xi}.$$

As

$$\int_1^N \xi^{-1}\mathrm{d}\xi = \log(N) \to \infty \quad \text{as } N \to \infty,$$

it follows that the integral

$$\int_{-\infty}^{\infty}\left(e^{-\xi^2/4}\int_2^{\xi} e^{s^2/4}\mathrm{d}s\right)\mathrm{d}\xi$$

cannot be finite. So for (16) to hold, we must take $A = 0$, as claimed. Then, since

$$\int_{-\infty}^{\infty} e^{-\xi^2/4}\mathrm{d}\xi = 2\sqrt{\pi},$$

we must also take $B = (2\sqrt{\pi})^{-1}$. Writing u_δ for the corresponding function in (15), we conclude that:

$$u_\delta(\tau, x) = \frac{1}{2\sqrt{\pi\tau}}e^{-x^2/4\tau}$$

is a solution of the Heat Equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

which satisfies the boundary conditions

$$u_\delta(0,x) = \delta(x) \text{ for all } x; \qquad \int_{-\infty}^{\infty} u(\tau,x)\mathrm{d}x = 1 \text{ for all } \tau.$$

It is also clear that for each $\tau \neq 0$ we have

$$u_\delta(\tau,x) \to 0 \text{ as } x \to \pm\infty. \tag{17}$$

Solution of the general initial value problem

We can use the fundamental solution $u_\delta(\tau,x)$ of the Heat Equation to build a solution satisfying any giving initial condition $u(0,x) = u_0(x)$. To do so, write the given function u_0 as a superposition of delta functions:

$$u_0(x) = \int_{-\infty}^{\infty} u_0(y)\delta(y-x)\mathrm{d}y.$$

By the linearity of the integration operator, the function

$$\begin{aligned} u(\tau,x) &= \int_{-\infty}^{\infty} u_0(y)u_\delta(\tau,y-x)\mathrm{d}y \\ &= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(y)e^{-(y-x)^2/4\tau}\mathrm{d}y \end{aligned}$$

is a solution of the Heat Equation satisfying the given initial condition. It might or might not be possible to simplify this integral analytically, depending on the function u_0.

Black-Scholes formula for the European Call Option

We now apply the method just described to solve the Heat Equation subject to the initial condition corresponding to the European call option. Here the initial value is

$$\begin{aligned} u_0(x) &= \max(e^{(k+1)x/2} - e^{(k-1)x/2}, 0) \\ &= \begin{cases} e^{(k+1)x/2} - e^{(k-1)x/2} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0. \end{cases} \end{aligned}$$

The solution should then be

$$\begin{aligned} u(\tau,x) &= \frac{1}{2\sqrt{\pi\tau}}\int_0^{\infty} (e^{(k+1)y/2} - e^{(k-1)y/2})e^{-(y-x)^2/4\tau}\mathrm{d}y \\ &= I_+ - I_-, \end{aligned}$$

where we have written

$$\begin{aligned} I_+ &= \frac{1}{2\sqrt{\pi\tau}}\int_0^{\infty} e^{(k+1)y/2}e^{-(y-x)^2/4\tau}\mathrm{d}y, \\ I_- &= \frac{1}{2\sqrt{\pi\tau}}\int_0^{\infty} e^{(k-1)y/2}e^{-(y-x)^2/4\tau}\mathrm{d}y. \end{aligned}$$

We evaluate I_+ in terms of the cumulative normal distribution function

$$N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\rho^2/2} \mathrm{d}\rho. \tag{18}$$

Note that by the symmetry of the normal distribution, we have

$$N(a) = 1 - N(-a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{\infty} e^{-\rho^2/2} \mathrm{d}\rho. \tag{19}$$

To relate I_+ to this function, we make a change of variable $z = (y-x)/\sqrt{2\tau}$, so that $\mathrm{d}y = \sqrt{2\tau}\,\mathrm{d}z$, and we then complete the square in the exponent:

$$\begin{aligned}
I_+ &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{(k+1)(\sqrt{2\tau}z+x)/2} e^{-z^2/2} \mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} e^{(k+1)x/2} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(z^2-(k+1)\sqrt{2\tau}z)} \mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} e^{(k+1)x/2} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}[z-(k+1)\sqrt{2\tau}/2]^2} e^{\frac{1}{2}\left(\frac{k+1}{2}\right)^2 2\tau} \mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} e^{(k+1)x/2} e^{(k+1)^2\tau/4} \int_{-x/\sqrt{2\tau}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\rho^2/2} \mathrm{d}\rho \\
&= e^{(k+1)x/2} e^{(k+1)^2\tau/4} N(d_1),
\end{aligned}$$

where in the last line we have used (19) and written

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}.$$

A similar calculation (replacing $k+1$ with $k-1$ throughout) gives

$$I_- = e^{(k-1)x/2} e^{(k-1)^2\tau/4} N(d_2)$$

with

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}.$$

Thus our solution is

$$u(\tau, x) = e^{(k+1)x/2} e^{(k+1)^2\tau/4} N(d_1) - e^{(k-1)x/2} e^{(k-1)^2\tau/4} N(d_2)$$

This determines the solution to the option pricing problem, but we must make the translation back into the financial variables. For the option price $V(t, S)$ we have

$$\begin{aligned}
V(t, S) &= Ev(\tau, x) \\
&= Ee^{\alpha x + \beta\tau} u(\tau, x)
\end{aligned}$$

with

$$\alpha = \frac{1-k}{2}, \quad \beta = -\frac{1}{4}(k+1)^2.$$

Thus our formula for $u(\tau, x)$ gives

$$V(t, S) = Ee^{x} N(d_1) - Ee^{-k\tau} N(d_2).$$

Now $S = Ee^x$ and $t = T - 2\tau/\sigma^2$, so also

$$k\tau = \frac{2r}{\sigma^2}(T-t)\frac{\sigma^2}{2} = r(T-t).$$

We therefore have

$$V(t,S) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{20}$$

with

$$\begin{aligned} d_1 &= \frac{x + \frac{1}{2}(2r/\sigma^2 + 1)2\tau}{\sqrt{2\tau}} \\ &= \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

and

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

Some Comments on the Black-Scholes formula

We now have an explicit solution (20) of the Black-Scholes equation, giving the price of a European call option at any time t prior to expiry. The formula expresses the option price in terms of the original financial variables, and involves nothing more complicated that the cumulative normal distribution function $N(a)$ (for which tables are readily available, and which is a built-in function on many pocket calculators).

We now make several observations about this formula:

(a) Verification of Boundary Conditions: We check that our formula does indeed satisfy the boundary conditions we found for a European call option.

Consider first the expiry-time condition $V(T,S) = \max(S-E, 0)$, which gave us our initial condition in terms of the reversed time variable τ. We cannot evaluate the formula (20) at $t = T$ directly, since finding d_1 and d_2 would involve division by 0. Instead, we investigate the limit of $V(t,S)$ as $t \to T$. (Remember that V is assumed to be smooth, and hence in particular V is continuous.) If $S > E$ then $\log(S/E) > 0$ and d_1, $d_1 \to \infty$ as $t \to T$. This means $N(d_1)$, $N(d_2) \to 1$, so that $V(t,S) \to S - E$. On the other hand, if $S < E$ then d_1, $d_2 \to -\infty$ as $t \to T$, so that $V(t,S) \to 0$. Combining the two cases, we have $V(t,S) \to \max(S-E,0)$ as $t \to T$, as required.

The other two boundary conditions are easier: as $S \to 0$ (with $t < T$ fixed) we have d_1, $d_2 \to -\infty$, and $V(t,S) \to 0$; and as $S \to \infty$ we have $V(t,S) \sim S - Ee^{-r(T-t)} \sim S$.

(b) Formula for the European Put Option: This can be obtained using the fundamental solution of the Heat Equation, by a similar calculation to that for the call option. However, since we already have a formula for the call option, it is easier to price the put option using the Put-Call Parity Formula

$$P(t,S) = C(t,S)Ee^{-r(T-t)} - S,$$

where $P(t,S)$ is the value of the put option, and $C(t,S)$, the value of the call option, is the quantity $V(t,S)$ in (20). This gives

$$P(t,S) = S(N(d_1) - 1) + Ee^{-r(T-t)}(1 - N(d_2)).$$

Using (19), this reduces to

$$P(t,S) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1). \tag{21}$$

Notice how similar this is to the formula (20) for a call option: the two terms are interchanged and d_1, d_2 are replaced by $-d_1$, $-d_2$ respectively.

(c) The Delta of a European Call Option: Recall that the Delta

$$\Delta = \frac{\partial V}{\partial S}$$

gives the (varying) quantity of the underlying asset to be held in the hedged portfolio. We can use the explicit Black-Scholes formulae to obtain an expression for the Delta, although some calculation is required. We do this now for the call option.

Differentiating the formula

$$V(t,S) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

with respect to S (and remembering that d_1, d_2 themselves depend on S), we obtain

$$\frac{\partial V}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}.$$

Since

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-s^2/2}\mathrm{d}s,$$

we have

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

so from

$$d_1,\, d_2 = \frac{\log(S/E) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

we find

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{\sigma\sqrt{T-t}} \times \frac{1}{S/E} \times \frac{1}{E} = \frac{1}{\sigma S\sqrt{T-t}}.$$

Thus

$$\frac{\partial V}{\partial S} = N(d_1) + \frac{1}{\sigma S\sqrt{T-t}}\left(SN'(d_1) - Ee^{-r(T-t)}N'(d_2)\right).$$

We claim that

$$SN'(d_1) = Ee^{-r(T-t)}N'(d_2) \tag{22}$$

so that the Delta of the option is given by the very simple formula:

$$\Delta = \frac{\partial V}{\partial S} = N(d_1).$$

To verify the claim (22) by direct calculation is rather messy, but we can avoid this by a trick. We have

$$d_1 + d_2 = \frac{2(\log(S/E) + r(T-t))}{\sigma\sqrt{T-t}}, \qquad d_1 - d_2 = \sigma\sqrt{T-t}.$$

Thus

$$d_1^2 - d_2^2 = (d_1 + d_2)(d_1 - d_2) = 2(\log(S/E) + r(T-t))$$

and

$$\begin{aligned}
\log\left(\frac{SN'(d_1)}{Ee^{-r(T-t)}N'(d_2)}\right) &= \log\left(\frac{Se^{-d_1^2/2}}{Ee^{-r(T-t)}e^{-d_2^2/2}}\right) \\
&= \log(S/E) - \frac{d_1^2}{2} + r(T-t) + \frac{d_2^2}{2} \\
&= \log(S/E) + r(T-t) - \frac{1}{2}(d_1^2 - d_2^2) \\
&= 0,
\end{aligned}$$

proving (22).

(d) Comparison with the Binomial Method: We obtained an explicit formula for the price of a European call option at time 0 under the Binomial Model:

$$C_0 = e^{-rT}\sum_{k=a}^{n}\binom{n}{k}p_*^k(1-p_*)^{n-k}(u^k d^{n-k}S_0 - E), \tag{23}$$

where

$$a = \left\lceil \frac{\log(E/S_0 d^n)}{\log(u/d)} \right\rceil.$$

Here the parameters p_* (the risk-neutral probability) and u, d (the multiplication factors for jumps up or down in the binomial tree) depend on the number of steps n.

As $n \to \infty$, the binomial model price C_0 should agree with the value given by the Black-Scholes formula (20) with $t = 0$, namely

$$C = S_0 N(d_1) - Ee^{-rT}N(d_2)$$

with

$$d_1,\, d_2 = \frac{\log(S_0/E) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

We will now indicate why this holds, without giving the calculations in full. Further details can be found in the paper of Cox, Ross and Rubenstein, *Option pricing: a simplified approach* (Journal of Financial Economics, 7 (1979), 229–263), in which the Binomial Model was introduced as a way of approximating the Black-Scholes formulae.

We split C_0 into two parts: $C_0 = C_0^{(1)} - C_0^{(2)}$, where

$$C_0^{(1)} = e^{-rT}\sum_{k=a}^{n}\binom{n}{k}p_*^k(1-p_*)^{n-k}u^k d^{n-k}S_0$$

and

$$C_0^{(2)} = Ee^{-rT} \sum_{k=a}^{n} \binom{n}{k} p_*^k (1-p_*)^{n-k}.$$

We first reinterpret $C_0^{(2)}$. Let the random variable H_n be the number of successes in n independent Bernouilli trials, where the probability of success in each trial is p. (In more concrete terms, H_n is the number of heads when a coin is tossed n times, the coin being biased so that it comes down heads with probability p each time.) As is well-known (and easily checked),

$$\mathbb{E}[H_n] = np, \qquad \mathrm{Var}[H_n] = np(1-p).$$

For a given integer b, write

$$\Phi(b, n, p) = \mathbb{P}[H_n \geq b] = \sum_{k=b}^{n} \binom{n}{k} p^k (1-p)^{n-k}.$$

By the Central Limit Theorem, we have

$$\frac{H_n - np}{\sqrt{np(1-p)}} \sim \mathcal{N}(0,1)$$

approximately, for n large, so that in terms of the cumulative normal distribution function N in (18), we have

$$\Phi(b, n, p) \approx 1 - N\left(\frac{b - np}{\sqrt{np(1-p)}}\right) = N\left(\frac{np - b}{\sqrt{np(1-p)}}\right).$$

Thus

$$C_0^{(2)} = Ee^{-rT}\Phi(a, n, p_*) \approx N\left(\frac{np_* - a}{\sqrt{np_*(1-p_*)}}\right)$$

for large n.

For $C_0^{(1)}$ we have

$$C_0^{(1)} = S_0 e^{-rT} \sum_{k=a}^{n} \binom{n}{k} (p_* u)^k ((1-p_*)d)^{n-k},$$

but since $(p_* u) + (1-p_*)d \neq 1$, we cannot use the same argument as above straight away. However, let λ and q be the solutions of

$$p_* u = \lambda q, \qquad (1-p_*)d = \lambda(1-q). \tag{24}$$

Then

$$C_0^{(1)} = S_0 e^{-rT} \lambda^n \sum_{k=a}^{n} \binom{n}{k} q^k (1-q)^{n-k} = S_0 e^{-rT} \lambda^n \Phi(a, n, q).$$

Now in the Binomial Model, the risk-neutral probability p_* is given by

$$p_* = \frac{e^{r\delta t} - d}{u - d},$$

where $\delta t = T/n$ is the timestep. It then follows from (24) that $\lambda = e^{r\delta t}$, so that $\lambda^n = e^{rT}$. Hence

$$C_0^{(1)} = S_0\Phi(a,n,q) \approx S_0 N\left(\frac{nq-a}{\sqrt{nq(1-q)}}\right)$$

for large n. By considering asymptotic expansions, one can check that

$$\frac{nq-a}{\sqrt{nq(1-q)}} \to d_1, \qquad \frac{np_*-a}{\sqrt{np_*(1-p_*)}} \to d_2 \qquad \text{as } n \to \infty.$$

Thus we find

$$C_0 = C_0^{(1)} - C_0^{(2)} \to S_0 N(d_1) - Ee^{-rT}N(d_2),$$

agreeing with the Black-Scholes formula.

(e) Validity of the formulae: In setting up the Black-Scholes equation and obtaining the explicit Black-Scholes formulae (20) and (21), we have made a number of financial assumptions. In particular, the No-Arbitrage Principle was the basis of the argument, but we also assumed that the price of the underlying asset follows a Geometric Brownian Motion, that the market is frictionless (there are no taxes, fees or other transaction costs associated with buying and selling), and that there is no restriction on short-selling, etc. These are of course somewhat idealised conditions, and cannot be expected to be a completely accurate description of the situation in real markets. Thus the Black-Scholes formulae should be thought of as approximations which ought to be valid when the effects of these neglected factors are not significant.

So far, we have only considered the Black-Scholes model for a rather restrictive range of financial instruments: namely European (rather than American) options on an underlying asset which does not pay dividends. Our next task is see how the Black-Scholes model applies to more complicated financial instruments.

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