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Accelerated Grids

Optimizing Solvers for Financial Partial Differential Equations

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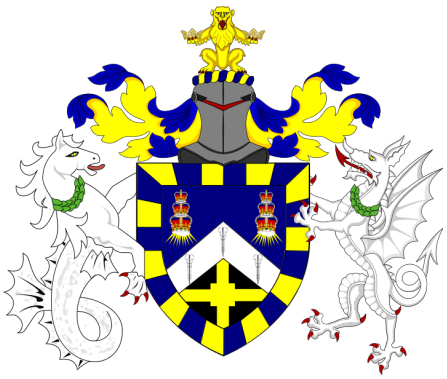

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Declaration of original work

This declaration is made on August 27, 2019.

Student's Declaration: I, Mustafa Berke Erdis, hereby declare that the work in this thesis is my original work. I have not copied from any other students' work, work of mine submitted elsewhere, or from any other sources except where due reference or acknowledgement is made explicitly in the text, nor has any part been written for me by another person.

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This work is dedicated to my family.

Acknowledgements

Here you thank people that have helped you in the journey.
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Abstract

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12th August 2019

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Chapter 1

Introduction

In Ancient Greece, Thales was scorned for his poverty. Later that year, Thales utilized his skills in astrology to forecast an increase in olive yields. Using his limited capital, he rented oil presses in winter. Months later, over the oil making season, many people rushed to the presses because of the high yields that Thales predicted. As he rented the presses over the winter, he forced the terms he pleased. Thales showed it was easy for philosophers to be rich if they chose it and practically used the first financial derivative product [1].

In the modern world, financial derivatives are contracts between two or more parties. The value of the contract depends on one or several underlying assets. Commonly the assets are currencies, equities, bonds, interest rates, market indices or commodities. The vanilla call option gives the right but not the obligation to buy the underlying asset at the expiry date at a previously agreed strike price. Essentially, Thales bought call options for oil presses. If the olive yields didn't come as Thales expected he didn't have the obligation to use the olive presses. On the other hand, the vanilla put option gives the right but not the obligation to sell the underlying asset at the expiry date at a previously agreed strike price. Practical applications of the options include hedging or speculating the future asset price. Hence, accurately pricing the options is crucial for an efficient and mature financial market. Merton and Scholes received the 1997 Nobel Prize in Economic Science for this work [21].

1.1 Motivation of the Project

Derivative pricing in the real world is a computationally intensive task. The existing numerical methods for partial differential equations are all constrained by the computational complexity. Being fast when evaluating new information is critical for the operations of hedge funds and investment banks. Therefore optimizing the existing numerical methods with hardware and software that can be installed on a trading floor is crucial. Goal of the project is to provide efficient methods for pricing options.

Purpose of this project to optimize numerical solutions of parabolic PDEs by testing tridiagonal system solvers, compilers and solution platforms. The idea of this project is to study how to take advantage of parallelism and explore how much faster we can make these calculations . Included in your Introduction section should be a clear summary of what you have achieved in the project work presented, such as any new results, generalisations, corollaries, examples, new connections, or computer investigations.

Chapter 2

Pricing Financial Derivatives

2.1 The Risk Neutral Approach

The Black-Scholes framework is a theoretical valuation formula for options. It reveals the relationship between the prices of the options and the underlying assets. Since almost all corporate liabilities can be viewed as combinations of options, the formula is applicable to common stocks and corporate bonds [3]. The Black-Scholes model makes the following assumptions:

- There does not exist any arbitrage opportunity in the financial market. The traders cant make instantaneous profit without any risk.
- The underlying asset value follows a geometric Brownian Motion $\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}B$ where μ denotes the average rate of growth of the underlying assets, σ denotes the volatility of the asset price and B is a Brownian Motion.
- The market is frictionless. This means there are no transaction fees, the interest rates for borrowing and lending money from and to the bank are the same, every party in market has immediate information and all entities are available at any time and in any size.

2.1.1 Black-Scholes Partial Differential Equation

The original model is used to price vanilla options, which is the simplest type of an option. The dividends can be included in the Black-Scholes formula. Presence of dividends can be included in the Black-Scholes formula. Since it doesn't effect the

performance, for the sake of simplicity we will assume there are no dividends paid. Under the assumptions of Black-Scholes framework, the call or put option price satisfies the parabolic partial differential equation.

$$\frac{\partial V}{\partial t} = rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \tag{2.1}$$

Framework shows the price $V(t, S)$ of a European option driven by one underlying asset that satisfies the PDE where r, σ, t, S respectively denotes the risk-free interest rate, volatility, time and the underlying price. It is assumed that r and σ are constants. In more complicated models such as stochastic volatility, they are modelled as a function. We will consider the PDE and conditions for the call options. In order to price a vanilla call option, the PDE needs to satisfy the following boundary and initial conditions.

$$C(0,t) = 0,\ C(S_{\max}, t) = S_{\max} - Ke^{-r(T-t)}, \quad 0 \le t \le T \tag{2.2}$$

$$C(S,T) = \max(S-K, 0), \quad 0 \le S \le S_{\max} \tag{2.3}$$

2.1.2 Derivation of the Black-Scholes Equation

Black-Scholes model takes advantage of the properties of the geometric Brownian motion and Itô's lemma.

Definition 2.1.1. Brownian Motion
Brownian motion (also known as Wiener Process) was discovered by botanist Robert Brown as he observed a chaotic motion of particles suspended in water [36]. A Brownian motion, $B(t)$, is a continuous-time stochastic process with the following properties:

- $B(0) = 0$.
- $B(t)$ is a continuous function of t.
- For $0 \le s < t$ the increment $B(t) - B(s)$ has normal distribution $\mathcal{N}(0, t-s)$.
- For $t_0 \le t_1 \le ... \le t_n$ the increments $B(t_k) - B(t_{k-1})$ where $k = 1, ..., n$ are independent random variables.

Brownian motion is the basic building block in stochastic calculus and geometric Brownian motion is used to model the stock prices in Black-Scholes model.

Lemma 2.1.2. *Itô's Lemma: Let $B(t)$ be a Brownian motion and $X(t)$ be an Ito process which satisfies the stochastic differential equation:*

$$\mathrm{d}X(t) = \mu(X(t), t)\mathrm{d}t + \sigma(X(t), t)\mathrm{d}B(t) \tag{2.4}$$

If $f(x, t)$ is twice continuously differentiable function then $f(X(t), t)$ is also an Ito drift-diffusion process [18], with its differential given by:

$$\mathrm{d}(f(X(t), t)) = \frac{\partial f}{\partial t}(X(t), t)\mathrm{d}t + f'(X(t), t)\mathrm{d}X + \frac{1}{2}f''(X(t), t)\mathrm{d}X(t)^2 \tag{2.5}$$

With $\mathrm{d}X(t)^2$ given by: $\mathrm{d}t^2 = 0$, $\mathrm{d}t\mathrm{d}B(t) = 0$ and $\mathrm{d}B(t)^2 = \mathrm{d}t$.

Theorem 2.1.3. *Assume that the asset price S follows a geometric Brownian motion. Under the assumptions of Black-Scholes framework, the call or put option price $V(t, S)$ satisfies the parabolic partial differential equation*

$$\frac{\partial V}{\partial t} = rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \tag{2.6}$$

Proof. Suppose an investor sets up a self-financing portfolio, $X(t)$, comprising one option and an Δ amount of the underlying asset. Therefore, value of the portfolio at time t is $X(t) = V(t) + \Delta S(t)$. Since the self-financing trading strategy has no capital influx or consumption, the value of portfolio change can be written as

$$\mathrm{d}X = \mathrm{d}V + \Delta \mathrm{d}S \tag{2.7}$$

Applying the Itô's Lemma to the option price V(t,S)

$$\mathrm{d}V = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}(S, t)\mathrm{d}S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(S, t)\mathrm{d}S^2 \tag{2.8}$$

Since the Black-Scholes model assumes that the stock price under the "market probability measure" follows a gBM.

$$\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}W \tag{2.9}$$

Putting 2.9 and 2.8 together yields

$$\mathrm{d}V = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \Delta\mu S)\mathrm{d}t + (\sigma S\frac{\partial V}{\partial S} + \Delta\sigma S)\mathrm{d}W \tag{2.10}$$

The fact that portfolio is risk-free implies that the second term involving the Brownian Motion, $\mathrm{d}W$, must be zero. This technique is known as delta-hedging , otherwise, we would have an arbitrage opportunity. Thus, $\Delta = -\frac{\partial V}{\partial S}$. Hence, the growth rate of the portfolio must be the risk free rate which can be summarized as $\mathrm{d}X = rX\mathrm{d}t$. Substituting Δ and $\mathrm{d}X$ yields

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = r(V - S\frac{\partial V}{\partial S}) \tag{2.11}$$

Rearranging the equation to get famous Black-Scholes equation:

$$\frac{\partial V}{\partial t} = rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV \tag{2.12}$$

□

Definition 2.1.4. The resulting partial differential equation can be solved analytically using the following boundary conditions and initial conditions for call options.

$$\frac{\partial C}{\partial t} = rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC \tag{2.13}$$

$$C(0,t) = 0,\ C(S_{\max},t) = S_{\max} - Ke^{-r(T-t)},\quad 0 \le t \le T \tag{2.14}$$

$$C(S,T) = \max(S-K,0),\quad 0 \le S \le S_{\max} \tag{2.15}$$

Theorem 2.1.5. *In order to solve the Black-Scholes equation analytically we need the Feynman - Kac Theorem [18].*

Suppose that x_t follows the process

$$\mathrm{d}x_t = \mu(x_t,t)\mathrm{d}t + \sigma(x_t,t)\mathrm{d}W_t^Q \tag{2.16}$$

.

Assume that there is a solution to the function $V(x_t,t)$ that follow the partial equation

$$\frac{\partial V}{\partial t} + \mu(x_t,t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(x_t,t)^2\frac{\partial^2 V}{\partial x^2} - r(t,x)V(x_t,t) = 0 \tag{2.17}$$

.

The solution to the function under the measure Q is

$$V(x_t, t) = E^Q[\exp(-\int_t^T r(X_u, u)\mathrm{d}u)V(X_T, T)|\mathcal{F}_t] \tag{2.18}$$

Theorem 2.1.6. *Solving the equations, the formulae [46] for European call is*

$$C = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{2.19}$$

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{2.20}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{2.21}$$

where $\Phi(x)$ denotes the cumulative normal distribution function.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \exp(-\frac{t^2}{2})\mathrm{d}t \tag{2.22}$$

Proof. Applying the Feynman - Kac formula 2.1.5 to the Black-Scholes equation with boundary conditions yields

$$C(S_t, t) = E^Q[e^{-\int_t^T r(X_u,u}\mathrm{d}u)C(S_T, T)|\mathcal{F}_t] \tag{2.23}$$

$$= e^{-r(T-t)}E^Q[(S_T - K)^+|\mathcal{F}_t] \tag{2.24}$$

Evaluating the expectation as integral

$$C(S_t, t) = e^{-r(T-t)}\int_K^\infty (S_T - K)\mathrm{d}F(S_t) \tag{2.25}$$

$$= e^{-r(T-t)}\int_K^\infty S_T\mathrm{d}F(S_t) - e^{-r(T-t)}\int_K^\infty K\mathrm{d}F(S_t) \tag{2.26}$$

As the stock price S follow log-normal distribution [15] the first integral becomes

$$\int_K^\infty S_T \mathrm{d}F(S_t) = E^Q[S_T|S_T > K] \tag{2.27}$$

$$= S_t e^{r(T-t)}\Phi(\frac{\log(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}) \tag{2.28}$$

$$= S_t e^{r(T-t)}\Phi(d_1) \tag{2.29}$$

Thus, second integral in 2.25 can be written as

$$e^{-r(T-t)}K\int_K^\infty \mathrm{d}F(S_t) = e^{-r(T-t)}K(1-F(K)) \tag{2.30}$$

$$= e^{-r(T-t)}K(1-\Phi(\frac{\ln(K/S_t) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}})) \tag{2.31}$$

$$= e^{-r(T-t)}K(1-\Phi(\sigma\sqrt{T-t} - d_1) \tag{2.32}$$

$$= e^{-r(T-t)}K\Phi(d_2) \tag{2.33}$$

Combining both integrals yield the analytical solution for the Black-Scholes equation with the given boundary conditions.

$$C = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{2.34}$$

□

The following parameters for Black-Scholes equation will be used for the research purposes of this project.

Parameter	Value
Strike Price (K)	1.0
Volatility (σ)	20 %
Risk Free Rate (r)	5 %
Time to Expiry (T)	2.0
Maximum Share Price (S_{max})	2.0

Definition 2.1.7. Black-Scholes PDE has coefficients that depend on the underlying price, S. Meaning that PDE is not space homogeneous. Log-spot Black-Scholes PDE is an economically intrinsic way of looking at numbers, if two assets are similar it is

conventional to investigate using the $x = lnS$ conversion. First step is to apply the chain rule to the first and second order derivatives $\frac{\partial C}{\partial S}$, $\frac{\partial^2 C}{\partial S^2}$.

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial S} = \frac{\partial C}{\partial x}\frac{1}{S} \tag{2.35}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}(\frac{\partial C}{\partial S}) = \frac{\partial}{\partial S}(\frac{\partial C}{\partial x}\frac{1}{S}) = -\frac{1}{S^2}\frac{\partial C}{\partial x} + \frac{\partial}{\partial S}\frac{\partial C}{\partial x}\frac{1}{S} = \tag{2.36}$$

$$= -\frac{1}{S^2}\frac{\partial C}{\partial x} + \frac{\partial^2 C}{\partial x^2}\frac{1}{S^2} \tag{2.37}$$

Substituting the transformed derivatives into the original PDE

$$\frac{\partial C}{\partial t} = (r - \frac{\sigma^2}{2})\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial x^2} - rC \tag{2.38}$$

The transformation creates a PDE with constant coefficients rather than coefficients that depend on S.

Remark 2.1.8. Untradable Assets
Modern financial engineering created derivatives using untradable assets as an underlying such as multi asset derivatives like equity baskets, weather derivatives, non-deliverable swaps and non-deliverable forwards. Non-deliverable forwards are for offshore investors that want to trade non-convertible currencies such as Brazilian Real, South Korean Won. The Black-Scholes model is still used in these cases [17] [5] but not entirely applicable to assets that cannot be hedged.

2.2 Partial Differential Equations

Since the foundation of the world humanity tried to understand and model the nature. Differential equations serves this purpose by enabling us to describe natural phenomena for instance, heat, sound and fluid flow. Differential equations can be classified in to two categories. Ordinary differential equations serve to model a movement space or plane, an example would be the trajectory of a projectile launched from a cannon follows a curve determined by an ordinary differential equation that is derived from Newton's second law.

On the other hand, partial differential equations modelles a function, a typical example is the heat distribution. This distinction usually makes PDEs much harder

to determine an analytical solution than ordinary differential equations. Therefore, we need to achieve a numerical solution to the problem. One of the most commong numerical method for partial differential equations is the finite difference methods. The methods consist of finding approximate solutions to the problem at a discrete set of points, normally on a rectangular grid of points. Finite difference methods are simple to construct and analyse but can compromise performance because of increased computational complexity when there are high dimensions.

Feynman-Kac theorem [18], establishes a link between partial differential equations and stochastic processes by writing the solution as a conditional expectation. Thanks to the theorem, Monte Carlo method is also utilized to find the numerical solutions to the partial differential equations. The convergence rate of Monte Carlo method for n simulations can be denoted as $\mathcal{O}(n^{\frac{-1}{2}})$ which holds for all dimensions (d). The error in d dimensional trapezoidal rule for twice continuously differentiable integrands is $\mathcal{O}(n^{\frac{-2}{d}})$ [13]. Thus, Monte Carlo is a method of choice when evaluating higher dimensions. In our calculations, we will test the case where $T = 0.06$ and $x_{max} = 1.0$.

2.2.1 Heat Equation

The heat equation is fundamental to financial engineering. Heat equation is a component in the Black-Schole equation and Black-Scholes equation can be transformed to the heat equation by changing variables [47]. Therefore, understanding heat equation is crucial to grasping concepts of partial differential equations. Heat equation will serve as a benchmark with the following initial and boundary conditions.

$$u_t(x,t) = u_{xx}(x,t) \tag{2.39}$$

$$u(0,t) = u(x_{max},t) = 0, \quad 0 \leq t \leq T \tag{2.40}$$

$$u(x,0) = \sin(\pi x), \quad 0 \leq x \leq x_{\max} \tag{2.41}$$

Heat equation $u(x,t)$ is a dissipative partial differential equation, describing the dissipation of heat in a region. Physically, as time progresses heat flows to cooler regions from the warmer regions. The temperature gradient, $\Delta u = u_x$, governs the heat flow. If there is an injustice distribution of heat at a certain point such that $\Delta u < 0$ or $\Delta u > 0$, heat flow will continue until there is no privilege and $\Delta u = 0$. This

phenomenon is also known as the tax system or robin hood principle.

Definition 2.2.1. Analytical Solution of Heat Equation

Certain kinds of partial differential equations allows us to find an analytical solution with the help of the Separation of Variables technique.

$$u(x,t) = X(x)T(t) \tag{2.42}$$

$$u_{xx}(x,t) = X^{''}(x)T(t) \tag{2.43}$$

$$u_t(x,t) = X(t)T^{'}(t) \tag{2.44}$$

Using the partial derivatives the equation $u_t = u_{xx}$ becomes

$$\frac{T^{'}(t)}{T(t)} = \frac{X^{''}(x)}{X(x)} \tag{2.45}$$

Right hand side only depends on x and the left hand side depends only on t. Therefore, the equation is valid only when each side is equal to a constant, which we set to λ. Rearranging terms gives us the following equations:

$$\frac{T^{'}(t)}{T(t)} = \frac{X^{''}(x)}{X(x)} = -\lambda \tag{2.46}$$

$$X^{''}(x) + \lambda X(x) = 0 \tag{2.47}$$

$$T^{'}(t) + \lambda T(t) = 0 \tag{2.48}$$

$$X(0) = X(1) = 0 \tag{2.49}$$

Solving for X(x) is an example case of Sturm-Liouville problem [20] with three cases.

- Let $\lambda < 0$ and $\lambda = -k^2$. Then the solution to 2.47 is

$$X = Ae^{kx} + Be^{-kx}$$

  Using the boundary conditions yield $X(0) = A+B = 0$ and $X(1) = Ae^{k}+Be^{-k} = 0$. Solving the equations $A = B = u = 0$ which is a trivial solution, thus

discarded.

- Let $\lambda = 0$, the solution to 2.47 is

$$X(x) = Ax + B$$

The boundary conditions imply $X(0) = B = 0$ and $X(1) = A = 0$. Thus this case is discarded too.

- Finally, let $\lambda > 0$,the solution to 2.47 is

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

The boundary conditions leads to $X(0) = A = 0$ and $X(1) = B\sin(\sqrt{\lambda}) = 0$. Since we do not want a trivial solution where $B = 0$, the equation reduces to

$$\sin(\sqrt{\lambda}) = 0 \tag{2.50}$$

Thus $\sqrt{\lambda} = n\pi$ for $n = 1, 2, 3, ...$ Solution to 2.47 becomes,

$$X_n = b_n \sin(n\pi x), \quad n = 1, 2, 3, ... \tag{2.51}$$

As we determined $\lambda = n^2\pi^2$ for $n = 1, 2, 3, ...$ Solving 2.48 for $T(t)$ gives the solution

$$T^{'}(t) = -n^2\pi^2 T(t) T_n = c_n \exp(-n^2\pi^2 t) \tag{2.52}$$

$$\tag{2.53}$$

where c_n's are integration constants.

Putting the solution of $T(t)$ and $X(x)$ together,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp(-n^2\pi^2 t)\sin(n\pi x) \tag{2.54}$$

where we have set $B_n = c_n b_n$. The initial condition gives

$$u(x,0) = \sin(\pi x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \tag{2.55}$$

which is a Fourier sine series. Solving for the B_ns, we use the orthogonality property for the eigenfunctions $\sin(n\pi x)$.

$$\int_0^1 \sin(m\pi x)\sin(\pi n x)\mathrm{d}x = \begin{cases} 0, & \text{if } m \neq n \\ 1/2, & \text{m = n} \end{cases} = 0.5\delta_{mn}$$

where δ_{mn} is the kronecker delta,

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{m = n} \end{cases}$$

Solving 2.55 for B_n, multiplying both sides with $\sin(m\pi x)$ and integrate from 0 to 1 and from the definition of kronecker delta yields

$$B_n = 2\int_0^1 \sin(\pi x)\sin(\pi n x)\mathrm{d}x = \frac{2\sin(\pi n)}{\pi - \pi n^2} \tag{2.56}$$

Combining the solutions

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2\sin(\pi n)}{\pi - \pi n^2} \exp(-n^2\pi^2 t)\sin(n\pi x) = \exp(-\pi^2 t)\sin(\pi x) \tag{2.57}$$

2.2.2 Two Dimensional Heat Equation

The natural extension of our study of the one-dimensional problem would now be to investigate partial differential equations with more than one space-like dimension. When more than one space dimensions are involved, we have to deal with equations such as two dimensional heat equation or multi-asset black-scholes equation. We will consider the following PDE and conditions for the purposes of research.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{2.58}$$

Initial and boundary condition

$$u(x,y,0) = 1, \quad 0 \le x \le x_{\max}, \quad 0 \le y \le y_{\max} \tag{2.59}$$

$$u(x,0,t) = u(x,y_{max},t) = 0, \quad 0 \le t \le T \tag{2.60}$$

$$u(0,y,t) = u(x_{max},y,t) = 0, \quad 0 \le t \le T \tag{2.61}$$

In the calculations, we will test the case where $T = 0.06$, $x_{max} = 1.0$ and $y_{max} = 1.0$.

Definition 2.2.2. Analytical Solution of Two Dimensional Heat Equation Similarly, applying separation of variables method to the equation

$$u(x,t) = X(x)Y(y)T(t) \tag{2.62}$$

$$X^{''}(x) - BX(x) = 0 \tag{2.63}$$

$$Y^{''}(y) - C(y) = 0 \tag{2.64}$$

$$T^{'}(t) - (B+C)T(t) = 0 \tag{2.65}$$

$$X(0) = X(1) = 0, \quad Y(0) = Y(1) = 0 \tag{2.66}$$

In 2.2.1, we have already seen that the solutions to X(x) and Y(y) are

$$X_m(x) = b_n \sin(m\pi x) \tag{2.67}$$

$$Y_n(x) = a_m \sin(n\pi y) \tag{2.68}$$

Using these values to solve for $T(t)$ gives

$$T_{mn}(t) = c_{mn} \exp(-\pi^2(m^2+n^2)t) \tag{2.69}$$

Substituting the solutions yields

$$u_{mn}(x,y,t) = X_m(x)Y_n(y)T_{mn}(t) \tag{2.70}$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin(m\pi x)\sin(n\pi y)\exp(-\pi^2(m^2+n^2)t) \tag{2.71}$$

where $A_{mn} = b_n a_m c_{mn}$. The initial condition gives

$$u(x, y, 0) = 1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi x) \sin(n\pi y) \tag{2.72}$$

which is a double Fourier sine series. Thus, the coefficient A_{mn} is chosen such that

$$A_{mn} = 4 \int_0^1 \int_0^1 \sin(\pi m x) \sin(\pi n y) \mathrm{d}x \mathrm{d}y = \frac{4(\cos(\pi n) - 1)(\cos(\pi m) - 1)}{\pi^2 mn} \tag{2.73}$$

Combining the solutions

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4(\cos(\pi n) - 1)(\cos(\pi m) - 1)}{\pi^2 mn} \sin(m\pi x) \sin(n\pi y) e^{-\pi^2(m^2+n^2)t} \tag{2.74}$$

2.3 Finite Difference Methods

2.3.1 Discretization

Essentially, solving a PDE is the problem of finding a function which depends on values at infinitely many points. Naturally, the finite difference methods first step is to make the problem discrete that we are able to solve [41]. As a result, we need to discretise the space dimensions and time dimension. The discretization procedure begins by replacing the domain $[0, x_{max}]$ x $[0, T]$ by a set of mesh points. In order to get a n x m equally spaced mesh points the step sizes are calculated as $\Delta t = \frac{T}{m}$, $\Delta x = \frac{x_{max}}{n}$.

In order to replace our PDE, we need to utilize finite difference approximations for the partial derivatives. Notationally, we will define u_i^n to be a function defined at the point $(i\Delta x, n\Delta t)$.

- Forward difference: $\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + \mathcal{O}(\Delta t)$
- Central difference: $\frac{\partial u}{\partial x} = \frac{u_{i+1}^n - u_{i-1}^n}{\Delta t} + \mathcal{O}(\Delta x)$
- Backwards difference: $\frac{\partial u}{\partial x} = \frac{u_i^n - u_{i-1}^n}{\Delta t} + \mathcal{O}(\Delta x)$
- Second order central difference: $\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + \mathcal{O}(\Delta x^2)$

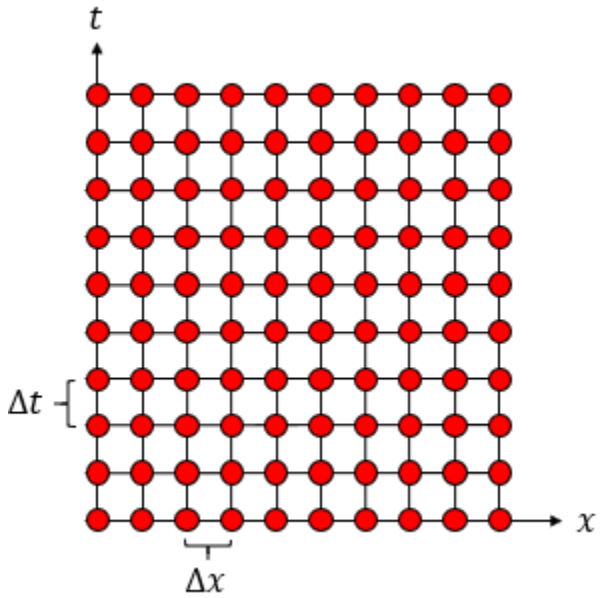

Figure 2.1: 10 x 10 grid.

We now have a grid that approximates our domain. Aiming to obtain a unique solution using numerical methods, we need initial and boundary conditions. Final step is applying the values given by such conditions.

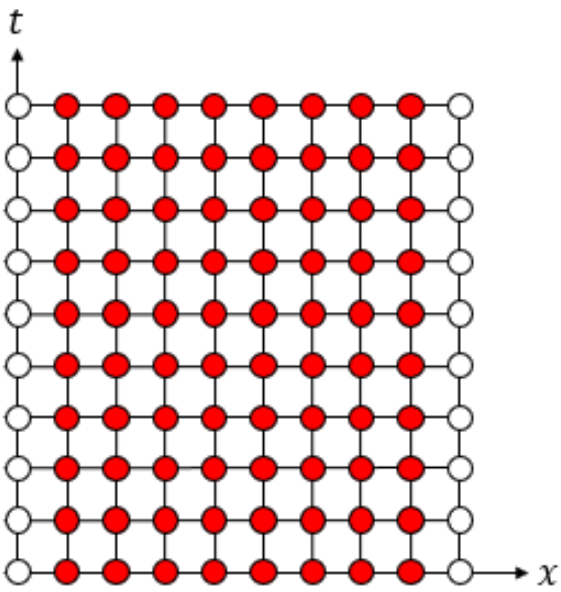

Figure 2.2: Boundary conditions.

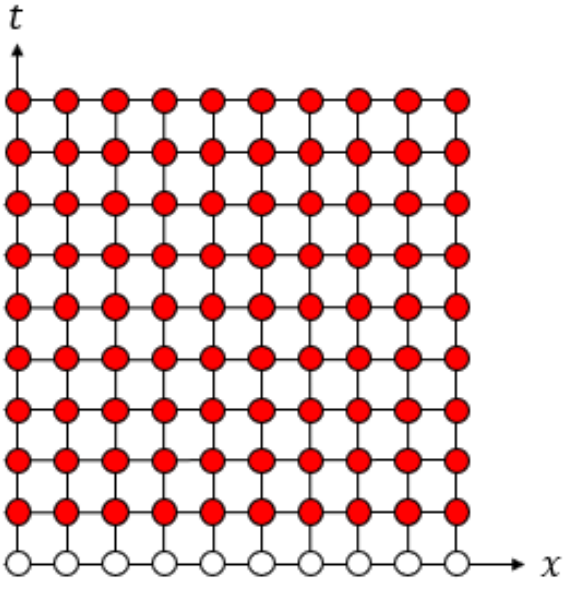

Figure 2.3: Initial condition.

2.3.2 Explicit Method

Explicit method generalises the parabolic partial differential equation by applying the forward difference to the time derivative and the centred second difference (FTCS scheme).

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u \tag{2.75}$$

We will be applying the finite differences to the equation 2.75 for the purposes of simplicity since heat equation and Black-Scholes equation can be generalized in the form for certain choices of coefficients. Applying the forward time and centred space differences where $r = \frac{\Delta t}{\Delta x^2}$.

$$\begin{aligned}
\frac{u_i^{n+1} - u_i^n}{\Delta t} &= a(x,t)\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + b(x,t)\frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} + c(x,t)u_i^n \\
u_i^{n+1} &= u_{i+1}^n(\frac{-rb(x,t)}{\Delta x} - ra(x,t)) \\
&+ u_i^n(1 + 2ra(x,t) - c(x,t)\Delta t) \\
&+ u_{i-1}^n(-ra(x,t) + \frac{rb(x,t)}{\Delta x})
\end{aligned}$$

We will replace the coefficients of $u_{i+1}^n, u_i^n, u_{i-1}^n$ terms with γ, α, β respectively. The formula reduces to

$$u_j^{n+1} = \gamma u_{j+1}^n + \beta u_j^n + \alpha u_{j-1}^n. \tag{2.76}$$

The formula expresses one unknown nodal value directly in terms of known nodal values [11]. It can be expanded as

$$\begin{aligned}
u_1^{n+1} &= \gamma u_2^n + \beta u_1^n + \alpha u_0^n \\
u_2^{n+1} &= \gamma u_3^n + \beta u_2^n + \alpha u_1^n \\
&\vdots \\
u_{j-1}^{n+1} &= \gamma u_j^n + \beta u_{j-1}^n + \alpha u_{j-2}^n
\end{aligned} \tag{2.77}$$

Using the boundary conditions and initial condition, the expanded formula can be

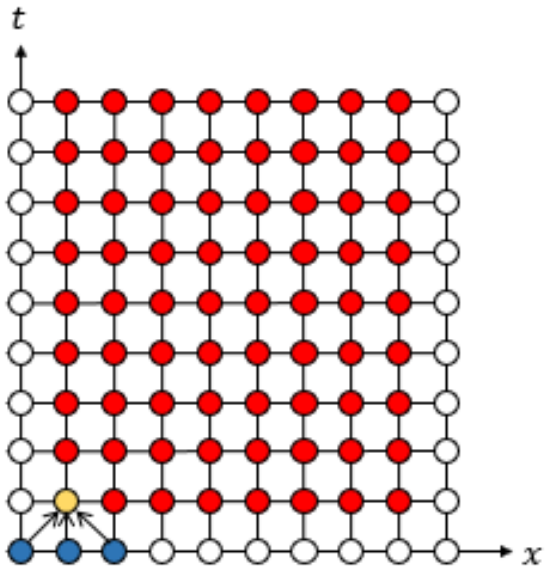

Figure 2.4: Computational stencil of heat equation.

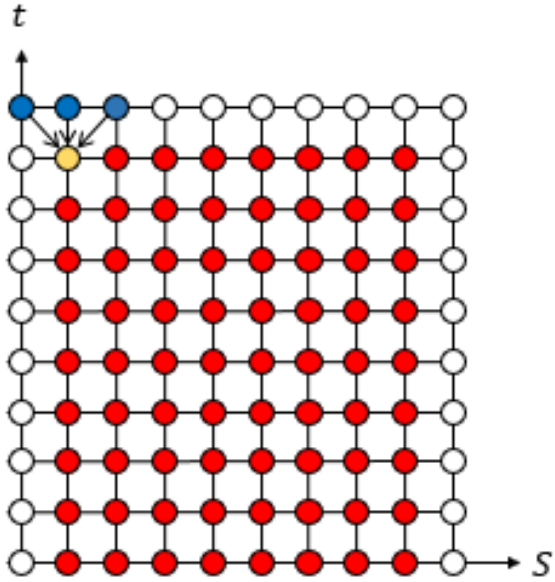

Figure 2.5: Computational stencil of Black-Scholes equation.

condensed in the following matrix form.

$$\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ . \\ . \\ . \\ u_{j-1}^{n+1} \end{bmatrix} = \begin{bmatrix} \alpha u_0^n \\ 0 \\ 0 \\ . \\ . \\ . \\ \gamma u_j^n \end{bmatrix} + \begin{bmatrix} \beta & \gamma & 0 & . & . & 0 \\ \alpha & \beta & \gamma & 0 & ... & . \\ 0 & \alpha & \beta & \gamma & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & \gamma \\ 0 & 0 & 0 & 0 & \alpha & \beta \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \\ u_3^n \\ . \\ . \\ . \\ u_{j-1}^n \end{bmatrix}$$

Deriving the coefficients γ, α, β in the case of heat equation yields

$$\alpha = r \quad \beta = 1 - 2r \quad \gamma = r \tag{2.78}$$

In the case of Black-Scholes formula, since the share price S_j increases linearly with Δx we can replace it as $S_j = j\Delta x$.

$$\alpha = \frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2} \quad \beta = 1 - \sigma^2 j^2 \Delta t - r\Delta t \quad \gamma = \frac{\sigma^2 j^2 \Delta t}{2} + \frac{rj\Delta t}{2} \tag{2.79}$$

Lastly, solving heat equation and Black-Scholes differs in time stepping. Black-Scholes formula is solved backwards in time. On the other hand, heat equation is solved forwards in time.

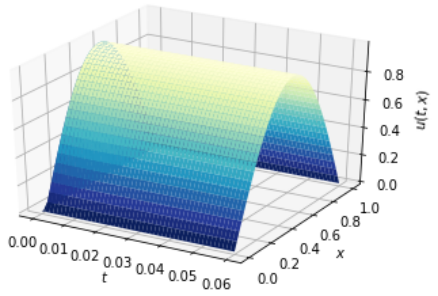

Figure 2.6: Output grid of heat equation using explicit scheme.

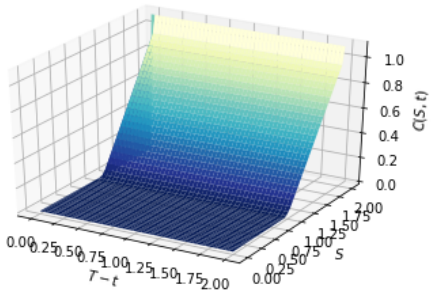

Figure 2.7: Output grid of Black-Scholes equation using explicit scheme.

2.3.3 Crank - Nicolson Method

The explicit method is computationally cheap. However, this brings a serious drawback, for explicit method to attain reasonable accuracy the step size must be kept small [37]. Thankfully, the Crank-Nicolson finite difference scheme was introduced by John Crank and Phyllis Nicolson [7]. Considering numerous articles and publications in the financial engineering literature use Crank-Nicolson as the de-facto scheme for time discretisation, the method has become one of the most popular finite difference schemes for approximating the solution of the Black - Scholes equation and its generalisations [38]. If we apply backwards time difference instead of forward time difference that Explicit method used and a central space approximation in space again, we get the BTCS scheme. Applying the BTCS to the base equation 2.75 yields

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a(x,t)\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + b(x,t)\frac{u_i^{n+1} - u_i^{n+1}}{\Delta t} + c(x,t)u_i^{n+1} \tag{2.80}$$

Crank - Nicolson method takes a weighted average of the FTCS and BTCS schemes.

Therefore, the approximations become

$$u(x,t) \approx \frac{1}{2}(u_i^{n+1} + u_i^n) \tag{2.81}$$

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} \tag{2.82}$$

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}^n - u_{i-1}^n + u_{i+1}^{n+1} - u_{i-1}^{n+1}}{4\Delta x} \tag{2.83}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n + u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2(\Delta x)^2} \tag{2.84}$$

Applying the new finite differences to the base partial differential equation equation 2.75 yields

$$(-A-B)u_{i+1}^{n+1}+(1+2A-C)u_i^{n+1}+(-A+B)u_{i-1}^{n+1} = (A+B)u_{i+1}^n+(1-2A+C)u_i^n+(A-B)u_{i-1}^n \tag{2.85}$$

where $A = a(x,t)\frac{\Delta t}{\Delta x^2}$, $B = b(x,t)\frac{\Delta t}{4\Delta x}$, $C = c(x,t)\frac{\Delta t}{2}$. Note that in contrast to the FTCS scheme, we now have three unknowns in this equation, the three values of u at the higher time level. We respectively denote the coefficients in the right hand side as γ, β, α and coefficients in the left hand side as λ, θ, ω for simplicity.

$$\lambda u_{i+1}^{n+1} + \theta u_i^{n+1} + \omega u_{i-1}^{n+1} = \gamma u_{i+1}^n + \beta u_i^n + \alpha u_{i-1}^n \tag{2.86}$$

The left hand side groups the unknowns and the right hand side groups knowns. The system of equations can be reduced to a matrix system.

$$\begin{bmatrix} \theta & \lambda & 0 & . & . & 0 \\ \omega & \theta & \lambda & 0 & ... & . \\ 0 & \omega & \theta & \lambda & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & \lambda \\ 0 & 0 & 0 & 0 & \omega & \theta \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ . \\ . \\ . \\ u_{j-1}^{n+1} \end{bmatrix} = \begin{bmatrix} \alpha u_0^n \\ 0 \\ 0 \\ . \\ . \\ . \\ \gamma u_j^n \end{bmatrix} + \begin{bmatrix} \beta & \gamma & 0 & . & . & 0 \\ \alpha & \beta & \gamma & 0 & ... & . \\ 0 & \alpha & \beta & \gamma & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & \gamma \\ 0 & 0 & 0 & 0 & \alpha & \beta \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \\ u_3^n \\ . \\ . \\ . \\ u_{j-1}^n \end{bmatrix}$$

The problem reduces to a tridiagonal matrix system. This system of equations can be solved by various algorithms such as Gaussian elimination or Thomas algorithm.

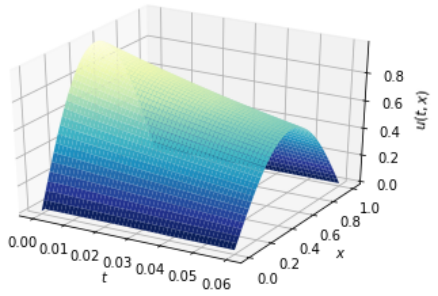

Figure 2.8: Output grid of heat equation using Crank - Nicolson scheme.

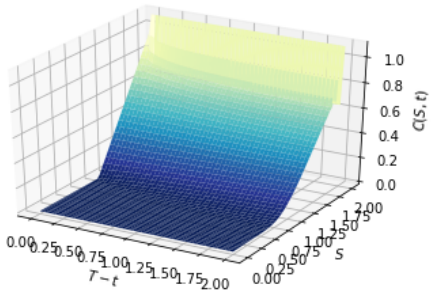

Figure 2.9: Output grid of Black-Scholes equation using Crank - Nicolson scheme.

2.3.4 Alternating Direction Implicit Method

Curse of dimensionality effects finite difference schemes as they tend to become more difficult to set up, understand and implement as the dimensionality of the space increases [9]. The alternating direction implicit (ADI) method is one of the most common techniques to numerically solve two-dimensional parabolic PDEs. The scheme was first proposed by Peaceman and Rachford in 1955 for oil reservoir modelling [33]. The method consists of splitting the time dimension and solving the two-dimensional problem as two consecutive one-dimensional problems. At each time step, the spatial dimensions are solved implicitly in one direction and explicitly in the other dimension. Using the alternating direction implicit scheme provides us with the advantages of implicit method. Computationally requires only solving tridiagonal systems. It is possible to use ADI in more than three dimensions which produce the same number of consecutive one-dimensional problems [8]. To develop a more compact notation, we introduce the finite difference operator notation δ^2.

$$\delta x^2 u_{i,j}^n = \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} \tag{2.87}$$

Explicit method in two space dimensions can be abbreviated as

$$\frac{u_{i,j}^{n+1} + u_{i,j}^n}{\Delta t} = \delta x^2 u_{i,j}^n + \delta y^2 u_{i,j}^n \tag{2.88}$$

and implicit method in two space dimensions can be written as

$$\frac{u_{i,j}^{n+1} + u_{i,j}^{n}}{\Delta t} = \delta x^2 u_{i,j}^{n+1} + \delta y^2 u_{i,j}^{n+1}. \tag{2.89}$$

Dividing each time step in half we introduce a temporary intermediate unknown $u_{i,j}^{n+1/2}$. Firstly, the two dimensional heat equation is approximating implicitly x and explicitly over y. The total work involved in one time step amounts to solving $N_{steps} - 1$ tridiagonal systems [30].

$$\frac{u_{i,j}^{n+1/2} + u_{i,j}^{n}}{0.5\Delta t} = \frac{\delta x^2 u_{i,j}^{n+1/2}}{\Delta x^2} + \frac{\delta y^2 u_{i,j}^{n}}{\Delta y^2} \tag{2.90}$$

Rearranging the set of equations yields a tridiagonal system which is solved for the temporary intermediate unknown $u_{i,j}^{n+1/2}$.

$$-r_x u_{i+1,j}^{n+1/2} + (1 + 2r_x) u_{i,j}^{n+1/2} - r_x u_{i,j}^{n+1/2} = r_y u_{i,j+1}^{n} + (1 + 2r_y) u_{i,j}^{n} + r_y u_{i,j-1}^{n} \tag{2.91}$$

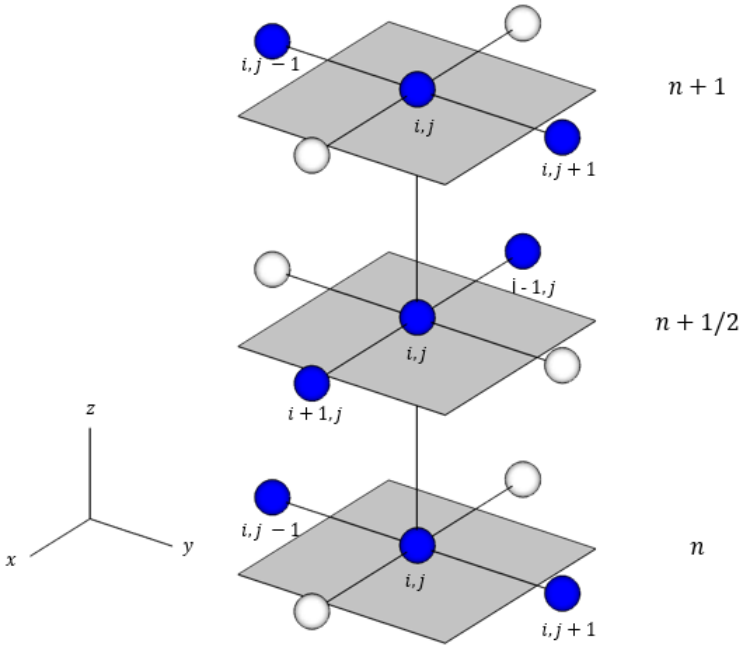

Figure 2.10: Computational stencil of alternating direction implicit method.

Next step of the grid $u_{i,j}^{n+1}$ is calculated by approximating explicitly x and implicitly

over y.

$$\frac{u_{i,j}^{n+1} + u_{i,j}^{n+1/2}}{0.5\Delta t} = \frac{\delta x^2 u_{i,j}^{n+1/2}}{\Delta x^2} + \frac{\delta y^2 u_{i,j}^{n+1}}{\Delta y^2} \tag{2.92}$$

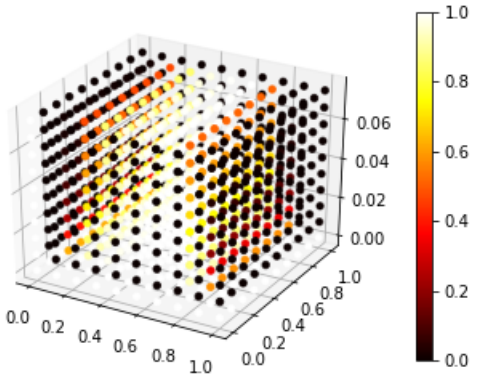

Figure 2.11: Output grid of two dimensional heat equation using ADI.

Rearranging the set of equations yields a tridiagonal system which can be solved using Gaussian elimination, cyclic reduction or Thomas algorithm.

$$-r_y u_{i,j+1}^{n+1} + (1+2r_y)u_{i,j}^{n+1} - r_y u_{i,j-1}^{n+1} = r_x * u_{i+1,j}^{n+1/2} + (1+2r_x)u_{i,j}^{n+1/2} + r_x u_{i,j}^{n+1/2} \tag{2.93}$$

Chapter 3

Optimizing Solvers

Attempting to progress in solving complex problems using numerical methods is impossible without applying optimizations. However, nowadays the problem is not to solve a given problem but rather solve it in a given computing environment while exploiting the resources in an optimal way. Thus, it is necessary to investigate methods that allow for efficient implementations. The aim of this section is to introduce practical optimization techniques that can be easily implemented on a regular trading floor. Main optimization techniques that will be tested are parallelizing tridiagonal solvers, Visual Studio optimization switches, compilers and solution platforms.

3.1 Solution Platforms

The CPU accesses data from RAM using the register that stores memory addresses. 32 bit and 64 bit refers to the amount of data the system can access. so a 32-bit system can address a maximum of 4 GB (4,294,967,296 bytes) of RAM where a 64-bit register can theoretically reference 18,446,744,073,709,551,616 bytes, or 17,179,869,184 GB (16 exabytes) of memory. Since 32 bit does not have access to more than 4 GB, if the system has more than 4 GB of RAM, it will be inaccesible by the CPU, thus A 64 bit system will be needed. The memory increase of 64 bit systems means it is capable of very fast processing of numerical quantities. One disadvantage of the 64 bit systems is more requirement of memory because addresses are 64 bits (8 bytes) wide instead of 32 bits (4 bytes) wide. Due to the increased size of pointers and data structures, 64-bit programs will occupy more memory than an 32-bit version. Visual Studio offers the the x86 and x64 solution platforms which corresponds to 32-bit and

64-bit respectively. The solution platforms will be tested against to determine the optimal solution platform.

3.2 Compilers

The software we write is translated into low level abstractions by a compiler. The quality of the translation plays a crucial role in how the software performs. Commonly, compilers are comprises of three stages, front end, optimization and the back end. First step is to understand the source code translate it into intermediate representation using data structures and formal language theory. The intermediate representation generated in the front end is later used by the back end. In the middle, optimizer is focused on efficiency. It transforms the intermediate representation by deriving knowledge about runtime behaviour and improve the behaviour. Finally, the back end map the functionality to the instruction set of the processor [43].

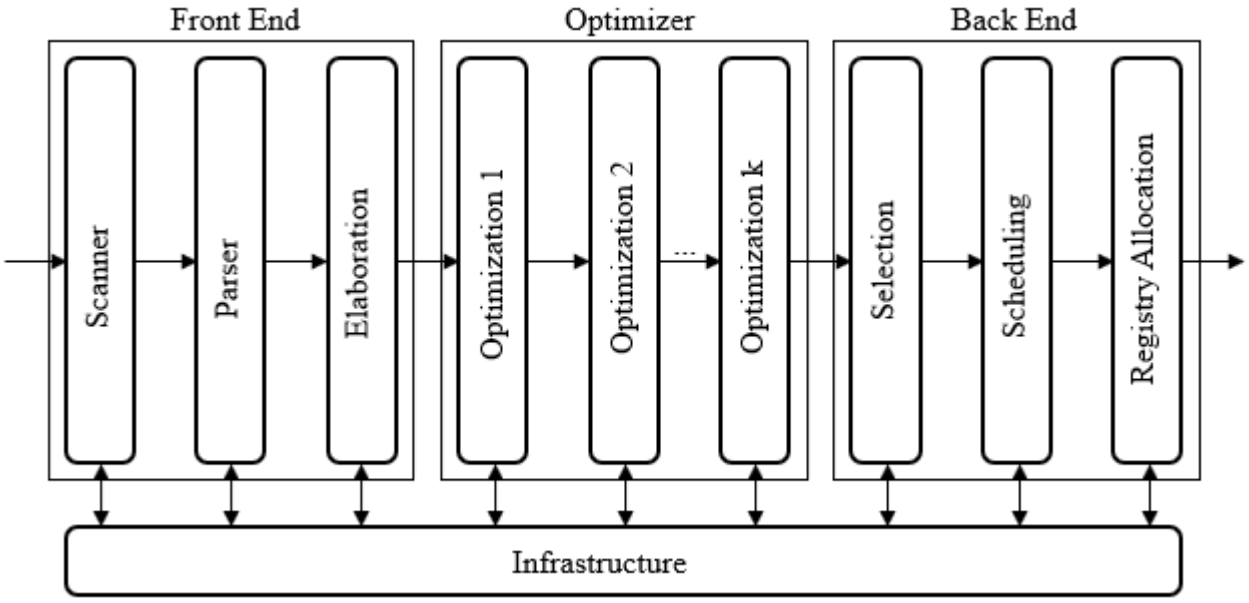

Figure 3.1: Basic structure of a compiler.

During the back end stage, the compiler approximates the allocation and scheduling. The speed and size of the code is a direct result of the ability to approximate correctly. This produces complex interactions that can lead to problematic results. Therefore, regardless of the implementation of the program, the performance can be different under varying compilers and they are an important factor for time constrained

tasks such as option pricing. Well designed and implemented compilers savings accumulate over time. It should be able to produce well-optimized code and let us focus on the process of writing programs rather than struggling with the inadequacies of the compiler. In this project Visual C++ and Intel C++ compiler will be tested against each other.

3.3 Visual Studio Optimization Switches

Visual Studio Optimization Switches, also known as /O options controls various optimizations to be chosen according to the needs of the project. There are various switches for different goals such as minimizing the size of the code (/O1) but since the scope of this project is limited with speed optimizations. Speed optimization flags are /O2 and /Ox. /O2 is a combination of /Og, /Oi, /Ot, /Oy, /Ob2, /GF and /Gy flags. /Ox is a subset of /O2 without the /GF and /Gy flags. These additional options applied by /O2 can cause pointers to strings or to functions to share a target address, which can affect debugging and strict language conformance [27].

- /Og: Enables local and global optimizations (subexpression elimination), automatic-register allocation, and loop optimization [28].
- /Oi: Generates intrinsic functions for appropriate function calls. Compiler may not replace the function call with an intrinsic if it will result in better performance [25].
- /Ot: Favors optimizations for speed over optimizations for size by instructing the compiler to reduce many C and C++ constructs to functionally similar sequences of machine code. If /Ot is used, /Og must be specified to optimize the code [26].
- /Oy: Suppresses the creation of frame (base) pointers on the call stack for quicker function calls. Frees one register for general usage [29].
- /Ob2: Controls inline expansion of functions. Under /O2 and /Ox, allows the compiler to expand any function including the ones that are not explicitly marked for no inlining. Function-calling-overheads are saved thus inline functions run faster than the normal functions with a memory penalty [24].

- /GF: Enables the compiler to create a single copy of identical strings in the program image and in memory during execution. This is an optimization called string pooling that can create smaller programs. Under this flag, strings are pooled as read-only, trying to modify strings throws an error [22].

- /Gy: Enables function-level linking. Allows the compiler to package individual functions in the form of packaged functions (COMDATs) or order individual functions in a DLL or .exe file [23].

/O2 and /Ox flags are tested for maximum speed against the /Od flag which disables all the optimizations.

3.4 Tridiagonal Solvers

Tridiagonal solvers are the most demanding part of the solvers. Hence, development and improvement of such solvers is of great interest [39] [6] [31] [2] concerned with this problem. Large tridiagonal systems appear in many numerical analysis applications. In our work, they arise in the Crank-Nicolson and Alternating Direction Implicit schemes. Solving tridiagonal systems is the most computationally intensive part of the schemes. Therefore, choosing efficient tridiagonal solvers is crucial for the speed of the solver. In this experiment two implementations of Thomas Algorithm and Cyclic Reduction will be tested.

3.4.1 Thomas Algorithm

Thomas Algorithm is the most commonly used method for solving tridiagonal system of equations. The method is used to solve a tridiagonal matrix system invented by Llewellyn Thomas [42]. The algorithm is equivalent to Gaussian elimination without pivoting.

The system equations can be written as

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & ... & 0 \\ a_2 & b_2 & c_2 & 0 & ... & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & c_{k-1} \\ 0 & 0 & 0 & 0 & a_k & b_k \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ . \\ . \\ . \\ f_k \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ . \\ . \\ d_k \end{bmatrix}$$

The method begins by calculating coefficients c_i^* and d_i^* replacing a_i, b_i and c_i [10].

$$c_i^* = \begin{cases} \frac{c_1}{b_1} & ; i = 1 \\ \frac{c_i}{b_i - c_{i-1}^* a_i} & ; i = 2, 3, ..., k-1 \end{cases}$$

$$d_i^* = \begin{cases} \frac{d_1}{b_1} & ; i = 1 \\ \frac{d_i - d_{i-1}^* a_i}{b_i - c_{i-1}^* a_i} & ; i = 2, 3, ..., k-1 \end{cases}$$

The equations can be rewritten as

$$\begin{bmatrix} 1 & c_1^* & 0 & 0 & ... & 0 \\ 0 & 1 & c_2^* & 0 & ... & 0 \\ 0 & 0 & 1 & c_3^* & 0 & 0 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & c_{k-1}^* \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ . \\ . \\ . \\ f_k \end{bmatrix} = \begin{bmatrix} d_1^* \\ d_2^* \\ d_3^* \\ . \\ . \\ . \\ d_k^* \end{bmatrix}$$

The last step is to work in reverse with the following equations.

$$f_k = d_k^*, \qquad f_i = d_k^* - c_i^* x_{i+1}, \qquad i = k-1, k-2, ..., 2, 1$$

Notice that when solving Thomas Algorithm, it cannot take advantage of parallelism as each step depends on the other element.

3.4.2 Intel Math Kernel Library

Intel Math Kernel Library implements routines for solving systems of linear equations from the standard LAPACK library which is a software package provided by University of Tennessee. Variety of matrix types are supported by the routines. Specifically gtsv function is utilized from the package. Using Gaussian elimination with partial pivoting, gstv computes the solution to the system of linear equations with a tridiagonal coefficient matrix [16]. Gaussian elimination with partial pivoting starts by determining the pivot by finding the largest absolute value at the left column. If necessary, row interchange is performed to ensure that the largest value is at the first row [44].

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & ... & 0 \\ a_2 & b_2 & c_2 & 0 & ... & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & c_{k-1} \\ 0 & 0 & 0 & 0 & a_k & b_k \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ . \\ . \\ . \\ f_k \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ . \\ . \\ d_k \end{bmatrix}$$

Assume that the first row is the pivot in the above system of equations. The pivot row is substituted from the second row to eliminate a_2. The process of selecting pivot and eliminating an element is repeated until the system of equations is in upper triangular form

$$\begin{bmatrix} b_1^* & c_1^* & 0 & 0 & ... & 0 \\ 0 & b_2^* & c_2^* & 0 & ... & 0 \\ 0 & 0 & b_3^* & c_3^* & 0 & 0 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & c_{k-1}^* \\ 0 & 0 & 0 & 0 & 0 & b_k^* \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ . \\ . \\ . \\ f_k \end{bmatrix} = \begin{bmatrix} d_1^* \\ d_2^* \\ d_3^* \\ . \\ . \\ . \\ d_k^* \end{bmatrix}$$

Once the system is in this form, it is easily solved by back substitution.

3.4.3 Cyclic Reduction

Cyclic reduction was proposed by R. W. Hockney in the 1960s for solving the resulting linear systems from the discretization of the Poisson equation [14]. Cyclic reduction consists of two stages, forward reduction and back substitution.

Given an n sized system, we will model the matrix as three vectors, lower diagonal a of size $n-1$, diagonal b of size n and upper diagonal c of size $n-1$.

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & ... & 0 \\ a_2 & b_2 & c_2 & 0 & ... & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & c_{k-1} \\ 0 & 0 & 0 & 0 & a_k & b_k \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ . \\ . \\ . \\ f_k \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ . \\ . \\ . \\ d_k \end{bmatrix}$$

During the forward reduction stage at each step s the odd number equations are reduced and even-indexed equations are updated. Thus, the number of unknows are reduced by half and new equations of the form

$$a_i^s f_{i-2^{s-1}}^s + f_i^s + c^s x_{i+2^{s-1}} = d_i^s \tag{3.1}$$

are generated where $i = 2^s, 2^s + 2^s, ..., n$ and $s = 1, 2, ..., \log_2 n$. The updated values at each step are

$$k_1 = \frac{a_i^{s-1}}{b_{i-1}^{s-1}}, \qquad k_2 = \frac{c_i^{s-1}}{b_{i+1}^{s-1}} \tag{3.2}$$

$$a_i^s = -a_{i-1}^{s-1} k_1, \qquad c_i^s = -c_{i+1}^{s-1} k_2 \tag{3.3}$$

$$b_i^s = b_i^{s-1} - c_{i-1}^{s-1} k_1 - a_{i+1}^{s-1} k_2 \tag{3.4}$$

$$d_i^s = d_i^{s-1} - d_{i-1}^{s-1} k_1 - d_{i+1}^{s-1} k_2 \tag{3.5}$$

The same procedure is applied recursively until there remains only one equation with one unknown where $i = 2^s, 2^s + 2^s, ..., n$ and $s = 1, 2, ..., \log_2 n$.

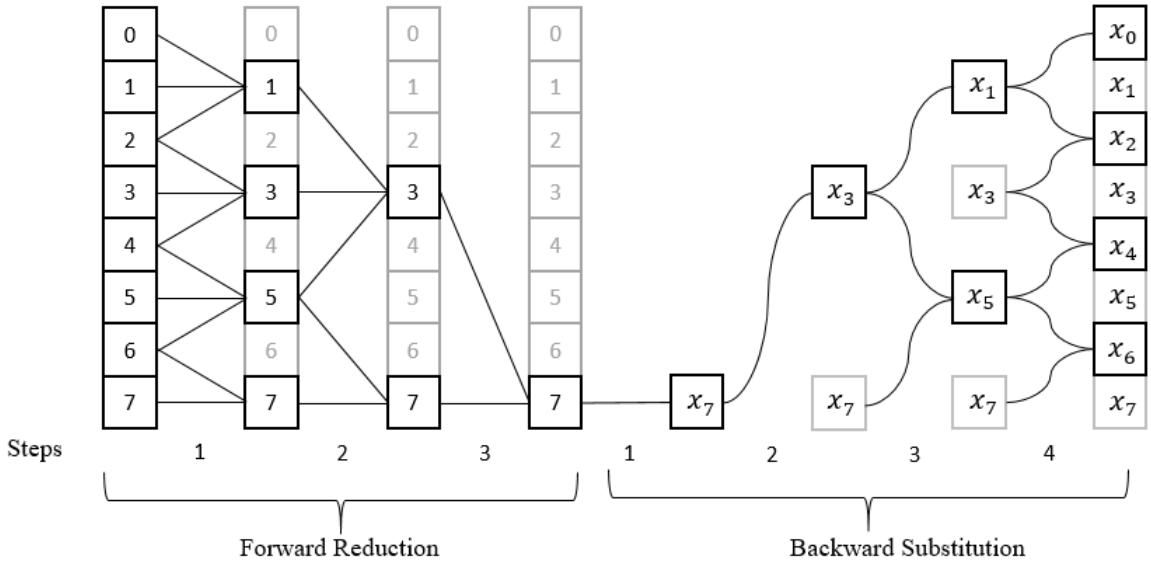

Figure 3.2: Cyclic reduction for an eight equation system.

Each step of backward substitution, we solve all rest of the unknowns by substituting the already solved value. As the name implies the steps go backwards as $s = \log_2 n - 1, ..., 0$ and $i = 2^s, 2^s + 2^{s+1}, ..., n$.

$$f_i = \frac{d_i^s - a_i^s f_{i-2^s} - c_i^s f_{i+2^s}}{b_i^s} \tag{3.6}$$

If serially computed, Thomas algorithm performs $8n$ operations while cyclic reduction performs $17n$ operations. On the other hand,if parallel computing is used with n cores, cyclic reduction requires $2\log_2 n - 1$ steps while the Thomas algorithm requires $2n$ steps [48]. The cyclic reduction algorithm was focused towards fine-grained parallelism which could be achieved using CPU parallelism [35].

3.5 Open Multi-Processing

Traditionally, programs are serially computed on a single processor. On the other hand, parallel computation is used to break our code execution in pieces so that it utilizes parallelism. Multithreading uses the CPUs cores to run calculations concurrently in each core. The concurrent programs are called a thread. If the code is executed on parallel processors, one of the biggest problem is the processors generally require

results that have been calculated on other processors. The main issue in this case is that processors clocks are not synchronized and execute the code at minimally different speeds.

In order to solve this problem, a group of major computer hardware and software vendors and major parallel computing user facilities joined forces to form The Open Multi-Processing Architecture Review Board (The OpenMP ARB)[40]. Open Multi-Processing (OpenMP) is an implementation of multithreading for C, C++ and Fortran and it was introduced to public in 1997. OpenMP is aiming to standardize high level parallelism that is performant, productive and portable.

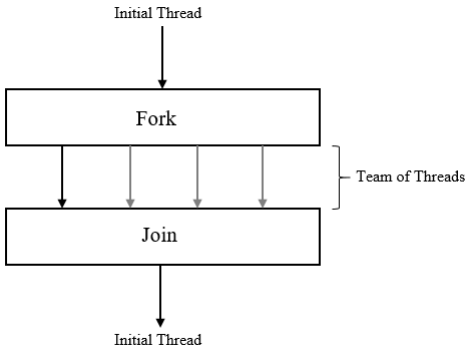

Figure 3.3: The fork-join programming model.

OpenMP approach to multithreading is the fork-join programming model. Firstly, the program start as a single thread of execution called the initial thread. The fork stage begins when the program encounters an OpenMP parallel construct. Parallel execution takes place and multiple threads are created in the parallel region. The initial thread becomes the master and collaborates with the newly created threads to execute the code dynamically. Finally, at the join stage all threads are synchronized, threads are terminated except the original thread [4].

Algorithms like cyclic reduction 3.4.3 uses nested loops to calculate the solution. Therefore nested parallelism is needed to build efficient programs. Nesting the parallel constructs results in nested parallelism. Each thread that encounters the next parallel region creates a new parallel region at runtime [45].

In order to specify and control the parallelization procedure, OpenMP uses compiler directives, runtime functions, and environment variables. OpenMP enables developers to just give a high-level specification of the parallelism by indicating the regions to be executed in parallel using compiler directives, runtime library routines, and envi-

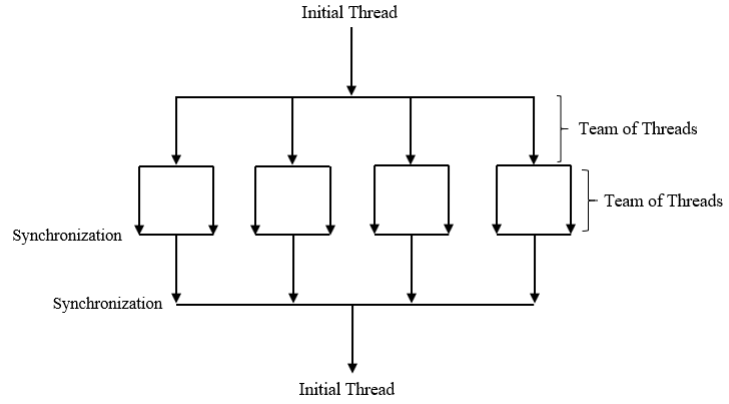

Figure 3.4: 4 threads create 2 threads, nested parallelism.

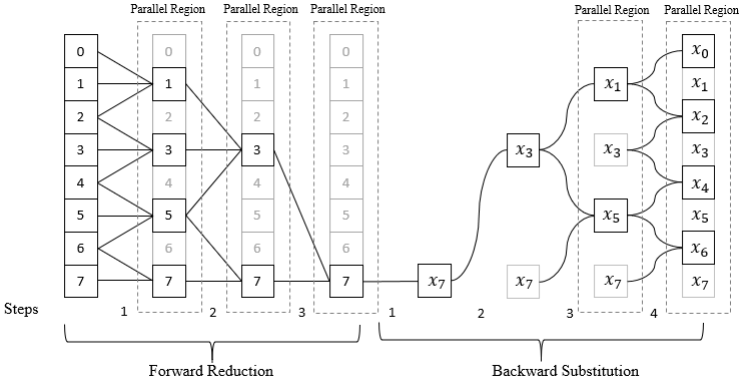

Figure 3.5: Implementation of cyclic reduction using OpenMP.

ronment variables. The details of the parallelism is up to the compiler which makes OpenMP comparatively easy to use.

3.6 Timing the Code

Timers and clocks might differ on distinct systems. Therefore, measuring execution time intervals accurately is an important task to compare the efficiency and speed of different environments and implementations.

3.6.1 Windows Application Programming Interface

Windows Application Programming Interface (API) is the lowest level of interaction between applications and the windows operating system. Thus every program is built

upon the API. Mostly, the interaction is hidden, the runtime and support libraries manage it in the background [34]. The APIs can be used in the C++ environment. Runtime can be calculated by "QueryPerformanceCounter" or "QueryPerformanceFrequency" functions. Respectively, the functions retrieve a high resolution time stamp and the frequency of the performance counter.

3.6.2 Chrono Library

Given the amount of work, using the Windows API for just timing the code is slightly excessive. Luckily, Chrono library was introduced part of the C++11s standard library. The Chrono library is intended to work effortlessly with date and time. The "high-resolution clock" provides the smallest possible tick period and with the now method, returns a value corresponding to the calls point in time.

```
auto Beginning = std::chrono::high_resolution_clock::now();
    //Portion of code to be timed
auto End = std::chrono::high_resolution_clock::now();
```

Once the start and end time of the code is recorded, the duration::count method is used to get the elapsed time.

```
std::chrono::duration<double> Elapsed = End - Beginning;
std::cout << "Elapsed time:" << Elapsed.count() << std::endl;
```

Chapter 4

Results and Discussion

This section documents the performance of attempted optimizations. Experiments are conducted at W307 computer laboratory, Queen Mary University of London. Each computer has Windows 10 Enterprise 64 bit, 16 GB of RAM, Intel Core i7 - 6700 CPU with 4 cores clocked at 3.40 GHz. The source code is written in C++ and compiled with Microsoft Visual Studio Enterprise 2017, Version 15.3.3 in the release mode. External tools utilized in the tests include Intel Compiler, version 18.0.3 and Intel Math Kernel Library. First step was solving the base cases 2.1.4, 2.39, 2.59 by hand and Excel. Following the simple implementations, the solvers are ported to C++ to measure and optimize the performance. Different solution platforms, compilers, optimization flags and tridiagonal solvers are tested against each other.

4.0.1 Base Case

Previously defined analytical solutions 2.1.6, 2.2.1, 2.2.2 are used to calculate errors for the solutions using different grid sizes. Space and time dimensions are discretized by 64 for Explicit and Crank - Nicolson schemes. In order to solve two dimensional heat equation using the ADI scheme two space dimensions and the time dimension are discretized by 32. All optimizations of Visual Studio (/Od flag) are disabled for the base case and an average of 1000 trials is taken to make sure the integrity of timings. During each trial a random number $0 < \epsilon < 10^{-}7$ is added to the step size to avoid automated optimizations.

Thomas Algorithm	Crank Nicolson	ADI
Visual Studio Compiler x86	0.04669	0.34690
Visual Studio Compiler x64	0.03403	0.26058
Intel Compiler x86	0.04387	0.32967
Intel Compiler x64	0.03436	0.26679
Intel Solver	**Crank Nicolson**	**ADI**
Visual Studio Compiler x86	0.04252	0.31399
Visual Studio Compiler x64	0.03243	0.24595
Intel Compiler x86	0.03723	0.29872
Intel Compiler x64	0.03265	0.24821
Cyclic Reduction	**Crank Nicolson**	**ADI**
Intel Compiler x86	0.04311	0.31157
Intel Compiler x64	0.03315	0.24346

	Explicit
Visual Studio Compiler x86	0.04212
Visual Studio Compiler x64	0.03238
Intel Compiler x86	0.03858
Intel Compiler x64	0.03178

Figure 4.1: Time taken to solve heat equation using the Explicit, Crank Nicolson and ADI schemes without any optimization.

Thomas Algorithm	Crank Nicolson
Visual Studio Compiler x86	0.04442
Visual Studio Compiler x64	0.03784
Intel Compiler x86	0.04583
Intel Compiler x64	0.03631
Intel Solver	**Crank Nicolson**
Visual Studio Compiler x86	0.04571
Visual Studio Compiler x64	0.03426
Intel Compiler x86	0.04432
Intel Compiler x64	0.03382
Cyclic Reduction	**Crank Nicolson**
Intel Compiler x86	0.04203
Intel Compiler x64	0.03519
	Explicit
Visual Studio Compiler x86	0.04205
Visual Studio Compiler x64	0.03443
Intel Compiler x86	0.04097
Intel Compiler x64	0.03314

Figure 4.2: Time taken to solve Black - Scholes equation using the Explicit and Crank Nicolson schemes without any optimization.

Following sections summarizes the performance increases by taking average of runtimes. For detailed timings, please refer to the Appendices D and C.

4.0.2 Solution Platforms

Switching the solution platform from x86 to x64 resulted in $17 - 22\%$ decrease in the computing time on average. The x86 CPU architecture provides 8 32 bit general registers and 8 SSE registers. Using a x64 architecture increases the number of both registers to 16. Therefore, calculations are more efficient and fast under 64 bit.

Therefore, we will prefer 64 bit for our best case.

4.0.3 Compilers

Switching to Intel Compiler provided $0.5 - 4.3\%$ performance increase on average.

The gtsv function from Intel Math Library was faster under the Intel compiler

Performance Increase by 64 Bit	Explicit	Crank - Nicolson	ADI
Not-Optimized (/Od)	19.5%	21.9%	21.0%
/O2 Flag	19.1%	20.9%	17.9%
/Ox Flag	17.1%	19.5%	18.4%

Figure 4.3: Average increase when switched to Intel Compiler under different optimization flags.

Not-Optimized (/Od)	Explicit	Crank - Nicolson	ADI
x86, 32 Bit	0.04093	0.04357	0.32017
x64, 64 Bit	0.03293	0.03402	0.25300

/O2 Flag	Explicit	Crank - Nicolson	ADI
x86, 32 Bit	0.03835	0.04037	0.29179
x64, 64 Bit	0.03103	0.03192	0.23946

/Ox Flag	Explicit	Crank - Nicolson	ADI
x86, 32 Bit	0.03835	0.04021	0.29354
x64, 64 Bit	0.03178	0.03237	0.23944

Figure 4.4: Average increase when switched to 64 bit under different optimization flags.

Performance Increase by Intel Compiler	Explicit	Crank - Nicolson	ADI
Not-Optimized (/Od)	4.3%	3.1%	3.0%
/O2 Flag	3.1%	3.4%	2.9%
/Ox Flag	0.5%	1.2%	3.0%

Figure 4.5: Average increase when switched to Intel compiler.

Not-Optimized	Explicit	Crank - Nicolson	ADI
Visual Studio Compiler	0.03774	0.03974	0.29185
Intel Compiler	0.03612	0.03849	0.28307

O2	Explicit	Crank - Nicolson	ADI
Visual Studio Compiler	0.03524	0.03703	0.27031
Intel Compiler	0.03414	0.03576	0.26250

Ox	Explicit	Crank - Nicolson	ADI
Visual Studio Compiler	0.03516	0.03678	0.26180
Intel Compiler	0.03497	0.03632	0.26962

Figure 4.6: Average timings comparing Intel Compiler and Visual Studio Compiler.

when compiler optimizations are disabled. However when we enable the /O2 and /Ox optimizations the difference was negligible. The OpenMP support of Visual Studio compiler was not supporting nested for loops. Therefore it is only tested using Intel Compiler. All in all, Intel compiler is faster and provides the support for new technologies such as parallelism. Thus, Intel compiler should be used when dealing with numerical calculations.

4.0.4 Visual Studio Optimizations

Figure 1 plots the relative performance of the computational kernels when compiled by the different compilers and run with a single thread. The performance values are normalizes so that the performance of G++ is equal to 1.0. The normalization constant is different for different kernels.

	Black - Scholes Equation		Heat Equation	
	Explicit (/O2)	Explicit (/Ox)	Explicit (/O2)	Explicit (/Ox)
Visual Studio Compiler x86	5.4%	3.9%	9.4%	13.8%
Visual Studio Compiler x64	7.5%	1.2%	3.7%	7.5%
Intel Compiler x86	5.1%	5.1%	5.1%	2.0%
Intel Compiler x64	6.8%	1.9%	4.9%	3.5%

Figure 4.7: Average increase when switched to intel compiler.

	Black - Scholes Equation		Heat Equation			
Thomas Algorithm	Crank Nicolson (/O2)	Crank Nicolson (/Ox)	Crank Nicolson (/O2)	Crank Nicolson (/Ox)	ADI (/O2)	ADI(/Ox)
Visual Studio Compiler x86	8.2%	4.5%	12.8%	15.5%	13.2%	15.6%
Visual Studio Compiler x64	14.3%	10.1%	6.0%	14.4%	6.1%	13.7%
Intel Compiler x86	11.0%	15.4%	7.0%	7.4%	8.5%	8.8%
Intel Compiler x64	8.5%	12.0%	7.3%	7.4%	8.2%	7.6%
Intel Solver	Crank Nicolson (/O2)	Crank Nicolson (/Ox)	Crank Nicolson (/O2)	Crank Nicolson (/Ox)	ADI (/O2)	ADI(/Ox)
Visual Studio Compiler x86	3.0%	3.7%	6.2%	14.1%	6.8%	8.7%
Visual Studio Compiler x64	0.6%	9.4%	1.6%	3.8%	1.3%	1.2%
Intel Compiler x86	3.4%	0.5%	-2.5%	5.3%	4.6%	5.1%
Intel Compiler x64	3.5%	15.2%	5.9%	1.0%	6.3%	2.5%
Cyclic Reduction	Crank Nicolson (/O2)	Crank Nicolson (/Ox)	Crank Nicolson (/O2)	Crank Nicolson (/Ox)	ADI (/O2)	ADI(/Ox)
Intel Compiler x86	6.6%	7.1%	16.6%	11.9%	10.6%	12.2%
Intel Compiler x64	9.0%	12.3%	7.4%	9.2%	4.6%	6.2%

Figure 4.8: Average increase when switched to intel compiler.

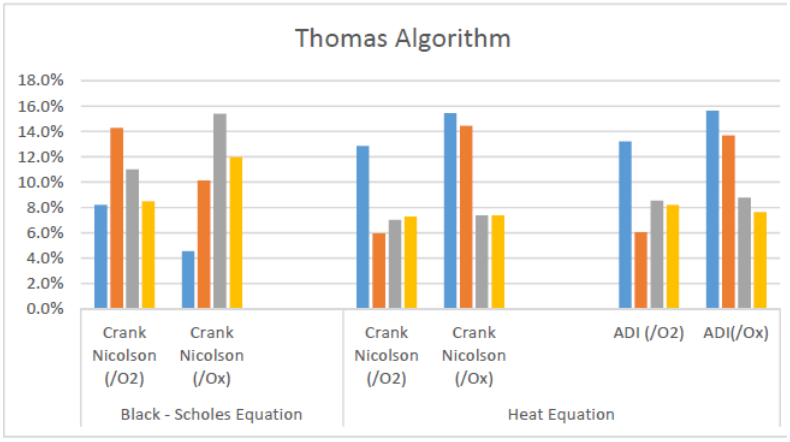

Figure 4.9: Average increase when switched to intel compiler.

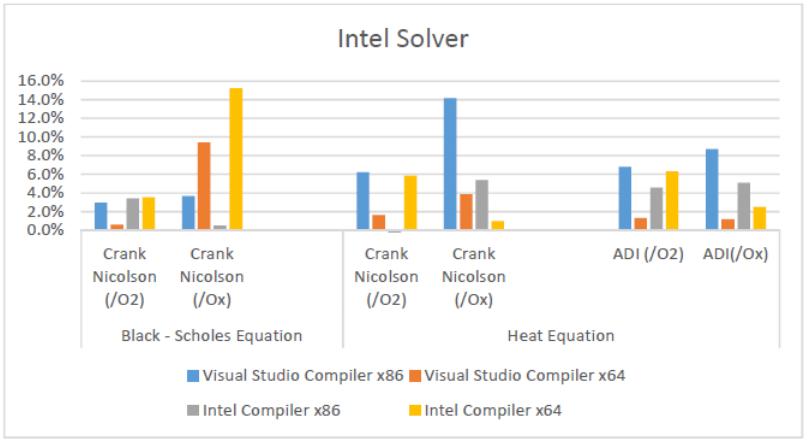

Figure 4.10: Average increase when switched to intel compiler.

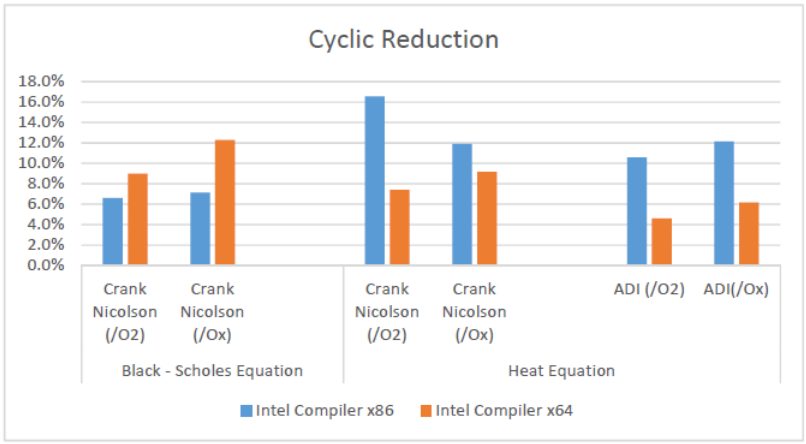

Figure 4.11: Average increase when switched to intel compiler.

4.0.5 Tridiagonal Solvers

The best performing tridiagonal solver is cyclic reduction with OpenMP followed by the gtsv function from Intel Math Library. The worst performer was the thomas algorithm as expected.

Average Percent Change	Thomas Algorithm	Intel Solver	Cyclic Reduction
Thomas Algorithm		5.8%	6.4%
Intel Solver	-5.8%		1.9%
Cyclic Reduction	-6.4%	-1.9%	

Figure 4.12: Average increase when between solvers.

Average Timings of Solvers	Thomas Algorithm	Intel Solver	Cyclic Reduction
Visual Studio Compiler x86	0.146002843	0.13407143	n/a
Visual Studio Compiler x64	0.110816247	0.104212217	n/a
Intel Compiler x86	0.13979119	0.126753987	0.13223676
Intel Compiler x64	0.1124857	0.104894493	0.10393382

Figure 4.13: Average increase when between solvers timings.

Our base case is using x86 platform, Visual Studio compiler, Thomas algorithm and no optimizations. Under these conditions heat equation was solved in 0.046 seconds and Black - Scholes equation was solved in 0.044 seconds using the Crank - Nicolson method. The fastest time we achieved is 0.030 for heat equation and 0.031 for Black - Scholes equation. Utilizing the Intel compiler, cyclic reduction with OpenMP and /Ox optimization flag resulted in 35.5% and 30.5% performance gain for heat equation and Black - Scholes equation. Under the same base case and fastest case conditions the ADI method for two-dimensional heat equation ran in 0.34 and 0.23 seconds, resulting in 34.1% faster solutions.

4.0.6 Different Sized Discretizations

In the numerical experiments above, the grid size is kept constant. Logically, the performance increase should be greater in case of bigger grids. Therefore, the grid size for Crank - Nicolson method was increase to 128 by 128 and 256 by 256. Only the base case and fastest case is tested in this section.

As expected, the bigger grid size benefits more from the performance gain. Similarly, the grid size of ADI method is increased to 64 and 128 from 32 for all dimensions.

	64x64		128x128		256x256	
	Base Case	Fastest Case	Base Case	Fastest Case	Base Case	Fastest Case
Crank Nicolson Heat	0.046686	0.030116	0.182665	0.115522	0.822998	0.459988
Crank Nicolson BS	0.044422	0.030871	0.158756	0.106463	0.653932	0.427301

Figure 4.14: Average increase when between solvers.

	128x128	128x128	256x256
	Base Case	Base Case	Base Case
Crank Nicolson Heat	35.5%	36.8%	44.1%
Crank Nicolson BS	30.5%	32.9%	34.7%

Figure 4.15: Average increase when between solvers timings.

Chapter 5

Conclusion

Numerical analysis and computer simulations are undertaken to put theory and observation together to gain insight into the workings of numerical solutions of partial differential equations.

A Critique of the Crank Nicolson Scheme Strengths and Weaknesses for Financial Instrument Pricing + rannacher AVX and Intrinsics CPUs are pipelining and use of SSE/SIMD kusswurm registers with Advanced Vector Extensions(AVX 512)GPGPU In the case of General Purpose GPUs, CUDA or Open Computing Language(OpenCL) can be utilized but can be challenging because of the requirement of delicate memory management. [32], cloud functions [12].

More type of options BSPde bdegistir callputflag duffy c++ 677 Generalize ADI (multi assset BS) Rannacher trick (168 foreign exchange pricing rannacher ve stencil) Burst cloud functions Rannacher trick

change black scholes to a different pricing pde like interest rate derivatives The HJM model [19] Parallel cyclic reductions is faster than thomas algorithm. 64 bit faster than 32 bit. Results concludela

Appendix A

Implementation of the `PDE` class

Parabolic partial differential equation can be denoted as

$$\frac{\partial u}{\partial t} = a(t,x)\frac{\partial^2 u}{\partial x^2} + b(t,x)\frac{\partial u}{\partial x} + c(t,x)u(t,x) + d(t,x)$$

$a(t,x)$ denotes diffusion coefficient, $b(t,x)$ convection coefficient, $c(t,x)$ reaction coefficient, $d(t,x)$ source coefficient analytic solution, initial conditions boundary conditions

Appendix B

Implementation of the `FiniteDifferenceMethod` class

```
void stepSize();
void initialConditions();
void boundaryConditions();
void innerDomain();
void timeMarch();
```

Appendix C

Heat Equation Timings

Thomas Algorithm	Crank Nicolson	ADI
Visual Studio Compiler x86	0.04069	0.30110
Visual Studio Compiler x64	0.03200	0.24480
Intel Compiler x86	0.04079	0.30157
Intel Compiler x64	0.03185	0.24491

Intel Solver	Crank Nicolson	ADI
Visual Studio Compiler x86	0.04002	0.29258
Visual Studio Compiler x64	0.03191	0.24274
Intel Compiler x86	0.03819	0.28508
Intel Compiler x64	0.03084	0.23253

Cyclic Reduction	Crank Nicolson	ADI
Intel Compiler x86	0.03597	0.27861
Intel Compiler x64	0.03069	0.23230

	Explicit
Visual Studio Compiler x86	0.03817
Visual Studio Compiler x64	0.03117
Intel Compiler x86	0.03662
Intel Compiler x64	0.03021

Figure C.1: Time taken to solve heat equation using the Explicit, Crank Nicolson and ADI schemes optimized with /O2 flag.

Thomas Algorithm	Crank Nicolson	ADI
Visual Studio Compiler x86	0.03947	0.29261
Visual Studio Compiler x64	0.02912	0.22494
Intel Compiler x86	0.04064	0.30079
Intel Compiler x64	0.03183	0.24639

Intel Solver	Crank Nicolson	ADI
Visual Studio Compiler x86	0.03650	0.28663
Visual Studio Compiler x64	0.03118	0.24304
Intel Compiler x86	0.03921	0.31398
Intel Compiler x64	0.03234	0.25438

Cyclic Reduction	Crank Nicolson	ADI
Intel Compiler x86	0.03797	0.27370
Intel Compiler x64	0.03012	0.22847

	Explicit
Visual Studio Compiler x86	0.03629
Visual Studio Compiler x64	0.02994
Intel Compiler x86	0.03782
Intel Compiler x64	0.03066

Figure C.2: Time taken to solve heat equation using the Explicit, Crank Nicolson and ADI schemes optimized with /Ox flag.

Appendix D

Black - Scholes Equation Timings

Thomas Algorithm	Crank Nicolson
Visual Studio Compiler x86	0.04078
Visual Studio Compiler x64	0.03243
Intel Compiler x86	0.04079
Intel Compiler x64	0.03323
Intel Solver	**Crank Nicolson**
Visual Studio Compiler x86	0.04436
Visual Studio Compiler x64	0.03405
Intel Compiler x86	0.04281
Intel Compiler x64	0.03263
Cyclic Reduction	**Crank Nicolson**
Intel Compiler x86	0.03925
Intel Compiler x64	0.03203
	Explicit
Visual Studio Compiler x86	0.03976
Visual Studio Compiler x64	0.03186
Intel Compiler x86	0.03887
Intel Compiler x64	0.03087

Figure D.1: Time taken to solve heat equation using the Explicit and Crank Nicolson schemes optimized with /O2 flag.

Thomas Algorithm	Crank Nicolson
Visual Studio Compiler x86	0.04240
Visual Studio Compiler x64	0.03401
Intel Compiler x86	0.03878
Intel Compiler x64	0.03196
Intel Solver	**Crank Nicolson**
Visual Studio Compiler x86	0.04403
Visual Studio Compiler x64	0.03748
Intel Compiler x86	0.04410
Intel Compiler x64	0.03897
Cyclic Reduction	**Crank Nicolson**
Intel Compiler x86	0.03903
Intel Compiler x64	0.03087
	Explicit
Visual Studio Compiler x86	0.04040
Visual Studio Compiler x64	0.03402
Intel Compiler x86	0.03890
Intel Compiler x64	0.03250

Figure D.2: Time taken to solve heat equation using the Explicit and Crank Nicolson schemes optimized with /Ox flag.

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