121st ASEE Annual Conference & Exposition
Indianapolis, IN
June 15-18, 2014

Paper ID #9196

Using Excel to Implement the Finite Difference Method for 2-D Heat Transfer in a Mechanical Engineering Technology Course

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Abstract:

Multi-dimensional heat transfer problems can be approached in a number of ways. Sometimes an analytical approach using the Laplace equation to describe the problem can be used. This involves finding the solution of differential equations, which may be reasonable for Mechanical Engineering Technology (MET) students. However, these students are not always particularly proficient in using this approach. Also, the analysis can get quite complex depending on boundary conditions, often involving advanced mathematics using Bessel functions, Fourier series and other special functions. Graphical methods might be used, but their usefulness extends primarily to discussions about the relationships between isotherms and heat flow paths. Shape factors and other approximations can also be useful in certain instances. None of these seem to provide an especially good approach for MET students.

A more practical approach for these students is the use of numerical methods. The finite difference method seems to provide a good approach for MET students. Using this method a student can model fairly complex two-dimensional problems with a variety of boundary conditions using a simple spreadsheet.

This paper presents information on how this method is used at Penn State Erie, The Behrend College in a first course in heat transfer for MET students. The method is used to aid in presenting the theory, as well as for a laboratory exercise. The basic equations for a variety of node types are included, as well as equation modifications that are used to account for several thermal loading and boundary conditions. The lectures are reinforced with homework practice problems before the more involved lab exercise. Finally, the lab exercise is included. The exercise is designed to give the students practice using the method.

Introduction:

The first course in heat transfer for Mechanical Engineering Technology (MET) students at Penn State Erie, The Behrend College focuses primarily on one-dimensional heat transfer with applications. Conduction, convection and radiation are introduced early. Conduction and convection are covered in some detail, including the calculation of convection coefficients using a variety of Nusselt correlations. Only the basics of radiation are included in the course. A section on transient heat transfer is also part of the course, including the lumped mass method, closed solutions for simple shapes and semi-infinite solids.

Very little is done with multi-dimensional conduction. Traditionally it was touched on during the treatment of fins and heatsinks. A laboratory exercise was developed to provide a little more coverage of the topic. The exercise requires the students to determine the temperature distribution across a plate with a variety of inputs and boundary conditions. This is done using finite difference formulations with a spreadsheet. Very little time is devoted to this during lectures, so the students are not being asked to develop the formulas that are used on the spreadsheet. Instead, one or two lectures are devoted to a simple treatment of the topic. The basic formula for an interior node is developed, and the others formulas that are needed are simply given to the students.

Theory:

The finite difference method is a numerical approach to solving differential equations. The fundamental equation for two-dimensional heat conduction is the two-dimensional form of the Fourier equation (Equation 1)[1,2]

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \text{Equation 1}$$

In order to approximate the differential increments in the temperature and space coordinates consider the diagram below (Fig 1).

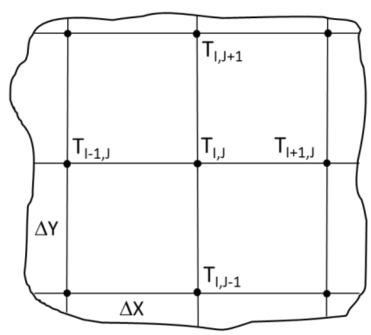


Fig 1

The temperature gradients become:

$$\left.\frac{\partial T}{\partial x}\right]_{I+1/2,J} \approx \frac{T_{I+1,J} - T_{I,J}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right]_{I-1/2,J} \approx \frac{T_{I,J} - T_{I-1,J}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right]_{I,J+1/2} \approx \frac{T_{I,J+1} - T_{I,J}}{\Delta y}$$

$$\left.\frac{\partial T}{\partial x}\right]_{I,J-1/2} \approx \frac{T_{I,J} - T_{I,J-1}}{\Delta y}$$

Using these gradients in the LaPlace equation the resulting relation between the temperatures can be determined:

$$\left.\frac{\partial^2 T}{\partial x^2}\right]_{I+1/2,J} \approx \frac{T_{I+1,J}-T_{I,J}}{\Delta x}\,\frac{\left.\frac{\partial T}{\partial x}\right]_{I+1/2,J}-\left.\frac{\partial T}{\partial x}\right]_{I-1/2,J}}{\Delta x} = \frac{T_{I+1,J}+T_{I-1,J}-2T_{I,J}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right]_{I,J+1/2} \approx \frac{T_{I,J+1}-T_{I,J}}{\Delta y}\,\frac{\left.\frac{\partial T}{\partial y}\right]_{I,J+1/2}-\left.\frac{\partial T}{\partial y}\right]_{I,J-1/2}}{\Delta y} = \frac{T_{I,J+1}+T_{I,J-1}-2T_{I,J}}{(\Delta y)^2}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right]_{I,J}+\left.\frac{\partial^2 T}{\partial y^2}\right]_{I,J}=0$$

$$\frac{T_{I+1,J}+T_{I-1,J}-2T_{I,J}}{(\Delta x)^2}+\frac{T_{I,J+1}+T_{I,J-1}-2T_{I,J}}{(\Delta y)^2}=0$$

If the node spacing is equal in the x and y directions ($\Delta X = \Delta Y$), then:

$$\boxed{T_{I+1,J}+T_{I-1,J}+T_{I,J+1}+T_{I,J-1}-4T_{I,J}=0} \qquad \text{Equation 2}$$

Some technology students may not have the math background to really understand this true finite difference method. Another path to the same result is to look at an energy balance[3]. Consider the diagram below (Fig 2).

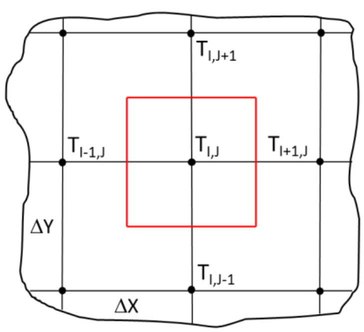


Fig 2

The boxed area around $T_{I,J}$ represents a control volume of uniform thickness, δ. Assume that heat is transferred into the control volume from each of the surrounding nodes (Fig 3).

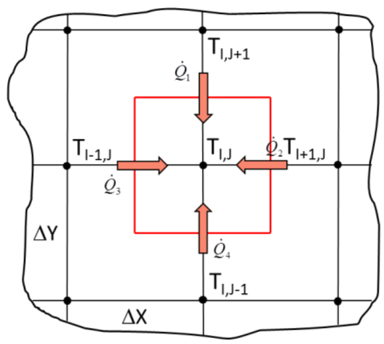


Fig 3

For steady state, the first law energy balance for the control volume is: (Equation 3)

$$\dot{Q}_1 + \dot{Q}_2 + \dot{Q}_3 + \dot{Q}_4 = 0 \qquad \text{Equation 3}$$

But:

$$\dot{Q}_1 = -\frac{kA\Delta T}{L} = \frac{kA(T_{I,J+1} - T_{I,J})}{L}$$

$$\dot{Q}_1 = -\frac{k\delta\Delta x(T_{I,J+1} - T_{I,J})}{\Delta y}$$

$$\dot{Q}_1 = -k\delta\left(T_{I,J+1} - T_{I,J}\right)$$

Writing similar equations for each of the conduction terms, assuming the same node spacing in the x and y directions, substituting into equation 3 and cancelling out k and δ yields equation 4:

$$\boxed{T_{I+1,J} + T_{I-1,J} + T_{I,J+1} + T_{I,J-1} - 4T_{I,J} = 0} \qquad \text{Equation 4}$$

Notice that this is the same as equation 2.

Equation 4 is the "basic" equation for an interior node. For the exercise described in this paper, this is the only equation that is actually derived for the students. All of the equations for other node types and other inputs and boundary conditions are presented for the students, but not derived. Equations similar to equation 4 are presented as "basic" equations for the node types, and then they are modified to account for a variety of inputs and boundary conditions. In all cases a uniform, square mesh is used.

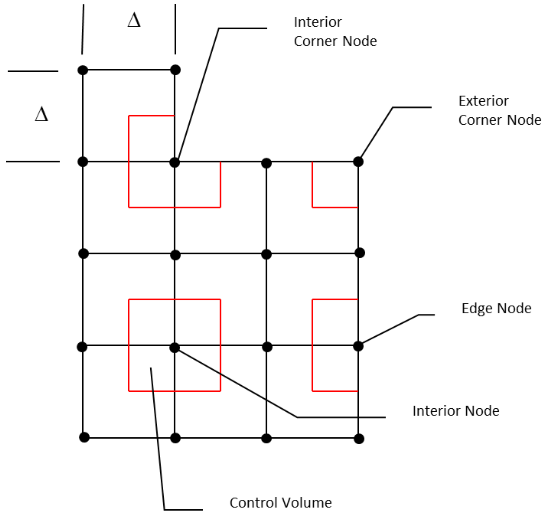


Fig 4

Figure 4 shows the various types of nodes that are possible (interior, edge, exterior corner and interior corner). Table 1 shows the basic equation for each of the 4 types. Temperature T is the node temperature and the subscripted temperatures are the surrounding nodes.

Interior Nodes	$T = \dfrac{T_1 + T_2 + T_3 + T_4}{4}$	Equation 4
Edge Nodes	$T = \dfrac{T_1 + .5T_2 + .5T_3}{2}$	Equation 5
Exterior Corner Nodes	$T = \dfrac{.5T_1 + .5T_2}{1}$	Equation 6
Interior Corner Nodes	$T = \dfrac{T_1 + T_2 + .5T_3 + .5T_4}{3}$	Equation 7

Table 1

These equations can be modified to account for a point heat source attached to the node or for internal heat generation in the control volume associated with the node. The following terms are added to the numerator of each of the equations in Table 1 is appropriate.

Point Heat Source: Add : $\dfrac{\dot{Q}_S}{\delta K}$ Equation 8

Where:
$\dot{Q}_S$ = Point heat source (W)
δ = Thickness of part (m)
K = Conductivity – (W/m–^{0}C)

Internal Heat Generation: $\text{Add}: \dfrac{\dot{q}\Delta^2}{K}$ Equation 9

Where:

$$\dot{q} = \text{Internal heat generation (W/m}^3\text{)}$$
$$\Delta = \text{Node spacing (m)}$$
$$K = \text{Conductivity (W/m}-{}^0\text{C)}$$

For nodes along the boundary that have internal heat generation the control volume is smaller than for an internal node, so the factor above needs to be modified to account for that. The factor is corrected by the ratio of the control volume size relative to the control volume size of an interior node. For example, the control volume for an edge node is half the size of one for an interior node, so the internal heat generation factor is multiplied by 0.5.

Both the point heat source and the internal heat generation factors can apply to any of the four node types. If both conditions are present then both factors are added to the numerator of the basic equation. The following boundary conditions only apply to three of them. Internal nodes do not see the boundary, so these conditions would never be added to them.

The most common boundary conditions that might be experienced are convection and heat flux.

Convection: $\text{Add}: \text{BiT}_\infty$ To the top of the equation; $\text{Add}: \text{Bi}$ To the bottom of the equation Equation 10

Where:

$$\text{Bi} = \text{Biot Number} = h\Delta/K \text{ (dimensionless)}$$
$$T_\infty = \text{Surrounding temperature (}^0\text{C)}$$

Example:

$$T = \frac{T_1 + .5T_2 + .5T_3 + BiT_\infty}{2 + Bi} \qquad \text{Equation 11}$$

Heat Flux: $Add: \dfrac{q\Delta}{K}$ To the top of the equation Equation 12

Where:

$$q = \text{heat flux (W/m}^2\text{)}$$

Example:

$$T = \frac{T_1 + .5T_2 + .5T_3 + \dfrac{q\Delta}{K}}{2} \qquad \text{Equation 13}$$

Other circumstances can exist on the boundaries. Any special cases are addressed as needed.

Application of this Method:

Once the equations have been presented to the students it is time to implement them for a two-dimensional heat transfer problem. The general procedure is to first break the part into a mesh. All of the meshes used are square. The intersection of the mesh lines are the nodes, which are the points at which temperatures are calculated. The appropriate equation is written for each of the nodes. All of the nodes are interrelated through the equations, so the result of writing the equations is a set of simultaneous equations equal to the number of nodes. There will be an the same number of unknowns as equations to calculate. For a simple case it is possible to set up a matrix to solve the equations, however, as the number of nodes increases so does the size and complexity of the matrix. The exercise addressed by this paper uses a spreadsheet to numerically solve the equations.

Each node is represented by a cell on a spreadsheet, and the equations for the nodes are entered into the corresponding cells on the spreadsheet. In order to avoid a circular reference error the spreadsheet should be set to enable iterative calculations. The default using EXCEL for maximum iterations is 100 which means that each time a change is made to a formula the spreadsheet will run 100 iterations. You can change this if you want to, but it is not necessary. The F9 key will reiterate the sheet 100 times each time it is hit. If the calculated values are formatted to two decimal places then the F9 key can be held down until the numbers in the cells stop changing indicating that the sheet has essentially converged. Note that if the cells are left to default to an unlimited number of decimal places then it is unlikely that the sheet will ever fully converge.

The students are asked to create spreadsheets for five different two-dimensional problems during lab. They have a very difficult time accomplishing this without practice, so two example problems, with solutions, are provided for them to try before lab.

<u>Example 1</u>: This is a simple example which only involves formulas for interior nodes. Notice that the temperature is given for some of the surrounding nodes. These values are simply entered directly into the spreadsheet. Figure 5 shows the mesh for this example.

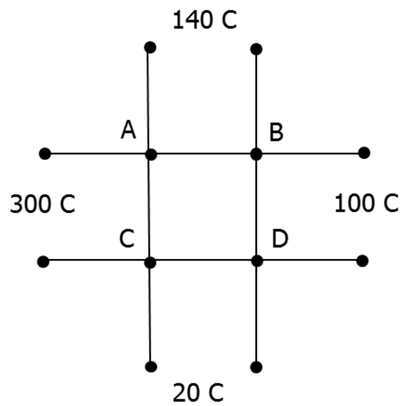


Fig 5

The problem is to solve for the temperatures at A, B, C and D.

First, an area on the spreadsheet is selected to represent the mesh (Figure 6).

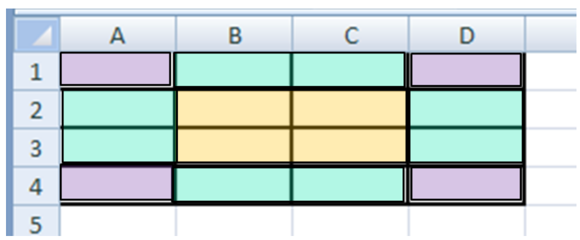
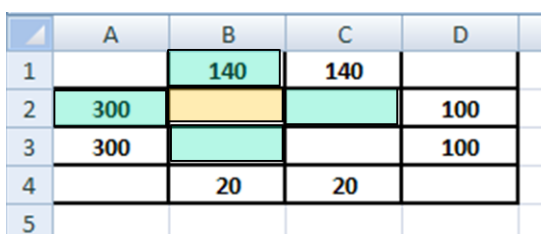


Fig 6

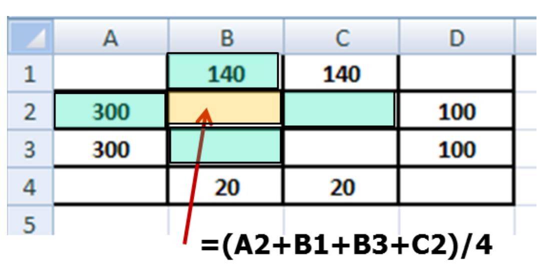
Next, the given fixed temperatures are entered into the appropriate cells (Figure 7).



	A	B	C	D
1		140	140	
2	300			100
3	300			100
4		20	20	
5				

Fig 7

The basic formula for one of the internal nodes is entered as shown (Figure 8).



	A	B	C	D
1		140	140	
2	300			100
3	300			100
4		20	20	
5				

Fig 8

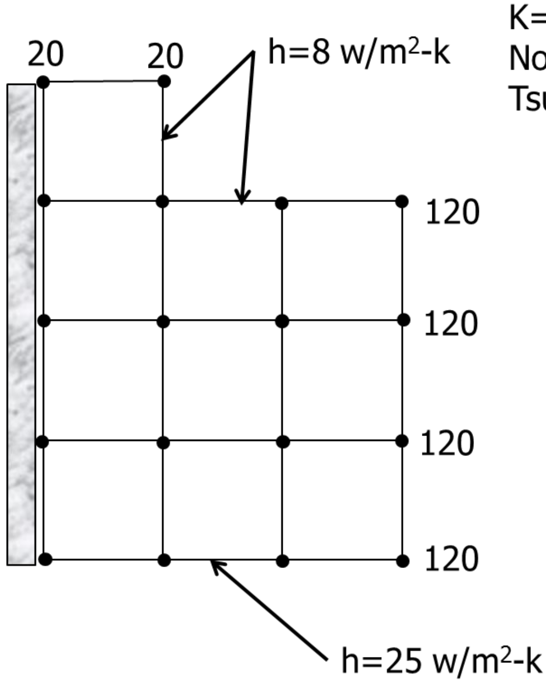
Finally, the remaining equations are entered yielding the results shown in Figure 9.

	A	B	C	D
1		140	140	
2	300	180.00	130.00	100
3	300	150.00	100.00	100
4		20	20	

Fig 9

The second homework problem is shown in Figure 10.

K=180 w/m-k
Node spacing = 0.1 m
Tsurr = 5 ^{0}C



	A	B	C	D	E
1	20	20			
2	59.62	68.55	98.41	120	
3	81.36	87.12	102.97	120	
4	91.59	95.59	106.34	120	
5	93.83	97.31	106.79	120	
6					
7					
8	h1=	8		Bi 1 =	0.00444
9	h2=	25		Bi 2 =	0.01389
10	K=	180			
11	Δ=	0.1			
12	Tsurr=	5			

Fig 10

This problem contains one node which is a special case. Notice that the node on the lower left hand corner (A5 on the spreadsheet) has convection on one surface, but the other surface is insulated. Since convection is only present on half of the control volume, then the Biot number is multiplied by 0.5 in both the numerator and denominator of the modified equation for that exterior corner node.

The Lab Exercise:

The actual lab exercise involves using a spreadsheet to determine the temperature distribution across five different two-dimensional parts. The requirements are that the students must set up their spreadsheets in an orderly fashion, solve for all of the unknown node temperatures and provide the final file for grading. This is done on an individual basis, although they are allowed to consult with each other during the lab period. Some students are able to finish all five cases within the two hour lab period, but many cannot. They are given one week to turn in the computer file. If they wish to leave early from lab they must demonstrate that they have correctly completed the first three cases. The five problems and their solutions are shown in the appendix.

This exercise provides the students with a hands on application of a numerical 2-dimensional analysis method (finite difference) for solving a variety of heat transfer problems. Another benefit of this project is its ability to satisfy the Accreditation Board For Engineering and Technology (ABET) student outcomes. ABET designates the student outcomes as a through k. The following ABET student outcomes could be satisfied with this project:

a. An ability to select and apply the knowledge, techniques, skills and modern tools of the discipline to engineering technology activities;
b. An ability to select and apply a knowledge of mathematics, science, engineering and technology to engineering technology problems that require the application of principles and applied procedures or methodologies;
c. An ability to analyze and interpret experimental results;
f. An ability to identify, analyze and solve engineering technology problems;
k. A commitment to quality, timeliness and continuous improvement.

References:

1 Holman, J.P., “Heat Transfer”, Tenth Edition, McGraw-Hill, 2010
2 Chapman, A.J., “Heat Transfer”, Fourth Edition, Macmillen Publishing, 1984
3 Hagen, K.D., “Heat Transfer with Applications”, First Edition, Prentice Hall, 1999

Appendix:

Lab Problem #1:

The first problem is a simple exercise similar to the first homework problem. It not only helps to get the students started during the lab period, but also helps to identify which students actually did the homework. Usually there are a couple of students in a lab section who did not feel that it was necessary to work on the practice problems, and they clearly stand out when they try this exercise. The grid is shown in figure 11.

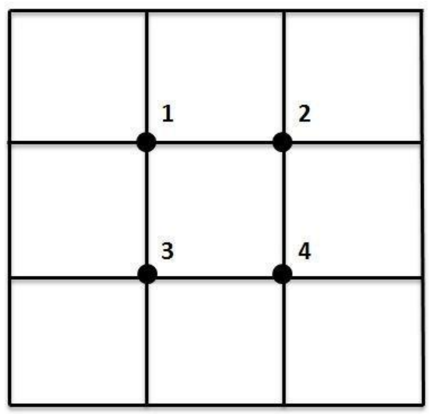


Fig 11

The problem is to determine the temperature at nodes 1, 2, 3 and 4. The four surrounding surfaces are at fixed temperatures: top: 300^0C, bottom: 100^0C, left: 500^0C and right: 150^0C. The solution is given in figure 12.

	A	B	C	D
1		300	300	
2	500	331.25	243.75	150
3	500	281.25	193.75	150
4		100	100	
5				

Fig 12

Lab Problem #2:

The second problem represents a fin attached to a wall on the left end. It serves two purposes. First, it adds a simple boundary condition (convection) for the students to deal with. The problem also provides a teaching moment during the lab session. This problem is symmetrical about the x-axis. This leads to a qualitative discussion of what the temperature profile should look like (decreasing temperatures from left to right, decreasing temperatures from center out and symmetrical). The other teaching point here is that this problem can be solved by slicing it along the axis of symmetry and only analyzing that portion. The axis of symmetry becomes an adiabatic surface for the analysis. The grid for this problem is shown in figure 13.

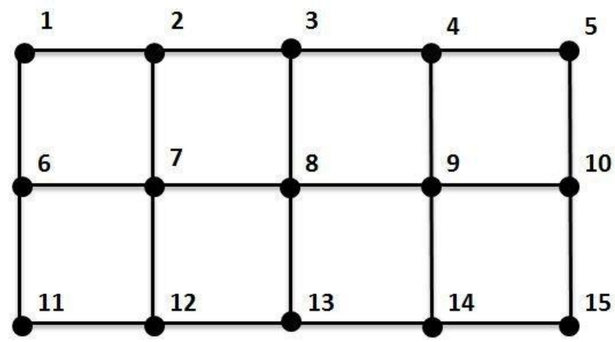


Fig 13

The temperature at the wall (left end) is 125^0C. All of the other boundary nodes are subjected to convection. The surrounding air is at 20^0C and the convection coefficient on all surfaces is 30 W/(m^2-^{0}C). The conductivity of the fin material is 6 W/(m-K) and the node spacing is 4 cm. The temperature at each of the labeled nodes is to be determined. Figure 14 gives the solution for this problem. The Biot # in cell A5 is calculated from the given information.

	A	B	C	D	E
1	125	86.53	64.35	51.20	44.08
2	125	91.69	68.70	54.42	46.59
3	125	86.53	64.35	51.20	44.08
4					
5	0.2	Biot #		h	30
6	20	Ambient		Δ	0.04
7				k	6
8					

Fig 14

Lab Problem #3:

Problem three adds a few other complications. Several of the surfaces are insulated while other surfaces are subjected to convection. Nodes 3 and 18 are insulated on one side while there other side has convection on the surface. Figure 15 shows the grid for this problem.

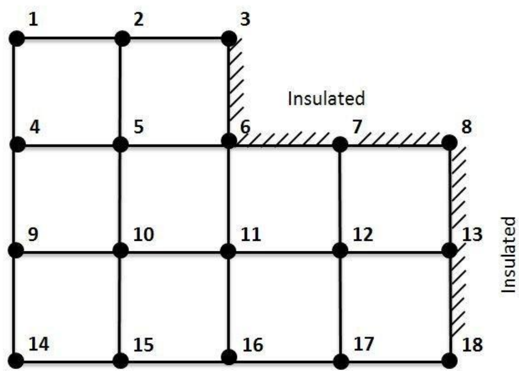


Fig 15

The following conditions apply to this problem:

<u>Nodes 1, 4, 9 and 14</u>: Fixed temperature at 85^0C
<u>Nodes 2, 3, and 15-18</u>: Subjected to 10^0C ambient air with a convection coefficient of 18 W/(m^2-^{0}C)

Node spacing = 5 cm
Conductivity = 11 W/(m-K)

Figure 16 shows the solution. Note that the Biot # in cell B11 is a calculated value.

	A	B	C	D	E
1	85	75.54	72.08		
2	85	77.90	73.70	70.06	68.88
3	85	77.37	72.13	68.82	67.71
4	85	74.46	68.63	65.38	64.32
5					
6					
7	h	18.00			
8	delta	0.05			
9	k	11			
10					
11	Bi	0.081818			
12					
13	t amb	10			
14					

Fig 16

Lab Problem #4:

Problem four is similar, but again, more complexity is added. This problem has one surface with an incoming heat flux, an insulated surface and two surfaces with convection. Problem four generally gives the students the most trouble, probably due to the fact that there are several special cases that they have to recognize and deal with. Figure 17 shows the grid for this problem.

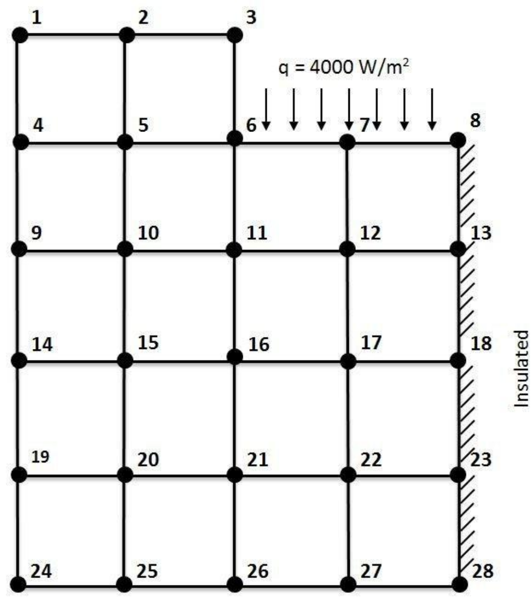


Fig 17

The following conditions apply to this problem:

<u>Nodes 1, 4, 9, 14, 19 and 24</u>: Fixed temperature at 120^0C
<u>Nodes 2, 3 and 6</u>: Subjected to 10^0C ambient air with a conv coef of 25 W/(m^2-^{0}C)
<u>Nodes 25-28</u>: Subjected to 50^0C ambient air with a conv coef of 40 W/(m^2-^{0}C)

Node spacing = 5 cm
Conductivity = 5 W/(m-K)

Figure 18 gives the solution for this problem. Note again that the Biot numbers in cells B11 and B15 are calculated values.

	A	B	C	D	E
1	120	99.75	91.84		
2	120	116.02	124.84	163.80	173.59
3	120	119.48	125.04	138.39	143.38
4	120	116.87	117.46	121.33	123.15
5	120	110.52	106.62	106.30	106.57
6	120	98.59	92.19	90.69	90.53
7					
8					
9	T1	10			
10	h1	25			
11	Bi 1	0.25			
12					
13	T2	50			
14	h2	40			
15	Bi 2	0.4			
16					
17	k	5			
18	delta	0.05			
19					

Fig 18

Lab Problem #5:

The final problem adds point heat sources. The exercise consists of a circuit board with several electronic components mounted on the surface. The devices are treated as point heat sources at the nodes they are attached to. Figure 19 shows the physical layout of the circuit board and the grid.

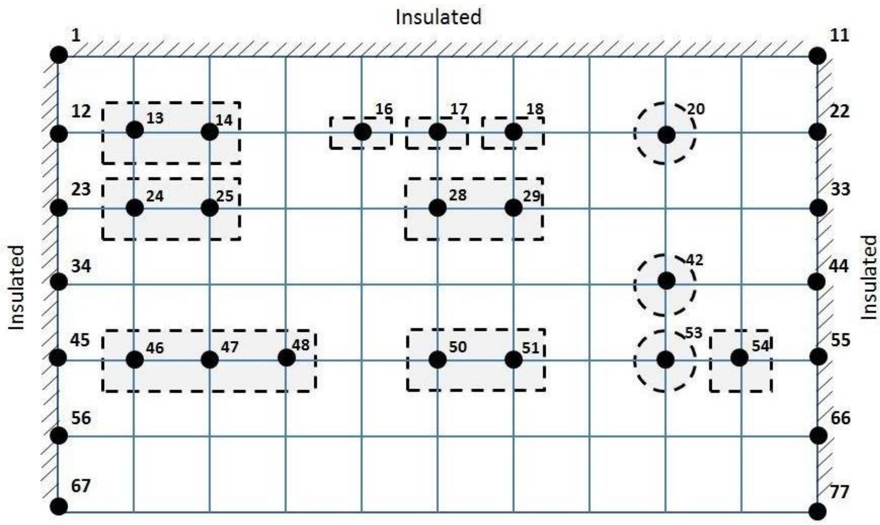


Fig 19

The dashed rectangles and circles represent components that are mounted to the board. The bottom edge is attached to a cold plate at 40^0C. The assumption is that all the nodes along the bottom are fixed at that temperature. All of the other edges are considered to be insulated. Table 2 shows the power that is dissipated from each component.

Nodes	Total power for each device
13, 14 and 24, 25	400 mW
16 and 17 and 18	40 mW
28, 29 and 50, 51	450 mW
46, 47, 48	3 W
20 and 42 and 53	150 mW
54	250 mW

Table 2

The board has a conductivity of 165 W/(m-K) and a thickness of 0.8 mm. The node spacing is 1.5 cm.

Figure 20 shows the solution for this problem. The highlighted areas show where the components are mounted.

	A	B	C	D	E	F	G	H	I	J	K
1	54.47	54.47	53.94	52.88	51.91	51.07	50.26	49.50	49.00	48.48	48.30
2	54.47	54.73	54.20	52.83	51.85	51.06	50.23	49.37	49.00	48.32	48.11
3	53.94	54.27	53.79	52.39	51.28	50.78	49.94	48.75	48.18	47.69	47.50
4	52.77	53.10	52.79	51.67	50.10	49.14	48.29	47.50	47.29	46.75	46.51
5	50.92	52.57	52.60	51.41	48.32	47.38	46.59	45.66	45.61	45.51	45.04
6	45.78	46.10	46.04	45.46	44.39	43.78	43.33	42.96	42.83	42.74	42.63
7	40	40	40	40	40	40	40	40	40	40	40
8											
9											
10	δ	0.0008	m								
11	k	165	W/m-K								
12	Δ	0.015	m								
13											

Fig 20