Mathematical Finance MSc Dissertation MTH775P, 2018/19

Accelerated Grids

Optimizing Solvers for Financial Partial Differential Equations

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A thesis presented for the degree of
Master in Sciences in *Mathematical Finance*

School of Mathematical Sciences
and *School of Economics and Finance*
Queen Mary University of London

Declaration of original work

This declaration is made on July 31, 2019.

Student's Declaration: I, Mustafa Berke Erdis, hereby declare that the work in this thesis is my original work. I have not copied from any other students' work, work of mine submitted elsewhere, or from any other sources except where due reference or acknowledgement is made explicitly in the text, nor has any part been written for me by another person.

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This work is dedicated to my family.

Acknowledgements

Here you thank people that have helped you in the journey.
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Abstract

Here you write a short summary, around 10 lines, of your work.
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Preface

Here you write a summary of the work. A paragraph on the motivation, previous work, then maybe a brief chapter by chapter summary.

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Queen Mary University of London
12^{th} August 2019

Contents

Chapter 1

Introduction

1.1 Pricing Financial Derivatives

1.1.1 The Risk Neutral Approach

$S(t) = S(0)exp((\alpha - \sigma^2/2)t + \sigma B(t))$

1.1.2 Solving Financial Partial Differential Equations

Theorem 1.1.1 ([P99, Theorem 2.3], see also [BS, pg. 45])**.** *The Gramm matrix for E_8 is:*

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Recall the theorem of Petri 1.1.1 Look at section **??**.

1.2 Finite Difference Methods

$\frac{\partial u}{\partial t} = a(t,x)\frac{\partial^2 u}{\partial x^2} + b(t,x)\frac{\partial u}{\partial x} + c(t,x)u(t,x) + d(t,x)$

a(t,x) Diffusion coefficient b(t,x) Convection coefficient c(t,x) reaction coefficient d(t,x) source coefficient

1st order Forward Difference $\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$

1st order Central Difference $\frac{\partial u}{\partial x} = \frac{u_{i+1}^n - u_i^n}{\Delta t}$

2nd order Central Difference $\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$

1.2.1 Explicit Method

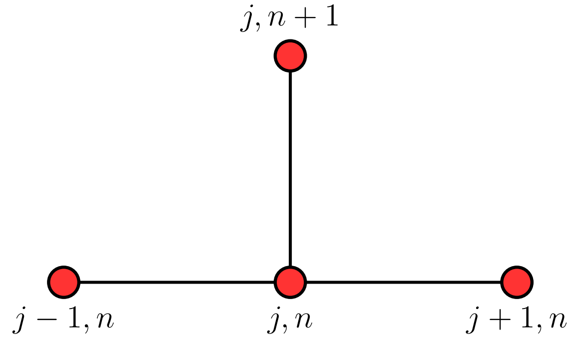
Explicit method is a forward time, central space scheme .

Explicit method can be generalized as: $u_j^{n+1} = \alpha u_{j-1}^n + \beta u_j^n + \gamma u_{j+1}^n$

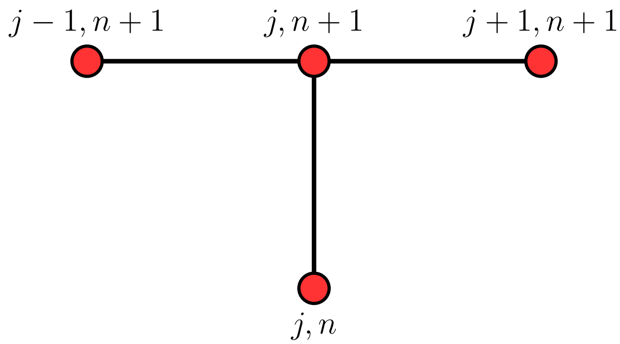
$r = \frac{\delta t}{\delta x^2}$

For heat equation: $\alpha = r, \beta = 1 - 2r, \gamma = r$

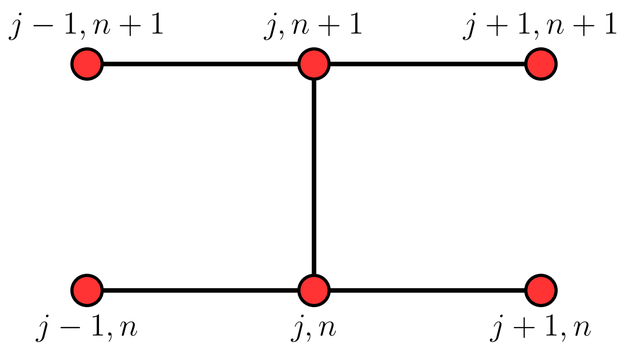
For black-scholes equation: $\alpha = \frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2}, \beta = 1 - \sigma^2 j^2 \Delta t - r\Delta t, \gamma = \frac{\sigma^2 j^2 \Delta t}{2} - \frac{rj\Delta t}{2}$



1.2.2 Implicit Method



1.2.3 Theta Method and Crank-Nicholson Method



1.2.4 Alternating Direction Implicit Method

1.2.5 Rannacher Trick

1.3 Optimizations

1.3.1 Compilers

1.3.2 32 bit and 64 bit

1.3.3 Optimization Switches

1.3.4 Tridiagonal Solvers

1.3.5 Threading

1.3.6 OpenMP

1.3.7 AVX and Intrinsics

1.3.8 GPGPU

1.3.9 Cloud Applications

Chapter 2

Optimization of Financial Partial Differential Equations

2.1 Timing the Code

2.1.1 Windows API

2.1.2 Chrono Library

Chapter 3

Conclusions

Appendix A

Implementation of the `FiniteDifferenceMethod` class

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Appendix B

Additional details on the Gundermanian determinant

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