ALTERNATING DIRECTION IMPLICIT METHOD SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

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ALTERNATING DIRECTION IMPLICIT METHOD SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

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Certification

This is to certify that this project was carried out by **IMAM, Mubarak Adeshina** with Matriculation Number: **09/55EB067** in the Department of Mathematics, Faculty of Science, University of Ilorin, Ilorin, Nigeria.

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Dedication

Dedicated to the Glory of the Almighty God, the noblest of mankind, Prophet Muhammad (PBUH), and my late father, Alhaji Shuaib Ayinde Imam.

Acknowledgements

All praise is due to God, the Almighty, the bestower of life, knowledge and all that is good, for giving me the rare opportunity of starting this academic sojourn in his name and concluding with his praises.

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Abstract

Alternating Direction Implicit method was first proposed by D.W.Peaceman and H.H. Rachford Jnr. in their paper entitled *"The numerical solution of parabolic and elliptic differential equations"* in 1955. Since then, series of works have been carried out on this method.

In this work, we use the Alternating Direction Implicit method to provide numerical solution to one of the most important elliptic partial equations in application: the Laplace equation in 2-Dimensions.

Our results are compared with the results obtained using the Finite difference method and the exact solution from solving using separation of variables.

Contents

Chapter 1

GENERAL INTRODUCTION

1.1 Introduction

Alternating Direction Method is a class of methods first introduced by Peaceman and Rachford for solving the time-dependent heat equation in two space dimensions. It was quickly recognized that the unconditional stability of the method might render it effective as a steady-state (hence, elliptic) solver due to the possibility of employing large time steps for pseudo-time marching to a steady state. At each pseudo-time step (iteration) the discrete equations are implicitly (line-by-line) solved first in one spatial direction, and then in the other, leading to the terminology *Alternating Direction Implicit*

1.2 Definition of Relevant Terms

Definition 1.2.1. Equation:

An equation describes the relationship between the dependent and independent variables.

Definition 1.2.2. Differential Equation:

An equation which involves differential coefficient(s) is called a differential equation.

Definition 1.2.3. Partial Differential Equation:

A partial differential equation involves derivatives of a function $u(x, y, \ldots)$ of more than one independent variable. Partial differential equations necessarily involve partial derivatives such as $\frac{\partial}{\partial x}$

Examples of second order, linear, partial differential equations are:

- Wave Equation: $u(x, t)$

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2} \tag{1.1}$$

- Heat Equation: $u(x, t)$

$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial u}{\partial t} \tag{1.2}$$

- Laplaces Equation: $u(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.3}$$

Definition 1.2.4. Order of a Differential Equation:

The order of a differential equation is the order of the highest derivative involved in the equation.

Definition 1.2.5. Linear Partial Differential Equation:

A PDE is linear if the dependent variable and its functions are all of first order.

Definition 1.2.6. Second Order Linear Partial Differential Equation:

A PDE is said to be a second order linear PDE if it is linear and of the second order, that is; it is of degree one in the highest order derivative in the equation.

Example:

$$u_{xx} + u_{yy} = 0.$$

Definition 1.2.7. Solution of a Partial Differential Equation:

A solution of a PDE in some region R of the space of independent variables is a function, which has all the derivatives that appear on the equation, and satisfies the equation everywhere in R. For example:

$$u = x^2 y^2$$

$$u = e^x cos(y), \text{ and}$$

$$u = \ln(x^2 + y^2)$$

are all solutions to the two-dimensional Laplace's equation (1.3) above.

1.3 Partial Differential Equations

A PDE is an equation which includes derivatives of an unknown function with respect to two or more independent variables.

PDE's describe the behavior of many engineering phenomena like:

- Wave propagation
- Fluid flow (air or liquid)
- Air around wings, helicopter blade, atmosphere
- Water in pipes or porous media
- Material transport and diffusion in air or water
- Weather: large system of coupled PDE's for momentum, pressure, moisture, heat, etc.
- Vibration
- Mechanics of solids: stress-strain in material, machine part, structure
- Heat flow and distribution
- Electric fields and potentials
- Diffusion of chemicals in air or water
- Electromagnetism and quantum mechanic

1.3.1 Classification of Partial Differential Equations

Consider the general second order linear partial differential equation in two variables.

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + = F(x,y,u,u_x,u_y)$$

We classify the equation by the sign of the discriminant. At a given point (x_0, y_0), the equation is classified as one of the following types:

$$b^2 - ac > 0 : \text{ hyperbolic}$$

$$b^2 - ac = 0 : \text{ parabolic}$$

$$b^2 - ac < 0 : \text{ elliptic}$$

If an equation has a particular type for all points (x, y) in a domain then the equation is said to be of that type in the domain. Each of these types has a canonical form that can be obtained through a change of independent variables.

The type of an equation indicates much about the nature of its solution.

1.4 Elliptic Partial Differential Equations

Elliptic Equations ($B^2 4AC < 0$) [steady-state in time] typically characterize steady-state systems (no time derivative), Examples are:

- temperature
- torsion
- pressure
- membrane displacement
- electrical potential

1.4.1 Boundary Conditions for Elliptic PDEs

Boundary Conditions for Elliptic PDE's:

1. Dirichlet: u provided along all of edge.

2. Neumann: $\frac{\partial u}{\partial \eta}$ provided along all of the edge (derivative in normal direction).

3. Mixed: u provided for some of the edge and $\frac{\partial u}{\partial \eta}$ for the remainder of the edge.

Elliptic PDE's are analogous to Boundary Value Ordinary Differential Equations.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we derive the ADI method using numerical methods for the solution of the Laplace equation.

2.2 The Laplace Equation

The Laplaces Equation in two dimension with Dirichlet boundary conditions is given as:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, 0 < x < G,\ 0 < y < H \\ u(x,0) = f(x), \quad u(x,H) &= g(x) \\ u(0,y) = g(y), \quad u(G,y) &= h(x) \end{aligned} \tag{2.1}$$

It is a very good example of an elliptic PDE, using analytical method of solution, the general solution $u(x, y)$ of this equation is usually given as:

$$u(x, y) = \sum_{n=1}^{\infty} [(a_n \cos(\lambda x) + b_n \sin(\lambda x)) (c_n \cosh(\lambda y) + d_n \sinh(\lambda y))] \quad (2.2)$$

where $a_n, b_n, c_n,$ and d_n are constants to be determined.

2.2.1 Numerical Solution of The Laplace Equation

In this section, we consider the Laplace's equation

$$\nabla^2 u = u_x x + u_y y = 0 \quad (2.3)$$

This is one of the most important elliptic PDEs in applications. To obtain a numeric solution, we replace the partial derivatives by corresponding difference quotients, as follows. By the Taylor formula,

$$\begin{aligned} u(x+h, y) &= u(x, y) + h u_x(x, y) + \frac{1}{2} h^2 u_{xx}(x, y) + \frac{1}{6} h^3 u_{xxx}(x, y) + \cdots \quad (a) \\ u(x-h, y) &= u(x, y) - h u_x(x, y) + \frac{1}{2} h^2 u_{xx}(x, y) - \frac{1}{6} h^3 u_{xxx}(x, y) + \cdots \quad (b) \end{aligned} \quad (2.4)$$

We subtract (2.4a) from (2.4b), neglect terms in $h^3, h^4, \cdots$, and solve for u_x. Then

$$u_x(x, y) \approx \frac{1}{2h} [u(x+h, y) - u(x-h, y)]. \quad (2.5)$$

Similarly,

$$u(x, y+k) = u(x, y) + k u_y(x, y) + \frac{1}{2} k^2 u_{yy}(x, y) + \cdots$$

and

$$u(x, y-k) = u(x,y) - ku_y(x,y) + \frac{1}{2}k^2 u_{yy}(x,y) + \cdots$$

also, by subtracting and neglecting terms in $k^3, k^4, \cdots$, and solve for u_y, we obtain

$$u_y(x,y) \approx \frac{1}{2k}[u(x, y+k) - u(x, y-k)]. \tag{2.6}$$

We now turn to second derivatives, adding (2.4a) and (2.4b), neglecting terms in $h^4, h^5, \cdots$ and solving for $u_x x$, we have

$$u_{xx}(x,y) \approx \frac{1}{h^2}[u(x+h, y) - 2u(x,y) + u(x-h, y)]. \tag{2.7}$$

Similarly,

$$u_{yy}(x,y) \approx \frac{1}{k^2}[u(x, y+k) - 2u(x,y) + u(x, y-k)]. \tag{2.8}$$

We now subtitute into the Laplace's equation (2.1), equation (2.7) and (2.8), and also choosing $k = h$ to obtain a simple formula given as:

$$u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x,y) = 0 \tag{2.9}$$

This is a difference equation corresponding to (2.1), h is called the mesh size. Equation (2.9) relates u at (x, y) to u at the four neighbouring points as shown in the Fig 2.1. It has a remarkable interpretation: u at (x, y) equals the mean of the value of u at the four neighbouring points. This is an analog of the mean value property of harmonic functions.

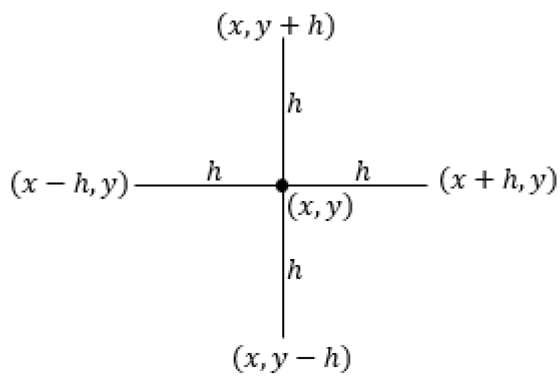

Figure 2.1: $u(x, y)$ and its neighbouring points.

2.3 Alternating Direction Implicit (ADI) Method

The ADI Method according to McDonough (2008), was first introduced by Peaceman and Rachford (1955) for solving the time-dependent heat equation in two space dimensions. It was quickly recognized that the unconditional stability of the method might render it effective as a steady-state (hence, elliptic) solver due to the possibility of employing large time steps for pseudo-time marching to a steady state. At each pseudo-time step (iteration) the discrete equations are implicitly (line-by-line) solved first in one spatial direction, and then in the other, leading to the terminology alternating direction implicit.

2.3.1 Description of the ADI Method

A matrix is called a tridiagonal matrix if at has all its non-zero entries on the main diagonal and on the two sloping parallels immediately below or above the diagonal.

$$u_{i-1,j} - 4u_{i,j} + u_{i+1,j} = -u_{i,j-1} - u_{i,j+1} \tag{2.10}$$

so that the left side belongs to y-row j only and the right side to x-column i, of course, we can also write in the form,

$$u_{i,j-1} - 4u_{i,j} + u_{i,j+1} = -u_{i-1,j} - u_{i+1,j} \tag{2.11}$$

so that the left side belongs to column i and the right side to row j.

In the **ADI method** we proceed by iteration. At every mesh point, we choose an arbitrary starting value $u_{i,j}^{(0)}$.

In each step, we compute new values at all mesh points, in one step we use an iteration formula resulting from (2.10) and in the next step, an iteration formula resulting from (2.11) and so on in **alternating** order.

In Detail: Suppose approximations $u_{i,j}^{(m)}$ have been computed, then to obtain the next approximation $u_{i,j}^{(m+1)}$, we subtitute the $u_{i,j}^{(m)}$ on the right side of (2.10) and solve for $u_{i,j}^{(m+1)}$ on the left side, that is, we use

$$u_{i-1,j}^{(m+1)} - 4u_{i,j}^{(m+1)} + u_{i+1,j}^{(m+1)} = -u_{i,j-1}^{(m)} - u_{i,j+1}^{(m)} \tag{2.12}$$

We use (2.12) for a fixed row (j) and for all internal mesh points in this row (j). This gives a linear system of N algebraic functions (N=number of internal mesh points per row) in N unknowns, the new approximations of u at these mesh points.

NOTE: that (2.12) involves not only approximations computed in the previous step but also given boundary values.

We solve the system (2.12)[j fixed!] by gauss elimination, then we go to the next row, obtain another systenm of N equations and solve it by Gauss, and repeat this until all rows are done.

In the next step, we alternate direction, that is, we compute the next approximate $u_{i,j}^{(m+2)}$ column by column from the $u_{i,j}^{(m+1)}$ and the given boundary values, using a formula obtained from (2.11) by subtituting the $u_{i,j}^{(m+1)}$ on the right.

$$u_{i,j-1}^{(m+2)} - 4u_{i,j}^{(m+2)} + u_{i,j+1}^{(m+2)} = -u_{i-1,j}^{(m+1)} - u_{i+1,j}^{(m+1)} \tag{2.13}$$

for each fixed i, that is, for each column, this is a system of M equations (M=number of internal mesh points per column) in M unknowns, which we solve by Gauss elimination, then we go to the next column, and so on until all columns are done.

2.3.2 Improving Convergence of the ADI Method

Additional improvement of the convergence of the ADI Method results from the following interesting idea. Introducing a parameter p, we can also write (2.9) in the form

$$\begin{aligned} u_{i-1,j} - (2+p)u_{ij} + u_{i+1,j} &= -u_{i,j-1} + (2-p)u_{ij} - u_{i,j+1} \\ u_{i,j-1} - (2+p)u_{ij} + u_{i,j+1} &= -u_{i-1,j} + (2-p)u_{ij} - u_{i+1,j} \end{aligned} \tag{2.14}$$

This gives the more general iteration formulas

$$\begin{aligned} u_{i-1,j}^{(m+1)} - (2+p)u_{ij}^{(m+1)} + u_{i+1,j}^{(m+1)} &= -u_{i,j-1}^{(m)} + (2-p)u_{ij}^{(m)} - u_{i,j+1}^{(m)} \qquad (a) \\ u_{i,j-1}^{(m+2)} - (2+p)u_{ij}^{(m+2)} + u_{i,j+1}^{(m+2)} &= -u_{i-1,j}^{(m+1)} + (2-p)u_{ij}^{(m+1)} - u_{i+1,j}^{(m+1)} \qquad (b) \end{aligned} \tag{2.15}$$

For $p = 2$, equations (2.15a) and (2.15b) becomes (2.12) and (2.13). The parameter p may be used for improving convergence. For positive p, the ADI method converges and the optimum value for maximum rate of convergence is

$$p_0 = 2 \sin \frac{\pi}{K}$$

where K is the larger of $M+1$ and $N+1$. Even better results can be obtained by letting p vary from step to step.

2.4 Finite Difference Method (FDM)

Laplace Equation is a second order partial differential equation(PDE) that appears in many areas of science an engineering, such as electricity, fluid flow, and steady heat conduction.

Solution of this equation, in a domain, requires the specification of certain conditions that the unknown function must satisfy at the boundary of the domain. As in the case of ordinary differential equation, the idea of finite-difference-method(FDM) is to discretize the PDE by replacing the partial derivatives with their approximations, that is, finite differences.

We will illustrate the scheme with Laplaces equation in the following.

Let us divide a two-dimensional region into small regions with increments in the x and y directions given as Δx and Δy, as shown in Fig. (2.2)

Each nodal point is designated by a numbering scheme i and j, where i indicates x increment and j indicates y increment, as shown in Fig. (2.2).

In a case study on temperature distribution, the temperature at each nodal

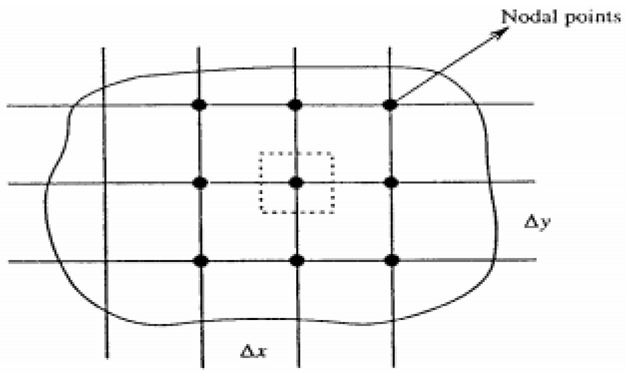

Figure 2.2: Finite differencing along x and y

point (x_i, y_j) is the average temperature of the surrounding hatched region.

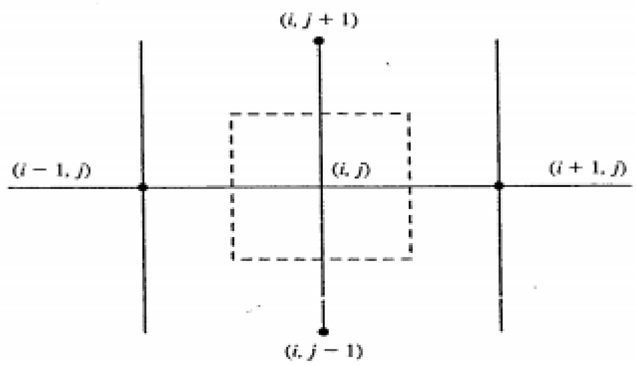

Figure 2.3: 5-point stencil for Laplace equation

A finite difference equation suitable for the interior nodes of a steady two-dimensional system can be obtained by considering Laplaces equation at the nodal point i, j as

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} + \left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j} = 0 \tag{2.16}$$

The second derivatives at the nodal point (i, j) can be approximated (derived from the Taylor series) as

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} \text{ and } \left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2} \tag{2.17}$$

then equation (2.16) becomes

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2}$$

and assuming $\Delta x = \Delta y$, the finite difference approximation of Laplaces equation for interior regions can be expressed as

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 0 \tag{2.18}$$

More accurate higher order approximations for interior nodes and boundary nodes are also obtained in a similar manner.

2.5 Method of Seperation of Variables

This is a method which provides analytical solution to partial differential equations, we illustrate this technique by providing a general solution to the Laplace equation in two dimensions. Given the boundary value problem with dirichlet boundary conditions.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < L, 0 < y < H \tag{2.19}$$

$$u(x, 0) = f(x) \qquad u(0, y) = g(y)$$

$$u(x, H) = p(x) \qquad u(L, y) = q(y)$$

Using the method of seperation of variables, we have

$$u(x,y) = X(x)Y(y) \tag{2.20}$$

which implies that

$$\frac{\partial^2 u}{\partial x^2} = X''(x)Y(y) \text{ and } \frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$$

and subtituting this into equation (2.19) gives

$$X''(x)Y(y) + X(x)Y''(y) = 0 \tag{2.21}$$

which gives

$$X''(x)Y(y) = -X(x)Y''(y)$$

which on dividing both sides by $X(x)Y(y)$, gives

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda^2, \quad \text{say} \tag{2.22}$$

and from equation (2.22), we can have two seperate ordinary differential equations given as

$$X''(x) - \lambda^2 X(x) = 0 \tag{2.23}$$

and

$$Y''(y) + \lambda^2 Y(y) = 0 \tag{2.24}$$

which when solved gives

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \tag{2.25}$$

and

$$Y(y) = C\cos(\lambda y) + D\sinh(\lambda y) \tag{2.26}$$

and subtituting equations (2.25) and (2.26) into (2.20) gives

$$u(x,y) = (A\cos(\lambda x) + B\sin(\lambda x))(C\cos(\lambda y) + D\sinh(\lambda y)) \tag{2.27}$$

where λ, A, B, C, D are solved for using the given boundary conditions.

Chapter 3

NUMERICAL SOLUTION TECHNIQUES

3.1 Introduction

In this chapter, we use the ADI method and the Finite Difference Method for the solution of the Laplace Equation in 2-Dimensions.

3.2 Demonstration of ADI method on some examples

Problem 1

Consider the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, 0 < y < 1$$

$$u(x,0) = u(0,y) = u(2,y) = 0$$

$$u(x,1) = x(2-x)$$

Exact solution is given by.

$$u(x,y) = \frac{32}{\pi^3 \sinh(\frac{\pi}{2})} \sin(\frac{\pi x}{2}) \sinh(\frac{\pi y}{2}).$$

Solution

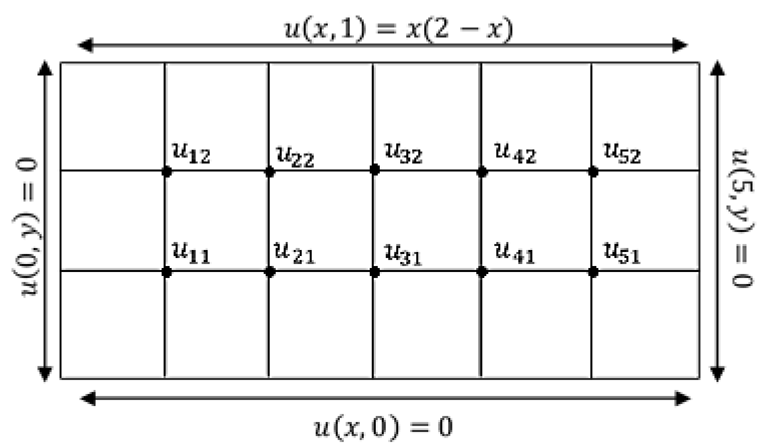

Figure 3.1: Grid showing the point $u(x,y)$

Using the ADI algorithm as discussed in section (2.3), and setting the initial values at 0, choosing $h = \frac{1}{3}$ we developed a program in MATHLAB see Appendix (1) which generated the following results.

for $m = 0$ we have

$$u_{11} = 0 \quad u_{12} = 0.237179487179487$$

$$u_{21} = 0 \quad u_{22} = 0.393162393162393$$

$$u_{31} = 0 \quad u_{32} = 0.446581196581197$$

$$u_{41} = 0 \quad u_{42} = 0.393162393162393$$

$$u_{51} = 0 \quad u_{52} = 0.237179487179487$$

for $m = 1$

$$u_{11}^{(2)} = 0.063247863247863 \quad u_{12}^{(2)} = 0.252991452991453$$

$$u_{21}^{(2)} = 0.104843304843305 \quad u_{22}^{(2)} = 0.419373219373219$$

$$u_{31}^{(2)} = 0.119088319088319 \quad u_{32}^{(2)} = 0.476353276353276$$

$$u_{41}^{(2)} = 0.104843304843305 \quad u_{42}^{(2)} = 0.419373219373219$$

$$u_{51}^{(2)} = 0.063247863247863 \quad u_{52}^{(2)} = 0.252991452991453$$

for $m = 2$

$$u_{11}^{(3)} = 0.109533201840894 \quad u_{21}^{(3)} = 0.264562787639711$$

$$u_{21}^{(3)} = 0.185141354372124 \quad u_{22}^{(3)} = 0.439447731755424$$

$$u_{31}^{(3)} = 0.211658996274381 \quad u_{32}^{(3)} = 0.499495945649792$$

$$u_{41}^{(3)} = 0.185141354372124 \quad u_{42}^{(3)} = 0.439447731755424$$

$$u_{51}^{(3)} = 0.109533201840894 \quad u_{52}^{(3)} = 0.264562787639711$$

We continued iteration up to $m = 20$, the results converges at the fourteenth iteration, a clear illustration is presented in the Tables of Results and Errors in section (4.2). We also worked on this example with a convergence parameter p, which was also solved with MATHLAB see(Appendix 1b), these results were found to counverge a $m = 11$, The table of results and errors is presented in section (4.2).

Problem 2.

Consider the value problem,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x, y < 4$$

$$u(x, 0) = u(0, y) = u(4, y) = 0, \quad u(x, 4) = \frac{1}{2} \sin\left(\frac{1}{4}\pi x\right).$$

Exact solution is given by,

$$u(x, y) = \frac{1}{2 \sinh \pi} \sin\left(\frac{1}{4}\pi x\right) \sinh\left(\frac{1}{4}\pi y\right).$$

Solution

Using the ADI method as explained, we use a grid of $1cm$, that is $h = 1$ with starting values 0.

Now, for $m = 0$, and starting with $j = 1$.

From

$$u_{i-1,j}^{(m+1)} - 4u_{i,j}^{(m+1)} + u_{i+1,j}^{(m+1)} = -u_{i,j-1}^{(m)} - u_{i,j+1}^{(m)}$$

Now for $i = 1$, we have,

$$u_{01}^{(1)} - 4u_{11}^{(1)} + u_{21}^{(1)} = -u_{10}^{(0)} - u_{12}^{(0)}$$

$$\Rightarrow \quad -4u_{11}^{(1)} + u_{21}^{(1)} = 0.$$

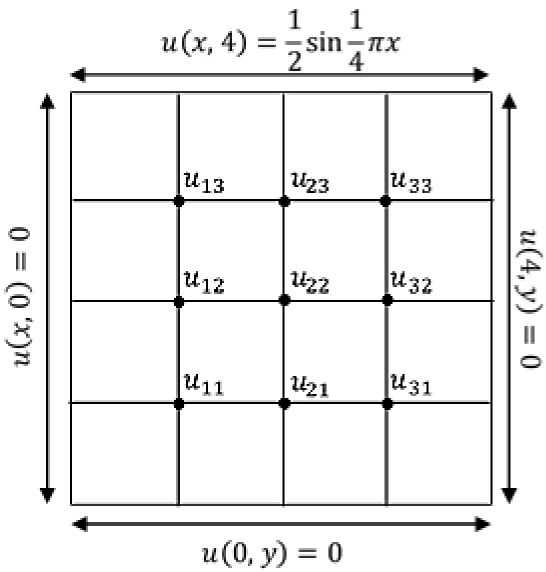

Figure 3.2: Grid showing the point $u(x, y)$

and for $i = 2$,

$$u_{11}^{(1)} - 4u_{21}^{(1)} + u_{31}^{(1)} = -u_{20}^{(0)} - u_{22}^{(0)}$$
$$\Rightarrow \quad -4u_{11}^{(1)} + u_{21}^{(1)} = 0.$$

also, for $i = 3$,we have,

$$u_{21}^{(1)} - 4u_{31}^{(1)} + u_{41}^{(1)} = -u_{30}^{(0)} - u_{32}^{(0)}$$
$$\Rightarrow \quad u_{21}^{(1)} - 4u_{31}^{(1)} = 0$$

$$\Rightarrow \quad \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11}^{(1)} \\ u_{21}^{(1)} \\ u_{31}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} u_{11}^{(1)} \\ u_{21}^{(1)} \\ u_{31}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

also for $j = 2$, $i = 1$

$$u_{02}^{(1)} - 4u_{12}^{(1)} + u_{22}^{(1)} = -u_{11}^{(0)} - u_{13}^{(0)}$$

$$\Rightarrow \quad -4u_{12}^{(1)} + u_{22}^{(1)} = 0.$$

and for $i = 2$,

$$u_{12}^{(1)} - 4u_{22}^{(1)} + u_{32}^{(1)} = -u_{21}^{(0)} - u_{23}^{(0)}$$

$$\Rightarrow \quad -4u_{12}^{(1)} + u_{22}^{(1)} = 0.$$

also, for $i = 3$,we have,

$$u_{22}^{(1)} - 4u_{32}^{(1)} + u_{42}^{(1)} = -u_{31}^{(0)} - u_{33}^{(0)}$$

$$\Rightarrow \quad u_{22}^{(1)} - 4u_{32}^{(1)} = 0$$

$$\Rightarrow \quad \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{12}^{(1)} \\ u_{22}^{(1)} \\ u_{32}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} u_{12}^{(1)} \\ u_{22}^{(1)} \\ u_{32}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

also for $j = 3$, $i = 1$

$$\begin{aligned} u_{03}^{(1)} - 4u_{13}^{(1)} + u_{23}^{(1)} &= -u_{12}^{(0)} - u_{14}^{(0)} \\ \Rightarrow \quad -4u_{13}^{(1)} + u_{23}^{(1)} &= -\frac{1}{2}\sin\left(\frac{1}{4}\pi\right). \end{aligned}$$

and for $i = 2$,

$$\begin{aligned} u_{13}^{(1)} - 4u_{23}^{(1)} + u_{33}^{(1)} &= -u_{22}^{(0)} - u_{24}^{(0)} \\ \Rightarrow \quad -4u_{13}^{(1)} + u_{23}^{(1)} &= -\frac{1}{2}\sin\left(\frac{1}{2}\pi\right). \end{aligned}$$

also, for $i = 3$,we have,

$$\begin{aligned} u_{23}^{(1)} - 4u_{33}^{(1)} + u_{43}^{(1)} &= -u_{32}^{(0)} - u_{34}^{(0)} \\ \Rightarrow \quad u_{23}^{(1)} - 4u_{33}^{(1)} &= -\frac{1}{2}\sin\left(\frac{3}{4}\pi\right). \end{aligned}$$

$$\Rightarrow \quad \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{13}^{(1)} \\ u_{23}^{(1)} \\ u_{33}^{(1)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sin\left(\frac{1}{4}\pi\right) \\ -\frac{1}{2}\sin\left(\frac{1}{2}\pi\right). \\ -\frac{1}{2}\sin\left(\frac{3}{4}\pi\right). \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} u_{13}^{(1)} \\ u_{23}^{(1)} \\ u_{33}^{(1)} \end{pmatrix} = \begin{pmatrix} 0.1367295401695 \\ 0.1933647700848 \\ 0.1367295401695 \end{pmatrix}$$

For the next iteration, we have $m = 1$ and alternate direction by choosing $i = 1$. Now for $j = 1$ we have,

$$\begin{aligned} u_{10}^{(2)} - 4u_{11}^{(2)} + u_{12}^{(2)} &= -u_{01}^{(1)} - u_{21}^{(1)} \\ \Rightarrow \quad -4u_{11}^{(1)} + u_{12}^{(1)} &= 0. \end{aligned}$$

and for $j = 2$,

$$u_{11}^{(2)} - 4u_{12}^{(2)} + u_{13}^{(2)} = -u_{02}^{(1)} - u_{22}^{(1)}$$
$$\Rightarrow \quad -4u_{11}^{(2)} + u_{12}^{(2)} = 0.$$

also, for $j = 3$,we have,

$$u_{12}^{(2)} - 4u_{13}^{(2)} + u_{14}^{(2)} = -u_{03}^{(1)} - u_{23}^{(1)}$$
$$\Rightarrow \quad u_{12}^{(2)} - 4u_{13}^{(2)} = -\frac{1}{2}\sin\left(\frac{1}{4}\pi\right) - u_{23}^{(1)}.$$

$$\Rightarrow \quad \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11}^{(2)} \\ u_{12}^{(2)} \\ u_{13}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}\sin\left(\frac{1}{4}\pi\right) - u_{23}^{(1)} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} u_{11}^{(2)} \\ u_{12}^{(2)} \\ u_{13}^{(2)} \end{pmatrix} = \begin{pmatrix} 0.0097663957264 \\ 0.0390655829056 \\ 0.1464959358959 \end{pmatrix}$$

Now, for $i = 2$, $j = 1$ we have,

$$u_{20}^{(2)} - 4u_{21}^{(2)} + u_{22}^{(2)} = -u_{11}^{(1)} - u_{31}^{(1)}$$
$$\Rightarrow \quad -4u_{21}^{(1)} + u_{22}^{(1)} = 0.$$

and for $j = 2$,

$$u_{21}^{(2)} - 4u_{22}^{(2)} + u_{23}^{(2)} = -u_{12}^{(1)} - u_{32}^{(1)}$$
$$\Rightarrow \quad -4u_{21}^{(2)} + u_{22}^{(2)} = 0.$$

also, for $j = 3$,we have,

$$u_{22}^{(2)} - 4u_{23}^{(2)} + u_{24}^{(2)} = -u_{13}^{(1)} - u_{33}^{(1)}$$
$$\Rightarrow \quad u_{22}^{(2)} - 4u_{23}^{(2)} = -\frac{1}{2}\sin\left(\frac{1}{2}\pi\right) - u_{13}^{(1)} - u_{33}^{(1)}$$

$$\Rightarrow \quad \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11}^{(2)} \\ u_{12}^{(2)} \\ u_{13}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}\sin\left(\frac{1}{2}\pi\right) - u_{13}^{(1)} - u_{33}^{(1)} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} u_{21}^{(2)} \\ u_{22}^{(2)} \\ u_{23}^{(2)} \end{pmatrix} = \begin{pmatrix} 0.0138117692918 \\ 0.0552470771671 \\ 0.2071765393765 \end{pmatrix}$$

And, for $i = 3$, $j = 1$ we have,

$$u_{30}^{(2)} - 4u_{31}^{(2)} + u_{32}^{(2)} = -u_{21}^{(1)} - u_{41}^{(1)}$$
$$\Rightarrow \quad -4u_{31}^{(1)} + u_{32}^{(1)} = 0.$$

and for $j = 2$,

$$u_{31}^{(2)} - 4u_{32}^{(2)} + u_{33}^{(2)} = -u_{22}^{(1)} - u_{42}^{(1)}$$
$$\Rightarrow \quad -4u_{31}^{(2)} + u_{32}^{(2)} = 0.$$

also, for $j = 3$,we have,

$$u_{32}^{(2)} - 4u_{33}^{(2)} + u_{34}^{(2)} = -u_{23}^{(1)} - u_{43}^{(1)}$$
$$\Rightarrow \quad u_{32}^{(2)} - 4u_{33}^{(2)} = -\frac{1}{2}\sin\left(\frac{3}{4}\pi\right) - u_{23}^{(1)}$$

$$\Rightarrow \quad \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11}^{(2)} \\ u_{12}^{(2)} \\ u_{13}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}\sin\left(\frac{3}{4}\pi\right) - u_{23}^{(1)} \end{pmatrix}.$$

$$\Rightarrow \quad \begin{pmatrix} u_{21}^{(2)} \\ u_{22}^{(2)} \\ u_{23}^{(2)} \end{pmatrix} = \begin{pmatrix} 0.0138117692918 \\ 0.0552470771671 \\ 0.2071765393765 \end{pmatrix}$$

using the above procedure,

for $m = 2$, we obtain

$$u_{11}^{(3)} = 0.0151078149135 \quad u_{12}^{(3)} = 0.0604312596541 \quad u_{13}^{(3)} = 0.1518373550830$$

$$u_{21}^{(3)} = 0.0213656767485 \quad u_{22}^{(3)} = 0.0854627069941 \quad u_{23}^{(3)} = 0.2147304468333$$

$$u_{31}^{(3)} = 0.0151078149135 \quad u_{32}^{(3)} = 0.0604312596541 \quad u_{33}^{(3)} = 0.1518373550830$$

for $m = 3$, we obtain

$$u_{11}^{(4)} = 0.0219753538684 \quad u_{12}^{(4)} = 0.0665357387251 \quad u_{13}^{(4)} = 0.1587048940379$$

$$u_{21}^{(4)} = 0.0310778434786 \quad u_{22}^{(4)} = 0.0940957440876 \quad u_{23}^{(4)} = 0.2244426135634$$

$$u_{31}^{(4)} = 0.0219753538684 \quad u_{32}^{(4)} = 0.0665357387251 \quad u_{33}^{(4)} = 0.1587048940379$$

for $m = 4$, we obtain

$$u_{11}^{(5)} = 0.0257313356420 \quad u_{12}^{(5)} = 0.0698743891905 \quad u_{13}^{(5)} = 0.1624608758115$$

$$u_{21}^{(5)} = 0.0363896038429 \quad u_{22}^{(5)} = 0.0988173088558 \quad u_{23}^{(5)} = 0.2297543739277$$

$$u_{31}^{(5)} = 0.0257313356420 \quad u_{32}^{(5)} = 0.0698743891905 \quad u_{33}^{(5)} = 0.1624608758115$$

for $m = 5$, we obtain

$$u_{11}^{(6)} = 0.0272218045998 \quad u_{12}^{(6)} = 0.0724976145562 \quad u_{13}^{(6)} = 0.1639513447693$$

$$u_{21}^{(6)} = 0.0384974452573 \quad u_{22}^{(6)} = 0.1025271097451 \quad u_{23}^{(6)} = 0.2318622153420$$

$$u_{31}^{(6)} = 0.0272218045998 \quad u_{32}^{(6)} = 0.0724976145562 \quad u_{33}^{(6)} = 0.1639513447693$$

for $m = 6$, we obtain

$$u_{11}^{(7)} = 0.0280369691407 \quad u_{12}^{(7)} = 0.0739323041483 \quad u_{13}^{(7)} = 0.1647665093102$$

$$u_{21}^{(7)} = 0.0396502620066 \quad u_{22}^{(7)} = 0.1045560672240 \quad u_{23}^{(7)} = 0.2330150320914$$

$$u_{31}^{(7)} = 0.0280369691407 \quad u_{32}^{(7)} = 0.0739323041483 \quad u_{33}^{(7)} = 0.1647665093102$$

and for $m = 7$, we obtain

$$u_{11}^{(8)} = 0.0285633325300 \quad u_{12}^{(8)} = 0.0746030681134 \quad u_{13}^{(8)} = 0.1652928726995$$

$$u_{21}^{(8)} = 0.4039465225050 \quad u_{22}^{(8)} = 0.1055046707206 \quad u_{23}^{(8)} = 0.2337594223352$$

$$u_{31}^{(8)} = 0.0285633325300 \quad u_{32}^{(8)} = 0.0746030681134 \quad u_{33}^{(8)} = 0.1652928726995$$

This results were also verified with the MATHLAB program see(Appendix 2a), where we did the iteration up to $m = 20$, these result was found to converge at $m = 14$.

The table of values comparing the results obtained with the exact solution is presented in section (4.2)

We also introduce a convergence parameter p for this problem which we solved with MATHLAB see(Appendix 2b), this result was found to converge at $m = 11$.

The table of values and errors for this solution is presented in section (4.2).

Problem 3.

Consider the boundary value problem,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 3, \quad 0 < y < 3.$$

$$u(x,0) = u(0,y) = u(3,y) = 0, \quad u(x,3) = \sin\left(\frac{1}{3}\pi x\right).$$

Exact solution is given by,

$$u(x,y) = \frac{1}{\sinh \pi} \sin\left(\frac{1}{3}\pi x\right) \sinh\left(\frac{1}{3}\pi y\right).$$

Solution

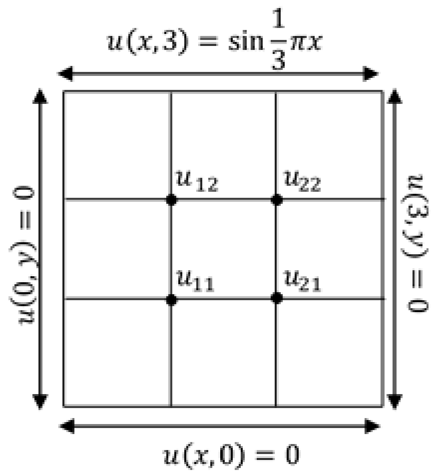

Figure 3.3: Grid showing the point $u(x, y)$

Using the ADI method as explained, we use a grid of $1cm$ with starting values 0, $m = 0$ and $j = 1$.

From

$$u_{i-1,j}^{(m+1)} - 4u_{i,j}^{(m+1)} + u_{i+1,j}^{(m+1)} = -u_{i,j-1}^{(m)} - u_{i,j+1}^{(m)}$$

Now for $i = 1$, we have,

$$u_{01}^{(1)} - 4u_{11}^{(1)} + u_{21}^{(1)} = -u_{10}^{(0)} - u_{12}^{(0)}$$

$$\Rightarrow \quad -4u_{11}^{(1)} + u_{21}^{(1)} = 0.$$

also, for $i = 2$,we have,

$$u_{11}^{(1)} - 4u_{21}^{(1)} + u_{31}^{(1)} = -u_{20}^{(0)} - u_{22}^{(0)}$$

$$\Rightarrow \quad u_{11}^{(1)} - 4u_{21}^{(1)} = 0.$$

$$\Rightarrow \quad u_{11}^{(1)} = u_{21}^{(1)} = 0.$$

also for $j = 2$. $i = 1$

$$u_{02}^{(1)} - 4u_{12}^{(1)} + u_{22}^{(1)} = -u_{11}^{(0)} - u_{13}^{(0)}$$

$$\Rightarrow \quad -4u_{12}^{(1)} + u_{22}^{(1)} = -\sin\left(\frac{\pi}{3}\right).$$

also, for $i = 2$,we have,

$$u_{12}^{(1)} - 4u_{22}^{(1)} + u_{32}^{(1)} = -u_{21}^{(0)} - u_{23}^{(0)}$$

$$\Rightarrow \quad u_{12}^{(1)} - 4u_{22}^{(1)} = -\sin\left(\frac{2\pi}{3}\right).$$

$$\Rightarrow \quad u_{12}^{(1)} = u_{22}^{(1)} = 0.2886751346.$$

For the next iteration, we have $m = 1$ and alternate direction by choosing $i = 1$. Now for $j = 1$ we have,

$$u_{10}^{(2)} - 4u_{11}^{(2)} + u_{12}^{(2)} = -u_{01}^{(1)} - u_{21}^{(1)}$$

$$\Rightarrow \quad -4u_{11}^{(2)} + u_{12}^{(2)} = 0.$$

also, for $j = 2$,we have,

$$u_{11}^{(2)} - 4u_{12}^{(2)} + u_{13}^{(2)} = -u_{02}^{(1)} - u_{22}^{(1)}$$

$$\Rightarrow \quad u_{11}^{(2)} - 4u_{12}^{(2)} = -\sin\left(\frac{\pi}{3}\right) - 0.2886751346.$$

$$\Rightarrow \quad u_{11}^{(2)} = 0.07698003589 \quad u_{12}^{(2)} = 0.3079201436$$

and for $i = 2$ we have, for $j = 1$

$$u_{20}^{(2)} - 4u_{21}^{(2)} + u_{22}^{(2)} = -u_{11}^{(1)} - u_{31}^{(1)}$$
$$\Rightarrow \quad -4u_{21}^{(2)} + u_{22}^{(2)} = 0.$$

also, for $j = 2$,we have,

$$u_{21}^{(2)} - 4u_{22}^{(2)} + u_{23}^{(2)} = -u_{12}^{(1)} - u_{32}^{(1)}$$
$$\Rightarrow \quad u_{21}^{(2)} - 4u_{22}^{(2)} = -\sin\left(\frac{2\pi}{3}\right) - 0.2886751346.$$

$$\Rightarrow \quad u_{21}^{(2)} = 0.07698003589 \quad u_{22}^{(2)} = 0.3079201436$$

and continuing iteration for $m = 2$, we have

$$u_{11}^{(3)} = 0.1026400479 \qquad u_{21}^{(3)} = 0.1026400479$$
$$u_{12}^{(3)} = 0.3143351466 \qquad u_{22}^{(3)} = 0.3143351466$$

and for $m = 3$, we have

$$u_{11}^{(4)} = 0.1060613828 \qquad u_{21}^{(4)} = 0.1060613828$$
$$u_{12}^{(4)} = 0.3216054833 \qquad u_{22}^{(4)} = 0.3216054833$$

The results obtained in this problem were also verified using a MATHLAB program, see (Appendix 3), where we did iterations up to $m = 20$, the results were found to converge at $m = 7$. Now introducing a convergence parameter p as discussed in section (2.3.2), where we have,

$$u_{i-1,j}^{(m+1)} - (2+p)u_{ij}^{(m+1)} + u_{i+1,j}^{(m+1)} = -u_{i,j-1}^{(m)} + (2-p)u_{ij}^{(m)} - u_{i,j+1}^{(m)}$$
$$u_{i,j-1}^{(m+2)} - (2+p)u_{ij}^{(m+2)} + u_{i,j+1}^{(m+2)} = -u_{i-1,j}^{(m+1)} + (2-p)u_{ij}^{(m+1)} - u_{i+1,j}^{(m+1)}$$

and

$$p_0 = 2\sin\frac{\pi}{K}$$

where K is the larger of $M+1$ and $N+1$, since $M = N = 2$, then we have $K = 1$, Hence

$$p_0 = 2\sin\frac{\pi}{3 = \sqrt{3}}$$

, subtituting $p = \sqrt{3}$ in equation (2.15), we have

$$u_{i-1,j}^{(m+1)} - (2+\sqrt{3})u_{ij}^{(m+1)} + u_{i+1,j}^{(m+1)} = -u_{i,j-1}^{(m)} + (2-\sqrt{3})u_{ij}^{(m)} - u_{i,j+1}^{(m)}$$

$$u_{i,j-1}^{(m+2)} - (2+\sqrt{3})u_{ij}^{(m+2)} + u_{i,j+1}^{(m+2)} = -u_{i-1,j}^{(m+1)} + (2-\sqrt{3})u_{ij}^{(m+1)} - u_{i+1,j}^{(m+1)}$$

and starting, also with starting values 0, setting $m = 0$ and $j = 1$,we have

$$fori = 1 \quad u_{01}^{(1)} - (2+\sqrt{3})u_{11}^{(1)} + u_{21}^{(1)} = -u_{10}^{(0)} + (2-\sqrt{3})u_{11}^{(0)} - u_{12}^{(0)}$$

$$\Rightarrow \quad -(2+\sqrt{3})u_{11}^{(1)} + u_{21}^{(1)} = 0$$

$$fori = 2 \quad u_{11}^{(1)} - (2+\sqrt{3})u_{21}^{(1)} + u_{31}^{(1)} = -u_{20}^{(0)} + (2-\sqrt{3})u_{21}^{(0)} - u_{22}^{(0)}$$

$$\Rightarrow \quad u_{11}^{(1)} - (2+\sqrt{3})u_{21}^{(1)} = 0$$

$$\Rightarrow \quad u_{11}^{(1)} = u_{21}^{(1)} = 0$$

Now for $j = 2$,

$$fori = 1 \quad u_{02}^{(1)} - (2+\sqrt{3})u_{12}^{(1)} + u_{22}^{(1)} = -u_{11}^{(0)} + (2-\sqrt{3})u_{12}^{(0)} - u_{13}^{(0)}$$

$$\Rightarrow \quad -(2+\sqrt{3})u_{12}^{(1)} + u_{22}^{(1)} = -\sin\left(\frac{\pi}{3}\right).$$

$$fori = 2 \quad u_{12}^{(1)} - (2+\sqrt{3})u_{22}^{(1)} + u_{32}^{(1)} = -u_{21}^{(0)} + (2-\sqrt{3})u_{22}^{(0)} - u_{23}^{(0)}$$

$$\Rightarrow \quad u_{12}^{(1)} - (2+\sqrt{3})u_{22}^{(1)} = -\sin\left(\frac{2\pi}{3}\right).$$

$$\Rightarrow \quad u_{11}^{(1)} = u_{21}^{(1)} = 0.3169872981$$

For the next iteration, we have $m = 1$ and alternate direction by choosing $i = 1$. Now for $j = 1$ we have,

$$u_{10}^{(2)} - (2+\sqrt{3})u_{11}^{(2)} + u_{12}^{(2)} = -u_{01}^{(1)} + (2-\sqrt{3})u_{11}^{(1)} - u_{21}^{(1)}$$

$$\Rightarrow \quad -(2+\sqrt{3})u_{11}^{(2)} + u_{12}^{(2)} = 0$$

for $j = 2$,

$$u_{11}^{(2)} - (2+\sqrt{3})u_{12}^{(2)} + u_{13}^{(2)} = -u_{02}^{(1)} + (2-\sqrt{3})u_{12}^{(1)} - u_{22}^{(1)}$$

$$\Rightarrow \quad u_{11}^{(2)} - (2+\sqrt{3})u_{12}^{(2)} = -\sin\left(\frac{\pi}{3}\right) + (2-\sqrt{3})u_{12}^{(1)} - u_{22}^{(1)}$$

$$\Rightarrow \quad u_{11}^{(2)} = 0.0849364905 \quad u_{12}^{(2)} = 0.3169872981$$

and repeating the procedure for $i = 2$, we have

$$u_{21}^{(2)} = 0.0849364905 \quad u_{22}^{(2)} = 0.3169872981$$

and continuing iteration for $m = 2$, we have

$$u_{11}^{(3)} = 0.1076951546 \qquad u_{21}^{(3)} = 0.1076951546$$
$$u_{12}^{(3)} = 0.3169872981 \qquad u_{22}^{(3)} = 0.3169872981$$

and for $m = 3$, we have

$$u_{11}^{(4)} = 0.1076951546 \qquad u_{21}^{(4)} = 0.1076951546$$
$$u_{12}^{(4)} = 0.3230854638 \qquad u_{22}^{(4)} = 0.3230854638$$

The results obtained in this problem were also verified using a MATHLAB program, see (Appendix 3b), where we did iterations up to $m = 20$, the results were found to converge at $m = 4$. The table of values comparing the results obtained with the exact solution is presented in section (4.2)

3.3 Demonstration of FDM on some examples

Problem 1

Consider the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, 0 < y < 1$$

$$u(x,0) = u(0,y) = u(2,y) = 0$$

$$u(x,1) = x(2-x)$$

Exact solution is given by.

$$u(x,y) = \frac{32}{\pi^3 \sinh(\frac{\pi}{2})} \sin(\frac{\pi x}{2}) \sinh(\frac{\pi y}{2}).$$

Solution

Following the FDM algorithm as explained in section (2.4), and using the mesh points as shown in Figure (3.1); we have

$$\begin{pmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \\ u_{41} \\ u_{42} \\ u_{51} \\ u_{52} \end{pmatrix} = - \begin{pmatrix} 0 \\ \frac{1}{3}\left(2-\frac{1}{3}\right) \\ 0 \\ \frac{2}{3}\left(2-\frac{2}{3}\right) \\ 0 \\ \frac{3}{3}\left(2-\frac{3}{3}\right) \\ 0 \\ \frac{4}{3}\left(2-\frac{4}{3}\right) \\ 0 \\ \frac{5}{3}\left(2-\frac{5}{3}\right) \end{pmatrix}$$

which when solved simultaneously gives the following results

$$\begin{array}{ll} u_{11} = 0.1270482604 & u_{12} = 0.2927048260 \\ u_{21} = 0.2154882155 & u_{22} = 0.4882154882 \\ u_{31} = 0.2466891134 & u_{32} = 0.5557800224 \\ u_{41} = 0.2154882155 & u_{42} = 0.4882154882 \\ u_{51} = 0.1270482604 & u_{52} = 0.2927048260 \end{array}$$

Problem 2.

Consider the value problem,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x, y < 4$$

$$u(x,0) = u(0,y) = u(4,y) = 0, \quad u(x,4) = \frac{1}{2}\sin\left(\frac{1}{4}\pi x\right).$$

Exact solution is given by,

$$u(x,y) = \frac{1}{2\sinh\pi}\sin\left(\frac{1}{4}\pi x\right)\sinh\left(\frac{1}{4}\pi y\right).$$

Solution

Using the FDM algorithm as explained in section (2.4), and mesh points as shown in Figure (3.2); we have

$$\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}\sin(\frac{1}{4}\pi) \\ 0 \\ 0 \\ \frac{1}{2}\sin(\frac{1}{2}\pi) \\ 0 \\ 0 \\ \frac{1}{2}\sin(\frac{3}{4}\pi) \end{pmatrix}$$

which when solved simultaneously gives the following results

$$u_{11} = 0.0291764907 \quad u_{12} = 0.0754441738 \quad u_{13} = 0.1659060308$$

$$u_{21} = 0.0412617888 \quad u_{22} = 0.1066941738 \quad u_{23} = 0.2346265589$$

$$u_{31} = 0.0291764907 \quad u_{32} = 0.0754441738 \quad u_{33} = 0.1659060308$$

Problem 3.

Consider the boundary value problem,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 3, \quad 0 < y < 3.$$

$$u(x,0) = u(0,y) = u(3,y) = 0, \quad u(x,3) = \sin\left(\frac{1}{3}\pi x\right).$$

Exact solution is given by,

$$u(x,y) = \frac{1}{\sinh \pi} \sin\left(\frac{1}{3}\pi x\right) \sinh\left(\frac{1}{3}\pi y\right).$$

Solution

Using the FDM algorithm as explained in section (2.4), and mesh points as shown in Figure (3.3); we have

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} = - \begin{pmatrix} 0 \\ \sin(\frac{1}{3}\pi) \\ 0 \\ \sin(\frac{2}{3}\pi) \end{pmatrix}$$

which when solved simultaneously gives the following results

$$u_{11} = 0.1082531755 \qquad u_{12} = 0.3247595264$$

$$u_{21} = 0.1082531755 \qquad u_{22} = 0.3247595264$$

Chapter 4

TABLES OF RESULTS AND ERRORS

4.1 Introduction

In this chapter, we present the tables of errors and results comparing the computed solution at the different values of m, with the solution obtained with FDM and the exact solution.

4.2 Results obtained by ADI Method on some examples

Table 1: of Results for Problem 1, at m=5,10,15,20

u_{ij}	Exact	FDM	$m = 5$	$m = 10$	$m = 15$	$m = 20$
u_{11}	0.12284621	0.12704826	0.12341137	0.12695526	0.12704470	0.12704816
u_{12}	0.28014791	0.29270483	0.28963036	0.29260604	0.29270121	0.29270472
u_{21}	0.21277587	0.21548822	0.20918779	0.21532713	0.21548204	0.21548804
u_{22}	0.48523041	0.48821549	0.48288951	0.48804438	0.48820923	0.48821531
u_{31}	0.24569242	0.24668911	0.23941533	0.24650311	0.24668198	0.24668891
u_{32}	0.56029581	0.55578002	0.54963108	0.55558245	0.55577280	0.55577982
u_{41}	0.21277587	0.21548822	0.20918779	0.21532713	0.21548204	0.21548804
u_{42}	0.48523041	0.48821549	0.48288951	0.48804438	0.48820923	0.48821531
u_{51}	0.12284621	0.12704826	0.12341137	0.12695526	0.12704470	0.12704816
u_{52}	0.28014791	0.29270483	0.28963036	0.29260604	0.29270121	0.29270472

Table 2: of Results for Problem 1, with $p = 1$ at m=5,10,15,20

u_{ij}	Exact	FDM	$m = 5$	$m = 10$	$m = 15$	$m = 20$
u_{11}	0.12284621	0.12704826	0.12517147	0.12729735	0.12705208	0.12704776
u_{12}	0.28014791	0.29270483	0.29458162	0.29245574	0.29270101	0.29270533
u_{21}	0.21277587	0.21548822	0.21219136	0.21592078	0.21549479	0.21548735
u_{22}	0.48523041	0.48821549	0.49151235	0.48778292	0.48820891	0.48821635
u_{31}	0.24569242	0.24668911	0.24288409	0.24718825	0.24669674	0.24668811
u_{32}	0.56029581	0.55578002	0.55958505	0.55528089	0.55577239	0.55578102
u_{41}	0.21277587	0.21548822	0.21219136	0.21592078	0.21549479	0.21548735
u_{42}	0.48523041	0.48821549	0.49151235	0.48778292	0.48820891	0.48821635
u_{51}	0.12284621	0.12704826	0.12517147	0.12729735	0.12705208	0.12704776
u_{52}	0.28014791	0.29270483	0.29458162	0.29245574	0.29270101	0.29270533

Table 3: of Results for Problem 2, at m=5,10,15,20

u_{ij}	Exact	FDM	$m=5$	$m=10$	$m=15$	$m=20$
u_{11}	0.02659351	0.02917649	0.02722180	0.02907697	0.02917165	0.02917625
u_{12}	0.07045202	0.07544417	0.07249761	0.07530546	0.07543734	0.07544384
u_{13}	0.16004926	0.16590603	0.16395134	0.16580651	0.16590119	0.16590579
u_{21}	0.03760891	0.04126179	0.03849745	0.04112104	0.04125495	0.04126145
u_{22}	0.09963420	0.10669417	0.10252711	0.10649801	0.10668451	0.10669370
u_{23}	0.22634384	0.23462656	0.23186222	0.23448581	0.23461972	0.23462622
u_{31}	0.02659351	0.02917649	0.02722180	0.02907697	0.02917165	0.02917625
u_{32}	0.07045202	0.07544417	0.07249761	0.07530546	0.07543734	0.07544384
u_{33}	0.16004926	0.16590603	0.16395134	0.16580651	0.16590119	0.16590579

Table 4: of Results for Problem 2 with $p = \sqrt{2}$, at m=5,10,15,20

u_{ij}	Exact	FDM	$m = 5$	$m = 10$	$m = 15$	$m = 20$
u_{11}	0.02659351	0.02917649	0.02888252	0.02916844	0.02917642	0.02917649
u_{12}	0.07045202	0.07544417	0.07474747	0.07542366	0.07544412	0.07544417
u_{13}	0.16004926	0.16590603	0.16566113	0.16589822	0.16590596	0.16590603
u_{21}	0.03760891	0.04126179	0.04084604	0.04125040	0.04126169	0.04126179
u_{22}	0.09963420	0.10669417	0.10570888	0.10666517	0.10669409	0.10669417
u_{23}	0.22634384	0.23462656	0.23428022	0.23461552	0.23462646	0.23462656
u_{31}	0.02659351	0.02917649	0.02888252	0.02916844	0.02917642	0.02917649
u_{32}	0.07045202	0.07544417	0.07474747	0.07542366	0.07544412	0.07544417
u_{33}	0.16004926	0.16590603	0.16566113	0.16589822	0.16590596	0.16590603

Table 5: of Results for Problem 3, at m=5,10,15,20

u_{ij}	Exact	FDM	$m=5$	$m=10$	$m=15$	$m=20$
u_{11}	0.09368846	0.10825318	0.10792411	0.10825200	0.10825317	0.10825318
u_{12}	0.29985682	0.32475953	0.32449461	0.32475826	0.32475952	0.32475953
u_{21}	0.09368846	0.10825318	0.10792411	0.10825200	0.10825317	0.10825318
u_{22}	0.29985682	0.32475953	0.32449461	0.32475826	0.32475952	0.32475953

Table 6: of Results for Problem 3 with $p=\sqrt{3}$, at m=5,10,15,20

u_{ij}	Exact	FDM	$m=5$	$m=10$	$m=15$	$m=20$
u_{11}	0.09368846	0.10825318	0.10813298	0.10825316	0.10825318	0.10825318
u_{12}	0.29985682	0.10825318	0.32471946	0.32475932	0.32475953	0.32475953
u_{21}	0.09368846	0.32475953	0.10813298	0.10825316	0.10825318	0.10825318
u_{22}	0.29985682	0.32475953	0.32471946	0.32475932	0.32475953	0.32475953

4.3 Tables of Errors obtained from ADI and FDM

Table 7: of Errors for Problem 1, at m=5,10,15,20

u_{ij}	FDM	$m = 5$	$m = 10$	$m = 15$	$m = 20$
u_{11}	4.20E-3	5.65E-4	4.11E-3	4.20E-3	4.20E-3
u_{12}	1.26E-2	9.48E-3	1.25E-2	1.26E-2	1.26E-2
u_{21}	2.71E-3	3.59E-3	2.55E-3	2.71E-3	2.71E-3
u_{22}	2.99E-3	2.34E-3	2.81E-3	2.98E-3	2.98E-3
u_{31}	9.97E-4	6.28E-3	8.11E-4	9.90E-4	1.00E-3
u_{32}	4.52E-3	1.07E-2	4.71E-3	4.52E-3	4.52E-3
u_{41}	2.71E-3	3.59E-3	2.55E-3	2.71E-3	2.71E-3
u_{42}	2.99E-3	2.34E-3	2.81E-3	2.98E-3	2.98E-3
u_{51}	4.20E-3	5.65E-4	4.11E-3	4.20E-3	4.20E-3
u_{52}	1.26E-2	9.48E-3	1.25E-2	1.26E-2	1.26E-2

Table 8: of Errors for Problem 1, with $p = 1$ at m=5,10,15,20

u_{ij}	FDM	$m = 5$	$m = 10$	$m = 15$	$m = 20$
u_{11}	4.20E-3	2.33E-2	4.45E-3	4.21E-3	4.20E-3
u_{12}	1.26E-2	1.44E-2	1.23E-2	1.26E-2	1.26E-2
u_{21}	2.71E-3	5.85E-4	3.14E-3	2.72E-3	2.71E-3
u_{22}	2.99E-3	6.28E-3	2.55E-3	2.98E-3	2.99E-3
u_{31}	9.97E-4	2.81E-3	1.50E-3	1.00E-3	9.96E-4
u_{32}	4.52E-3	7.11E-4	5.01E-3	4.52E-3	4.51E-3
u_{41}	2.71E-3	5.85E-4	3.14E-3	2.72E-3	2.71E-3
u_{42}	2.99E-3	6.28E-3	2.55E-3	2.98E-3	2.99E-3
u_{51}	4.20E-3	2.33E-2	4.45E-3	4.21E-3	4.20E-3
u_{52}	1.26E-2	1.44E-2	1.23E-2	1.26E-2	1.26E-2

Table 9: of Errors for Problem 2, at m=5,10,15,20

u_{ij}	FDM	$m = 5$	$m = 10$	$m = 15$	$m = 20$
u_{11}	2.58E-3	6.28E-4	2.48E-3	2.58E-3	2.58E-3
u_{12}	4.99E-3	2.05E-3	4.85E-3	4.99E-3	4.99E-3
u_{13}	5.86E-3	3.90E-3	5.76E-3	5.85E-3	5.86E-3
u_{21}	3.65E-3	8.89E-4	3.51E-3	3.65E-3	3.65E-3
u_{22}	7.06E-3	2.89E-3	6.86E-3	7.05E-3	7.06E-3
u_{23}	8.28E-3	5.52E-3	8.14E-3	8.28E-3	8.28E-3
u_{31}	2.58E-3	6.28E-4	2.48E-3	2.58E-3	2.58E-3
u_{32}	4.99E-3	2.05E-3	4.85E-3	4.99E-3	4.99E-3
u_{33}	5.86E-3	3.90E-3	5.76E-3	5.85E-3	5.86E-3

Table 10: of Errors for Problem 2 with $p = \sqrt{2}$, at m=5,10,15,20

u_{ij}	FDM	$m = 5$	$m = 10$	$m = 15$	$m = 20$
u_{11}	2.58E-3	2.29E-3	2.57E-3	2.58E-3	2.58E-3
u_{12}	4.99E-3	4.30E-3	4.99E-3	4.99E-3	4.99E-3
u_{13}	5.86E-3	5.61E-3	5.85E-3	5.86E-3	5.86E-3
u_{21}	3.65E-3	3.23E-3	3.64E-3	3.65E-3	3.65E-3
u_{22}	7.06E-3	6.07E-3	7.06E-3	7.06E-3	7.06E-3
u_{23}	8.28E-3	7.94E-3	8.27E-3	8.28E-3	8.28E-3
u_{31}	2.58E-3	2.29E-3	2.57E-3	2.58E-3	2.58E-3
u_{32}	4.99E-3	4.30E-3	4.99E-3	4.99E-3	4.99E-3
u_{33}	5.86E-3	5.61E-3	5.85E-3	5.86E-3	5.86E-3

Table 11: of Errors for Problem 3, at m=5,10,15,20

u_{ij}	FDM	$m=5$	$m=10$	$m=15$	$m=20$
u_{11}	1.46E-2	1.43E-2	1.46E-2	1.46E-2	1.46E-2
u_{12}	2.49E-2	2.46E-2	2.49E-2	2.49E-2	2.49E-2
u_{21}	1.46E-2	1.43E-2	1.46E-2	1.46E-2	1.46E-2
u_{22}	2.49E-2	2.46E-2	2.49E-2	2.49E-2	2.49E-2

Table 12: of Errors for Problem 3 with $p=\sqrt{3}$, at m=5,10,15,20

u_{ij}	FDM	$m=5$	$m=10$	$m=15$	$m=20$
u_{11}	1.46E-2	1.44E-2	1.46E-2	1.46E-2	1.46E-2
u_{12}	2.49E-2	2.49E-2	2.49E-2	2.49E-2	2.49E-2
u_{21}	1.46E-2	1.44E-2	1.46E-2	1.46E-2	1.46E-2
u_{22}	2.49E-2	2.49E-2	2.49E-2	2.49E-2	2.49E-2

Chapter 5

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

The Alternating Direction Implicit (ADI) method has been presented for the purpose of application to some selected problem and with a view to discussing the accuracy of the technique.

Several problems have been solved and the ... obtained from the tables of numerical results show that the method is accurate and is desirable for use in the solution of elliptic partial differential equation.

5.2 Recommendation

The accuracy of most numerical technique for solving mathematical problems improve as the meshes and/or steplenghts are refined.

It is recommended, therefore, that the mesh lenghts used in this study be reduced to smaller sizes. This will lead to better accuracy, however at a higher cost of computation.

REFERENCES

D. McMahon, (2007), *"Mathlab Demystified: A Self Teaching Guide"*, The McGraw-Hill Companies.

E. Kreyzig (2006), *"Advanced Engineering Mathematics"*, John Wiley and Sons Inc.

J. M. McDonough (2008) *"Lectures on Computational Numerical Analysis of Partial Differential Equations"*, Departments of Mechanical Engineering and Mathematics, University of Kentucky.

K.F. Riley, M.P. Hobson and S.J. Bence (2006) *"Mathematical Methods for Physics and Engineering"*, Cambridge University Press.

K.W. Morton, and D. F. Mayers (2005), *"Numerical Solution of Partial Differential Equations"*, Cambridge University Press.

P. Blomgren (2012), *"Numerical Solution of PDE"*, Lectures Notes #13, Department of Mathematics and Statistics, San Diego State University.

R.B. Adeniyi (2013), *"MAT402, Partial Differential Equations"*, Lecture Notes, Department of Mathematics, University of Ilorin.

APPENDIX

Appendix 1a

```
clc
disp('MATLAB Program for Solution of Problem 1, ADI Without p')
format long
A=[0,0,0,0,0;0,0,0,0,0]';
M=[-4 1 0 0 0;1 -4 1 0 0;0 1 -4 1 0;0 0 1 -4 1;0 0 0 1 -4];
N=[-4 1;1 -4];
B=[1/3:1/3:5/3];
B=[B.*(2-B)]';
D=A;
Exact=A;
x=[1/3:1/3:5/3];
    Exact(:,1)=(32/(pi^3*sinh(pi/2)))*sin(pi*x/2)*sinh(pi*(1/3)/2);
    Exact(:,2)=(32/(pi^3*sinh(pi/2)))*sin(pi*x/2)*sinh(pi*(2/3)/2);
Exact
for m=0: 1: 20
    marker=rem(m,2);
    if marker==0
        D(:,1)=-M\A(:,2);
        D(:,2)=-M\(A(:,1)+B);
    else
        D(1,:)=-N\[A(2,1); A(2,2)+B(1)];
        D(2,:)=-N\[A(1,1)+A(3,1);A(1,2)+A(3,2)+B(2)];
        D(3,:)=-N\[A(2,1)+A(4,1);A(2,2)+A(4,2)+B(3)];
        D(4,:)=-N\[A(3,1)+A(5,1);A(3,2)+A(5,2)+B(4)];
        D(5,:)=-N\[A(4,1);A(4,2)+B(5)];
    end
    A=D;
    disp('for')
    m
    disp('computed solution is')
    A
    disp('and computed error is')
    Error=Exact-A;
    Error=abs(Error)
end
```

```
Exact =

   0.122846207659390   0.280147907144299
   0.212775873183221   0.485230408808015
   0.245692415318781   0.560295814288599
   0.212775873183221   0.485230408808015
   0.122846207659390   0.280147907144299

for

m =

     0

computed solution is

A =

                   0   0.237179487179487
                   0   0.393162393162393
                   0   0.446581196581197
                   0   0.393162393162393
                   0   0.237179487179487

and computed error is

Error =

   0.122846207659390   0.042968419964812
   0.212775873183221   0.092068015645621
   0.245692415318781   0.113714617707402
   0.212775873183221   0.092068015645621
   0.122846207659390   0.042968419964812

for

m =

     1

computed solution is

A =

   0.063247863247863   0.252991452991453
   0.104843304843305   0.419373219373219
   0.119088319088319   0.476353276353276
   0.104843304843305   0.419373219373219
   0.063247863247863   0.252991452991453

and computed error is

Error =

   0.059598344411527   0.027156454152846
   0.107932568339916   0.065857189434795
   0.126604096230462   0.083942537935322
   0.107932568339916   0.065857189434795
   0.059598344411527   0.027156454152846

for

m =

     2

computed solution is

A =

   0.109533201840894   0.264562787639711
   0.185141354372124   0.439447731755424
   0.211658996274381   0.499495945649792
   0.185141354372124   0.439447731755424
   0.109533201840894   0.264562787639711

and computed error is

Error =

   0.013313005818496   0.015585119504589
   0.027634518811098   0.045782677052591
   0.034033419044400   0.060799868638807
   0.027634518811098   0.045782677052591
   0.013313005818496   0.015585119504589

for

m =

     3

computed solution is

A =

   0.115704580319965   0.277676966907736
   0.195847760975966   0.462198845788589
   0.224001753232522   0.525724304185843
   0.195847760975966   0.462198845788589
   0.115704580319965   0.277676966907736
```

```
Exact =

   0.122846207659390   0.280147907144299
   0.212775873183221   0.485230408808015
   0.245692415318781   0.560295814288599
   0.212775873183221   0.485230408808015
   0.122846207659390   0.280147907144299

for

m =

     0

computed solution is

A =

                   0   0.237179487179487
                   0   0.393162393162393
                   0   0.446581196581197
                   0   0.393162393162393
                   0   0.237179487179487

and computed error is

Error =

   0.122846207659390   0.042968419964812
   0.212775873183221   0.092068015645621
   0.245692415318781   0.113714617707402
   0.212775873183221   0.092068015645621
   0.122846207659390   0.042968419964812
for

m =

     1

computed solution is

A =

   0.063247863247863   0.252991452991453
   0.104843304843305   0.419373219373219
   0.119088319088319   0.476353276353276
   0.104843304843305   0.419373219373219
   0.063247863247863   0.252991452991453

and computed error is

Error =

   0.059598344411527   0.027156454152846
   0.107932568339916   0.065857189434795
   0.126604096230462   0.083942537935322
   0.107932568339916   0.065857189434795
   0.059598344411527   0.027156454152846

for

m =

     2

computed solution is

A =

   0.109533201840894   0.264562787639711
   0.185141354372124   0.439447731755424
   0.211658996274381   0.499495945649792
   0.185141354372124   0.439447731755424
   0.109533201840894   0.264562787639711

and computed error is

Error =

   0.013313005818496   0.015585119504589
   0.027634518811098   0.045782677052591
   0.034033419044400   0.060799868638807
   0.027634518811098   0.045782677052591
   0.013313005818496   0.015585119504589

for

m =

     3

computed solution is

A =

   0.115704580319965   0.277676966907736
   0.195847760975966   0.462198845788589
   0.224001753232522   0.525724304185843
   0.195847760975966   0.462198845788589
   0.115704580319965   0.277676966907736
```

```
and computed error is

Error =

   0.003390659414375   0.011659602102749
   0.001307058922388   0.001430990438101
   0.000626088578657   0.006310425424133
   0.001307058922388   0.001430990438101
   0.003390659414375   0.011659602102749

for

m =

     8

computed solution is

A =

   0.126652617602069   0.292347067837089
   0.214802961161229   0.487595848719034
   0.245897827796731   0.555064506044548
   0.214802961161229   0.487595848719034
   0.126652617602069   0.292347067837089

and computed error is

Error =

   0.003806409942679   0.012199160692789
   0.002027087978008   0.002365439911019
   0.000205412477950   0.005231308244051
   0.002027087978008   0.002365439911019
   0.003806409942679   0.012199160692789

for

m =

     9

computed solution is

A =

   0.126824216594634   0.292493905217306
   0.215100149624382   0.487850153098727
   0.246241025781860   0.555358180804982
   0.215100149624382   0.487850153098727
   0.126824216594634   0.292493905217306
```

```
and computed error is

Error =

   0.003978008935243   0.012345998073006
   0.002324276441161   0.002619744290712
   0.000548610463079   0.004937633483617
   0.002324276441161   0.002619744290712
   0.003978008935243   0.012345998073006

for

m =

    10

computed solution is

A =

   0.126955258966196   0.292606038268031
   0.215327130647477   0.488044380921933
   0.246503110524984   0.555582446906432
   0.215327130647476   0.488044380921933
   0.126955258966196   0.292606038268031

and computed error is

Error =

   0.004109051306805   0.012458131123731
   0.002551257464255   0.002813972113919
   0.000810695206203   0.004713367382167
   0.002551257464255   0.002813972113919
   0.004109051306805   0.012458131123731

for

m =

    11

computed solution is

A =

   0.126993897271160   0.292648458437162
   0.215394056801871   0.488117857716306
   0.246580387134912   0.555667287244695
   0.215394056801871   0.488117857716306
   0.126993897271160   0.292648458437162
```

```
and computed error is

Error =

   0.004147689611769   0.012500551292863
   0.002618183618650   0.002887448908291
   0.000887971816131   0.004628527043904
   0.002618183618650   0.002887448908291
   0.004147689611769   0.012500551292863

for

m =

    12

computed solution is

A =

   0.127023406465965   0.292680855951384
   0.215445167426698   0.488173970978821
   0.246639405524523   0.555732082273138
   0.215445167426698   0.488173970978821
   0.127023406465965   0.292680855951384

and computed error is

Error =

   0.004177198806575   0.012532948807085
   0.002669294243477   0.002943562170806
   0.000946990205742   0.004563732015460
   0.002669294243477   0.002943562170806
   0.004177198806575   0.012532948807085

for

m =

    13

computed solution is

A =

   0.127034013082745   0.292690884904280
   0.215463538338358   0.488191341362942
   0.246660618758082   0.555752140178931
   0.215463538338358   0.488191341362942
   0.127034013082745   0.292690884904280
```

```
and computed error is

Error =

   0.004187805423354   0.012542977759981
   0.002687665155136   0.002960932554928
   0.000968203439301   0.004543674109668
   0.002687665155136   0.002960932554928
   0.004187805423354   0.012542977759981

for

m =

    14

computed solution is

A =

   0.127042113351743   0.292698544011601
   0.215477568502691   0.488204607408105
   0.246676819296078   0.555767458393573
   0.215477568502691   0.488204607408105
   0.127042113351743   0.292698544011601

and computed error is

Error =

   0.004195905692352   0.012550636867302
   0.002701695319470   0.002974198600091
   0.000984403977297   0.004528355895026
   0.002701695319470   0.002974198600091
   0.004195905692352   0.012550636867302

for

m =

    15

computed solution is

A =

   0.127044695798295   0.292701214690489
   0.215482041459023   0.488209233188272
   0.246681984189182   0.555772799751348
   0.215482041459023   0.488209233188272
   0.127044695798295   0.292701214690489
```

```
and computed error is

Error =

   0.004198488138904   0.012553307546190
   0.002706168275802   0.002978824380257
   0.000989568870402   0.004523014537251
   0.002706168275802   0.002978824380257
   0.004198488138904   0.012553307546190

for

m =

    16

computed solution is

A =

   0.127046668041755   0.292703254317976
   0.215485457476533   0.488212765918052
   0.246685928676103   0.555776879006322
   0.215485457476533   0.488212765918052
   0.127046668041755   0.292703254317976

and computed error is

Error =

   0.004200460382365   0.012555347173676
   0.002709584293311   0.002982357110038
   0.000993513357322   0.004518935282277
   0.002709584293311   0.002982357110038
   0.004200460382365   0.012555347173676

for

m =

    17

computed solution is

A =

   0.127047343425316   0.292703916224731
   0.215486627272308   0.488213912371374
   0.246687279443224   0.555778202819832
   0.215486627272308   0.488213912371374
   0.127047343425316   0.292703916224731
```

```
and computed error is

Error =

   0.004201135765925   0.012556009080432
   0.002710754089087   0.002983503563359
   0.000994864124443   0.004517611468767
   0.002710754089087   0.002983503563359
   0.004201135765925   0.012556009080432

for

m =

    18

computed solution is

A =

   0.127047859220222   0.292704421727312
   0.215487520656158   0.488214787928376
   0.246688311033037   0.555779213824994
   0.215487520656158   0.488214787928376
   0.127047859220222   0.292704421727312

and computed error is

Error =

   0.004201651560832   0.012556514583012
   0.002711647472937   0.002984379120361
   0.000995895714256   0.004516600463605
   0.002711647472937   0.002984379120361
   0.004201651560832   0.012556514583013

for

m =

    19

computed solution is

A =

   0.127048028407238   0.292704592972792
   0.215487813696949   0.488215084534536
   0.246688649407068   0.555779556315955
   0.215487813696949   0.488215084534536
   0.127048028407238   0.292704592972792
```

```
and computed error is

Error =

   0.004201820747847   0.012556685828493
   0.002711940513728   0.002984675726521
   0.000996234088287   0.004516257972644
   0.002711940513728   0.002984675726521
   0.004201820747847   0.012556685828493

for

m =

    20

computed solution is

A =

   0.127048157616792   0.292704723754414
   0.215488037494376   0.488215311054862
   0.246688907826177   0.555779817879198
   0.215488037494376   0.488215311054862
   0.127048157616792   0.292704723754414

and computed error is

Error =

   0.004201949957402   0.012556816610115
   0.002712164311155   0.002984902246848
   0.000996492507396   0.004515996409401
   0.002712164311155   0.002984902246848
   0.004201949957402   0.012556816610115

>>
```

Appendix 1b

```
clc
disp('MATLAB Program for Solution of Problem 1, ADI With p=1')
format long
A=[0,0,0,0,0;0,0,0,0,0]';
M=[-3 1 0 0 0;1 -3 1 0 0;0 1 -3 1 0;0 0 1 -3 1;0 0 0 1 -3];
N=[-3 1;1 -3];
B=[1/3:1/3:5/3];
B=[B.*(2-B)]';
D=A;
Exact=A;
x=[1/3:1/3:5/3];
    Exact(:,1)=(32/(pi^3*sinh(pi/2)))*sin(pi*x/2)*sinh(pi*(1/3)/2);
    Exact(:,2)=(32/(pi^3*sinh(pi/2)))*sin(pi*x/2)*sinh(pi*(2/3)/2);
Exact
for m=0: 1: 20
    marker=rem(m,2);
    if marker==0
        D(:,1)=M\(A(:,1)-A(:,2));
        D(:,2)=M\(A(:,2)-A(:,1)-B);
    else
        D(1,:)=(N\((A(1,:)-A(2,:)-[0,B(1)])'))';
        D(2,:)=(N\((A(2,:)-A(1,:)-A(3,:)-[0,B(2)])'))';
        D(3,:)=(N\((A(3,:)-A(2,:)-A(4,:)-[0,B(3)])'))';
        D(4,:)=(N\((A(4,:)-A(3,:)-A(5,:)-[0,B(4)])'))';
        D(5,:)=(N\((A(5,:)-A(4,:)-[0,B(5)])'))';
    end
    A=D;
    disp('for')
    m
    disp('computed solution is')
    A
    disp('and computed error is')
    Error=Exact-A;
    Error=abs(Error)
end
```

```
Exact =

   0.122846207659390   0.280147907144299
   0.212775873183221   0.485230408808015
   0.245692415318781   0.560295814288599
   0.212775873183221   0.485230408808015
   0.122846207659390   0.280147907144299

for

m =

     0

computed solution is

A =

                   0   0.419753086419753
                   0   0.703703703703704
                   0   0.802469135802469
                   0   0.703703703703704
                   0   0.419753086419753

and computed error is

Error =

   0.122846207659390   0.139605179275454
   0.212775873183221   0.218473294895689
   0.245692415318781   0.242173321513870
   0.212775873183221   0.218473294895689
   0.122846207659390   0.139605179275454

for

m =

     1

computed solution is

A =

   0.104938271604938   0.314814814814815
   0.175925925925926   0.527777777777778
   0.200617283950617   0.601851851851852
   0.175925925925926   0.527777777777778
   0.104938271604938   0.314814814814815

and computed error is

Error =

   0.017907936054452   0.034666907670516
   0.036849947257295   0.042547368969763
   0.045075131368164   0.041556037563253
   0.036849947257295   0.042547368969763
   0.017907936054452   0.034666907670515

for

m =

     2

computed solution is

A =

   0.162551440329218   0.257201646090535
   0.277777777777778   0.425925925925926
   0.318930041152263   0.483539094650206
   0.277777777777778   0.425925925925926
   0.162551440329218   0.257201646090535

and computed error is

Error =

   0.039705232669828   0.022946261053764
   0.065001904594557   0.059304482882089
   0.073237625833482   0.076756719638393
   0.065001904594557   0.059304482882089
   0.039705232669828   0.022946261053764

for

m =

     3

computed solution is

A =

   0.133744855967078   0.286008230452675
   0.226851851851852   0.476851851851852
   0.259773662551440   0.542695473251029
   0.226851851851852   0.476851851851852
   0.133744855967078   0.286008230452675
```

```
and computed error is

Error =

   0.010898648307688   0.005860323308376
   0.014075978668631   0.008378556956163
   0.014081247232659   0.017600341037570
   0.014075978668631   0.008378556956163
   0.010898648307688   0.005860323308376

for

m =

     4

computed solution is

A =

   0.116598079561043   0.303155006858711
   0.197530864197531   0.506172839506173
   0.225994513031550   0.576474622770919
   0.197530864197531   0.506172839506173
   0.116598079561043   0.303155006858711

and computed error is

Error =

   0.006248128098348   0.023007099714411
   0.015245008985690   0.020942430698158
   0.019697902287231   0.016178808482320
   0.015245008985690   0.020942430698158
   0.006248128098348   0.023007099714411

for

m =

     5

computed solution is

A =

   0.125171467764060   0.294581618655693
   0.212191358024691   0.491512345679012
   0.242884087791495   0.559585048010974
   0.212191358024691   0.491512345679012
   0.125171467764060   0.294581618655693
```

```
and computed error is

Error =

   0.002325260104670   0.014433711511393
   0.000584515158530   0.006281936870998
   0.002808327527286   0.000710766277625
   0.000584515158530   0.006281936870998
   0.002325260104670   0.014433711511393

for

m =

     6

computed solution is

A =

   0.130029721079104   0.289723365340649
   0.220679012345679   0.483024691358025
   0.252686328303612   0.549782807498857
   0.220679012345679   0.483024691358025
   0.130029721079104   0.289723365340649

and computed error is

Error =

   0.007183513419713   0.009575458196350
   0.007903139162458   0.002205717449990
   0.006993912984831   0.010513006789742
   0.007903139162458   0.002205717449990
   0.007183513419713   0.009575458196350

for

m =

     7

computed solution is

A =

   0.127600594421582   0.292152491998171
   0.216435185185185   0.487268518518518
   0.247785208047554   0.554683927754915
   0.216435185185185   0.487268518518518
   0.127600594421582   0.292152491998171
```

```
and computed error is

Error =

    0.004754386762192    0.012004584853872
    0.003659312001964    0.002038109710504
    0.002092792728773    0.005611886533683
    0.003659312001964    0.002038109710504
    0.004754386762192    0.012004584853872

for

m =

     8

computed solution is

A =

    0.126181222374638    0.293571864045115
    0.213991769547325    0.489711934156379
    0.244960752934004    0.557508382868465
    0.213991769547325    0.489711934156379
    0.126181222374638    0.293571864045115

and computed error is

Error =

    0.003335014715248    0.013423956900816
    0.001215896364104    0.004481525348364
    0.000731662384777    0.002787431420134
    0.001215896364104    0.004481525348364
    0.003335014715248    0.013423956900816

for

m =

     9

computed solution is

A =

    0.126890908398110    0.292862178021643
    0.215213477366255    0.488490226337449
    0.246372980490779    0.556096155311690
    0.215213477366255    0.488490226337449
    0.126890908398110    0.292862178021643
```

```
and computed error is

Error =

    0.004044700738720    0.012714270877344
    0.002437604183034    0.003259817529434
    0.000680565171998    0.004199658976908
    0.002437604183034    0.003259817529434
    0.004044700738720    0.012714270877344

for

m =

    10

computed solution is

A =

    0.127297350505512    0.292455735914241
    0.215920781893004    0.487782921810700
    0.247188246202307    0.555280889600163
    0.215920781893004    0.487782921810700
    0.127297350505512    0.292455735914241

and computed error is

Error =

    0.004451142846122    0.012307828769941
    0.003144908709783    0.002552513002685
    0.001495830883526    0.005014924688436
    0.003144908709783    0.002552513002685
    0.004451142846122    0.012307828769941

for

m =

    11

computed solution is

A =

    0.127094129451811    0.292658956967942
    0.215567129629630    0.488136574074074
    0.246780613346543    0.555688522455926
    0.215567129629630    0.488136574074074
    0.127094129451811    0.292658956967942
```

```
and computed error is

Error =

    0.004247921792421    0.012511049823643
    0.002791256446408    0.002906165266060
    0.001088198027762    0.004607291832672
    0.002791256446408    0.002906165266060
    0.004247921792421    0.012511049823643

for

m =

    12

computed solution is

A =

    0.126976113058646    0.292776973361107
    0.215363511659808    0.488340192043896
    0.246544977476333    0.555924158326136
    0.215363511659808    0.488340192043896
    0.126976113058646    0.292776973361107

and computed error is

Error =

    0.004129905399256    0.012629066216808
    0.002587638476587    0.003109783235881
    0.000852562157552    0.004371655962463
    0.002587638476587    0.003109783235881
    0.004129905399256    0.012629066216807

for

m =

    13

computed solution is

A =

    0.127035121255229    0.292717965164524
    0.215465320644719    0.488238383058985
    0.246662795411438    0.555806340391031
    0.215465320644719    0.488238383058985
    0.127035121255229    0.292717965164524
```

```
and computed error is

Error =

    0.004188913595838    0.012570058020225
    0.002689447461498    0.003007974250970
    0.000970380092657    0.004489473897568
    0.002689447461498    0.003007974250970
    0.004188913595838    0.012570058020225

for

m =

    14

computed solution is

A =

    0.127069035532637    0.292684050887116
    0.215524262688615    0.488179441015089
    0.246730690118941    0.555738445683528
    0.215524262688615    0.488179441015089
    0.127069035532637    0.292684050887116

and computed error is

Error =

    0.004222827873246    0.012536143742817
    0.002748389505393    0.002949032207075
    0.001038274800160    0.004557368605070
    0.002748389505393    0.002949032207075
    0.004222827873246    0.012536143742817

for

m =

    15

computed solution is

A =

    0.127052078393933    0.292701008025820
    0.215494791666667    0.488208912037037
    0.246696742765189    0.555772393037280
    0.215494791666667    0.488208912037037
    0.127052078393933    0.292701008025820
```

```
and computed error is

Error =

   0.004205870734542   0.012553100881521
   0.002718918483445   0.002978503229022
   0.001004327446409   0.004523421251319
   0.002718918483445   0.002978503229022
   0.004205870734542   0.012553100881521

for

m =

    16

computed solution is

A =

   0.127042251044801   0.292710835374952
   0.215477823502515   0.488225880201189
   0.246677099092373   0.555792036710096
   0.215477823502515   0.488225880201189
   0.127042251044801   0.292710835374952

and computed error is

Error =

   0.004196043385410   0.012562928230653
   0.002701950319294   0.002995471393174
   0.000984683773593   0.004503777578503
   0.002701950319294   0.002995471393174
   0.004196043385410   0.012562928230653

for

m =

    17

computed solution is

A =

   0.127047164719367   0.292705921700386
   0.215486307584591   0.488217396119113
   0.246686920928781   0.555782214873688
   0.215486307584591   0.488217396119113
   0.127047164719367   0.292705921700386
```

```
and computed error is

Error =

   0.004200957059976   0.012558014556087
   0.002710434401370   0.002986987311098
   0.000994505610001   0.004513599414911
   0.002710434401370   0.002986987311098
   0.004200957059976   0.012558014556087

for

m =

    18

computed solution is

A =

   0.127049992134201   0.292703094285553
   0.215491219421582   0.488212484282122
   0.246692577596024   0.555776558206445
   0.215491219421582   0.488212484282122
   0.127049992134201   0.292703094285552

and computed error is

Error =

   0.004203784474810   0.012555187141253
   0.002715346238361   0.002982075474107
   0.001000162277243   0.004519256082153
   0.002715346238361   0.002982075474107
   0.004203784474810   0.012555187141253

for

m =

    19

computed solution is

A =

   0.127048578426784   0.292704507992969
   0.215488763503086   0.488214940200617
   0.246689749262402   0.555779386540067
   0.215488763503086   0.488214940200617
   0.127048578426784   0.292704507992969
```

```
and computed error is

Error =

   0.004202370767393   0.012556600848670
   0.002712890319865   0.002984531392603
   0.000997333943622   0.004516427748532
   0.002712890319865   0.002984531392603
   0.004202370767393   0.012556600848670

for

m =

    20

computed solution is

A =

   0.127047759685198   0.292705326734555
   0.215487349489407   0.488216354214297
   0.246688112085493   0.555781023716976
   0.215487349489407   0.488216354214297
   0.127047759685198   0.292705326734555

and computed error is

Error =

   0.004201552025807   0.012557419590256
   0.002711476306186   0.002985945406282
   0.000995696766712   0.004514790571623
   0.002711476306186   0.002985945406282
   0.004201552025807   0.012557419590256

>>
```

Appendix 2a

```
clc
disp('MATLAB Program for Solution of Problem 2, ADI Without p')
format long
A=[0, 0, 0;0, 0, 0; 0, 0, 0]';
N=[-4, 1, 0;1, -4, 1;0, 1,-4];
B=[1;2;3];
B=0.5*sin((1/4)*pi*B);
I=[1,0,0;0,1,0;0, 0, 1];
D=A;
Exact=A;
x=[1,2,3];
    Exact(:,1)=(1/(2*sinh(pi)))*sin(pi*x/4)*sinh(pi/4);
    Exact(:,2)=(1/(2*sinh(pi)))*sin(pi*x/4)*sinh(2*pi/4);
    Exact(:,3)=(1/(2*sinh(pi)))*sin(pi*x/4)*sinh(3*pi/4);
Exact
for m=0: 1: 20
    marker=rem(m,2);
    if marker==0
        D(:,1)=N\(-A(:,2));
        D(:,2)=N\(-A(:,1)-A(:,3));
        D(:,3)=N\(-A(:,2)-B);
    else
        D(1,:)=N\((-A(2,:)-[0,0,B(1)])');
        D(2,:)=N\((-A(1,:)-A(3,:)-[0,0,B(2)])');
        D(3,:)=N\((-A(2,:)-[0,0,B(1)])');
    end
    A=D;
    disp('for')
    m
    disp('computed solution is')
    A
    disp('and computed error is')
    Error=Exact-A;
    Error=abs(Error)
end
```

```
Exact =

   0.026593514170037   0.070452021169566   0.160049261024727
   0.037608908410427   0.099634203834597   0.226343835588960
   0.026593514170037   0.070452021169566   0.160049261024727

for

m =

     0

computed solution is

A =

                   0                   0   0.136729540169507
                   0                   0   0.193364770084753
                   0                   0   0.136729540169507

and computed error is

Error =

   0.026593514170037   0.070452021169566   0.023319720855220
   0.037608908410427   0.099634203834597   0.032979065504207
   0.026593514170037   0.070452021169566   0.023319720855220

for

m =

     1

computed solution is

A =

   0.009766395726393   0.039065582905573   0.146495935895900
   0.013811769291768   0.055247077167072   0.207176539376521
   0.009766395726393   0.039065582905573   0.146495935895900

and computed error is

Error =

   0.016827118443644   0.031386438263993   0.013553325128827
   0.023797139118659   0.044387126667524   0.019167296212439
   0.016827118443644   0.031386438263993   0.013553325128827
```

```
for

m =

     2

computed solution is

A =

   0.015107814913526   0.060431259654105   0.151837355083033
   0.021365676748531   0.085462706994125   0.214730446833285
   0.015107814913526   0.060431259654105   0.151837355083033

and computed error is

Error =

   0.011485699256511   0.010020761515461   0.008211905941694
   0.016243231661896   0.014171496840472   0.011613388755676
   0.011485699256511   0.010020761515462   0.008211905941694

for

m =

     3

computed solution is

A =

   0.021975353868411   0.066535738725113   0.158704894037918
   0.031077843478655   0.094095744087568   0.224442613563409
   0.021975353868411   0.066535738725113   0.158704894037918

and computed error is

Error =

   0.004618160301626   0.003916282444453   0.001344366986809
   0.006531064931772   0.005538459747029   0.001901222025552
   0.004618160301626   0.003916282444453   0.001344366986809
for

m =

     4

computed solution is

A =

   0.025731335642002   0.069874389190527   0.162460875811508
   0.036389603842893   0.098817308855779   0.229754373927646
   0.025731335642002   0.069874389190527   0.162460875811508
```

```
and computed error is

Error =

   0.000862178528035   0.000577631979039   0.002411614786782
   0.001219304567535   0.000816894978817   0.003410538338686
   0.000862178528035   0.000577631979039   0.002411614786782

for

m =

     5

computed solution is

A =

   0.027221804599776   0.072497614556209   0.163951344769282
   0.038497445257273   0.102527109745088   0.231862215342026
   0.027221804599776   0.072497614556209   0.163951344769282

and computed error is

Error =

   0.000628290429739   0.002045593386643   0.003902083744556
   0.000888536846845   0.002892905910492   0.005518379753066
   0.000628290429739   0.002045593386643   0.003902083744556

for

m =

     6

computed solution is

A =

   0.028036969140709   0.073932304148252   0.164766509310216
   0.039650262006627   0.104556067223951   0.233015032091380
   0.028036969140709   0.073932304148252   0.164766509310216

and computed error is

Error =

   0.001443454970672   0.003480282978686   0.004717248285489
   0.002041353596199   0.004921863389354   0.006671196502420
   0.001443454970672   0.003480282978686   0.004717248285489
```

```
for

m =

     7

computed solution is

A =

   0.028563332529997   0.074603068113363   0.165292872699504
   0.040394652250495   0.105504670720562   0.233759422335248
   0.028563332529997   0.074603068113363   0.165292872699504

and computed error is

Error =

   0.001969818359960   0.004151046943797   0.005243611674777
   0.002785743840068   0.005870466885965   0.007415586746288
   0.001969818359960   0.004151046943797   0.005243611674777

for

m =

     8

computed solution is

A =

   0.028851210226715   0.074969921107411   0.165580750396222
   0.040801772793498   0.106023479200141   0.234166542878252
   0.028851210226715   0.074969921107411   0.165580750396222

and computed error is

Error =

   0.002257696056678   0.004517899937845   0.005531489371495
   0.003192864383071   0.006389275365545   0.007822707289291
   0.002257696056678   0.004517899937845   0.005531489371495

for

m =

     9

computed solution is

A =

   0.028997150753117   0.075186830218971   0.165726690922624
   0.041008163865236   0.106330235007511   0.234372933949989
   0.028997150753117   0.075186830218971   0.165726690922624
```

```
and computed error is

Error =

   0.002403636583080   0.004734809049405   0.005677429897897
   0.003399255454808   0.006696031172915   0.008029098361029
   0.002403636583080   0.004734809049405   0.005677429897897

for

m =

    10

computed solution is

A =

   0.029076968277385   0.075305461751299   0.165806508446892
   0.041121042890570   0.106498005329456   0.234485812975324
   0.029076968277385   0.075305461751299   0.165806508446892

and computed error is

Error =

   0.002483454107348   0.004853440581733   0.005757247422165
   0.003512134480143   0.006863801494859   0.008141977386364
   0.002483454107348   0.004853440581733   0.005757247422165

for

m =

    11

computed solution is

A =

   0.029122265504375   0.075368019126928   0.165851805673881
   0.041185102843317   0.106586474818496   0.234549872928070
   0.029122265504375   0.075368019126928   0.165851805673881

and computed error is

Error =

   0.002528751334338   0.004915997957362   0.005802544649155
   0.003576194432889   0.006952270983900   0.008206037339110
   0.002528751334338   0.004915997957362   0.005802544649155
```

```
for

m =
    12

computed solution is

A =

   0.029147039380443   0.075402232891744   0.165876579549950
   0.041220138394846   0.106634860388719   0.234584908479599
   0.029147039380443   0.075402232891744   0.165876579549950

and computed error is

Error =

   0.002553525210406   0.004950211722178   0.005827318525223
   0.003611229984418   0.007000656554122   0.008241072890639
   0.002553525210406   0.004950211722178   0.005827318525223

for

m =

    13

computed solution is

A =

   0.029160353866972   0.075421277073042   0.165889894036479
   0.041238967922271   0.106661792928195   0.234603738007024
   0.029160353866972   0.075421277073042   0.165889894036479

and computed error is

Error =

   0.002566839696935   0.004969255903476   0.005840633011752
   0.003630059511843   0.007027589093599   0.008259902418064
   0.002566839696935   0.004969255903476   0.005840633011752

for

m =

    14

computed solution is

A =

   0.029167635801455   0.075431692681650   0.165897175970961
   0.041249266132776   0.106676522823149   0.234614036217530
   0.029167635801455   0.075431692681650   0.165897175970961
```

```
and computed error is

Error =

   0.002574121631418   0.004979671512084   0.005847914946235
   0.003640357722349   0.007042318988552   0.008270200628569
   0.002574121631418   0.004979671512084   0.005847914946235

for

m =

    15

computed solution is

A =

   0.029171651965983   0.075437341731155   0.165901192135490
   0.041254945847121   0.106684511785574   0.234619715931874
   0.029171651965983   0.075437341731155   0.165901192135490

and computed error is

Error =

   0.002578137795946   0.004985320561589   0.005851931110763
   0.003646037436693   0.007050307950977   0.008275880342914
   0.002578137795946   0.004985320561589   0.005851931110763

for

m =

    16

computed solution is

A =

   0.029173848479300   0.075440431298920   0.165903388648806
   0.041258052186043   0.106688881094209   0.234622822270797
   0.029173848479300   0.075440431298920   0.165903388648806

and computed error is

Error =

   0.002580334309263   0.004988410129354   0.005854127624080
   0.003649143775616   0.007054677259612   0.008278986681836
   0.002580334309263   0.004988410129354   0.005854127624080
```

```
for

m =

    17

computed solution is

A =

   0.029175045000563   0.075442127816211   0.165904585170070
   0.041259744322642   0.106691280331970   0.234624514407396
   0.029175045000563   0.075442127816211   0.165904585170070

and computed error is

Error =

   0.002581530830526   0.004990106646645   0.005855324145343
   0.003650835912215   0.007057076497373   0.008280678818435
   0.002581530830526   0.004990106646645   0.005855324145343

for

m =

    18

computed solution is

A =

   0.029175699399772   0.075443055672327   0.165905239569279
   0.041260669782879   0.106692592518673   0.234625439867632
   0.029175699399772   0.075443055672327   0.165905239569279

and computed error is

Error =

   0.002582185229735   0.004991034502761   0.005855978544552
   0.003651761372451   0.007058388684076   0.008281604278672
   0.002582185229735   0.004991034502761   0.005855978544552

for

m =

    19

computed solution is

A =

   0.029176057987121   0.075443562165605   0.165905598156628
   0.041261176901970   0.106693308808337   0.234625946986724
   0.029176057987121   0.075443562165605   0.165905598156628
```

```
and computed error is

Error =

   0.002582543817084   0.004991540996039   0.005856337131901
   0.003652268491543   0.007059104973740   0.008282111397764
   0.002582543817084   0.004991540996039   0.005856337131901

for

m =

    20

computed solution is

A =

   0.029176254105054   0.075443839175978   0.165905794274561
   0.041261454254611   0.106693700560162   0.234626224339365
   0.029176254105054   0.075443839175978   0.165905794274561

and computed error is

Error =

   0.002582739935017   0.004991818006412   0.005856533249834
   0.003652545844184   0.007059496725566   0.008282388750405
   0.002582739935017   0.004991818006412   0.005856533249834

>>
```

Appendix 2b

```
clc
disp('MATLAB Program for Solution of Problem 2, ADI Without p')
format long
A=[0, 0, 0;0, 0, 0; 0, 0, 0]';
N=[-(2+sqrt(2)), 1, 0;1, -(2+sqrt(2)), 1;0, 1,-(2+sqrt(2))];
B=[1;2;3];
B=0.5*sin((1/4)*pi*B);
I=[1,0,0;0,1,0;0, 0, 1];
D=A;
Exact=A;
x=[1,2,3];
    Exact(:,1)=(1/(2*sinh(pi)))*sin(pi*x/4)*sinh(pi/4);
    Exact(:,2)=(1/(2*sinh(pi)))*sin(pi*x/4)*sinh(2*pi/4);
    Exact(:,3)=(1/(2*sinh(pi)))*sin(pi*x/4)*sinh(3*pi/4);
Exact
for m=0: 1: 20
    marker=rem(m,2);
    if marker==0
        D(:,1)=N\(-A(:,2)+((2-sqrt(2))*A(:,1)));
        D(:,2)=N\(-A(:,1)-A(:,3)+((2-sqrt(2))*A(:,2)));
        D(:,3)=N\(-A(:,2)-B+((2-sqrt(2))*A(:,3)));
    else
        D(1,:)=N\((-A(2,:)+((2-sqrt(2))*A(1,:))-[0,0,B(1)])');
        D(2,:)=N\((-A(1,:)-A(3,:)+((2-sqrt(2))*A(2,:))-[0,0,B(2)])');
        D(3,:)=N\((-A(2,:)+((2-sqrt(2))*A(3,:))-[0,0,B(1)])');
    end
    A=D;
    disp('for')
    m
    disp('computed solution is')
    A
    disp('and computed error is')
    Error=Exact-A;
    Error=abs(Error)
end
```

```
Exact =

   0.026593514170037   0.070452021169566   0.160049261024727
   0.037608908410427   0.099634203834597   0.226343835588960
   0.026593514170037   0.070452021169566   0.160049261024727

for

m =

     0

computed solution is

A =

                   0                   0   0.176776695296637
                   0                   0   0.250000000000000
                   0                   0   0.176776695296637

and computed error is

Error =

   0.026593514170037   0.070452021169566   0.016727434271910
   0.037608908410427   0.099634203834597   0.023656164411040
   0.026593514170037   0.070452021169566   0.016727434271910

for

m =

     1

computed solution is

A =

   0.015165042944955   0.051776695296637   0.161611652351682
   0.021446609406726   0.073223304703363   0.228553390593274
   0.015165042944955   0.051776695296637   0.161611652351682

and computed error is

Error =

   0.011428471225082   0.018675325872929   0.001562391326955
   0.016162299003701   0.026410899131234   0.002209555004314
   0.011428471225082   0.018675325872929   0.001562391326955
```

```
for

m =

     2

computed solution is

A =

   0.021446609406726   0.073223304703363   0.155330085889911
   0.030330085889911   0.103553390593274   0.219669914110089
   0.021446609406726   0.073223304703363   0.155330085889911

and computed error is

Error =

   0.005146904763311   0.002771283533797   0.004719175134816
   0.007278822520517   0.003919186758677   0.006673921478871
   0.005146904763311   0.002771283533797   0.004719175134816

for

m =

     3

computed solution is

A =

   0.026650429449553   0.073223304703363   0.162689398770626
   0.037689398770626   0.103553390593274   0.230077554195743
   0.026650429449553   0.073223304703363   0.162689398770626

and computed error is

Error =

   0.000056915279516   0.002771283533797   0.002640137745899
   0.000080490360198   0.003919186758677   0.003733718606783
   0.000056915279516   0.002771283533797   0.002640137745899

for

m =

     4

computed solution is

A =

   0.028805922287441   0.073223304703363   0.165737725975565
   0.040737725975565   0.103553390593274   0.234388539871519
   0.028805922287441   0.073223304703363   0.165737725975565
```

```
and computed error is

Error =

   0.002212408117404   0.002771283533797   0.005688464950838
   0.003128817565137   0.003919186758677   0.008044704282559
   0.002212408117404   0.002771283533797   0.005688464950838

for

m =

     5

computed solution is

A =

   0.028882515367181   0.074747468305833   0.165661132895824
   0.040846044947717   0.105708883431162   0.234280220899367
   0.028882515367181   0.074747468305833   0.165661132895824

and computed error is

Error =

   0.002289001197144   0.004295447136267   0.005611871871098
   0.003237136537290   0.006074679596565   0.007936385310406
   0.002289001197144   0.004295447136267   0.005611871871098

for

m =

     6

computed solution is

A =

   0.028914241259594   0.075378797541251   0.165629407003412
   0.040890912135045   0.106601717798213   0.234235353712039
   0.028914241259594   0.075378797541251   0.165629407003412

and computed error is

Error =

   0.002320727089557   0.004926776371685   0.005580145978685
   0.003282003724618   0.006967513963616   0.007891518123079
   0.002320727089557   0.004926776371685   0.005580145978685
```

```
for

m =

     7

computed solution is

A =

   0.029093710008905   0.075378797541251   0.165819762357886
   0.041144719274344   0.106601717798213   0.234504556836006
   0.029093710008905   0.075378797541251   0.165819762357886

and computed error is

Error =

   0.002500195838868   0.004926776371685   0.005770501333159
   0.003535810863916   0.006967513963616   0.008160721247046
   0.002500195838868   0.004926776371685   0.005770501333159

for

m =

     8

computed solution is

A =

   0.029168048398892   0.075378797541251   0.165898610127379
   0.041249849633668   0.106601717798213   0.234616064420986
   0.029168048398892   0.075378797541251   0.165898610127379

and computed error is

Error =

   0.002574534228855   0.004926776371685   0.005849349102653
   0.003640941223241   0.006967513963616   0.008272228832026
   0.002574534228855   0.004926776371685   0.005849349102653

for

m =

     9

computed solution is

A =

   0.029168435242496   0.075423664728579   0.165898223283776
   0.041250396713139   0.106665169583038   0.234615517341516
   0.029168435242496   0.075423664728579   0.165898223283776
```

```
and computed error is

Error =

   0.002574921072459   0.004971643559013   0.005848962259049
   0.003641488302712   0.007030965748441   0.008271681752555
   0.002574921072459   0.004971643559013   0.005848962259049

for

m =

    10

computed solution is

A =

   0.029168595478363   0.075442249326076   0.165898063047908
   0.041250623320876   0.106691452172869   0.234615290733779
   0.029168595478363   0.075442249326076   0.165898063047908

and computed error is

Error =

   0.002575081308326   0.004990228156510   0.005848802023182
   0.003641714910448   0.007057248338272   0.008271455144819
   0.002575081308326   0.004990228156510   0.005848802023182

for

m =

    11

computed solution is

A =

   0.029174011288816   0.075442249326076   0.165903533842618
   0.041258282433469   0.106691452172869   0.234623027605854
   0.029174011288816   0.075442249326076   0.165903533842618

and computed error is

Error =

   0.002580497118779   0.004990228156510   0.005854272817891
   0.003649374023042   0.007057248338272   0.008279192016894
   0.002580497118779   0.004990228156510   0.005854272817891
```

```
for

m =

    12

computed solution is

A =

   0.029176254590957   0.075442249326076   0.165905799919984
   0.041261454941781   0.106691452172869   0.234626232323198
   0.029176254590957   0.075442249326076   0.165905799919984

and computed error is

Error =

   0.002582740420920   0.004990228156510   0.005856538895257
   0.003652546531354   0.007057248338272   0.008282396734238
   0.002582740420920   0.004990228156510   0.005856538895257

for

m =

    13

computed solution is

A =

   0.029176256544762   0.075443570092754   0.165905797966178
   0.041261457704879   0.106693320019018   0.234626229560100
   0.029176256544762   0.075443570092754   0.165905797966178

and computed error is

Error =

   0.002582742374725   0.004991548923188   0.005856536941452
   0.003652549294452   0.007059116184421   0.008282393971140
   0.002582742374725   0.004991548923188   0.005856536941452


for

m =

    14

computed solution is

A =

   0.029176257354055   0.075444117172225   0.165905797156886
   0.041261458849392   0.106694093706226   0.234626228415587
   0.029176257354055   0.075444117172225   0.165905797156886
```

```
and computed error is

Error =

   0.002582743184018   0.004992096002659   0.005856536132159
   0.003652550438964   0.007059889871629   0.008282392826627
   0.002582743184018   0.004992096002659   0.005856536132159

for

m =

    15

computed solution is

A =

   0.029176417451069   0.075444117172225   0.165905957531606
   0.041261685260761   0.106694093706226   0.234626455219691
   0.029176417451069   0.075444117172225   0.165905957531606

and computed error is

Error =

   0.002582903281032   0.004992096002659   0.005856696506879
   0.003652776850334   0.007059889871629   0.008282619630731
   0.002582903281032   0.004992096002659   0.005856696506879

for

m =

    16

computed solution is

A =

   0.029176483765424   0.075444117172225   0.165906023960990
   0.041261779043421   0.106694093706226   0.234626549165027
   0.029176483765424   0.075444117172225   0.165906023960990

and computed error is

Error =

   0.002582969595387   0.004992096002659   0.005856762936263
   0.003652870632994   0.007059889871629   0.008282713576067
   0.002582969595387   0.004992096002659   0.005856762936263
```

```
for

m =

    17

computed solution is

A =

   0.029176483775292   0.075444156051966   0.165906023951122
   0.041261779057376   0.106694148690483   0.234626549151072
   0.029176483775292   0.075444156051966   0.165906023951122

and computed error is

Error =

   0.002582969605255   0.004992134882400   0.005856762926395
   0.003652870646949   0.007059944855886   0.008282713562112
   0.002582969605255   0.004992134882400   0.005856762926395

for

m =

    18

computed solution is

A =

   0.029176483779379   0.075444172156482   0.165906023947034
   0.041261779063157   0.106694171465707   0.234626549145291
   0.029176483779379   0.075444172156482   0.165906023947034

and computed error is

Error =

   0.002582969609342   0.004992150986916   0.005856762922307
   0.003652870652729   0.007059967631111   0.008282713556331
   0.002582969609342   0.004992150986916   0.005856762922307

for

m =

    19

computed solution is

A =

   0.029176488495582   0.075444172156482   0.165906028664639
   0.041261785732874   0.106694171465707   0.234626555816992
   0.029176488495582   0.075444172156482   0.165906028664639
```

```
and computed error is

Error =

   0.002582974325545   0.004992150986916   0.005856767639912
   0.003652877322447   0.007059967631111   0.008282720228032
   0.002582974325545   0.004992150986916   0.005856767639912

for

m =

    20

computed solution is

A =

   0.029176490449096   0.075444172156482   0.165906030618735
   0.041261788495561   0.106694171465707   0.234626558580501
   0.029176490449096   0.075444172156482   0.165906030618735

and computed error is

Error =

   0.002582976279059   0.004992150986916   0.005856769594008
   0.003652880085134   0.007059967631111   0.008282722991541
   0.002582976279059   0.004992150986916   0.005856769594008

>>
```

Appendix 3a

```
clc
disp('MATLAB Program for Solution of Problem 3, ADI Without')
format long
A=[0,0;0,0]';
N=[-4 1;1 -4];
B=[1;2];
B=sin((1/3)*pi*B);
D=A;
Exact=A;
x=[1,2];
    Exact(:,1)=(1/(sinh(pi)))*sin(pi*x/3)*sinh(pi/3);
    Exact(:,2)=(1/(sinh(pi)))*sin(pi*x/3)*sinh(pi*2/3);
Exact
for m=0: 1: 20
    marker=rem(m,2);
    if marker==0
        D(:,1)=-N\A(:,2);
        D(:,2)=-N\(A(:,1)+B);
    else
        D(1,:)=-N\([A(2,1); A(2,2)+B(1)]);
        D(2,:)=-N\([A(1,1);A(1,2)+B(2)]);
    end
    A=D;
    disp('for')
    m
    disp('computed solution is')
    A
    disp('and computed error is')
    Error=Exact-A;
    Error=abs(Error)
end
```

```
Exact =

   0.093688459920522   0.299856822258377
   0.093688459920522   0.299856822258377

for

m =

     0

computed solution is

A =

                   0   0.288675134594813
                   0   0.288675134594813

and computed error is

Error =

   0.093688459920522   0.011181687663564
   0.093688459920522   0.011181687663564

for

m =

     1

computed solution is

A =

   0.076980035891950   0.307920143567800
   0.076980035891950   0.307920143567800

and computed error is

Error =

   0.016708424028572   0.008063321309424
   0.016708424028572   0.008063321309424
for

m =

     2

computed solution is
 A =

   0.102640047855933   0.314335146558796
   0.102640047855933   0.314335146558796

and computed error is

Error =

   0.008951587935411   0.014478324300420
   0.008951587935411   0.014478324300420

for

m =

     3

computed solution is

A =

   0.106061382784465   0.321605483281925
   0.106061382784465   0.321605483281925

and computed error is

Error =

   0.012372922863942   0.021748661023548
   0.012372922863942   0.021748661023548

for

m =

     4

computed solution is

A =

   0.107201827760642   0.324028928856301
   0.107201827760642   0.324028928856301

and computed error is

Error =

   0.013513367840120   0.024172106597925
   0.013513367840120   0.024172106597925
for

m =
     5

computed solution is

A =

   0.107924109578887   0.324494610554907
   0.107924109578887   0.324494610554907
```

and computed error is

Error =

```
0.014235649658365   0.024637788296530
0.014235649658365   0.024637788296530
```

for

m =

6

computed solution is

A =

```
0.108164870184969   0.324649837787775
0.108164870184969   0.324649837787775
```

and computed error is

Error =

```
0.014476410264447   0.024793015529399
0.014476410264447   0.024793015529399
```

for

m =

7

computed solution is

A =

```
0.108222314820806   0.324724389098255
0.108222314820806   0.324724389098255
```

and computed error is

Error =

```
0.014533854900284   0.024867566839878
0.014533854900284   0.024867566839878
```

for

m =

8

computed solution is

A =

```
0.108241463032752   0.324749239535082
0.108241463032752   0.324749239535082
```

and computed error is

Error =

```
0.014553003112230   0.024892417276705
0.014553003112230   0.024892417276705
```

for

m =

9

computed solution is

A =

```
0.108249366363368   0.324756002420722
0.108249366363368   0.324756002420722
```

and computed error is

Error =

```
0.014560906442846   0.024899180162346
0.014560906442846   0.024899180162346
```

for

m =

10

computed solution is

A =

```
0.108252000806907   0.324758256715936
0.108252000806907   0.324758256715936
```

and computed error is

Error =

```
0.014563540886385   0.024901434457559
0.014563540886385   0.024901434457559
```

for

m =

11

computed solution is

A =

```
0.108252777581867   0.324759109520560
0.108252777581867   0.324759109520560
```

```
and computed error is

Error =

   0.014564317661345   0.024902287262184
   0.014564317661345   0.024902287262184

for

m =

    12

computed solution is

A =

   0.108253036506853   0.324759393788769
   0.108253036506853   0.324759393788769

and computed error is

Error =

   0.014564576586331   0.024902571530392
   0.014564576586331   0.024902571530392

for

m =

    13

computed solution is

A =

   0.108253129573375   0.324759481786645
   0.108253129573375   0.324759481786645

and computed error is

Error =

   0.014564669652853   0.024902659528269
   0.014564669652853   0.024902659528269

for

m =

    14

computed solution is

A =

   0.108253160595548   0.324759511119271
   0.108253160595549   0.324759511119271
```



```
and computed error is

Error =

   0.014564700675026   0.024902688860895
   0.014564700675026   0.024902688860895

for

m =

    15

computed solution is

A =

   0.108253170485727   0.324759521347359
   0.108253170485727   0.324759521347359

and computed error is

Error =

   0.014564710565205   0.024902699088983
   0.014564710565205   0.024902699088983

for

m =
    16

computed solution is

A =

   0.108253173782453   0.324759524756722
   0.108253173782453   0.324759524756722

and computed error is

Error =

   0.014564713861931   0.024902702498345
   0.014564713861931   0.024902702498345

for

m =

    17

computed solution is

A =

   0.108253174911398   0.324759525863140
   0.108253174911398   0.324759525863140
```

```
and computed error is

Error =

   0.014564714990876   0.024902703604763
   0.014564714990876   0.024902703604763

for

m =

    18

computed solution is

A =

   0.108253175287713   0.324759526231946
   0.108253175287713   0.324759526231946

and computed error is

Error =

   0.014564715367191   0.024902703973569
   0.014564715367191   0.024902703973569

for

m =

    19

computed solution is

A =

   0.108253175411149   0.324759526356883
   0.108253175411149   0.324759526356883

and computed error is

Error =

   0.014564715490627   0.024902704098507
   0.014564715490627   0.024902704098507

for

m =

    20

computed solution is

A =

   0.108253175452294   0.324759526398529
   0.108253175452294   0.324759526398529

and computed error is

Error =

   0.014564715531772   0.024902704140153
   0.014564715531772   0.024902704140153

>>
```

Appendix 3b

```
clc
disp('MATLAB Program for Solution of Problem 3, ADI With p=3^(1/2)')
format long
A=[0,0;0,0]';
N=[-(2+sqrt(3)) 1;1 -(2+sqrt(3))];
B=[1;2];
B=sin((1/3)*pi*B);
I=[1,0;0,1];
D=A;
Exact=A;
x=[1,2];
    Exact(:,1)=(1/(sinh(pi)))*sin(pi*x/3)*sinh(pi/3);
    Exact(:,2)=(1/(sinh(pi)))*sin(pi*x/3)*sinh(pi*2/3);
Exact
for m=0: 1: 20
    marker=rem(m,2);
    if marker==0
        D(:,1)=N\((2-sqrt(3))*A(:,1)-A(:,2));
        D(:,2)=N\((2-sqrt(3))*A(:,2)-A(:,1)-B);
    else
        D(1,:)=N\(((2-sqrt(3))*A(1,:)-A(2,:)-[0,B(1)])');
        D(2,:)=N\(((2-sqrt(3))*A(2,:)-A(1,:)-[0,B(2)])');
    end
    A=D;
    disp('for')
    m
    disp('computed solution is')
    A
    disp('and computed error is')
    Error=Exact-A;
    Error=abs(Error)
end
```

```
Exact =

   0.093688459920522   0.299856822258377
   0.093688459920522   0.299856822258377

for

m =

     0

computed solution is

A =

                   0   0.316987298107781
                   0   0.316987298107781

and computed error is

Error =

   0.093688459920522   0.017130475849404
   0.093688459920522   0.017130475849404

for

m =

     1

computed solution is

A =

   0.084936490538903   0.316987298107781
   0.084936490538903   0.316987298107781

and computed error is

Error =

   0.008751969381619   0.017130475849404
   0.008751969381619   0.017130475849404

for

m =

     2

computed solution is

A =

   0.107695154586736   0.316987298107781
   0.107695154586736   0.316987298107781
```



```
and computed error is

Error =

   0.014006694666214   0.017130475849404
   0.014006694666214   0.017130475849404

for

m =

     3

computed solution is

A =

   0.107695154586736   0.323085463760209
   0.107695154586736   0.323085463760209

and computed error is

Error =

   0.014006694666214   0.023228641501832
   0.014006694666214   0.023228641501832

for

m =

     4

computed solution is

A =

   0.107695154586736   0.324719462322088
   0.107695154586736   0.324719462322088

and computed error is

Error =

   0.014006694666214   0.024862640063711
   0.014006694666214   0.024862640063712

for

m =

     5

computed solution is

A =

   0.108132983181825   0.324719462322088
   0.108132983181825   0.324719462322088
```

```
and computed error is

Error =

   0.014444523261303   0.024862640063712
   0.014444523261303   0.024862640063712

for

m =

     6

computed solution is

A =

   0.108250299000303   0.324719462322088
   0.108250299000303   0.324719462322088

and computed error is

Error =

   0.014561839079781   0.024862640063711
   0.014561839079781   0.024862640063712

for

m =

     7

computed solution is

A =

   0.108250299000303   0.324750897000908
   0.108250299000303   0.324750897000908

and computed error is

Error =

   0.014561839079781   0.024894074742532
   0.014561839079781   0.024894074742532

for

m =

     8

computed solution is

A =

   0.108250299000303   0.324759319897713
   0.108250299000303   0.324759319897713
```



```
and computed error is

Error =

   0.014561839079781   0.024902497639336
   0.014561839079781   0.024902497639336

for

m =

     9

computed solution is

A =

   0.108252555908699   0.324759319897713
   0.108252555908699   0.324759319897713

and computed error is

Error =

   0.014564095988177   0.024902497639336
   0.014564095988177   0.024902497639336

for

m =

    10

computed solution is

A =

   0.108253160645482   0.324759319897713
   0.108253160645482   0.324759319897713

and computed error is

Error =

   0.014564700724960   0.024902497639336
   0.014564700724960   0.024902497639336

for

m =

    11

computed solution is

A =

   0.108253160645482   0.324759481936445
   0.108253160645482   0.324759481936445
```

```
and computed error is

Error =

   0.014564715536070   0.024902704155300
   0.014564715536070   0.024902704155300

for

m =

    18

computed solution is

A =

   0.108253175472661   0.324759526413677
   0.108253175472661   0.324759526413677

and computed error is

Error =

   0.014564715552139   0.024902704155300
   0.014564715552139   0.024902704155300

for

m =

    19

computed solution is

A =

   0.108253175472661   0.324759526417983
   0.108253175472661   0.324759526417983

and computed error is

Error =

   0.014564715552139   0.024902704159606
   0.014564715552139   0.024902704159606

for

m =

    20

computed solution is

A =

   0.108253175472661   0.324759526419136
   0.108253175472661   0.324759526419136

and computed error is

Error =

   0.014564715552139   0.024902704160760
   0.014564715552139   0.024902704160760

>>
```

```
and computed error is

Error =

   0.014564715536070   0.024902704155300
   0.014564715536070   0.024902704155300

for

m =

    18

computed solution is

A =

   0.108253175472661   0.324759526413677
   0.108253175472661   0.324759526413677

and computed error is

Error =

   0.014564715552139   0.024902704155300
   0.014564715552139   0.024902704155300

for

m =

    19

computed solution is

A =

   0.108253175472661   0.324759526417983
   0.108253175472661   0.324759526417983

and computed error is

Error =

   0.014564715552139   0.024902704159606
   0.014564715552139   0.024902704159606

for

m =

    20

computed solution is

A =

   0.108253175472661   0.324759526419136
   0.108253175472661   0.324759526419136

and computed error is

Error =

   0.014564715552139   0.024902704160760
   0.014564715552139   0.024902704160760

>>
```