The Black-Scholes equation for weather derivatives

Stephen Jewson[1]*and Mihail Zervos[2]
[1]RMS, London, United Kingdom
[2]King's College, London, United Kingdom

August 16, 2003

Abstract

We show how the Black and Scholes (1973) and Black (1976) partial differential equations can be adapted for the pricing of weather options that are hedged using weather swaps.

1 Introduction

The Black and Scholes (1973) (BS) model shows how, under certain assumptions, the price of options can be determined from the price of the underlying asset. The assumptions are:

- That the asset can be traded continuously with no transaction costs
- That the asset price follows geometric brownian motion with drift, and is uninfluenced by the trading of the asset that is undertaken to hedge the option
- That market dynamics acts so as to immediately remove any possibility of arbitrage

Based on these assumptions, the option price V is shown to be deterministic, and to obey the following partial differential equation (PDE) in terms of the asset price S, volatility σ, interest rate r and time t:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

In this article we show how this PDE can be adapted to the pricing of weather options, following the discussions on this topic in Jewson (2002a), Jewson et al. (2002) and Jewson (2002c).
The approach we take is:

- We assume that there is a liquidly traded linear weather swap contract
- This swap contract is used to hedge options on the same index
- We assume that the strike for the swap contract follows a Brownian motion

2 The Black-Scholes derivation

We start by recalling the derivation of the BS PDE, equation 1. We imagine that we trade an equity option, and are hedging the risk in that option by using the underlying equity.
We assume that the equity price S follows a stochastic differential equation given by:

$$dS=\mu Sdt+\sigma SdW \tag{2}$$

where μ is the drift, t is time, σ is the volatility and W is Brownian motion. The first term on the right hand side denotes a slow upward drift, while the second term denotes random fluctuations driven by the arrival of new information in the market and fluctuations in supply and demand. The effect of new information is random because otherwise if it were not random, it would predictable, and already included in the price. Note that the changes in the share price are proportional to the share price itself. This means that absolute changes in the share price are lognormally distributed while relative changes are normally distributed.

* *Correspondence address*: RMS, 10 Eastcheap, London, EC3M 1AJ, UK. Email: `x@stephenjewson.com`

We can also consider how the discounted value of the share price changes in time. The value of the share at time t, discounted to time t_0, is:

$$S_d = e^{r(t_0 - t)} S \tag{3}$$

and the stochastic process for the discounted value (considering t_0 as fixed) is then:

$$dS_d = (\mu - r) S_d dt + \sigma S_d dW \tag{4}$$

We can 'solve' equation 2 to give the share price in terms of the Brownian motion. The solution is:

$$S = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W} \tag{5}$$

Note that when differentiating equation 5 to give equation 2 we have to use Ito's formula for evaluating derivatives of functions $f = f(W, t)$ of a stochastic variable W which is:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial W} dW + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} dt \tag{6}$$

The extra term on the right hand side arises because although W is continuous it is not differentiable, and makes jumps of size $dt^{\frac{1}{2}}$ in time dt.
Standard arbitrage pricing theory can be derived mathematically in a number of different ways. We will use a partial differential equation approach similar to the original derivation of Black and Scholes (1973).

2.1 Delta hedging and the Black-Scholes PDE

Imagine that at time t we own, in addition to the option position, a short position Δ in shares, and an amount of cash cB invested in a risk free bond with interest rate r. The total value of our holding, Π, is then given by:

$$\Pi = V - \Delta S + cB \tag{7}$$

where $V(S, t)$ is the unknown value of the option, S is the value of one share, Δ is the number of shares being held and B is the value of the bond.
Moving forward an infinitesimal time step, the value of our holding changes by:

$$d\Pi = dV - \Delta dS - S d\Delta + c dB + B dc \tag{8}$$

Thus the value of our portfolio changes because the value of the option changes, the share price changes, the number of shares we are holding changes, the value of the bond changes, and the number of bonds we are holding changes.
If we assume that the number of shares only changes because we bought or sold them with or for cash, and the amount of cash only changes because we used it to buy or sell shares, then we see that the changes in value due to changes in the number of shares are cancelled by the change in value due to a change in the number of bonds held. This is known as self-financing of the portfolio, and is written as:

$$c dB = S d\Delta \tag{9}$$

Thus the change in our portfolio reduces to:

$$d\Pi = dV - \Delta dS + c dB \tag{10}$$

The bond increases at interest rate r, so:

$$dB = rB dt \tag{11}$$

and hence:

$$d\Pi = dV - \Delta dS + crB dt \tag{12}$$

We can expand dV in terms of dS and dt (taking care to use Ito's lemma, equation 6) so that:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \tag{13}$$

If we now expand S using the model for the share price given in equation 2 we get:

$$dV = (\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S}) dt + (\sigma S \frac{\partial V}{\partial S}) dW \tag{14}$$

The change in the portfolio value then becomes:

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - \mu S \Delta + crB)dt + (\sigma S \frac{\partial V}{\partial S} - \sigma S \Delta)dW \tag{15}$$

This change has a deterministic component (the coefficient of dt) and a random component (the coefficient of dW). If we now choose Δ as

$$\Delta = \frac{\partial V}{\partial S} \tag{16}$$

then the random term in dW and the drift term in dt both cancel out and the total portfolio change becomes:

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + crB)dt \tag{17}$$

The cancellation of the random and drift terms is the essence of continuously hedging with shares.
Since changes in the value of the portfolio are now deterministic, the change must be the same as that which would be earned by putting the same money into safe bonds with interest rate r. If this were not the case someone would be able to make a risk free profit by either buying the option and hedging, or selling the option and hedging. This equality between the return on our portfolio and the return on safe bonds can be written as:

$$d\Pi = \Pi r dt \tag{18}$$

Equation 18, equation 17 and equation 7 give:

$$(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} + crB)dt = (V - \Delta S + cB)rdt \tag{19}$$

Rearranging terms we get the famous Black-Scholes partial differential equation for the option price, as a function of the share price and time:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{20}$$

In terms of the greeks defined by $\theta = \frac{\partial V}{\partial t}$ (with S held constant), $\gamma = \frac{\partial^2 V}{\partial S^2}$ and $\delta = \frac{\partial V}{\partial S}$ this equation can be written as:

$$\theta + \frac{1}{2}\sigma^2 S^2 \gamma + rS\delta - rV = 0 \tag{21}$$

Given appropriate boundary conditions, which specify the final payoff structure, this is equation has an analytical solution. For an unlimited call option contract with strike at K this solution is:

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{22}$$

where

$$\begin{aligned} d_1 &= \frac{log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \\ d_2 &= \frac{log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \end{aligned} \tag{23}$$

2.2 The Black (76) model

Rather than hedge an option with shares, one could imagine hedging an option with forward contracts on the shares. A simple static hedging argument gives the forward price in terms of the share price as:

$$F = e^{r(T-t)}S \tag{24}$$

Following Black (1976) we can then rewrite equation 20 in terms of F rather than S using the following relations:

$$\frac{\partial V}{\partial t}|_S = \frac{\partial V}{\partial t}|_F - \frac{\partial V}{\partial F}|_t \frac{\partial F}{\partial t}|_S \tag{25}$$
$$\frac{\partial V}{\partial S}|_t = \frac{\partial V}{\partial F}|_t \frac{\partial F}{\partial S}|_t \tag{26}$$
$$\frac{\partial^2 V}{\partial S^2}|_t = \frac{\partial^2 V}{\partial F^2}|_t \left(\frac{\partial F}{\partial S}|_t\right)^2 \tag{27}$$

The Black-Scholes equation then becomes:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} + rV = 0 \tag{28}$$

In terms of the greeks $\theta = \frac{\partial V}{\partial t}$ (now with F held constant) and $\gamma = \frac{\partial^2 V}{\partial F^2}$ this can be written as:

$$\theta + \frac{1}{2}\sigma^2 F^2 \gamma + rV = 0 \tag{29}$$

We note that the $rS\frac{\partial V}{\partial S}$ term has disappeared. One way to understand this is that this term relates to changes in the value of the portfolio due to increase in the value of the share holding at the risk free rate. When one is hedging with forwards this term disappears because no money changes hands until the end of the contract.
If we now write the options price in terms of the discounted value at time T (which involves discounting *forward* in time):

$$V_T = e^{r(T-t)}V \tag{30}$$

then equation 28 simplifies even more to:

$$\frac{\partial V_T}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V_T}{\partial F^2} = 0 \tag{31}$$

3 Weather swap price processes

Having given a brief overview of standard option pricing theory, we now move on to see how such theories can be developed in the case of hedging a weather option with a weather swap. The key stage is to develop a price process for the swap. Once we have that, it will be easy to apply slightly modified versions of the standard theory to derive an option price.
We will assume that all swaps are linear, without caps, and are based on cumulative average temperature (CAT) or linear degree days. This greatly simplifies the analysis, and is a reasonably good model for most commonly traded contracts.
We will also use a mathematical trick of dealing with swap *prices* rather than strikes, in order to make the similarity with the standard BS theory (where shares have prices) more clear. In other words, rather than entering into a costless swap with tick 1 and strike K that pays $x - K$ if the index settles at x, we will imagine paying a premium to buy a swap that will pay us x at settlement. The premiums that would be paid for such premium-based swaps are given by a static hedging argument in terms of the strikes of the costless swaps as:

$$\text{premium} = S = Ke^{r(t-T)} \tag{32}$$

These imaginary premium-based swaps are to equities as costless swaps are to equity forwards. In other words, we can think of these fictional premium-based swaps as analogous to equities, and real costless swaps as analogous to forwards on these equities.

3.1 Balanced market model

In order to develop a price process for the swap we will make the assumption that the swap market is balanced in terms of supply and demand. This leads to our imaginary premium-based swaps trading at the discounted expected payoff, and the costless swap trading at the expected index. If our estimate of the discounted expected payoff does not change, then the swap price will only grow at the risk-free rate. This argument would not make sense for equities, since the fundamental reason to buy equities is as an

investment: no-one would invest unless there was a good chance of the equity growing at faster than the risk free rate. However, the fundamental reason to trade swaps is as a hedging instrument, and one does not expect capital growth from a hedge.
How are the expected index and the expected payoff calculated? We assume that all market participants use the same historical data and forecasts to estimate the expected index of the swap.
As shown by Jewson (2002b) the expected index then changes as a Wiener process given by:

$$d\mu = \sigma dW \tag{33}$$

By equation 32, the *price* (of the premium-based swap) is given by:

$$S = e^{r(t-T)}\mu \tag{34}$$

and so:

$$\begin{aligned} dS &= rSdt + e^{r(t-T)}\sigma dW \\ &= rSdt + \sigma_s dW \end{aligned} \tag{35}$$

where $\sigma_s = e^{r(t-T)}\sigma$ has been defined so as the remove the discounting term.
Discounting the price at time t back to time t_0:

$$S_d = e^{r(t_0-t)}S \tag{36}$$

and so:

$$\begin{aligned} dS_d &= e^{r(t_0-T)}\sigma dW \\ &= \sigma_d dW \end{aligned} \tag{37}$$

where $\sigma_d = e^{r(t_0-T)}\sigma$ has also been defined to swallow the discounting term. We see that the discounted swap price is a martingale.
Equations 35 and 37 are the stochastic processes for the swap price and the discounted swap price that follow from our assumptions. Unlike the share price process in equation 2, the random part of the change in the swap price is not related to the swap price. This is because the random part is entirely driven by changes in forecasts and temperatures. Note also that although temperature shows significant autocorrelations and long memory these have entirely cancelled out in the swap price process. This is because the autocorrelation of temperature is known about by the forecast and hence has been included in our estimate of the index.
Finally we note that the swap price can go negative. This would be unacceptable for shares: if a share price goes negative, then one can buy the share (for a negative amount...i.e. by receiving money), throw it away and make a risk free profit. However, buying a swap at a negative price still involves committing to the possibility of having to pay out at the end of the swap term, and so no such risk free profit is available.
One of the assumptions made in the derivation of the swap price was that all market players use the same forecasts and data to estimate the expected index. This is, of course, not true since one of the main ways that secondary market participants seek to gain advantage is by using more accurately cleaned historical data, or more accurate forecasts. However, we argue that the swap price given by equation 35 still holds. The rationale for this is that market participants will rationalise the price in terms of the data they have available. Price *changes* will still be driven by the changes in the forecasts and data as described above.

3.2 Option pricing in the balanced market model

Given the swap price process in equation 35, we can now price options on the same index based on the assumption that the swap is tradable without transaction costs and is used to continuously delta hedge the option. Replacing equation 2 with our new price process 35 we can rederive equation 20. This gives:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_s^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{38}$$

which is the equation analogous to the BS equation for weather swaps trading with premium S. Note that the only difference from the actual BS equation is the coefficient in front of the second derivative term.

We showed above that the Black-Scholes equation can be rewritten as a relation between the Greeks. The same is true of equation 38, which can be rewritten as:

$$\theta + \frac{1}{2}\sigma_s^2\gamma - rS\Delta - rV = 0 \tag{39}$$

We must now transform this equation so that V is a function of the swap strike K and t rather than S and t, since K is what we actually observe in the swap market. This is analogous to the set of transformations that gave the Black(76) model in section 2.2.
Equation 38 becomes:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial K^2} + rV = 0 \tag{40}$$

We have lost the $rS\frac{\partial V}{\partial S}$ term, and the σ_s has become a σ again.
This equation is the same as equation 28 but with the coefficient in front of the second derivative changing from $\sigma^2 S^2$ to σ^2. In terms of the greeks this equation can be written as:

$$\theta + \frac{1}{2}\sigma^2\gamma + rV = 0 \tag{41}$$

This is the PDE satisfied by the price of the weather option. If we write this price in terms of the discounted value of the option at time T

$$V_T = e^{r(T-t)}V \tag{42}$$

then the equation simplifies to:

$$\frac{\partial V_T}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V_T}{\partial K^2} = 0 \tag{43}$$

In weather derivatives it is more common to deal with the standard deviation of the settlement index σ_x rather than the daily volatility σ since σ_x can be easily evaluated from historical meteorological data. The simplest reasonable model for the relationship between σ_x and σ is:

$$\sigma_x = \begin{cases} (T-t_0)^{\frac{1}{2}}\sigma & \text{if } t \le t_0 \\ (T-t)^{\frac{1}{2}}\sigma & \text{if } t_0 \le t \le T \end{cases} \tag{44}$$

where t_0 is the start of the contract, T is the end of the contract and σ is constant in time.
Using σ_x rather than σ, equation 43 simplifies to:

$$\frac{\partial V_T}{\partial \sigma_x} = \sigma_x\frac{\partial^2 V_T}{\partial K^2} \tag{45}$$

or

$$\zeta = \sigma_x\gamma \tag{46}$$

where ζ is defined as $\frac{\partial V_T}{\partial \sigma_x}$.
More generally, one could include the effects of the overlap of forecasts with the contract period, which gives a σ that varies deterministically in time. Jewson (2002b) proposes a simple trapezium-shape model for σ^2 that captures this effect. For contracts longer than a month, one might also want to include seasonal effects on the size of σ.
When σ varies in time deterministically, then equation 43 becomes:

$$\frac{\partial V_T}{\partial t} + \frac{1}{2}\overline{\sigma}^2\frac{\partial^2 V_T}{\partial K^2} = 0 \tag{47}$$

where $\overline{\sigma}^2$ is the average value of σ^2 during the remaining period (see Wilmott (1999), page 120).
Like the Black-Scholes equation, equation 43 can also be solved analytically. We derive the solution by noting that the Green's function solution of this equation is:

$$V_T = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}(T-t)^{-\frac{1}{2}} exp\left(-\frac{(x-K)^2}{2\sigma^2(T-t)}\right) \tag{48}$$

We can verify that this is a solution by calculating the derivatives:

$$\frac{\partial V_T}{\partial t} = V_T\left[-\frac{(x-K)^2}{2\sigma^2(T-t)^2}+\frac{1}{2(T-t)}\right] \tag{49}$$

$$\frac{\partial V_T}{\partial K} = V_T\left[\frac{(x-K)}{\sigma^2(T-t)}\right] \tag{50}$$

$$\frac{\partial^2 V_T}{\partial K^2} = V_T\left[\frac{(x-K)^2}{\sigma^4(T-t)^2}-\frac{1}{\sigma^2(T-t)}\right] \tag{51}$$

and substituting into equation 43.
Letting $t \to T$ in 48 we get the boundary condition:

$$V_T = \delta(x-K) \tag{52}$$

(note that 48 is one of many functions that converge onto the Dirac delta function: see Arfken (1985), page 481).
Since 43 is a linear PDE, to satisfy a more general boundary condition $V(K,T) = p(K)$ we just have to superimpose solutions, and so the more general solution is:

$$V_T = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}(T-t)^{-\frac{1}{2}}\int_{-\infty}^{\infty} p(x)exp\left(-\frac{(x-K)^2}{2\sigma^2(T-t)}\right)dx \tag{53}$$

Letting $\sigma_x^2 = \sigma^2(T-t)$ gives:

$$V_T = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_x}\int_{-\infty}^{\infty} p(x)exp\left(-\frac{(x-K)^2}{2\sigma_x^2}\right)dx \tag{54}$$

which is just the expected value of $p(x)$ under the normal distribution with expected value K and standard deviation σ_x. The price of the option V is then just the discounted expected value. We note that this arbitrage-free price is equivalent to the standard definition of the actuarial fair price.

4 Summary

We have shown how, under certain assumptions, one can derive analogies of the Black and Scholes (1973) and Black (1976) equations for weather options. In particular we have shown that the arbitrage-free price of weather options is just the expected payoff under *objective* probabilities. In other words, the real world is risk-neutral in this case. This is because of the particular stochastic process we propose for weather swap prices.

References

G Arfken. *Mathematical Methods for Physicists.* Academic Press, 1985.

F Black. The pricing of commodity contracts. *Journal of Financial Economics*, 3:167–179, 1976.

F Black and M Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

S Jewson. Arbitrage pricing for weather derivatives. In *Climate Risk and the Weather Market*, pages 314–316. Risk Books, 2002a.

S Jewson. Weather derivative pricing and risk management: volatility and value at risk. *SSRN*, 2002b.

S Jewson. Weather option pricing with transaction costs. *Energy, Power and Risk Management*, 2002c.

S Jewson, A Brix, and C Ziehmann. Risk management. In *Weather Risk Report*, pages 5–10. Global Reinsurance Review, 2002.

P Wilmott. *Derivatives.* Wiley, 1999.