Fast pricing of Barrier options

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To my lovely late grandmother,
Paulina Kalomho Wanghondeli (1907-2010)

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Abstract

The Black-Scholes model is the most popular model for option pricing in finance. This model has been widely used, especially in pricing plain vanilla options, for the last four decades. The invention of this model has resulted in high option trading activities for about the past two decades due to the availability of closed-form solutions and the trivial nature of the model in assessing of options price to the price in the market. However, a rapid increase in these trading activities has resulted in some discrepancies which the model cannot accommodate. Researchers had to develop alternative models that can handle these discrepancies, such as jump diffusions and volatility smile/skew.

Barrier options are playing an increasing role amongst other exotic options that are very common, especially in the foreign exchange market. There are several universal numerical methods for pricing barrier options. This research is limited to comparing Fourier cosine (FCOS) series expansion method and Monte Carlo simulation as benchmark method to price down-out and down-in (call) barrier options. Accuracy and speed of pricing barrier options can then be investigated under exponential Lévy processes.

The results of this thesis show that despite the rigorous mathematics involved in the Fourier cosine method, it is faster and more accurate. Thus, FCOS is an efficient method of pricing barrier options when compared to the Monte Carlo method.

KEYWORDS: Lévy processes, Fourier transform, barrier options.
AMS Classification: 60H10 · 90A06.

Opsomming

Die Black-Scholes-model is die gewildste model vir die prysbepaling van opsies in die finansiewese. Hierdie model is die afgelope vier dekades algemeen gebruik, veral in die prysbepaling van standaard opsies. Die uitvinding van hierdie model het die afgelope twee dekades ho opsiehandelbedrywighede tot gevolg gehad te danke aan die beskikbaarheid van oplossings in geslote vorm en die triviale aard van die model in die assessering van opsiepryse in verhouding tot die markprys. Vinnige toename in hierdie handelsbedrywighede het egter teenstrydighede tot gevolg gehad wat nie deur die model geakkommodeer kan word nie. Navorsers moes alternatiewe modelle ontwikkel wat hierdie teenstrydighede, soos sprongdiffusie en die onbestendigheidsglimlag/krulvlak, kan hanteer.

Versperringsopsies speel toenemende rol in ander algemene eksotiese opsies, veral in die buitelandsevalutamark. Daar is verskeie universele numeriese metodes vir die prysbepaling van versperringsopsies. Hierdie navorsing is beperk tot vergelyking van die Fourier-kosinus-reeksuitbreidingsmetode en Monte Carlo-simulasie as maatstaf vir die prysbepaling van af-en-uit- en af-en-in- (koop-) versperringsopsies. Die akkuraatheid en spoed waarmee die prys van versperringsopsies bepaal word, kan dan volgens eksponensile Lévy-prosesse ondersoek word. Die resultate van hierdie tesis dui aan dat, alhoewel die Fourier-kosinus-metode gestrenge wiskunde behels, dit vinniger en meer akkuraat is. Die Fourier-kosinusmetode is dus doelmatige metode vir die prysbepaling van versperringsopsies as dit met die Monte Carlo-metode vergelyk word.

SLEUTELWOORDE: Lévy-prosesse, Fourier-transform, versperringsopsies.
AMS-klassifikasie: 60H10 90A06.

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Chapter 1

Introduction

Barrier options are path-dependent exotic options that are playing an increasing role in the financial market, according to Chiarella et al. [2012] and Stolte [2007]. “Path-dependent“ means that the option payoff at maturity date T does not only depend on the maturity underlying asset price but in addition it also depends on the path taken by the underlying asset's price during the option lifespan. Barrier options are similar (in terms of their payoff) to plain vanilla options, which can take two forms, either knock-in or knock-out. A knock-in barrier option exists if the underlying asset price touches or crosses the barrier line. The payoff function for a down-in barrier option (on a call option) is given by

$$\mathcal{C}_{\downarrow in}(S,T,H) = \max\{S_T - K, 0\}\mathbb{I}_{\min_{t\leq T} S_t \leq H}, \tag{0.1}$$

while a knock-out option cease to exist as soon as the stock price or underlying touches the barrier; its corresponding payoff function is given by

$$\mathcal{C}_{\downarrow out}(S,T,H) = \max\{S_T - K, 0\}\mathbb{I}_{\min_{t\leq T} S_t > H}. \tag{0.2}$$

The payoff functions given above in equation 0.1 and 0.2 are for down options. For every down and up options there are corresponding call or put options. This means barrier options can be either up-in (UI), up-out (UO), down-in (DI) or down-out (DO) call or put.

A down-in call option has the same payoff as the standard vanilla option if the underlying asset price touches the barrier line H during the option lifespan. As given in equation 0.1, $\mathbb{I}$ denotes the indicator function, which becomes one if the minimum of S_t is less or equal to H or else zero.

On the contrary, knock-out options immediately become worthless when the underlying asset price touches or crosses the barrier before or on the maturity date T. In the case of such knock-out event there is usually a cash rebate paid to the option holder.

Barrier options gained popularity over the past two decades, especially in over-the-counter (OTC) market, for two reasons:

1. Barrier options can meet specific hedging needs and investors have a precise view of their market movement. Investors can simply use them to gain exposure to future market than the one taking a bullish position in plain vanilla options. For example, knock-in options are for investors who believe that the underlying asset price is too volatile and there is a high probability of crossing the barrier.

2. Barrier options' premium is relatively lower in comparison with the standard plain vanilla option. For example for knock-out options, if the spot price is near to the barrier line (also known as a threshold), there is a higher chance for the underlying to cross the barrier and being knocked out. Therefore the premium is less due to the high risk associated with it.

1.1 Problem Statement : Efficient methods for pricing Barrier options.

Let $\mathcal{C}_{\downarrow_{in}}(S,T,H)$, $\mathcal{C}_{\downarrow_{out}}(S,T,H)$ and $\mathcal{C}(T,S)$ be the price of an European down-out call option (DO), a down-in (DI) call option and a plain vanilla call option respectively as functions of an underlying asset price S, expiration time T, and barrier level H. We make use of Lévy models by letting X_t be a Lévy process. By following Schoutens [2003] and Tankov [2005] the dynamic of the underlying asset S can be modelled by an exponential Lévy processes with risk neutral dynamics

$$dS_t = S_{t^-}rdt + S_{t^-}dX_t. \tag{1.3}$$

Under a risk neutral measure $\mathbb{Q}$, we would like to find the fair price $\mathcal{C}_{\downarrow_{in}}(S,T,H)$ which is given by

$$\mathcal{C}_{\downarrow_{in}}(S,0,H) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max\{S_T - K, 0\}\mathbb{I}_{\min_{t\leq T} S_t \leq H}], \tag{1.4}$$

where $\mathbb{I}$ and K are the indicator function and option strike price respectively. The price of down-in(DI) call can be obtained via in-out parity. For our case, the in-out parity can be seen as having a portfolio of down-in and down-out call options at time t with the same strike price, same maturity date and on the same underlying variable. If the underlying asset touches or crosses the barrier, the down-in call option becomes a vanilla call option while the down-out call option becomes zero. On the other hand, if the underlying asset does not cross the barrier line, i.e. $\min_{t\leq T} S_t > H$ for every $t \leq T$, the down-out call becomes a vanilla call while the down-in call expires with the value zero. Then at maturity date T, the option holder will receive exactly $\mathcal{C}(K, S)$. These reduce to the up-in parity equation 1.5

$$\mathcal{C}_{\downarrow in}(S, 0, H)+\mathcal{C}_{\downarrow out}(S, 0, H) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max\{S_T-K, 0\} \left(\mathbb{I}_{\min_{t\leq T} S_t\leq H} + \mathbb{I}_{\min_{t\leq T} S_t>H}\right)] = \mathcal{C}(0, S). \tag{1.5}$$

Since the seminal work of Merton [1973] on European option valuation under Black-Scholes model, the analytical solutions to equation 1.4 are available whereas in Lévy- driven models it is difficult to find closed-form formulas for barrier options. Equation 1.4 can be evaluated via the characteristic function whereby one has to apply numerical methods. Errors will always occur due to the fact that numerical methods are approximation techniques. Method with minimal error is therefore preferred. Methods to be used in this project are FCOS and the Monte Carlo simulation method as the -benchmark method-. Monte Carlo simulation has proven to be flexible and user friendly in pricing barrier options regardless of the option pricing model. The Monte Carlo simulation is actually one of the best choices in finance when it comes to utilising when analytical estimation to the mathematical expectation given in equation 1.4 is not possible ([Huynh et al., 2011]). The natural question is: which of the two is the most efficient method?

The result of this thesis will be derived from a standard model (Black-Scholes) and extend it to more general Lévy model, i.e. normal inverse Gaussian (NIG) model.

1.2 Structure of the thesis

The thesis is structured in the following way: In the remaining section of this chapter, the literature review will be discussed briefly. In chapter 2, mathematical preliminaries and notations to be used in this project will be introduced. Chapter 3, discusses barrier options pricing using two methods, the FCOS and the Monte Carlo simulation. These methods will then be compared in terms of speed and accuracy. In chapter 4, the result will be discussed. Finally, concluding remarks follow in chapter 5.

1.3 Literature review

The main concern is to price barrier options following the discussion in Section 1.1. Therefore this section gives a review on the subject of pricing exotic options, i.e barrier options.

The first analytical pricing formula for barrier options (down-out call option) in the Black-Scholes environment was proposed by Merton [1973]. The closed-form solution was derived by Carr et al. [1998]; they also discussed the replication method for barrier options. To use such formula one must consider the usual assumption that the probability distribution for the underlying asset's price at some future date is lognormally distributed. This means that the log of the return must be normally distributed. It is now a well-known fact that the return of the underlying asset's price is not necessary normal, as discussed by Chourdakis, Linghao [2010] and Yongqiang [2007]. This is partially due to rapid growth as well as the high leverage in the derivative market. Therefore there is a response from the market as more options are traded. This is accompanied by some stylised features such as high kurtosis (leptokurtosis),jumps and other stochastic volatility (volatility smiles) which are not accounted for in Black-Scholes model.[1] Thus, literature suggests new models with an ability of taking into account these leptokurtosis features associated with the asset price return; these models include **stochastic volatility (SV) models** and **Lévy processes**. For an introduction to these models the reader is referred to Stein and Stein [1991], Heston [1993] and Schoutens [2003].

1.3.1 Literature review of Lévy processes

Lévy models are amongst the most popular and tractable models in finance. This is partially due to their mathematical tractability in the computation of modern financial modelling.

Option pricing under Lévy models form a strong field of research. For instance to evaluate barrier options under Lévy models is a very difficult subject in the mathematics of finance. The main issue is to find the joint distribution of the S_T and minimum process $\min_{t \leq T} S_t$ under a risk-neutral measure $\mathbb{Q}$; we also have to observe whether the underlying asset price has hit the barrier line either discretely or continuously. This is not an easy task since under Lévy models (density of *jump*) an underlying asset may cross the barrier line without actually hitting the barrier line (Schoutens [2004]). This joint distribution is actually not yet known in a Lévy-driven (Tankov [2005]).

Due to the complexity of these problems a number of recent works focus on the pricing of barrier options modelled by Lévy-driven models. This thesis will consider the major contributions by Schoutens [2003], Schoutens [2004], Tankov [2005] and Yongqiang [2007]. Schoutens [2003] introduced the theory behind Lévy processes and its application to finance. In the same paper he proved that these models incorporate the stylised features associated with finance time series. His overall results agree with the survey conducted by Deville [2008]. In Schoutens [2004], he discussed option pricing in cases where the underlying is driven by Lévy processes. Chourdakis, Yongqiang [2007] and Tankov [2005], considered five continuous-time Lévy models, standard Brownian motion, NIG, Meixner, variance gamma and CGMY processes and their results provide an empirical motivation in applying Lévy

[1]The Black-Scholes model assumes that the underlying asset's price return is normal and volatility is constant.

models in finance compared to the traditional standard Brownian motion driven model. Lévy driven models describe the market much better than Brownian motion in the presence of jumps diffusion and stochastic volatility. This has been verified by fitting a histogram of normalised daily log returns along with the standard normal density function (Linghao [2010]).

Lévy processes are valuable tools in financial modelling as they provide a good fit with real market data. The modelling of economic indices, especially for barrier options in a Lévy-driven market has drawn many interests. Boyarchenko and Levendorskii [2002]- considered an equivalent martingale measure(EMM) for stock price return which is driven by Lévy model and they came up explicit formulae for European type barrier options. Kudryavtsev and Levendorskii [2009], considered different types of Lévy processes to accurately price barrier options. In the same paper, they observed the behaviour of option prices near the barrier level and their results show that if the underlying asset price is modelled by Lévy processes and has no diffusion component to it, then there are two possibilities: either the option price is discontinuous or option delta is not bounded near the barrier line. For more references on barrier option price modelling under Lévy processes, see Schoutens [2003], Schoutens and Wilmott [2005] and Tankov [2005].

1.3.2 Literature review of pricing methods

There are numerous numerical methods for pricing exotic options, e.g. partial integro-differential equations (PIDEs), techniques of Monte Carlo (MC) simulation and numerical integration. In this paper we shall restrict ourselves to Fourier-cosine series expansion (FCOS) and Monte Carlo simulation.

1.3.2.1 FCOS

The Fourier-cosine series expansion method is a widely used alternative numerical method for pricing exotic options which depends on a Fourier series. To use this method, instead of the density function of a random variable X simply substitute it‘s Fourier cosine series expansion. This method was introduced to finance by Fang and Oosterlee [2008], when they outlined that the series coefficients and characteristic functions are related by a simple relationship. In another paper Fang and Oosterlee [2010], they showed that the FCOS method can be applied provided we know the underlying characteristic function. In the same paper, they demonstrated how the FCOS method can speed up the pricing of European plain vanilla options. They also discussed the FCOS method as one of the most efficient methods for pricing early exercise options and discretely monitored barrier options under exponential

Lévy processes (Fang and Oosterlee [2009]).

1.3.2.2 Monte Carlo simulation

The Monte Carlo simulation is mostly used in finance when there are no analytic formulas. The Monte Carlo method can evaluate the mathematical expectation in equation 1.4 without the need for analytical solution. This method was extensively used by [Schoutens and Symens, 2002] in pricing exotic options under various models. The authors remarked that although this method takes dozens of hours, it is reliable and it has been used as a benchmark tool for most of the option pricing models in finance. This is because the Monte Carlo method offers many advantages. For example, in Lévy model, *jumps* can be controlled well through the Monte Carlo method. For basic application of Monte Carlo method regarding the pricing of path-dependent options refer to Glasserman [2003].

Chapter 2

Mathematical preliminaries

This chapter introduces some useful mathematical terminology, definitions, results and notations to be used throughout this paper. The main focus is to show the link between characteristic functions, option pricing theory and Lévy-market models. The theory of mathematics involved in most of these models and option pricing methods is very complex and therefore this chapter is as self-contained as possible.

2.1 Basic notations and terminology

If not stated, the following notations will be fixed.

Table 2.1: Notations

Symbol	**Definition**	**Symbol**	**Definition**	**Symbol**	**Definition**
$\mathbb{R}$	Real numbers	$\mathcal{F}_t$	Filtration	H	Barrier level
$\mathbb{N}$	Natural numbers	$(X_t)_t$	Lévy processes	K	Strike price
$\mathbb{C}$	Complex numbers	$\mathbb{I}$	Indicator function	$\mathcal{C}$	call option
$\mathcal{F}$	Sigma algebra	S_t	Stock price	$\mathcal{C}_{\downarrow out}$	Down-out option

2.2 Mathematical formulation

This section introduces the mathematical requirements for understanding the work.

Definition 2.2.1 (Characteristic function). *If X is a $\mathbb{R}^d$ random variable then its characteristic function is known to be a function $\varphi_X : \mathbb{R}^d \to \mathbb{R}$ defined by:*

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)] = \int_{-\infty}^{\infty} e^{itx} f_X(x)dx,$$

where f_X represent a density function of a random variable X.

From the basic theory of Fourier transform, one can easily notice that the characteristics function φ_X of a random variable X is the same as the Fourier transform of a density function $f_X(x)$.

Theorem 2.2.2 (Lévy inversion). *If two random variables are assumed to have the same characteristic functions then they must have the same distribution and vice versa.*

The following observations were made regarding the characteristic function.

Remark 2.2.3. *(i)* $\varphi_X(0) = 1$*, for any distribution*

(ii) $\varphi_X(u)$ *is always continuous by Lebesgue's dominated convergence theorem (LDCT). This is because* $\left|e^{iuX}\right|$ *is a continuous and bounded function for all finite real* u *and* X*.*

(iii) $\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$*,*

(iv) $|\varphi_X(t)| = \underbrace{|\mathbb{E}[e^{itX}]| \leq \mathbb{E}[|e^{itX}|]}_{\textit{Jensen Inequality}}$.

The characteristic function φ_X of a random variable X has the adequate information necessary to obtain the probability distribution of X. This can be achieved fairly easily via the Fourier inversion (see Section 3.1).

2.3 Lévy processes

Most of the basic models in finance rely on the assumption that the underlying asset returns are normally distributed. Recent studies have proved that this assumption is not true in general, simply because the returns have features such as jumps, fatter tails and skewness. These abnormal features are commonly known as leptokurtosis. Ultimately, these features are hardly captured by some basic models in finance, e.g. the Black-Scholes model. This is mainly because Black-Scholes assumes constant volatility and therefore does not allow discontinuities and jumps in the underlying asset price process.

Lévy processes play a central role in finance especially, in modelling option pricing. This is because they generalise the Black-Scholes model by allowing extension of jumps in the underlying asset price while preserving the independence and stationarity of the asset price returns. There exists a number of Lévy processes such as diffusion processes, Poisson processes and jump diffusion processes. The definitions of Lévy processes were followed closely by Chourdakis, Kienitz and Wetterau, Cont and Tankov [2004] and Tankov [2005].

Definition 2.3.1 (Càdlàg process). *A process X_t is called càdlàg process if it is right continous with left hand limit (RCLL), i.e.*

$$X_{t^-} = \lim_{s\uparrow t} X_s \quad \text{exists for all } s \neq t \text{ and } \quad X_{t^+} = \lim_{s\downarrow t} X_s = X_t.$$

Definition 2.3.2 (Lévy processes). *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where we may adapt the notation $\mathcal{F} = \mathcal{F}_T$ and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration defined on the time horizon $T = [0, +\infty]$.*
An adapted, càdlàg, stochastic process $X = (X_t)_{0\leq t\leq T}$ with the following properties is called a Lévy process.

(i) X_t is continuous in probability, i.e. for every $0 \leq s< \leq T$. and $\epsilon>0$,

$$\lim_{s\to t} \mathbb{P}(|X_t - X_s|>0) = 0.$$

(ii) X has stationary increments. This means that the distribution of $X_{t+s} - X_t$ is independent of t, for all $0 \leq s<t \leq T$.

(iii) X has independent increments, i.e. $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s \ \forall \ 0 \leq s<t \leq T$.

(iv) $X_0 = 0$ almost surely.

Lévy processes are semi-martingale and admit càdlàg (RCLL) modification. This means sample paths of X are continuous almost everywhere, i.e. X has càdlàg paths. Due to the independence and stationarity of the increments of X, the *characteristic function* of any Lévy process is guaranteed to have the form of this nature

$$\mathbb{E}[e^{-iuX_t}] = e^{-t\psi(u)},$$

where ψ is known as the *characteristic exponent* of X. The most common Lévy processes includes Poisson, compound Poisson processes and standard Brownian motion.

Definition 2.3.3 (Infinitely divisible laws). *We say that a law $\mathbb{P}_X$ of a random variable X is infinitely divisible, if for all $n \in \mathbb{N}$ there is i.i.d. random variables $X_1^{(\frac{1}{n})}, \cdots, X_n^{(\frac{1}{n})}$ such that*

$$X \overset{d}{=} X_1^{(\frac{1}{n})} + \cdots + X_n^{(\frac{1}{n})}. \tag{3.1}$$

Strictly speaking, there is a simple relationship connecting Lévy processes and infinitely divisible distribution. This relation assert that for every infinitely divisible distribution, a *Lévy process* $(X_t)_t$ exists.

We shall first give the characterisation lemma for infinitely divisible distibution of a random variable.

Lemma 2.3.4. *Suppose $\mathbb{P}_X$ has infinitely divisible distibution then for every $n \in \mathbb{N}$, there is a random variable $X^{(\frac{1}{n})}$ such that the characteristic function of X take the form*

$$\varphi_X(u) = \left(\varphi_{X^{(\frac{1}{n})}}(u)\right)^n. \tag{3.2}$$

We adopt the Lévy-Khintchine formula from Kienitz and Wetterau in caese of multi-dimension.

Theorem 2.3.5 (Lévy-Khintchine formula)**.** *Suppose $(X_t)_t$ is a Lévy process. Consider a random variable X_1 and denote its characteristic function by $\mathbb{E}[e^{-iuX_1}] = e^{-\psi(u)}, \quad u \in \mathbb{R}$. The Lévy-Khintchine formula states that for any Lévy process $(X_t)_t$, the function $\psi(u)$ is given by:*

$$\psi(u) = iau - \frac{1}{2}u^T\sigma^2 u + \int_{\mathbb{R}} (1 - e^{iux} + iux\mathbb{I}_{\{|x|<1\}})\nu(dx) \tag{3.3}$$

for a vector a, $\sigma \geq 0$ is a definite matrix and ν is a finite measure on $\mathbb{R} \; - \{0\}$ with the following integrability condition

$$\int_{\mathbb{R}} \inf\{1, x^2\}\nu(dx) = \int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty.$$

ν is often called a Lévy measure defined by:

$$\nu[A] := \mathbb{E}[\#\{t \geq 1 : \Delta_t X \in A - \{0\}\}], \quad \textit{for any set } A. \tag{3.4}$$

The triplet (σ^2, ν, a) is known as the Lévy triplet which can be assigned for any infinitely divisible random variable. Here ν is the Lévy measure satisfying $\nu(\{0\}) = 0$.

In the case of one dimension the Lévy-Khintchine formula is given by

$$\mathbb{E}\left[e^{iuX}\right] = e^{iau - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}}(1 - e^{iux} + iux\mathbb{I}_{\{|x|<1\}})\nu(dx)} \tag{3.5}$$

The main interested in modelling asset price using Lévy-type model. Most of the detailed discussions of these models can be followed up on the work of Cont and Tankov [2004] and Tankov [2005]. The exponential Lévy model is used under equivalent martingale measure $\mathbb{Q}$.

For exponential Lévy processes, the risk-neutral dynamics are given by

$$S_t = S_0 \exp\{X_t\}. \tag{3.6}$$

Before modelling option prices with Lévy models, firstly, define how one construct a Lévy process from other Lévy process using what is termed Brownian subordination.

Theorem 2.3.6 (Subordinator). *An increasing Lévy process is known as subordinator. i.e. if $S = (S_t)_{t\in[0,T]}$ is a Lévy process with Lévy triplet (c, ν, a) satisfying*

$$\nu(-\infty, 0) = 0, \quad c = 0, \quad \int_{(0,1)} x\nu(dx) < \infty \quad \textit{and} \quad a = b + \int_{(0,1)} x\nu(dx) > 0,$$

then S is called a subordinator, where $\nu(dx)$ is the jump of size density of X.

According to theorem 2.3.6 a Lévy process X is said to be a subordinator if it has almost surely non-decreasing sample paths. Therefore under any circumstances such process cannot have a diffusion component.

One can then characterise all those Lévy processes that can be written as a linear combination of a subordinator in the theorem 2.3.7. To be more precise, this theorem says that every Lévy process can be written as a linear combination of a Brownian motion (deterministic drift) and pure jump process which is independent of that Brownian motion.

Theorem 2.3.7. *For every Lévy measure ν on $\mathbb{R}$ and $\mu \in \mathbb{R}$ fixed, there exists a Lévy process $(X_t)_t$ such that $X_t = B_t(S) + \mu S$, where B_t and S is a Brownian motion and subordinator respectively. Further $B_t \perp\!\!\!\perp S$ iff:*

1. *ν is absolutely continuous with density $\nu(x)$,*
2. *$\nu(x)e^{-\mu x} = \nu(-x)e^{\mu x} \quad \forall x$,*
3. *$\nu(\sqrt{u})e^{-\mu\sqrt{u}}$ is a completely monotonic function on $\mathbb{R}^+$.*

For detail proof, the reader is referred to the work of [Cont and Tankov, 2004].

2.3.1 Black-Scholes model: An introduction

The Black-Scholes model is one of the major concepts in the theory of finance. It is an exponential Lévy model with continuous sample paths and is driven by Brownian motion. It was pioneered in finance by Black and Scholes [1973]. This model assumes that the daily changes in the underlying price S over a very small time horizon is normally distributed with expected return μ and volatility σ, i.e

$$\frac{\Delta S}{S} \sim N\left(\mu\Delta t, \sigma^2\Delta t\right)$$

where $N(a, b)$ represents a normal distribution function where a and b denote the mean and variance respectively.

Under the Black-Scholes model, there are a number of assumptions that were imposed in order to be able to derive the valuation formulae. These include the following:

1. Stock prices follow a Brownian motion with constant μ and σ.
2. Short selling of securities is allowed.
3. No arbitrage opportunities exist.
4. No dividend is paid during the option lifespan and there is no transaction cost.
5. Risk-free rate r of interest is constant for all maturities.
6. The market is perfectly liquid.

If S_t and dS_t is the stock price and change in S_t at time t, then using the above assumption 1, the price movement is modelled by the following dynamic:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

where W_t is a Wiener process.

Recall that $W_t \sim N(0, t)$ for $t>0$, i.e W_t is normally distributed.

Using Itö formula on $\ln S_t$, we get

$$d \ln S_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} d[S]_t = rdt + \sigma dW_t - \frac{\sigma^2}{2} dt = \left(r - \frac{\sigma^2}{2}\right) dt - \sigma dW_t.$$

Therefore,

$$\ln S_t - \ln S_0 = \int_0^t \left(r - \frac{\sigma^2}{2}\right) ds + \int_0^t dW_s = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

According to stochastic calculus,

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r - \frac{\sigma^2}{2}\right) t, (\sigma W_t)^2\right) = N\left(\left(r - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right).$$

This means the log return $\ln\left(\frac{S_t}{S_0}\right)$ is normally distributed and the probability density function for a normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{(x - \nu t)^2}{2\sigma^2 t}\right\},$$

where $x = \ln\left(\frac{S_t}{S_0}\right)$ and $\nu = \left(r - \frac{\sigma^2}{2}\right)$. The probability density function (pdf) can be standardised by letting $\nu = 0$ and $\sigma t = 1$. The cumulative distribution function $N(x)$ of a standardise (pdf) is given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz. \tag{3.7}$$

Solving for a characteristic function for Black-Scholes model, we get

$$\varphi_{\mathrm{BS}}(u,t) = \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{(x-\nu t)^2}{2\sigma^2 t}\right\} dx = e^{\left(iu\mu t - \frac{1}{2}\sigma^2 u^2 t\right)}.$$

2.3.2 Normal Inverse Gaussian (NIG) process

A normal inverse Gaussian (NIG) process is a pure jump Lévy processes whose increments have a *normal inverse Gaussian* distribution. It is particular case for generalised hyperbolic distribution. The probability density function of the NIG process is given by

$$f_{\mathrm{NIG}}(x,\alpha,\beta,\mu,\delta) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}} \tag{3.8}$$

where K_1 represents the Bessel function (see [Schoutens and Wilmott, 2005]) given by

$$K_1(u) = \frac{1}{2}\int_{\mathbb{R}+} e^{-\frac{1}{2}u\left(t+\frac{1}{t}\right)} dt, \tag{3.9}$$

and with parameters $\alpha, \delta, |\beta| \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$. With these parameters, the NIG process can incorporate skewness and high kurtosis in the underlying asset return which a standard Brownian motion (Gaussian) process cannot. This property is the main reason for its widely application in financial modelling.

The characteristic function for NIG is given by

$$\varphi_{\mathrm{NIG}}(u,\alpha,\beta,\delta,\mu) = e^{iu\mu-\delta\left(\sqrt{\alpha^2-(\beta+iu)^2})+\sqrt{\alpha^2-\beta^2}\right)}. \tag{3.10}$$

The inverse Gaussian (IG) tell us the time it takes for a Brownian motion with a drift $b>0$ to get to some level $a>0$. It has a mean $\frac{a}{b}$ and variance $\frac{a}{b^3}$. The IG process is only defined for $x>0$ because the time it takes for a Brownian motion to reach a positive level a is always positve.

The inverse Gaussian process has a probability density function given by

$$f_{\text{IG}}(x,a,b) = \sqrt{\left(\frac{a}{2\pi x^3}\right)} e^{\frac{-a(x-b)}{2b^2 x}}. \tag{3.11}$$

Define the notation for the NIG process by

$$X^{\text{NIG}} = \{X_t^{\text{NIG}} : t>0\}. \tag{3.12}$$

With $X_0^{\text{NIG}} = 0$, this is a stationary and independent NIG distributed increment. Schoutens [2003] showed that a NIG process can be obtained by using theorem 2.3.7, i.e. by time-change Brownian motion $B = \{B_t : t \geq 0\}$ with a drift given by the IG process $I^{\text{IG}} = \{I_t^{\text{IG}} : t \geq 0\}$, $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$.

Using theorem 2.3.7, NIG process X^{NIG} can then be expressed as

$$X_t{}^{\text{NIG}} = \mu + \beta\delta^2 I_t^{\text{IG}} + \delta B_{I_t^{\text{IG}}} \tag{3.13}$$

Theorem 2.3.7 will be used extensively in the next section regarding the discussion of the NIG process. Figure 2.1 shows the NIG subordination. It is usually said that the NIG process has been subordinated. Clearly the jumps are continuous and hence we can model asset price very easily. Therefore, subordination makes it easy to generate the NIG process we will be eventually be used in the assessment of option prices.

The NIG distribution was pioneered by [Barndorff-Nielsen, 1996].

Let us assume we have a process $\{X(t) : 0 \leq t<\infty\}$. In our context, this is a stock price movement modelled by a Lévy processes. The distribution of X_t is uniquely determined by its marginal distribution. If fact, X_1 is infinitely divisible and its characteristic function is given by Lévy-Khintchine formula 3.3.

We remark that under Lévy models especially, in pure jump process such as NIG, stock price may cross the barrier line without actual hitting the barrier level. This makes it difficult for one to monitor the option at barrier line.

However, under the Black-Scholes framework, we are able to trace the option paths because the Black-Scholes model does not permit jumps in asset prices. Hence we would like to find the maximum and minimum process of the underlying variable(stock price).

We follow closely the definition by Björk [2009].

Definition 2.3.8 (Running maximum/minimum process). *Let any $y \in \mathbb{R}$, then the hitting time of y, denoted by $\tau(X,y)$ or $\tau(y)$ or τ_y, is defined by*

$$\tau(y) = \inf\{t \geq 0 | X(t) = y\}. \tag{3.14}$$

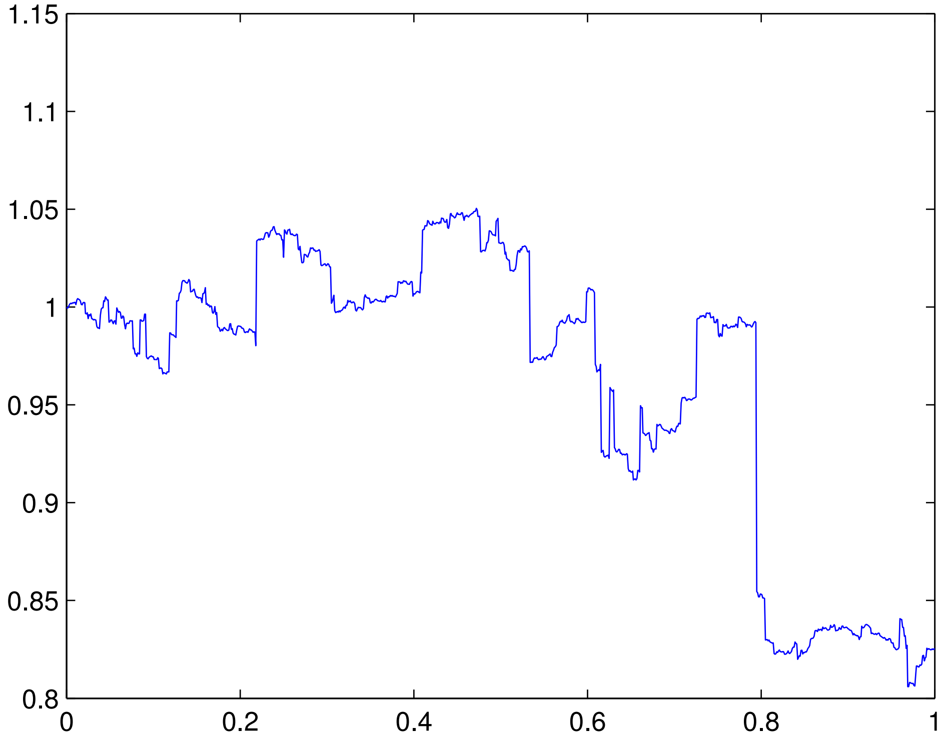

Figure 2.1: NIG subordination process with $\alpha = 15, \beta = -5$ and $\delta = 0.5$

The X process absorbed at y is defined to be $X_y(t) = X(t \wedge \tau) = X(\min(t, \tau))$. The running maximum and minimum processes of a process $X(t)$ is defined by,

$$M_X(t) = \sup_{0 \le s \le t} X(s) \quad \textit{and} \quad m_X(t) = \inf_{0 \le s \le t} X(s) \qquad \textit{respectively.} \tag{3.15}$$

Under the Black-Scholes model one can be able to find the joint distribution of a Brownian motion and its running minimum. Ultimately, we can find the joint distribution of M_X and X. On the other hand, under pure *Jump* Lévy processes, this joint distribution is not yet known (an open problem).

2.4 Chapter summary

Summary of chapter 2. *In this chapter, The concept of characteristic function of a random variable X and its relationship with Lévy processes via the Lévy-Khintchine formula was introduced. Two Lévy processes, the Black-Scholes and the NIG model were considered. Essentially, this chapter introduced the neccessary mathematical theory for understanding the work in the following chapter.*

Chapter 3

Barrier options pricing

Recall that barrier options are path-dependent options with an unusual payoff. At maturity, the payoff depends on the whether the underlying asset has crossed a barrier line and whether the underlying asset price exceedes the strike price (in case of a call option). Barrier options can be seen as an extension of vanilla options with an additional parameter H, a barrier level of which option can be activated or deactivated upon hitting this barrier line. Barrier options can take different forms. Barrier options that are activated once an underlying asset hits the barrier line are known as *knock-in barrier options* and the ones that cease to exist upon hitting the barrier line are called *knock-out barrier options*. For each type (knock-in and knock-out options), there are two types of barrier options assigned according to the spot price of the underlying (stock price). Suppose S_0 is the current stock price. If $S_0>H$, then the underlying barrier option is named a *down barrier option* and if $S_0<H$, then the barrier option is called an *up barrier option*.

In this chapter, we restrict ourselves to *Down barrier options*. The option value for down-out barrier is given in equation 1.4.

In this section we are going to implement two methods of pricing barrier options. The first method is based on Fourier series (Fourier-cosine series expansion) and the second one is the well-known Monte Carlo method.

3.1 Option pricing by Transform methods

In this section, the theory behind option pricing via characteristic functions and direct integration is outline. The concept of Fourier transform which plays a major role in option pricing and then show its relationship to option pricing will be introduced.

In this section, we fix $x = \log S_0, y = \log S_T$ and $k = \log K$ to be the log asset price at $t = 0$, log price at T and log strike price respectively.

Recall from definition 2.2.1 that the characteristic function of a random variable X is the same as the Fourier transform of its density function f_X. We revise the definition 2.2.1 as a Fourier transform:

Definition 3.1.1 (Fourier transform). *Let $f : \mathbb{R} \to \mathbb{R}$ be a well-defined function, then the Fourier transform of a function f is given by*

$$\hat{f}(t) = \int_{\mathbb{R}} e^{itx} f(x)dx$$

usually denoted as $\mathcal{F}(f) = \hat{f}$. Denote inverse Fourier transform *of f by*

$$\mathcal{F}^{-1}(\hat{f})(x) = \int_{\mathbb{R}} e^{-iux} f(t)dt.$$

Now it is easier to see that if f represents the density of the probability distribution, then the characteristic function $\varphi(t)$ of f is

$$\varphi(t) = \hat{f}$$

and the cumulative distribution function (CDF) of f is

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

based on the fact that a Fourier transform is invertible, it follows that

$$f_X = f_Y \iff \varphi_X = \varphi_Y.$$

Using the inversion of Fourier transform, one can recover the probability density function f_X from φ_X in definition 2.2.1 using the following theorem:

Theorem 3.1.2 (FT theorem). *Let f_X be a density function of a random variable X and let φ_X be the corresponding characteristic function. Then*

$$\int_{x}^{\infty} f(t)dt = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} \frac{1}{t} Im\left(e^{-itx}\varphi_X(t)\right) dt$$

The density function of a random variable X is given by

$$f_X(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\varphi_X(t)dt. \tag{1.1}$$

Remark 3.1.3. *If f_X is a density function of a random variable X then*

$$\mathbb{P}(X \geq x) = \int_0^\infty f_X(t)dt = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{1}{t} Im\left(e^{-itx}\varphi_X(t)\right) dt.$$

For an European option with a payoff function h, we can evaluate option value $V(0, S_0)$ using direct integration as given below:

$$V(h,T) = \int_0^\infty h(x) f_X(x) dx. \tag{1.2}$$

Using the same technique as in theorem 3.1.2, we can also recover option prices from the characteristic function. Essentially, this is the main theorem that enable us to use characteristic functions to recover option prices.

3.2 The Black-Scholes model

This section assume the reader is acquainted with the Black-Scholes framework. We are would like to model prices of the underlying variables following geometric Brownian motion, i.e. its risk-neutral dynamics are given by:

$$dS_t = S_t r dt + S_t \sigma d\hat{W} \tag{2.3}$$

where r denotes the risk-free rate and $\hat{W}$ represents the risk-neutral Brownian Motion as given by the Girsanov theorem.

We can derive an analytic formula for barrier options under Black-Scholes in a similar approach for plain vanilla options, i.e. they satisfy the same PDE but with different boundary conditions.

Suppose we want to find the analytical formula for a down-in barrier option. Consider a portfolio π consisting of n shares and one derivative, i.e. $\mathcal{C}_{\downarrow in}(S,t,H)$, H fixed. By Itö formula, it follows that

$$d\mathcal{C}_{\downarrow in} = \frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} dt + \frac{\partial \mathcal{C}_{\downarrow in}}{\partial S} dS + \frac{1}{2}\left[\frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial t^2} dt^2 + \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S^2} dS^2\right] = \frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} dt + \frac{\partial \mathcal{C}_{\downarrow in}}{\partial S}\left(S_t r dt + S_t \sigma d\hat{W}\right)$$

$$+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S_t^2} dt = \left[\frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} + r S_t \frac{\partial \mathcal{C}_{\downarrow in}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S_t^2}\right] dt + S_t \sigma \frac{\partial \mathcal{C}_{\downarrow in}}{\partial S} d\hat{W}.$$

The value of the portfolio is

$$\pi = \mathcal{C}_{\downarrow in} - nS \iff d\pi = d\mathcal{C}_{\downarrow in} - ndS_t = \left[\frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} + rS_t\left(\frac{\partial \mathcal{C}_{\downarrow in}}{\partial S} - n\right) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S_t^2}\right] dt$$

$$+S_t\sigma\left(\frac{\partial \mathcal{C}_{\downarrow in}}{\partial S} - n\right) d\hat{W} = \left[\frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S_t^2}\right] dt$$

when

$$n = \frac{\partial \mathcal{C}_{\downarrow in}}{\partial S}.$$

This portfolio is riskless, so it must earn the same risk-free rate as the bank account, i.e. $d\pi = r\pi dt$ by no arbitrage. Therefore,

$$d\pi = r\pi dt = \left[\frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S^2}\right] dt$$

$$\Updownarrow$$

$$r\pi = r\left(\mathcal{C}_{\downarrow in} - S\frac{\partial \mathcal{C}_{\downarrow in}}{\partial S}\right) = \frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S_t^2} \Longrightarrow \frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} + rS\frac{\partial \mathcal{C}_{\downarrow in}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S^2} - r\mathcal{C}_{\downarrow in} = 0.$$

The boundary condition of this partial differential equation is:

$$\begin{cases} \frac{\partial \mathcal{C}_{\downarrow in}}{\partial t} + rS\frac{\partial \mathcal{C}_{\downarrow in}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}_{\downarrow in}}{\partial S^2} - r\mathcal{C}_{\downarrow in} = 0 \\ \Phi(S_T, K, H) = \mathcal{C}_{\downarrow in}(S, T, H) = \begin{cases} \max\{S_T - K, 0\} & \text{if } \min_{t\in[0,T]} S_t \le H \\ 0 & \text{if } \min_{t\in[0,T]} S_t > H. \end{cases} \end{cases} \tag{2.4}$$

We adopt the analytical solution to the above boundary condition derived in Appendix A 5. See also Björk [2009] and Hull [2012]. The analytical solution to equation 2.4 is given by

$$\mathcal{C}_{\downarrow in} = S_0 e^{-qT}\left(\frac{H}{S_0}\right)^{2\lambda} N(y) - Ke^{-rT}\left(\frac{H}{S_0}\right)^{2\lambda-2} N(y - \sigma\sqrt{T}) \tag{2.5}$$

where

$$\lambda = \frac{r - q + \frac{\sigma^2}{2}}{\sigma^2} \quad \text{and} \quad y = \frac{\ln\frac{H^2}{S_0 K}}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}.$$

The corresponding analytical formula for a down-out is obtained from up-in parity in equation 1.5.

The main objective in this subsection is to make use of the risk-neutral valuation and find the arbitrage-free price of the barrier options using Monte Carlo and FCOS; then we compare our results with the analytical formula given in equation 2.5. Recall from the previous discussion that the payoff of the down-in call option is given by:

$$\mathcal{C}_{\downarrow_{in}}(S,T,H) = \begin{cases} \max\{S_T - K, 0\} & \text{if } \min_{t\in[0,T]} S_t \leq H \\ 0 & \text{if } \min_{t\in[0,T]} S_t > H. \end{cases} \tag{2.6}$$

One of the most common assumptions made under the Black-Scholes model is that the log returns are normally distributed. The main challenge in pricing barrier options is to find the distribution of running minimum and maximum processes with asset price process S_t from definition 2.3.8. Hence following model under discussion it is possible to generate $\ln\left(\frac{S_T}{S_0}\right)$, since it is known that is in normally distributed, i.e.

$$\ln\left(\frac{S_T}{S_0}\right) \sim N\left(rT, \delta\sqrt{T}\right).$$

Based on the well-known paper of Babsiri and Noel [1998], it can be verified that,

$$\mathbb{P}\left(\min_{t\in[0,T]} \ln\left(\frac{S_t}{S_0}\right) \leq y \mid \ln\left(\frac{S_T}{S_0}\right) = x\right) = e^{\frac{2y(x-y)}{\sigma^2 T}}, \tag{2.7}$$

where $y \leq x$ and $y \leq 0$ following the previous discussion in Section 2.3.1. Now considering the conditional distribution function m_Y of minimum y of $\ln\left(\frac{S_T}{S_0}\right)$ it can be seen that for each y, $m_Y(y)$ is picked from a uniform distribution ranging from $[0,1]$.

We let $u = m_Y(y)$ so that

$$y = m_Y^{-1}(u) = \frac{x - \sqrt{x^2 - 2\sigma^2 T \ln(u)}}{2}, \tag{2.8}$$

according to Huynh et al. [2011]. We can also see that $y \leq 0$ because u is taken from a uniform distribution over the interval $[0,1]$.

3.2.1 The Monte Carlo method

The assumption is that we model asset prices following the geometric Brownian motion given by equation 2.3. Now by using the *Itô* lemma, we can express equation 2.3 in the form

$$S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma \hat{W}_t\}. \tag{2.9}$$

To implement the Monte Carlo simulation one has to discretise the time interval $[0,T]$ into N sub-steps $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N = T$ and we denote $S_{n+1} := S_{t_{n+1}}$ for

$n = 0, 1, \cdots, N-1$. Equation 2.9 can be rewritten in the form

$$S_{n+1} = S_0 \exp\{(r - \frac{1}{2}\sigma^2)\Delta t_n + \sigma \Delta \hat{W}_n\}, \tag{2.10}$$

where $\Delta t_n = t_{n+1} - t_n$ and $\Delta \hat{W}_n = \hat{W}_{n+1} - \hat{W}_n$.

The procedure for calculating the price of a down-in barrier option via the Monte Carlo simulation is given as follows:

Algorithm *Down-in barrier option*(N, M, S_0, H)
(∗ Pricing a down-in barrier (call) option ∗)
1. **for** $i \leftarrow 1$ **to** M
2. **for** $n \leftarrow 1$ **to** $N-1$
3. **do** generate random number $z_n = randn(N, 1)$;
4. set $S_{n+1} = S_0 e^{(r-\frac{1}{2}\sigma^2)\Delta t_n + \sigma\sqrt{\Delta t} z_n}$
5. set $x = \ln\left(\frac{S_{n+1}}{S_0}\right)$
6. **do** generate uniform variate $u_n = rand(N, 1)$
7. $\min = y = \frac{x - \sqrt{x^2 - 2\sigma^2 T \ln(u_n)}}{2}$
8. **endfor**
9. **if** $y = \min_{0 \le n \le N-1} S_n \le H$
10. **then** $V_i = e^{-r(T-t)} \max\{S_T - K, 0\}$
11. **else** $V_i = 0$
12. **endif**
13. **endfor**
14. Finally, compute $\mathcal{C}_{\downarrow in} = \frac{1}{M}\sum_{i=1}^{M} V_i$.

Figure 3.1 depicts the Monte Carlo price for a barrier option. One can see that if we increase N, then the Monte Carlo price and Black-Scholes-Merton formula converge.

Table 3.1: MC prices for down-in call option as $N \to \infty$.

N	**Down-in**	**Error (%)**	**Time (s)**
1000	4.4722	35.78	0.000844
2500	4.8510	24.42	0.001062
5000	4.7414	17.90	0.002245
10000	4.6769	12.11	0.003748
100000	4.4593	3.75	0.031004
1000000	4.5208	1.20	0.426470
10000000	4.5093	0.38	11.896412

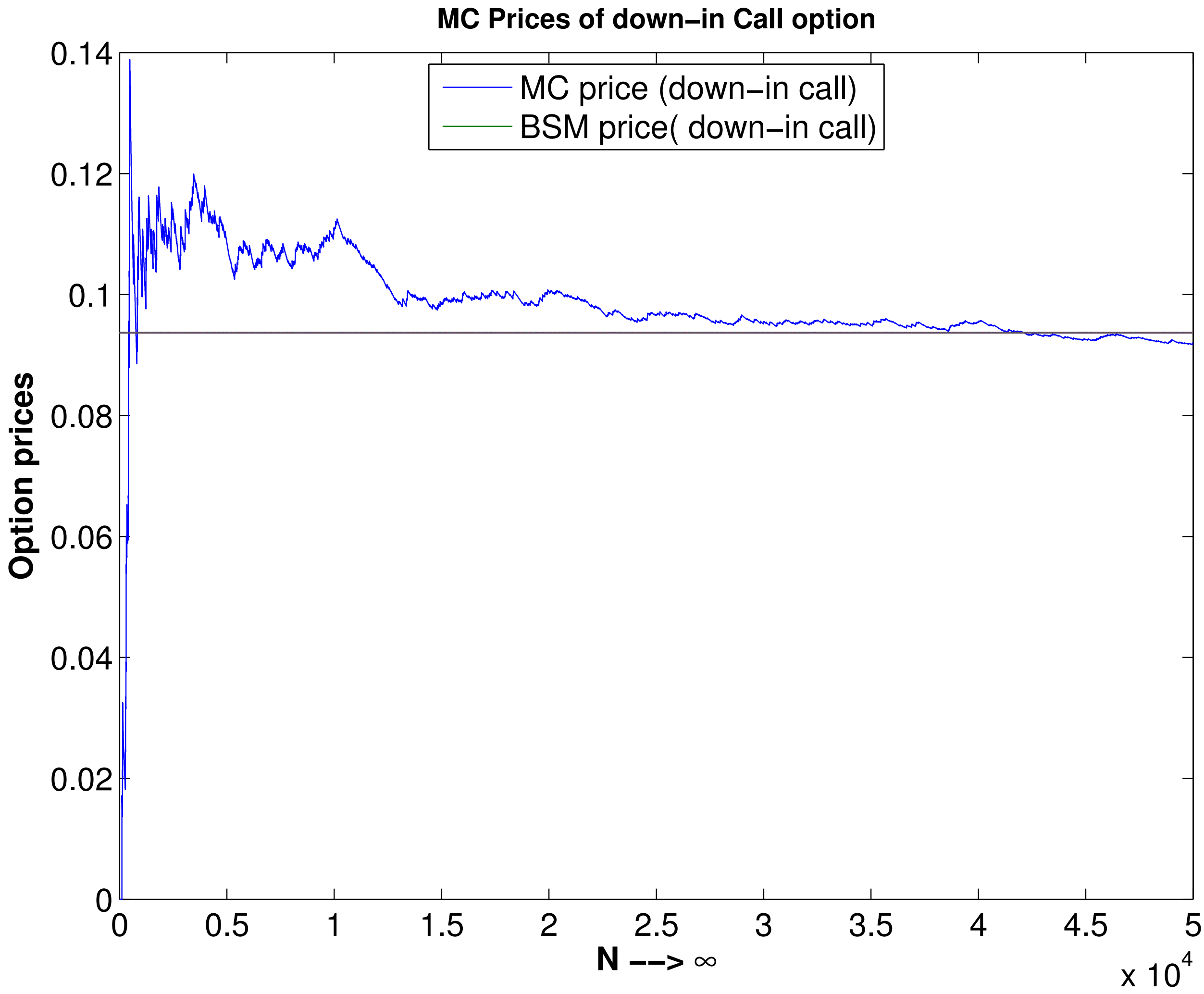

Figure 3.1: Monte Carlo barrier option pricing

3.2.2 Fourier Cosine series expansion (FCOS) method

Recall from chapter 2 that the European option type allow you to buy an option now and wait until the expiring date T, whereby you may decide to exercise or not. In this section we consider a barrier option of European type which is discretely monitored,[1] say M-times. We follow closely Fang and Oosterlee [2008], Fang and Oosterlee [2009], Fang and Oosterlee [2010] and Deng and Chong [2011].

We consider the log asset prices $x = \ln\left(\frac{S_0}{K}\right)$ and $y = \ln\left(\frac{S_T}{K}\right)$. We can write the payoff

[1] this means checking whether the underlying asset price has crossed a barrier line using interval length of time.

for a call option and denote it by

$$\mathcal{C}(y,T) = \max\{K\left(e^y - 1\right), 0\}. \tag{2.11}$$

Using the risk-neutral evaluation procedures, we have

$$\mathcal{C}(x,t_0) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\mathcal{C}(y,T)|x] = e^{-r(T-t)}\int_{-\infty}^{\infty}\mathcal{C}(y,T)f(y|x)dy \tag{2.12}$$

where

$$f(y|x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iuy}\varphi(u)du$$

represents the conditional probability density function of y conditioned on x. Now we can replace the infinite integration range by a finite range $[a,b]$ of the integral in equation 2.12 by means of truncation to obtain

$$\mathcal{C}(x,t_0) = e^{-r(T-t)}\int_a^b \mathcal{C}(y,T)f(y|x)dy + \epsilon \tag{2.13}$$

where the integral interval $[a,b] = \left[(c_1+x_0) - L\sqrt{c_2+\sqrt{c_4}}\ ,\ (c_1+x_0) + L\sqrt{c_2+\sqrt{c_4}}\right]$ for cumulant c_1, c_2 and c_4, $x_0 = \ln\left(\frac{S_0}{K}\right)$ and domain length L depends on the user-defined tolerance level and ϵ is the minimal truncation error. Fang and Oosterlee [2008] suggested that for accurate truncation, we must choose $L = 10$. Cumulant for various Lévy processes can be found in the book by Kienitz and Wetterau, page 223.

The main motivation for using characteristic function in option pricing is that the density function $f(y|x)$ is not always ready available, while the characteristic function of y is relatively easy to find. We therefore interchange the density function with the cosine series expansion of y, which can be done in the following manner:

$$f(y|x) = {\sum_{k=0}^{+\infty}}' A_k(x)\cos\left(k\pi\frac{y-a}{b-a}\right) \tag{2.14}$$

"where $\sum'$ means that the first component of the summation of the series is weighted by half" (Fang and Oosterlee [2009]), with

$$A_k(x) = \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy.$$

After replacing the density function with its COS approximation, one then interchange the integration by the summation in equation 2.13 to get

$$\mathcal{C}(x, t_0) = e^{-r(T-t)} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{ \phi\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi\frac{a}{b-a}} \right\} H_k. \tag{2.15}$$

The series coefficients (H_k) of the payoff function are analytic according to Fang and Oosterlee [2009]. This means that for a call option with a payoff function in definition 2.11, we can express H_k as

$$H_k = \frac{2}{b-a}\int_0^b \mathcal{C}(y,T)\cos\left(k\pi\frac{y-a}{b-a}\right)dy = \frac{2}{b-a}K(\chi_k(0,b) - \psi_k(0,b)),$$

where

$$\chi_k(0,b) = \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos(k\pi)e^b - \cos\left(k\pi\frac{a}{b-a}\right) + \frac{k\pi}{b-a}\sin(k\pi)e^b + \frac{k\pi}{b-a}\sin\left(k\pi\frac{a}{b-a}\right)\right]$$

and

$$\psi_k(0,b) = \begin{cases} \left[\sin(k\pi) + \sin\left(k\pi\frac{a}{b-a}\right)\right]\frac{b-a}{k\pi} & \text{if } k \neq 0 \\ b & \text{if } k = 0. \end{cases}$$

According to Fang and Oosterlee [2009], we can also find an expression of H_k for exotic options like barrier options. We use the expression of H_k provided by Fang and Oosterlee [2009].

$$H_k(t_m) = C_k(a,h,t_m) = \frac{2}{b-a}\int_a^h \mathcal{C}(x,t_m)\cos\left(k\pi\frac{x-a}{b-a}\right)dx = \frac{e^{-rT}}{\pi}\operatorname{Im}((M_c+M_s)\mathbf{u}) \quad \forall \mathbf{u} \in \mathbb{R}, \tag{2.16}$$

where

$$M_c = \left(M^c_{k,j}(x_1,x_2)\right)_{k,j=0}^{N-1} \iff M^c_{k,j}(x_1,x_2) = \begin{cases} i\pi\frac{x_2-x_1}{b-a} & \text{if } k=j=0 \\ \frac{\exp\left(i(j-k)\pi\frac{x_2-a}{b-a}\right)-\exp\left(i(j+k)\pi\frac{x_1-a}{b-a}\right)}{j+k}, & \text{otherwise} \end{cases}$$

and

$$M_s = \left(M^s_{k,j}(x_1,x_2)\right)_{k,j=0}^{N-1} \iff M^s_{k,j}(x_1,x_2) = \begin{cases} i\pi\frac{x_2-x_1}{b-a} & \text{if } k=j \\ \frac{\exp\left(i(j-k)\pi\frac{x_2-a}{b-a}\right)-\exp\left(i(j-k)\pi\frac{x_1-a}{b-a}\right)}{j-k}, & \text{if } k \neq j. \end{cases}$$

Recall that down-out barrier options expires worthless if the underlying asset price cross the barrier line at pre-specified dates (discretely monitored), i.e if $\min S_t \leq H$.

Knock-out barrier options may have a rebate (Rb) associated with. We consider a case when $Rb = 0$.

Since we discretely monitored the movement of the underlying asset, we let $\mathcal{T}$ be the set consisting of monitoring dates given by

$$\mathcal{T} = \{t_1, \cdots t_M\}, \quad \text{with} \ \ t_1 < t_2 < \cdots < t_{M-1} < T_M = T. \tag{2.17}$$

We let $h = \ln\left(\frac{H}{K}\right)$ and $m = M, M-1, M-2, \cdots, 2$.

The price of the down-out barrier option must satisfy the following recursive definition according to Fang and Oosterlee [2009].

$$\begin{cases} \mathcal{C}_{\downarrow out}(x, t_{m-1}) = e^{-r(t_m - t_{m-1})} \int_{-\infty}^{\infty} \mathcal{C}_{\downarrow}(y, t_m) f(y|x) dy \\ \mathcal{C}_{\downarrow out}(y, t_{m-1}) = \begin{cases} 0 & \text{if} \quad \min y \leq h \\ \mathcal{C}(x, t_{m-1}) & \text{if} \quad \min y > h \end{cases} \end{cases} \tag{2.18}$$

Since the barrier line H is known in advance, the pricing of barrier options is simplified. Thus pricing a discretely monitored barrier option we need two steps (Kienitz and Wetterau):

1. Recovery of the Fourier cosine series expansion coefficients $H_k(t_1)$ of the option value at time t_1 recursively from $H_k(t_M), H_k(t_{M-1}), \cdots, H_k(t_2)$ in $O((M-1)N\log_2(N))$ operations.

2. Insert $H_k(t_1)$ in the FCOS formula in equation 2.15 to get the option value at time t_0 in $O(N)$ operations.

Below we give the procedure for computing the price of a down-out call option using MATLAB/ GNU Octave version 3.6.4.

Algorithm *FCOS down-out barrier option*$(S_0, H, K, T, r, \sigma)$
(∗ Pricing down-out barrier (call) option adopted from Fang and Oosterlee [2009] ∗)
1. Compute $H_k(t_{N-1})$ in equation 2.16
2. Set $x_1 \leftarrow h, x_2 \leftarrow b, c \leftarrow a$ and $d \leftarrow h$
3. Compute M_s and M_c
4. Set $\mathcal{F}_s \leftarrow \mathcal{F}(M_s)$ and $\mathcal{F}_s \leftarrow \mathcal{F}(M_s)$
5. Set $G \leftarrow \frac{2}{b-a}\psi_k(c, d)$
6. Backward induction to recover $\mathcal{C}_{\downarrow}(x, t_{m-1})$
7. **for** $m = N$ **to** 0
8. **do** $\mathbf{u} \leftarrow (u_j)_{j=0}^{N-1}$, $u_j \leftarrow \varphi\left(\frac{j\pi}{b-a}\right) H_j(t_{m+1})$

9. Compute $\mathbf{u}_s$ by extending vector $\mathbf{u}$ with N zeros

10. **do** $\mathbf{M_{su}} :=$ The first N elements from $\mathcal{F}^{-1}(\mathcal{F}_s\mathcal{F}(\mathbf{u_s}))$

11. Calculate $\mathbf{M_{cu}} :=$ The first N elements from $\mathcal{F}^{-1}(\mathcal{F}_c\mathcal{F}(\mathbf{u_s}))$ and reversing them

12. Compute the continuation value $C(x_1, x_2, t_0)$ given in equation 2.16

13. **endfor**

14. Finally, compute $\mathcal{C}_{\downarrow_{out}}(x, t_0)$ using series coefficients $H_k(t_1)$.

Table 3.2 and Figure 3.2 below show the output of the algorithm *FCOS down-out barrier option*.

Table 3.2: Barrier option pricing using Fourier-cosine method. $S = 100, K = 90, r = 0.05, q = 0.02, \sigma = 0.2993, H = 90, T = 1$

N	Down-out	Error (%)	Time(s)
2^8	10.58	1.48	0.070404

From Figure 3.2 we observe the inverse relation between the strike price and the barrier price, i.e. as the strike price increases, the barrier option price decreases and vice versa. Figure 3.3 depicts the price of down-in, down-out and vanilla call using Fourier-cosine method under the NIG model.

3.3 The normal inverse Gaussian model

Lévy processes can be categorised into two forms:

- Jump diffusion models,e.g. Merton model
- Infinite activity pure jump model.

In this section we will discuss the pricing of barrier options under a Lévy infinite activity model name the normal inverse Gaussian (NIG) following the same procedure as in Section 3.2. There is a large amount of literature on pricing options under the NIG model. We utilise the parameters set by Fang and Oosterlee [2009].

In Black-Scholes model, recall that we constructed a risk-neutral measure via the Qirsarnov theorem. We, however, could not use the same approach under the Lévy models. We therefore use the concept known as mean-correcting martingale measure. The interested reader is referred to Schoutens [2004] and Cont and Tankov [2004].

The characteristic function for a NIG model is given by:

$$\phi(\mu, t, \alpha, \beta, \delta) = \exp\left\{-t\delta\left(\sqrt{\beta^2 - (\beta + \mu i)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right\} \tag{3.19}$$

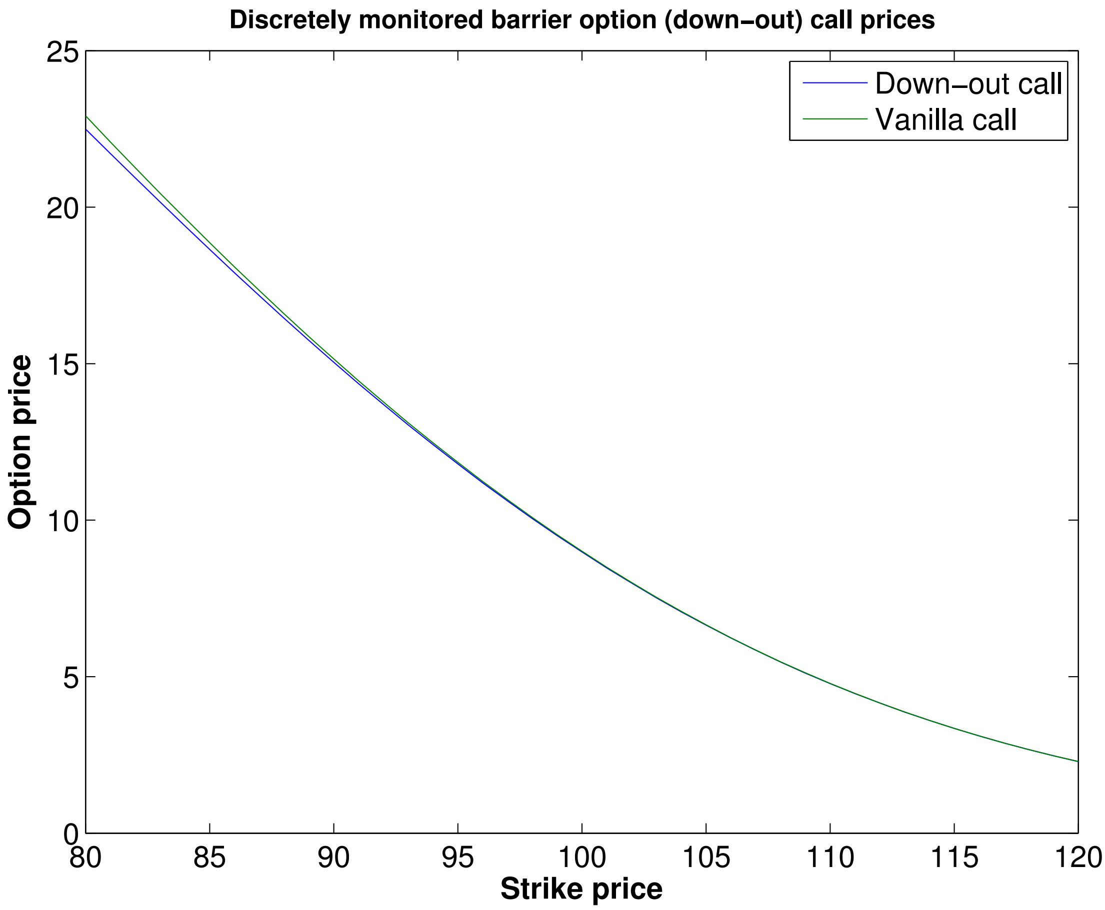

Figure 3.2: Fourier-cosine prices under NIG model

with three model parameters α, β and δ (see Table 3.3).

Recall from definition 2.3.4 that this characteristic function in definition 3.19 is infinitely divisible. We are interested in setting up a model for asset price (of the form $S_t = S_0 \exp\{\omega t + \psi_t(u)\}$) under risk-neutral measure. According to Cont and Tankov [2004], we can decompose the characteristic function into two parts, namely the drift term and the driftless term. We can achieve this by doing simple algebra:

$$\mathbb{E}\left[e^{\omega T+\psi_T(u)}|\mathcal{F}_0\right] = e^{\omega t}\mathbb{E}\left[e^{\psi_T(u)}|\mathcal{F}_0\right] = e^{rt} \Longleftrightarrow \mathbb{E}\left[e^{\psi_T(u)}|\mathcal{F}_0\right] = e^{t(r-\omega)}. \tag{3.20}$$

From equation 3.20, we can easily make ω the subject by

$$\omega = r - \frac{1}{t}\ln \mathbb{E}\left[e^{\psi_T(u)}|\mathcal{F}_0\right].$$

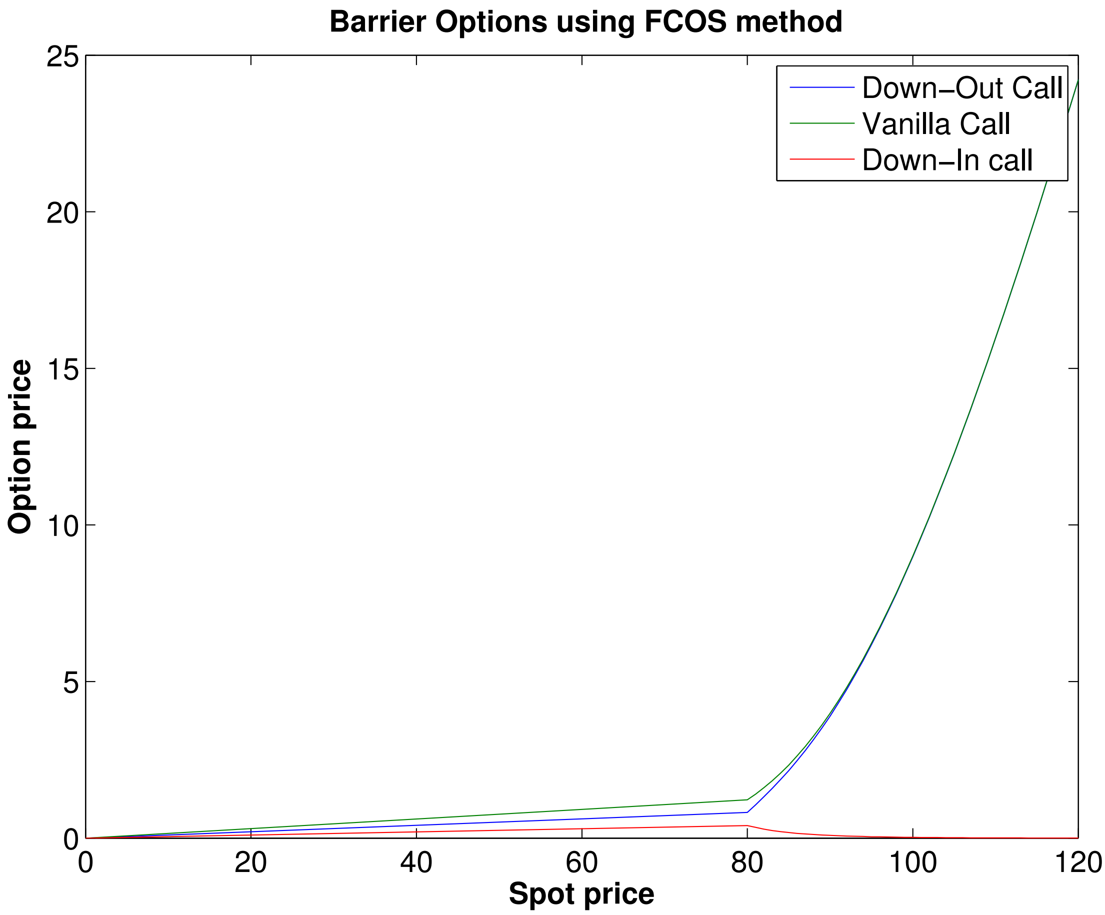

Figure 3.3: Fourier-cosine method under the NIG Model

To exclude arbitrage opportunities, we impose the condition that

$$\omega = r - \frac{1}{t} \ln \mathbb{E}\left[e^{\psi_T(u)} | \mathcal{F}_0\right] = r - \psi(-i).$$

For NIG, the cumulant for the drift term is given by

$$\omega = \psi_t^{drift}(\alpha, \beta, \delta) = i(r - q - \psi(-i))ut = i\left(r - q + \delta\left(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right) ut. \tag{3.21}$$

and for the driftless term

$$\psi_t^{driftless}(\alpha, \beta, \delta) = -\delta t\left(\sqrt{\alpha^2 - (\beta + ut)^2} - \sqrt{\alpha^2 - \beta^2}\right). \tag{3.22}$$

Table 3.3: NIG parameters α, β and δ.

Parameter	**Description**	**Value**
α	α>0, tail heaviness	$\alpha = 15$
β	$-\alpha$<β<α, asymmetry parameter	$\beta = -5$
δ	δ>0, scale parameter	$\delta = 0.5$
μ	for location	$\mu = 0.3$

Putting equation 3.21 and 3.22 together we obtain the risk-neutral characteristic function of the NIG model:

$$\phi^{RN}(u) = \exp\left\{i\omega ut - t\delta\left(\sqrt{\beta^2 - (\beta + ui)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right\}. \tag{3.23}$$

The reader is further referred to Chourdakis.

To be able to use the Monte Carlo method as in Section 3.2.1, we have to simulate a NIG process. To achieve this we first need to simulate the inverse Gaussian (IG) processes and use the use the NIG subordination discussed in definition 2.3.6.

To generate IG process X, we use the algorithm introduced by ([Michael et al., 1976]).

Algorithm *IG generator*(a, b)
(∗ Generating I^{IG} and X^{NIG} ∗)refer to Section 2.3.2 for parameter a,b
1. Generate normal random variable N
2. Set $Z \leftarrow N^2$
3. Set $X_1 \leftarrow \frac{a}{b} + \frac{Z}{2b^2} - \sqrt{4abZ + \frac{Z^2}{2b^2}}$
4. Generate uniform random variable $U \in [0, 1]$
5. **if** $U \leq \frac{a}{a+bX_1}$
6. **then** $X \leftarrow X_1$
7. **else** $X = \frac{a^2}{b^2X_1}$
8. **endif**
9. $X \leftarrow \frac{a^2}{b^2X_1}$
10. Simulation of the process $(X_{t_1}, \cdots, X_{t_n})$ for fixed $t_i, i = 1, 2, 3, \cdots$
11. Simulate n i.i.d. IG random variables I_t with parameter (ah, b) (h is the discretisation step)
12. Set $a \leftarrow 1$ and $b = \sqrt{\alpha^2 - \beta^2}$
13. The discrete trajectory: $X_{t_i} = X_{t_{i-1}} + I_{t_i}$

We can compute the stock price process modelled by the NIG model using theorem 2.3.7, i.e. $X_t = \beta\delta^2 IG + W_{IG}$ for a Brownian motion W. The risk-neutral process for stock price is given by

$$S_t = S_0 \exp\{\omega t + X_t\}.$$

3.4 Chapter summary

SUMMARY OF CHAPTER 3. *In this chapter, we considered the application of characteristic functions to option pricing, most particulary barrier options. We included the pseudocodes for pricing down-in and down-out barrier options using the Monte Carlo and FCOS method respectively. Generating a NIG process is not an obvious task, so we used the Brownian motion subordination approach. We will summarise the numerical results from the algorithm Down-in barrier option* and *FCOS down-out barrier option in the chapter 4.*

Chapter 4

Results and discussion

We have partially presented our results from the Black-Scholes model in Section 3.2. In this section we will present the full results and compare our methods. We will do some extension to the Monte Carlo method to improve the simulation and finally draw a conclusion.

4.1 Input parameters

We have adopted the input from Fang and Oosterlee [2009] and Feng and Linetsky [2007]. We

Table 4.1: Pricing data inputs and model parameters

Model	S_0	**K**	**r**	**q**	**T**	**H**	**Model parameters**
BS	100	100	0.05	0.02	1	80	$\sigma = 20\%$
NIG	100	100	0.05	0.02	1	80	$\alpha = 15, \beta = -5, \delta = 0.5$

descretely monitored whether the barrier level has been closed or not. We do this monthly; $m = 12$, where m is the monitoring points.

4.2 Numerical results

In this section, we shall discuss the numerical result from the algorithms for *Down-in barrier option* and *FCOS down-out barrier option*. We do this in order to answer the question as to which method is more suitable in pricing barrier options.

We have run $N = 500000$ paths for the Monte Carlo and 2^{10} Fourier Cosine series expansion method. We used a computer with the following specifications: Inspiron 15, with 2nd Generation Intel® Core™ and dual core processors i3-3227U CPU 1.90$GHz \times 4$ and up to

4GB ram memory. Programs scripts were written in MATLAB 12.04 Student version and in GNU Octave, version 3.6.4.

Table 4.2 shows our numerical results. $\mathcal{C}^{\hat{\epsilon}}$, $\mathcal{C}^{\hat{\epsilon}}_{\downarrow in}$, $\mathcal{C}^{\hat{\epsilon}}_{\downarrow out}$ and τ represent the standard error

Model	**Method**	$\mathcal{C}_{\downarrow out}$	$\mathcal{C}^{\hat{\epsilon}}_{\downarrow out}$	$\mathcal{C}_{\downarrow in}$	$\mathcal{C}^{\hat{\epsilon}}_{\downarrow in}$	$\mathcal{C}$	$\mathcal{C}^{\hat{\epsilon}}$	$\tau(s)$
BS	MC	9.2018	0.0218	0.0355	0.0066	9.2381	0.0653	5.303795
	FCOS	9.1927	0.04128	0.0343	0.0594	9.2270	$5.5 \times e^{-6}$	1.784137
NIG	MC	8.9959	0.0627	0.0240	0.0065	9.0199	0.0597	179.210257
	FCOS	8.9831	$6.034 \times e^{-6}$	0.0247	$1.607 \times e^{-2}$	9.0078	$2.71 \times e^{-5}$	2.672359

Table 4.2: Prices, errors and time taken for pricing of monthly monitored down barrier options under the Black-Schole's model and NIG model

for vanilla call, down-in call, down-out call and the CPU time taken respectively.

Following from Section 3.2, we can use the exact analytical formula to compare accuracy. Using the Black-Scholes-Merton formula, we get:

$$\mathcal{C} = \boxed{9.1896}, \quad \mathcal{C}_{\downarrow in} = \boxed{0.0337} \text{ and } \mathcal{C}_{\downarrow out} = \boxed{9.2044}$$

Errors for NIG (where analytical formulae are not available) are calculated with respect to reference values provided by Feng and Linetsky [2007], page 371, table 7.1 and Fang and Oosterlee [2009] page 23, table 6. We can see from Table 4.2 that the Fourier-cosine series expansion method is the better method than the Monte Carlo method. Although the Monte Carlo method is a reliable tool in option pricing, it takes more time to compute the actual option value as compared to the FCOS. This corresponds with the observations in a well-recognised paper by Fang and Oosterlee [2009]. The general conclusion of this study is that the FCOS method is quite fast and very efficient, and we can get accurate price by running a very small number of simulations. With the Monte Carlo method, one has to run a very large number of simulations which takes time. The Monte Carlo simulation is user-friendly, however.

Figure is the same as Figure 3.1, except that we increase N to observe the convergence of the Monte Carlo method price with the exact price.

For a Monte Carlo simulation the errors shown in the Table 3.1 are the standard deviation of the vector of the number of simulations. We can improve our simulation's accuracy by performing more simulations, as seen in Table 3.1.

We will consider two variance reduction techniques, namely anthithetic variables and

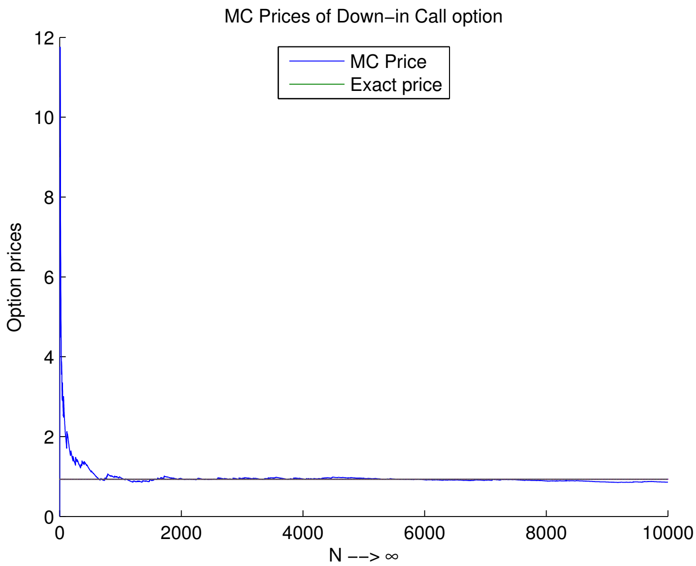

Figure 4.1: Monte-Carlo barrier option pricing $N = 10000$

control variables in order to improve the accuracy in Monte Carlo simulation. For a basic understanding of these two techniques, the reader is referred to Schoutens and Symens [2002] and Glasserman [2003].

4.2.1 Antithetic variables

The Table 4.3 shows the Monte Carlo results after using anthithetic variables.

Table 4.3: Monte Carlo prices and times taken for pricing monthly monitored down barrier options under the Black-Scholes' model and th NIG model using anthithetic variables

Model	$\mathcal{C}_{\downarrow out}$	$\mathcal{C}^{\hat{\epsilon}}_{\downarrow out}$	$\mathcal{C}_{\downarrow in}$	$\mathcal{C}^{\hat{\epsilon}}_{\downarrow in}$	$\mathcal{C}$	$\mathcal{C}^{\hat{\epsilon}}$	$\tau(s)$
BS	9.1846	0.0154	0.0324	0.0014	9.2170	0.0153	1561.313553
NIG	9.0002	0.0141	0.0240	0.0013	9.0242	0.0141	2691.755495

Now from Tables 4.2 and 4.3, one can observe that the errors in each case have decreased and the actual option value in Black-Scholes model are nearly equal to the one given above by using the analytical formula provided by Hull [2012]. Hence, using antithetic variables, we can see the improvement in option values.

4.2.2 Control variables

Suppose X denotes the payoff process for a down-out call option and let Y be a stochastic process that is correlated with X with known expectation, i.e. $\mathbb{E}[Y]$ is known in advance. Define another payoff by

$$\tilde{X} = X + \alpha\left(Y - \mathbb{E}[Y]\right).$$

Clearly

$$\mathbb{E}[\tilde{X}] = \mathbb{E}\left[X + \alpha\left(Y - \mathbb{E}[Y]\right)\right] = \mathbb{E}[X] + \alpha\left(\mathbb{E}[Y] - \underbrace{\mathbb{E}\left[\mathbb{E}[Y]\right]}_{\mathbb{E}[Y]}\right) = \mathbb{E}[X]$$

and

$$\mathrm{Var}(\tilde{X}) = \mathrm{Var}(X) + 2\alpha\underbrace{\mathrm{Cov}(X,Y)}_{\mathbb{E}[XY - X\mathbb{E}[Y]]} + \alpha^2\mathrm{Var}(Y).$$

We want to minimise the varience of $\tilde{X}$. We take the derivative of $\mathrm{Var}(\tilde{X})$ with respect to α.

$$\frac{d\mathrm{Var}(\tilde{X})}{d\alpha} = 2\mathrm{Cov}(X,Y) + 2\alpha\mathrm{Var}(Y) = 0 \Longleftrightarrow \alpha = -\frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}.$$

With the given α value, we can now compute the option value using the payoff $\tilde{X}$, choosing our control variable to be the payoff of a vanilla call option. This is because there is a correlation[1] between a down-out and a vanilla call option.

Table 4.4 shows that performing Monte Carlo using control variables another way to improve the results in Table 4.2. We can now see the improvement on Monte Carlo performance

Table 4.4: Monte Carlo prices and time taken for pricing monthly monitored down barrier options under the Black-Schole's model and the NIG model using control variables

Model	$\mathcal{C}_{\downarrow out}$	$\mathcal{C}^{\hat{\epsilon}}_{\downarrow out}$	$\mathcal{C}_{\downarrow in}$	$\mathcal{C}^{\hat{\epsilon}}_{\downarrow in}$	$\mathcal{C}$	$\mathcal{C}^{\hat{\epsilon}}$	$\tau(s)$
BS	9.1771	0.0081	0.0343	$9.2935 \times e^{-4}$	9.2114	0.0080	3.600764
NIG	9.0009	0.0082	0.0262	$8.4228 \times e^{-4}$	9.0271	0.0081	8.859271

compared with the analytical formulae results above.

[1] A down-out call option become vanilla call option if the underlying asset did not cross the barrier.

4.3 Chapter summary

SUMMARY OF CHAPTER 4. *Our results show that the FCOS method is fast and efficient compare to the results presented by Feng and Linetsky [2007]. Variance reduction techniques were carried out in order to improve the Monte Carlo simulation.*

Chapter 5

Conclusion

This thesis aimed to price barrier options, one of the fastest growing exotic option in the over-the-counter market, using the Fourier-cosine method and the Monte Carlo simulation.

Our primary focus was the exponential Lévy processes, namely the Black-Scholes and the NIG model. We have used the empirical motivation of numerous researchers that exponential Lévy processes is the best model for describing the market. We discussed the normal inverse Gaussian(NIG) model as one of the Lévy models with infinite activity or a pure jump model.

We began with the Black-Scholes model, in which only a single parameter is required and the model itself is relatively easy to handle. Closed-formulas are readily available and we can always compare our results to determine the accuracy. We have used the Monte Carlo simulation and the Fourier-cosine series expansion method. To obtain an acceptable result with the Monte Carlo method one needs to do large amount of simulations of which takes dozens of hours to produce the output (see 5.2). We performed variance reduction techniques as a way of improving the Monte Carlo results.

Pricing barrier options according to NIG model it is more difficult to know whether the method is accurate or not. This is because with NIG model no analytical formulas are available. Hence we nee to obtain market data. However, since barrier options, pricing data are obtained at a higher fee, we have used model parameters by recognised authors such as Feng and Linetsky [2007] andFang and Oosterlee [2009] to compare our methods.

Our results showed that barrier options are relatively cheaper than to ordinary vanilla options when keeping all input parameter the same. In fact, deep-in-the-Money barrier options (down-out call options) are cheaper as shown in figure **??**. Based on our computation, we concluded that although the Monte Carlo simulation is a very flexible, user-friendly tool, the Fourier-cosine method is the best approach to pricing barrier options under Lévy models. In essence, to get the Monte Carlo price to converge to the exact price, we have to generate as many paths as possible which eventually takes longer. We include an excel sheet in Appendix A that illustrates the time it take for both the Monte Carlo and the FCOS to converge to the exact solution.

A possible extension of this work is to calculate the barrier option price from a portfolio of different combinations of vanilla options under the NIG model, known as static hedge or static replication. Model risk with barrier options is also an interesting further work of this thesis. This will be very interesting as there are no joint dynamics for the asset prices and the barrier hitting time is nearly impossible to observe.

Appendix A

5.1 Black-Scholes formulae

In this section, we will derive the formula for the down-out option under the Black-Scholes model. Recall from section 3.2 that the risk-neutral dynamic of a process S_t, following a Brownian motion, is given by

$$dS_t = S_t r dt + S - t\sigma d\hat{W}_t. \tag{1.1}$$

where r is the risk-free rate and $\hat{W}$ represents the risk-neutral Brownian motion as given by the Girsanov theorem.

We will derive analytical formula for vanilla options under Black-Scholes.

Suppose we want to find the analytical formula for an European call option. Consider a portfolio π consisting n shares and one derivative, i.e. $\mathcal{C}$. We use Itô formula found in basic textbooks of financial Mathematics. Clearly $\mathcal{C} = f(S,t)$, so it follows that

$$d\mathcal{C} = \frac{\partial \mathcal{C}}{\partial t}dt + \frac{\partial \mathcal{C}}{\partial S}dS + \frac{1}{2}\left[\frac{\partial^2 \mathcal{C}}{\partial t^2}dt^2 + \frac{\partial^2 \mathcal{C}}{\partial S^2}dS^2\right] = \frac{\partial \mathcal{C}}{\partial t}dt + \frac{\partial \mathcal{C}}{\partial S}\left(S_t r dt + S_t \sigma d\hat{W}\right)$$

$$+\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}}{\partial S_t^2}dt = \left[\frac{\partial \mathcal{C}}{\partial t} + rS_t\frac{\partial \mathcal{C}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}}{\partial S_t^2}\right]dt + S_t\sigma\frac{\partial \mathcal{C}}{\partial S}d\hat{W}.$$

The value of the portfolio is

$$\pi = \mathcal{C} - nS \iff d\pi = d\mathcal{C} - ndS_t = \left[\frac{\partial \mathcal{C}}{\partial t} + rS_t\left(\frac{\partial \mathcal{C}}{\partial S} - n\right) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 \mathcal{C}}{\partial S_t^2}\right]dt$$

$$+S_t\sigma\left(\frac{\partial \mathcal{C}}{\partial S} - n\right)d\hat{W} = \left[\frac{\partial \mathcal{C}}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 \mathcal{C}}{\partial S_t^2}\right]dt$$

when

$$n = \frac{\partial \mathcal{C}}{\partial S}.$$

This portfolio is riskless, so it must earn the same risk-free rate as the bank account, i.e. $d\pi = r\pi dt$ by no arbitrage. Therefore,

$$d\pi = r\pi dt = \left[\frac{\partial \mathcal{C}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}}{\partial S^2}\right] dt$$

$$\Updownarrow$$

$$r\pi = r\left(\mathcal{C} - S\frac{\partial \mathcal{C}}{\partial S}\right) = \frac{\partial \mathcal{C}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}}{\partial S_t^2} \Longrightarrow \frac{\partial \mathcal{C}}{\partial t} + rS\frac{\partial \mathcal{C}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}}{\partial S^2} - r\mathcal{C} = 0.$$

The boundary condition of this partial differential equation is:

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} + rS\frac{\partial \mathcal{C}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{C}}{\partial S^2} - r\mathcal{C} = 0 \\ \Phi(S_T, K) = \mathcal{C}(S, T) = \max\{S_T - K, 0\} \end{cases} \tag{1.2}$$

Barrier options are assentially vanilla options with an additional condition. For the down-out option, for example, we impose a condition that it is a vanilla call provided the stock price S_t has not crossed the barrier line from above during the life of the option, i.e. for all $t \leq T$. Hence barrier options must also satisfy the boundary PDE in equation 1.2 above with the additional condition that $\mathcal{C}(H, t) = 0$.

Let us do some change of variables and use the method of images to derive analytical formulae for barrier options. Let $S = He^x$ so that $\mathcal{C} = f(He^x, t)$, hence

$$\frac{\partial \mathcal{C}}{\partial x} = \frac{\partial \mathcal{C}}{\partial S}\frac{\partial S}{\partial x} = S\frac{\partial \mathcal{C}}{\partial S} \tag{1.3}$$

and

$$\frac{\partial^2 \mathcal{C}}{\partial x^2} = \frac{\partial}{\partial x}\underbrace{\frac{\partial \mathcal{C}}{\partial x}}_{1.3} = \frac{\partial}{\partial x}\left[S\frac{\partial \mathcal{C}}{\partial S}\right] = \frac{\partial S}{\partial x}\frac{\partial \mathcal{C}}{\partial S} + S\frac{\partial}{\partial x}\left[\frac{\partial \mathcal{C}}{\partial S}\right] = S\frac{\partial \mathcal{C}}{\partial S} + S^2\frac{\partial^2 \mathcal{C}}{\partial S^2} \Leftrightarrow S^2\frac{\partial^2 \mathcal{C}}{\partial S^2} = \frac{\partial^2 \mathcal{C}}{\partial x^2} - \frac{\partial \mathcal{C}}{\partial x}. \tag{1.4}$$

Now substituting equation 1.3 and 1.4 in into the above PDE and collect like terms we get the following boundary condition

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial \mathcal{C}}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \mathcal{C}}{\partial x^2} - r\mathcal{C} = 0, \\ \Phi(x,T) = \begin{cases} \mathcal{C}(x,T) = \max\{He^x - K, 0\}, \\ \mathcal{C}(0,t) = 0. \end{cases} \end{cases} \tag{1.5}$$

Let $t = T - \frac{2\tau}{\sigma^2} \iff \frac{\partial t}{\partial \tau} = -\frac{2}{\sigma^2}$, substituting it into equation 1.5, we get the following new boundary condition in terms of τ.

$$\begin{cases} -\frac{1}{2}\sigma^2\frac{\partial \mathcal{C}}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial \mathcal{C}}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \mathcal{C}}{\partial x^2} - r\mathcal{C} = 0, \\ \Phi(x,T) = \begin{cases} \mathcal{C}(x,0) = \max\{He^x - K, 0\}, \\ \mathcal{C}(0,\tau) = 0. \end{cases} \end{cases} \tag{1.6}$$

Let $\mathcal{C} = He^{ax+b\tau}u(x,\tau)$, then

$$\frac{\partial \mathcal{C}}{\partial \tau} = bHe^{ax+b\tau}u(x,t) + He^{ax+b\tau}\frac{\partial u}{\partial t} = \left(bu + \frac{\partial u}{\partial t}\right)He^{ax+b\tau},$$

$$\frac{\partial \mathcal{C}}{\partial x} = aHe^{ax+b\tau}u(x,t) + He^{ax+b\tau}\frac{\partial u}{\partial x} = \left(au + \frac{\partial u}{\partial x}\right)He^{ax+b\tau},$$

$$\frac{\partial^2 \mathcal{C}}{\partial x^2} = \left(a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right)He^{ax+b\tau} + aH\left(au + \frac{\partial u}{\partial x}\right)e^{ax+b\tau} = \left[a^2u + 2a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right]He^{ax+b\tau}.$$

Substitute these partial derivatives of $\mathcal{C}$ into equation 1.6 to get

$$\left\{-\frac{1}{2}\sigma^2\left(bu + \frac{\partial u}{\partial t}\right) + \left(r - \frac{1}{2}\sigma^2\right)\left(au + \frac{\partial u}{\partial x}\right) + \frac{1}{2}\sigma^2\left[a^2u + 2a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right] - ru\right\}He^{ax+b\tau} = 0.$$

Then clearly

$$-\frac{1}{2}\sigma^2\left(bu + \frac{\partial u}{\partial t}\right) + \left(r - \frac{1}{2}\sigma^2\right)\left(au + \frac{\partial u}{\partial x}\right) + \frac{1}{2}\sigma^2\left[a^2u + 2a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right] - ru = 0. \tag{1.7}$$

Our aim is to find the values of a and b. Rearrange equation 1.7 and set the coefficient of u and $\frac{\partial u}{\partial x}$ to zero.

$$-\frac{1}{2}\sigma^2\frac{\partial u}{\partial t}+\underbrace{\left[-\frac{1}{2}\sigma^2 b+ra-\frac{1}{2}\sigma^2 a+\frac{1}{2}\sigma^2 a^2-r\right]}_{=0}u+\underbrace{\left[r-\frac{1}{2}\sigma^2+2a\frac{1}{2}\sigma^2\right]}_{=0}\frac{\partial u}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}=0,$$

then

$$-\frac{1}{2}\sigma^2 b+\left(r-\frac{1}{2}\sigma^2\right)a+\frac{1}{2}\sigma^2 a^2-r=0,$$

$$r-\frac{1}{2}\sigma^2+2a\frac{1}{2}\sigma^2=0.$$

Solving the two equations we get

$$a=-\frac{r}{\sigma^2}+\frac{1}{2}=\frac{-\left(r-\frac{\sigma^2}{2}\right)}{\sigma^2}=-\frac{\lambda}{\sigma^2}\quad\text{and}\quad b=-\frac{r^2}{\sigma^4}-\frac{1}{4}-\frac{r}{\sigma^2}.$$

Now, since $u(x,\tau)=\frac{\mathcal{C}}{H}e^{-(ax+b\tau)}$, one can verify that with the given values of a and b, we get the heat equation with its boundary value problem:

$$\begin{cases}\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}, \\ \begin{cases} u(x,0)=U(x)=\max\{e^{x-ax}-\frac{K}{H}e^{-ax},0\}, \\ u(0,\tau)=0.\end{cases}\end{cases} \tag{1.8}$$

The heat eqaution is invariant under reflection (see Wilmot et al.). This means if $u(x,\tau)$ is a solution to equation 1.8 then $u(-x,\tau)$ is also a solution. Hence u can be expressed as

$$u(x,0)=\begin{cases}U(x), & x>0, \\ U(-x), & x<0\end{cases} \tag{1.9}$$

or

$$u(x,0)=U(x)-U(-x).$$

We would like to derive the equation for the down-out barrier option for the case $H<K$.

$$\mathcal{C}_{\downarrow out}=He^{ax+b\tau}u(x,\tau)=He^{ax+b\tau}(U(x)-U(-x))$$

$$= He^{ax+b\tau}\left(\frac{\mathcal{C}(S,t,K)}{H}e^{-(ax+b\tau)} - \frac{\mathcal{C}(He^{-x},t,K)}{H}e^{ax-b\tau}\right) = \mathcal{C}(S,t,K) - e^{2xa}\mathcal{C}(He^{-x},t,K).$$

Recall that $S = He^x \iff \left(\frac{S}{H}\right)^{2a} = e^{2ax}$ and $\left(\frac{H}{S}\right) = e^{-x}$ hence

$$\mathcal{C}_{\downarrow_{out}}(S,t,H) = \mathcal{C}(S,t,K) - e^{2xa}\mathcal{C}(He^{-x},t,K) = \mathcal{C}(S,t,K) - \left(\frac{S}{H}\right)^{2a}\mathcal{C}\left(\frac{H^2}{S},t,K\right). \quad (1.10)$$

Simplifying equation 1.10 further, we get

$$\mathcal{C}_{\downarrow_{out}}(S,t,H) = \mathcal{C}(S,t,K) - \left(\frac{S}{H}\right)^{\frac{-2\lambda}{\sigma^2}}\left(\frac{H^2}{S_0}e^{-qT}N(d_1) - Ke^{-rT}N(d_2)\right).$$

Now using the relation in equation 1.5,

$$\mathcal{C}_{\downarrow_{in}}(S,t,H) = \mathcal{C}(S,t,K) - \mathcal{C}_{\downarrow_{out}}(S,t,H) = \left(\frac{S}{H}\right)^{\frac{-2\lambda}{\sigma^2}}\mathcal{C}\left(\frac{H^2}{S},t,K\right) \quad (1.11)$$

where

$$d_1 = \frac{\ln\left(\frac{H^2}{S_0K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

We can simplify equation 1.11 further to get

$$\mathcal{C}_{\downarrow_{in}}(S,t,H) = \left(\frac{S}{H}\right)^{2a}\left[\frac{H^2}{S}N(d_1) - Ke^{-rT}N(d_2)\right].$$

5.2 Continuity correction

We shall use the following theorem to correctly price barrier options in a discrete case from a continuous case. Interested reader is referred to Broadie and Glasserman [1997].

Theorem 5.2.1. *Let $\mathcal{C}_{\downarrow_m}(H)$ be the price of a discretely monitored down-in or down-out call with barrier H . Let $\mathcal{C}_{\downarrow}(H)$ be the price of the corresponding continuously monitored barrier option. Then*

$$\mathcal{C}_{\downarrow_m}(H) = \mathcal{C}_{\downarrow}\left(He^{-\beta\sigma\sqrt{\frac{T}{m}}}\right) + 0\left(\frac{1}{\sqrt{m}}\right),$$

where $\beta = \frac{\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi}} = 0.5826$, for some zeta function ζ.

Proof. Referred to Broadie and Glasserman [1997]. ◻

Appendix B

All programs were written in GNU Octave 3.6.4, and executed in MATLAB 13.04. Codes can be accessed on the author's website: https://www.researchgate.net/profile/Mesias_Alfeus/info/.

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