

Dimension Reduction in Polynomial Chaos Surrogates using Adaptive DGSM

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POLYNOMIAL CHAOS EXPANSION

$$\mathcal{M}(x) \approx \mathcal{M}^{PC}(x) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} \Psi_{\alpha}(x)$$

Projection-based

$$c_{\alpha} = \mathbb{E}[\Psi_{\alpha}(x) \mathcal{M}(x)]$$

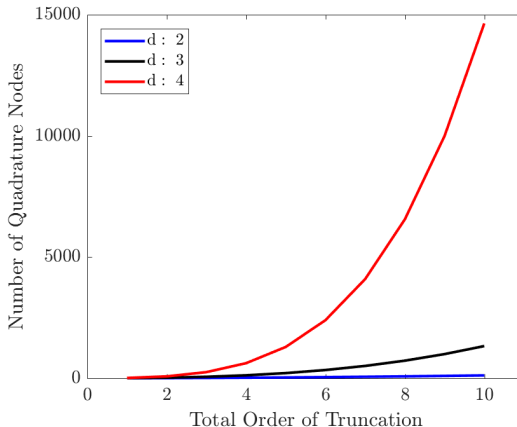
- ▶ Gaussian Quadrature
- ▶ Smolyak Sparse Quadrature
- ▶ Nested Quadrature

Regression-based

$$\hat{c} = \operatorname{argmin} \mathbb{E} \left[\left(c^T \Psi(x) - \mathcal{M}(x) \right)^2 \right]$$

- ▶ Ordinary Least-Squares
- ▶ Least Angle Regression
- ▶ Orthogonal Matching Pursuit

MOTIVATION: DIMENSION REDUCTION



MOTIVATION: DGSM

- Sensitivity analysis based on Sobol indices is commonly used to determine relative importance of the parameters.
- Sobol sensitivity indices are compute intensive:

$$\mathcal{T}(\theta_i) = \frac{\mathbb{E}_{\boldsymbol{\theta} \sim i} [\mathbb{V}_{\theta_i}(\mathcal{G} | \boldsymbol{\theta}_{\sim i})]}{\mathbb{V}(\mathcal{G})}$$

- Bounds on Sobol indices can be computed easily using DGSM and are shown to converge at a much faster rate.

DGSM

- DGSM for Randomly distributed parameters [Sobol and Kucherenko, 2009]:

$$\mu_i = \mathbb{E} \left[\left(\frac{\partial G(\mathbf{x})}{\partial x_i} \right)^2 \right]$$

where,

$$\frac{\partial G(\mathbf{x}^*)}{\partial x_i} = \lim_{\delta \rightarrow 0} \frac{[G(x_1^*, \dots, x_{i-1}^*, x_i^* + \delta, x_{i+1}^*, \dots, x_d^*) - G(\mathbf{x}^*)]}{\delta}$$

- Total number of model realizations required to compute μ_i using N samples is $N(d + 1)$.

BOUNDS ON SOBOL INDICES

- Upper bound (UB_i) on Sobol Total Effect index (ST_i) [Sobol and Kucherenko, 2009]:

$$ST_i \leq \frac{C_i \mu_i}{V} (\propto \hat{C}_i \mu_i)$$

$$\hat{C}_i \mu_i = \frac{C_i \mu_i}{\sum_i C_i \mu_i}$$

C : Poincaré Constant V : Variance

- The Poincaré Constant is specific to a given probability distribution:

$\mathcal{U}[a, b]$	$(b - a)^2 / \pi^2$
$\mathcal{N}(\mu, \sigma^2)$	σ^2

ALGORITHM: PARAMETER SCREENING

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1 Generate  $n_1$  points in  $\mathbb{R}^d$ ;
2 Compute  $UB_i$  for parameters  $\theta_i$  using  $n_1$  points;
  %  $NF = n_1(d+1)$ ,  $NF$ : Number of model realizations;
3 Rank Parameters ( $\theta_i$ ) based on  $UB_i$  estimates ( $\mathcal{R}^{old}$ );
4 set  $k = 1$  %  $k$ : Iteration counter;
5 repeat
6   Generate  $\beta n_1$  new points in  $\mathbb{R}^d$  ( $\beta n_1 \in \mathbb{Z}$ );
7   Compute  $UB_i^{new}$  using  $(1+\beta k)n_1$  points;
  %  $NF = (1+\beta k)n_1(d+1)$ ;
8   Rank Parameters based on  $UB_i^{new}$  estimates ( $\mathcal{R}^{new}$ );
9   if ( $\mathcal{R}^{new} = \mathcal{R}^{old}$ ) then
10    Compute:  $r_i = \frac{UB_i^{new}}{\max(UB_i^{new})}$ ;
11    Construct a set  $s = \{\theta_i \ni r_i < \alpha\}$ ;
12    Exit the loop;
13  end
14  set  $k = k + 1$ ;
15 until  $\mathcal{R}^{new} \neq \mathcal{R}^{old}$ ;
16 Construct a validation set:  $(\theta_j, \mathcal{M}(\theta_j))$ ,  $j=1, 2, \dots, NF$ ;

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REDUCED MORRIS FUNCTION

$$f(x) = \sum_{i=1}^4 b_i x_i + \sum_{i \leq j}^4 b_{ij} x_i x_j + \sum_{i \leq j \leq k=4}^4 b_{ijk} x_i x_j x_k$$

$$x_i \sim \mathcal{U}[0, 1]$$

$$b_i = \begin{bmatrix} 0.05 \\ 0.59 \\ 10 \\ 0.21 \end{bmatrix}, \quad b_{ij} = \begin{bmatrix} 0 & 80 & 60 & 40 \\ 0 & 30 & 0.73 & 0.18 \\ 0 & 0 & 0.64 & 0.93 \\ 0 & 0 & 0 & 0.06 \end{bmatrix}, \quad b_{ijk} = \begin{bmatrix} 0 & 10 & 0.98 & 0.19 \\ 0 & 0 & 0.49 & 50 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

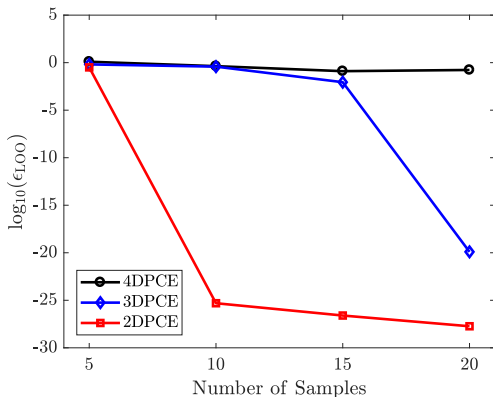
PARAMETER SCREENING

UB_i : 5 Samples	UB_i : 10 Samples	ST_i : 10^5 Samples
x_1 : 1.4539	x_1 : 0.6926	x_1 : 0.4323
x_2 : 1.3303	x_2 : 0.6804	x_2 : 0.4051
x_4 : 0.3761	x_4 : 0.2531	x_4 : 0.1300
x_3 : 0.3374	x_3 : 0.1443	x_3 : 0.0928

PCE CONVERGENCE

$$\epsilon_{LOO} = \frac{\sum_{i=1}^N (\mathcal{M}(\mathbf{x}^{(i)}) - \mathcal{M}^{PCE \setminus i}(\mathbf{x}^{(i)}))^2}{\sum_{i=1}^N (\mathcal{M}(\mathbf{x}^{(i)}) - \hat{\mu}_Y)^2}$$

$$\hat{\mu}_Y = \frac{1}{N} \sum_{i=1}^N \mathcal{M}(\mathbf{x}^{(i)})$$



- The PCE is constructed using Least Angle Regression with Latin Hypercube Sampling.

PCE VERIFICATION

- Generate an independent set of validation points in the 4D input parameter space.
- Compute the following error:

$$\epsilon_{val} = \frac{\left[\sum_{i=1}^N (\mathcal{M}(\mathbf{x}^{(i)}) - \mathcal{M}^{PCE}(\mathbf{x}^{(i)}))^2 \right]^{\frac{1}{2}}}{\left[\sum_{i=1}^N (\mathcal{M}(\mathbf{x}^{(i)}))^2 \right]^{\frac{1}{2}}}$$

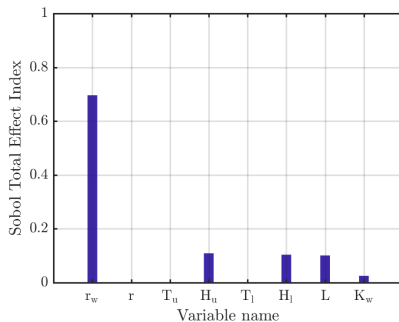
- ϵ_{val} was found to be $\mathcal{O}(10^{-1})$ in the case of 4D PCE and 2D PCE constructed using 15 and 5 training points respectively.

THE BOREHOLE FUNCTION

$$Q = \frac{2\pi T_u (H_u - H_l)}{\ln\left(\frac{r}{r_w}\right) \left(1 + \frac{2LT_u}{\ln\left(\frac{r}{r_w}\right) r_w^2 K_w} + \frac{T_u}{T_l}\right)}$$

Q : Discharge of water through a borehole

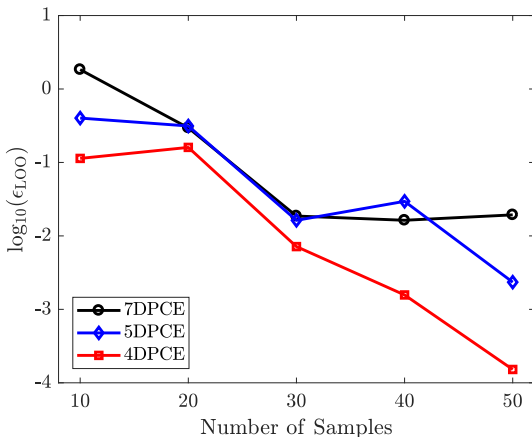
r_w	$\mathcal{N}(0.1, 0.016)$	Radius of the borehole (m)
r	$\log \mathcal{N}(7.71, 1.006)$	Radius of influence (m)
T_u	$\mathcal{U}[63070, 115600]$	Transmissivity of upper aquifer (m ² /yr)
H_u	$\mathcal{U}[990, 1110]$	Potentiometric head of upper aquifer (m)
T_l	$\mathcal{U}[63.1, 116]$	Transmissivity of lower aquifer (m ² /yr)
H_l	$\mathcal{U}[700, 820]$	Potentiometric head of lower aquifer (m)
L	$\mathcal{U}[1120, 1680]$	Length of borehole (m)
K_w	$\mathcal{U}[9855, 12045]$	Hydraulic conductivity of borehole (m/yr)



PARAMETER SCREENING

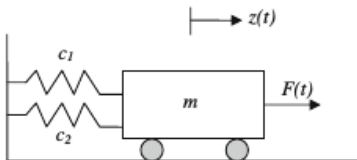
UB_i : 5 Samples	UB_i : 10 Samples	UB_i : 15 Samples	ST_i : 10^5 Samples
x_1 : 0.2124	x_1 : 0.8671	x_1 : 0.8384	x_1 : 0.6942
x_6 : 0.0406	x_3 : 0.1505	x_3 : 0.1472	x_3 : 0.1062
x_3 : 0.0393	x_5 : 0.1505	x_5 : 0.1472	x_5 : 0.1059
x_5 : 0.0393	x_6 : 0.1466	x_6 : 0.1437	x_6 : 0.1026
x_7 : 0.0089	x_7 : 0.0337	x_7 : 0.0360	x_7 : 0.0250
x_2 : 0.0000	x_2 : 0.0000	x_2 : 0.0000	x_2 : 0.0000
x_4 : 0.0000	x_4 : 0.0000	x_4 : 0.0000	x_4 : 0.0000

PCE CONVERGENCE AND VERIFICATION



- ϵ_{val} was found to be $\mathcal{O}(10^{-2})$ in the case of 7D PCE and 4D PCE constructed using 60 and 30 training points respectively.

NON-LINEAR OSCILLATOR

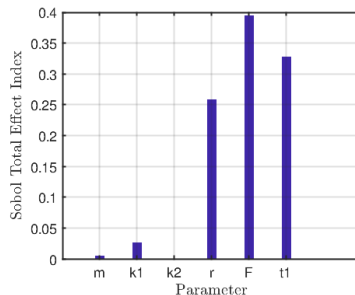


m	$\mathcal{N}(1,0.05)$	Mass
k_1	$\mathcal{N}(1,0.1)$	Spring Constant
k_2	$\mathcal{N}(0.1,0.01)$	Spring Constant
r	$\mathcal{N}(0.5,0.05)$	Displacement
F	$\mathcal{N}(1,0.2)$	Force
t_1	$\mathcal{N}(1,0.2)$	Time

Limit State Function:

$$g(\mathbf{X}) = 3r - \left| \frac{2F}{m\omega_0^2} \sin\left(\frac{\omega_0 t_1}{2}\right) \right|$$

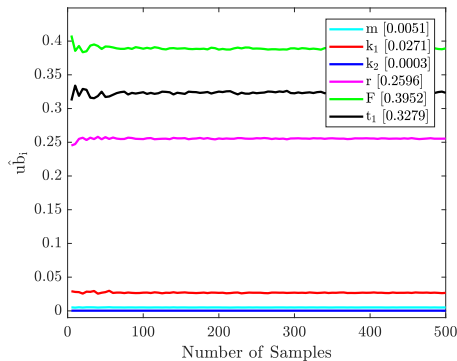
$$\mathbf{X} = \{m, k_1, k_2, r, F, t_1\} \quad \omega_0 = \sqrt{\frac{c_1 + c_2}{m}}$$



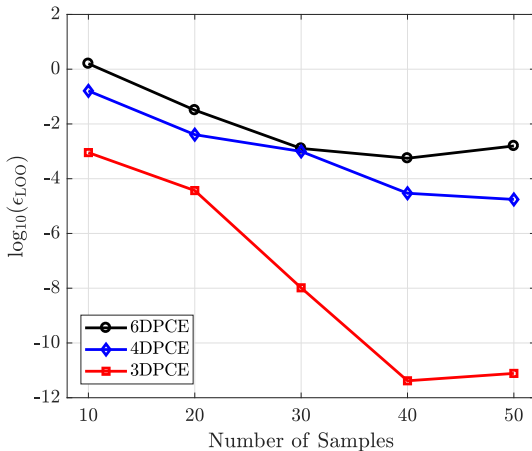
PARAMETER RANK METRIC

$$\hat{C}_i \mu_i = \frac{C_i \mu_i}{\sum_i C_i \mu_i}$$

C : Poincaré Constant μ_i : DGSM



PCE CONVERGENCE

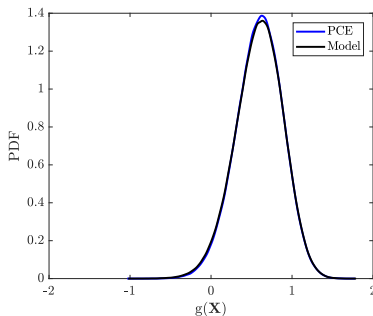


PCE VERIFICATION

- Relative L-2 norm of the error is computed:

$$\epsilon_{val} = \frac{\left[\sum_{i=1}^N \left(\mathcal{M}(x^{(i)}) - \mathcal{M}^{PCE}(x^{(i)}) \right)^2 \right]^{\frac{1}{2}}}{\left[\sum_{i=1}^N \left(\mathcal{M}(x^{(i)}) \right)^2 \right]^{\frac{1}{2}}} \approx 8.35 \times 10^{-2}$$

- Comparison of PDFs:

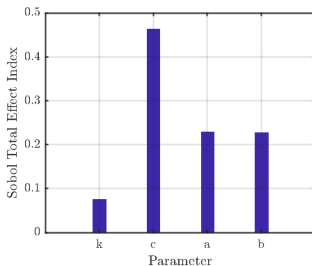
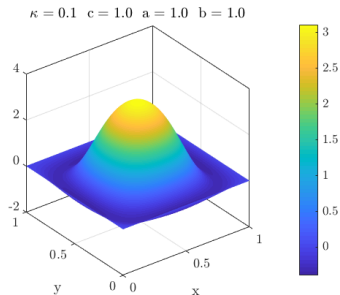


ELLIPTIC PDE

$$-\kappa \Delta u + cu^3 = q$$

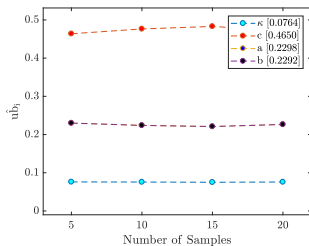
$$q = (-2\pi^2)(a \cos(2\pi x) \sin^2(\pi y) + b \cos(2\pi y) \sin^2(\pi x))$$

κ	$\mathcal{U}[0.09, 0.11]$
c	$\mathcal{U}[0.9, 1.1]$
a	$\mathcal{U}[0.9, 1.1]$
b	$\mathmathcal{U}[0.9, 1.1]$

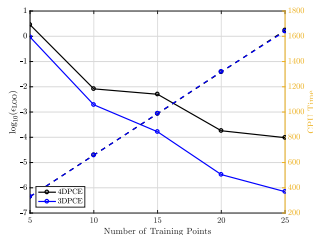


ANALYSIS

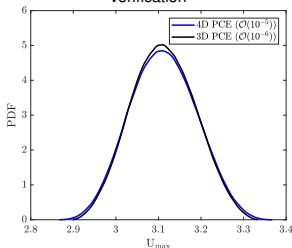
Parameter Screening



Convergence



Verification



H₂/O₂ REACTION MECHANISM

■ Reactions rates:

$$k = AT^n \exp(-E_a/RT)$$

k : Reaction rate, A : Pre-Exponent

E_a : Activation Energy

■ Uniform prior marginals:

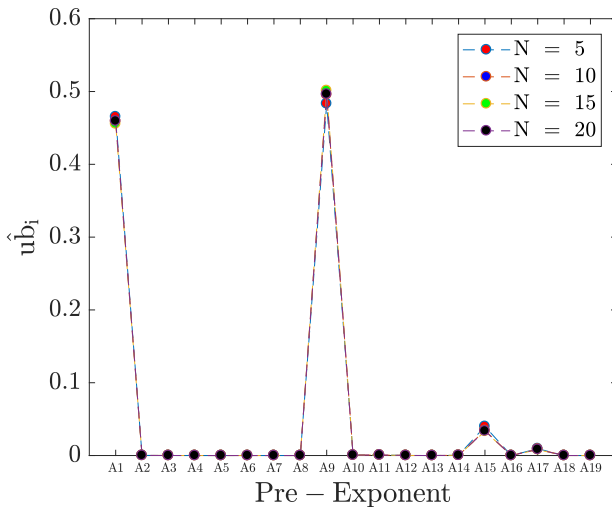
$$A_i \sim \mathcal{U}[0.9A_{i,\text{nom}}, 1.1A_{i,\text{nom}}]$$

$A_{i,\text{nom}}$: Nominal value of the i^{th} Pre-Exponent

■ QoI: Ignition Delay

1. $\text{H} + \text{O}_2 \rightleftharpoons \text{O} + \text{OH}$
2. $\text{O} + \text{H}_2 \rightleftharpoons \text{H} + \text{OH}$
3. $\text{H}_2 + \text{OH} \rightleftharpoons \text{H}_2\text{O} + \text{H}$
4. $\text{OH} + \text{OH} \rightleftharpoons \text{O} + \text{H}_2\text{O}$
5. $\text{H}_2 + \text{M} \rightleftharpoons \text{H} + \text{H} + \text{M}$
6. $\text{O} + \text{O} + \text{M} \rightleftharpoons \text{O}_2 + \text{M}$
7. $\text{O} + \text{H} + \text{M} \rightleftharpoons \text{OH} + \text{M}$
8. $\text{H} + \text{OH} + \text{M} \rightleftharpoons \text{H}_2\text{O} + \text{M}$
9. $\text{H} + \text{O}_2 + \text{M} \rightleftharpoons \text{HO}_2 + \text{M}$
10. $\text{HO}_2 + \text{H} \rightleftharpoons \text{H}_2 + \text{O}_2$
11. $\text{HO}_2 + \text{H} \rightleftharpoons \text{OH} + \text{OH}$
12. $\text{HO}_2 + \text{O} \rightleftharpoons \text{O}_2 + \text{OH}$
13. $\text{HO}_2 + \text{OH} \rightleftharpoons \text{H}_2\text{O} + \text{O}_2$
14. $\text{HO}_2 + \text{HO}_2 \rightleftharpoons \text{H}_2\text{O}_2 + \text{O}_2$
15. $\text{H}_2\text{O}_2 + \text{M} \rightleftharpoons \text{OH} + \text{OH} + \text{M}$
16. $\text{H}_2\text{O}_2 + \text{H} \rightleftharpoons \text{H}_2\text{O} + \text{OH}$
17. $\text{H}_2\text{O}_2 + \text{H} \rightleftharpoons \text{HO}_2 + \text{H}_2$
18. $\text{H}_2\text{O}_2 + \text{O} \rightleftharpoons \text{OH} + \text{HO}_2$
19. $\text{H}_2\text{O}_2 + \text{OH} \rightleftharpoons \text{HO}_2 + \text{H}_2\text{O}$

DERIVATIVE-BASED SENSITIVITY MEASURES



PCE VERIFICATION

- Relative L-2 norm of the error is computed:

$$\epsilon_{val} = \frac{\left[\sum_{i=1}^N \left(\mathcal{M}(\mathbf{x}_{19D}^{(i)}) - \mathcal{M}^{PCE}(\mathbf{x}_{2D}^{(i)}) \right)^2 \right]^{\frac{1}{2}}}{\left[\sum_{i=1}^N \left(\mathcal{M}(\mathbf{x}_{19D}^{(i)}) \right)^2 \right]^{\frac{1}{2}}} \approx 3.35 \times 10^{-2}$$