IEM 5013 Intro to Optimization
Homework 2 Solution

HW Policy: This assignment can be done individually or with a partner i.e., maximum team size is two. A student can be part of exactly one submission. In order to include your name in the team's submission you must have made a substantial contribution by actively and continuously participating in the development of your team's assignment submission. Any violation will be treated as academic dishonesty.

Problems:

1. (25 points) Consider the problem of locating a new machine to an existing layout consisting of four machines. These four machines (approximated as points) are located at the following (x_1, x_2) coordinates: $\binom{3}{0}$, $\binom{0}{-3}$, $\binom{-2}{1}$ and $\binom{1}{4}$. Let the coordinates of the new machine be $\binom{x_1}{x_2}$. Formulate the problem of finding an optimal location as a linear optimization model for each of the following cases.

   (a) The sum of the distances from the new machine to the four existing machines is minimized. Use street distance; for example, the distance from $\binom{x_1}{x_2}$ to the first machine located at $\binom{3}{0}$ is $|x_1 - 3| + |x_2 - 0|$ (note that absolute value function is not linear!).

   (b) Because of various amounts of flow between the new machine and the existing machines, reformulate the problems where the sum of the weighted street distances is minimized, where weights corresponding to the four machines are 5,7,3 and 1 respectively.

   (c) In order to avoid congestion, suppose that the new machine must be located in the square $\{(x_1, x_2) : -1 \leq x_1 \leq 2, 0 \leq x_2 \leq 1\}$. Formulate (a) with this additional restriction.

   (d) Suppose that the new machine must be located so that its distance from the first machine does not exceed 3/2. Formulate (a) with this additional restriction.

Solution. Grading key: (a) Correctly linearized objective (5 points); correctly linearized constraints (5 points); non-negativity not essential. (b) Weighted objective (5 points). (c) Additional location constraints (5 points). (d) Additional distance constraint (5 points).

Parameters: Let the coordinates of the four machines be denoted by $\binom{a_i}{b_i}$ for $i = 1, 2, 3, 4$.

Decision variables: (x_1, x_2) denotes the coordinates of the new machine.

Unconstrained Nonlinear Formulation.

$$\min \sum_{i=1}^{4} (|x_1 - a_i| + |x_2 - b_i|) \quad \text{s.t.} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

Linearization. Note that $|g(x)| = \max\{g(x), -g(x)\}$. Introduce variables u_i, v_i for each term $i = 1, 2, 3, 4$ in the summation.

$$\min_{x \in \mathbb{R}^2} \sum_{i=1}^{4} (|x_1 - a_i| + |x_2 - b_i|),$$

is equivalent to,

$$
\begin{aligned}
& \min \sum_{i=1}^{4} (u_i + v_i) \\
s.t. \quad & \left.\begin{array}{rcl}
u_i & \geq & x_1 - a_i \\
u_i & \geq & -x_1 + a_i \\
v_i & \geq & x_2 - b_i \\
v_i & \geq & -x_2 + b_i \\
u_i, v_i & \geq & 0
\end{array}\right\} \quad \forall i = 1, \ldots, 4 \\
& x_1, x_2 \in \mathbb{R}
\end{aligned}
$$

(b) Replace the previous objective with,

$$\min \sum_{i=1}^{4} \alpha_i (u_i + v_i),$$

where $\alpha_i, i = 1, 2, 3, 4$ denote the weights provided. The transformation is valid as the weights are all nonnegative.

(c) Add the additional constraints, $-1 \leq x_1 \leq 2, 0 \leq x_2 \leq 1$.

(d) Add the additional constraint $u_1 + v_1 \leq 3/2$.

Note that you still have unrestricted (in sign) variables x_1 and x_2 in the formulation. (If you wanted to, how can you replace free variables with nonnegative variables?)

2. Completing a project X requires tasks A, B, C, D, and E to be completed, and for each of these tasks, the normal duration to complete them are known. Some of these tasks have predecessors that must be completed before the task can be started. Formulate a linear optimization model to find out the earliest completion date of project X if all tasks are completed according to their regular duration.

Task	Duration (weeks)	Predecessors
A	2	-
B	13	A
C	7	A
D	8	A,B
E	10	B,C,D

Solution. Grading key: Start time DV definition (5 points); correctly linearized objective function (5 points); precedence constraints (7 * 2 points = 14 points); non-negativity constraints (1 point).

Decision variables: Let x_i denote the start time of task $i \in T := \{A, B, C, D, E\}$.

The project is completed, when the last of the tasks is completed.

Let z denote the completion time of the last task to finish.

$$
\begin{aligned}
\min\ & z \\
s.t.\quad & z \geq x_i + t_i & \forall i \in T \\
& x_B \geq x_A + t_A \\
& x_C \geq x_A + t_A \\
& x_D \geq x_A + t_A \\
& x_D \geq x_B + t_B \\
& x_E \geq x_B + t_B \\
& x_E \geq x_C + t_C \\
& x_E \geq x_D + t_D \\
& x_i \geq 0 & \forall i \in T
\end{aligned}
$$

Note that I am not using the observation that is specific to this instance that E is the last job to finish when writing the objective function. I want my model to work regardless of the instance. More compactly, we can write precedence constraints as: $x_j \geq x_i + t_i$ for every task precedence pair (i, j).

You can alternately represent this information as network $(\mathcal{N}, \mathcal{A})$ with node set $\mathcal{N}$ representing the tasks, arc set $\mathcal{A}$ in which each arc $(i, j) \in \mathcal{A}$ represents that task i is a predecessor of task j. The precedence constraints would then be written for every arc in this network. The formulation can then be written compactly as follows.

$$
\begin{aligned}
\min\ & z \\
s.t.\quad & z \geq x_i + t_i & \forall i \in \mathcal{N} \\
& x_j \geq x_i + t_i & \forall (i, j) \in \mathcal{A} \\
& x_i \geq 0 & \forall i \in \mathcal{N}
\end{aligned}
$$