IEM 5013 Intro to Optimization
Homework 1 Solutions

HW Policy: This assignment can be done individually or with a partner i.e., maximum team size is two. A student can be part of exactly one submission. In order to include your name in the team's submission you must have made a substantial contribution by actively and continuously participating in the development of your team's assignment submission. Any violation will be treated as academic dishonesty.

Problems:

1. (25 points) A company sells many household products through an online catalog. The company needs substantial warehouse space for storing its goods. Plans are now being made for leasing warehouse storage space over the next 5 months. Just how much space will be required in each of these months is listed in the table below.

Month	Space required in sq-ft
1	30,000
2	20,000
3	40,000
4	10,000
5	50,000

Leasing period in months	Cost per sq-ft leased
1	\$65
2	\$100
3	\$135
4	\$160
5	\$190

Assume that several leases of different durations, over different time windows, and different amounts warehouse space can be held concurrently, over the planning horizon. Formulate a linear optimization model to help the company identify an optimal combination of leases that meets its needs.

Hint. It is useful to understand the cost trade-off in this problem. On the one hand, as the space requirements are quite different month-to-month, one could lease only the amount needed each month on a month-to-month basis, i.e., using only one month leases. On the other hand, the monthly cost of leasing space for multiple consecutive months at once, is less compared to an equivalent collection of multiple one month leases. The other extreme is to lease the maximum amount needed for the entire 5 months in a single lease. Of course, these are two extreme approaches to meeting the demand, and we can envision many combinations of leases of different durations that are held concurrently to meet demand when aggregated.

Solution. Grading key: Decision variable definition—precise meaning and indices (10 points); correct objective function (6 points); requirement coverage constraints (8 points); Non-negativity constraints (1 point).

Parameters: Let the space requirement in month i be denoted by d_i, for $i = 1, 2, \ldots, 5$; and let the cost of a lease that is j months in duration be c_j, for $j = 1, 2, \ldots, 5$.

Decision Variables: Let x_{ij} denote the amount of space leased (in sq. ft) in a lease starting in month i and ending in month j (both included) for each $i = 1, 2, \ldots, 5$ and $j = i, i+1, \ldots, 5$.

Note how we control the indices of the decision variable from taking improper values (e.g. $x_{3,1}$) as the lease can only terminate the same month it began or after. Furthermore, for such i, j pairs, the duration of the lease is given by $j - i + 1$. We can use the duration to determine the cost of that lease— a lease x_{ij} will incur a unit cost of c_{j-i+1}.

$$
\begin{aligned}
\min 65(x_{1,1}+x_{2,2}+x_{3,3}+x_{4,4}+x_{5,5})& \\
+100(x_{1,2}+x_{2,3}+x_{3,4}+x_{4,5})& \\
+135(x_{1,3}+x_{2,4}+x_{3,5})+160(x_{1,4}+x_{2,5})+190x_{1,5}& \\
\text{subject to.}\qquad x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4}+x_{1,5}&\geq d_1 \\
x_{1,2}+x_{1,3}+x_{1,4}+x_{1,5}+x_{2,2}+x_{2,3}+x_{2,4}+x_{2,5}&\geq d_2 \\
x_{1,3}+x_{1,4}+x_{1,5}+x_{2,3}+x_{2,4}+x_{2,5}+x_{3,3}+x_{3,4}+x_{3,5}&\geq d_3 \\
x_{1,4}+x_{1,5}+x_{2,4}+x_{2,5}+x_{3,4}+x_{3,5}+x_{4,4}+x_{4,5}&\geq d_4 \\
x_{1,5}+x_{2,5}+x_{3,5}+x_{4,5}+x_{5,5}&\geq d_5 \\
x_{ij}&\geq 0 \qquad \forall i=1,2,\ldots,5 \text{ and } j=i,i+1,\ldots,5
\end{aligned}
$$

The trick in the constraints is in recognizing that in any given month k, the only active leases are those that started in or before month k (that is in months $1,2,\ldots,k$) and ended in or after month k (i.e., in months $k,k+1,\ldots,5$). Any lease that ends before k or starts after k, cannot help satisfy the space requirement in month k.

With that in mind, we can now write this formulation more compactly (ready for coding) for a general n-month planning horizon as follows.

$$
\begin{aligned}
&\min \sum_{i=1}^{n}\sum_{j=i}^{n} c_{j-i+1}x_{ij} \\
&\text{subject to.}\quad \sum_{i=1}^{k}\sum_{j=k}^{n} x_{ij}\geq d_k && \forall k=1,2,\ldots,n \\
&\qquad x_{ij}\geq 0 && \forall i=1,2,\ldots,n \text{ and } j=i,i+1,\ldots,n
\end{aligned}
$$

Can you see how the space requirement constraint for an arbitrary month k in the planning horizon captures all the active leases during month k in the LHS double summation?

Alternately, you can define the decision variables as stated below and still develop a valid formulation for this problem. Try this out!

Alternate Decision Variables: Let x_{ij} denote the amount of space leased (in sq. ft) in a lease starting in month i, for a period of j months; for each $i=1,2,\ldots,5$ and $j=1,\ldots,6-i$; more generally, for n-period planning models, we will have $i=1,2,\ldots,n$ and $j=1,\ldots,n+1-i$. Note that the longest duration permissible (largest value of j), depends on when the lease starts (month i).

$$
\begin{aligned}
&\min \sum_{i=1}^{n}\sum_{j=1}^{n+1-i} c_j x_{ij} \\
&\text{subject to.}\quad \sum_{i=1}^{k}\sum_{j=k+1-i}^{n+1-i} x_{ij}\geq d_k && \forall k=1,2,\ldots,n \\
&\qquad x_{ij}\geq 0 && \forall i=1,2,\ldots,n \text{ and } j=1,\ldots,n+1-i
\end{aligned}
$$

The objective function coefficients are different because j in x_{ij} directly gives us the duration of the lease, arguably, easier to model than in the previous formulation. The coverage constraints are considerably different—to cover any arbitrary month k, the lease must start in one of these months $i \in \{1, 2, \dots, k\}$, but what is the shortest duration j that is long enough to cover month k? We need: $i + (j - 1) \geq k$. In other words, k "must be reached" by a j-month long lease starting in month i (which includes month i, hence, the $j - 1$ term). That inequality among these subscripts tells us that $j \geq k + 1 - i$, the minimum duration of a lease that covers month k and starts in month i. We then use this observation in the inner summation in the coverage constraint.

2. (25 points) A round-the-clock manufacturing company has minimal daily requirements for workers in each of its 4-hour periods as listed in Table 1. Workers may be employed to work in a shift consisting of either two consecutive periods, or three consecutive periods. Period 1 follows immediately after period 6. Formulate a linear optimization model to find a schedule that minimizes total labor cost, assuming worker salary is proportional to the number of periods worked in a shift.

Table 1: Requirements for the worker scheduling problem

Time of day	Period	Minimum # required
2:00–6:00	1	20
6:00–10:00	2	50
10:00–14:00	3	80
14:00–18:00	4	100
18:00–22:00	5	40
22:00–2:00	6	30

Solution. Grading key: Decision variable definition—precise meaning and indices (10 points); correct objective function (6 points); requirement coverage constraints (8 points); Non-negativity constraints (1 point).

Parameters: Let d_i denote the minimum required number of workers in slot $i = 1, 2, \dots, 6$

Decision Variables:
Let x_i denote the number of workers starting a 8-hr shift in slot $i = 1, \dots, 6$
Let y_i denote the number of workers starting a 12-hr shift in slot $i = 1, \dots, 6$

$$
\begin{aligned}
& \min \sum_{i=1}^{6} (2x_i + 3y_i) \\
\textit{subject to:} \quad & x_6 + x_1 + y_5 + y_6 + y_1 \geq d_1 \\
& x_1 + x_2 + y_1 + y_2 + y_6 \geq d_2 \\
& x_2 + x_3 + y_1 + y_2 + y_3 \geq d_3 \\
& x_3 + x_4 + y_2 + y_3 + y_4 \geq d_4 \\
& x_4 + x_5 + y_3 + y_4 + y_5 \geq d_5 \\
& x_5 + x_6 + y_4 + y_5 + y_6 \geq d_6 \\
& x_i, y_i \geq 0 \ \forall i = 1, \ldots, 6
\end{aligned}
$$

Note that the objective coefficients of x_i and y_i are different because they represent shifts of different durations. Although we don't know what the exact labor costs are, we can assume that it is proportional to the duration of the shift, and we have captured that difference in the objective function.