IEM 5013: Introduction to Optimization

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August 31, 2023

Chapters

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What is Operations Research (O.R.)?

Operations Research, or Operational Research in British usage, is the discipline that deals with the application of *advanced analytical methods* to help make better decisions.

- Optimization Modeling and Algorithms
- Stochastic Modeling and Analysis
- Simulation Modeling and Analysis

Video link: ORigin stORy

Video link: What is O.R.?

INFORMS link: Impact of O.R.

O.R. in Action

- **Airlines**: Crew scheduling, flight planning, demand forecasting, pricing and revenue management
- **Healthcare**: Radiation therapy, organ transplant, wait-times, layout
- **Insurance**: Fraud detection, credit classification, risk analysis
- **Manufacturing**: Master/plant scheduling, production planning, inventory management, demand forecasting
- **Railways**: Meet-pass planning, train scheduling, locomotive planning, crew scheduling
- **SCM and Logistics**: Supply chain/production-distribution network design, facility location, routing

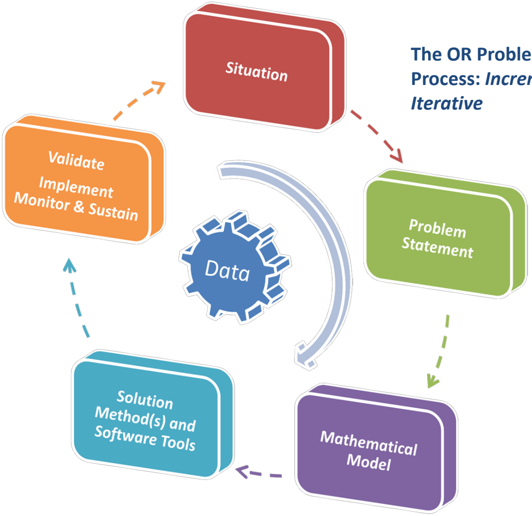
`https://www.informs.org/Explore/Operations-Research-Analytics`

Optimization in Operations Research & Analytics

The field of O.R. has a lot in common with analytics, data science, and machine learning—fields that have emerged more recently.

- Descriptive Analytics
  - What has happened?
  - Visualizing historical data, data mining, basic measures of statistics
- Predictive Analytics
  - What is going to happen?
  - Statistical learning, Bayesian models, simulation, regression and forecasting techniques
- Prescriptive Analytics
  - What should we do?
  - Optimization, simulation, stochastic modeling

Problem-solving with O.R.



The OR Problem Solving Process: *Incremental and Iterative*

Mathematical Optimization

Solve the following problem:

$$\min\{f(x) : x \in \mathscr{X}\},$$

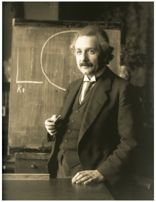
where $\mathscr{X} \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Noteworthy items:

- Sense of optimization
- Objective function, constraints, feasible region
- Linearity (assumes proportionality and additivity) vs nonlinearity
- Continuous vs discrete variables
- Single vs multiple objectives
- Certainty vs uncertainty

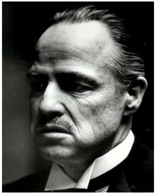
Mathematical Modeling

"As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."



"All models are wrong, but some are useful."

"I'll make him an offer he can't refuse."



Weapon Mix Problem

A military commander has been assigned the task of defending an asset from enemy air attack. He has two types of air defense missiles; 5 missiles each of type I and type II are available for deployment.

Each type I missile costs 7 units for installation and each type II missile costs 8.5 units for installation. The total budget available is 60 units.

Each type I missile requires 6 persons for handling whereas each type II requires 2 persons. There are only 32 trained persons to handle the missiles at the site.

If the site is not defended the enemy aircrafts are estimated to destroy 95% of the asset value. If one type I missile is deployed on the site, it is expected to save 13% of the asset value. Similarly, deployment of one type II missile is estimated to save 9% of the asset value.

Weapon Mix Problem –contd.

In other words, the enemy aircrafts which were earlier capable of destroying 95% of the asset value are able to destroy 82% of the asset value in the presence of one missile of type I and 86% of the asset value in the presence of one missile of type II.

Determine the mix of missiles that provides maximum protection to the asset against an attack of enemy aircrafts aiming simultaneously.

Weapon Mix Problem

Decision variables.

x_i for $i = 1, 2$ denotes the number of type i missiles deployed

$$\max 13x_1 + 9x_2$$

subject to:

$$\begin{aligned} x_1 &\leq 5 \\ x_2 &\leq 5 \end{aligned}$$

$$\begin{aligned} 7x_1 + 8.5x_2 &\leq 60 \\ 6x_1 + 2x_2 &\leq 32 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Feed Mix Problem

An animal feed mix manufacturer would like to determine the optimal combination of the three basic ingredients in their final product. These three basic ingredients, their nutrient content (as a percentage value), and the unit costs are shown in the table below.

Ingredient	Calcium	Protein	Fiber	Unit cost (cents/kg)
Limestone	38%	0%	0%	10.0
Corn	0.1%	9%	2%	30.5
Soybean meal	0.2%	50%	8%	90.0

The mixture must meet the following restrictions.

- Calcium: at least 0.8% but not more than 1.2%
- Protein: at least 22%
- Fiber: at most 5%

Formulate a linear optimization model to find the composition of the feed mix that satisfies these requirements and minimizes cost.

Round-the-clock Staffing Problem

A round-the-clock manufacturing company has minimal daily requirements for workers in each of its 4-hour slot as listed in the table. A work shift consists of two contiguous slots, and period 1 follows immediately after period 6. Formulate a linear optimization model to find an optimal staffing plan.

Time of day	Period	Minimum # required
2:00–6:00	1	20
6:00–10:00	2	50
10:00–14:00	3	80
14:00–18:00	4	100
18:00–22:00	5	40
22:00–2:00	6	30

Round-the-clock Staffing Problem

Parameters:

d_i denotes the daily requirement of workers in slot $i = 1, \ldots, 6$

Decision variables:

x_i denotes the number of workers starting in slot $i = 1, \ldots, 6$

$$\min \sum_{i=1}^{6} x_i$$

$$\begin{array}{rrcl} \textit{subject to:} & x_6 + x_1 & \geq & d_1 \\ & x_1 + x_2 & \geq & d_2 \\ & x_2 + x_3 & \geq & d_3 \\ & x_3 + x_4 & \geq & d_4 \\ & x_4 + x_5 & \geq & d_5 \\ & x_5 + x_6 & \geq & d_6 \\ & x_i & \geq & 0 \ \forall i = 1, \ldots, 6 \end{array}$$

Employee Hiring-Training Problem

An airline company has to decide how many flight-attendants to hire and train over the next four months. The requirements expressed as the number of flight-hours needed are 8000 in September, 9000 in October, 7000 in November, and 10,000 in December.

It takes one month of training before a flight-attendant can be put on a regular flight. So an individual must be hired a month before he or she is actually needed. A trainee requires 100 hours of supervision by an experienced flight-attendant during the month of training so 100 hours less are available for flight service by a regular (experienced) flight-attendant.

Employee Hiring-Training Problem

Each experienced flight-attendant can work up to 150 hours in a month, and the airline will have 60 regular flight-attendants at the beginning of September.

If the maximum time available from experienced flight-attendants exceeds a month's flying and training requirement, they work fewer than 150 hours. None are laid off.

By the end of each month, 10% of experienced flight-attendants quit their jobs for various reasons (better jobs, finding a soul-mate, etc.). The airline company pays a regular flight-attendant effectively \$25,000 per month and a trainee, \$12,000.

Formulate the hiring and training problem as a linear optimization model.

Employee Hiring-Training Problem

Decision Variables:

x_i denotes the number of trainees hired in month $i = 1, 2, 3, 4$

$$\min 12000(x_1 + x_2 + x_3 + x_4) + 25000\Big\{60 + 0.9(60 + x_1)$$
$$+0.9[0.9(60 + x_1) + x_2] + 0.9\{0.9[0.9(60 + x_1) + x_2] + x_3\}\Big\}$$

subject to:

$$\begin{aligned}
60 \times 150 - 100x_1 &\geq 8000 \\
0.9 \times (60 + x_1) \times 150 - 100x_2 &\geq 9000 \\
0.9 \times \{0.9 \times (60 + x_1) + x_2\} \times 150 - 100x_3 &\geq 7000 \\
0.9 \times [0.9 \times \{0.9 \times (60 + x_1) + x_2\} + x_3] \times 150 - 100x_4 &\geq 10000 \\
x_i &\geq 0 \ \forall i = 1, \ldots, 4
\end{aligned}$$

Question. Can x_4 take a (strictly) positive value in any optimal solution to this LP? How can we address this "discontinuity" at the end of the planning horizon?

Employee Hiring-Training Problem

Same formulation, different notation.

Parameters:

d_i denotes the number of flight-hours required in month $i = 1, 2, 3, 4$

Decision Variables:

x_i denotes the number of trainees hired in month $i = 1, 2, 3, 4$

y_i denotes the number of experienced flight-attendants in month $i = 1, 2, 3, 4$

$$\min 12000 \sum_{i=1}^{4} x_i + 25000 \sum_{i=1}^{4} y_i$$

subject to:

$$\begin{aligned} 150 \times y_i - 100 \times x_i &\geq d_i & \forall i = 1, 2, 3, 4 \\ y_1 &= 60 & \\ y_{i+1} &= 0.9 \times (y_i + x_i) & \forall i = 1, 2, 3 \\ x_i, y_i &\geq 0 & \forall i = 1, \ldots, 4 \end{aligned}$$

Workforce Planning Problem

A firm produces two types of products (A and B). The firm has 60 experienced workers and would like to increase its workforce to 90 workers during the next eight weeks. Each experienced worker can train 3 new employees in a period of two weeks during which the workers involved virtually produce nothing. Production rates are 10 units per hour and 6 units per hour for A and B respectively. A work week is 40 hours long and the weekly demands (in 1000 units) are summarized below.

	Week							
Product	1	2	3	4	5	6	7	8
A	12	12	12	16	16	20	20	20
B	8	8	10	10	12	12	12	12

Suppose that a trainee receives full salary as an experienced worker. Further suppose that due to perishable nature of the products,inventory is limited to one week. How should the company hire and train its new employees so that the labor cost is minimized? Formulate the problem as a linear optimization model.

Workforce Planning Problem

Decision Variables:

x_i denotes the no. of new employees hired in week i, for $i = 1, \ldots, 8$

$$\min \sum_{i=1}^{8} (9-i)x_i$$

subject to:

$$\begin{aligned}
(60 - \tfrac{1}{3}x_1)40 &\geq \frac{12000}{10} + \frac{8000}{6} \\
(60 - \tfrac{1}{3}x_1 - \tfrac{1}{3}x_2)40 &\geq \frac{12000}{10} + \frac{8000}{6} \\
(60 + x_1 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3)40 &\geq \frac{12000}{10} + \frac{10000}{6} \\
(60 + x_1 + x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4)40 &\geq \frac{16000}{10} + \frac{10000}{6} \\
(60 + x_1 + x_2 + x_3 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5)40 &\geq \frac{16000}{10} + \frac{12000}{6}
\end{aligned}$$

-contd.

Workforce Planning Problem

-contd.

$$\begin{aligned}
(60+x_1+x_2+x_3+x_4-\frac{1}{3}x_5-\frac{1}{3}x_6)40 &\geq \frac{20000}{10}+\frac{12000}{6}\\
(60+x_1+x_2+x_3+x_4+x_5-\frac{1}{3}x_6-\frac{1}{3}x_7)40 &\geq \frac{20000}{10}+\frac{12000}{6}\\
(60+x_1+x_2+x_3+x_4+x_5+x_6-\frac{1}{3}x_7-\frac{1}{3}x_8)40 &\geq \frac{20000}{10}+\frac{12000}{6}\\
60+\sum_{i=1}^{8} x_i &\geq 90\\
x_i &\geq 0,\ \forall i=1,\ldots,8
\end{aligned}$$

Machining Problem

A lathe is used to reduce the diameter of a steel shaft whose length is 36in. from 14in. to 12in. The speed x_1 (in rpm), the depth feed x_2 (in inches per minute), and the length feed x_3 (in inches per minute) must be determined. The duration of the cut is given by $36/x_2x_3$.

The compression and side stresses exerted on the cutting tool are given by $30x_1 + 400x_2$ and $40x_1 + 6000x_2 + 6000x_3$ pounds per square inch respectively. The temperature (in degrees Fahrenheit) at the tip of the cutting tool is $200 + 0.5x_1 + 150(x_2 + x_3)$. The maximum compression stress, side stress, and temperature allowed are 150,000 psi, 100,000 psi, and 800° F. It is desired to determine the speed (which must be in the range from 600 to 800 rpm), the depth feed, and the length feed such that the duration of the cut is minimized.

In order to use a linear model the following approximation is made. Since $36/x_2x_3$ is minimized if and only if x_2x_3 is maximized, it was decided to replace the objective by maximization of the minimum of x_2 and x_3. Formulate the problem as a linear optimization model.

Machining problem

Decision variables:

x_1, x_2, x_3 as defined in the problem;
We require $z = \min\{x_2, x_3\}$

$$
\begin{aligned}
\max z & & \\
\textit{subject to:} & & \\
z &\leq & x_2 \\
z &\leq & x_3 \\
30x_1 + 400x_2 &\leq & 150,000 \\
40x_1 + 6000x_2 + 6000x_3 &\leq & 100,000 \\
200 + 0.5x_1 + 150(x_2 + x_3) &\leq & 800 \\
x_1 &\leq & 800 \\
x_1 &\geq & 600 \\
x_j &\geq & 0, \forall j = 1,2,3 \\
z &\geq & 0
\end{aligned}
$$

Absolute Value Minimization

Consider the following optimization problem, where $|..|$ denotes the absolute value function.

$$\begin{aligned} \min |2x_1 - 3x_2| + 5|3x_1 - 2x_2| & \\ \text{s.t.} \qquad -x_1 + 3x_2 &\leq 10 \\ 2x_1 - 5x_2 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Reformulate this problem as a linear optimization problem. What if the objective was $|2x_1 - 3x_2| - 5|3x_1 - 2x_2|$? Can you still reformulate?

Absolute Value Minimization

$$\begin{aligned}\min |2x_1 - 3x_2| + 5|3x_1 - 2x_2| & & & \\ \text{s.t.} \quad & -x_1 + 3x_2 & \leq & \ 10 \\ & 2x_1 - 5x_2 & \leq & \ 15 \\ & x_1, x_2 & \geq & \ 0\end{aligned}$$

- As stated the objective function is nonlinear (it is piecewise linear), and therefore the problem needs to be reformulated as a linear optimization problem.
- We make the following observation: for any real number y, $|y| = \max\{y, -y\}$.
- So, $|2x_1 - 3x_2| = \max\{2x_1 - 3x_2, -2x_1 + 3x_2\}$ and $|3x_1 - 2x_2| = \max\{3x_1 - 2x_2, -3x_1 + 2x_2\}$

Absolute value minimization -contd.

$$\begin{aligned}
\min & \Big\{ \max\{2x_1 - 3x_2, -2x_1 + 3x_2\} + 5\max\{3x_1 - 2x_2, -3x_1 + 2x_2\} \Big\} \\
\text{s.t.} \quad & -x_1 + 3x_2 \leq 10 \\
& 2x_1 - 5x_2 \leq 15 \\
& x_1, x_2 \geq 0
\end{aligned}$$

- We now introduce two new variables z_1 and z_2, one for each term in the objective
- If we minimize $z_1 + 5z_2$ while making sure $z_1 \geq \max\{2x_1 - 3x_2, -2x_1 + 3x_2\}$ and $z_2 \geq \max\{3x_1 - 2x_2, -3x_1 + 2x_2\}$, they will actually become equal
- Note that if we just set them to be $=$ and not $\geq$, we simply move the nonlinearity from the objective to the constraints! We wish to remove it!

Absolute value minimization -contd.

$$
\begin{aligned}
\min z_1 + 5z_2 & \\
\text{s.t.} \qquad z_1 &\geq 2x_1 - 3x_2 \\
z_1 &\geq -2x_1 + 3x_2 \\
z_2 &\geq 3x_1 - 2x_2 \\
z_2 &\geq -3x_1 + 2x_2 \\
-x_1 + 3x_2 &\leq 10 \\
2x_1 - 5x_2 &\leq 15 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

- We now have (two) more variables than in the original problem, but now we have a linear optimization problem!
- Notice that our argument breaks-down (where?) if the coefficient in front of the second term was "-5" instead! So not all such problems can be reformulated!

Production Planning Problem

The demand for a particular product in the next four months is provided in the table.

Month	Demand (in units)
Sep	800
Oct	900
Nov	1100
Dec	700

Operating during regular hours the production capacity is 800 units/month. But overtime capacity is an additional 200 units/month. The selling price is the same irrespective of production mode, and the unit production cost are \$10 and \$12 for regular time production and overtime production, respectively. In addition, there exists a storage cost of \$1 per unit per month for each unsold unit held over. Formulate this problem as a linear optimization model to determine a production plan that minimizes total production and storage costs.

Production Planning Problem

Parameters:

n denotes number of months in the planning horizon

d_i for $i = 1, 2 \ldots, n$ denotes the demand in month i

u denotes regular-time capacity

v denotes overtime capacity

c denotes unit production cost in regular-time

f denotes unit production cost in overtime

h denotes unit holding cost per month

$\bar{z}_0$ denotes the initial inventory

Decision Variables:

x_i denotes the number of units produced in regular-time in month $i = 1, 2 \ldots, n$

y_i denotes the number of units produced in overtime in month $i = 1, 2 \ldots, n$

z_i denotes the number of units carried over as inventory from month $i = 0, 1 \ldots, n$ to $i + 1$

Note: z_0 and z_n variables denote beginning and ending inventory, respectively

Production Planning Problem

$$\min_{x,y,z} \sum_{i=1}^{n} cx_i + \sum_{i=1}^{n} fy_i + \sum_{i=1}^{n} hz_i$$

subject to:

$$x_i + y_i + z_{i-1} = d_i + z_i \quad \forall i = 1, \ldots, n$$

$$z_0 = \bar{z}_0$$

$$x_i \leq u \quad \forall i = 1, \ldots, n$$

$$y_i \leq v \quad \forall i = 1, \ldots, n$$

$$x_i, y_i, z_i \geq 0 \quad \forall i = 1, \ldots, n$$

Production Planning Problem

1. We assume a flat rate of \$1 per unit of unsold product held over from one month to the next, irrespective of when it was actually produced during the month.
2. We have assumed, c, f, h, u, v do not vary by month; it is easy to accommodate if these parameters vary monthly.
3. This model does not have any closing inventory requirement; if it did, we would impose that constraint on z_n.
4. This model does not have inventory capacity (a limit on how much can be carried from month to month, say due to space restrictions). This can be captured by placing an upper-bound on the z_i variables.
5. We could have also used x_{ij} for $j = 1, 2$ instead of the pair x_i, y_i for each month. This choice would be preferable if there were several production modes instead of just two (regular-time and overtime), or if we were producing this product in several facilities and pooling the inventory in a single warehouse. Note that our formulation will work for both variants of the problem statement with x_{ij} variables with index j indicating production mode/facility.

Production Planning with Back-orders

Reformulate the production planning problem assuming that unfulfilled demand in any month can be satisfied in a future month, but it incurs a back-order cost of \$10 per unit per month late.

Production Planning with Back-orders

Additional Parameters:

b denotes unit backordering cost per month

$\bar{w}_0$ denotes opening back-orders

Additional Decision Variables:

- w_i denotes the number of units of demand back-ordered from month $i = 0, 1, \ldots, n$ to $i + 1$ Note: w_0 denotes the opening back-orders, w_n denotes closing back-orders

Production Planning with Back-orders

$$\min \sum_{i=1}^{n} cx_i + \sum_{i=1}^{n} fy_i + \sum_{i=1}^{n} hz_i + \sum_{i=1}^{n} bw_i$$

subject to:

$$x_i + y_i + z_{i-1} - z_i = d_i + w_{i-1} - w_i \quad \forall i = 1, \ldots, n$$

$$z_0 = \bar{z}_0$$

$$w_0 = \bar{w}_0$$

$$x_i \leq u \quad \forall i = 1, \ldots, n$$

$$y_i \leq v \quad \forall i = 1, \ldots, n$$

$$x_i, y_i, z_i, w_i \geq 0 \quad \forall i = 1, \ldots, n$$

Production Planning Problem—Perishable Product

Reformulate the production planning problem assuming that inventory is perishable, and each unit can be held for at most one more month beyond the month in which it is produced.

Basic Production Planning Problem: Alternate Formulation

This formulation is for the conventional production planning with inventory; no back-orders are permitted and inventory is non-perishable.

Parameters:

n denotes the number of months in the planning horizon
m denotes the number of production modes (regular, over-time)
d_i denotes the demand in month $i = 1, \ldots, n$
u_j denotes monthly production capacity limit for mode $j = 1, \ldots, m$
c_j unit production cost for mode $j = 1, \ldots, m$
h unit storage cost per month

Decision Variables:

x_{ijk} denotes the number of units produced in month i in mode j to satisfy demand of month k, where $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $k = i, \ldots, n$.

Basic Production Planning Problem: Alternate Formulation

$$
\begin{aligned}
\min \sum_{j=1}^{m} c_j \sum_{i=1}^{n} \sum_{k=i}^{n} x_{ijk} \quad &+ \quad \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=i}^{n} h(k-i) x_{ijk} \\
\textit{subject to:} & \\
\sum_{j=1}^{m} \sum_{i=1}^{k} x_{ijk} \quad &= \quad d_k \;\; \forall \; k = 1, \ldots, n \\
\sum_{k=i}^{n} x_{ijk} \quad &\leq \quad u_j \;\; \forall \; j = 1, \ldots, m, \;\; i = 1, \ldots, n \\
x_{ijk} \quad &\geq \quad 0 \;\; \forall \; i = 1, \ldots, n, j = 1, \ldots, m, k = i, \ldots, n
\end{aligned}
$$

Production Planning with Perishable Inventory

Decision Variables:

x_{ijk} denotes the number of units produced in month i in mode j to satisfy demand of month k, where $i = 1, \ldots, n, j = 1, \ldots, m$ and $k = i, i+1$.

$$
\begin{aligned}
\min \sum_{j=1}^{m} c_j \sum_{i=1}^{n} \sum_{k=i}^{i+1} x_{ijk} \quad &+ \quad \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=i}^{i+1} h(k-i) x_{ijk} \\
\textit{subject to:} & \\
\sum_{j=1}^{m} x_{1j1} \quad &= \quad d_1 \\
\sum_{j=1}^{m} \sum_{i=k-1}^{k} x_{ijk} \quad &= \quad d_k \ \ \forall\ k = 2, \ldots, n \\
\sum_{k=i}^{i+1} x_{ijk} \quad &\leq \quad u_j \ \ \forall\ j = 1, \ldots, m,\ \ i = 1, \ldots, n \\
x_{n,j,n+1} \quad &= 0 \quad \forall j = 1, \ldots, m \\
x_{ijk} \quad &\geq \quad 0 \ \ \forall\ i = 1, \ldots, n, j = 1, \ldots, m, k = i, i+1
\end{aligned}
$$

Petroleum Blending Problem

An oil refinery produces four types of raw gasoline: alkylate, catalytic-cracked, straight-run, and isopentane. Two important characteristics of each gasoline are its performance number PN (indicating antiknock properties) and its vapor pressure RVP (indicating volatility). These two characteristics, together with the production levels in barrels per day, are as follows:

	PN	RVP	Barrels produced per day
Alkylate	107	5	3,814
Catalytic-cracked	93	8	2,666
Straight-run	87	4	4,016
Isopentane	108	21	1,300

These gasolines can be sold either raw, at \$4.83 per barrel, or blended into aviation gasoline (Avgas A and/or Avgas B). Quality standards impose certain requirements on the aviation gasolines; these requirements, together with the selling prices, are as follows:

	PN	RVP	Price per barrel
Avgas A	at least 100	at most 7	\$6.45
Avgas B	at least 91	at most 7	\$5.91

The PN and RVP of a blend are the volume weighted averages of the PNs and RVPs of its constituents. The refinery aims for a plan that yields the largest possible profit. Formulate the problem as a LP.

Petroleum Blending Problem

Index sets:

$I = \{1, 2, 3, 4\} \equiv \{$Alkylate, Catalytic-cracked, Straight-run, Isopentane$\}$ for the types of gasoline

$J = \{1, 2, 3\} \equiv \{$Avi gas A, Avi gas B, Unblended$\}$ for the types of blends

Data:

p_i denotes performance number of type $i \in I$ gas
r_i vapor pressure of type $i \in I$ gas
U_i number of barrels of type $i \in I$ produced per day
P_j minimum PN required of product $j = 1, 2$
R_j maximum RVP required of product $j = 1, 2$
S_j selling price in dollar per barrel of product $j \in J$

Decision Variables:

x_{ij} denotes the number of barrels/day of type $i \in I$ gas used to make product $j \in J$

Petroleum Blending Problem

$$
\begin{aligned}
\max \sum_{j \in J} S_j \sum_{i \in I} x_{ij} & & \\
\textit{subject to:} \qquad \sum_{j \in J} x_{ij} & \leq & U_i, \forall i \in I \\
\sum_{i \in I} r_i x_{ij} - R_j \sum_{i \in I} x_{ij} & \leq & 0, \forall j = 1, 2 \\
\sum_{i \in I} p_i x_{ij} - P_j \sum_{i \in I} x_{ij} & \geq & 0, \forall j = 1, 2 \\
x_{ij} & \geq & 0, \forall i \in I, j \in J
\end{aligned}
$$