

# SECTION 1

## RECURSION

in recursion we just call the function in on itself till a condition is met, this results in the program firstly processing *ALL* the data and then computing one by one

```
(define (factorial n)
  (if (= n 1)
      1
      (* n (factorial (- n 1)))))

;(factorial 6)
;(* 6 (factorial 5))
;(* 6 (* 5 (factorial 4)))
;(* 6 (* 5 (* 4 (factorial 3))))
;(* 6 (* 5 (* 4 (* 3 (factorial 2)))))
;(* 6 (* 5 (* 4 (* 3 (* 2 (factorial 1))))) <-- accumulates all the processes
;(* 6 (* 5 (* 4 (* 3 (* 2 1)))))
;(* 6 (* 5 (* 4 (* 3 2))))
;(* 6 (* 5 (* 4 6)))
;(* 6 (* 5 24))
;(* 6 120)
;720
```

## ITERATION

Iteration means going forward as opposed to the recursion's going backwards at first, to the very lowest level, and then going forward calculating the result on the way back up, you *change* the value of the variable at every step:

```
(define (factorial n)
  (define (iter product counter)
    (if (> counter n)
        product
        (iter (* counter product) (+ counter 1))))
  (iter 1 1))

;(fact 4)
;(iter 1 1 4)
;(iter 1 2 4)
;(iter 2 3 4)
;(iter 6 4 4)
;(iter 24 5 4)
;24
```

Product	Input	Counter
1	4	1
1	4	2
2	4	3
6	4	4
24	4	5

## DIFF BETWEEN THEM

One of the primary ways to differentiate between an iterative and recursive process is to imagine what'd happen if you turned the computer off, then resumed the current computation.

In a recursive process we can never know how deep into the recursion we are.

In the factorial example, this means that we can't return the right value.

In an iterative process, all state information is contained by the process.

## Exercise 1

What is the result printed by the interpreter in response to each expression?

```
10
;10
(+ 5 3 4)
;12
(- 9 1)
;8
(/ 6 2)
;3
(+ (* 2 4) (- 4 6))
;6

(define a 3)
(define b (+ a 1))
(+ a b (* a b))
;(+ 3 (+ 3 1) (* 3 (+ 3 1)))
;(+ 3 4 12)
;(19)

(= a b)
;FALSE (amogus)
(if (and (> b a) (< b (* a b)))
    b
    a)
;(if (and true true) b a)
;(if true b a)
;(b)
;4

(cond ((= a 4) 6)
      ((= b 4) (+ 6 7 a))
      (else 25))
;(+ 6 7 3)
;16
```

```

(+ 2 (if (> b a) b a))
;(+ 2 (if true b a))
;(+ 2 4)
;(+ 6)

(* (cond ((> a b) a)
      ((< a b) b)
      (else -1))
  (+ a 1))
;(* 4 (+ 3 1))
;(* 4 4)
;16

```

## Exercise 2

Translate the following expression into prefix

$$\frac{5 + 4 + (2 - (3 - (6 + \frac{4}{5})))}{3(6 - 2)(2 - 7)}.$$

```

(/ (+ 5 4 (- 2 (- 3 (+ 6 (/ 4 5))))) (* 3 (- 6 2) (- 2 7)))
; -37/150

```

## Exercise 3

Define a procedure that takes three numbers as arguments and returns the sum of the squares of the two larger numbers

```

(define (f1 a b c)
  (if (< a b)
      (if (< a c)
          (+ (* b b) (* c c))
          (+ (* b b) (* a a)))
      (if (< b c)
          (+ (* a a) (* c c))
          (+ (* a a) (* b b)))))

```

## Exercise 4

Observe that our model of evaluation allows for combinations whose operators are compound expressions. Use this observation to describe the behavior of the following procedure:

```
(define (a-plus-abs-b a b)
  ((if (> b 0) + -) a b))
```

```
;this is just a+|b| by performing a+b if b>0, else a-b
(a-plus-abs-b 3 4)
; (7)
(a-plus-abs-b 3 -4)
```

## Exercise 5

Ben Bitdiddle has invented a test to determine whether the interpreter he is faced with is using applicative-order evaluation or normal-order evaluation. He defines the following two procedures:

```
(define (p) (p))
(define (test x y) (if (= x 0) 0 y))
```

Then he evaluates the expression `(test 0 (p))`. What behavior will Ben observe with an interpreter that uses applicative-order evaluation? What behavior will he observe with an interpreter that uses normal-order evaluation? Explain your answer. (Assume that the evaluation rule for the special form `if` is the same whether the interpreter is using normal or applicative order: The predicate expression is evaluated first, and the result determines whether to evaluate the consequent or the alternative expression.)

evaluating `(p)` will result in an infinite loop for applicative order because internal reduction happens **BEFORE** the values are evaluated, like:

```
(test 0 (p))
(if (= 0 0) 0 (p)) ;<---- here the program will go for an infinite loop
because it can't evaluate `(p)`
```

for normal order eval this will happen:

```
(test 0 (p))
(if (= 0 0)
    0 ;<--- here the program will stop
    (p)) ;<--- the program will actually never reach this phase
```

## Exercise 6

Alyssa P. Hacker doesn't see why if needs to be provided as a special form. "Why can't I just define it as an ordinary procedure in terms of cond?" she asks. Alyssa's friend Eva Lu Ator claims this can indeed be done, and she defines a new version of if:

```
(define (new-if predicate then-clause else-clause)
  (cond (predicate then-clause)
        (else else-clause)))
```

Eva demonstrates the program for Alyssa:

```
(new-if (= 2 3) 0 5)
5
```

```
(new-if (= 1 1) 0 5)
0
```

Delighted, Alyssa uses new-if to rewrite the square-root program:

```
(define (sqrt-iter guess x)
  (new-if (good-enough? guess x)
          guess
          (sqrt-iter (improve guess x) x)))
```

What happens when Alyssa attempts to use this to compute square roots? Explain.

in the old (if cond A B) the B condition would only be evaluated if cond was false BUT in the new (new-if cond A B) as per the definition the argument B needs to be evaluated regardless of if the cond was false because when a function is called the first thing that Scheme does with the argument list is evaluating every single argument

do you remembah? the scheme uses applicative-order for function in septembah?

## Exercise 7

The good-enough? test used in computing square roots will not be very effective for finding the square roots of very small numbers. Also, in real computers, arithmetic operations are almost always performed with limited precision. This makes our test inadequate for very large numbers. Explain these statements, with examples showing how the test fails for small and large numbers. An alternative strategy for implementing good-enough? is to watch how guess changes from one iteration to the next and to stop when the change is a very small fraction of the guess. Design a square-root procedure that uses this kind of end test. Does this work better for small and large numbers?

if  $y$  is our guess for  $\sqrt{x}$  then,  $y = \sqrt{x} + \delta y$  and  $|y^2 - x| > \varepsilon$ , then for small values:

$$\varepsilon = |y^2 - x|$$

$$\varepsilon = |(\sqrt{x} + \delta y)^2 - x|$$

$$\varepsilon = |(x + 2\delta y)\sqrt{x} - x|$$

$$\varepsilon = 2|\delta y|\sqrt{x}$$

```
(define (average x y) (/ (+ x y) 2))
```

```
(define (improve guess x) (average guess (/ x guess)))
```

```
(define (square x) (* x x))
```

```
(define (abs x)
  (cond ((= x 0) 0)
        (< x 0) (-x)
        (> x 0) (x))))
```

```
(define (good-enough? guess x)
  (< (abs (- (square guess) x)) 0.001))
```

```
(define (sqrt-iter guess x)
  (if (good-enough? guess x) guess
      (sqrt-iter (improve guess x) x)))
```

```
(define (my-sqrt x)
  (sqrt-iter 1.0 x))
```

; Given the last and 2nd last guesses, check if the fractional change is good enough.

; I'm choosing the formula  $|guessL - guessLL| / guessL < 0.001$

```
(define (good-enough2? guessL guessLL)
  (< (/ (abs (- guessL guessLL)) guessL) 0.01))
```

```
(define (sqrt-iter2 guessL guessLL x)
  (if (good-enough2? guessL guessLL)
      guessL
      (sqrt-iter2 (improve guessL x) guessL x)))
```

```
(define (my-sqrt2 x)
  (sqrt-iter2 1.0 1.1 x))
```

## Exercise 8

Newton's method for cube roots is based on the fact that if  $y$  is an approximation to the cube root of  $x$ , then a better approximation is given by the value

$$\frac{\frac{x}{y^2} + 2y}{3}$$

Use this formula to implement a cube-root procedure analogous to the square-root procedure.

```
(define (improve y x)
  (/ (+ (/ x (* y y)) (* 2 y)) 3))

; Fractional good-enough like in ex 1-7
(define (good-enough? guessL guessLL)
  (< (/ (abs (- guessL guessLL)) guessL) 0.01))

(define (curt-iter guessL guessLL x)
  (if (good-enough? guessL guessLL)
      guessL
      (curt-iter (improve guessL x) guessL x)))

(define (my-curt x)
  (curt-iter 1.0 1.1 x))

; 1.4422495703074083
(my-curt 3)
```

## Exercise 9

Each of the following two procedures defines a method for adding two positive integers in terms of the procedures `inc`, which increments its argument by 1, and `dec`, which decrements its argument by 1.

```
(define (+ a b)
  (if (= a 0)
      b
      (inc (+ (dec a) b))))

(define (+ a b)
  (if (= a 0)
      b
      (+ (dec a) (inc b))))
```

Using the substitution model, illustrate the process generated by each procedure in evaluating `(+ 4 5)`. Are these processes iterative or recursive?

```

(define (+ a b)
  (if (= a 0)
      b
      (inc (+ (dec a) b))))
;recursive process because it doesn't change the value of the variable
;(+ 4 5)
;(inc (+ 3 5))
;(inc (inc (+ 2 5)))
;(inc (inc (inc (+ 1 5))))
;(inc (inc (inc (inc (+ 0 5)))))
;(inc (inc (inc (inc 5))))
;(inc (inc (inc 6)))
;(inc (inc 7))
;(inc 8)
;9

(define (+ a b)
  (if (= a 0)
      b
      (+ (dec a) (inc b))))
;iterative because it changes the value of the variable at every step
;(+ 4 5)
;(+ 3 6)
;(+ 2 7)
;(+ 1 8)
;(+ 0 9)
;9

```



## Exercise 10

The following procedure computes a mathematical function called Ackermann's function.

```
(define (A x y)
  (cond ((= y 0) 0)
        ((= x 0) (* 2 y))
        ((= y 1) 2)
        (else (A (- x 1)
                   (A x (- y 1))))))
```

What are the values of the following expressions?

```
(A 1 10)
(A 2 4)
(A 3 3)
```

Consider the following procedures, where A is the procedure defined above:

```
(define (f n) (A 0 n))
(define (g n) (A 1 n))
(define (h n) (A 2 n))
(define (k n) (* 5 n n))
```

Give concise mathematical definitions for the functions computed by the procedures f, g, and h for positive integer values of n. For example, (k n) computes  $5n^2$ .

```
(A 1 10)
;1024
(A 2 4)
;65536
(A 3 3)
;65536
```

```
(define (f n) (A 0 n))
;this is just x 0 always so it is just a function that doubles the argument
i.e n*2
(define (g n) (A 1 n))
;(A 1 0) is 0
;(A 1 1) is 2
;(A 1 2) is (* 2 2) = 4
;(A 1 3) is (* 2 (* 2 2)) = 8
;(A 1 n) is 2^n
(define (h n) (A 2 n))
;(A 2 0) is 0
;(A 2 1) is 2
;(A 2 2) is 4
;(A 2 3) is 16
;this is (A 1 (A 2 (- n 1)))
;or square of (A 2 (- n 1))
;so this is just square of square of square of square and so on
```

## Exercise 11

A function  $f$  is defined by the rule that  $f(n) = n$  if  $n < 3$  and  $f(n) = f(n-1) + 2f(n-2) + 3f(n-3)$  if  $n \geq 3$ . Write a procedure that computes  $f$  by means of a recursive process. Write a procedure that computes  $f$  by means of an iterative process.

recursive is pretty easy

```
(define (f n)
  (if (< n 3)
      n
      (+ (f (- n 1)) (* (f (- n 2)) 2) (* (f (- n 3)) 3))))
```

for iterative. We need three pieces of state to track the current, the last and the values of  $f$ . (that is,  $f(n-1)$ ,  $f(n-2)$  and  $f(n-3)$ .)

```
(define (f n)
  (f-it-helper 2 1 0 n))

(define (iterator a b c counter)
  (if (= counter 0)
      c
      (iterator (+ a (* 2 b) (* 3 c))
                 a
                 b
                 (- counter 1))))
```

;think like this

```
;f(0) = 0 |
;f(1) = 1 | all known values
;f(2) = 2 |
```

```
;f(3) = f(2) + 2f(1) + 3f(0)
;f(4) = f(3) + 2f(2) + 3f(1)
;f(5) = f(4) + 2f(3) + 3f(2)
;f(6) = f(5) + 2f(4) + 3f(3)
```

## Exercise 12

Write a recursive procedure that computes elements of Pascal's triangle.

```
(define (pascal row col)
  (cond ((= col 0) 1)
        ((= col row) 1)
        (else (+ (pascal (- row 1) col) (pascal (- row 1) (- col 1))))))
;idk how to display it :P
```

### Exercise 13

Prove that  $\text{fib}(n)$  is the closest integer to  $\frac{\varphi^n}{\sqrt{5}}$  where  $\varphi = \frac{1+\sqrt{5}}{2}$ .

Hint: Let  $\psi = \frac{1-\sqrt{5}}{2}$ . Use induction and the definition of the Fibonacci numbers to prove that  $\text{Fib}(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$ .

TODO

## Exercise 14

Draw the tree illustrating the process generated by the count-change procedure of 1.2.2 in making change for 11 cents. What are the orders of growth of the space and number of steps used by this process as the amount to be changed increases?

```
;the code was this
(define (count-change amount)
  (cc amount 5))
(define (cc amount kinds-of-coins)
  (cond ((= amount 0) 1)
        ((or (< amount 0) (= kinds-of-coins 0)) 0)
        (else (+ (cc amount
                      (- kinds-of-coins 1))
                  (cc (- amount
                        (first-denomination kinds-of-coins))
                      kinds-of-coins)))))
(define (first-denomination kinds-of-coins)
  (cond ((= kinds-of-coins 1) 1)
        ((= kinds-of-coins 2) 5)
        ((= kinds-of-coins 3) 10)
        ((= kinds-of-coins 4) 25)
        ((= kinds-of-coins 5) 50)))
(count-change 11)
```

$$\sum_{i=a}^b \frac{1}{i(i+2)}$$

## Exercise 15

The sine of an angle (specified in radians) can be computed by making use of the approximation  $\sin(x) = x$  if  $x$  is sufficiently small, and the trigonometric identity to reduce the size of the argument of  $\sin$ . (For purposes of this exercise an angle is considered “sufficiently small” if its magnitude is not greater than 0.1 radians.) These ideas are incorporated in the following procedures:

we are using the identity  $\sin(3x) = 3 \sin(x) - 4 \sin^3(x)$

```
(define (cube x) (* x x x))
(define (p x)
  (- (* 3 x) (* 4 (cube x))))
(define (sine angle)
  (if (not (> (abs angle) 0.1))
      angle
      (p (sine (/ angle 3.0)))))
```

- How many times is the procedure `p` applied when `(sine 12.15)` is evaluated?
- What is the order of growth in space and number of steps (as a function of  $a$ ) used by the process generated by the `sine` procedure when `(sine a)` is evaluated?

- the procedure `p` is evaluated 5 times

$$\frac{12.15}{3^n} = 0.1$$

$$n = \frac{\log\left(\frac{12.5}{0.1}\right)}{\log(3)} \approx 4.4$$

;also the output is this

```
(sine 12.5)
-0.060813768577286265
```

- the order of growth of steps is just  $\Theta(\log(a))$  and similar in space because there is no recursion

## Exercise 16

Design a procedure that evolves an iterative exponentiation process that uses successive squaring and uses a logarithmic number of steps, as does `fast-expt`.

(Hint: Using the observation that  $b^{(\frac{n}{2})^2} = (b^2)^{\frac{n}{2}}$  keep, along with the exponent  $n$  and the base  $b$ , an additional state variable  $a$ , and define the state transformation in such a way that the product  $ab^n$  is unchanged from state to state. At the beginning of the process  $a$  is taken to be 1, and the answer is given by the value of  $a$  at the end of the process. In general, the technique of defining an invariant quantity that remains unchanged from state to state is a powerful way to think about the design of iterative algorithms.)

```

(define (exponent base n)
  (helperiter 1 base n))
(define (helperiter acc base n)
  (cond ((= n 0) acc)
        ((= (remainder n 2) 0) (helperiter acc (* base base) (/ n 2))) ;if the
                                exponent is a multiple of 2 then do b^n = (b*b)^(n/2)
        (helperiter (* acc base) base (- n 1))))

```

## Exercise 17

The exponentiation algorithms in this section are based on performing exponentiation by means of repeated multiplication. In a similar way, one can perform integer multiplication by means of repeated addition. The following multiplication procedure (in which it is assumed that our language can only add, not multiply) is analogous to the `expt` procedure:

```

(define (* a b)
  (if (= b 0) 0
      (+ a (* a (- b 1)))))

```

this algorithm takes a number of steps that is linear in  $b$ . Now suppose we include, together with addition, operations `double`, which doubles an integer, and `half`, which divides an (even) integer by 2. Using these, design a multiplication procedure analogous to `fast-expt` that uses a logarithmic number of steps.

we can do something like  $ab = (2a)\left(\frac{b}{2}\right)$  for even  $b$  (because we are given the procedures `double` and `half`)

and the previous same procedure for odd  $b$

```

(define (even? n)
  (= (remainder n 2) 0))

(define (* a b)
  (cond ((= b 0) 0)
        ((even? b) (* (double a) (half 2)))
        (else (+ a (* a (- b 1))))))

```

## Exercise 18

Using the results of Exercise 1.16 and Exercise 1.17, devise a procedure that generates an iterative process for multiplying two integers in terms of adding, doubling, and halving and uses a logarithmic number of steps.

we know that in iterative processes we need an accumulator a counter and some number of other arguments for operations, so let's make one of the multiplication arguments as the counter, something like

```

(define (multiply a b)
  (define (iterator acc a b)
    (cond ((= b 0) acc)
          ((even? b) (iterator acc (double a) (half b)))
          (else (iterator (+ acc a) a (- b 1)))))
  (iterator 0 a b))

```

## Exercise 19

There is a clever algorithm for computing the Fibonacci numbers in a logarithmic number of steps. Recall the transformation of the state variables  $a$  and  $b$  in the fib-iter process of 1.2.2:

$$a \leftarrow a + b$$

$$b \leftarrow a$$

Call this transformation  $T$ , and observe that applying  $T$  over and over again  $n$  times, starting with 1 and 0, produces the pair  $\text{Fib}(n+1)$  and  $\text{Fib}(n)$ . In other words, the Fibonacci numbers are produced by applying  $T^n$ , the  $n^{\text{th}}$  power of the transformation  $T$ , starting with the pair  $(1, 0)$ .

Now consider  $T$  to be the special case of  $p = 0$  and  $q = 1$  in a family of transformations  $T_{pq}$  where  $T_{pq}$  transforms the pair  $(a, b)$  according to:

$$a \leftarrow bq + aq + ap$$

$$b \leftarrow bp + aq$$

Show that if we apply such a transformation  $T_{pq}$  twice, the effect is the same as using a single transformation  $T_{p'q'}$  of the same form, and compute  $p'$  and  $q'$  in terms of  $p$  and  $q$ . This gives us an explicit way to square these transformations, and thus we can compute  $T^n$  using successive squaring.

```

(define (fib n)
  (fib-iter 1 0 0 1 n))
(define (fib-iter a b p q count)
  (cond ((= count 0) b)
        ((even? count)
         (fib-iter a
                   b
                   <??> ; compute p'
                   <??> ; compute q'
                   (/ count 2)))
        (else (fib-iter (+ (* b q) (* a q) (* a p))
                        (+ (* b p) (* a q))
                        p
                        q
                        (- count 1)))))

```

let us say there is a matrix  $T_{pq}$ , then according to the question this matrix *transforms* the matrix (in question called pair)  $\begin{pmatrix} a \\ b \end{pmatrix}$  as follows:

$$T_{pq} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} bq + aq + ap \\ bp + aq \end{bmatrix}$$

$$T_{pq} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a(p+q) + bq \\ aq + bp \end{bmatrix}$$

$$T_{pq} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} p+q & q \\ q & p \end{bmatrix}$$

$$T_{pq} = \begin{bmatrix} p+q & q \\ q & p \end{bmatrix}$$

and according to question we need to show that  $T_{pq}^2$  ( $T_{pq}$  used 2 times) is equivalent to some  $T_{p'q'}$ , so:

$$T_{pq}^2 = \begin{bmatrix} p+q & q \\ q & p \end{bmatrix}^2$$

$$T_{pq}^2 = \begin{bmatrix} p^2 + 2pq + 2q^2 & 2pq + q^2 \\ 2pq + q^2 & q^2 + p^2 \end{bmatrix}$$

$$T_{pq}^2 = T_{(2pq+q^2)(q^2+p^2)}$$

now just shove it in the code and *noice*

```
(define (fib n)
  (fib-iter 1 0 0 1 n))
(define (fib-iter a b p q count)
  (cond ((= count 0) b)
        ((even? count)
         (fib-iter a
                   b
                   (+ (* 2 p q) (* q q)) ; compute p'
                   (+ (* q q) (* p p)) ; compute q'
                   (/ count 2)))
        (else (fib-iter (+ (* b q) (* a q) (* a p))
                          (+ (* b p) (* a q))
                          p
                          q
                          (- count 1)))))
```



## Exercise 20

The process that a procedure generates is of course dependent on the rules used by the interpreter. As an example, consider the iterative gcd procedure given above. Suppose we were to interpret this procedure using normal-order evaluation, as discussed in 1.1.5. (The normal-order-evaluation rule for if is described in Exercise 1.5.) Using the substitution method (for normal order), illustrate the process generated in evaluating (gcd 206 40) and indicate the remainder operations that are actually performed. How many remainder operations are actually performed in the normal-order evaluation of (gcd 206 40)? In the applicative-order evaluation?

- for normal order evaluation it will be

```
;i AM NOT WRITING THAT WTF it will take ages bro
```

- for applicative order of evaluation

```
; ok fuck this i saw everywhere and everyone has a different answer? wtf?
```

## Exercise 21

Use the smallest-divisor procedure to find the smallest divisor of each of the following numbers: 199, 1999, 19999.

```
(define (smallest-divisor n) (find-divisor n 2))

(define (divides? a b)
  (= (remainder b a) 0))

(define (find-divisor n test-divisor)
  (cond ((> (* test-divisor test-divisor) n) n)
        ((divides? test-divisor n) test-divisor)
        (else (find-divisor n (+ test-divisor 1)))))

(smallest-divisor 199)
;199
(smallest-divisor 1999)
;;1999
(smallest-divisor 7)
;7
```

## Exercise 22

Most Lisp implementations include a primitive called `runtime` that returns an integer that specifies the amount of time the system has been running (measured, for example, in microseconds). The following `timed-prime-test` procedure, when called with an integer prints and checks to see if is prime. If is prime, the procedure prints three asterisks followed by the amount of time used in performing the test.

```
(define (timed-prime-test n)
  (newline)
  (display n)
  (start-prime-test n (runtime)))

(define (start-prime-test n start-time)
  (if (prime? n)
      (report-prime (- (runtime)
                       start-time))))

(define (report-prime elapsed-time)
  (display " *** ")
  (display elapsed-time))
```

Using this procedure, write a procedure `search-for-primes` that checks the primality of consecutive odd integers in a specified range. Use your procedure to find the three smallest primes larger than 1000; larger than 10,000; larger than 100,000; larger than 1,000,000. Note the time needed to test each prime. Since the testing algorithm has order of growth of  $\Theta(\sqrt{n})$  you should expect that testing for primes around 10,000 should take about  $\sqrt{10}$  times as long as testing for primes around 1000. Do your timing data bear this out? How well do the data for 100,000 and 1,000,000 support the  $\Theta(\sqrt{n})$  prediction? Is your result compatible with the notion that programs on your machine run in time proportional to the number of steps required for the computation?