On the logic of belief with propositional quantifiers

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Background I

- Many normal modal logics, especially "applied modal logics", are "Kripke complete": they are the logics of some (intuitively interesting) classes of Kripke frames.
- Example: the logic for deductively closed beliefs KD45 is the logic of serial, transitive, and Euclidean frames.

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\begin{array}{llll}
\mathrm{K} & (\Box p \land \Box(p \to q)) \to \Box q & \mathrm{D} & \neg\Box(p \land \neg p) \\
4 & \Box p \to \Box\Box p & 5 & \neg\Box p \to \Box\neg\Box p.
\end{array}
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- The logic is simple and each axiom is well motivated. But once we think carefully, there are a few strong properties of Kripke frames not picked up by this logic.
- There are Kripke incomplete modal logics.

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- All normal modal logics are algebraically complete: they are the logics of classes of Boolean algebras with operators (BAOs) $\langle \mathcal{B}, \Box \rangle$.
- Indeed, a single Lindenbaum algebra is enough to refute all non-theorems.
- Kripke frames corresponds to BAOs with three properties:
    - **Atomicity**: every proposition (element in $\mathcal{B}$) is above a minimal consistent proposition which decides every proposition.
    - **Completeness**: every set of proposition has a meet.
    - **Complete multiplicativity**: $\Box \bigwedge a_i = \bigwedge \Box a_i$.
- For KD45, none of the above three properties matter: we can assume them with out changing the logic. In fact, assuming them becomes the natural thing to do.

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We have expressions that quantify over propositions:

- "Everything I believe is true."
- "For everything that I don't believe, I believe in its negation."
- "There is a proposition, say p, such that I believe it and that for any proposition that I believe is compatible with p, I in fact believe that it is entailed by p."

The sentences can be formalized.

- $\forall p(\Box p \to p)$.
- $\forall p(\neg\Box p \to \Box\neg p)$.
- $\exists q(\Box q \wedge \forall p(\Diamond(q \wedge p) \to \Box(q \to p)))$.

Can we pick up the three properties now? Will them now generate more controvercial principles (since the modal operator is to be interpreted as belief)? What logics do we get?

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Here's the full language.

Definition

Let $\mathcal{L}\Pi$ be the language with the following grammar

$$\varphi ::= p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall p \varphi$$

where $p \in \mathsf{Prop}$, a countably infinite set of propositional *variables*. Other Boolean connectives, $\bot$, and $\Diamond$ are defined as usual.

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Every subset is a proposition!

- A pointed model $\langle W, R, V\rangle, w$ makes $\forall p\varphi$ true iff for all $X \subseteq W$, $\langle W, R, V[p \mapsto X]\rangle, w$ makes φ true.
- Equivalently, $[\![\forall p\varphi]\!]^{\mathcal{M}} = \bigcap_{X \subseteq \mathcal{M}} [\![\varphi]\!]^{\mathcal{M}[p \mapsto X]}$.

Example: $\forall p(\Box p \to p)$ is true at w iff wRw. In other words, $[\![\forall p(\Box p \to p)]\!]^{\mathcal{M}}$ does not depend on V and is precisely the set of reflexive points in $\mathcal{M}$.

Another example:

$$[\![\Diamond p \wedge \forall q(\Box(p \to q) \vee \Box(p \to \neg q))]\!]^{\mathcal{M}}$$

is the set of points that can access to exactly one element in $V(p)$.

Call this formula $atom(p)$.

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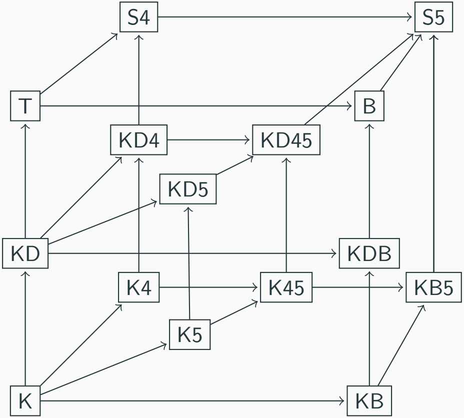

For S5: the logic is decidable.
For those below S4 or B: second-order arithmetic.
Question: what's the boundary between the complex and the simple?

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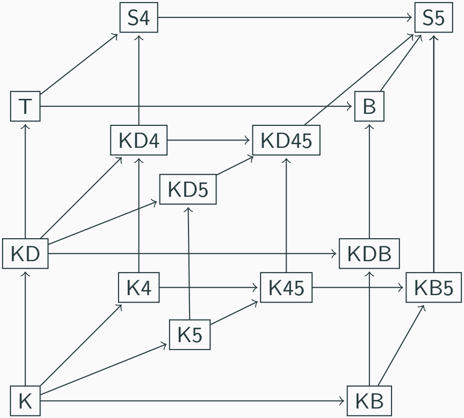

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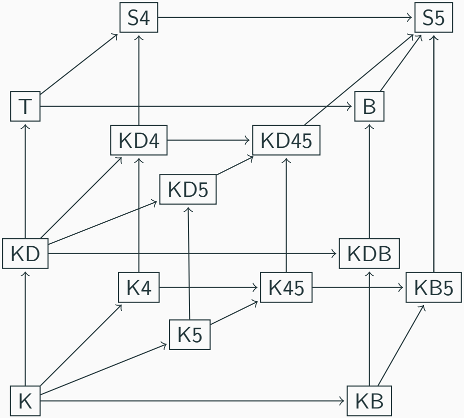

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An old-fasioned agenda

- I'll present
  - an algebraic semantics based on algebras of propositions, and
  - logics sound and complete with respect to algebras, possibly with special properties.
  - Decidability follows.

Algebraic Semantics

Algebraic semantics: reasons

- It brings the assumptions made in Kripke frames to the foreground: Kripke frames corresponds to complete, atomic, completely multiplicative modal algebras.
- It is natural for $\mathcal{L}\Pi$. Order-theoretically, $\forall p \varphi$ is the weakest proposition that entails all instances of φ.
- It helps raising intersting questions. What if we drop atomicity? What if we drop complete multiplicativity? How much lattice-completeness do we need for the semantics to be well-defined?

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Algebraic semantics: the belief case

Let B be a complete Boolean algebra and F a proper filter on B. We call $\langle B, F\rangle$ a complete proper filter algebra. Given a valuation $V : \mathrm{Prop} \to B$, it can be extended to an $\mathcal{L}\Pi$-*valuation* $\widehat{V} : \mathcal{L}\Pi \to B$ by setting

- $\widehat{V}(\Box\varphi) = \begin{cases} \top & \widehat{V}(\varphi) \in F \\ \bot & \widehat{V}(\varphi) \notin F; \end{cases}$
- $\widehat{V}(\forall p\varphi) = \bigwedge\{\widehat{V[p \mapsto b]}(\varphi) \mid b \in B\}$.

A formula $\phi \in \mathcal{L}\Pi$ is *valid* on a proper filter algebra $\langle B, F\rangle$, written as $\langle B, F\rangle \vDash \phi$, if for all valuations V on B, $\widehat{V}(\phi) = \top$.

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Algebraic semantics: the belief case

Let B be a complete Boolean algebra and F a proper filter on B. We call $\langle B, F\rangle$ a complete proper filter algebra. Given a valuation $V : \mathrm{Prop} \to B$, it can be extended to an $\mathcal{L}\Pi$-*valuation* $\widehat{V} : \mathcal{L}\Pi \to B$ by setting

- $\widehat{V}(\Box\varphi) = \begin{cases} \top & \widehat{V}(\varphi) \in F \\ \bot & \widehat{V}(\varphi) \notin F; \end{cases}$
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Interpretation of $\langle B, F\rangle$:

- B is not (always) the algebra of all propositions. Even if p is believed, $p \vee \neg p$ and $\Box p$ need not express/mean the same proposition though $\widehat{V}(p \vee \neg p)$ and $\widehat{V}(\Box p)$ are the same.
- One way of understanding: B is the algebra of all propositions quotiented by knowledge/veridical certainty. It the algebra of propositions "for" the ideal agent.
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We can use a larger class of algebras: complete pseudo-monadic algebras. We can also use semantics based on possibilities, which are like partial worlds.

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Definition

For any Boolean algebra B and $\Box : B \to B$, $\langle B, \Box \rangle$ is a pseudo-monadic algebra if

- $\Box\top = \top$
- $\Box(a \wedge b) = \Box a \wedge \Box b$
- $\Box a = \Box\Box a$
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It is complete if B is lattice-complete.

Pseudo-monadic algebras generalize proper filter algebras.
Semantics is almost trivial.

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Logic

Definition

Let $\mathsf{KD45}\Pi 4^{\forall}$ be the logic in $\mathcal{L}\Pi$ axiomatized by

K	$\Box(p \to q) \to (\Box p \to \Box q)$	D	$\neg\Box(p \land \neg p)$
4	$\Box p \to \Box\Box p$	5	$\neg\Box p \to \Box\neg\Box p$
Nec	$\varphi/\Box\varphi$	MP	$\varphi, \varphi \to \psi/\psi;$
Dist	$\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$		
Inst	$\forall p\varphi(p) \to \varphi(\psi)$		
Tri	$\varphi \to \forall p\varphi$ when p is not free		
Gen	$\varphi/\forall p\varphi;$		
$4^{\forall}$	$\forall p\Box\varphi \to \Box\forall p\Box\varphi.$		

Theorem

- $\mathrm{KD45\Pi4}^{\forall}$ *is sound with respect to all complete pseudo-monadic algebras.*
- $\mathrm{KD45\Pi4}^{\forall}$ *has a conservative extension with "almost" quantifier elimination.*
- $\mathrm{KD45\Pi4}^{\forall}$ *is complete with respect to all complete proper filter algebras.*

Corollary

$\mathrm{KD45\Pi4}^{\forall}$ *is the logic of all complete proper filter algebras. It is also the logic of all complete pseudo-monadic algebras.*

Characterization

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$\mathsf{KD45\Pi4}^{\forall}$ *is the logic of all complete proper filter algebras. It is also the logic of all complete pseudo-monadic algebras.*

Proof strategy

Let z abbreviate $\forall p(\Box p \to p)$.

Lemma

On any complete proper filter algebra $\langle B, F\rangle$, z evaluates to the meet of F (which we also call z).

Then we can split $\langle B, F\rangle$ into the direct product of two parts: $\langle B|_z, F|_z\rangle$ and $\langle B|_{\neg z}, F|_{\neg z}\rangle$.

- The modality in $\langle B|_z, F|_z\rangle$ is S5.
- $\langle B|_{\neg z}, F|_{\neg z}\rangle$ has a special property: the meet of $F|_{\neg z}$ is the bottom element.

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Syntactically, if we assume $\forall p(\Box p \to p)$, $\Box$ becomes an S5 modality. We know how to deal with this (by quantifier elimination).

If we assume $\neg\forall p(\Box p \to p)$, we can show that we are in fact talking about the quotient $B_{\neg z}/F_{\neg z}$ in a first-order way (by translation).

Theorem

The first-order logic of the non-trivial quotients of complete Boolean algebras is just the first-order logic of all non-trivial Boolean algebras.

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Theorem (Vermeer 1996)

Every $(2^{\omega_0})^+$-field of sets is a quotient of a complete Boolean algebra.

Theorem (Tarski)

Two Boolean algebras are elementarily equivalent iff they have the same invariant.

But for each invariant, there is a $(2^{\omega_0})^+$-field of sets having that invariant.

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Stronger logics

Theorem

- *For the class of complete proper* **principal** *filter algebras, the logic is* $\mathsf{KD45\Pi4}^{\forall} + \Box\forall p(\Box p \to p)$*. This logic is also* $\mathsf{KD45\Pi} + \forall\Box p\varphi \to \Box\forall p\varphi$*.*
- *For the class of complete and* **atomic** *proper principal filter algebras, the logic is* $\mathsf{KD45\Pi4}^{\forall} + \forall p(\Box p \to p) \to \exists p(p \wedge at(p))$ *where* $at(p)$ *expresses that the valuation of* p*, when restricted to the intersection of the proper filter, is atomic.*
- *For the class of complete* **atomic** *proper* **principal** *filter algebras, the logic is* $\mathsf{KD45\Pi4}^{\forall} + \Box\forall p(\Box p \to p) + \forall p(\Box p \to p) \to \exists p(p \wedge at(p))$*.*

Decidability

Theorem

All the logics mentioned – $\mathsf{KD45\Pi4}^{\forall}$ *and the three extensions – are decidable.*

This uses the decidability of the first-order logic of Boolean algerbas.

Extensions

- More concrete semantics: the logic of credence 1.
- Inconsistent beliefs: a vast literature.
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Thank you!