Convex Analysis

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Outline of this section

- Definition and properties of convex sets and related concepts;
- Applications in optimization, approximation and separation.

Convex Sets

- For any $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y$ is the **line segment** between x and y.
- A set S is **convex** if all line segments of any two elements of S belongs to S:
$$\forall x, y \in S,\ \forall \lambda \in [0,1] : \lambda x + (1-\lambda)y \in S$$
- S is **strictly convex** if $\forall x, y \in S$ and $\forall \lambda \in (0,1)$ we have $\lambda x + (1-\lambda)y \in int(S)$.
- Intersection of any collection of convex sets is convex.
- Union of convex sets may not be convex.

Convex combination

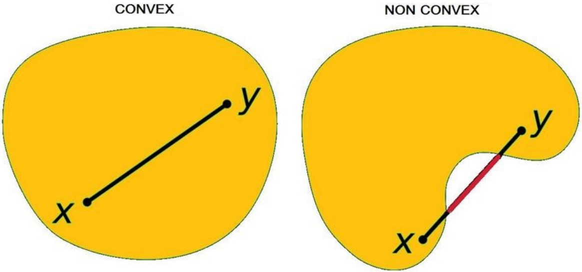
- A convex combination of a set S is a vector

$$s = \lambda_1 s_1 + \lambda_2 s_2 + \ldots + \lambda_n s_n$$

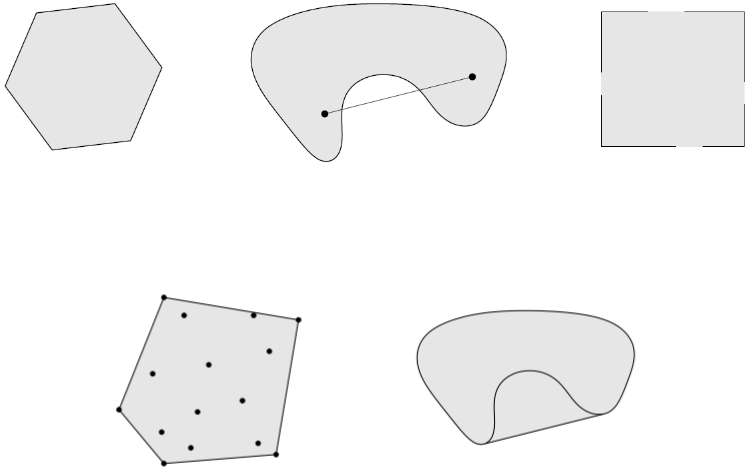
  where $s_i \in S$, $\lambda_i \geq 0$ and $\sum_1^n \lambda = 1$.
- A set S is convex if and only if it contains all convex combinations of S.

Convex Hull

- **Convex hull** of set $S \subset X$ is the set of all convex combinations of all elements in S;
- Convex hull of S is the intersection of all convex sets containing S;
- Obviously, the convex hull of a convex set is the set itself.



Examples of Convex hull



Extreme Points

- x is an extreme point of a convex set $S \subset X$ if it could not be written as a convex combination of any two distinct points in S. That is, whenever there exists $\lambda \in (0, 1)$ and $x_!, x_2 \in S$ such that
$$x = \lambda x_1 + (1 - \lambda)x_2$$
we must have $x_1 = x_2 = x$
- Let $ext\{S\}$ denote the set of extreme points of S. $ext\{S\}$ can be empty.
- If S is a convex and compact subset of X, then $ext\{S\} \neq \emptyset$.

Extreme points and Convex sets

- Let $P_n = \{\alpha \in \mathbb{R}^n \,|\alpha_i \geq 0, \sum \alpha_i = 1\}$
- **Caratheodory's Theorem:** For any set $S \in \mathbb{R}^n$,$conv(S) = \{\sum_{i=1}^{n+1} \alpha_i x_i | x_i \in S, \forall i, \alpha \in P_{n+1}\}$.
- **Krein-Milman Theorem** Let X be a normed linear space and S be a compact and convex subset of X, then S is the closed convex hull of its extreme points.
- These two theorems imply that any points in a convex and compact subset in $\mathbb{R}^n$ can be written as a convex combination of at most (n+1) of its extreme points.

Extreme points and optimization

- Extreme points can be useful for characterizing the solutions for some optimization problems.
- **Bauer Maximization Principle:** Let X be a normed linear space and $S \subset X$ be a non-empty,convex and compact subset. Suppose that $f : S \to \mathbb{R}$ is upper semicontinuous and convex. Then:

$$\arg\max_{x \in S} f(x) \cap \text{ext}(S) \neq \emptyset$$

Approximation in Hilbert spaces

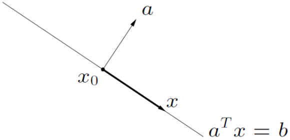
- Let X be a Hilbert space and M be a nonempty, closed and convex subset of X. For any $x \in X$ there exists a unique $y \in M$ such that $||x - y|| \leq ||x - y'||$ for all $y' \in M$.
- Furthermore, if M is a closed subspace of X, a necessary and sufficient condition for $y \in M$ to be the unique minimizing vector is that for all $y' \in M$, $x - y$ and y' be orthogonal, that is $(x - y) \cdot y' = 0$.

Affine manifold and hyperplane

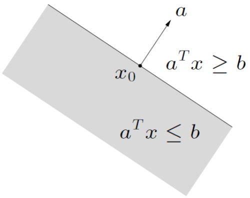
- A subset of Y of the vector space X is called an **affine manifold** if $Y = Z + x^*$ for some $x^* \in X$ and Z is a subspace of X.
- Y is called a **hyperplane** if Z is a $\supseteq$-maximal proper linear subspace of X.
- A **hyperplane** in $\mathbb{R}^n$ is characterized by a vector $p \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$ such that $H(p, \alpha) = \{x \in \mathbb{R}^n \,|x \cdot p = \alpha\}$.
    - It is a set constructed by shifting an $(n-1)$-dimensional subspace of $\mathbb{R}^n$.

Hyperplanes and Halfspaces on $\mathbb{R}^n$

- **Hyperplane**: $\{x| < a, x >= b\}$ $(a \neq 0)$ $\rightarrow$ affine and convex

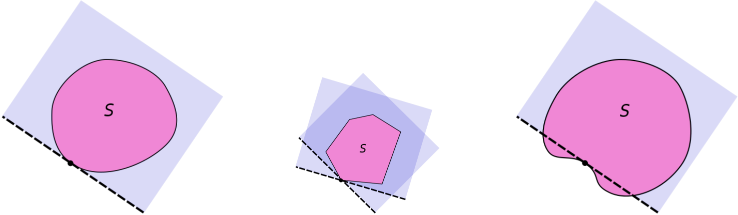


- **Halfspace**: $\{x| < a, x > \leq b\}$ $(a \neq 0)$ $\rightarrow$ convex



Supporting Hyperplane Theorem

- **Supporting Hyperplane Theorem:** Let $S \subseteq \mathbb{R}^n$ be a convex set and $\text{int}S \neq \emptyset$. Then for any $x \in bd(S)$, there exists a hyperplane $H(p, \alpha)$ such that $x \in H(p, \alpha)$ and $\forall y \in S$, $\langle y, p \rangle \geq \alpha$.



Separating hyperplane theorem

- **Separating Hyperplane Theorem:** Let $S, T \subseteq \mathbb{R}^n$ be disjoint nonempty convex sets. There exists a hyperplane $H(p, \alpha)$ such that $\forall x \in S$ and $\forall y \in T$, $\langle x, p \rangle \geq \alpha \geq \langle y, p \rangle$.