Linear Algebra(II)

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Linear Operators

- A **linear operator** is a function $T : V \to W$ between vector spaces V and W such that $\forall x, y \in V$ and $\forall \alpha \in \mathbb{R}$: $T(\alpha x + y) = \alpha T(x) + T(y)$.
- The set of all linear operators from V to W is denoted by $\mathcal{L}(V, W)$.
- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ between Euclidean spaces can only) be represented by an $m \times n$ matrix A:

$$T(x) = y \quad \Leftrightarrow \quad \forall i,\ y_i = \sum_{j=1}^{n} A_{ij} x_j \Leftrightarrow y = Ax$$

Null space

- Let $\mathcal{L}(V, W)$ be the set of linear operators from V to W. The null space of $T \in (T)$, is the subset of V consisting of vectors that T maps to zero:

$$\operatorname{null} T := \{u \in V : Tu = 0\}$$

- $\operatorname{null} T$ is a subspace of V.
- $\mathbf{0} \in \operatorname{null} T$.
- T is injective if and only if $\operatorname{null} T = \{\mathbf{0}\}$
- The **image** or range of T is given by $T(V) = \{T(x) \,|x \in V\}$.
- $T(V)$ is a subspace of W.

Fundamental theorem of Linear operators

- Suppose V is finite dimensional and $T \in \mathcal{L}(V, W)$, then

$$\dim(\text{null}(T)) + \dim(T(V)) = \dim(V)$$

Implications:

- A map to a smaller dimensional space is not injective;
- A map to a larger dimensional space is not surjective.

Matrices

- An $m \times n$ **matrix** is a table of scalars. It can be interpreted as a stack of m row vectors or n column vectors.
- An example of a 2×3 matrix:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 8.1 \end{bmatrix}$$

  - Matrix A has 2 row vectors: $(2, 3, 4)$ and $(1, 5, 8.1)$.
  - Matrix A has 3 column vectors: $(2, 1)$, $(3, 5)$, and $(4, 8.1)$.
- A_{ij} denotes the entry at i-th row and j-th column.
- If the number of the rows is equal to the number of the columns, then the matrix is called a square matrix.

Basic Matrix Operations

- Let A and B be matrices with same dimensions. $A + B$ is also a matrix where $(A + B)_{ij} = A_{ij} + B_{ij}$.
- Let A be a matrix and $\alpha \in \mathbb{R}$ be a scalar. αA is also a matrix where $(\alpha A)_{ij} = \alpha A_{ij}$.
- Let A be a matrix with dimension $m \times n$ and B be a matrix with dimension $n \times k$. The matrix multiplication of A and B is also a matrix where $(AB)_{ij} = \langle A_{i\cdot}, B_{\cdot j} \rangle$.
    - $A_{i\cdot}$ denotes the i-the row vector of A and $B_{\cdot j}$ denotes the j-th column vector of B.
- Note that $AB \neq BA$ in general.

Identity Matrix and Symmetric Matrices

- **Transpose** A' of matrix A is defined such that $A'_{ij} = A_{ji}$.
- If $A = A'$, then A is a **symmetric matrix**.
- n-dimensional **identity matrix** I is defined such that $I_{ij} = 1$ if $i = j$ and $I_{ij} = 0$ otherwise.
- Let A be a square matrix: $AI = IA = A$.
- **Inverse** of a square matrix A is denoted by A^{-1} and defined such that $AA^{-1} = A^{-1}A = I$.
  - A^{-1} is unique if it exists but it may not exist.

Rank of a Matrix

- **Row(column) rank** of a matrix is the dimension of the span of row (column) vectors of the matrix.
  - If the column(row) vectors of a matrix are linearly independent, column(row) rank of the matrix is equal to the number of columns(rows).
  - Row rank = Column rank
- A matrix is called **full rank** if all its rows or all its columns are linearly independent.
- A square matrix is called **singular** if it is not full rank.

Determinant

- Determining the rank of a matrix can be burdensome when the size of the matrix is large.
- Determinant is a useful tool to determine if a square matrix is full rank or not and to solve linear systems of equations.
- Let A be an $n \times n$ matrix. $|A|$ denotes the determinant of the matrix A. When $n = 1$, we have that $|A| = A_{11}$.
- Let A^{ij} denote the ij-th **principal** sub-matrix which is the matrix generated by deleting i-th row and j-th column of A.
- Given the principal sub-matrices, the **determinant** of A can be calculated iteratively:

$$|A| = \sum_{i=1}^{n}(-1)^{i+j}A_{ij}|A^{ij}|,\ \forall j \text{ OR } |A| = \sum_{j=1}^{n}(-1)^{i+j}A_{ij}|A^{ij}|,\ \forall i$$

Determinant

- For a square matrix A, if $|A| = 0$, then A is called **singular**.
  - For an $n \times n$ matrix, if $|A| \neq 0$, then A is full rank: $\text{rank}(A) = n$.
- If a square matrix A is non-singular (full rank), then A^{-1} exists such that: $AA^{-1} = A^{-1}A = I$.
- Some properties of determinant:
  - $|I| = 1$
  - $|A'| = |A|$
  - $|A^{-1}| = \frac{1}{|A|}$
  - $|AB| = |A||B|$
  - $|\alpha A| = \alpha^n |A|$

Trace

- **Trace** of a square matrix A is simply the sum of its diagonal entries: $\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$.
- Determinant of A can be computed by multiplying its eigenvalues: $|A| = \prod_{i=1}^{n} \lambda_i$.
- Trace of A can be computed by summing its eigenvalues: $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

Eigenvectors

- **Eigenvalues** of a square matrix A are the values of λ which solve the equation $Ax = \lambda x$ for some nonzero vector x.
- Eigenvalues of A can be found directly by solving the equation $|A - \lambda I| = 0$.
- For each eigenvalue λ, any vector x satisfying $Ax = \lambda x$ is called the **eigenvector** associated with eigenvalue λ.

Positive Definiteness

- Let A be an $n \times n$ square matrix.
- A is said to be **positive semi-definite** if $\forall x \in \mathbb{R}^n$, $x'Ax \geq 0$.
- A is said to be **positive definite** if $\forall x \in \mathbb{R}^n$ such that $x \neq 0$, $x'Ax > 0$.
- A is said to be **negative (semi-)definite** if $-A$ is positive (semi-)definite.
- A is said to be **negative semi-definite** if $\forall x \in \mathbb{R}^n$, $x'Ax \leq 0$.
- A is said to be **negative definite** if $\forall x \in \mathbb{R}^n$ such that $x \neq 0$, $x'Ax < 0$