Real Analysis

Shuoqi Sun

Tepper School of Business, CMU

Plan

- Preliminaries;
- Metric spaces and its properties;
- Basic Functional Analysis
- Application: Theorem of Maximum, Fixed point theory

Sets

- A **set** is a collection of objects. $A = \{x : P(x)\}$
    - Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
    - Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
    - Rational numbers: $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$
    - Real numbers: $\mathbb{R}$

- Basic Concepts
    - **Cardinality:** $|A| =$ number of elements of A
    - **Empty Set:** $\emptyset$
    - **Subset:** $A \subseteq B$ if $x \in A \Rightarrow x \in B$
    - **Equality:** $A = B$ if $A \subseteq B$ and $B \subseteq A$
    - **Proper Subset:** $A \subsetneq B$ if $x \in A \Rightarrow x \in B$ and $A \neq B$
    - **Power Set:** $2^A = \{T : T \subseteq A\}$

Set Operations

- **Union:** $A \cup B = \{x | x \in A \text{ or } x \in B\}$.
- **Intersection:** $A \cap B = \{x | x \in A \text{ and } x \in B\}$.
  - **Disjoint Sets:** $A \cap B = \emptyset$
- **Complement:** $A^c = \{x | x \notin A\}$.
- **Difference:** $B \setminus A = \{x | x \in B \text{ and } x \notin A\}$.

Properties of Set Operations

- Union and intersection operators are commutative, associative and distributive.
  - **Commutative:** $A \cup B = B \cup A$, $A \cap B = B \cap A$
  - **Associative:** $(A \cup B) \cup C = A \cup (B \cup C)$
  - **Associative:** $(A \cap B) \cap C = A \cap (B \cap C)$
  - **Distributive:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - **Distributive:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - **De Morgan's Law:** $(A \cap B)^c = A^c \cup B^c$
  - **De Morgan's Law:** $(A \cup B)^c = A^c \cap B^c$
    - $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
    - $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
  - $A \subseteq \Rightarrow |A| \leq |B|$

Binary relations

- **Cartesian Product:** $A \times B = \{(x, y) | x \in A, y \in B\}$.
  - $A = \{1, 2, 3\},\ B = \{4, 5\}$
  - $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$
- A subset R of $A \times B$ is called a binary relation from A to B;if $A = B$, then we say R is a relation on A;
- A relation R on a nonempty set A is
  - **reflexive** if xRx for all $x \in A$;
  - **complete** if either xRy or yRx holds for each x, y;
  - **symmetric** if for any x, y, xRy implies yRx;
  - **antisymmetric** if for any $x, y \in A$, xRy and yRx imply $x = y$;
  - **transitive** if xRy and yRz implies xRz.

Order relations

- Transitivity is the defining feature of any order relation;
- **preorder** a relation that is both transitive and reflexive;
- **partial order**: an antisymmetric preorder;
- **Equivalence**: a symmetric preorder.
- **linear/total order**: a partial order that is complete.

The greatest and the least

Let A be a subset of a partially ordered set("poset") P with binary relation $\leq$.

- The maximum element of the set A: $\max(A)$
  - $\max(A) \in A$ and for each $x \in A$, $\max(A) \geq x$.
- The minimum element of the set A: $\min(A)$
  - $\min(A) \in A$ and for each $x \in A$, $\min(A) \leq x$.
- The **supremum** of A is the least upper bound.
  - For each $x \in A$, $\sup(A) \geq x$ and for each $y \in P$ with the same property, we have $\sup(A) \leq y$.
- The **infimum** of A is the largest lower bound.
  - For each $x \in A$, $\inf(A) \leq x$ and for each $y \in P$ with the same property, we have $\inf(A) \geq y$.

Real numbers

- $\mathbb{R}$ is a **complete** ordered field($\mathbb{Q}$ is not!);
- **The Completeness Axiom:** Every non empty subset of $\mathbb{R}$ that is bounded from above has a supremum in $\mathbb{R}$.
- **The Archimedean property**: For any $(a, b) \in \mathbb{R}_{++} \times \mathbb{R}$, there exists an $m \in \mathbb{N}$ such that $b < ma$.
  - One can find a natural number larger than any real number.
- For any $a, b \in \mathbb{R}$ such that $a < b$ there exists a $q \in \mathbb{Q}$ such that $a < q < b$.

Exercises

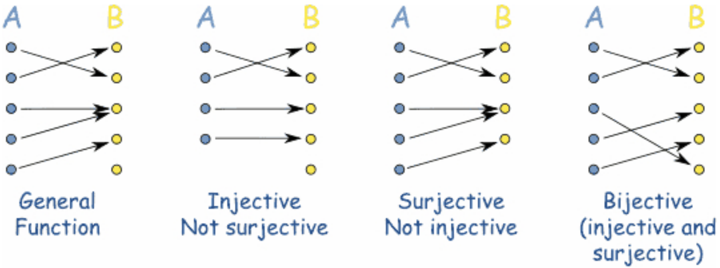
1. Let $A, B \subseteq \mathbb{R}$ be sets that are bounded from above. Show that
   - $A \subseteq B$ implies $\sup(A) \leq \sup(B)$
   - $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$
   - $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$
   - $\sup(\{a + b | (a, b) \in A \times B\}) = sup(A) + \sup(B)$
2. Show that
$$\sup\{q \in \mathbb{Q} : q^2 < 1\} = 1$$

Functions

- Given two sets, a **function** maps each member of one to a member of the other. $f : A \to B$
- **Domain** of f: A
- **Co-domain** of f: B
- **Range** of f: $f(A) = \{y \in B : \exists x \in A, f(x) = y\}$
- **Image** of $C \subseteq A$: $f(C) = \{y \in B : \exists x \in C, f(x) = y\}$
- **Preimage** or **inverse image** of $y \in B$ under f: $f^{-1}(y) = \{x \in A : f(x) = y\}$
- Inverse image of $D \subseteq B$: $f^{-1}(D) = \{x \in A : f(x) \in D\}$

Injections, Surjections, and Bijections

- If for all $x_1, x_2 \in A$, $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$, then f is said to be **one-to-one** or **injective**.
- If for every $y \in B$, there exists $x \in A$ such that $f(x) = y$ then f is said to be **onto** or **surjective**.
- If f is both injective and surjective, then f is **bijective**.
- If f is bijective, then $f^{-1} : B \to A$ is also a bijective function.



Exercises

- $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
- $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ and if f is injective, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$
- $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
- $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$
- $A_1 \subseteq A_2 \;\Rightarrow\; f(A_1) \subseteq f(A_2)$
- $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$
- $f(f^{-1}(B)) \subseteq B$
- $A \subseteq f^{-1}(f(A))$

Finite and infinite Sets

- Two sets A and B have equal cardinality if there is a bijection between A and B.
- An initial segment of $\mathbb{N}$ is the set $P_n = \{i \in \mathbb{N} : i \leq n\}$.
- A set A is **finite** if it is empty or there exists a bijection $f : A \rightarrow P_n$ for some $n \in \mathbb{N}$.
- Let B be a proper subset of a finite set A. There does not exist a bijection $f : A \rightarrow B$.
    - $\mathbb{N}$ is not finite.
- A set A is **infinite** if it is not finite. It is **countably infinite** if there exists a bijection $f : \mathbb{N} \rightarrow A$.
- A set is countable if it is finite or countably infinite. A set that is not countable is **uncountable**.

Real sequences

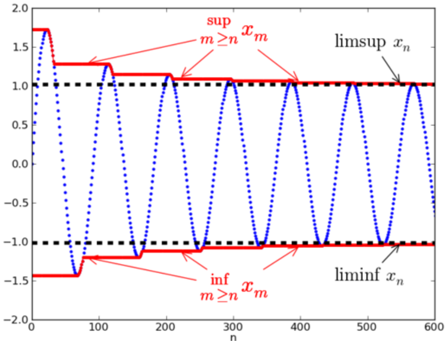
- A real **sequence** $\{x_n\}$ on X is a function from $\mathbb{N}$ to $\mathbb{R}$.
- $\{x_n\}$ **converges** to x if $\forall \varepsilon > 0,\ \exists n \in \mathbb{N}$ such that $\forall k \geq n,\ |x_k - x| < \varepsilon$.
  - If so, x is called the **limit** of the sequence: $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.
- If $\{x_n\}$ has a limit, then it is said to be **convergent**. If a sequence is not convergent, it is said to be **divergent**.
- Given a sequence $\{n_k\}$ on $\mathbb{N}$, $\{x_{n_k}\}$ is a sub-sequence of $\{x_n\}$.
- $\{x_n\}$ is **bounded from above(bounded from below)** if there exists $K \in \mathbb{R}$ such that $\forall n = 1, 2, 3, ..., x_n \leq (\geq) K$.

Bolzano-Weierstrass Theorem

- Every monotonic sequence that is bounded above or below converges;
- Every real sequence has a monotonic subsequence;
- These two lemma leads to **Bolzano-Weierstrass Theorem**: In finite dimensional Euclidean spaces ($\mathbb{R}^n$), every bounded sequence has a convergent sub-sequence.

Subsequential limits

- Let $\{x_n\}$ be a sequence of real numbers. We write $x = \limsup x_m$ if
  - For any $\epsilon > 0$, there exists M such that for all $m \geq M$, $x_m < x + \epsilon$
  - For any $\epsilon > 0$ and $M \in \mathbb{N}$, there exists a $k > M$ such that $x_k > x - \epsilon$.



Useful facts about limsup and liminf

Let $\{x_n\}$ be a sequence of real numbers. Then

- $$\limsup x_m = \inf(\sup\{x_n, x_{n+1}, ...|n = 1, 2, 3, ...\})$$
- $\limsup x_m$ is the greatest subsequential limit of (x_m)
- $$\liminf_{k\to\infty} x_k \leq \limsup_{k\to\infty} x_k$$
- $\lim_{n\to\infty} x_n = x \in \mathbb{R} \cup \{\pm\infty\}$ if and only if

$$\liminf_{k\to\infty} x_k = \limsup_{k\to\infty} x_k = x$$

Metric Spaces

- Given a set X, a **metric** on X is a function $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$:
  - $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$
  - Symmetry: $d(x, y) = d(y, x)$
  - Triangle inequality: $d(x, y) \leq d(x, z) + d(y, z)$
- If d is a metric on X, we say (X, d) is a metric space.
- Euclidean spaces can be described as metric spaces where $X = \mathbb{R}^n$ and $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$(known as Euclidean metric).

Metrizing $\mathbb{R}^n$

- $(\mathbb{R}^n, d_p)$ is a metric space for each $1 \leq \infty$, where $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is defined by

$$d_p(x, y) := (\sum_{1=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}}, \forall \leq p < \infty$$

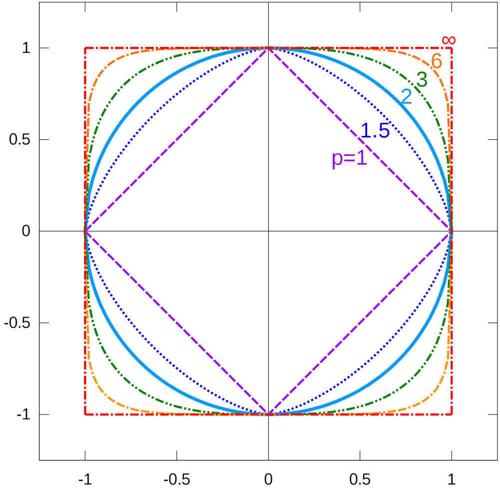
$$d_\infty(x, y) := \max\{|x_i - y_i| : 1 = 1, ..., n\}$$

- **(Minkowski's Inequality):** For any $n \in \mathbb{N}$, $a_i, b_i \in \mathbb{R}$, $i = 1, \cdots, n$ and any $1 \leq p < \infty$:

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}$$

- We call $\mathbb{R}^n$ endowed with d_2 metric **n-dimensional Euclidean space**.

d_p metric on $\mathbb{R}^2$



l_p and L_p Spaces for $p \in [1, \infty)$

- $l_p = \{\{x_m\} \in \mathbb{R}^\infty \mid \sum_{m=1}^{\infty} |x_m|^p < \infty\}$
- $d_p(\{x_m\}, \{y_m\}) = (\sum_{m=1}^{\infty} |x_m - y_m|^p)^{\frac{1}{p}}$

- $l_\infty = \{\{x_m\} \in \mathbb{R}^\infty \mid \sup |x_m| < \infty\}$
- $d_\infty(\{x_m\}, \{y_m\}) = \sup\{|x_m - y_m| | m \in \mathbb{N}\}$

- $L_p = \{f \in \mathbb{R}^T \mid \int_T |f|^p < \infty\}$
- $d_p(f, g) = \left(\int_T |f - g|^p\right)^{\frac{1}{p}}$

- $L_\infty = \{f \in \mathbb{R}^T \mid \sup |f| < \infty\}$
- $d_\infty(f, g) = \sup |f - g|$

Examples

- $D_\infty : C^1[a,b] \times C^1[a,b] \to \mathbb{R}_+$

$$: D_\infty(f,g) := d_\infty(f,g) + d_\infty(f',g')$$

- Discrete metric. $d : \mathbb{R} \times \mathbb{R} \to \{0,1\}$

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

- Kullback-Leiber distance: Let f and g be two probability density functions. $D(f||g) := \int_{\mathbb{R}} log(\frac{f(x)}{g(x)})f(x)dx$

Open and Closed Sets

- An **open ball** centered at $x \in X$ with radius $\varepsilon > 0$:

$$B_\varepsilon(x) = \{y \in X | d(y, x) < \varepsilon\}$$

- $S \subset X$ is an **open set** if $\forall x \in S$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S$.
- $S \subset X$ is a **closed set** if $X \setminus S$ is an open set.
- S is **bounded** if there exists an open ball $B_\varepsilon(x)$ for some $x \in X$ such that $S \subset B_\varepsilon(x)$.

Useful properties

- Union of open sets is open;
- Intersection of closed sets is closed;
- Intersection of finite number of open sets is open.
- Union of finite number of closed sets is closed;

Interior, closure and boundary

- $x \in S$ is an **interior point** if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S$.
  - The set of all interior points of S is called the **interior** of S: $int(S)$.
- The smallest closed set that contains S is called the **closure** of S (relative to X)
- $x \in X$ is a **boundary point** of S if for every $\varepsilon > 0$, $B_\varepsilon(x)$ contains points in S and in $X \setminus S$.
  - The set of all boundary points of S is called the **boundary** of S: $bd(S) = cl(S) \setminus int(S)$.

Exercise

Convince yourself the following statements are true:

- A set can be both closed and open.
- A set can be neither closed nor open.
- A set $S \subset Y \subset X$ may be open(closed) in metric space (Y, d) but not opened(closed) in (X, d) and/or (Y, d').
- Any subset of a discrete metric space is both open and closed.

Sequential characterization of a closed set

- Recall that a sequence $\{x_m\}$ is a function from $\mathbb{N}$ to metric space (X, d);
- $\{x_n\}$ **converges** to x if $\forall \varepsilon > 0,\ \exists n \in \mathbb{N}$ such that $\forall k \geq n,\ |x_k, x| < \varepsilon$.
- Let S be a subset of the metric space X. S is closed if and only if every convergent sequence in S converges to a point in S.

Compactness

- An **open covering** $\mathcal{O} = \{O_n\}$ of set S is a collection of open sets such that $S \subset \bigcup_{O_n \in \mathcal{O}} O_n$.
- S is **compact** if every open covering $\mathcal{O}$ of set S has a finite subcover $\{O_{n_k}\}_{k=1}^{K}$: $S \subset \bigcup_{k=1}^{K} O_{n_k}$.
- A subset S of the metric space (X, d) is **sequentially compact** if every sequence in S has a convergent sub-sequence which converges in S.
- For metric spaces, compactness=sequential compactness.
- **Every compact set is closed and bounded.**
  - The converse is not always true.(counter example?)
  - **The Heine-Borel Theorem:** In finite-dimensional Euclidean spaces ($\mathbb{R}^n$), every closed and bounded set is compact.

Cauchy Sequences and Completeness

- A sequence $\{x_n\}$ is a **Cauchy sequence** if $\forall \varepsilon > 0,\ \exists n \in \mathbb{N}$ such that $\forall k, l \geq n,\ d(x_k, x_l) < \varepsilon$.
  - If $\{x_n\}$ is convergent, then it is Cauchy.
- A metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges to a point in X.
  - Any closed subset of a complete metric space is complete.
- The set of rational numbers, $\mathbb{Q}$, is not complete.
  - Consider $x_n = \sum_{i=1}^{n} \frac{1}{i!}$. For every n, $x_n \in \mathbb{Q}$ and $\{x_n\}$ is a Cauchy sequence but $x_n \to e \notin \mathbb{Q}$.
  - $\mathbb{R}$ can be defined as the completion of $\mathbb{Q}$.

Topological spaces

- Topological space is a generalization of metric space.
- Let X be a nonempty set, a collection of subsets of X, $\mathcal{T}$, is a topology if:
  - $X, \phi \in \mathcal{T}$;
  - For any $\mathcal{F} \subseteq \mathcal{T}, \cup_{O \in \mathcal{F}} O \in \mathcal{T}$
  - For any $O_k{}_{k=1}^{n} \subset \mathcal{T}$, $\cap_{k=1}^{n} O_k \in \mathcal{T}$
- Given a topology $\mathcal{T}$ on X, we say that $(X, \mathcal{T})$ is a topological space.
- Any metric space is a topological space, where the topology is the collection of open sets.

Fixed points and contraction mappings

- $x \in X$ is a **fixed point** of the function $f : X \to X$ if $f(x) = x$.
- $x \in X$ is a fixed point of the correspondence $\Gamma : X \rightrightarrows X$ if $x \in \Gamma(x)$.
- Function $f : X \to X$ is a **contraction mapping** with modulus $\delta \in [0, 1)$ if $\forall x, y \in X$, $d(f(x), f(y)) \leq \delta d(x, y)$.

Checking contraction

(Blackwell's Sufficiency lemma) Let $X \subseteq \mathbb{R}^K$ and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$ with the sup metric, d_∞.
Let $T : B(X) \to B(X)$ be an operator satisfying:

- (monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.
- (discounting) there exists some $\beta \in (0, 1)$ such that: $[T(f + a)](x) \leq (Tf)(x) + \beta a$, for all $f \in B(X)$, $a \geq 0, x \in X$.

Then T is a contraction.

Contraction Mapping Theorem

Banach Fixed Point Theorem (Contraction Mapping Theorem): Let $f : X \to X$ be a contraction and X be **complete**. Then f has a unique fixed point x^*. Furthermore, for any $x_0 \in X$, the sequence $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$, $\ldots$ converges to x^*.

Continuity of functions

Let (X, d_X) and (Y, d_Y) be two metric spaces. Then the following four statements are equivalent:

- A function $f : X \to Y$ is **continuous** at $x \in X$.
- $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$ (equivalently, $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$)
- For any $x \in X$ and $\{x_m\}$, $x_n \to x$ implies $f(x_n) \to f(y)$.
- The inverse image of every open set is open(also the definition of continuity on general topological spaces).

Uniform Continuity and Lipschitz Continuity

- A function $f : X \to Y$ is **uniformly continuous** if $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)),\ \forall x \in X$.
- Example: $f : (0, 1] \to \mathbb{R}, f(x) = \frac{1}{x}$. f is continuous but not uniformly continuous.
- A function $f : X \to Y$ is **Lipschitz continuous** if there exists a real number $K \in \mathbb{R}+$ such that

$$d(f(x), f(y)) \leq Kd(x, y)$$

Extreme Value Theorem

- Let $f : X \to Y$ be continuous. If $S \subseteq X$ is compact, then $f(S) \subseteq Y$ is also compact.
- **Extreme Value Theorem(Weierstrass)** Let $f : X \to \mathbb{R}$ be a continuous function and X be a compact set. Then $\exists x \in X$ such that $f(x) \geq f(x')$ for all $x' \in X$. Similarly, $\exists y \in X$ such that $f(y) \leq f(x')$ for all $x' \in X$.a

Semi-continuity

- Continuity of functions have no jumps, whereas semi-continuous functions can only jump in one direction.
- A function $f : X \to Y$ is **upper semi-continuous** at $x \in X$ if for every $\epsilon > 0$ there exists an open ball U of x such that $\forall x' \in U,\ f(x') < f(x) + \epsilon$.
- A function $f : X \to Y$ is **lower semi-continuous** at $x \in X$ if for every $\epsilon > 0$ there exists an open ball U of x such that $\forall x' \in U,\ f(x') > f(x) + \epsilon$.
- A function is continuous iff it is both upper and lower semi-continuous.
- Extreme value theorem also holds for semi-continuous functions.

Brouwer's Fixed Point Theorems

- **Brouwer's Fixed Point Theorem (for single dimension):** $f : [a, b] \to [a, b]$ has a fixed point if f is continuous.
- **Brouwer's Fixed Point Theorem:** Let $S \subseteq \mathbb{R}^n$ be a nonempty, compact and convex subset of $\mathbb{R}^n$. If $f : S \to S$ is continuous, then f has a fixed point.

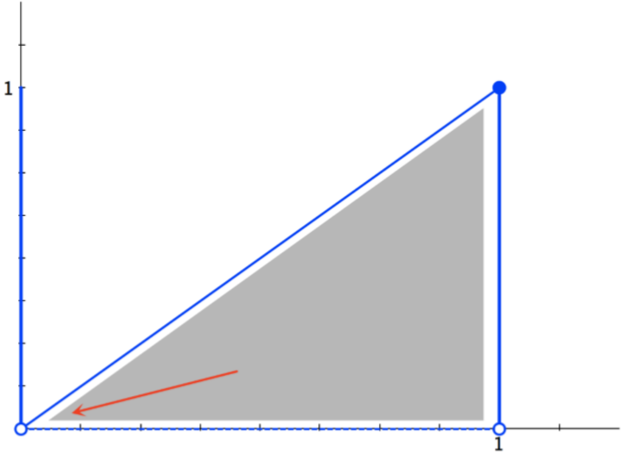
Correspondences

- A **correspondence** is simply a set-valued mapping $\Gamma : X \rightrightarrows Y$.
- Image of $S \subseteq X$ is defined as $\Gamma(S) = \bigcup_{x \in S} \Gamma(x)$.
- $\Gamma : X \rightrightarrows Y$ is said to be compact-valued (closed-valued) [convex-valued] if for every $S \subseteq X$, $\Gamma(S)$ is compact (closed) [convex].
- $\Gamma : X \rightrightarrows Y$ is said to have a closed graph if the set $\{(x, y) \in X \times Y : y \in \Gamma(x)\}$ is closed, that is for any (x_m, y_m) with $y_m \in \Gamma(x_m)$ and $(x_m, y_m) \to (x, y)$, we have $y \in \Gamma(x)$.

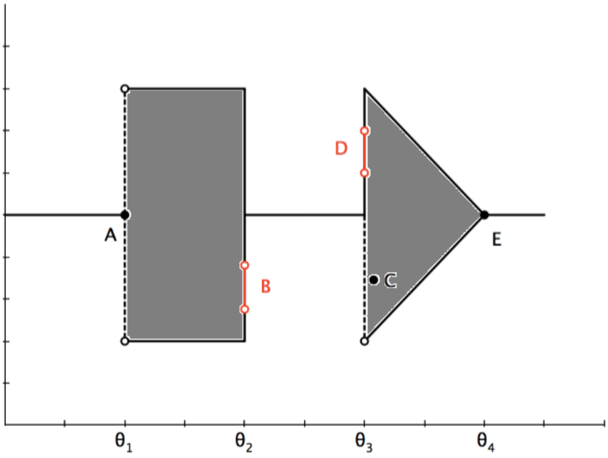
Hemicontinuity

- $\Gamma : X \rightrightarrows Y$ is **lower hemicontinuous** at $x \in X$ if for every open subset O of Y with $\Gamma(x) \cap O \neq \emptyset$, there exists some $\delta > 0$ with $\Gamma(x') \cap O \neq \emptyset$ for each $x' \in B_\delta(X)$.
- Γ is lower hemicontinuous at $x \in X$ if and only if for any $\{x_m\} \in X^\infty$ with $x_m \to x$ and any $y \in \Gamma(x)$, there exists a sequence $\{y_m\} \in Y^\infty$ with $y_m \to y$ and $y_m \in \Gamma(x_m)$ for each m.
- $\Gamma : X \rightrightarrows Y$ is **upper hemicontinuous** at $x \in X$ if for every open subset O of Y with $\Gamma(x) \subseteq O$, there exists some $\delta > 0$ with $\Gamma(B_\delta(x)) \subseteq O$.
- Γ is upper hemicontinuous at $x \in X$ if for any $\{x_m\} \in X^\infty$ and $\{y_m\} \in Y^\infty$ with $x_m \to x$ and $y_m \in \Gamma(x_m)$ for each m, there exists a sub-sequence of $\{y_m\}$ that converges in $\Gamma(x)$.
    - If Γ is compact-valued, then the converse is also true.

Example



Example



The Maximum Theorem

(The Maximum Theorem) Let $\Gamma : \Theta \rightrightarrows X$ be a compact-valued correspondence and $f : X \times \Theta \to \mathbb{R}$ be a continuous function. Define $\sigma(\theta)$ and $f^*(\theta)$ such that:

$$\sigma(\theta) = \arg\max\{f(x,\theta)|x \in \Gamma(\theta)\}$$
$$f^*(\theta) = \max\{f(x,\theta)|x \in \Gamma(\theta)\}$$

and assume that Γ is continuous. Then:

- σ is compact-valued, upper hemicontinuous and has a closed graph.
- f^* is continuous.

Application: the canonical consumer's problem

Kakutani's Fixed Point Theorem

- **Kakutani's Fixed Point Theorem:** Let $X \subseteq \mathbb{R}^n$ be a nonempty, compact and convex subset of $\mathbb{R}^n$. If $\Gamma : X \rightrightarrows X$ is convex-valued and has a closed graph then Γ has a fixed point.
- Let $\Gamma : X \rightrightarrows Y$ be a correspondence:
  - If Γ has a closed graph and Y is compact then Γ is upper hemicontinuous.
  - If Γ is upper hemicontinuous and closed-valued then Γ has a closed graph.
- Application: the existence of Nash equilibrium