Linear Algebra(I)

Shuoqi Sun

Tepper School of Business, CMU

Plan

The following topics will be covered:

- Linear spaces, Norms and inner product;
- Span, basis and dimensions
- Affinity and hyperplane;
- Linear operators and linear functionals;
- Matrix algebra on $\mathbb{R}^n$.

Vector Spaces

Vector space (over a field F) is a set V, whose elements are called vectors,together with two binary operation rules that satisfy certain axioms:

- A rule, called vector addition, such that $\forall \alpha, \beta, \gamma \in V$:
  - $\alpha + \beta \in V$
  - $\alpha + \beta = \beta + \alpha, \quad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
  - $\exists 0 \in V$ such that $\forall \alpha \in V, \quad 0 + \alpha = \alpha$
  - $\forall \alpha \in V$, $\exists (-\alpha) \in V$ such that $\alpha + (-\alpha) = 0$
- A rule, called scalar multiplication, such that $\forall c, c_1, c_2 \in F$ and $\forall \alpha \in V$:
  - $c \cdot \alpha \in V$
  - $\exists 1 \in F$ such that $\forall \alpha \in V, \quad 1 \cdot \alpha = \alpha$
  - $(c_1 \times c_2) \cdot \alpha = c_1 \cdot (c_2 \cdot \alpha)$, $c \cdot (\alpha + \beta) = c \cdot \alpha + c \cdot \beta$, $(c_1 + c_2) \cdot \alpha = c_1 \cdot \alpha + c_2 \cdot \alpha$

Vector spaces

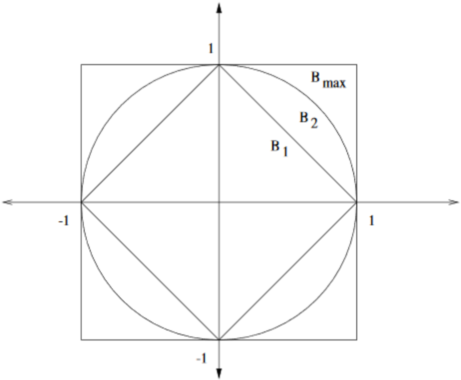
- A **real vector space** is a space where the scalars are real numbers.
- A subset Y of the vector space X is called a **subspace** of X if $\forall x, y \in Y$ we have $x + y \in Y$ and $\forall \alpha \in \mathbb{R}$, $\alpha x \in Y$.

Normed vector spaces

- Let X be a real vector space. **Norm** $||\cdot|| : X \to \mathbb{R}$ is a function such that $\forall x \in X,\ \forall \alpha \in \mathbb{R}$:
  - $||x|| \geq 0$ and $||x|| = 0 \Leftrightarrow x = 0$
  - $||\alpha x|| = |\alpha|\ ||x||$
  - $||x + y|| \leq ||x|| + ||y||$
- A vector space is said to be a normed vector space if it admits a norm.
- Every normed space is also a metric space with the metric $d : X \times X \to \mathbb{R}$ where $d(x, y) = ||x - y||$, but the converse is not always true(counterexample?)
- A complete normed space is called a **Banach Space**.

Euclidean Spaces

- The vector space $\mathbb{R}^n$ with the usual addition and multiplication allows for several norms:
  - $||(x_1, \ldots, x_n)||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \; p \in [1, \infty)$
  - $||(x_1, \ldots, x_n)||_\infty = \max\{|x_1|, \ldots, |x_n|\}$



Inner Product Spaces

- Let X be a real vector space. **Inner product** $< \cdot, \cdot >: X \times X \to \mathbb{R}$ is a function such that $\forall x, y, z \in X, \ \forall \alpha \in \mathbb{R}$:
  - $< x, x > \geq 0$ and $< x, x >= 0 \Leftrightarrow x = 0$
  - Symmetry: $< x, y >=< y, x >$
  - Linearity: $< \alpha x, y >= \alpha < x, y >$ and $< x + y, z >=< x, z > + < y, z >$
- A vector space is said to be an inner product space if it admits an inner product.
- Every inner product space is also a normed space with the norm $\|x\| = \sqrt{< x, x >}$.
- A complete inner product space is called a **Hilbert Space**.
  - Every Hilbert space is a Banach space.

Euclidean Vector Spaces

- In Euclidean vector spaces, the usual inner product (dot product) is defined as:

$$< x, y >= x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

- In Euclidean vector spaces, the usual norm is defined as:

$$||x|| = \sqrt{< x, x >} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

- All Euclidean spaces are Hilbert Spaces with the usual inner product.

Normed and Inner Product Spaces

- **Cauchy Schwartz Inequality:** Let X be an inner product space and $x, y \in X$. If $x = 0$ and $y = 0$ then the inequality holds trivially. Assume $x \neq 0$ or $y \neq 0$:

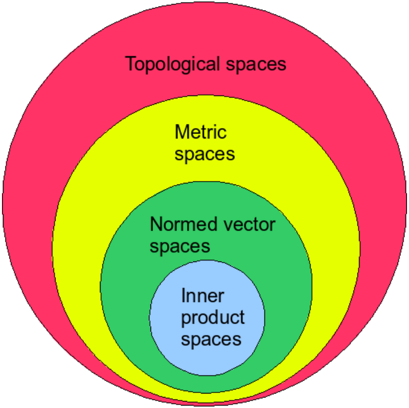
$$| < x, y > | \leq \|x\| \|y\|$$

- A normed vector space X is an inner product space if and only if for every $x, y \in X$:

$$2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2$$

- l_p and L_p are Hilbert space if and only if $p = 2$.

Hierarchy of mathematical spaces



Linear Dependence

- Given a set of vectors $\{x_1, ..., x_k\} \subset X$ and a set of scalars $\{\alpha_1, ..., \alpha_k\}$, a **linear combination** of these vectors is $x = \sum_{i=1}^{k} \alpha_i x_i$.
- A set of vectors $\{x_1, ..., x_k\} \subset X$ is called **linearly dependent** if one of them can be written as a linear combination of the others.
    - There exist numbers $c_1, c_2, \ldots, c_k$, not all equal to zero, such that $\sum_{i=1}^{k} c_i x_i = 0$.
- A set of vectors $\{x_1, ..., x_k\} \subset X$ is called **linearly independent** if none of them can be written as a linear combination of the others.

Span

- Take a set of vectors $V = \{x_1, ..., x_k\} \subset X$. **Span** of V is the set of all linear combinations of V:

$$\text{span}(V) = \{x \in X | \exists \alpha_1, ..., \alpha_k \subset \mathbb{R}, x = \sum_{i=1}^{k} \alpha_i x_i\}$$

- Span of any set of vectors forms a vector subspace.(Proof?)
- The span of V is the smallest linear subspace that contains V.
- If every vector of a vector space X can be written as a linear combination of vectors in set V, we say that V spans X.

Basis and Dimension

- Assume a set of vectors span the subspace Y of the vector space X. If the set of vectors are linearly independent, then it is called a **basis** of the subspace of Y.
- Any basis of a vector space contains the same number of elements.
  - The number of elements of the basis of a vector space is called the **dimension** of the vector space.
- A basis for $\mathbb{R}^3$ is $\{(1,2,0),(0,1,2),(2,0,1)\}$. Therefore dimension of $\mathbb{R}^3$ is 3.

Orthogonal and Orthonormal Basis

- If the inner product of two vectors gives zero, these vectors are said to be **orthogonal**.
  - Elements of $\{(2,0,0),(0,2,0),(0,0,2)\}$ are orthogonal to each other.
- If the vectors of the basis of a vector space are orthogonal, we call it an orthogonal basis.
- If the vectors of the basis of a vector space are orthogonal and the norm of each is 1, we call it an **orthonormal** basis.
  - $\{(1,0,0),(0,1,0),(0,0,1)\}$ is an orthonormal basis of $\mathbb{R}^3$.
- Any set of basis vectors can be converted into an orthonormal basis by the **Gram-Schmidt** process.

Orthogonal Complement

- Given a set of vectors S in the Hilbert space X, $S^{\perp}$ denotes the **orthogonal complement** of S:

$$S^{\perp} = \{x \in X | \forall s \in S, < x, s >= 0\}$$

- Orthogonal complement of any set is a linear subspace.
- Orthogonal complement of k linearly independent vectors in an n-dimensional vector space is of dimension $n - k$.
- If S is a linear subspace, then $S \cap S^{\perp} = \{0\}$

Orthogonal projection

- Let S be a linear subspace of $\mathbb{R}^n$.Then $x \in V$ can be **uniquely** defined as
$$x = s_x + s_x^\perp$$
We call s_x the orthogonal decomposition of x onto S.
- s_x is the closest element in S to x in the sense that:
$$\|s_x - x\| \leq \|s - x\| \forall s \in S$$