The dynamics of relationship and opinion co-evolution on social networks

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Abstract

This paper investigates how opinions and relationships evolve over time on various social networks where agents repeatedly interact with their peers to revise opinion and relationship. We make the key behavioral assumption that depending on how tolerant they are, agents choose to revise either their opinion or friendship with whom they exchange opinion with. Experimenting on four different real-world network datasets, including Facebook ego data, our simulated study shows that global consensus can be reached at a relatively broad range of tolerant levels, and this result is robust to various network structures. However, when tolerant level falls below certain threshold, social segregation emerges, and become increasingly severe as level of tolerance falls down. Our model provides a novel explanation to the persistence of disagreement in social networks, suggesting that belief polarization can arise even if people are well connected in the beginning and information flows freely among individuals.

Keywords: Opinion dynamics, network formation, social learning, agent-based modelling.

1 Introduction

People form opinions about uncertain events in many contexts: are global warming happening? who will win the next presidential election? Although in most cases people are willing to interact with their social peers to exchange their opinions and share the information individually observed, there are long-standing disagreement on many issues, even on matters of fact. As a contradiction, economists' model on social learning almost always predict global consensus in the long run. Behind these theoretical results lies some strong assumptions, one of which is that the connectedness of network structure should be stable over time. However, in reality, people are constantly adjusting their connections with others, in particular, they may choose to break up with a friend if their disagreement on certain issues is irreconcilable. Therefore, it would be interesting to understand the driving force of long-run disagreement through the co-evolution of relationship and belief in social networks.

In this paper, I study a heuristic behavioral rule about how agents in a social network update their beliefs and friendship[1] through communications with their social peers. Following the recent focus of theoretical literature, the proposed model belongs to the case of observational learning, where the state of the world is fixed over time and agents observe private information only once, prior to the onset of interaction. That is, after agents form their initial beliefs, there is no further information beyond their social peers' opinion that may affect an agent's opinion, and agents repeatedly exchange opinions with their peers. On top of this infinite communication procedure, I impose a regulation parameter, namely, tolerance level, which is defined as the cutting off value by which agents choose to revise either their opinion or friendship, after each round of interaction. A detailed model characterization is presented in section 2.1.

The main object of this study is to provide an alternative model in which disagreement, or social groups segregated by opinion disparities can be generated in the long run. While a handful of research has proposed models which can generate disagreement by assuming

[1] Throughout the paper we will interchangeably use "connection", "link" and "friendship".

disconnected network structure, or by introducing false information among which agents choose to believe, my work aims to understand this issue from the view of network evolution.

Broadly speaking, this study is also related to the sociological discussion on the origin of social segregation. It is well-observed that people get segregated in different ways. There is segregation by gender, race, language, income, state, etc. The way segregation by opinion differs from other driven force of segregation is that, unlike those stable, fixed characteristics such as race and gender, people's opinions per se are evolving over time. While previous studies have revealed the surprising fact that large-scale segregation may happen even if people are fairly tolerable towards people with different characteristics, it remains unclear whether the same prediction holds when the driving force is changing over time.

Methodologically, this paper uses Agent-based modeling (ABM) to simulate the model dynamics using NetLogo. ABM is a powerful computer simulation technique in which agents interact according to behavioral rules, resulting in the emergence of complex aggregate-level behavior. This method is particularly suitable for this study because of the dual evolution of opinion and link, which cannot be described by classical Markov chains.

This paper is organized as follows: Section 2 summarizes recent economic work on observational learning, with an emphasis on models that could generate disagreement. A brief discussion of Schelling's segregation model is also provided. Section 3 presents the core model, as well as a characterization of the initial network structures used for simulation. Section 4 described how I implemented this model in NetLogo. Section 5 discusses the main simulation results and its implications on our theoretical hypothesis. Section 6 discusses the main limitations of the model, and points out a few directions for future work.

2 Literature Review

This paper is primarily related to the scholar work on social learning, while it also aims to shed light on the origin of social segregation.

Theoretical literature on social learning takes two approaches: Bayesian and non-Bayesian learning. In the Bayesian model, agents are fully rational and update their opinion by making full inferences according to Bayes rule. This approach fails to model belief updating procedure in realistic manner because the inferences involved could be extremely complex. The non-Bayesian approach, on the other hand, relaxes the assumption to the extent that agents are willing to make inferences using some simplified heuristic rules. The existing economics literature on non-Bayesian learning mainly relies on the DeGroot[5] model: the opinion of an agent in a given period is a weighted[2] average of the last period opinions the agent observed. The main conclusion is that if the network is strongly connected and primitive[3], then all agents' opinion will converge to the same limit in the long run.

Since then, a lot of studies have extended the benchmark Degroot model in a variety ways. Some earlier work includes introducing time-dependent weights, permitting each agent to have a persistent "original" estimates on which she always put some weight, as well as assuming discrete set of opinions. Other work (Mueller, 2015) generalized the linear updating rule by characterizing all revise functions that satisfy certain axiomatic properties.

The common underlying assumption of these models, however, is that the network structure remains strongly connected all time, despite the possibly changing weighting matrix. For non strongly-connected networks, since it can always be divided to several strongly-connected sub graphs, the properties of strongly connected networks would apply to each sub graph. Long-run consensus can only be reached within each strongly-connected sub graphs, thus producing segregation by opinion polarization on the entire network. This approach is not fully satisfying, in that it fails to consider the case where the connectedness of

[2]This can be summarized as an $n * n$ non-negative weight matrix.

[3]A strongly connected network is primitive if and only if the limit of each agents' opinion exists and the value of the opinion is independent of the individual agent. See Seneta (2006) for a formal definition.

network itself is evolving over time.

Despite these variations, the rationale behind DeGroot model, that is, under which circumstances does real people behave in this way, remains to be an open question. A couple of microeconomic literature have provided some justifiable behavioral foundations of DeGroot model. DeMarzo et al. (2003) describe a case where DeGroot updating rule might arise in as Bayesian way from an imperfect optimization. An agent optimally pools others' belief with her own past belief by taking linear combination of all those estimates, where the weights assigned to others' belief account for the different precisions of others' information. Another model (Golub and Jackson, 2012) incorporates DeGroot-type learning in a coordination game. The key driving force is that it is costly to make a choice that differs from that of one's neighbors. This leads to an equilibrium strategy of selecting information from their surroundings and attempt to coordinate with their neighbors in each period. Such a rule gives rise to exactly the dynamics of the benchmark DeGroot model. These models rationalize DeGroot model in the economic framework of utility optimization, and they rely on very strong assumptions on decision makers' cognitive mode, on which empirical evidence is still required.

Scholars has long realized DeGroot model's unsatisfying ability to generate persistent disagreement. As mentioned previously, an instant solution would be assuming disconnected network structure in the beginning, so that long-run opinion segregation follows inevitably. However, such assumptions would not always be appropriate. Speed of convergence in DeGroot-type model has also been viewed as a contributing factor, for even consensus can be reached in the long run, it would not be observed in reality if it takes too long to do so. Other research attributes the persistence of disagreement to the way individuals process information. Sadler (2018) models a case in which individuals can encounter false information, and they cannot distinguish true propositions from the false. The conclusion is that global consensus cannot be achieved if agents update their opinion according to rules that satisfy "willingness to learn" and "non-manipulability" under this scenario. In other words, when agents anticipate that there are some false information which they cannot distinguish from the truth, they refuse to believe what others tell them. Essentially, Sadler's paper conceptualized a circumstance under which agents rationally choose not to update by others' information, which has the similar spirit to our study.

Schelling's spacial proximity model is the pioneering work of revealing the underlying mechanism of social segregation. It illustrates how spatial segregation at large scale can be produced after rounds of movement despite that people are not intended to completely separate themselves from people of different races. In other words, there is a mismatch between micro-level incentives and macro-level system pattern. The Schelling model gives an example about how fixed characteristics such as race and ethnicity can be correlated with mutable characteristics such as decisions on where to live. On the other hand, Schelling's model leaves an open question about how segregation may be shaped by mutable characteristics, which is what we want to explore in this study.

Outside economic and sociology literature, there are a couple of models that is closely related to our study, namely, models of continuous opinion dynamics with bounds of confidence (Hegselman et al, 2002; Deffuant el at, 2002). The key idea is that agents repeatedly averaging their opinions under bounded confidence, so that opinions too far away from one's own are weighted less or not at all.[4] These model well captures individuals' selectivity on which information to be used in updating, however they fail to consider the impact of intolerable disagreement on network structure.

[4]See Lorenzo(2007) for a survey of this type of models.

3 The model

3.1 Setup

Let $\Theta = \{\theta_1, \theta_2\}$ be a binary, unknown state of the world. Define $\Delta(\Theta) = \{p|p_1 + p_2 = 1, 0 \leq p_j \leq 1\}$ be the set of probabilistic opinions/beliefs on Θ.[5]

In the initial period t_0, each agent i holds a belief p_i randomly drawn from $\Delta(\Theta)$. Then the interaction procedure begins, and agents repeatedly interact with their link neighbors according to the following behavioral rules:

1. At each period, an agent i from the population is randomly selected. He is then asked to randomly select one of his friends(one who are directly linked with) to talk with, denoted as agent j;

2. If the level of disagreement exceeds the tolerance level ,they break up their friendship, and each of them reach out and get connected to a new agent whom they have never befriended with previously[6], whereas their own opinion stay unchanged; otherwise, both of them update their opinion by a heuristic linear updating rule with equal weights. That is, $p_{i,t+1} = p_{j,t+1} = \frac{p_{i,t}+p_{j,t}}{2}$.

It's worth mentioning that under this scenario relationship revision and opinion revision are essentially two complementary process. This feature differs itself from existing models of relationship and behavior/belief co-evolution (Franzese et al 2012). It's also worth mentioning that I omit the specification of the strength of connections for simplicity.

[5]This setup is sometimes referred to as "continuous opinion dynamics" in contrast with "binary opinion dynamics".

[6]In this setting we only allow agents to build new friendships when one of their existing friendships die. The reason is that maintaining social relationships can be costly, so one only has the incentive to replace an old one with a new.

3.2 Measurement of disagreement

I use Bhattacharyya distance as a measurement of disagreement as it satisfies a couple of feasible properties[28]. Intuitively, the formula insures that disagreement level is zero if and only if two beliefs are the same; our measure of disagreement do not capture any aspect of the state space; the disagreement cannot increase after two states are merged, along with a few other properties irrelevant to our setting.

The formula is written as:

$$D_B(p, q) = -log(\sum_j p_j q_j)$$

where p and q are two opinions, p_j and q_j are their j-th components. In our binary- state case, the can be simplified as:

$$D_B(p, q) = -log[p_j q_j + (1 - p_j)(1 - q_j)]$$

Its range is $[0, +\infty)$, where 0 denotes perfect agreement, $+\infty$ denotes totally disagreement. A tolerance level t is a numeric value on $(0, +\infty)$. Agents choose to break up the tie when $D_B(p, q) > t$.

Figure 1 plots the disagreement level as a function of both parties' beliefs. Note that when either of the two beliefs reach 1, the function value will sharply increase towards infinity. This raises the need to appropriately set up the beliefs to numerically implement the model. More details will be covered in Section 4.

3.3 Initial network topology

In the experiment results below, I specify four network structures as the initial state. Three of them are generated according to network literature, from Erdos-Renyi model,small world model and preferential attachment model respectively. The last one is empirically

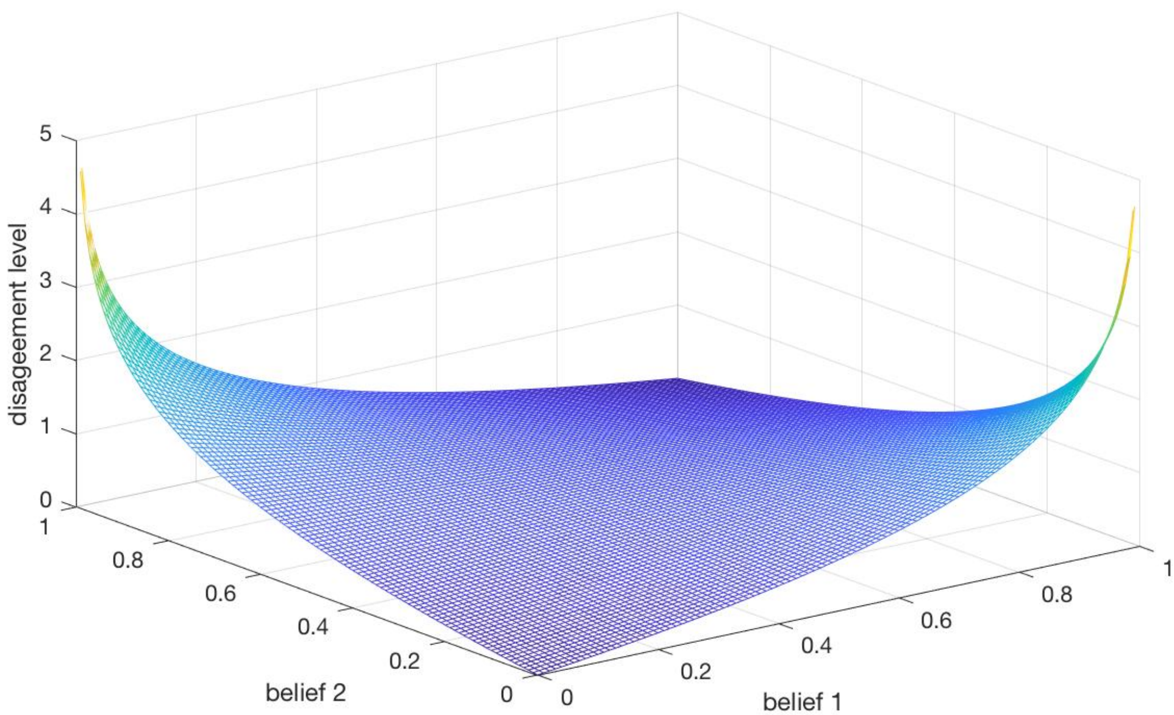


Figure 1: Disagreement level as a function of both agents' beliefs

derived from Facebook user network from Stanford Large Network Dataset Collection[16]. In all cases, the number of nodes in the networks is approximately 300. For generated networks, we chose parameters so as to yield a similar network density with that of the Facebook network. Specifically, the networks are:

1. **Random network:** This is an Erdos-Renyi graph with a uniform probability ρ of having an edge between two arbitrary nodes. In our case the probability is 0.05.

2. **Small world:** This is generated by Watts-Strogatz algorithm. It starts from a lattice network, and randomly rewiring some of the edges. In our case, the degree is 10 and the rewiring probability is 0.02.

3. **Preferential attachment:** This network is generated by the preferential attachment mechanism embedded in NetLogo. Nodes are incrementally added to the existing graph in a way that is biased towards more connected individuals. In our case, 9 edges were created per added node.

4. **Facebook ego network:** This is extracted from SNAP large network data collection[16]. It is created by first randomly selecting one of the User IDs, and then add the other nodes closest to the starting node using bread-first search algorithm. This results in a network for a total of 303 users.

It's worthwhile to notice that all the four networks above are are connected component graph (also a special case of strongly connected directed graphs), a network structure of which long-run consensus can be reached for sure[11].

Table 1 lists the summary statistics of the four initial network structure. Figure 2 depicts the networks for analysis using the igraph package in R. Figure 3 shows the initial degree distribution. Note that the maximum value of the x-coordinate varies a lot across different graphs. With this in mind, we could see that the Facebook network and the preferential attachment networks display more skewed distributions, indicating that a very small number

Table 1: Summary statistics of initial network structures

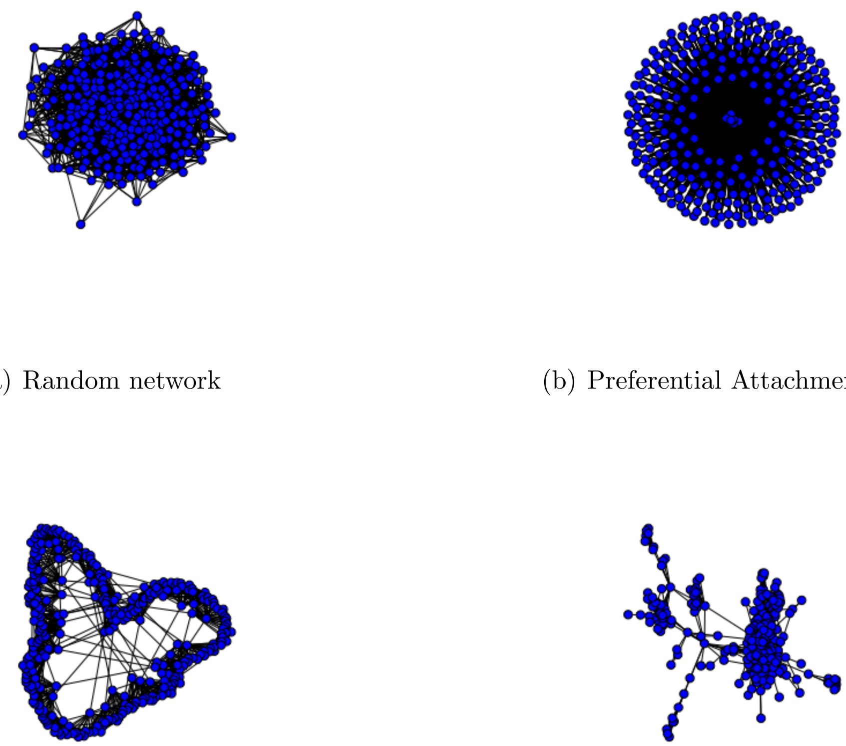
	Random	Small world	Preferential attachment	Facebook ego
Number of nodes	300	300	300	303
Number of edges	2776	3000	2695	2493
Average shortest path length	2.3593	3.3163	1.9461	3.5467
Average Local Clustering Coefficient	0.0493	0.6696	0.9713	0.5082
Number of Components	1	1	1	1

of agents have very large number of connections, whereas random network and small world have less fatter tails, meaning that in general agents are more evenly connected.

3.4 Focus of analysis

Before moving on to the implementation, it would be worthwhile to summarize our main focus of analysis for this computational model. I'm primarily interested in what the steady state looks like, and how long the model takes to reach it, in response to different parameter values, and across different network structures.

Specifically, I'm aiming to answering the following questions: Does network become segregated in the long run? Does opinion converge in each segregated groups whereas diverge across different groups? Is it possible that there exists some less than perfect tolerance level(i.e.$t > 0$) such that global consensus can be reached? If so, how does it vary across different network structures? Intuitively, the co-evolutionary outcome depends on the setting of parameter values. In an extreme case where each agent is perfectly tolerant (i.e. they linearly combine their belief with their neighbour regardless of the disagreement level), global consensus is sure to be achieved in the long run. In another extreme case where each agents are absolutely stubborn, no belief updating could ever happen and each agent is segregated from every other one in the network. The more interesting case lies in between, which is what I want to focus on.



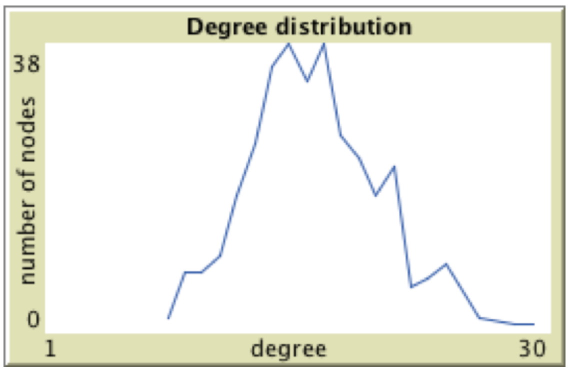
(a) Random network

(b) Preferential Attachment

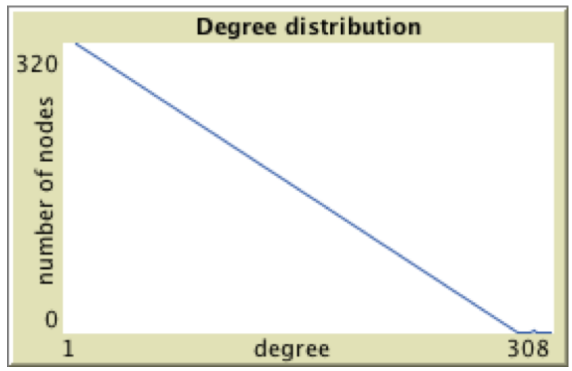
(c) Small world

(d) Facebook user network

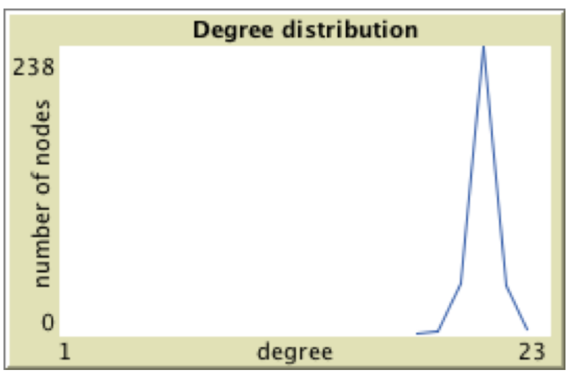
Figure 2: Visualization of initial network structures



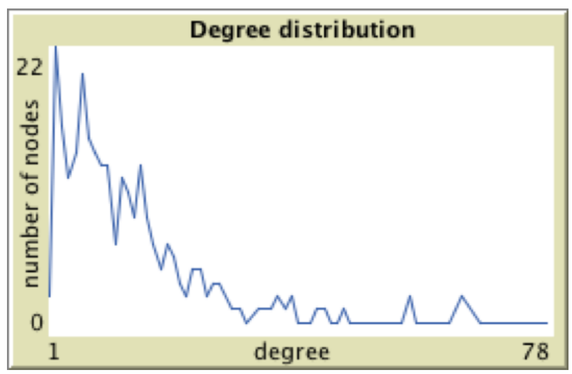
(a) Random network



(b) Preferential Attachment



(c) Small world



(d) Facebook user network

Figure 3: Initial degree distributions

4 Model implementation

I constructed the theoretical model using NetLogo. The three generated networks are created by built-in algorithms in NetLogo's network extension. The Facebook network is represented in the form of an adjacency matrix converted from the original edge list before being loaded into NetLogo. For convenience of tractability, I created two link breeds, namely, existing friendship breed and dead friendship breed. Any link belongs to either of the two breeds, depending on the historical outcome of the interaction of the two agents on the two ends. I then generate initial beliefs for each agent, which is a random float number on $[0, 1)$ interval. This limits the upper bound of our measure of disagreement to $-ln(0.01 \times 1 + 0) = 4.60517$. Therefore, I set the maximum value of tolerance level slider to be 4.6, as shown on the interface page.

Agents behave in a sequence of rounds. At each round, a random agent is selected from the agent set and behave according the rules characterized in Section 3.1. It will only move on to the next round until all the agents have had the chance to interact with one of his link neighbors. If no such neighbors exist, then we skip the current agent and move on to the next.

This procedure stops when none of the agents form new links any longer, so that the number of links in the network becomes stable over time. This is the end of the link formation process, as well as the steady states of the model. I keep track of the number of existing and dead friendships after each round of interaction is completed.

I also record the distribution of opinions, and the size of groups if segregation happens. This is done by executing a NetLogo command of exporting weak-component-clusters after dropping out dead-friendship links on the steady-state network.

5 Results and discussions

Figure 4 shows the case when agents are fully tolerant (t=4.6). No friendship is ever dead, and agents quickly reach to an consensus of 0.47503, close to the median opinion. In the mean time agents have no incentive to create new links, so the steady state network remains the same density. This confirmed our prediction that global consensus can be achieved very quickly when people are fully tolerant and willing to pool everybody's opinion together.

Figure 5 shows the case where agents are completely intolerant. All existing friendships eventually died (except for the rare cases where some agents happened to be the same belief) despite they always had the chance to form link with new agents. This again confirmed our hypothesis that segregation by opinion will surely happen when agents refuse to learn from others. Note that the time spend to reach steady state varies dramatically across different network structures. It takes Preferential Attachment and Facebook user network much more rounds for the link formation process to complete. This is because the distribution of number of social connections across different agents is more uneven, so much more rounds are needed for those initially less connected agents to complete the link formation process.

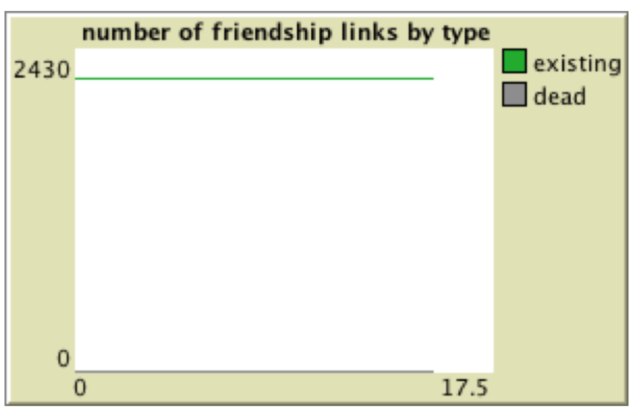
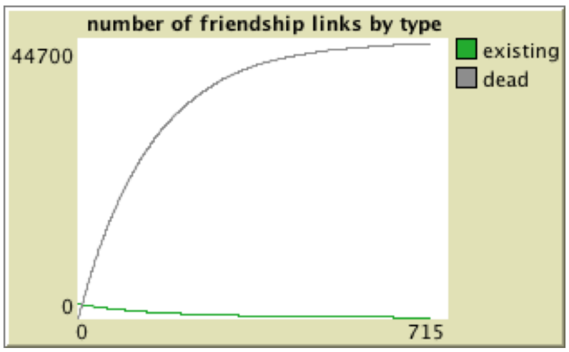
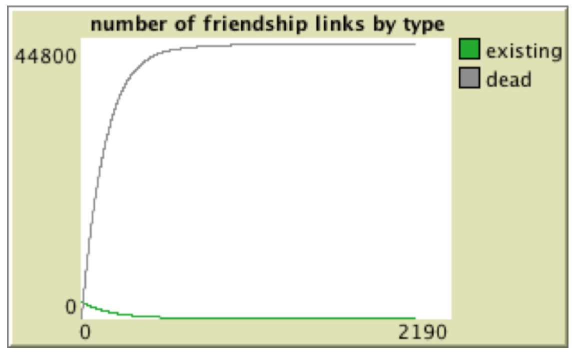


Figure 4: Model dynamics when agents are fully tolerant

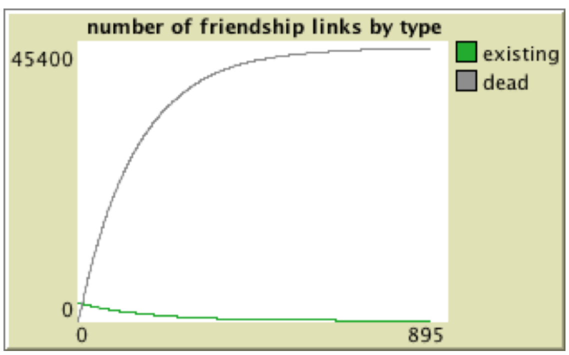
As we reduce the tolerance level to some extent, we find that global consensus can still be achieved in the long run for a relatively wide range of parameter values, although there emerges a couple of pairwise segregated individuals. Figure 6 shows the dynamics of Facebook ego network at tolerance level of 0.8. After about 11 rounds the agents reached to an



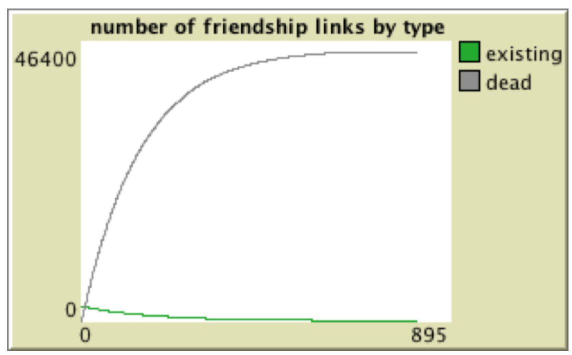
(a) Random network



(b) Preferential Attachment



(c) Small world



(d) Facebook user network

Figure 5: Number of friendship links when agents are completely intolerant

consensus of 0.4655, which is also roughly the median opinion. Beliefs in the network converge asymptotically to the expectation of a random one. In the mean time, approximately 500 dead-friendships persist due to disagreement in previous rounds. Intuitively, even though agents are not perfectly tolerant towards different opinions (in other words, they reserve the right to break up with someone holding opposite view), there is still reasonable chance to achieve global consensus. This may be explained by agents' linear updating rule: that choosing to average one's own opinion with his peer's reduces the overall variance of opinions when the cohort size is large, after sufficiently many rounds of interaction. This pattern is very different from the predictions of Schelling's model. It seems that, apart from being tolerance towards moderately different opinions, the process of constant opinion averaging process also contributes to the converge.

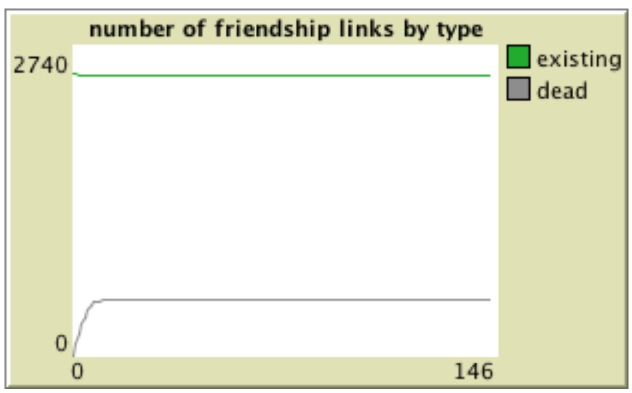


Figure 6: Model dynamics with t=0.8, initial network is Facebook ego. Similar patterns arise for the other three networks.

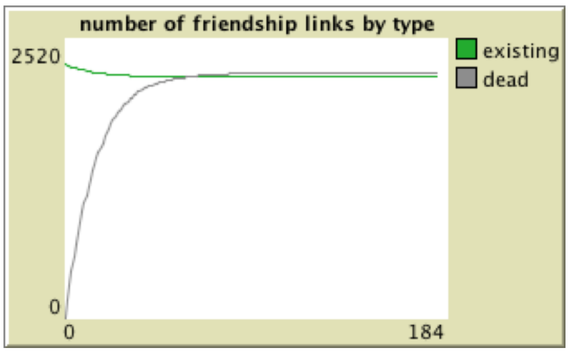
However, when we further reduce the tolerance level to 0.7, opinion polarization accompanied by social segregation finally arise. Figure 7 shows the model dynamics at the tolerance level of 0.7. After several rounds of interaction, all four networks have been separated into two opinion groups. Notably, all four networks eventually split into two opinion groups of roughly the equal size, and the consensus within two subgroups are also similar across different networks, at roughly 0.25 and 0.75 respectively. This is exactly the midpoints of interval $[0, 0.5]$ and $[0.5, 1]$.

Things becomes more complicated when we further reduce the tolerance level under 0.7.

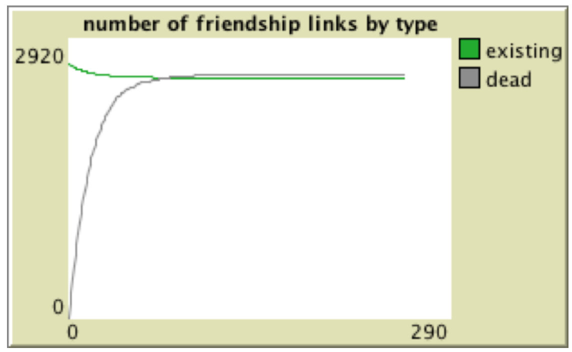
Figure 8 shows the model dynamics at various segregation-inducing tolerance levels. Note that the lower tolerance level is, the longer time it takes the model to reach steady state. This is because when agents are less tolerant, they break up with their current partners more often and they are more engaged in seeking new friends. Another notable finding is that the number of opinion groups grows very fast as tolerance level decreases. This again confirmed our hypothesis that segregation emerges as agents' become less tolerant towards difference of opinions. However, it remains unclear how the pattern of segregation changes with respect to parameter values.

Using the Behavioral Space tool integrated with NetLogo, I ran the model with every tolerance level range in $[0, 0.7]$ in increments of 0.05 (this procedure is also known as parameter sweeping). Figure 9 shows the steady-state number of opinion groups as a function of tolerance level. The growth rate of segregated opinion groups increase sharply as we slide down the tolerance level. And this pattern is almost the same in Facebook user network and generated random networks. The implication is that, when agents are very intolerant towards different opinions, segregation become increasingly severe, regardless of the initial network structure. [7]

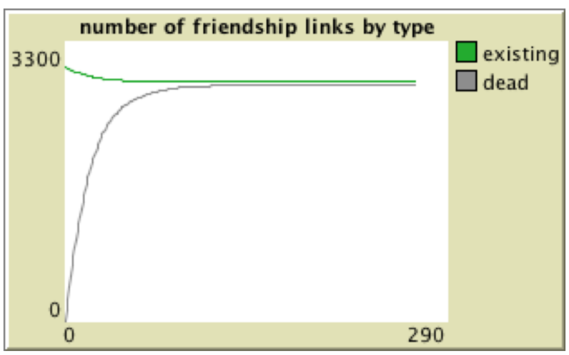
[7]However, one should be careful to interpret the absolute value of tolerance level. If we take a look back to the graph of disagreement function, we should notice that low level of disagreement appears more frequently than the higher ones under this measurement. Intuitively, as we slide down the tolerance level, the speed of actual tolerance decrease is even faster.



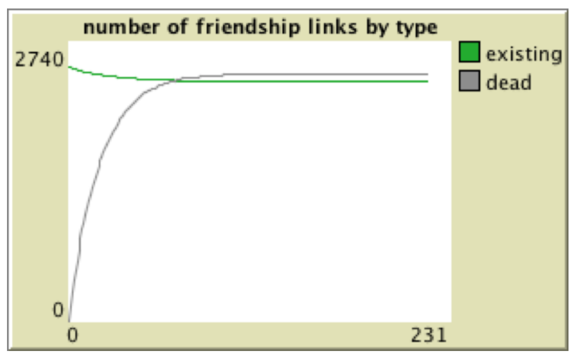
(a) Random network



(b) Preferential Attachment

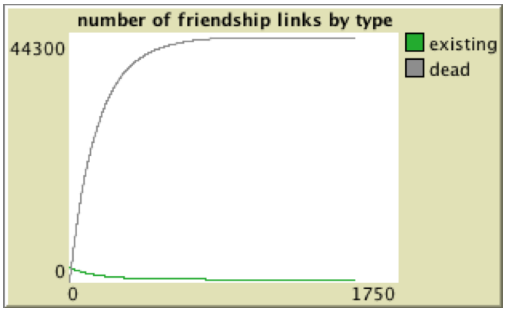


(c) Small world

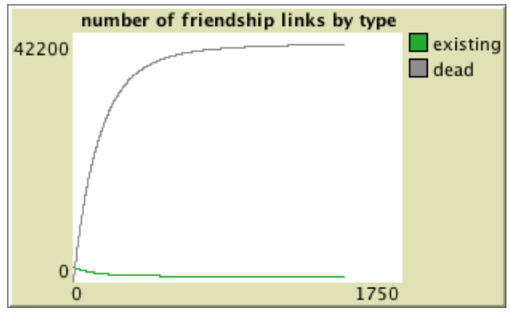


(d) Facebook user network

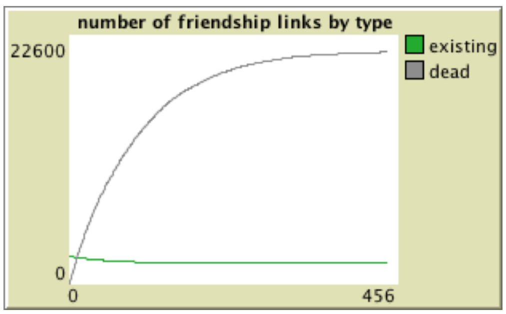
Figure 7: Model dynamics when t=0.7



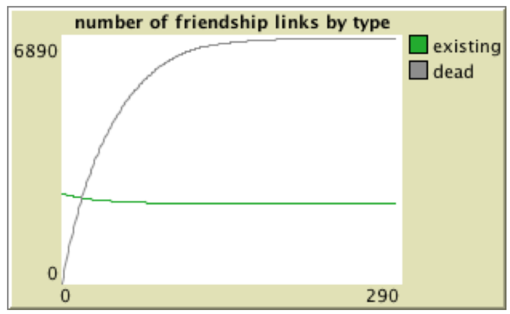
(a) t=0.1



(b) t=0.2



(c) t=0.4



(d) t=0.6

Figure 8: Model dynamics at segregation-inducing tolerance levels. Initial network = Facebook

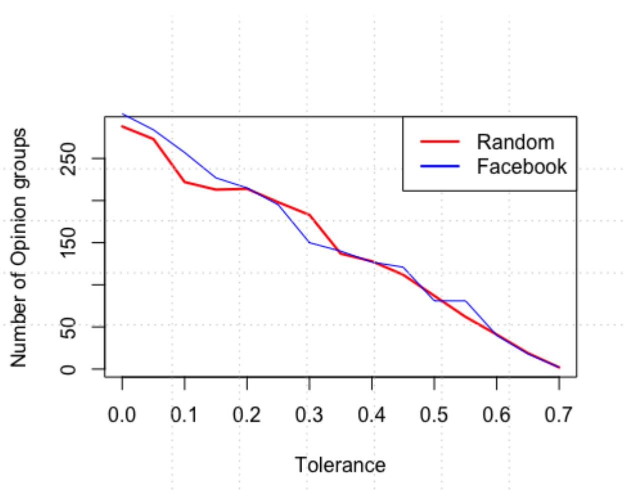


Figure 9: Steady state number of opinion groups as a function of tolerance level.

6 Concluding remarks

In this paper, I developed a behavioral model in which agents repeatedly interact with their social peers to update beliefs and adjust their social contacts if necessary. Using NetLogo simulation, we find that this model may or may not generate persistent disagreement and social segregation in the long run, depending on to what extent agents are tolerant towards others with different opinions.

As final note, we have yet to give a full rationale of why agents behave in this way. While a formal characterization would be out of the scope of this paper, here I give an informal narrative: People heavily relies on others' beliefs to revise their own. However, the volume of outside information can be overwhelmingly high, so that they tend to seek some cognitively less demanding way to do the updating. That's why they only interact with one agent at a time. In the meanwhile, it would be efficient to stay away from those who holds extremely different opinions, since these people are considered to be less reliable source of information. That's why they sometimes choose to break up with each other. In the meanwhile, people need to maintain a reasonable size of social contacts, so they replace the dead friendship with a new one.

6.1 Limitations

This study is at a very preliminary stage, and for convenience of computation I make some over-simplified assumptions which may undermine the model's external validity. Below I list some limitations currently on my mind:

To begin with, we should be aware that the initial beliefs and network structure are set independently in the model. In other words, it is assumed that agents' initial beliefs are exogenous of his standing in the social network. This assumption is questionable considering the fact that in reality like-minded people tend to be together.

Another problem lies in the randomization of probabilistic beliefs. Note that we generate random float number as agents' initial beliefs, which means that there can be infinitely many different opinions. However, in reality, differences in opinions can sometimes be too small to distinguish. If there are only 100 distinguishable opinions towards an issue, then any group with more than 100 agents will have at least one pair sharing the same opinion. In other words, small "cliques" will emerge despite all of them are completely intolerable.

It also deserves closer contemplation on how to appropriately incorporate the formation of new connections in the model. Currently, the model only allows agents to seek new social connections when they just break up with an existing friend. It would be more realistic to introduce some probabilistic element into this new link formation process.

Finally, we should note that the the discussion has been carried out in terms of simulations rather than mathematical analysis, which is due to the complexity of the model. We look forward to the potentiality of a mathematical characterization in the future.

6.2 Model extensions

Here I propose two potential extensions to the baseline model for future study.

6.2.1 Heterogeneous agents

In the baseline model we assume that every agent has the same tolerance levels, that is $t_i = t$. We can relax this assumption by setting heterogeneous tolerance level among different agents. Under this scenario, pairwise communications result in a breakup of tie if the disagreement exceed $min\{t_i, t_j\}$.

It may also be interesting to model another source of heterogeneity, type of agents, through variation of belief updating function. For example, there may exist a kind of "forceful" agents in the population, who want to influence the opinions of other agents they meet, but do not want to change their own opinions. This can be modeled by a belief updating function in which forceful agents put significant heavier weight on his own opinion.

However, it's worth mentioning that tolerant levels and type can be correlated. Agents with higher tolerant level are more likely to be forceful type, since they care more about influencing other people's opinion, rather than staying in touch with people they disagree with.

6.2.2 Multiple, weighted issues

In our previous model each agent holds one belief towards one issue, whereas in reality people tend to hold beliefs towards multiple issues. It is also reasonable to assume that people give different priority towards different issues. The more important the issue is, the more it will influence the agent's tie revision. Thus, it would be interesting to accommodate multiple issues with different weighted level of disagreement into our baseline model.

Formally, $D_B(p, q) = -\sum_i w_i log(\sum p_{i,j} q_{i,j})$, where w_i denotes the weight assigned to issue i.

A potential problem arise when we try to model this scenario. Do agents exchange their opinions towards different issues concurrently or sequentially? In the latter case, the network dynamics become dependent on the order of issues agents discuss, since broken social ties cannot be reconnected in our setting.

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