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Array-based lists

part 1

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Array-based implementations

of the **List** and **Queue** interfaces

	get(i) / set(i,x)	add(i,x) / remove(i)
ArrayStack	$O(1)$	$O(1+n-i)$
ArrayDeque	$O(1)$	$O(1+\min\{i,n-i\})$
DualArrayDeque	$O(1)$	$O(1+\min\{i,n-i\})$
RootishArrayStack	$O(1)$	$O(1+n-i)$

Interface - List

List – represents an indexed sequence of elements

size()	– returns the number of elements on the list ($\boldsymbol{n}$)	$\boldsymbol{O(1)}$
isEmpty()	– returns whether list is empty	
get($\boldsymbol{i}$)	– returns the element at position $\boldsymbol{i}$	
set($\boldsymbol{i}, \boldsymbol{x}$)	– update the element at position $\boldsymbol{i}$ to be $\boldsymbol{x}$	
add($\boldsymbol{i}, \boldsymbol{x}$)	– add the element $\boldsymbol{x}$ to position $\boldsymbol{i}$	$\boldsymbol{O(n-i+1)}$ **amortized time**
remove($\boldsymbol{i}$)	– remove element at position $\boldsymbol{i}$	

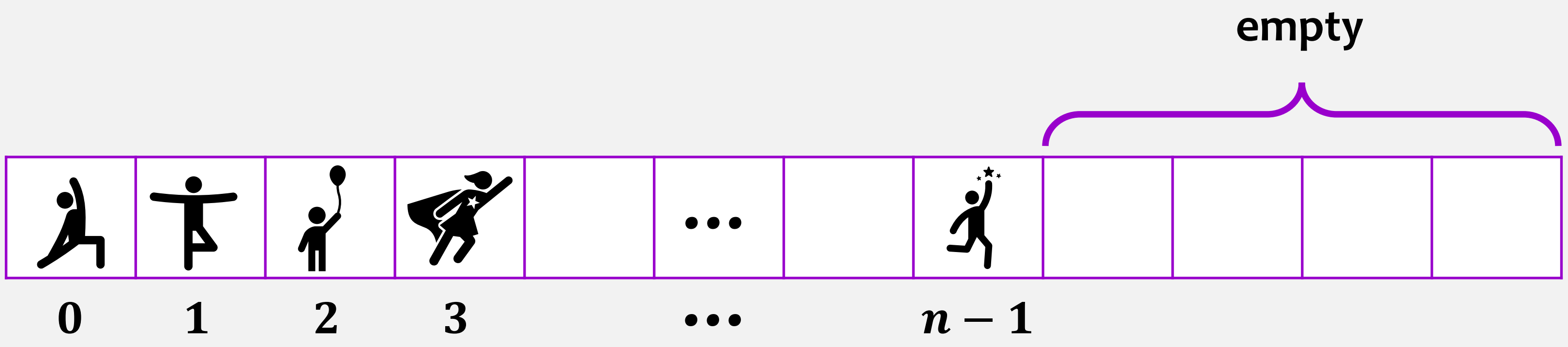
The Java Collections Framework documentation about List interface:
https://docs.oracle.com/javase/8/docs/api/java/util/List.html

ArrayStack (aka ArrayList)

ArrayStack is an array-based List implementation. It is equivalent to the **ArrayList** in the Java Collections Framework (JCF). It has the same performance.

Array $\boldsymbol{a}$ of type **T** and of size $\boldsymbol{a}.\mathbf{length}$

T[] $\boldsymbol{a}$ →



int $\boldsymbol{n}$ ← size of the list (can be different from $\boldsymbol{a}.\mathbf{length}$)

char[] $\boldsymbol{a}$

0	1	2	3	4	5	6	7	8	9	10	11
$\boldsymbol{c}$	$\boldsymbol{b}$	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$\boldsymbol{q}$						

$\boldsymbol{n} = \mathbf{6}$

$\boldsymbol{a}.\mathbf{length} = \mathbf{12}$

ArrayStack

Constructor **ArrayStack()**: $\boldsymbol{a} = \textbf{new T}[\mathbf{1}];$
$\boldsymbol{n} = \mathbf{0};$

0

notice, not $\boldsymbol{i} > \boldsymbol{a}.\mathbf{length}$

returns the element at position $\boldsymbol{i}$

T get($\boldsymbol{i}$):
if $(\boldsymbol{i} < \mathbf{0} \;||\; \boldsymbol{i} \geq \boldsymbol{n})$ then throw new IndexException;
return $\boldsymbol{a}[\boldsymbol{i}];$

$\boldsymbol{O(1)}$

char[] $\boldsymbol{a}$

$\boldsymbol{c}$	$\boldsymbol{b}$	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$\boldsymbol{q}$						
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**

$\boldsymbol{n} = \mathbf{6}$

$\boldsymbol{a}.\mathbf{length} = \mathbf{12}$

ArrayStack

- saves the element at position $\boldsymbol{i}$;
- updates the element at position $\boldsymbol{i}$ to be $\boldsymbol{x}$;
- returns the element that was at position $\boldsymbol{i}$

T set($\boldsymbol{i}, \boldsymbol{x}$):

if $(\boldsymbol{i} < \mathbf{0} \;||\; \boldsymbol{i} \geq \boldsymbol{n})$ then throw new IndexException;
T $\boldsymbol{y} = \boldsymbol{a}[\boldsymbol{i}]$;
$\boldsymbol{a}[\boldsymbol{i}] = \boldsymbol{x}$;
return $\boldsymbol{y}$;

$\boldsymbol{O(1)}$

set($4, \boldsymbol{w}$):

char[] $\boldsymbol{a}$

$\boldsymbol{c}$	$\boldsymbol{b}$	$\boldsymbol{x}$	$\boldsymbol{y}$	$\cancel{\boldsymbol{z}}$ $\boldsymbol{w}$	$\boldsymbol{q}$						
0	1	2	3	4	5	6	7	8	9	10	11

$\boldsymbol{n} = \mathbf{6}$

$\boldsymbol{a}.\mathbf{length} = \mathbf{12}$

ArrayStack

we do not return $\boldsymbol{a}.\mathbf{length}$

returns the number of elements on the list $(\boldsymbol{n})$

```
int size():
        return n;
```

$\boldsymbol{O(1)}$

char[] $\boldsymbol{a}$

0	1	2	3	4	5	6	7	8	9	10	11
c	b	x	y	w	q						

$\boldsymbol{n = 6}$

$\boldsymbol{a}.\mathbf{length} = \mathbf{12}$

ArrayStack

- check bounds
- if $\boldsymbol{n}$ is equal to $\boldsymbol{a}.\mathbf{length}$ then resize the array $\boldsymbol{a}$
- shift elements $\boldsymbol{i}$ through $\boldsymbol{n}-\mathbf{1}$ up an index (to indices $\boldsymbol{i}+\mathbf{1}, \dots, \boldsymbol{n}$)
- insert the element $\boldsymbol{x}$ to position $\boldsymbol{i}$
- increase $\boldsymbol{n}$ by $\mathbf{1}$ to reflect the update

i can be equal n

$O(1+n-i)$

void **add**($\boldsymbol{i}, \boldsymbol{x}$):

if $(\boldsymbol{i}<\mathbf{0} \,\|\, \boldsymbol{i}>\boldsymbol{n})$ then throw IndexException;
if $(\boldsymbol{n}+\mathbf{1}>\boldsymbol{a}.\mathbf{length})$ then **resize()**;
for $(\boldsymbol{j}=\boldsymbol{n};\ \boldsymbol{j}>\boldsymbol{i};\ \boldsymbol{j}--)$
    $\boldsymbol{a}[\boldsymbol{j}]=\boldsymbol{a}[\boldsymbol{j}-\mathbf{1}];$
$\boldsymbol{a}[\boldsymbol{i}]=\boldsymbol{x};$
$\boldsymbol{n}++;$

shift

we will see this function later

after a call to **resize()**, we can be sure that $\boldsymbol{a}.\mathbf{length}>\mathbf{n}$.

add$(\mathbf{2}, \boldsymbol{h})$:

char[] $\boldsymbol{a}$

c	b	h	y	w	q						
0	1	2	3	4	5	6	7	8	9	10	11

$\boldsymbol{a}.\mathbf{length} = \mathbf{12}$

$\boldsymbol{n} = \cancel{6}\ 7$

ArrayStack

- check bounds;
- save the element at position i;
- shift elements $i+1$ through $n-1$ down an index (to indices $i, \dots, n-2$);
- decrease n by 1;
- if a is 3 times longer than necessary then resize the array;
- return the element that was at position i.

$O(1+n-i)$

T remove(i):

```
if (i < 0 || i ≥ n) then throw IndexException;
T x = a[i];
for (j = i; j < n − 1; j++)
    a[j] = a[j + 1];
n−−;
a[n] = null;
if (3n ≤ a.length) then resize();
return x;
```

$$\left.\begin{array}{l}\text{for } (j = i;\ j < n-1;\ j++)\\ \quad a[j] = a[j+1];\end{array}\right\}\ \text{shift}$$

remove(3):

char[] a

c	b	h	x	y	w	q					
0	1	2	3	4	5	6	7	8	9	10	11

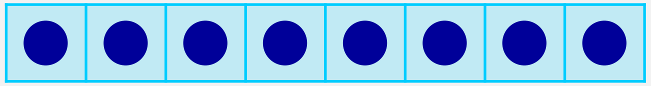
$a.\text{length} = 12$

$n = \cancel{7}\,6$

ArrayStack – resize()

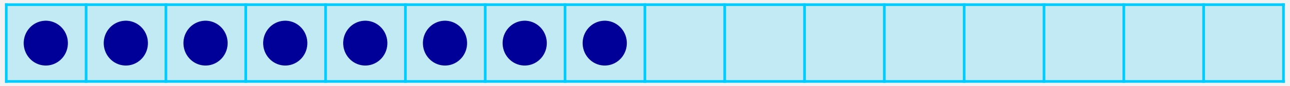
There are two situations that trigger **resize()**:

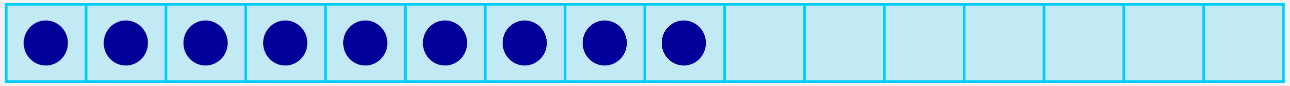
1. Array is full and we want to add another element



add(i, x):

resize():

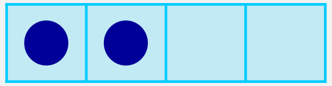




2. We remove an element, and the array becomes 2/3 empty

remove(i):

resize():



ArrayStack – resize()

to avoid 0-length array

void **resize()**:

$$\mathrm{T}[\,]\ \boldsymbol{b} = \text{new array}(\mathbf{max}\{\mathbf{2}\boldsymbol{n}, \mathbf{1}\})$$

$$\text{executes } \boldsymbol{n} \text{ times}\ \left\{ \begin{array}{l} \text{for } (\boldsymbol{j} = \mathbf{0};\, \boldsymbol{i} < \boldsymbol{n};\ \boldsymbol{i}++) \\ \qquad \boldsymbol{b}[\boldsymbol{i}] = \boldsymbol{a}[\boldsymbol{i}]; \end{array} \right.$$

$$\boldsymbol{a} = \boldsymbol{b};$$

$$\boxed{\boldsymbol{O}(\boldsymbol{n})}$$

- create new array $\boldsymbol{b}$;
- copy everything from $\boldsymbol{a}$ to $\boldsymbol{b}$;
- the new array becomes $\boldsymbol{a}$.

ArrayStack – resize()

void **resize()**:

$$\begin{aligned}
&\text{T[]}\ \boldsymbol{b} = \text{ new array of size } \boldsymbol{s} \\
&\text{for } (\boldsymbol{j} = \mathbf{0}; \boldsymbol{i} < \boldsymbol{n};\ \boldsymbol{i}++) \\
&\qquad \boldsymbol{b}[\boldsymbol{i}] = \boldsymbol{a}[\boldsymbol{i}]; \\
&\boldsymbol{a} = \boldsymbol{b};
\end{aligned}$$

- take $\boldsymbol{s} \gg \boldsymbol{n}$
  - \+ you never have to **resize()** again (fast!)
  - – wastes a lot of space
- take $\boldsymbol{s} = \boldsymbol{n} + \mathbf{1}$
  - \+ does not waste any space
  - – wastes too much time in **resize()** over many calls
- take $\boldsymbol{s} = \mathbf{max}\{\mathbf{2}\boldsymbol{n}, \mathbf{1}\}$
  - – wastes $\boldsymbol{O}(\boldsymbol{n})$ space
  - \+ gives us $\boldsymbol{O}(\mathbf{1})$ **amortized** run time

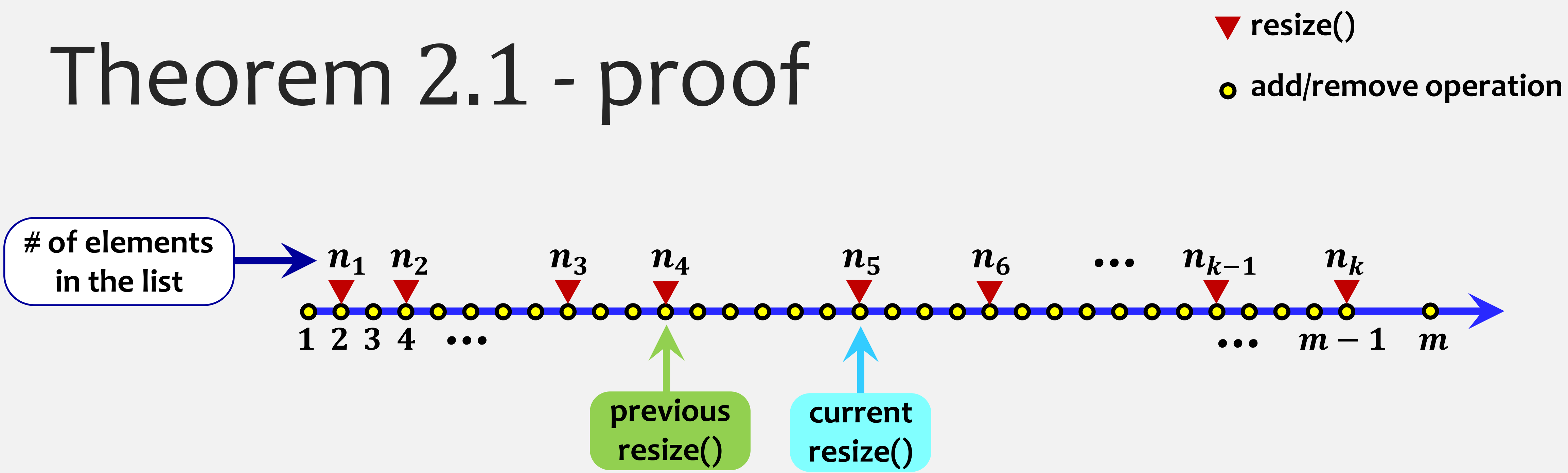
Theorem 2.1

An **ArrayStack** implements the **List** interface. Ignoring the cost of calls to **resize()**, an **ArrayStack** supports the operations

- **get**$(\boldsymbol{i})$ and **set**$(\boldsymbol{i}, \boldsymbol{x})$ in $\boldsymbol{O}(\mathbf{1})$ time per operation; and
- **add** $(\boldsymbol{i}, \boldsymbol{x})$ and **remove**$(\boldsymbol{i})$ in $\boldsymbol{O}(\mathbf{1} + \boldsymbol{n} - \boldsymbol{i})$ time per operation.

Furthermore, beginning with an empty **ArrayStack** and performing any sequence of $\boldsymbol{m}$ **add** $(\boldsymbol{i}, \boldsymbol{x})$ and **remove**$(\boldsymbol{i})$ operations results in a total of $\boldsymbol{O}(\boldsymbol{m})$ time spent during all calls to **resize()**.

Theorem 2.1 - proof



The total # of elements copied by all the calls to **resize()** is at most

$$n_1 + n_2 + n_3 + \cdots + n_k$$

Theorem 2.1 - proof

We resize in two cases:

- on **add**, when $\boldsymbol{n+1 > a.\mathrm{length}}$ (when $\boldsymbol{a.\mathrm{length} = n}$)
- on **remove**, when $\boldsymbol{3n \leq a.\mathrm{length}}$

We resize to $\boldsymbol{max\{2n, 1\}}$ (i.e., $\boldsymbol{2n}$ unless $\boldsymbol{n = 0}$)

Right after the **resize()** (in either of the two cases) our array is half full

$$\boldsymbol{\frac{a.\mathrm{length}}{2} - 1 \leq n \leq \frac{a.\mathrm{length}}{2}}$$

Suppose we have $\boldsymbol{m}$ calls to **add** $\boldsymbol{(i, x)}$ and **remove**$\boldsymbol{(i)}$. We want to show that the total time spent during all calls to **resize()** is $\boldsymbol{O(m)}$.

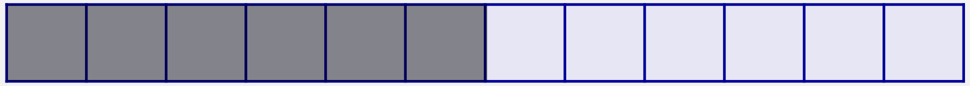
Theorem 2.1 - proof

$\boldsymbol{n}$ is the size of the list **NOW**

Consider two consecutive **resize()** operations: current $(\boldsymbol{j})$ and previous $(\boldsymbol{j}-\mathbf{1})$.

There are no **resize()** operations in between. Therefore, the size of the array right after the $(\boldsymbol{j}-\mathbf{1})$-st **resize()** and just before the $\boldsymbol{j}$-th **resize()** is the same.

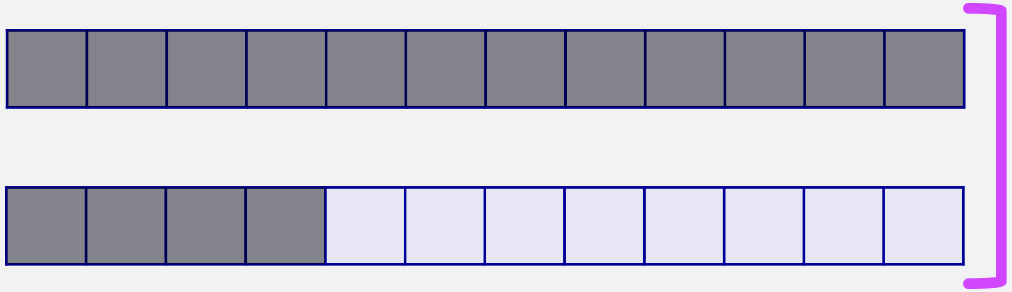
array right after the $(\boldsymbol{j}-\mathbf{1})$-st **resize()**



$$\frac{\boldsymbol{a}.\mathbf{length}}{\mathbf{2}} - \mathbf{1} \le \boldsymbol{n} \le \frac{\boldsymbol{a}.\mathbf{length}}{\mathbf{2}}$$

array just before the $\boldsymbol{j}$-th **resize()** (triggered by either add or remove operation):

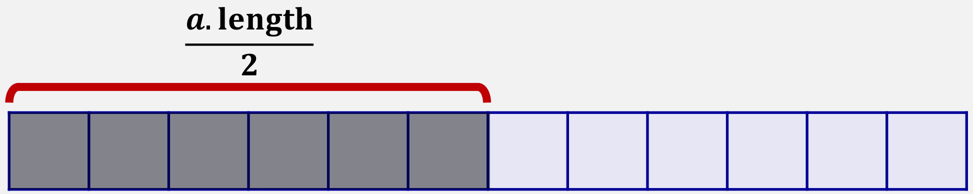
1. **add** $(\boldsymbol{i}, \boldsymbol{x})$ and $\boldsymbol{a}.\mathbf{length} = \boldsymbol{n}$
2. **remove**$(\boldsymbol{i})$ and $\boldsymbol{a}.\mathbf{length} \ge \mathbf{3}\boldsymbol{n}$



Theorem 2.1 - proof

$\boldsymbol{n}$ is the size of the list **NOW**

array right after the $(\boldsymbol{j}-\mathbf{1})$-st **resize()**



THEN

array just before the $\boldsymbol{j}$-th **resize()** (triggered by either add or remove operation):

1. **add** $(\boldsymbol{i}, \boldsymbol{x})$ and $\boldsymbol{a}.\mathbf{length} = \boldsymbol{n}$

NOW

We added at least (# elements at $\boldsymbol{j}$-th **resize()**) – (# elements at $(\boldsymbol{j}-\mathbf{1})$-st **resize()**)

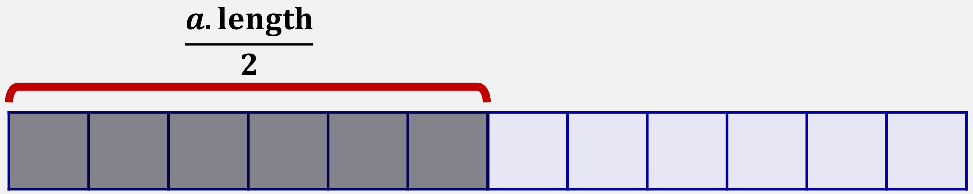
We added $\geq \quad \boldsymbol{a}.\mathbf{length} - \dfrac{\boldsymbol{a}.\mathbf{length}}{\mathbf{2}} = \dfrac{\boldsymbol{a}.\mathbf{length}}{\mathbf{2}} = \dfrac{\boldsymbol{n}}{\mathbf{2}}$

We have at least $\boldsymbol{n}/\mathbf{2}$ **add** $(\boldsymbol{i}, \boldsymbol{x})$ operations between these two **resize()** operations (maybe more if we also have **remove**$(\boldsymbol{i})$ operations)

Theorem 2.1 - proof

$\boldsymbol{n}$ is the size of the list **NOW**

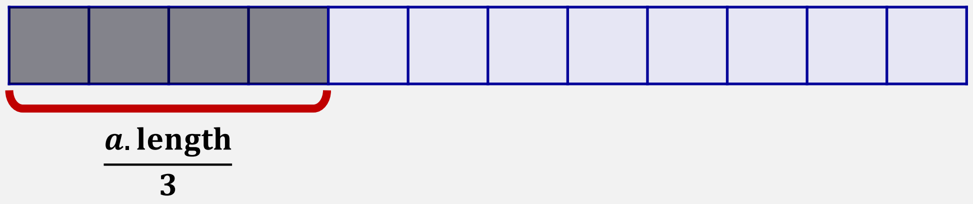
array right after the $(\boldsymbol{j}-\mathbf{1})$-st **resize()**



THEN

array just before the $\boldsymbol{j}$-th **resize()** (triggered by either add or remove operation):

2. $\mathbf{remove}(\boldsymbol{i})$ and $\boldsymbol{a}.\mathbf{length} \geq \mathbf{3}\boldsymbol{n}$



NOW

We removed at least (# elements at $(\boldsymbol{j}-\mathbf{1})$-st **resize()**) $-$ (# elements at $\boldsymbol{j}$-th **resize()**)

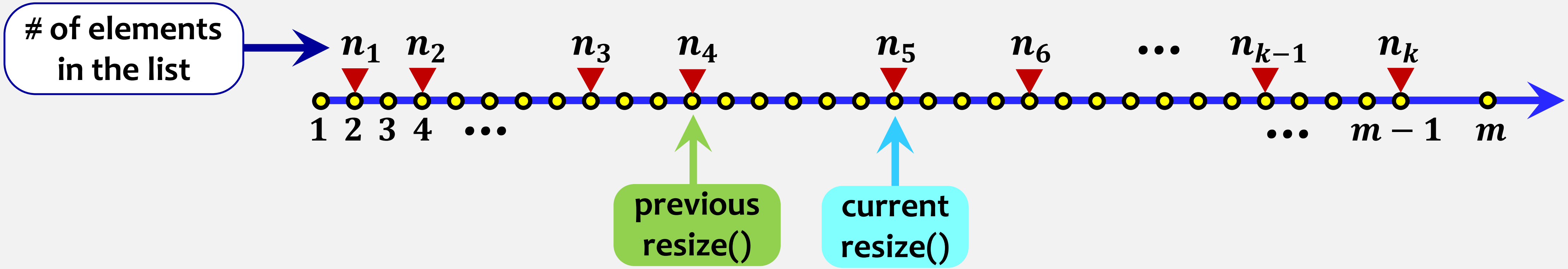
$$\text{We removed} \geq \frac{a.\text{length}}{2} - 1 - \frac{a.\text{length}}{3} = \frac{a.\text{length}}{6} - 1 \geq \frac{3n}{6} - 1 = \frac{n}{2} - 1$$

We have at least $\boldsymbol{n}/\mathbf{2}-\mathbf{1}$ $\mathbf{remove}(\boldsymbol{i})$ operations between these two **resize()** operations (maybe more if we also have **add** $(\boldsymbol{i}, \boldsymbol{x})$ operations)

Theorem 2.1 - proof

▼ resize()

● add/remove operation



The total # of elements copied by all the calls to **resize()** is at most

$$n_1 + n_2 + n_3 + \cdots + n_k$$

$$m \geq \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} + \cdots + \frac{n_k}{2} = \frac{1}{2}\left(\begin{array}{c}\text{total \# of elements copied} \\ \text{by all the calls to } \textbf{resize()}\end{array}\right)$$

$$2m \geq \left(\begin{array}{c}\text{total \# of elements copied} \\ \text{by all the calls to } \textbf{resize()}\end{array}\right)$$

Total running time of all the calls to **resize()** is $O(m)$.

ArrayStack

The **ArrayStack** is an efficient way to implement a **Stack.**

In particular, we can implement:

- **push**$(\boldsymbol{x})$ as **add**$(\boldsymbol{n}, \boldsymbol{x})$
- **pop()** as **remove**$(\boldsymbol{n} - \mathbf{1})$

in which case these operations will run in $\boldsymbol{O}(\mathbf{1})$ amortized time.

Why?

Recall:

add $(\boldsymbol{i}, \boldsymbol{x})$ and **remove**$(\boldsymbol{i})$ run in $\boldsymbol{O}(\mathbf{1} + \boldsymbol{n} - \boldsymbol{i})$ time per operation (ignoring **resize()**).