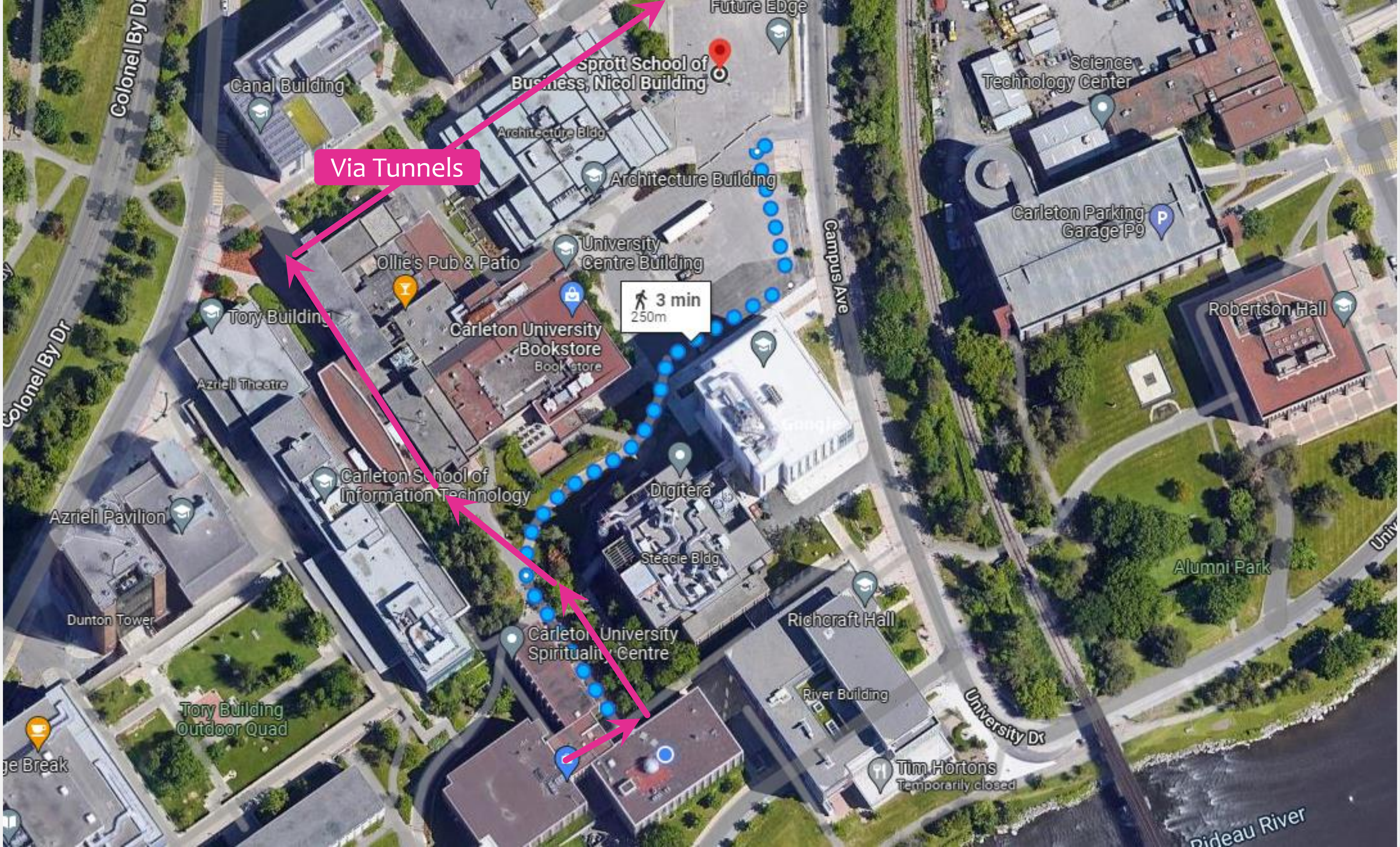
Carleton University
Alina Shaikhet

Shortest Paths

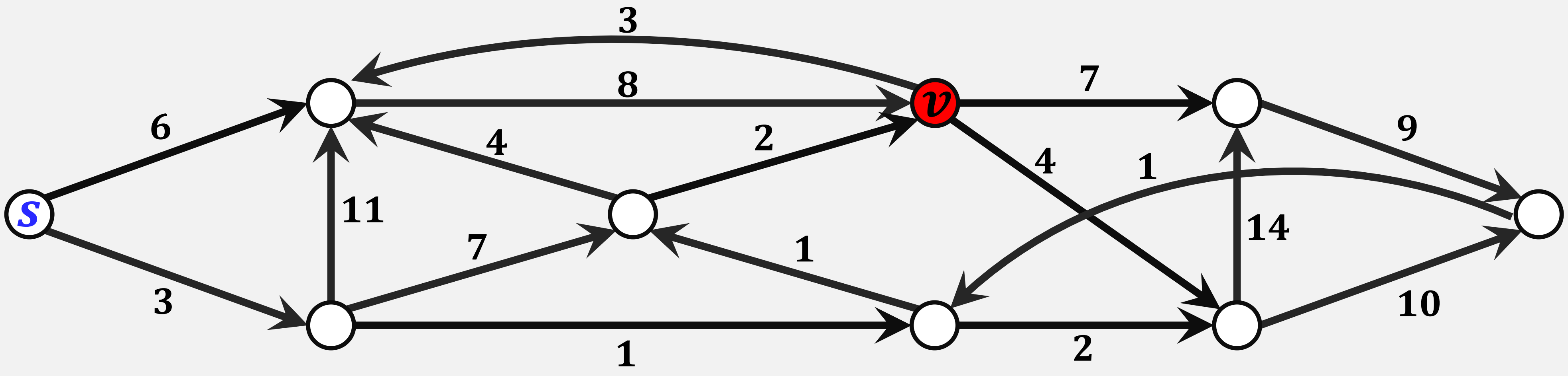

Via Tunnels
3 min
250m
Future EDge
Sprott School of Business, Nicol Building
Canal Building
Architecture Bldg
Architecture Building
University Centre Building
Ollie's Pub & Patio
Tory Building
Carleton University Bookstore
Book store
Azrieli Theatre
Carleton School of Information Technology
Azrieli Pavilion
Digitera
Steacie Bldg
Dunton Tower
Carleton University Spirituality Centre
Tory Building Outdoor Quad
Richcraft Hall
River Building
Tim Hortons
Temporarily closed
University Dr
Campus Ave
Colonel By Dr
Science Technology Center
Carleton Parking Garage P9
Robertson Hall
Alumni Park
Rideau River

Shortest Paths

Input: $G = (V, E)$ – directed graph,
each edge $(u, v) \in E$ has weight $wt(u, v) > 0$,
fixed vertex s (source).

$d(v_1, v_2)$ = length of a path from v_1 to v_2.

Output: for each vertex v:
$\delta(s, v)$ = length of a **shortest** path from s to v.



Shortest Paths

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each edge $(u, v) \in E$ has weight $wt(u, v) > 0$,
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Output: for each vertex v:
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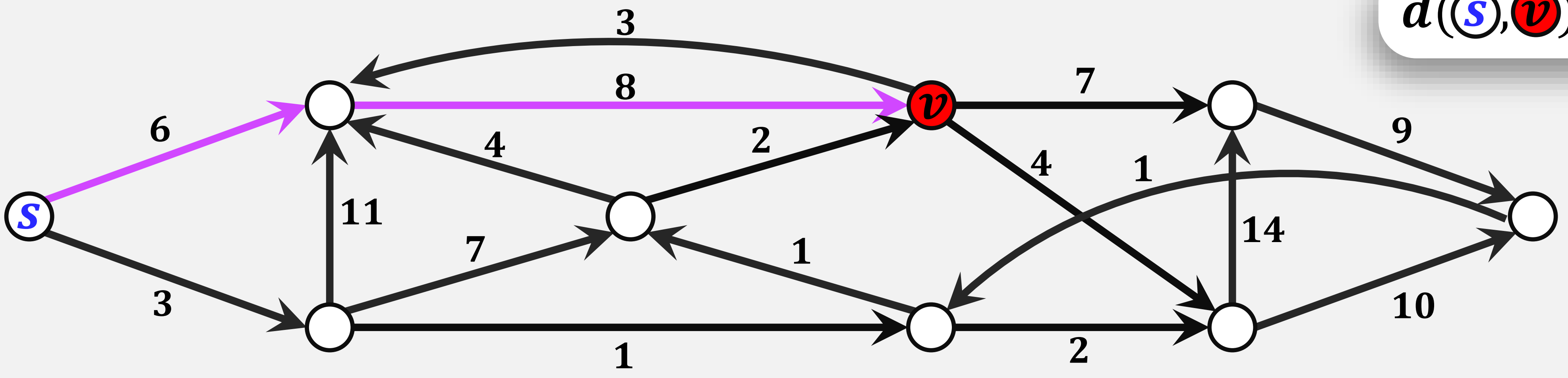
$d(s, v) = 14$



Shortest Paths

Input: $G = (V, E)$ – directed graph,
each edge $(u, v) \in E$ has weight $wt(u, v) > 0$,
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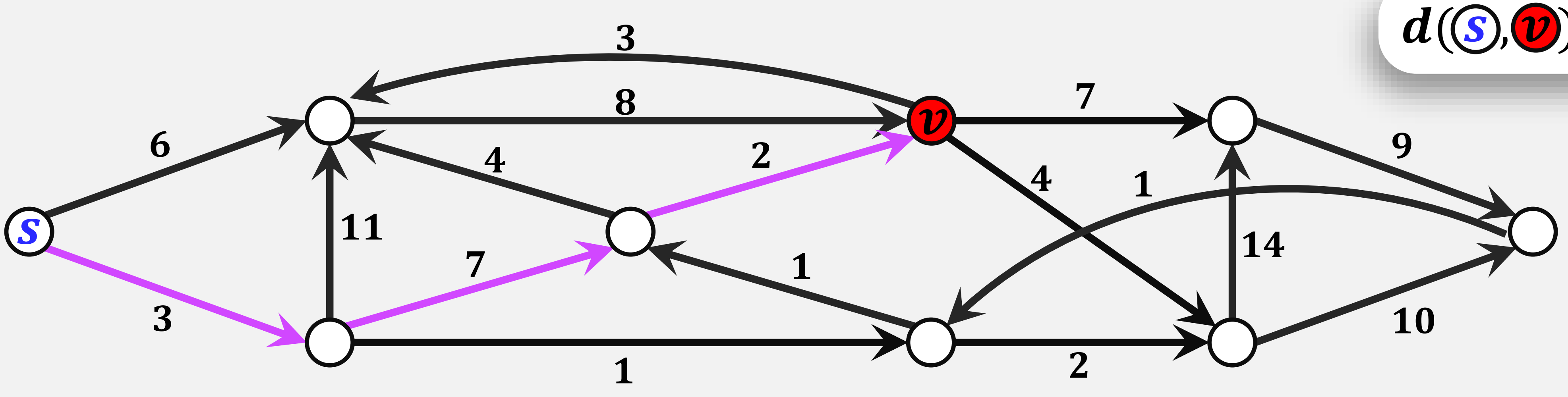
$d(v_1, v_2)$ = length of a path from v_1 to v_2.

Output: for each vertex v:
$\delta(s, v)$ = length of a **shortest** path from s to v.

$d(s, v) = 12$



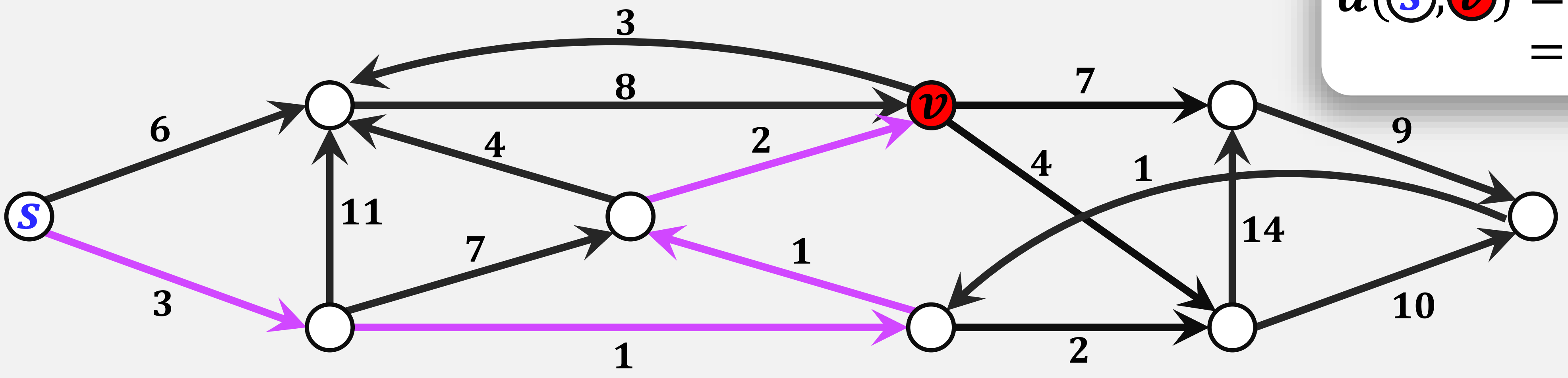
Shortest Paths

Input: $G = (V, E)$ – directed graph,
each edge $(u, v) \in E$ has weight $wt(u, v) > 0$,
fixed vertex s (source).

$d(v_1, v_2)$ = length of a path from v_1 to v_2.

Output: for each vertex v:
$\delta(s, v)$ = length of a **shortest** path from s to v.

$d(s, v) = 7$
$= \delta(s, v)$



How to Find the Shortest Paths?

For each vertex v, maintain a variable:

$d(v)$ = length of a shortest path from s to v found so far.

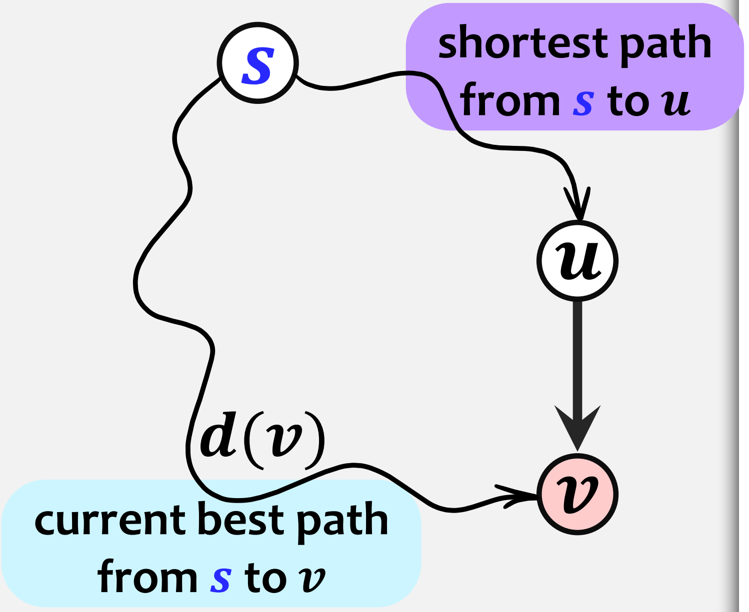
Start: $d(s) = 0,$

$d(v) = \infty$ for every vertex $v \neq s$

Loop: Pick a vertex u for which $d(u) = \delta(s, u)$

For each edge (u, v):

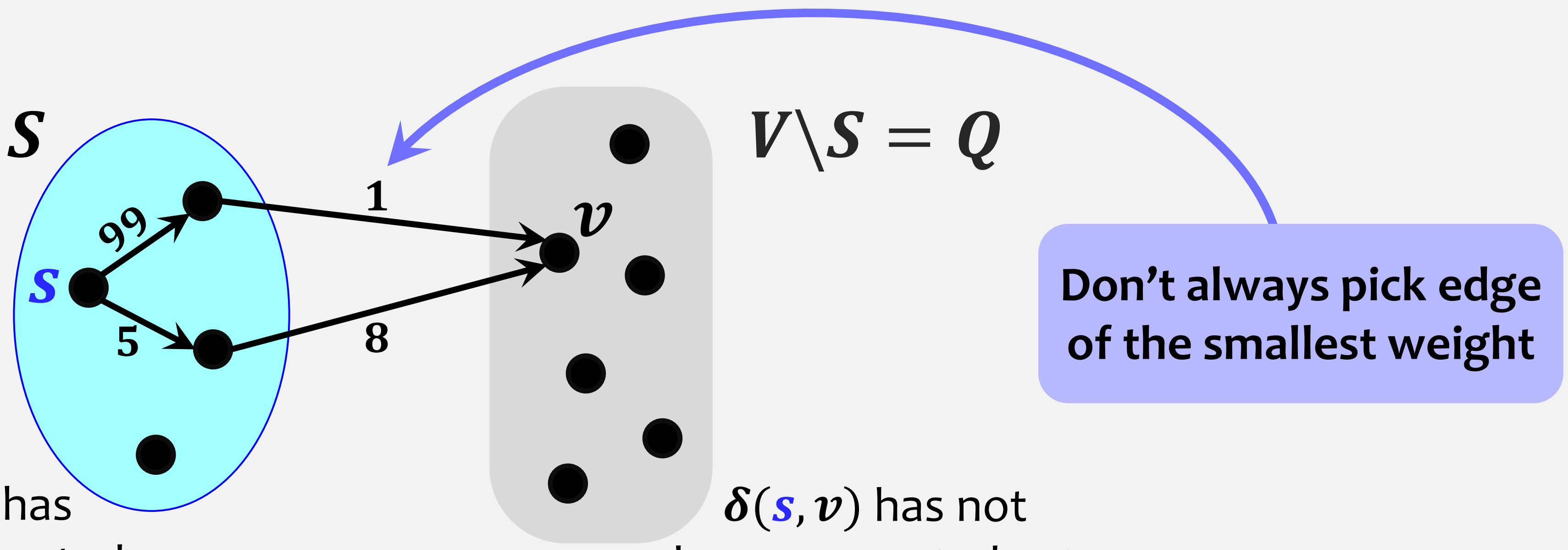
$$d(v) = \min\big(d(v), d(u) + wt(u, v)\big)$$



How do we pick u? How do we know that $d(u) = \delta(s, u)$?

Shortest Paths in Directed Graphs

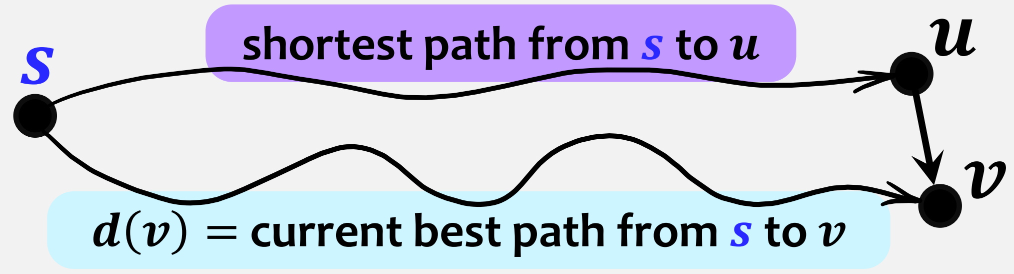
Maintain $\boldsymbol{S} \subseteq \boldsymbol{V}$ such that for all $\boldsymbol{v} \in \boldsymbol{S}$: $\boldsymbol{d}(\boldsymbol{v}) = \boldsymbol{\delta}(\boldsymbol{s}, \boldsymbol{v})$
(i.e. we know the shortest path from $\boldsymbol{s}$ to $\boldsymbol{v}$ for all $\boldsymbol{v} \in \boldsymbol{S}$)

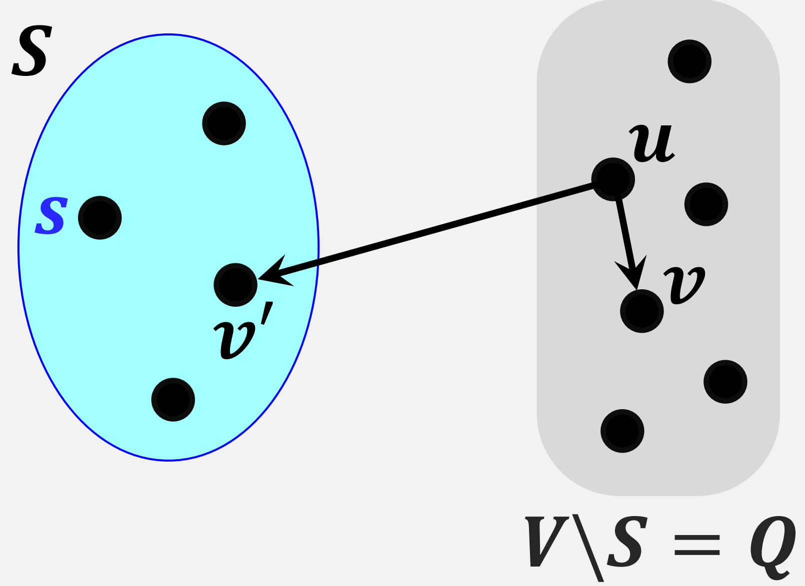


$\boldsymbol{\delta}(\boldsymbol{s}, \boldsymbol{v})$ has been computed

$\boldsymbol{\delta}(\boldsymbol{s}, \boldsymbol{v})$ has not been computed yet

Shortest Paths in Directed Graphs





Start: $S = \emptyset,\ Q = V$

$d(s) = 0,\ \ d(v) = \infty$ for each vertex $v \neq s$.

Loop: Grow S by moving one vertex u from Q to S.

Which vertex u do we move?

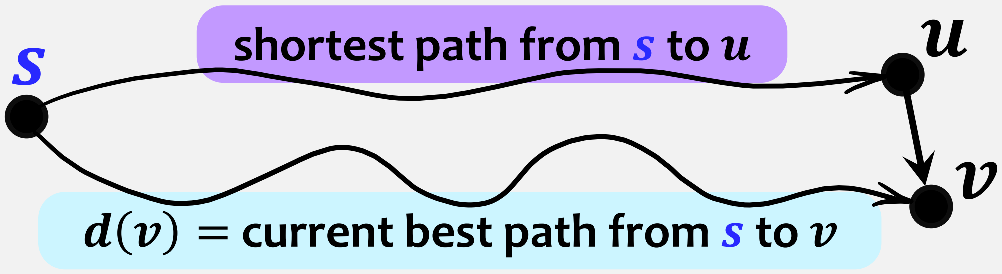
$u =$ vertex of Q for which $d(u)$ is minimum.

Need to prove that for this vertex u, $d(u) = \delta(s, u)$.

Once we move u from Q to S, for each edge (u, v) update

$$d(v) = \min\big(d(v), d(u) + wt(u, v)\big).$$

Dijkstra (1959)

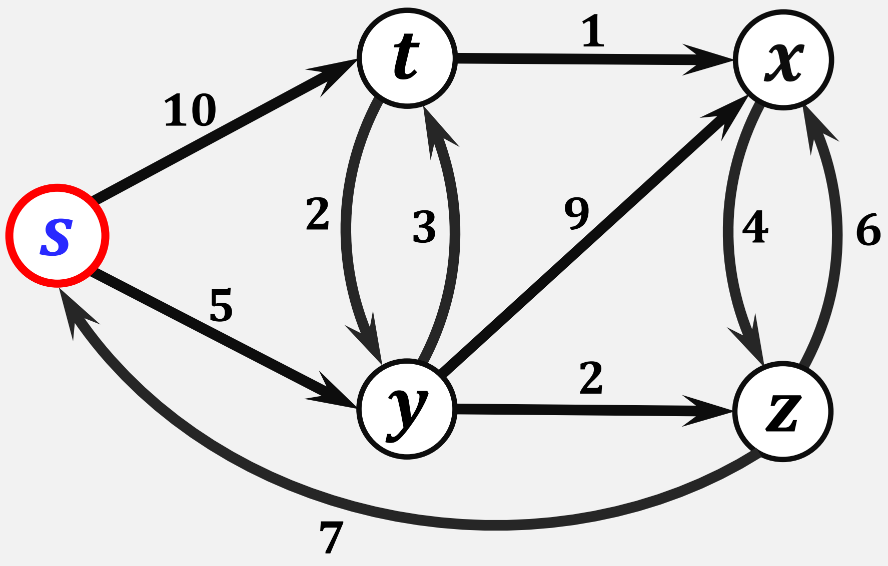


Input: $G = (V, E)$ – arbitrary directed graph (not necessarily acyclic),
each edge $(u, v) \in E$ has weight $wt(u, v) > 0$,
fixed vertex s (source).

Algorithm:

for each vertex $v \in V$: set $d(v) = \infty$.
$d(s) = 0$, $S = \emptyset$, $Q = V$
while $Q \neq \emptyset$:
  u = vertex of Q with minimum $d(u)$. ← $d(u) = \delta(s, u)$
  delete u from Q; add u to S.
  for each edge (u, v):
    if $d(v) > d(u) + wt(u, v)$ then
      $d(v) = d(u) + wt(u, v)$.

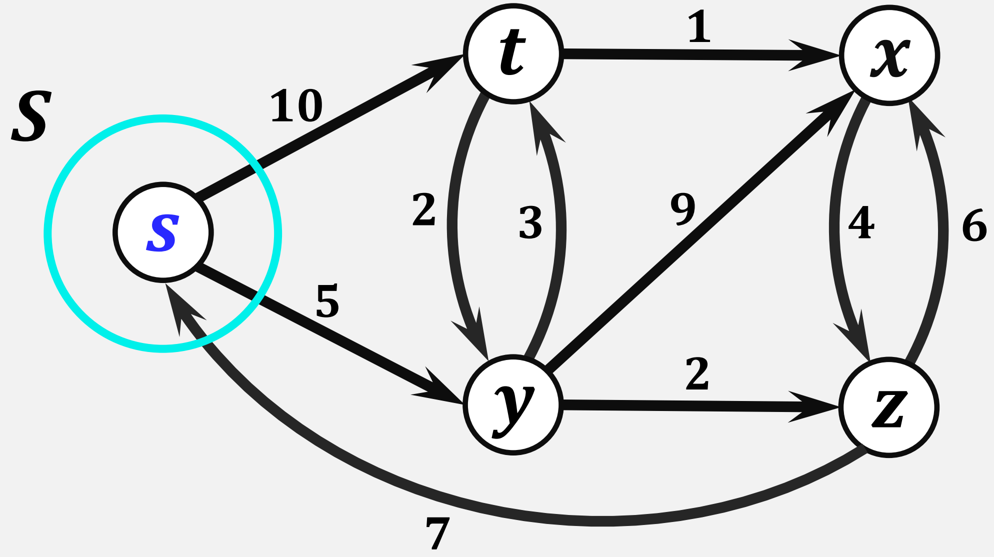
Dijkstra – Example



Q	s	t	x	y	z
d	0	∞	∞	∞	∞

- $u = s$
- $\delta(s, s) = d(s) = 0$
- delete s from Q
- update $d(t)$ and $d(y)$

Dijkstra – Example

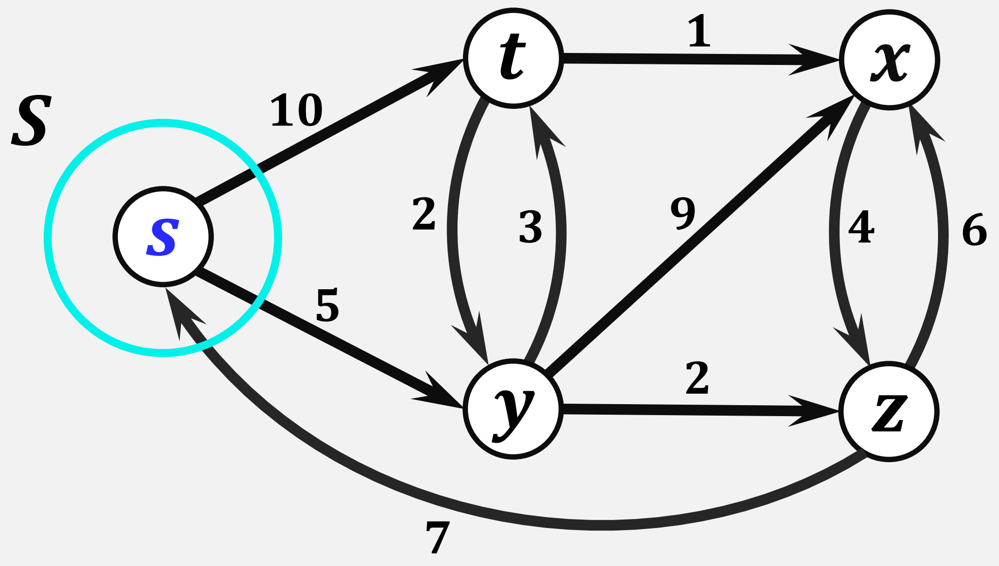


Q	s	t	x	y	z
d	0	∞	∞	∞	∞

- $u = s$
- $\delta(s, s) = d(s) = 0$
- delete s from Q
- update $d(t)$ and $d(y)$

$\delta(s, s) = 0$

Dijkstra – Example



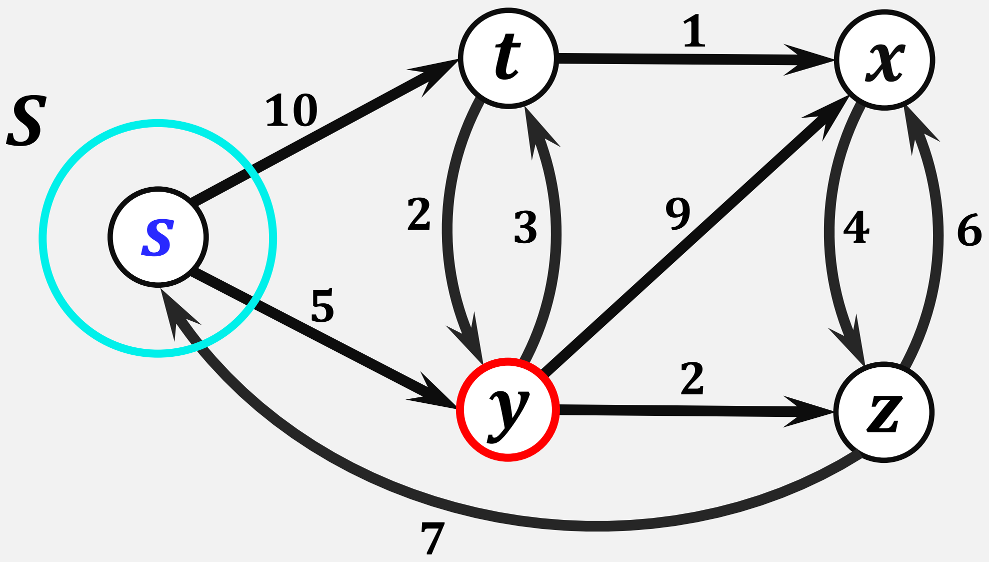
Q	s	t	x	y	z
d	0	∞	∞	∞	∞

Q	t	x	y	z
d	10	∞	5	∞

- $u = s$
- $\delta(s, s) = d(s) = 0$
- delete s from Q
- update $d(t)$ and $d(y)$

$\delta(s, s) = 0$

Dijkstra – Example

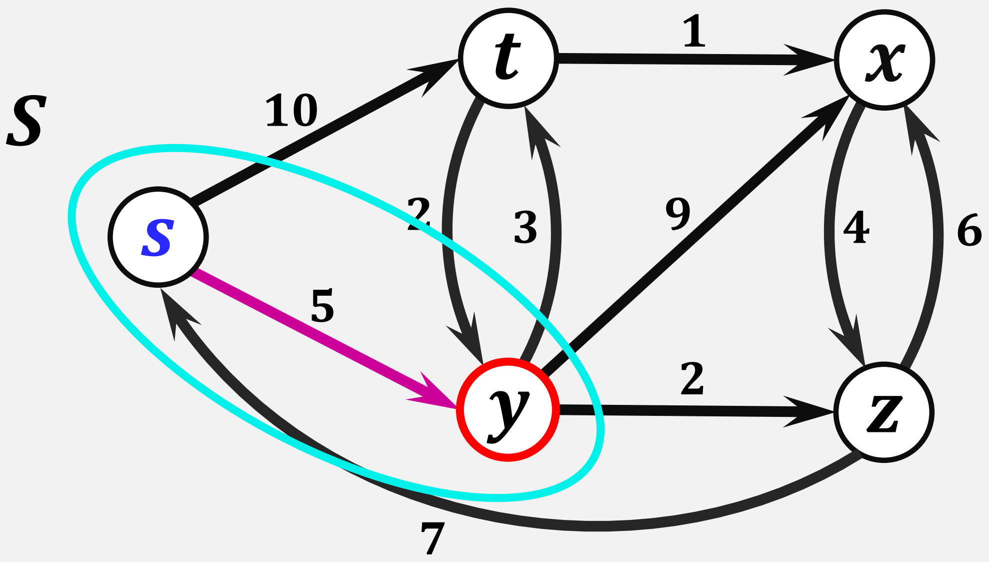


Q	t	x	y	z
d	**10**	∞	**5**	∞

- $u = y$
- $\delta(s, y) = d(y) = 5$
- delete y from Q
- update $d(t), d(x)$, and $d(z)$

$\delta(s, s) = 0$

Dijkstra – Example

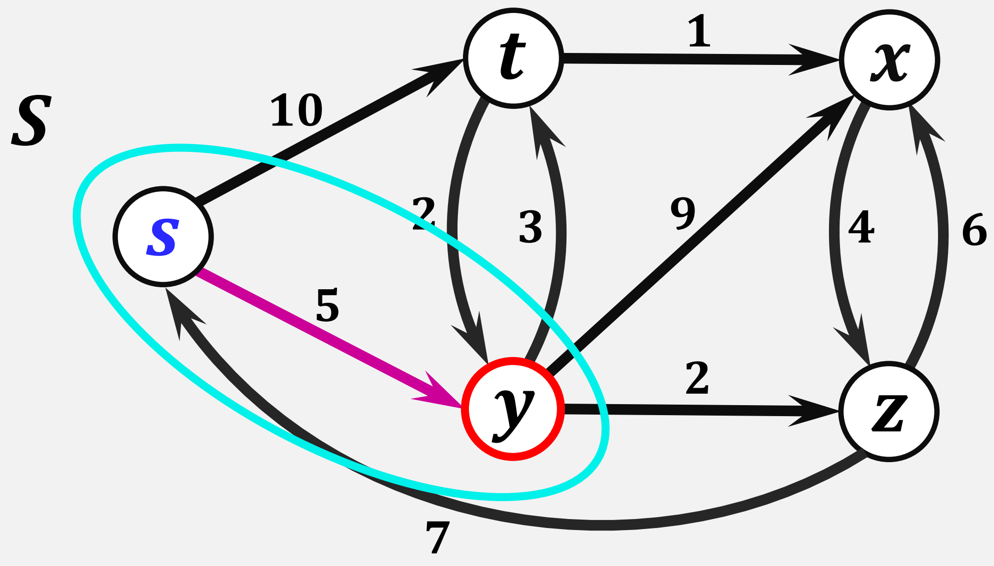


Q	t	x	y	z
d	10	∞	5	∞

- $u = y$
- $\delta(s, y) = d(y) = 5$
- delete y from Q
- update $d(t), d(x)$, and $d(z)$

$\delta(s, y) = 5$
$\delta(s, s) = 0$

Dijkstra – Example



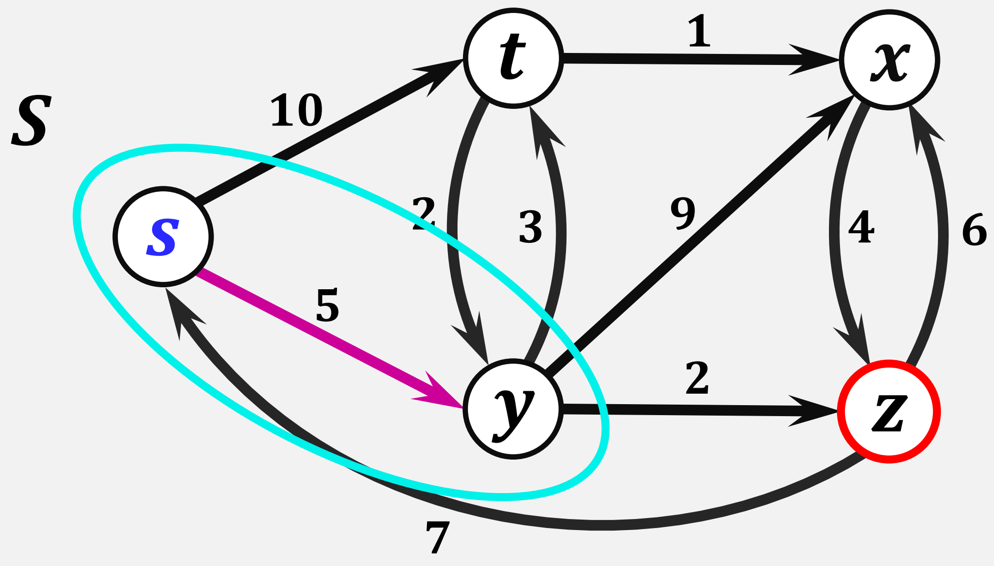
Q	t	x	y	z
d	10	∞	5	∞

Q	t	x	z
d	8	14	7

- $u = y$
- $\delta(s, y) = d(y) = 5$
- delete y from Q
- update $d(t), d(x)$, and $d(z)$

$\delta(s, y) = 5$
$\delta(s, s) = 0$

Dijkstra – Example

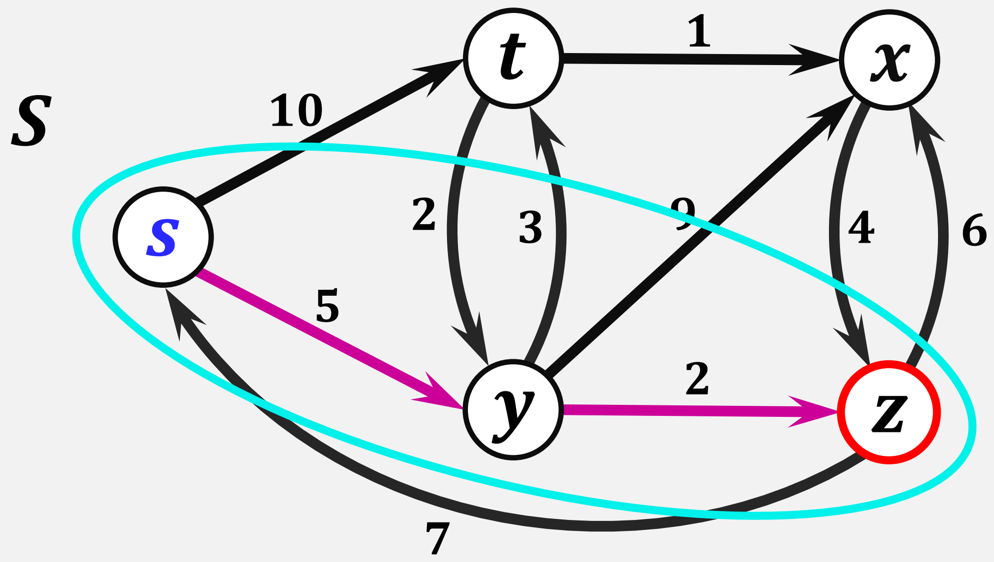


Q	t	x	z
d	8	14	7

- $u = z$
- $\delta(s, z) = d(z) = 7$
- delete z from Q
- update $d(x)$ and $d(s)$

$\delta(s, y) = 5$
$\delta(s, s) = 0$

Dijkstra – Example

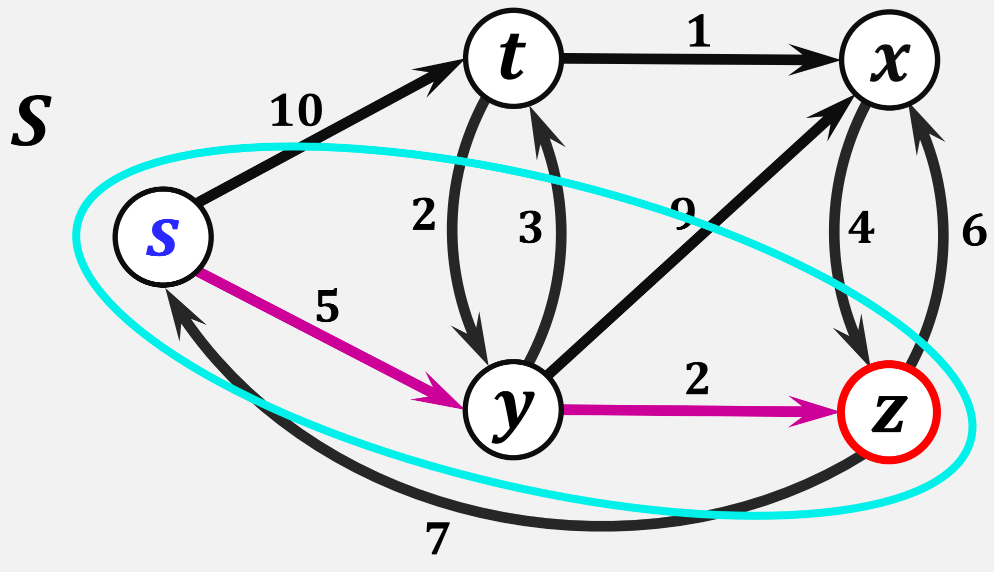


Q	t	x	z
d	8	14	7

- $u = z$
- $\delta(s, z) = d(z) = 7$
- delete z from Q
- update $d(x)$ and $d(s)$

$\delta(s, z) = 7$
$\delta(s, y) = 5$
$\delta(s, s) = 0$

Dijkstra – Example



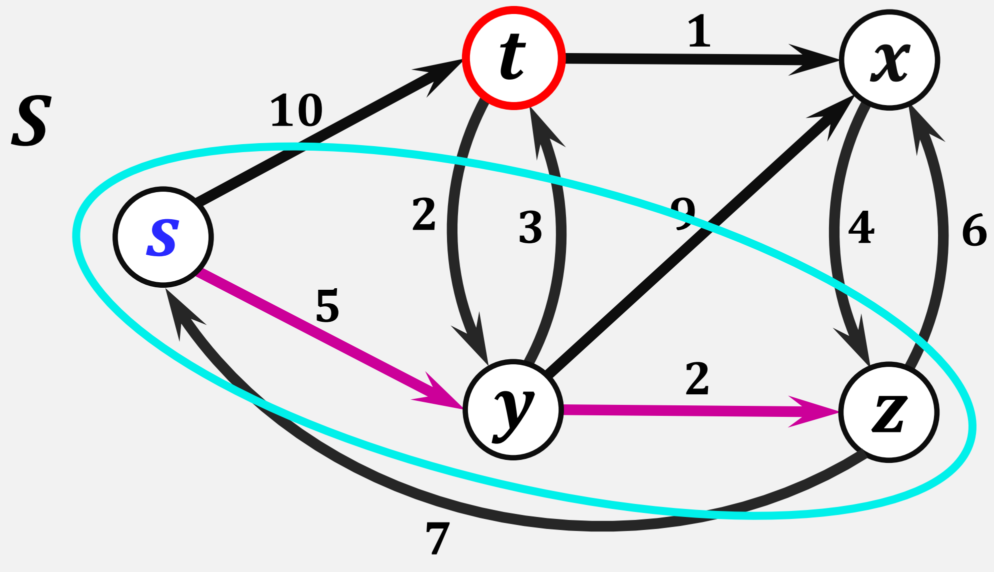
$\delta(s,z) = 7$
$\delta(s,y) = 5$
$\delta(s,s) = 0$

Q	t	x	z
d	8	14	7

Q	t	x
d	8	13

- $u = z$
- $\delta(s,z) = d(z) = 7$
- delete z from Q
- update $d(x)$ and $d(s)$

Dijkstra – Example

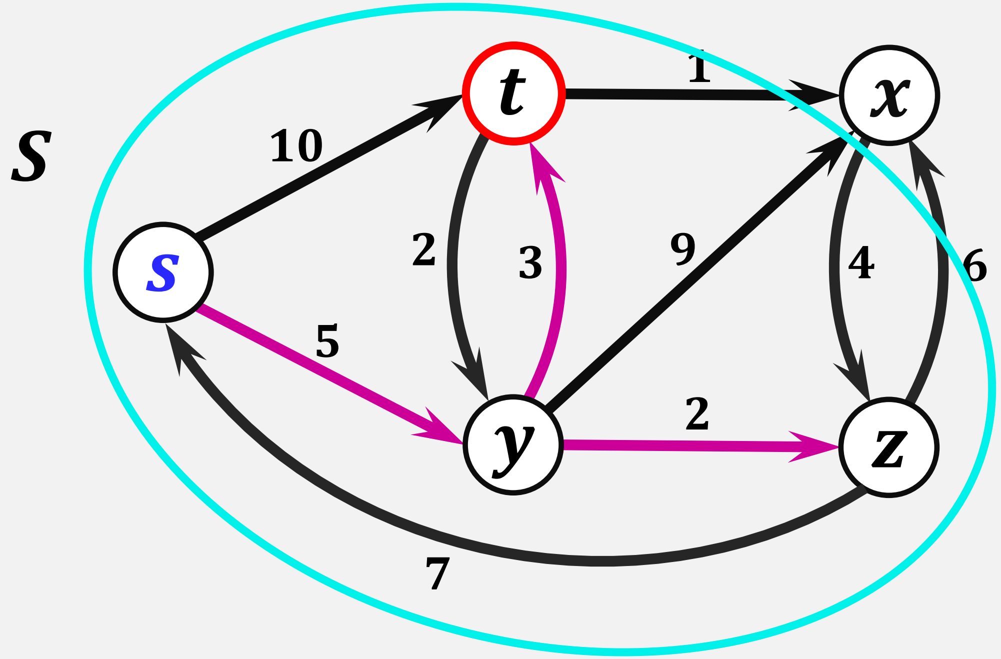


Q	t	x
d	8	13

- $u = t$
- $\delta(s,t) = d(t) = 8$
- delete t from Q
- update $d(x)$ and $d(y)$

$\delta(s,z) = 7$
$\delta(s,y) = 5$
$\delta(s,s) = 0$

Dijkstra – Example

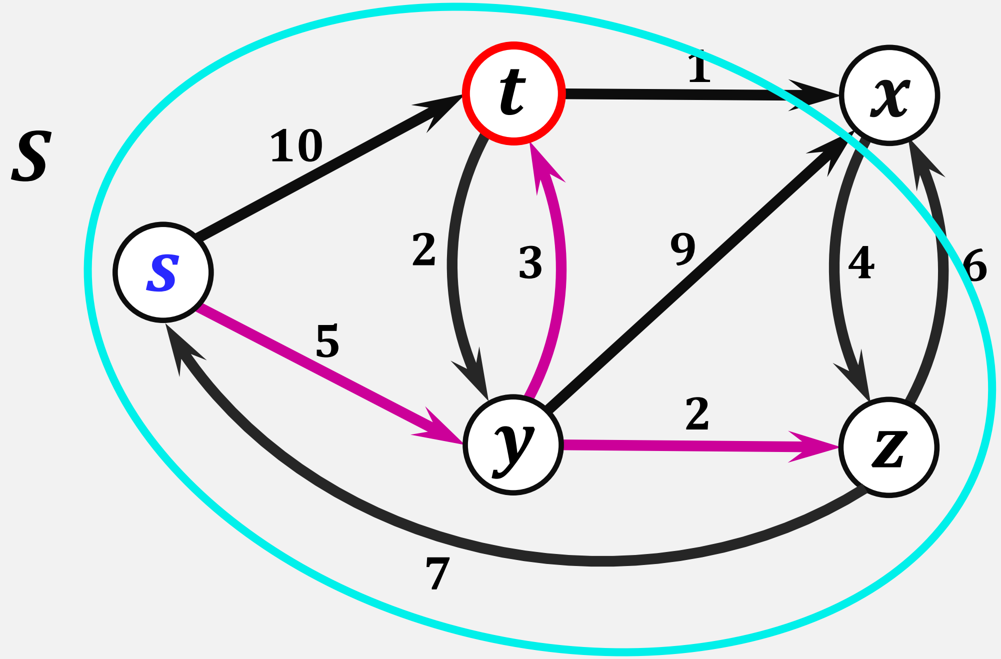


Q	t	x
d	8	13

- $u = t$
- $\delta(s, t) = d(t) = 8$
- delete t from Q
- update $d(x)$ and $d(y)$

$\delta(s, t) = 8$
$\delta(s, z) = 7$
$\delta(s, y) = 5$
$\delta(s, s) = 0$

Dijkstra – Example



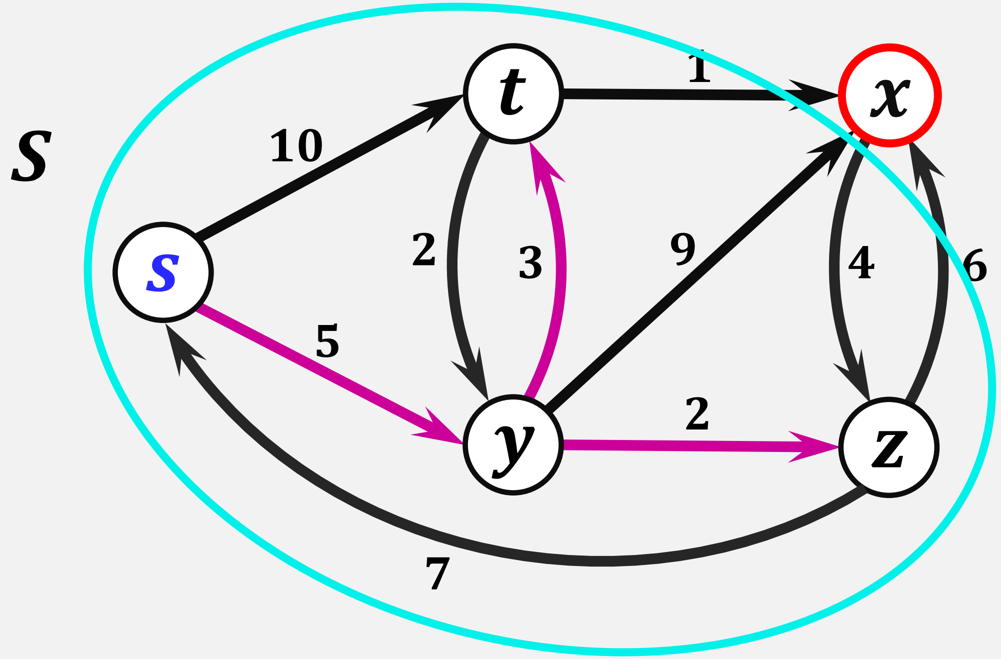
Q	t	x
d	8	13

Q	x
d	9

- $u = t$
- $\delta(s, t) = d(t) = 8$
- delete t from Q
- update $d(x)$ and $d(y)$

$\delta(s, t) = 8$
$\delta(s, z) = 7$
$\delta(s, y) = 5$
$\delta(s, s) = 0$

Dijkstra – Example

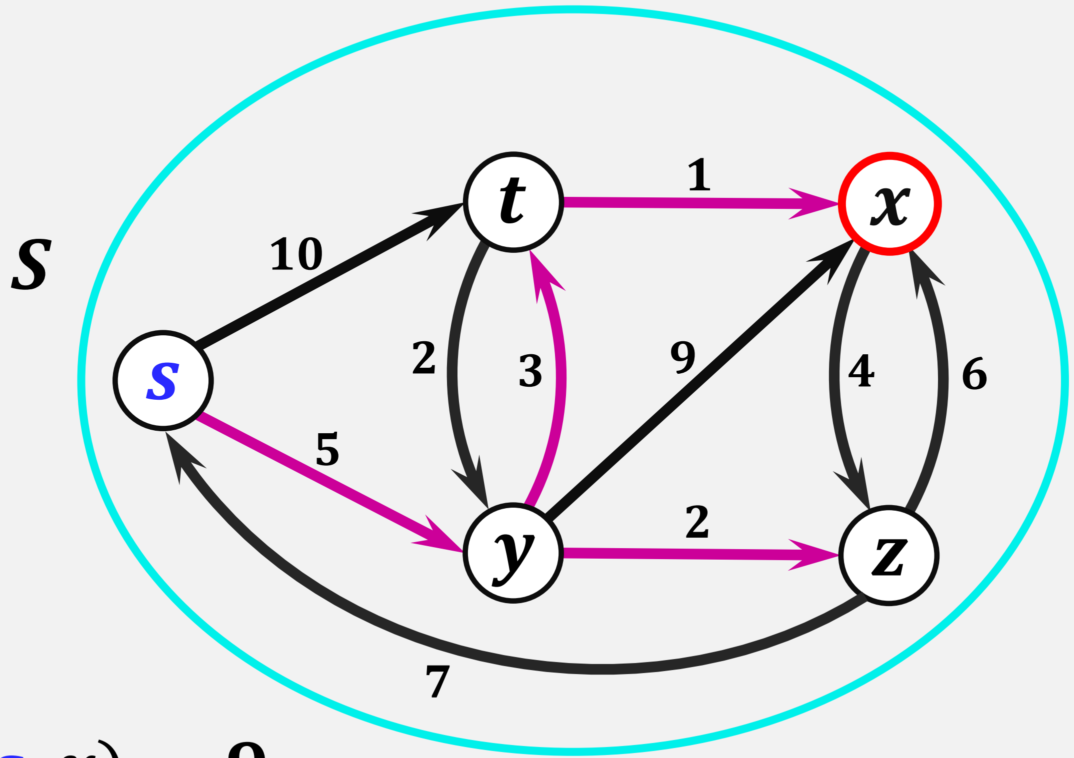


Q	x
d	9

$\delta(s, t) = 8$
$\delta(s, z) = 7$
$\delta(s, y) = 5$
$\delta(s, s) = 0$

- $u = x$
- $\delta(s, x) = d(x) = 9$
- delete x from Q
- update $d(z)$

Dijkstra – Example

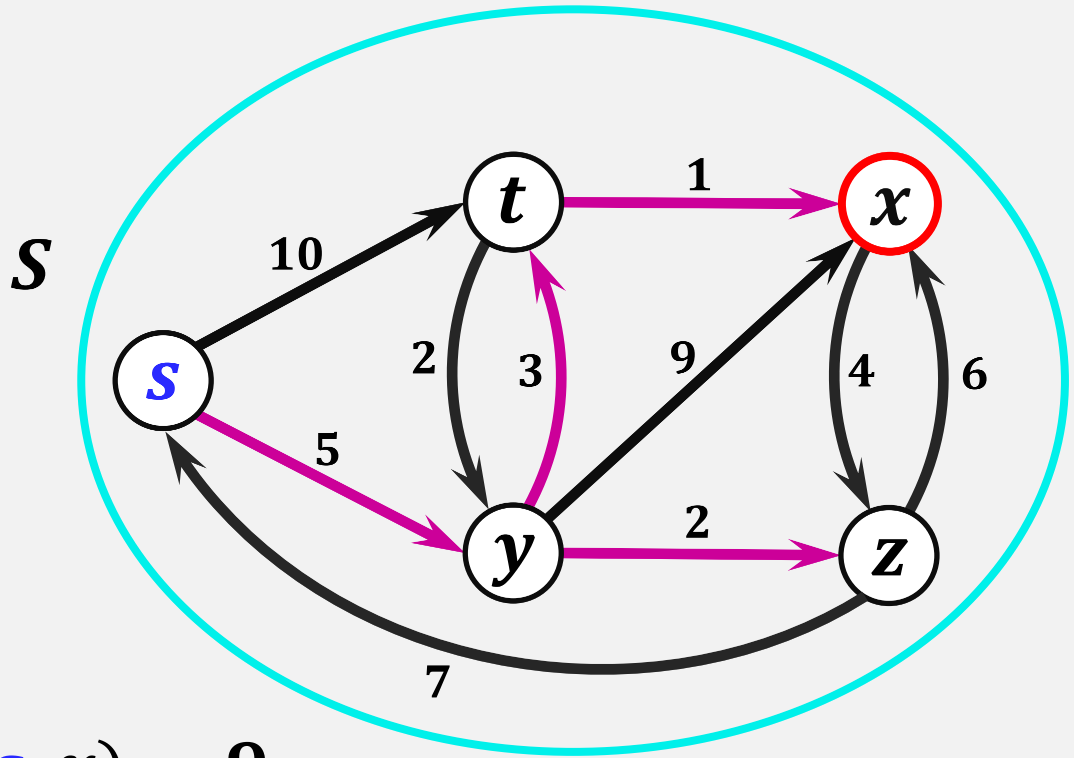


Q	x
d	9

- $u = x$
- $\delta(s, x) = d(x) = 9$
- delete x from Q
- update $d(z)$

$\delta(s, x) = 9$
$\delta(s, t) = 8$
$\delta(s, z) = 7$
$\delta(s, y) = 5$
$\delta(s, s) = 0$

Dijkstra – Example



Q	x
d	9

Q
d

$$\delta(s,x) = 9$$
$$\delta(s,t) = 8$$
$$\delta(s,z) = 7$$
$$\delta(s,y) = 5$$
$$\delta(s,s) = 0$$

- $u = x$
- $\delta(s,x) = d(x) = 9$
- delete x from Q
- update $d(z)$

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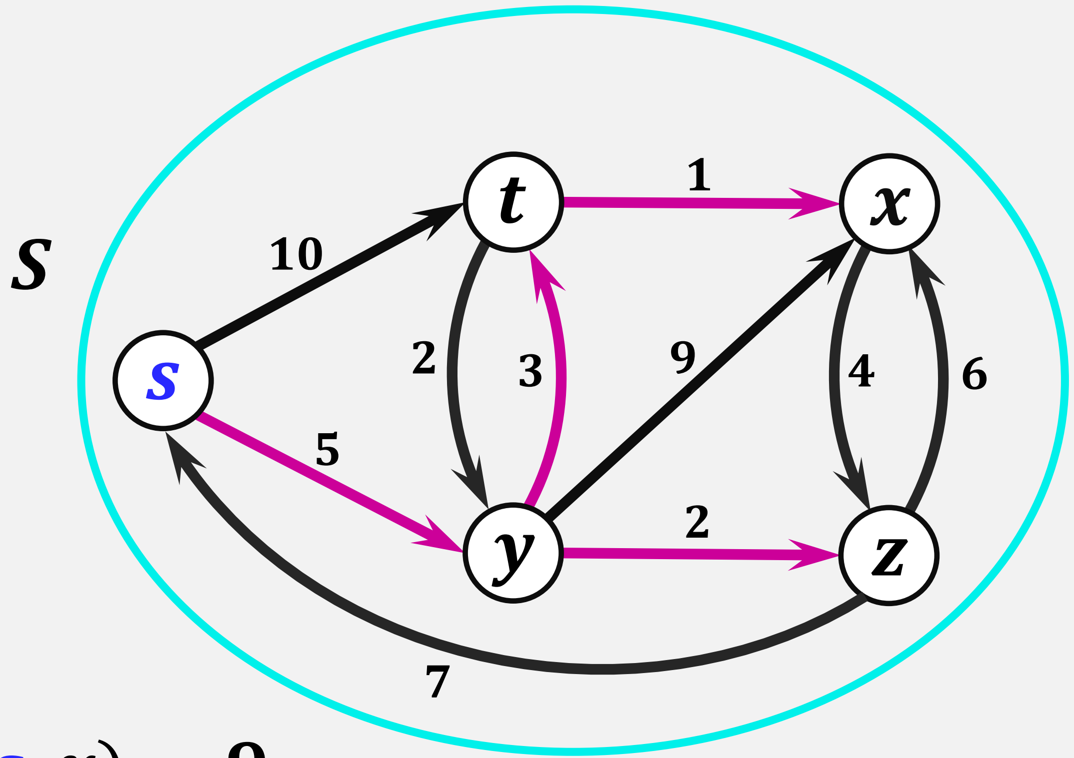
Dijkstra – Example



$$\delta(s, x) = 9$$
$$\delta(s, t) = 8$$
$$\delta(s, z) = 7$$
$$\delta(s, y) = 5$$
$$\delta(s, s) = 0$$

Q
d

- $Q = \emptyset$
- we are done!