Formal Solutions of the Radiative Transfer Equation

The integral formulation

The transfer equation for specific intensity I^+ in the direction of increasing τ is

$$\frac{dI^+}{d\tau} = S - I^+ \ , \tag{1}$$

while for I^-, in the direction of decreasing τ it is

$$\frac{dI^-}{d\tau} = I^- - S \ . \tag{2}$$

Changing variables to

$$Q^+ = I^+ - S \tag{3}$$

$$Q^- = I^- - S \tag{4}$$

we have

$$\frac{dQ^+}{d\tau} = -Q^+ - \frac{dS}{d\tau} \tag{5}$$

$$\frac{dQ^-}{d\tau} = Q^- - \frac{dS}{d\tau} . \tag{6}$$

These have the same form as the native RTE, but with a source term $\frac{dS}{d\tau}$ rather than *S*.

For an assumed quadratic source function *S*, the source function derivative is a linear function, and it is easy to find the analytical solution, going from one point to the next:

$$Q^+(\tau_i) = Q^+(\tau_{i-1})e^{-\Delta\tau} + S'(\tau_{i-1})(1-e^{-\Delta\tau}) + S''(\tau_{i-1})\left[(1-e^{-\Delta\tau}) - \Delta\tau\, e^{-\Delta\tau}\right], \tag{7}$$

where

$$\Delta\tau = \tau_i - \tau_{i-1} \tag{8}$$

and in the opposite direction

$$Q^-(\tau_{i-1}) = Q^-(\tau_i)e^{-\Delta\tau} - S'(\tau_i)(1-e^{-\Delta\tau}) + S''(\tau_i)\left[(1-e^{-\Delta\tau}) - \Delta\tau\, e^{-\Delta\tau}\right]. \tag{9}$$

Both the forward and reverse computations across a zone need the same three coefficients:

$$a_i = e^{-\Delta\tau} \tag{10}$$

$$b_i = (1-e^{-\Delta\tau}) \tag{11}$$

$$c_i = (1-e^{-\Delta\tau}) - \Delta\tau\, e^{-\Delta\tau} \ . \tag{12}$$

To compute these it is sufficient to evaluate the exponential function only once.

The Feautrier formulation

The transfer equation for specific intensity I^+ in the direction of increasing τ is

$$\frac{dI^+}{d\tau} = S - I^+ \ , \qquad (13)$$

while for I^- , in the direction of decreasing τ it is

$$\frac{dI^-}{d\tau} = I^- - S \ . \qquad (14)$$

Defining

$$P = (I^+ + I^-)/2 \qquad (15)$$

and

$$R = (I^- - I^+)/2 \qquad (16)$$

we have

$$\frac{dP}{d\tau} = R \qquad (17)$$

and

$$\frac{dR}{d\tau} = P - S \ , \qquad (18)$$

which leads to

$$\frac{d^2P}{d\tau^2} = P - S \ . \qquad (19)$$

This formulation has the great advantage, relative to the native form (1), that it is stable for large values of optical depth increments. Commonly used boundary conditions are vanishing in-falling radiation, for which R=P and hence

$$\frac{dP}{d\tau} = P \ , \qquad (20)$$

and I=S so

$$P = S \ . \qquad (21)$$

The net-heating Feautrier formulation

At large optical depths the Feautrier formulation becomes susceptible to numerical round-off, since P approaches S. If one desires the difference P-S, e.g. to compute the heating / cooling due to radiation, it is advantageous to write the equation directly in terms of the difference $Q=P-S$, for which the RTE becomes

$$\frac{d^2Q}{d\tau^2} = Q - \frac{d^2S}{d\tau^2}. \tag{22}$$

At large optical depths, where Q is small, this goes to the diffusion approximation,

$$Q = \frac{d^2S}{d\tau^2}, \tag{23}$$

without round-off error.

The boundary condition corresponding to no in-falling intensity at the surface is

$$\frac{dQ}{d\tau} = \frac{dP}{d\tau} - \frac{dS}{d\tau} = P - \frac{dS}{d\tau} = Q + S - \frac{dS}{d\tau}. \tag{24}$$

With in-falling intensity I^+_0 at the surface it is

$$\frac{dQ}{d\tau} = R - \frac{dS}{d\tau} = P - I_1^+ - \frac{dS}{d\tau} = Q + (S - \frac{dS}{d\tau}) - I_1^+, \tag{25}$$

while for reverse in-falling intensity I^-_N, the condition is

$$\frac{dQ}{d\tau} = R - \frac{dS}{d\tau} = I_N^- - P - \frac{dS}{d\tau} = -Q - (S + \frac{dS}{d\tau}) + I_N^-. \tag{26}$$

Written with Q-operators on the LHS, these become

$$\frac{dQ}{d\tau} - Q = S - \frac{dS}{d\tau} - I_1^+ \tag{27}$$

and

$$\frac{dQ}{d\tau} + Q = -(S + \frac{dS}{d\tau}) + I_N^-. \tag{28}$$

The simple boundary condition corresponding to $I=S$ is

$$Q = 0. \tag{29}$$

Discretization of the Feautrier formulation

At interior points the second order formulation of the Feautrier equation gives matrix elements

$$a_{i,i-1} = 1/(\Delta\tau_{i-1/2}\Delta\tau_i) \tag{30}$$

$$a_{i,i} = 1 - 1/(\Delta\tau_{i-1/2}\Delta\tau_i) - 1/(\Delta\tau_{i+1/2}\Delta\tau_i) \tag{31}$$

$$a_{i,i+1} = 1/(\Delta\tau_{i+1/2}\Delta\tau_i) \tag{32}$$

where

$$\Delta\tau_{i-1/2} = \tau_i - \tau_{i-1} \tag{33}$$

$$\Delta\tau_i = ½(\Delta\tau_{i-1/2} + \Delta\tau_{i+1/2}) \tag{34}$$

At the first boundary (i=1) we need a 2nd order accurate relation connecting points i=1 and i=2, so we use a Taylor expansion:

$$Q_2 = Q_1 + \Delta\tau\, Q'_1 + ½\Delta\tau^2\, Q''_1\,, \tag{35}$$

where we use

$$Q'_1 = Q_1 + S_1 - S'_1 - I_1^+ \tag{36}$$

$$Q''_1 = Q_1 - S''_1 \tag{37}$$

to get

$$Q_2 = Q_1 + \Delta\tau\,(Q_1 + S_1 - S'_1 - I_1^+) + ½\Delta\tau^2\,(Q_1 - S''_1) \tag{38}$$

or

$$Q_2 - Q_1(1 + \Delta\tau + ½\Delta\tau^2) = \Delta\tau(S_1 - S'_1 - I_1^+) - ½\Delta\tau^2 S''_1\,. \tag{39}$$

At the second boundary (i=n)

$$Q_{n-1} = Q_n - \Delta\tau\, Q'_n + ½\Delta\tau^2\, Q''_n\;\;, \tag{40}$$

where we use

$$Q'_n = -Q_n - S_n - S'_n + I_n^+ \tag{41}$$

$$Q''_n = Q_n - S''_n \tag{42}$$

to get

$$Q_{n-1} = Q_n + \Delta\tau\,(Q_n + S_n + S'_n - I_n^-) + ½\Delta\tau^2\,(Q_n - S''_n)\,, \tag{43}$$

or

$$Q_{n-1} - Q_n(1 + \Delta\tau + ½\Delta\tau^2) = \Delta\tau(S_n + S'_n - I_n^-) + ½\Delta\tau^2 S''_n\,. \tag{44}$$