Computational Astrophysics

3a. Magnetohydrodynamics part I: The Induction Equation

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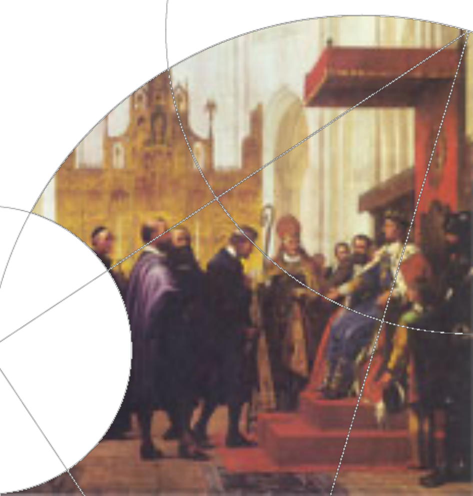
Slides adapted from P Benítez-Llambay



Topics today

- Plasmas and MHD
- Equations of MHD
- The induction equation
- A simple solver based on advection
- Dissipation, flux expulsion, and reconnection in 2D

- **Assignment 3a:**
  - Play around with the advection and reconnection aspects of the induction equation

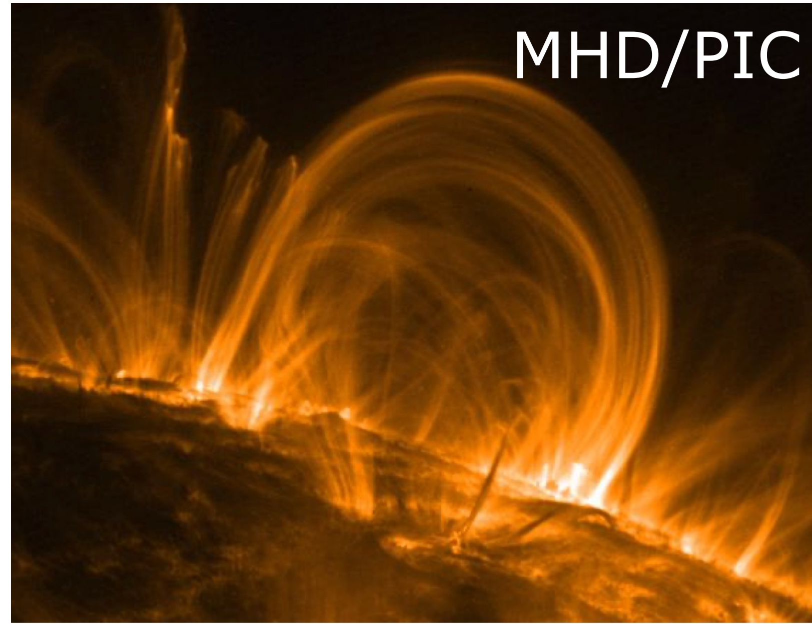
Dynamics of Plasmas and MHD

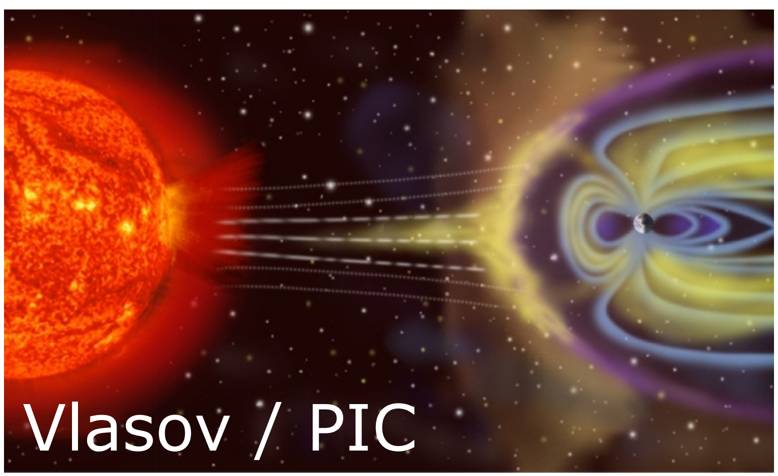
There are several ways to study plasma dynamics

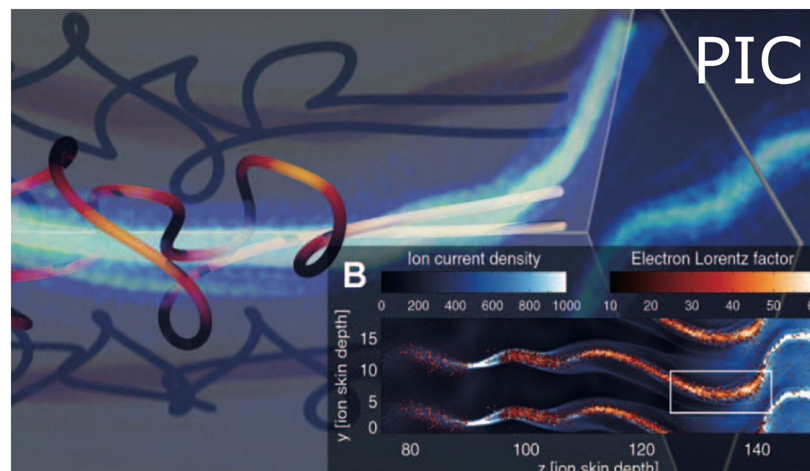
- By solving the equation of motion of particles within a volume under the presence of electromagnetic fields. This is, however, very expensive due to the number of particles involved (Newton equations, Lorentz Force, Maxwell equations), and the constraint from EM waves.
- By solving for a distribution function in phase-space (x,v). This is known as plasma kinetic theory, and it provides the most accurate description of plasmas (Vlasov equation, but 3D+3V+1T!!).
- The MHD approximation, which treats the plasma collectively as a neutral (single) "fluid", subject to electric and magnetic forces (MHD equations, similar to hydrodynamics).

MHD assumes frequent collisions and in the simplest form perfect coupling between fields and fluids. It is a simplified but frequently used model to describe the dynamics of plasmas.

Dynamics of Plasmas and MHD







Equations of Magnetohydrodynamics

- The MHD equations describing the collective behaviour of the plasma gives a new force term in the fluid equation
  In conservative form:

$$\partial_t \rho \boldsymbol{v} + \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{v} \otimes \boldsymbol{v} + P_{tot} \mathbb{I} + \frac{1}{4\pi} \boldsymbol{B} \otimes \boldsymbol{B} \right] = 0, P_{tot} = P_{gas} + \frac{1}{8\pi} B^2$$

  In Lagranian form:

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{\rho} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{\rho}$$

  where $\boldsymbol{B}$, $\boldsymbol{J} = \frac{c}{4\pi} \boldsymbol{\nabla} \times \boldsymbol{B}$ are the magnetic field and current density respectively.
- The magnetic field evolves according to Faraday's law of induction, which in the MHD approximation is:

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B})$$

  where we have neglected the displacement current, suppressing electromagnetic waves.
- In addition, the evolution needs to satisfy $\nabla \cdot \boldsymbol{B} = 0$
- The new ingredients are the induction equation and the Lorentz force (see chap 8 and chap 7 in the Theoretical Astrophysics notes)

The resistive induction equation

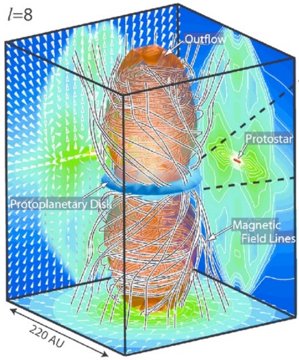
- Today we will talk about the induction equation

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B})$$

- If the magnetic field is not perfectly coupled to the fluid, the equation can also contain resistivity:

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B} - \eta\, \boldsymbol{\nabla} \times \boldsymbol{B})$$

with η being the resistivity (the inverse of the conductivity).



- If η is constant we can rewrite the last term

$$-\boldsymbol{\nabla} \times (\eta\, \boldsymbol{\nabla} \times \boldsymbol{B}) = -\eta\, \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B} = -\eta(\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}) = \eta \boldsymbol{\nabla}^2 \boldsymbol{B}$$

showing that resistivity diffuses the magnetic field

- A part from Ohmic resistivity, the Induction Equation can also contain more terms that arises due to non-perfect coupling between the gas and the magnetic field (Ambipolar diffusion and Hall effect) and are so-called non-ideal MHD effects.
- Non-ideal effects are important in fluids with few free charges, such as accretion disks around young stars or the coldest phases of the interstellar medium; the molecular clouds.

B evolves due to the perpendicular velocity

- The magnetic field evolves due to

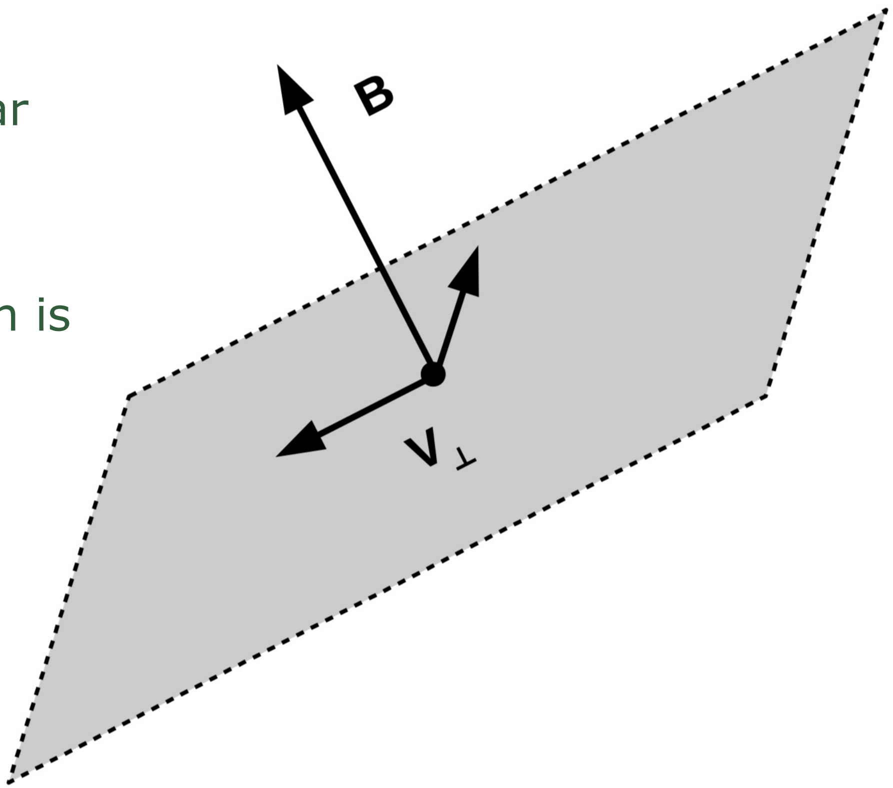
$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

- We can decompose the velocity in components parallel and perpendicular to the magnetic field

$$\boldsymbol{v} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}$$

- The rhs of the induction equation then is

$$\begin{aligned} \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) &= \nabla \times (\boldsymbol{v}_{\parallel} \times \boldsymbol{B}) + \nabla \times (\boldsymbol{v}_{\perp} \times \boldsymbol{B}) \\ &= \nabla \times (\boldsymbol{v}_{\perp} \times \boldsymbol{B}) \end{aligned}$$



and the induction equation becomes

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{v}_{\perp} \times \boldsymbol{B})$$

B only evolves due to the velocity perpendicular to it

Induction is almost advection

- The term $\nabla\times(\boldsymbol{v}\times\boldsymbol{B})$ can also be expanded as

$$\nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = \boldsymbol{v}\,(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}\,(\nabla\cdot\boldsymbol{v}) + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\,\mathbf{B}$$

$$= -\boldsymbol{B}\,(\nabla\cdot\boldsymbol{v}) + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\,\mathbf{B}$$

- This gives the induction equation

$$\partial_t\boldsymbol{B} = -\boldsymbol{B}\,(\nabla\cdot\boldsymbol{v}) + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\,\mathbf{B}$$

- Collecting terms we get an advection equation for the magnetic field with a source term:

$$\partial_t\boldsymbol{B} + \nabla\cdot(\,\boldsymbol{B}\otimes\boldsymbol{v}) = (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{v}, \qquad \nabla\cdot(\,\boldsymbol{B}\otimes\boldsymbol{v}) = \partial_j(\boldsymbol{B}\boldsymbol{v}_j)$$

Question: Can we use our finite volume hydro-solver for this equation?

Sources of magnetic fields

- We can also rewrite the equation in the Lagranian frame to see how the magnetic field changes while moving along the velocity streamlines

$$D_t\boldsymbol{B} = \partial_t\boldsymbol{B} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\mathbf{B} = -\boldsymbol{B}\,(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}$$

- We can get two insights from this

1. Magnetic field is not conserved along fluid stream lines (even in the incompressible limit, $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$)

2. Velocity gradients in the direction of ***B*** are sources of magnetic field

Alfvén frozen in theorem

- In a non-resistive fluid, the magnetic field is "frozen" into the fluid. In other words, magnetic field lines move attached to the fluid.
- This is also formulated as

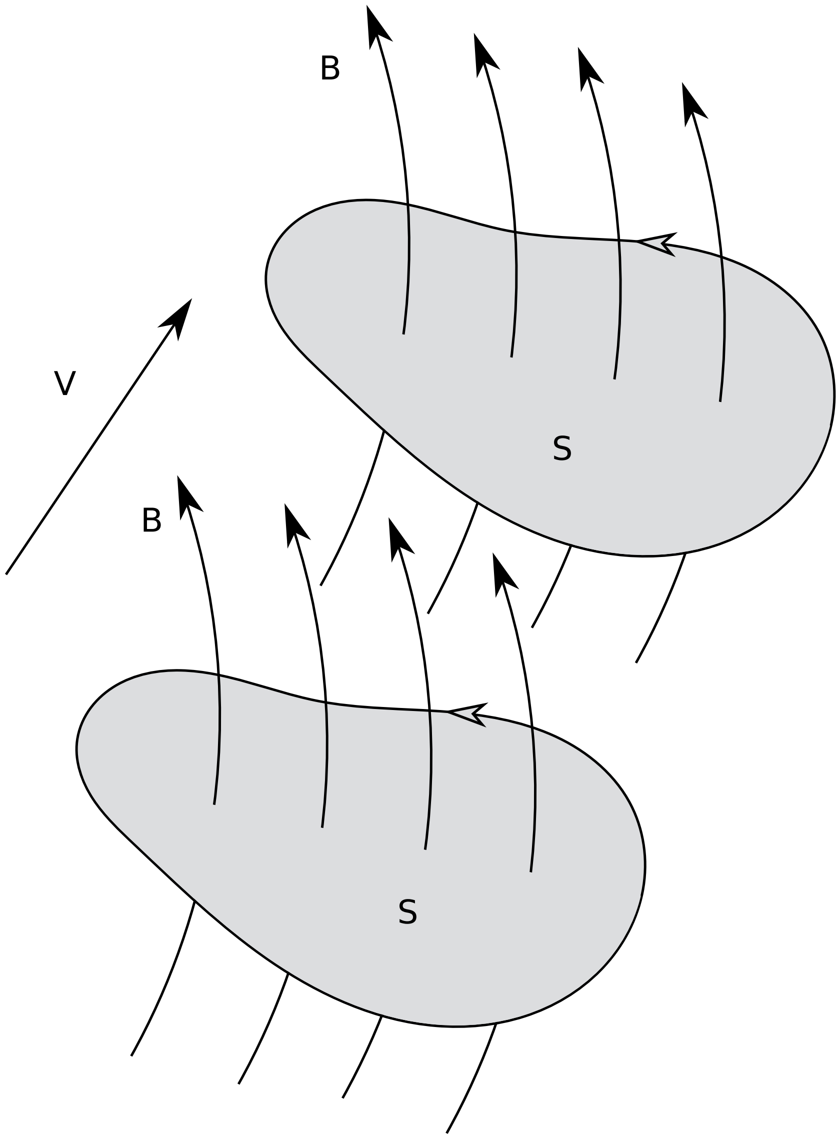
The magnetic flux through a surface moving with the fluid is conserved.

$$\frac{D\Phi_B}{Dt} = 0$$

where the flux is defined as

$$\Phi_B = \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$$

You can find a derivation of this in the theoretical astrophysics notes chap 8.4



Energy balance determines the dynamics

Induction equation as an advection equation

- The induction equation is

$$\partial_t \boldsymbol{B} + \boldsymbol{\nabla} \cdot (\boldsymbol{B} \otimes \boldsymbol{v}) = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} \equiv \frac{d\boldsymbol{v}}{d\boldsymbol{B}}$$

  so, if $(\boldsymbol{B} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} = \boldsymbol{0}$, induction becomes advection. This is exactly true if the velocity $\boldsymbol{v}$ does not vary along $\boldsymbol{B}$.

- Let us rotate the axes such that $\boldsymbol{B} = B_z(x,y,z)\boldsymbol{e}_z$ and $\boldsymbol{v} = \boldsymbol{v}(x,y)$ (i.e. $\boldsymbol{v}$ does not vary along $\boldsymbol{B}$), then:

$$\partial_t B_z + \boldsymbol{\nabla} \cdot (B_z\, \boldsymbol{v}) = 0$$

$$\partial_t B_z + \partial_x (B_z\, v_x) + \partial_y \left(B_z\, v_y\right) + \cancel{v_z(x,y)\partial_z(B_z)} = 0$$

- However, the solenoidal constraint implies $\partial_z B_z = 0$, so, the induction equation reduces to a 2D advection equation.

- Question: Can we solve this equation numerically with the same methods used for hydrodynamics?

Induction equation as an advection equation for the vector potential

- Another interesting case is when both the velocity and the magnetic fields are co-planar.
- Configuration: $\boldsymbol{B} = B_x(x,y)\boldsymbol{e_x} + B_y(x,y)\boldsymbol{e_y}$ and $\boldsymbol{v} = v_x(x,y)\boldsymbol{e_x} + v_y(x,y)\boldsymbol{e_y}$
- In this case we can rewrite the equations in terms of the vector potential: $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$
- For this configuration the vector potential can be written $\boldsymbol{A} = A(x,y)\,\boldsymbol{e_z}$, and the induction equation is

$$\partial_t(\boldsymbol{\nabla}\times\boldsymbol{A}) + \boldsymbol{\nabla}\times[(\boldsymbol{\nabla}\times\boldsymbol{A})\times\boldsymbol{v}] = 0$$

$$\boldsymbol{\nabla}\times[\,\partial_t\boldsymbol{A} + (\boldsymbol{\nabla}\times\boldsymbol{A})\times\boldsymbol{v}\,] = 0$$

$$\boldsymbol{\nabla}\times[\,(\partial_t A + \boldsymbol{v}\cdot\nabla A)\,\mathbf{e_z}\,] = 0$$

- The term in the parenthesis is the z-component of the gradient of a scalar, since $\boldsymbol{\nabla}\times\nabla\phi = 0$, such that

$$\partial_t A + \boldsymbol{v}\cdot\nabla A = \partial_z\,\phi$$

- However, the problem is 2D, so the r.h.s. must be zero, implying

$$\partial_t A + \boldsymbol{v}\cdot\nabla A = 0$$

We conclude that the vector potential is conserved along fluid lines

Dissipation, flux expulsion, and reconnection

- This 2D configuration has an important property called *"flux expulsion"*.
- The equation of motion:

$$\partial_t A + \boldsymbol{v} \cdot \nabla A = 0$$

- Admits steady-state solutions given by:

$$\boldsymbol{v} \cdot \nabla A = 0$$

- Geometrically it means that the gradient of *A* should be orthogonal to the velocity field, or, equivalently, the iso-contours of *A* should be parallel to the streamlines.
- Isocontours of A are the magnetic fieldlines for this problem, so the steady-state solution that the magnetic field will evolve towards is a configuration where ***v*** and ***B*** are parallel, also known as a force-free configuration.
- This can also easily be seen from the original induction equation:

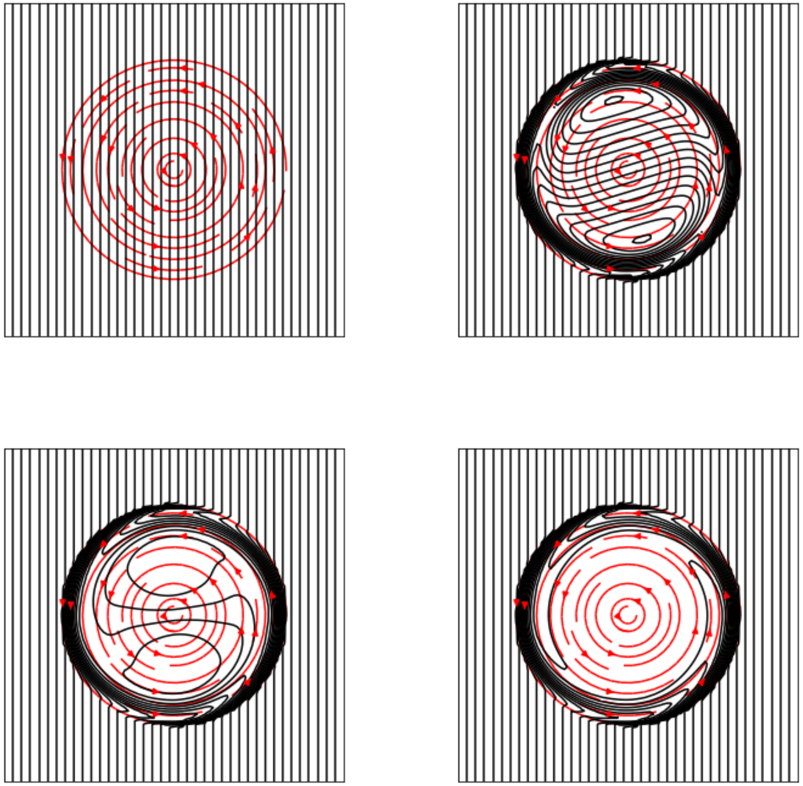
$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) = 0, \text{ if } \boldsymbol{v} \parallel \boldsymbol{B}$$

Dissipation, flux expulsion

- Numerical algorithms used to solve the equations introduce finite resistivity into the system.
- Hence, the steady-state solution is, in practice, the solution of the form

$$\boldsymbol{v} \cdot \nabla A = \eta \, \nabla^2 A$$

- This allows field lines to diffuse out at areas with a large divergence, effectively expulsing the magnetic field from regions that are completely enclosed by streamlines
- This is illustrated on the right with the loop-streamline geometry.
- In general, it is true for incompressible fluids ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$).
- This can be shown by integrating the steady-state solution along a streamline



Red: velocity field streamlines
Black: magnetic fieldlines

Summary

- The induction equation:

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B})$$

  dictates the motion of magnetic filed lines
- Which dynamics that wins, inertia or magnetic fields, depends on the energy balance (E_{kin} vs E_{mag})
- Evolving the induction equation forward in time can be seen as advecting the magnetic field with source terms
- For simple geometries the problem reduces to two dimensions
- In general, the magnetic field also has to satisfy the solenoidal constraint

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

- We will investigate how constrained transport can help in maintaining the constraint on Thursday.
- The method of characteristics or solving the finite volume Riemann problem can be used for the hydrodynamic part of the equations.

Assignment 3a:

- Design a simple finite difference solver to advect magnetic fields.
- This is a warmup for Thursday where we will continue with the full magnetohydrodynamical problem.