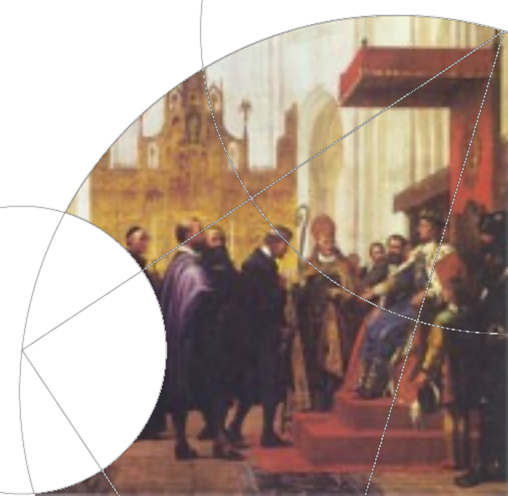
Computational Astrophysics

3b. Magnetohydrodynamics part II: Constrained transport and Full MHD

Troels Haugbølle
Niels Bohr Institute
University of Copenhagen

Some slides by P Benítez-Llambay



Topics today

- The induction equation and constrained transport
- Full MHD equations
- Wave nature of MHD

- **Assignment 3b:**
  - Use Fargo-3D to play around with the Orzag-Tang test and investigate the magneto-rotational-instability

Summary from Tuesday

- The induction equation:

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B})$$

  dictates the motion of magnetic filed lines
- Which dynamics that wins, inertia or magnetic fields, depends on the energy balance (E_{kin} vs E_{mag})
- Evolving the induction equation forward in time can be seen as advecting the magnetic field with source terms
- For simple geometries the problem reduces to two dimensions
- In general, the magnetic field also has to satisfy the solenoidal constraint

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

- We will investigate how constrained transport can help in maintaining the constraint on Thursday.
- The method of characteristics or solving the finite volume Riemann problem can be used for the hydrodynamic part of the equations.

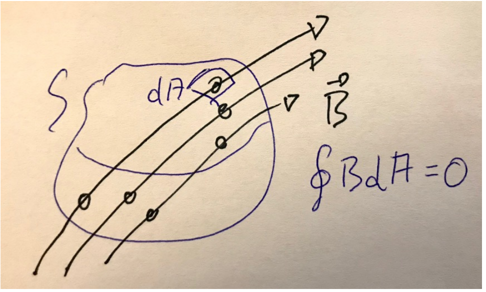
Induction equation numerically

- The solution of the induction equation is problematic mainly because of the solenoidal constraint:

$$\nabla \cdot \boldsymbol{B} = 0$$

- This equation implies that the magnetic flux through a closed surface must be zero:

$$\iiint_V dV\, \nabla \cdot \boldsymbol{B} = \oiint_{\partial\Omega} \boldsymbol{B} \cdot \boldsymbol{dS} = 0$$



- and this should be satisfied numerically for any cell within a hydrodynamic mesh.

Induction equation numerically

- Possible approaches to solve the induction equation numerically:
  - Solve it using any method and check if the errors accumulated in $\nabla \cdot B$ are significant. *Advantage:* error in $\nabla \cdot B$ should decrease as long as resolution increases (convergence). *Downside:* existence of magnetic monopoles within the domain (unphysical).
  - Fix the magnetic field every time step, e.g., by evolving two components of B and determining the third one from solenoidal constraint (but which??). Another possibility is to apply "flux-cleaning" methods.
  - Use as variable the vector potential $B = \nabla \times A$ (we did this in the exercise on Tuesday). *Downside:* Lorentz force involves second derivatives.
  - Constrained transport: solve for the magnetic flux using a staggered magnetic field (we will study it today).

Constrained Transport

The induction equation can be written in terms of the magnetic flux as:

$$\partial_t \phi_S = \oint_{\partial S} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$$

with ϕ_S the flux through surface S and bounded by ∂S, i.e.,

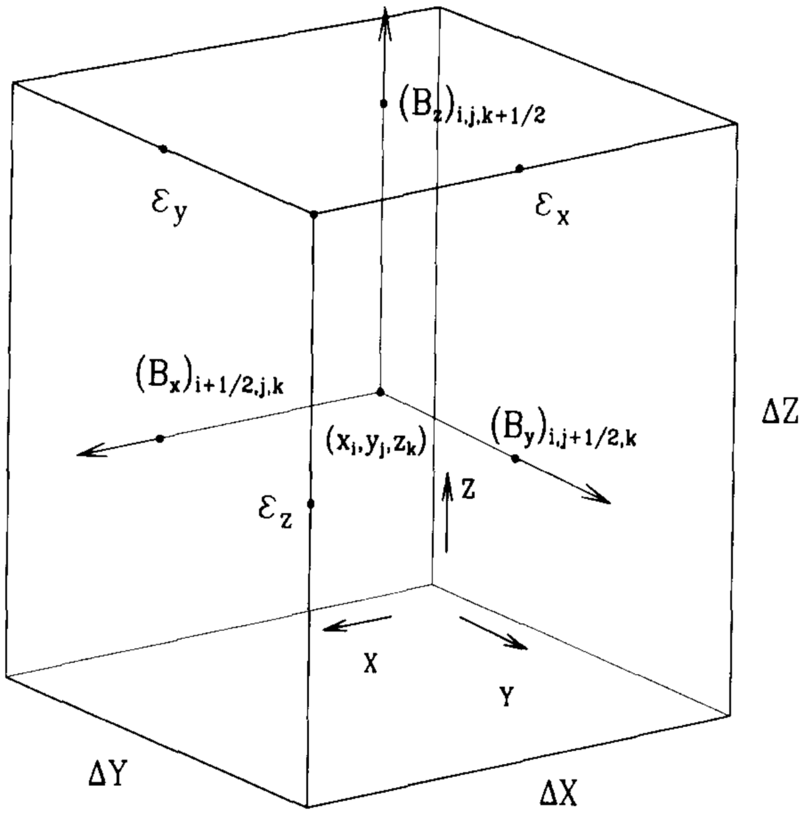
$$\phi_S = \int_S \mathbf{B} \cdot d\mathbf{S} \,.$$

For a given cell, the flux through the faces is:

$$\phi_{x_{i+1/2}} = \int_{\Delta y} \int_{\Delta z} B_{x_{i+1/2,j,k}} \, dydz$$

$$\phi_{y_{j+1/2}} = \int_{\Delta x} \int_{\Delta z} B_{y_{i,j+1/2,k}} \, dxdz$$

$$\phi_{z_{k+1/2}} = \int_{\Delta x} \int_{\Delta y} B_{z_{i,j,k+1/2}} \, dxdy$$

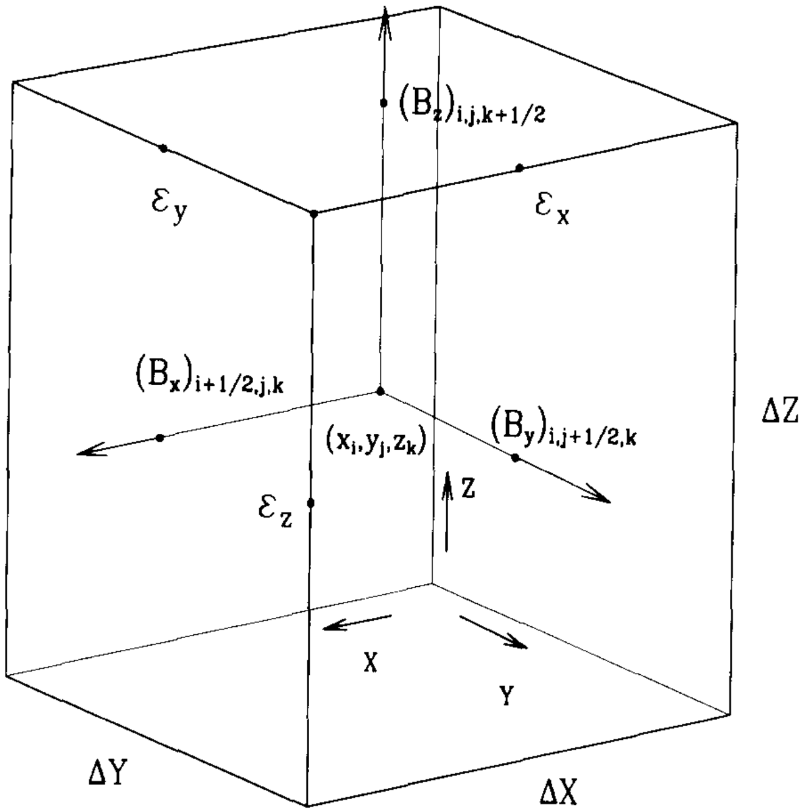


Constrained Transport

Remember that $\nabla \cdot \mathbf{B} = 0$ implies zero net flux through a cell, i.e.,

$$\phi_{x_{i+1/2}} - \phi_{x_{i-1/2}} + \phi_{y_{j+1/2}} - \phi_{y_{j-1/2}} + \phi_{z_{k+1/2}} - \phi_{z_{k-1/2}} = 0 \,.$$

The scheme developed to solve the induction equation should satisfy this condition.



Constrained Transport

In finite differences, along each coordinate axis, the integral form of the induction equation reads

$$\frac{\phi^{n+1}_{x_{i-1/2}} - \phi^{n}_{x_{i-1/2}}}{\Delta t} = \oint_{\Delta y \Delta z} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$$

$$\simeq \left(\mathcal{E}_{y\,k+1/2}^{\;*} - \mathcal{E}_{y\,k-1/2}^{\;*} \right) \Delta y - \left(\mathcal{E}_{z\,j+1/2}^{\;*} - \mathcal{E}_{z\,j-1/2}^{\;*} \right) \Delta z$$

where $\mathcal{E} = \mathbf{v} \times \mathbf{B}$ is a three component vector, and are called the EMFs (Electromotive forces).

Similarly, for the other faces we have:

$$\frac{\phi^{n+1}_{y_{j-1/2}} - \phi^{n}_{y_{j-1/2}}}{\Delta t} \simeq \left(\mathcal{E}_{z\,i+1/2}^{*} - \mathcal{E}_{z\,i-1/2}^{\;*} \right) \Delta z - \left(\mathcal{E}_{x\,k+1/2}^{\;*} - \mathcal{E}_{x\,k-1/2}^{\;*} \right) \Delta x$$

$$\frac{\phi^{n+1}_{z_{k-1/2}} - \phi^{n}_{z_{k-1/2}}}{\Delta t} \simeq \left(\mathcal{E}_{x\,j+1/2}^{\;*} - \mathcal{E}_{x\,j-1/2}^{\;*} \right) \Delta x - \left(\mathcal{E}_{y\,i+1/2}^{\;*} - \mathcal{E}_{y\,i-1/2}^{\;*} \right) \Delta y$$

Note the (increasing) "cyclic permutation" of the quantities ($x \to y \to z$).

Constrained Transport

The magnetic flux through a face can be calculated numerically as:

$$\phi_y \simeq B_y S_{xz} = B_y \Delta x \Delta z$$

The explicit form of the EMFs is:

$$\mathcal{E} = (v_y B_z - v_z B_y)\,\mathbf{e}_x + (v_z B_x - v_x B_z)\,\mathbf{e}_y + (v_x B_y - v_y B_x)\,\mathbf{e}_z\,.$$

And finally, the evolution equation for the magnetic field reads

$$\begin{aligned}\frac{B^{n+1}_{z_{i,j,k-1/2}} - B^{n}_{z_{i,j,k-1/2}}}{\Delta t}\Delta x \Delta y = {} & \left[(v_x B_y - v_y B_x)^*_{i,j+1/2,k-1/2} - (v_x B_y - v_y B_x)^*_{i,j-1/2,k-1/2}\right]\Delta x \\ & + \left[(v_z B_x - v_x B_z)^*_{i+1/2,j,k-1/2} - (v_z B_x - v_x B_z)^*_{i-1/2,j,k-1/2}\right]\Delta y\end{aligned}$$

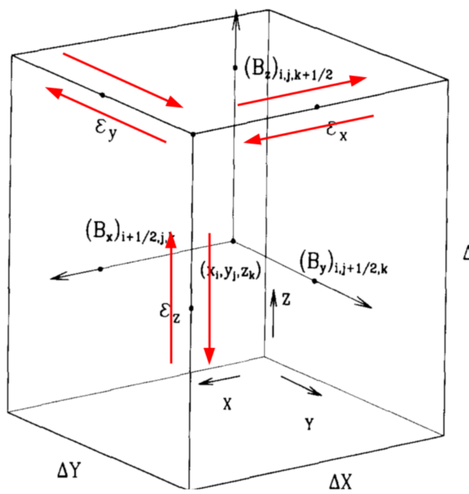
Constrained transport

No matter what are the EMFs, constrained transport ensures:

$$\frac{1}{\Delta t}\left(\frac{B_x^{n+1}-B_x^n}{\Delta x}+\frac{B_y^{n+1}-B_y^n}{\Delta y}+\frac{B_z^{n+1}-B_z^n}{\Delta z}\right)=0$$

i.e.,

$$\frac{\partial \nabla\cdot\mathbf{B}}{\partial t}=0$$



The key question is how to determine the time-averaged values of the EMFs.

Note that this problem is analogous to that of determining "Physical fluxes" to update hydrodynamical quantities from conservative equations.

Induction equation is not isolated

$$\partial_t \rho + \nabla \cdot (\mathbf{v}\rho) = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla P}{\rho} + \frac{\mathbf{J} \times \mathbf{B}}{\rho}$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

- The evolution of **v** and **B** are coupled.
- This coupling adds more degrees of freedom into the equations.
- New waves can propagate in the system.

Alfvén waves are transversal and propagate in the direction of **B**, which makes the problem different to the hydro case.

In other words, there are more characteristics in the system.

Simple analysis of potential issues

To understand the source of issues with a simple MHD solver, it is illustrative to assume a 1D system in which $\partial_y = \partial_z = 0$, and B_x is uniform (solenoidal constraint).

For this simple configuration, the induction equation is:

$$\partial_t B_y = -\partial_x \left(v_x B_y - v_y B_x \right)$$

since B_x is uniform, we have:

$$\partial_t B_y + \partial_x \left(v_x B_y \right) = B_x \partial_x \left(v_y \right)$$

The orange terms are the advection equation for the magnetic field. However, the source terms in the R.H.S. when coupled to the Lorentz force leads to transverse waves.

Simple analysis of potential issues

$$\partial_t B_y + \partial_x (v_x B_y) = B_x \partial_x (v_y) \tag{24}$$

If the magnetic field does not influence the dynamics of the fluid (i.e., magnetic fields are weak), we can just use the velocity of the fluid v_x as the characteristic speed, for both B_y, B_x and v_y.

This approach is also valid if the Alfvén speed is much slower that v_x.

Let's see how we can fix the issue...

Lorentz force is:

$$\mathbf{F}_{\mathrm{L}} = \frac{\nabla \times \mathbf{B} \times \mathbf{B}}{\mu_0}$$

using vector calculus identities, we can write:

$$\mathbf{F}_{\mathrm{L}} = -\nabla \left(\frac{B^2}{2\mu_0} \right) + \frac{(\mathbf{B} \cdot \nabla)\,\mathbf{B}}{\mu_0}$$

Thus, the momentum equation is:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla\left(P_{\mathrm{G}} + P_{\mathrm{M}}\right)}{\rho} + \frac{(\mathbf{B} \cdot \nabla)\,\mathbf{B}}{\mu_0 \rho}$$

The term P_M is called the "magnetic pressure" and can be treated similarly to the pressure in the hydrodynamical case. The term that introduces complications for upwind methods is the "Magnetic tension".

Momentum equation

Let's explore again the simple case in which B_x is uniform and $\partial_y = 0$. If we ignore the magnetic pressure (we already know how to account for it), the momentum equation reads:

$$\partial_t v_y + \partial_x (v_x v_y) = \frac{B_x}{\mu_0 \rho} \partial_x B_y \tag{25}$$

which is also and advection equation plus a source term (the transverse force).

Clearly, there is a feedback between momenta and induction due to transverse terms that needs be treated carefully.

Method of characteristics

The goal is to solve the coupled system:

$$\partial_t v_y + v_x \partial_x (v_y) = \frac{B_x}{\mu_0 \rho} \partial_x B_y \tag{26}$$

$$\partial_t B_y + v_x \partial_x (B_y) = B_x \partial_x v_y \tag{27}$$

where we have also assumed incompressibility, so we will only get as solution noncompressive Alfvén waves.

We can combine ($\pm$) these two equations together and show that they satisfy:

$$\partial_t \psi_\pm + C_\pm \partial_x \psi_\pm = 0 \tag{28}$$

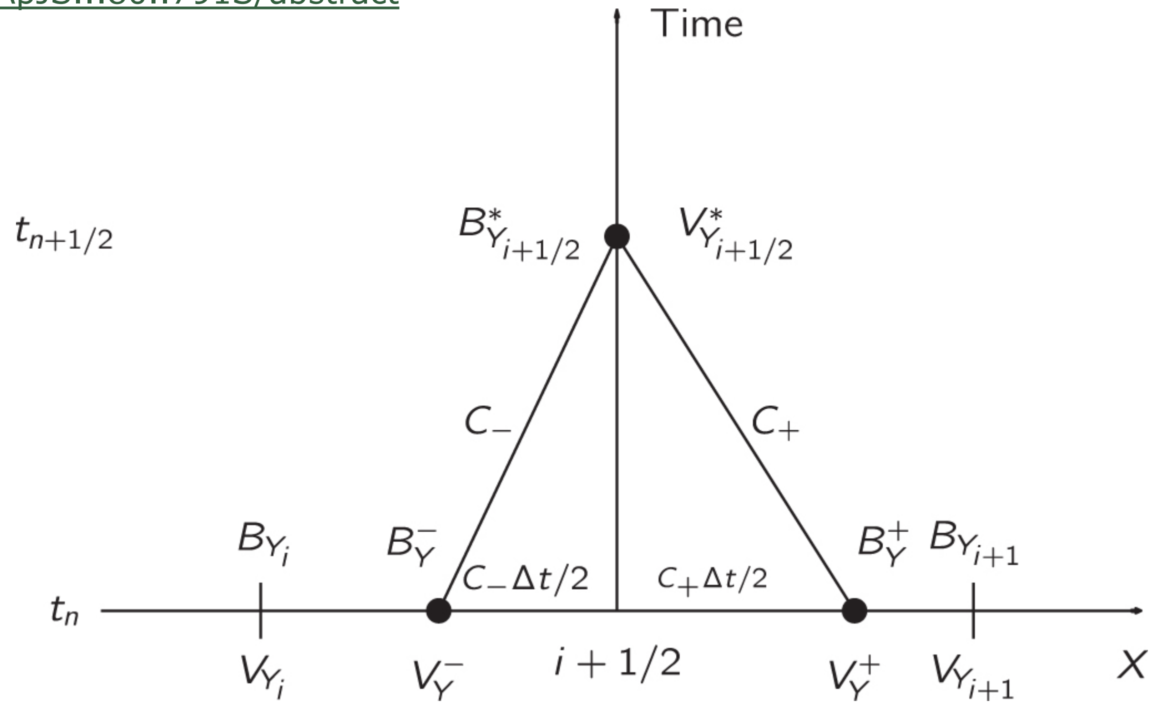
with $\psi_\pm = v_y \mp B_y / \sqrt{\mu_0 \rho}$, and $C_\pm = v_x \mp B_x / \sqrt{\mu_0 \rho}$.

$\psi_\pm$ is a conserved quantity along $C_\pm$.

https://ui.adsabs.harvard.edu/abs/1992ApJS...80..791S/abstract

$$\partial_t v_y + v_x \partial_x (v_y) = \frac{B_x}{\mu_0 \rho} \partial_x B_y$$

$$\partial_t B_y + v_x \partial_x (B_y) = B_x \partial_x v_y$$

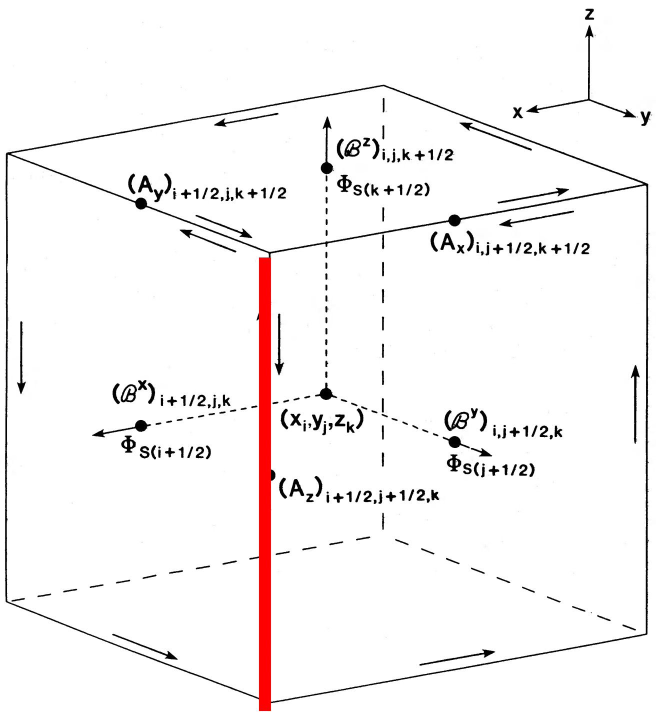


Once we propagate $\psi_\pm$ along the characteristics, we can obtain the "advanced" (or extrapolated in time, interpolated in space) values for B_y and v_y, that we denote as B_y^*, v_y^*.

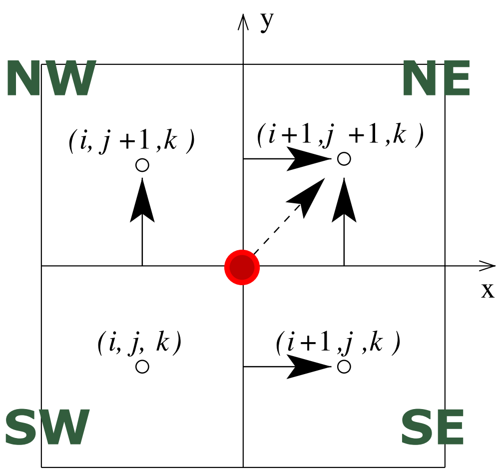
$$B_y^* = \frac{\sqrt{\mu_0 \rho}}{2} (\psi_+ - \psi_-); \quad v_y^* = \frac{1}{2} (\psi_+ + \psi_-);$$

Alternative Riemann way of computing the EMF

https://ui.adsabs.harvard.edu/abs/2006A%26A...457..371F/abstract



- In principle just ***v*** *x* ***B***
- …but much harder because of *ringing at steep gradients*
- need to slope interpolate values of fields at edges (corners in 2D)
- Solution: solve the 2D Riemann problem for the EMF only. Requires four 1D Riemann solves between states next to edge



The Full MHD Equations

Equations of Magnetohydrodynamics

- The MHD equations describing the collective behaviour of the plasma gives a new force and energy terms in the fluid equation
  In conservative form:

$$\partial_t \rho + \boldsymbol{\nabla} \cdot [\rho \boldsymbol{v}] = 0,$$

$$\partial_t \rho \boldsymbol{v} + \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{v} \otimes \boldsymbol{v} + P_{tot} \mathbb{I} + \frac{1}{4\pi} \boldsymbol{B} \otimes \boldsymbol{B}\right] = 0, \qquad P_{tot} = P_{gas} + \frac{1}{8\pi} B^2$$

$$\partial_t E_{tot} + \boldsymbol{\nabla} \cdot [(E_{tot} + P_{tot}) \boldsymbol{v} - (\boldsymbol{B} \cdot \boldsymbol{v}) \boldsymbol{B}] = 0, \qquad E_{tot} = \rho e + \frac{1}{2} \rho v^2 + \frac{1}{8\pi} B^2$$

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B})$$

$$\nabla \cdot \boldsymbol{B} = 0$$

- We have talked at length about how to solve the induction equation for the magnetic fields:
  - Place the magnetic fields at the interface of the cells
  - Update the magnetic field using the EMFs at the edges of the cell

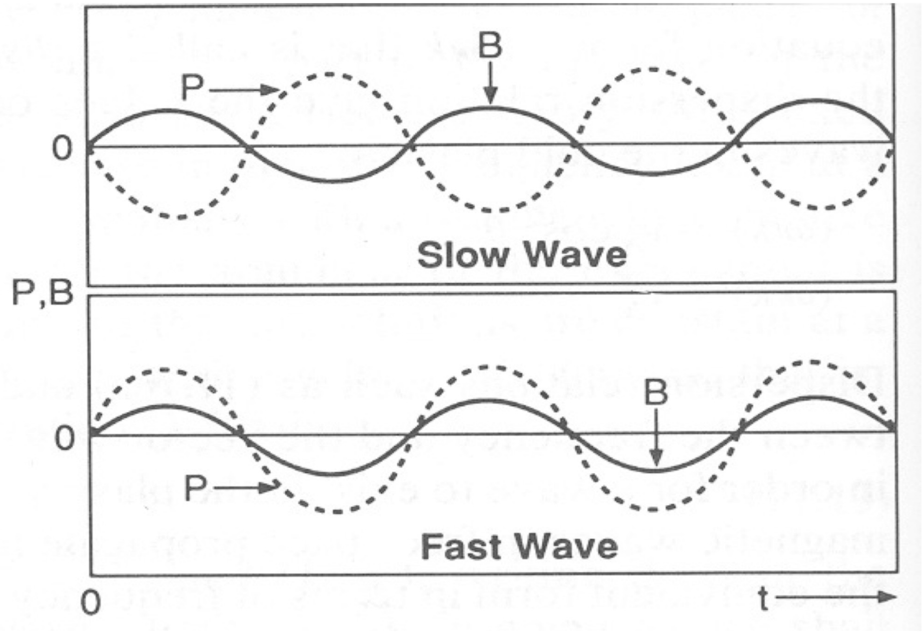
- Compared to last week we have additional Hoop stress terms, magnetic pressure, and a magnetic energy flux term, for momentum and energy

- Question: How would you (numerically) solve for the hydrodynamics?

Surfing the MHD equations

- Because of the new degree's of freedom, the fluid allow more ways to carry energy, and this gives rise to new wave phenomena, so-called Alfvén waves – you may have seen them in theoretical astrophysics.

- In addition, an Alfven wave can combine positively or negatively with a soundwave, and make magneto-sonic waves, that propagate with the slow or fast magneto sonic speed.



Surfing the MHD equations

- Because of the new degree’s of freedom, the fluid allow more ways to carry energy, and this gives rise to new wave phenomena, so-called Alfvén waves – you may have seen them in theoretical astrophysics.

- In addition, an Alfven wave can combine positively or negatively with a soundwave, and make magneto-sonic waves, that propagate with the slow or fast magneto sonic speed.

- Wave solutions are found by linearizing the MHD equations with small perturbations

$$\frac{\partial}{\partial t}\delta\rho + \rho_0 \nabla\cdot\delta\boldsymbol{v} = 0$$
$$\rho_0\frac{\partial}{\partial t}\delta\boldsymbol{v} + \nabla\delta P = \frac{1}{\mu_0}(\nabla\times\delta\boldsymbol{B})\times\boldsymbol{B}_0$$
$$\frac{\partial}{\partial t}\delta\boldsymbol{B} = \nabla\times(\delta\boldsymbol{v}\times\boldsymbol{B}_0)$$
$$\frac{\partial}{\partial t}\left(\frac{\delta P}{P_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0,$$

$$\Longrightarrow \quad \delta\rho(\boldsymbol{r},t) = \delta\hat{\rho}\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t),$$

$$-i\omega\delta\hat{\rho} + \rho_0 i\boldsymbol{k}\cdot\delta\hat{\boldsymbol{v}} = 0$$
$$-i\omega\rho_0\delta\hat{\boldsymbol{v}} + i\boldsymbol{k}\delta\hat{P} = \frac{1}{\mu_0}(i\boldsymbol{k}\times\delta\hat{\boldsymbol{B}})\times\boldsymbol{B}_0$$
$$-i\omega\delta\hat{\boldsymbol{B}} = i\boldsymbol{k}\times(\delta\hat{\boldsymbol{v}}\times\boldsymbol{B}_0)$$
$$-i\omega\left(\frac{\delta\hat{P}}{P_0} - \gamma\frac{\delta\hat{\rho}}{\rho_0}\right) = 0,$$

Surfing the MHD equations

- Because of the new degree’s of freedom, the fluid allow more ways to carry energy, and this gives rise to new wave phenomena, so-called Alfvén waves – you may have seen them in theoretical astrophysics.

- In addition, an Alfven wave can combine positively or negatively with a soundwave, and make magneto-sonic waves, that propagate with the slow or fast magneto sonic speed.

- Wave solutions are found by linearizing the MHD equations with small perturbations

- The end results is a dispersion relation

$$\left(\omega^2 - k^2 c_{\mathrm{A}}^2 \cos^2\theta\right)\left[\omega^4 - \omega^2 k^2 (c_{\mathrm{A}}^2 + c_{\mathrm{s}}^2) + k^4 c_{\mathrm{A}}^2 c_{\mathrm{s}}^2 \cos^2\theta\right] = 0,$$

- Containing the characteristic speeds

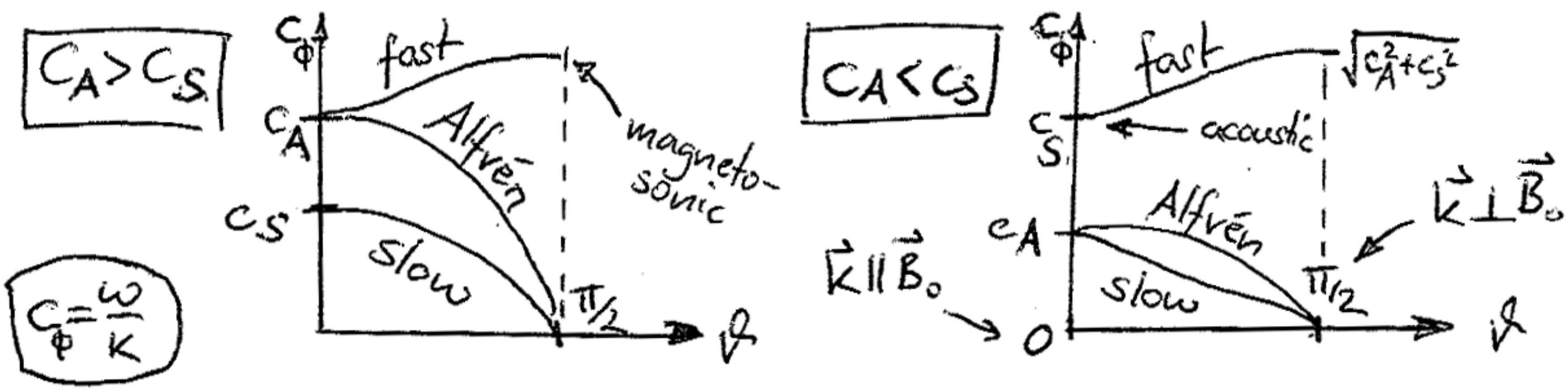
$$c_{\mathrm{s}} \equiv \sqrt{\gamma \frac{p_0}{\rho_0}} \qquad c_{\mathrm{A}} \equiv \sqrt{\frac{B_0^2}{\rho_0 \mu_0}}$$

Surfing the MHD equations

- Because of the new degree's of freedom, the fluid allow more ways to carry energy, and this gives rise to new wave phenomena, so-called Alfvén waves – you may have seen them in theoretical astrophysics.

- In addition, an Alfven wave can combine positively or negatively with a soundwave, and make magneto-sonic waves, that propagate with the slow or fast magneto sonic speed.

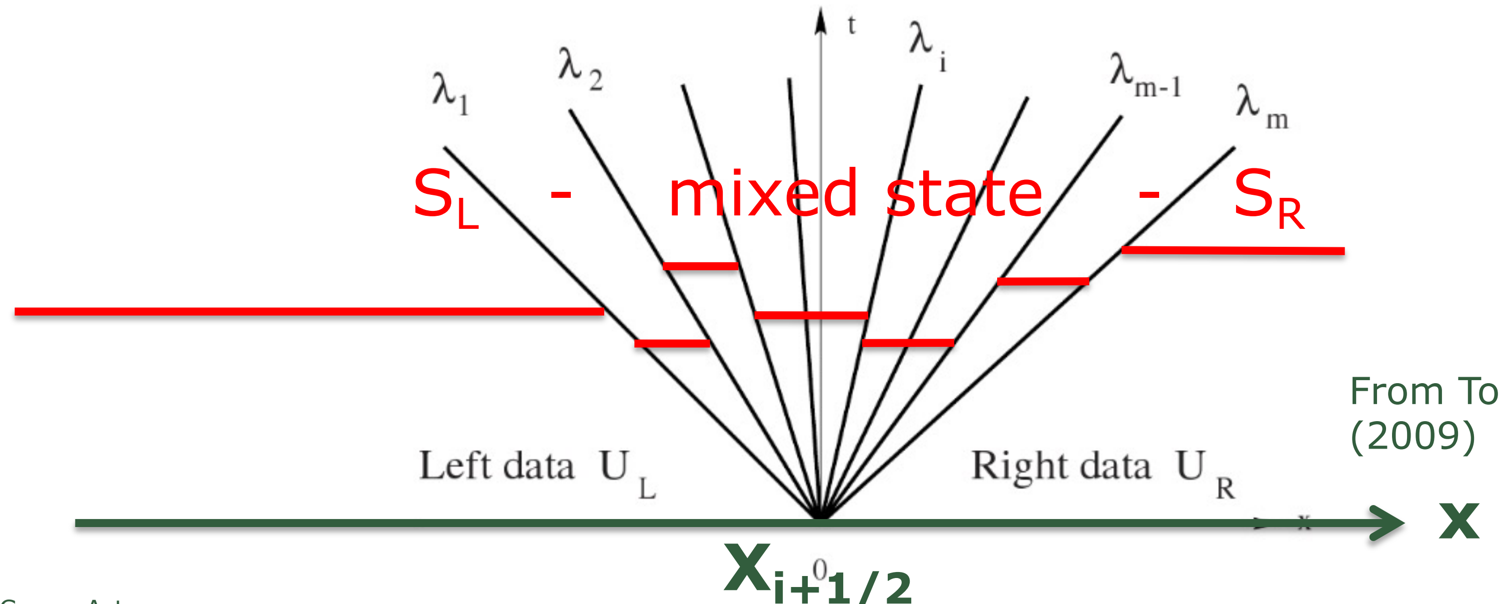
- Wave solutions are found by linearizing the MHD equations with small perturbations. The speed depend on the angle between the background magnetic field and the direction of propagation



Surfing the MHD equations

- Because of the new degree's of freedom, the fluid allow more ways to carry energy, and this gives rise to new wave phenomena, so-called Alfvén waves – you may have seen them in theoretical astrophysics.

- The waves makes the Riemann problem for MHD is much richer than for HD (7 MHD versus 3 HD waves), and only recently the full problem was solved (iteratively). Nonetheless, several approximate solvers exist. HLL is popular for its simplicity. The so-called HLLD solver is somewhat more complicated, but very stable.



From Toro (2009)

Surfing the MHD equations

- Because of the new degree's of freedom, the fluid allow more ways to carry energy, and this gives rise to new wave phenomena, so-called Alfvén waves – you may have seen them in theoretical astrophysics.

- The waves makes the Riemann problem for MHD is much richer than for HD (7 MHD versus 3 HD waves), and only recently the full problem was solved (iteratively). Nonetheless, several approximate solvers exist. HLL is popular for its simplicity. The so-called HLLD solver is somewhat more complicated, but very stable.

- *Consequence:* when solving the MHD equations the signal speed and CFL timestep condition is set by the bulk plus fast magnetosonic speed:

$$c_f = \sqrt{c_s^2 + c_A^2}, \qquad c_s = \sqrt{\gamma \frac{P}{\rho}}, \qquad c_A = \frac{B}{\sqrt{4\pi\rho}}$$

$$dt = C_{dt} \frac{\Delta x}{c_f + |v|}$$

A MUSCL algorithm for solving MHD

The basic method we will use is the MUSCL (Monotonic Upwind Scheme for Conservation Laws) scheme. It contains a number of key steps, most of which we already did last week for the hydrodynamics. The algorithm is as follows (this is closely mirrored in the python code below):

1. Use the Courant condition to find the largest permissible timestep dt
2. Compute *primitive* variables (ρ, ρe_{int}, $\mathbf{v}$, $\mathbf{B}_{\mathrm{centered}}$)
3. Compute slopes
4. Compute predicted states for $t + \Delta t/2$

Repeat 5+6 for all coordinate directions:

5. Compute left and right face values and NE, SE, SW, NW states around an edge
6. Compute fluxes and EMFs, using Riemann solver or the MOC-CT algorithm

7. Update the conserved variables

Summary

- The induction equation:

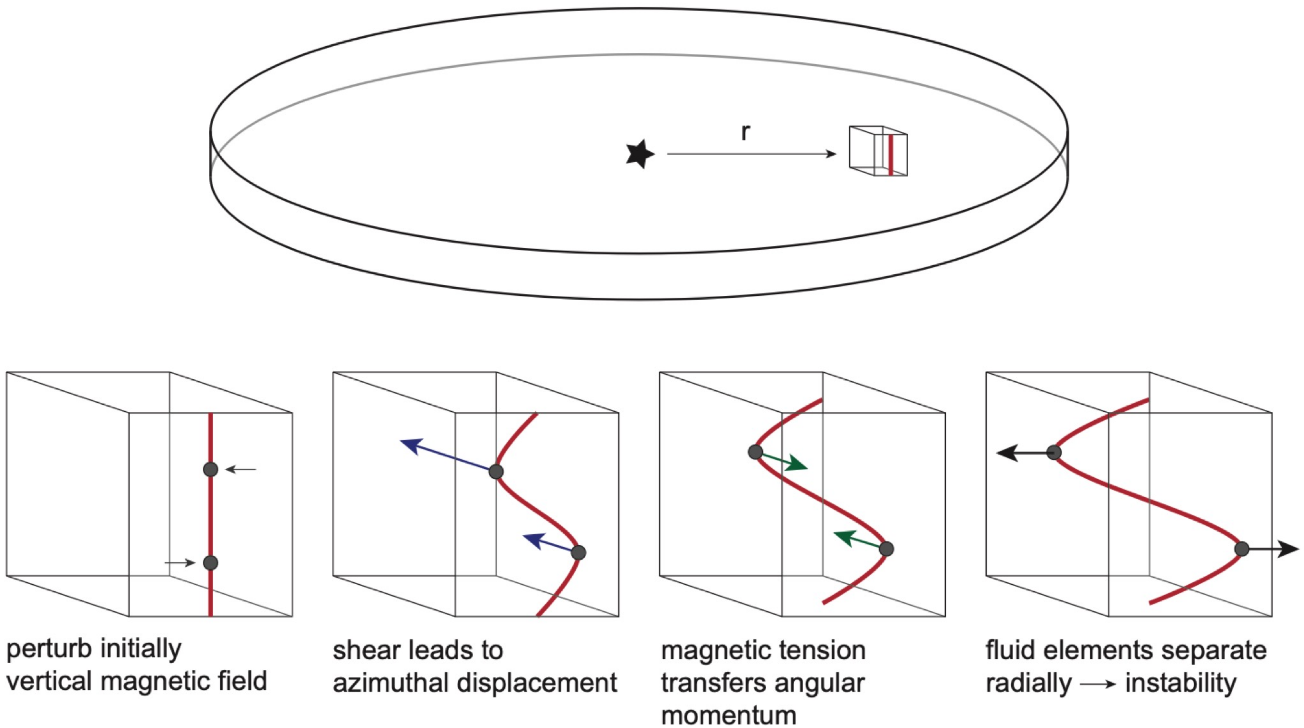
$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B})$$

  can be solved while respecting $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ by placing the magnetic fields at the interfaces.
- Constrained transport then takes advantage of Stokes theorem
- Hydro variables can still be integrated using a finite volume method
- MHD allows for new wave types, giving a richer dynamics and a more restrictive timestep
- Because of the added equations and the complexity of the resulting algorithm, we will not use our own code, but instead try to run a “real” production code, Fargo-3D

Assignment 3b: Fargo-3D

- Part a: Run a magnetised Orzag-Tang experiment with Fargo-3D
- Part b: Explore the magneto-rotational-instability

Magnetorotational instability



Credit: P. Armitage

The magnetorotational instability (MRI) can be derived using the shearing-sheet approximation, which consists of a local expansion of the equations around a fiducial radius in a rotating frame within a disk.

Magnetorotational instability

This instability shows how attractive forces can create an unstable system due to non-inertial forces (which might be counter intuitive).

The derivation of the instability is easy from the shearing-box equations:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -2\boldsymbol{\Omega} \times \mathbf{v} + 3\Omega^2 \mathbf{e}_x - \frac{\nabla B^2}{2\mu_0 \rho} + \frac{(\mathbf{B} \cdot \nabla)\, \mathbf{B}}{\mu_0 \rho} \tag{29}$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{30}$$

This treatment allows us to neglect the curvature terms and simplify the equations while mantainting important physical properties of the system (like the shear and non-inertial forces in the case of an astrophysical disk).

These equations admit steady-state solution equal to $\mathbf{v} = -\frac{3}{2}\Omega x$, $\mathbf{B} = B_z \mathbf{e}_z$.

Magnetorotational instability

We linearize equations in the incompressible limit and constant background magnetic field, assuming perturbations perpendicular to $\mathbf{B}_0$.

By writing the perturbations in Fourier space ($\delta \propto \hat{\delta} e^{ikz+\omega t}$), we get:

$$\begin{aligned}
\tilde{\omega}\delta v_x &= 2\delta v_y + i\tilde{k}\delta b_x, \\
\tilde{\omega}\delta v_y &= -\frac{1}{2}\delta v_x + i\tilde{k}\delta b_y, \\
\tilde{\omega}\delta b_x &= i\tilde{k}\delta v_x, \\
\tilde{\omega}\delta b_y &= -\frac{3}{2}\delta b_x + i\tilde{k}\delta v_y,
\end{aligned}$$

where $\delta b_{x,y} = \delta B_{x,y}/\sqrt{\mu_0\rho}$, $\tilde{k} = k v_A/\Omega_0$ with $v_A = B_z/\sqrt{\mu_0\rho}$ the Alfvén speed and $\tilde{\omega} = \omega/\Omega_0$. For simplicity, we have omitted the "hat" symbols, but remember that perturbations correspond to Fourier amplitudes (which are in general complex quantities).

Magnetorotational instability

By defining the vector $\vec{\delta} \equiv [\delta v_x, \delta v_y, \delta b_x, \delta b_y]$, the system can be written as:

$$L\delta = \tilde{\omega}\delta \tag{31}$$

with L given by:

$$L = \begin{bmatrix} 0 & 2 & i\tilde{k} & 0 \\ -1/2 & 0 & 0 & i\tilde{k}_n \\ i\tilde{k} & 0 & 0 & 0 \\ 0 & i\tilde{k} & -3/2 & 0 \end{bmatrix}, \tag{32}$$

In this way, we have reduced the problem to an eigenvalue problem (whose solution gives us the dispersion relation of the problem, i.e., the pair k, ω such that non-trivial solution for the perturbations exists.

Once we found the corresponding eigenvectors, we thus have the full solution of the problem.

If we find that, for a given $\mathbf{k}$, the eigenvalue ω is positive, then the system is unstable.