Computational Astrophysics

4b. Gravitational Collapse

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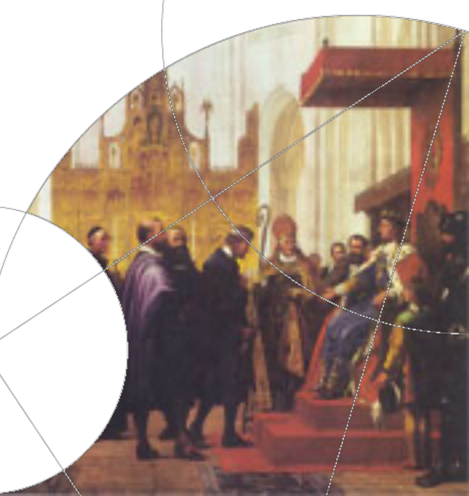
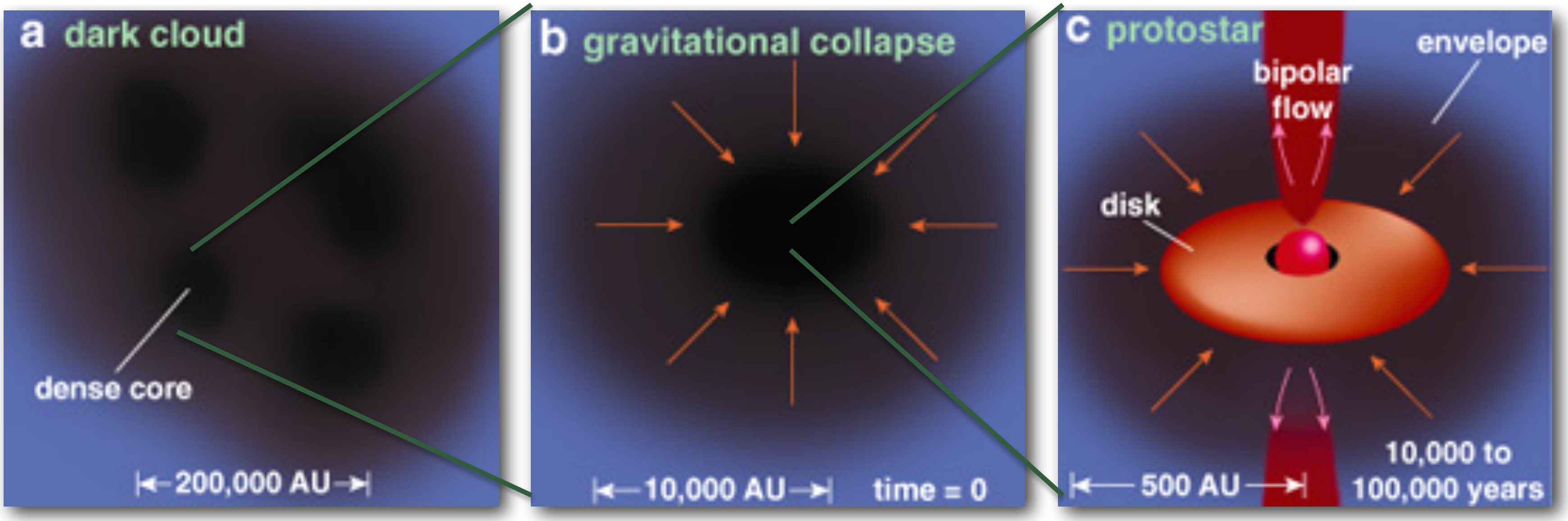
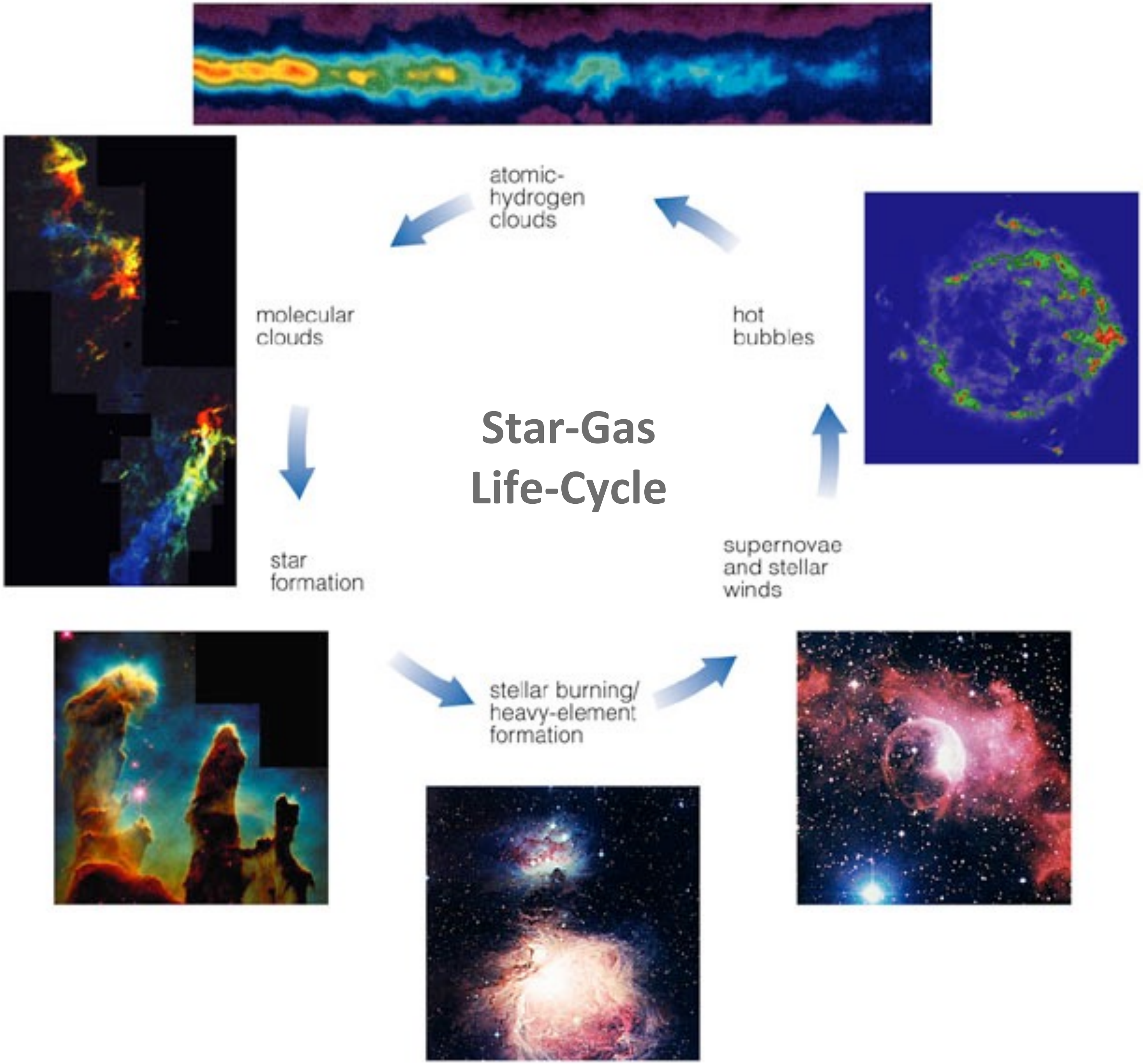


Self-gravity in Proto-Stellar Collapse



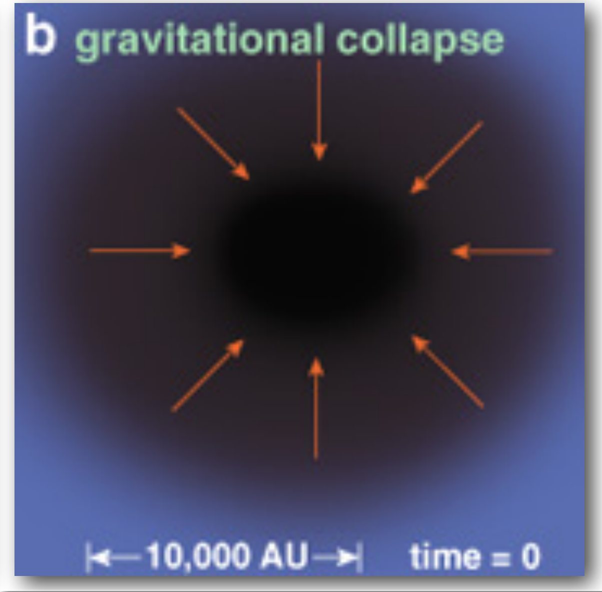
There are two main players in star formation [credit: Spitzer Science Center]

- **Energy input** by stars, compact objects, and galactic rotation
  - Injects energy into the interstellar and intergalactic media
  - Keeps an (intermittent) balance with cooling processes
  - Maintains the turbulent cascade
- **Self-gravity**
  - Collapse on large scales
  - Energy source and responsible for the formation of protostellar systems
  - Takes over from turbulence when density is high enough



[Bennett et al: Stars, Galaxies, and Cosmolog

Cloud Collapse



- Assume no pressure support; cloud collapse in free-fall :
  - We can look at a single mass-shell at distance r
  - Assume no shell crossing
  - Assume homogenous sphere
- Result:

$$t_{ff} = \sqrt{\frac{3\pi}{32G\rho}}$$

- No dependence on distance!
- Example: Put typical density into the free-fall formula (1 M_{sun}, 10^4 AU). How long does it take for the core to collapse ?
- How does it compare to the crossing time of a cloud core ? (10 K, roughly sonic speeds)

Bonnor-Ebert spheres = Hydrostatic, isothermal, self-gravitating spheres

In spherical coordinates:

$$\nabla f = \frac{df}{dr}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right)$$

Equation of hydrostatic equilibrium:

$$\frac{1}{\rho(r)}\frac{dP(r)}{dr} = -\frac{d\Phi(r)}{dr}$$

Equation of state:

$$P = \rho c_s^2$$

Equation for potential:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi(r)}{dr}\right) = 4\pi G \rho(r)$$

Python: https://github.com/aishapectyo/Bonnor-Ebert-Spheres/blob/master/BEsphere.py

Cloud Collapse

Collapse of (isothermal) Bonner-Ebert Spheres (Shu '77, Larson '69) :

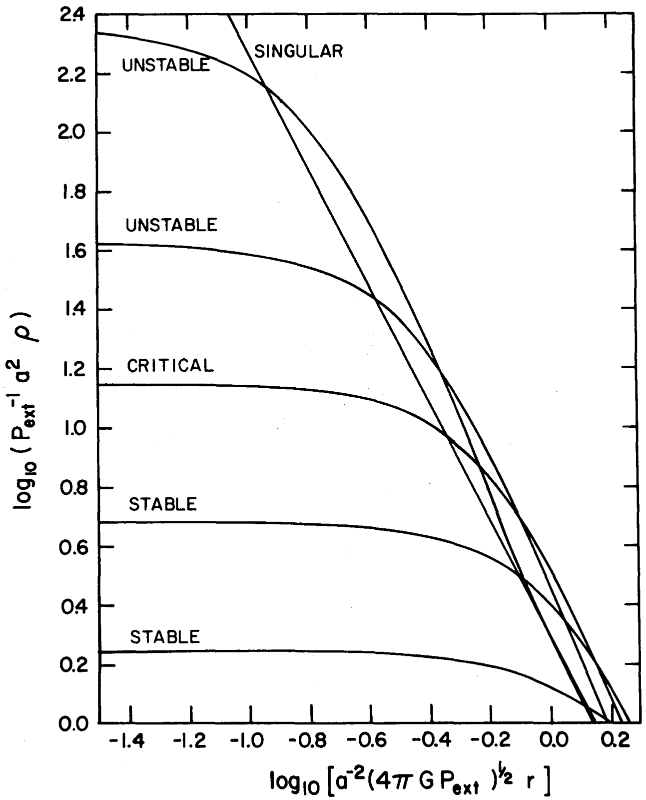


FIG. 1.—Density distributions of bounded isothermal spheres. The outer radius of each sphere is given by the intercept of the corresponding curve with the abscissa. The curve marked "critical" denotes the sphere with the maximum mass consistent with hydrostatic equilibrium at a given external pressure. Hydrostatic spheres which are less centrally concentrated than the critical Bonnor-Ebert state are gravitationally stable; those which are more centrally concentrated are gravitationally unstable. In the limit of infinite central concentration, the latter spheres approach the singular solution.

$$M_{crit} = 1.18 \frac{c_s^3}{G^{3/2} \rho_{external}^{1/2}}$$

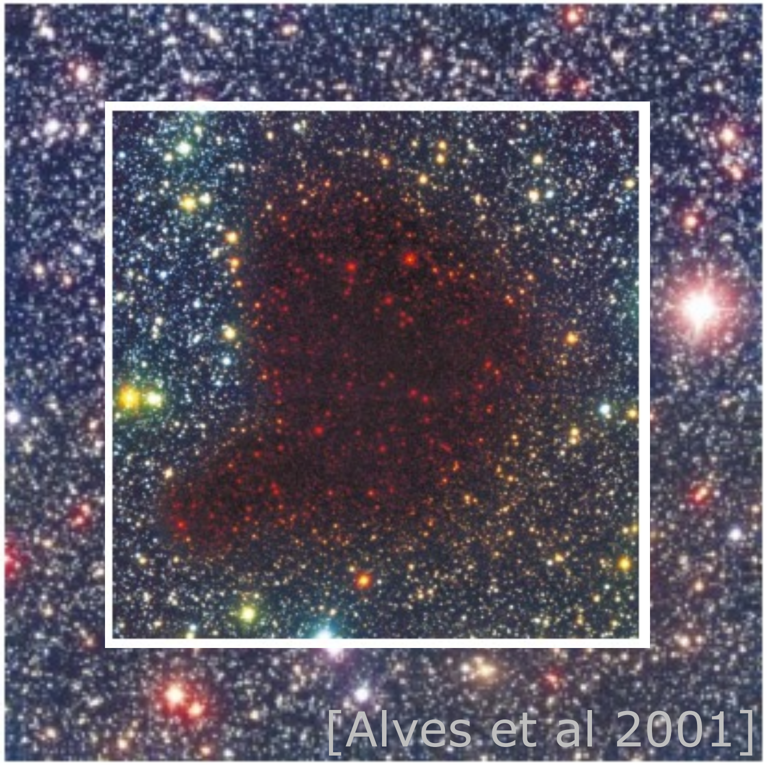
What can make a cloud-core collapse ?

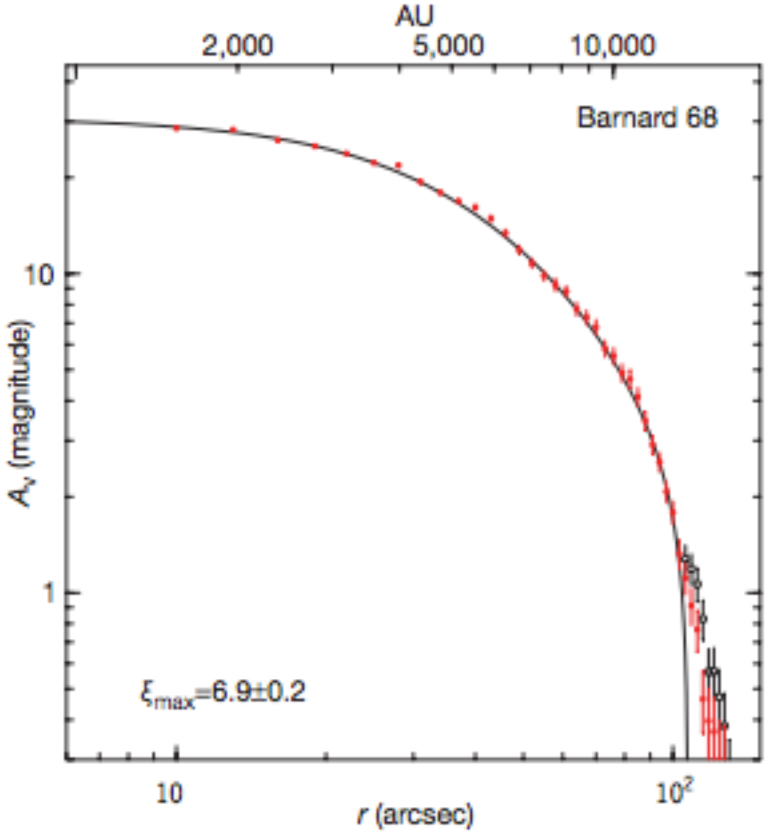
- ❑ Accretion of mass
- ❑ Change in external pressure (compression)
- ❑ Change in temperature / internal pressure support
- ❑ Core becomes unstable when

  $\rho_{central} / \rho_{edge} > 14.1$

Does this exist in nature ?

- A famous nearby dark cloud is Barnard 68: a dark starless core:





- While this is a text-book example, it is also atypical, by being isolated. In more crowded fields inertia and dynamics may be more important.

Bonnor-Ebert spheres: Theory versus reality

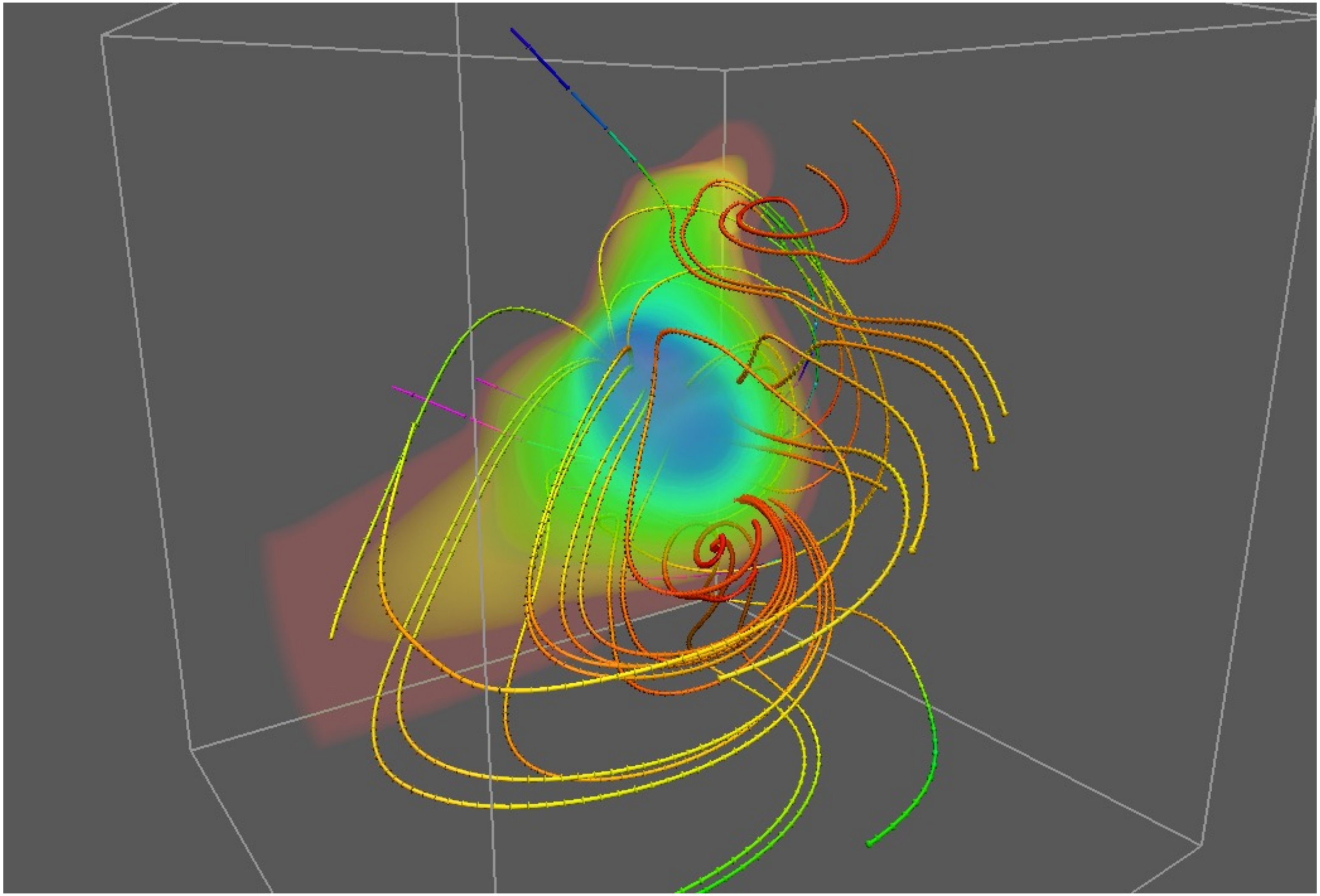
Properties of theoretical cores

- Round, self-gravitating “cores”, in hydrostatic equil.
- Size = Bonnor-Ebert sphere
- Surrounded by hot gas – otherwise would not be stable
- As time goes by (~ 0.1 – 1 Myr), the core grows (for no good reason) until collapse

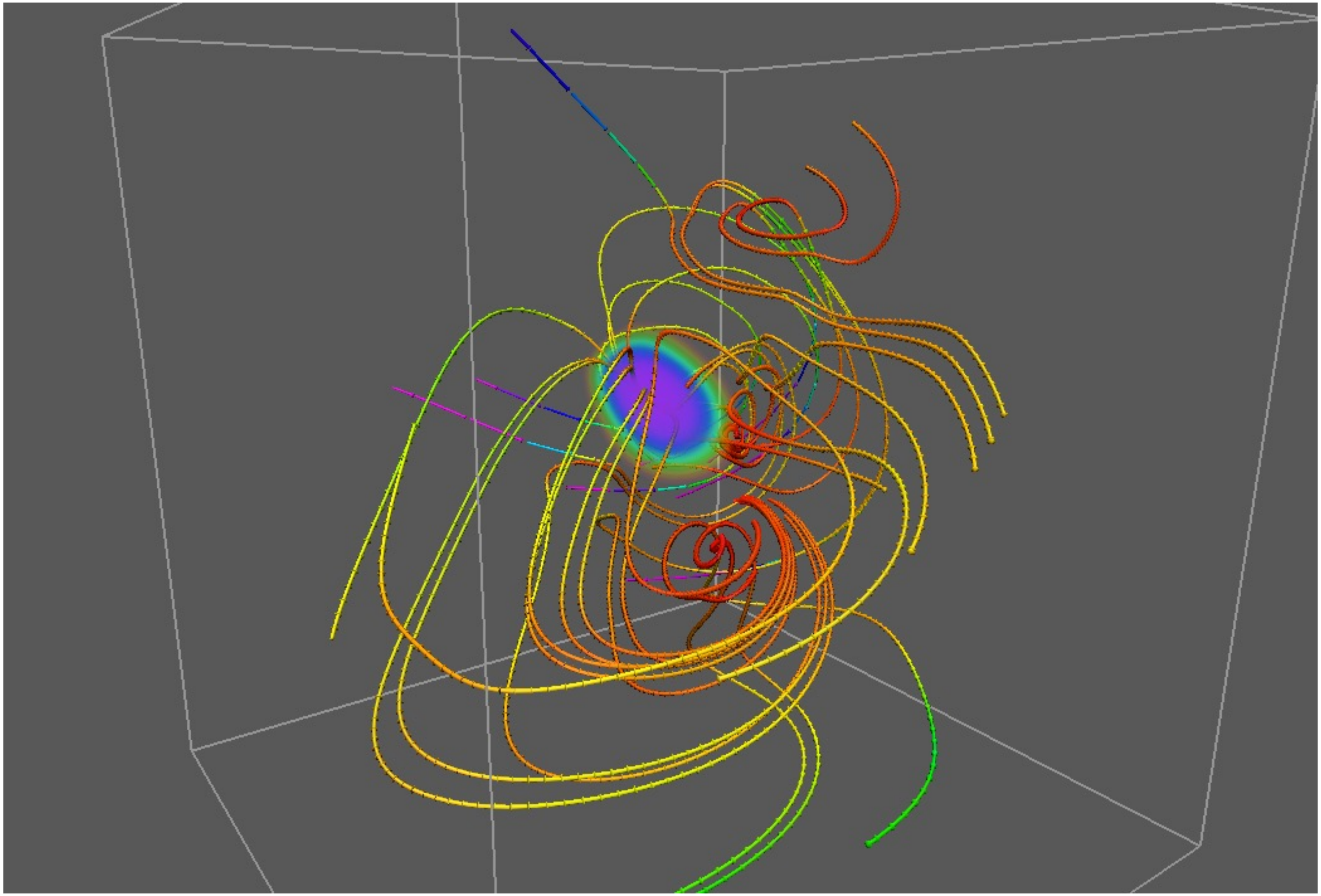
Properties of real cores

- Elongated, self-gravitating structures, in near hydrostatic equil.
- Size = only a small fraction of a Bonnor-Ebert sphere
- Surrounded by ~freely streaming gas, in filamentary form
- As time goes by (~ 0.1 – 1 Myr), the core grows until collapse
- Ongoing accretion from larger scales, while collapse proceeds
- Pressure contribution from magnetic fields
- Small net rotation from turbulent motions

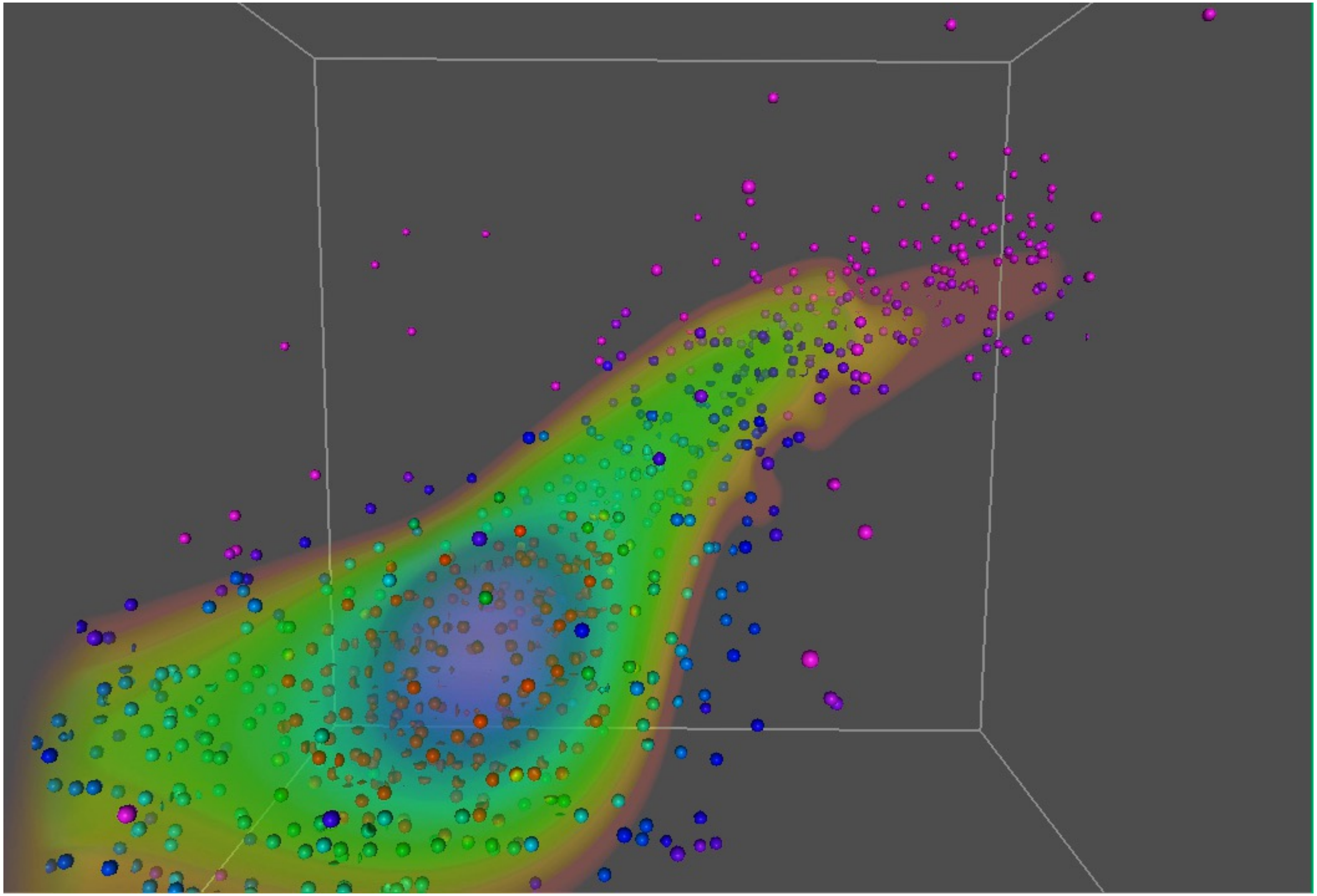
Density and velocity flow lines (scale ~ 30 kAU)



Phase: Pre-stellar core



Color: gravitational potential (red = core bound)



Coupling Hydrodynamics with Gravity

Source Terms

Hydrodynamics with selfgravity

- Hydrodynamics describes the evolution of a fluid
- The Euler equations are governed by three conservations principles for mass, momentum, and energy

$$\partial_t \rho + \nabla \cdot [\rho \boldsymbol{v}] = 0$$
$$\partial_t \rho \boldsymbol{v} + \nabla \cdot [\rho\, \boldsymbol{v} \otimes \boldsymbol{v} + P\, \mathbb{I}] = -\boldsymbol{\rho \nabla \Phi}$$
$$\partial_t E + \nabla \cdot [(E + P)\, \boldsymbol{v}] = -\boldsymbol{\rho}\, (\boldsymbol{v} \cdot \boldsymbol{\nabla \Phi})$$
$$\boldsymbol{\nabla^2 \Phi = 4\pi \underline{G} \rho}$$

- E is total energy density: $E = \rho e_{int} + \frac{1}{2}\rho v^2$, with eq of state $P(\rho, T)$
- We have now added two new terms that describes the ***change in momentum due to gravitational acceleration.*** The change in total energy density is due to the change in kinetic energy density.
- Tuesday, we discussed how to solve the Poisson equation, but how does it change our hydrodynamics scheme?

Hydrodynamics with selfgravity – Source terms

- Hydrodynamics describes the evolution of a fluid
- The Euler equations are governed by three conservations principles for mass, momentum, and energy

$$\partial_t \rho + \nabla \cdot [\rho \boldsymbol{v}] = 0$$
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$$\partial_t E + \nabla \cdot [(E + P)\, \boldsymbol{v}] = -\boldsymbol{\rho}\, (\boldsymbol{v} \cdot \boldsymbol{\nabla\Phi})$$
$$\nabla^2 \Phi = 4\pi G \rho$$

- We can in principle solve the equation using our *master equation* for finite volume dynamics

$$u(x,t+\Delta t) - u(x,t) = \Delta t \left[\tilde{S}_i(t+\Delta t/2) - \frac{\tilde{F}_{i+1}(t+\Delta t/2) - \tilde{F}_i(t+\Delta t/2)}{\Delta x} \right]$$

- Where the flux and source terms are found as the time averaged flux and source contribution through the interface and in the cell

$$\tilde{F}_i(t+\Delta t/2) = \frac{1}{\Delta t} \int_t^{t+\Delta t} dt\ F_i \ , \quad \tilde{S}_i(t+\Delta t/2) = \frac{1}{\Delta x \Delta t} \int_t^{t+\Delta t} \int_x^{x+\Delta x} dt\, dx\ S(x,t)$$

Selfgravity source terms

- The source terms are in the momentum and energy equation

$$\partial_t \rho \boldsymbol{v} + \nabla \cdot [\rho\, \boldsymbol{v} \otimes \boldsymbol{v} + P\, \mathbb{I}] = -\boldsymbol{\rho \nabla \Phi}$$
$$\partial_t E + \nabla \cdot [(E + P)\, \boldsymbol{v}] = -\boldsymbol{\rho}\, (\boldsymbol{v} \cdot \boldsymbol{\nabla \Phi})$$

- The flux and source terms are found as the time averaged contribution

$$\tilde{F}_i(t + \Delta t/2) = \frac{1}{\Delta t} \int_t^{t+\Delta t} dt\ F_i\ , \quad \tilde{S}_i(t + \Delta t/2) = \frac{1}{\Delta x \Delta t} \int_t^{t+\Delta t} \int_x^{x+\Delta x} dt\, dx\ S(x,t)$$

- The challenge is to ***solve the two contributions (Flux and Source) such that detailed balance, in particular hydrostatic equilibrium, is respected.*** This entails
  - Changing the predictor in the MUSCL scheme to include gravity
  - Evaluating the source term to second order
- There exists several alternatives to this, including
  - I: Well-balanced schemes with explicit support for H.E.
  - II: Reformulating the equations to become flux based

Selfgravity source terms – solution I

- The flux and source terms are found as the time averaged contribution

$$\tilde{F}_i(t+\Delta t/2) = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt\ F_i\ , \quad \tilde{S}_i(t+\Delta t/2) = \frac{1}{\Delta x \Delta t}\int_t^{t+\Delta t}\int_x^{x+\Delta x} dt\, dx\ S(x,t)$$

- **Problem I:** the *Riemann problem could include artifical fluxes*
  - **Solution:** change predictor in MUSCL scheme to include gravity

- Remember how we specify left and right states at an interface:

$$q_L\left(x_i + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) = q(x_i, t) + \frac{1}{2}\Delta_x q(x_i) + \frac{1}{2}\Delta_t q(x_i)$$

where $\Delta_t q(x_i)$ is the predicted change of q a time step forward.

- The prediction of q has to include gravitational acceleration to avoid spurious fluxes for a system in hydrostatic equilibrium:

$$\Delta_t \boldsymbol{v} = \Delta t \times \left[-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \frac{1}{\rho}\nabla P \ - \boldsymbol{\nabla\Phi}\ \right]$$

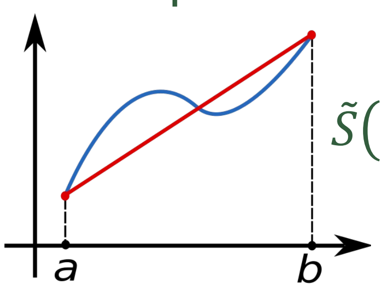
Selfgravity source terms – solution I

- The flux and source terms are found as the time averaged contribution

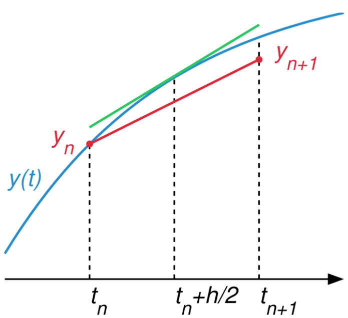
$$\tilde{F}_i(t+\Delta t/2) = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt\ F_i\ , \quad \tilde{S}_i(t+\Delta t/2) = \frac{1}{\Delta x \Delta t}\int_t^{t+\Delta t}\int_x^{x+\Delta x} dt\, dx\ S(x,t)$$

- **Problem II:** Source terms are evaluated in the volume by integrating both in time and space. Have to be as precise as flux
  - **Solution:** improve prediction of source term to 2nd order.

- To perform the integral of *S* we can either use the midpoint rule or the Trapezoidal rule



$$\tilde{S}\left(t+\frac{\Delta t}{2}\right) \approx \frac{1}{2}[S(t) + S(t+\Delta t)]$$



$$\tilde{S}\left(t+\frac{\Delta t}{2}\right) \approx S\left(t+\frac{\Delta t}{2}\right)$$

- Turns out that since we know the potential Φ at times [t, t+Δt, t+2Δt, …] trapezoidal rule is usually the easiest.

Selfgravity source terms – solution II

- The flux and source terms are found as the time averaged contribution

$$\tilde{F}_i(t+\Delta t/2)=\frac{1}{\Delta t}\int_t^{t+\Delta t} dt\ F_i\ ,\quad \tilde{S}_i(t+\Delta t/2)=\frac{1}{\Delta x\Delta t}\int_t^{t+\Delta t}\int_x^{x+\Delta x} dt\,dx\ S(x,t)$$

- **Problem:** Equilibrium is not respected due to interpolation

- **Solution:**
  - Reconstruct pressure, density, and gravitational acceleration such that (partial) hydrostatic. equillibriumis respected at cell interfaces.

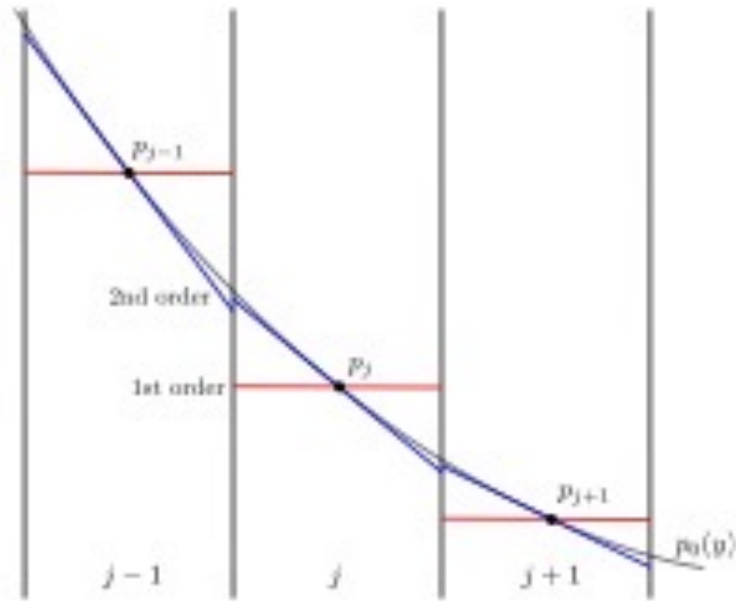


Fig. 2. First (red lines) and second (blue lines) order standard reconstruction in a hydrostatic pressure distribution. The solid black line represents the equilibrium hydrostatic pressure $p_0(y)$.

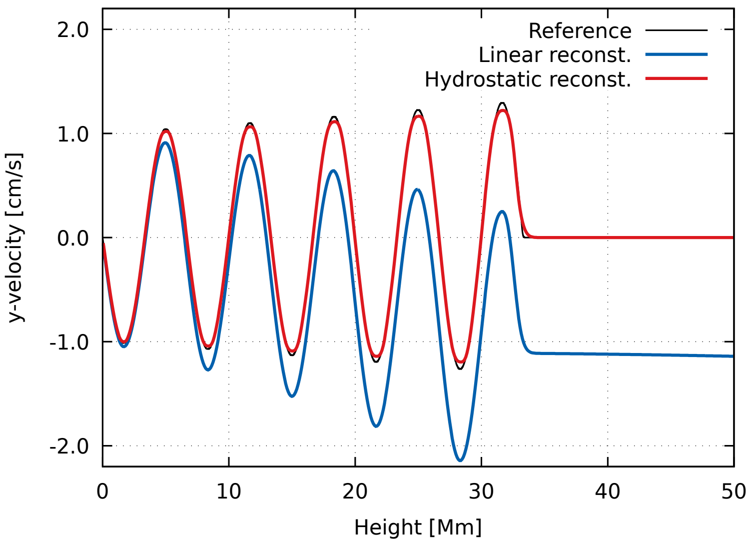
[G. Krause 2019]

Selfgravity source terms – solution II

- The flux and source terms are found as the time averaged contribution

$$\tilde{F}_i(t+\Delta t/2) = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt\ F_i\ ,\quad \tilde{S}_i(t+\Delta t/2) = \frac{1}{\Delta x \Delta t}\int_t^{t+\Delta t}\int_x^{x+\Delta x} dt\,dx\ S(x,t)$$

- **Problem:** Equilibrium is not respected due to interpolation



Wave propagating in a Solar atmosphere

- **Solution:**
  - Reconstruct pressure, density, and gravitational acceleration such that (partial) hydrostatic equillibrium is respected at cell interfaces.
- **Well-balanced scheme:**
  - Integrate source term such that ***Fluxes*** and ***Sources*** exactly cancel for hydrostatic equilibrium

[G. Krause 2019]

Selfgravity source terms – solution III

- The flux and source terms are found as the time averaged contribution

$$\tilde{F}_i(t+\Delta t/2) = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt\ F_i\ , \quad \tilde{S}_i(t+\Delta t/2) = \frac{1}{\Delta x \Delta t}\int_t^{t+\Delta t}\int_x^{x+\Delta x} dt\,dx\ S(x,t)$$

- <u>**Problem:**</u> Energy, incl potential, may not be conserved
- **Solution:** Rewrite equations of motion to become flux based

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v} + \mathbf{P} + \mathbf{T_g}) = 0,$$

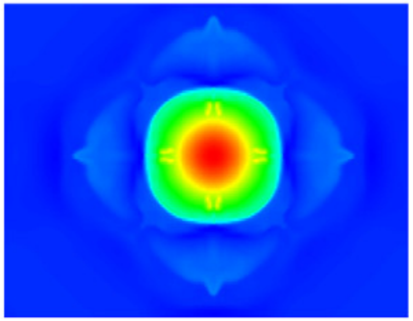
$$\frac{\partial}{\partial t}\left(E + \frac{1}{2}\rho\phi\right) + \nabla\cdot\left[(E+P)\mathbf{v} + \mathbf{F_g}\right] = 0.$$

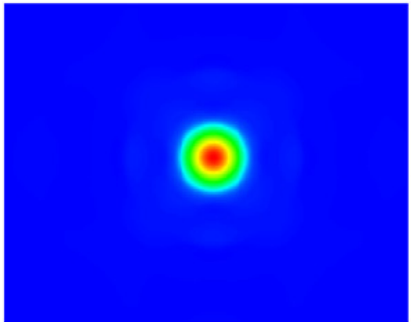
where the gravitational tensor $\mathbf{T_g}$ is

$$\mathbf{T_g} = \frac{1}{4\pi G}\left[\nabla\phi\nabla\phi - \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi)\mathbf{I}\right],$$

$$\mathbf{F_g} = \frac{1}{8\pi G}\left(\phi\nabla\dot{\phi} - \dot{\phi}\nabla\phi\right) + \rho\mathbf{v}\phi.$$

Gravity included with a source term





Gravity included through fluxes

Bonnor-Ebert sphere

[Jiang+…+Stone, 2013]

Selfgravity source terms – solution III

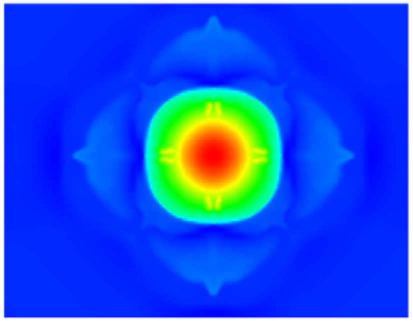
- The flux and source terms are found as the time averaged contribution

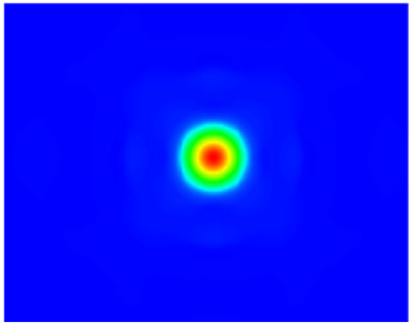
$$\tilde{F}_i(t+\Delta t/2) = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt\; F_i \;, \quad \tilde{S}_i(t+\Delta t/2) = \frac{1}{\Delta x \Delta t}\int_t^{t+\Delta t}\int_x^{x+\Delta x} dt\, dx\; S(x,t)$$

- **Problem:** Energy, incl potential, may not be conserved
- **Solution:** Rewrite equations of motion to become flux based

- **Caveat:** if potential energy dominates total energy, then changes in internal energy requires high precision. (rescue: Virial Theorem)

Gravity included with a source term





Gravity included through fluxes

Bonnor-Ebert sphere [Jiang+…+Stone, 2013]

Summary

- Self-gravity plays a crucial role in many astrophysical applications; from cosmological structure formation to planet formation
- The stability of a system depends on the detailed balance between gravity, external, and internal pressures.
- Rotation and magnetic fields can both help in stabilizing against gravitational collapse
- During collapse, mass is concentrated at the bottom of the potential well, releasing energy that is the ultimate source powering from proto-stellar systems to galactic rotation
- Adding gravity as a source term is non-trivial because the gravitational force is often balanced by other forces in a detailed balance; e.g. hydrostatic equilibrium. Source balances fluxes.