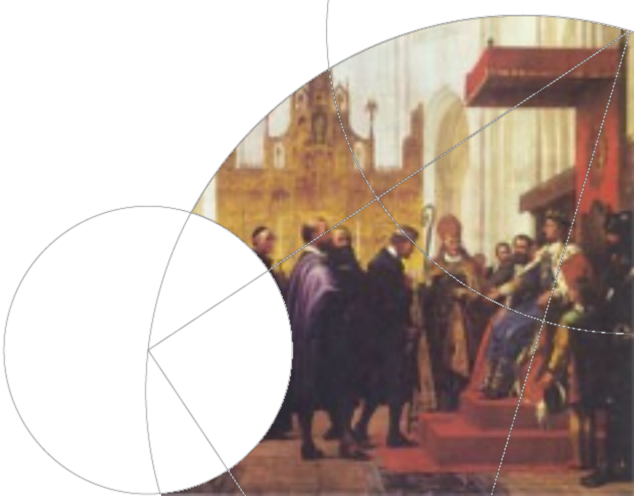
Computational Astrophysics

2a. Hydrodynamics

Troels Haugbølle
Niels Bohr Institute
University of Copenhagen



Topics today

- Overview of hydrodynamics
- Shocks and artificial viscosity
- A simple solver based on advection
- Godunov method
- Riemann solvers

- **Assignment 2a:**
  - Play around with the very simplest HD: Soundwaves

Hydrodynamics

- Hydrodynamics describes the evolution of a fluid.
- Many formulations exist depending on the subject, but in astrophysics we normally consider the compressible Euler equations
- These are governed by three conservations principles for mass, momentum, and energy

$$\partial_t \rho + \nabla \cdot [\rho \, \mathbf{v}] = 0$$
$$\partial_t \rho \, \mathbf{v} + \nabla \cdot [\rho \, \mathbf{v} \otimes \mathbf{v} + P\mathsf{I}] = 0$$
$$\partial_t E + \nabla \cdot [(E + P) \, \mathbf{v}] = 0$$

- E is the total energy: $E = \rho e_{int} + \frac{1}{2}\rho v^2$
- The difficulty with solving these equations comes predominantly from the Navier-Stokes equation that is non-linear in the velocity. This is what generates the rich structure seen in e.g. turbulent flows and enables shocks.

Hydrodynamics

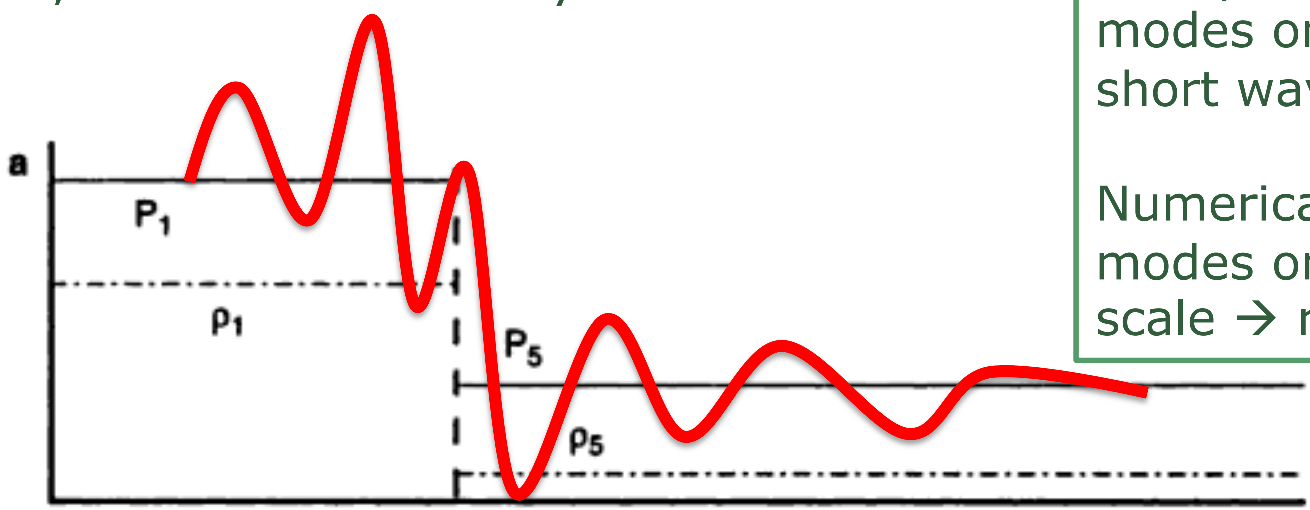
- The problem with hydrodynamics is that the equations allow not only for transport of material (advection)
- HD also support the propagation of waves
- The basic wave, working through an interaction between density, pressure, and velocity is a soundwave
- This modifies both the flux and the Courant conditions. Basically information can be transported with the three speeds:

$$u - c_{sound},\ u,\ u + c_{sound}$$

and we have to modify the courant condition correspondingly

Solving Hydrodynamics I – artificial viscosity

- Many different solvers exist in the literature.
- One option is to consider a finite difference formulation
- This turns out to be inherently unstable near large discontinuities, such as shocks (c.f. exercise 1c)
- A solution to this problem is artificial viscosity as discussed in chapter 6.1.4. This has historically been very popular, but is less so today



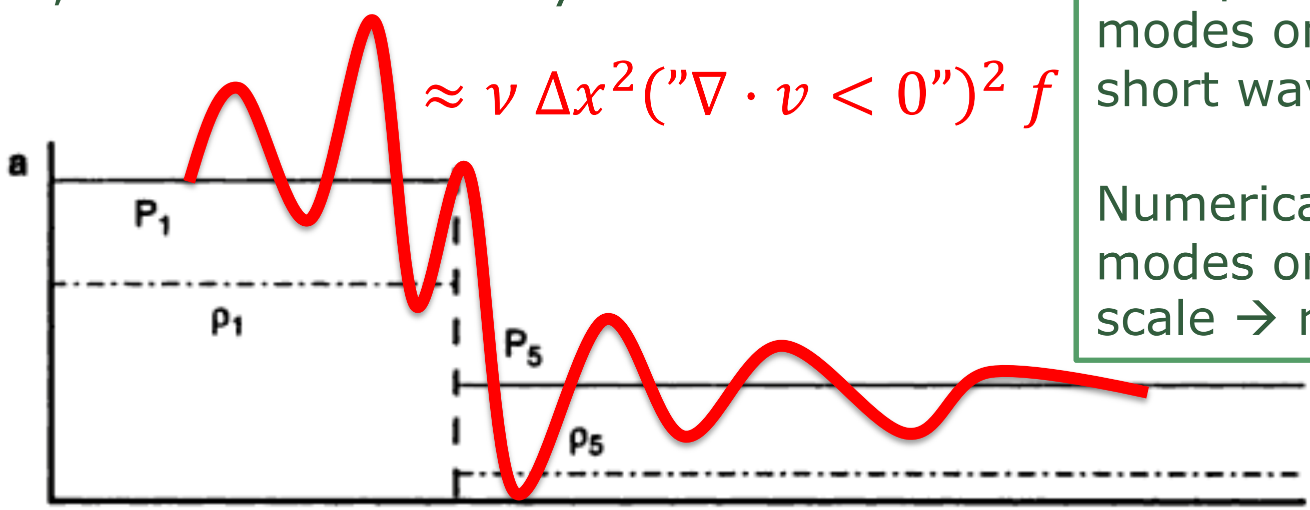
A step function has modes on infinitely short wavelengths.

Numerically we get modes on the grid scale → ringing

Solving Hydrodynamics I – artificial viscosity

- Many different solvers exist in the literature.
- One option is to consider a finite difference formulation
- This turns out to be inherently unstable near large discontinuities, such as shocks (c.f. exercise 1c)
- A solution to this problem is artificial viscosity as discussed in chapter 6.1.4. This has historically been very popular, but is less so today

$$\approx \nu\, \Delta x^2 (\text{"}\nabla \cdot v < 0\text{"})^2\, f$$

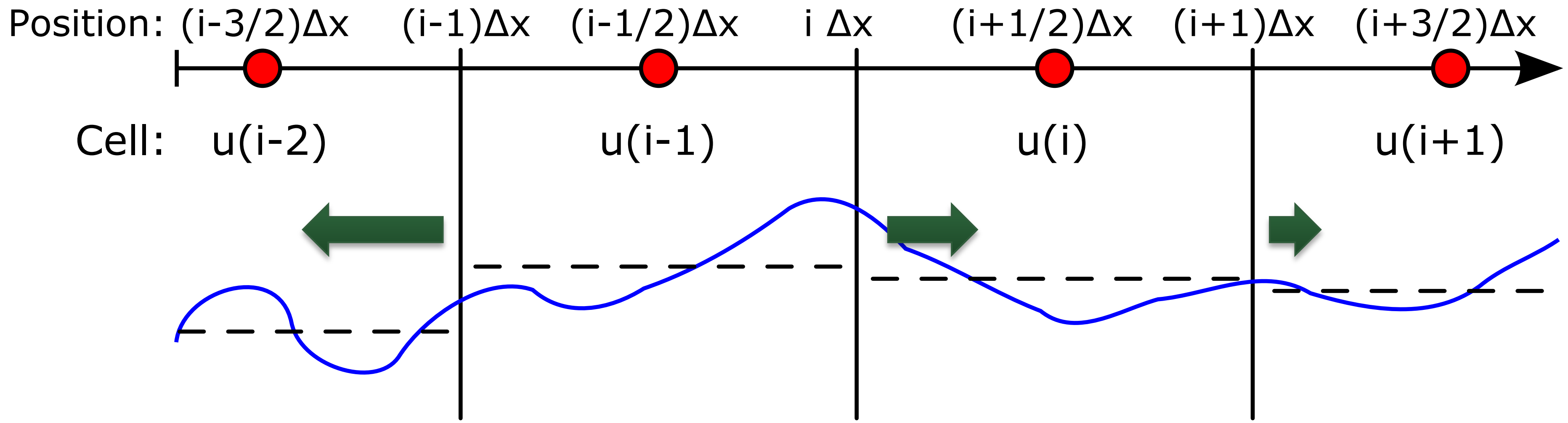


A step function has modes on infinitely short wavelengths.

Numerically we get modes on the grid scale → ringing

Finite Volume Method – viz from last week

- One popular approach to solver hydrodynamics is to use the finite volume method:



- ***u(x$_i$,t)*** is the average value in the interval [$\boldsymbol{x_{i-1/2}}$, $\boldsymbol{x_{i+1/2}}$] at time ***t***
- To find the solution to the volume average we have to consider the ***flux through the surface*** of each cell, resulting in the ***master equation***

$$u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2}^{n+1/2} - \tilde{F}_{i-1/2}^{n+1/2}\right), \qquad \tilde{F}_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt\, F_{i+1/2}[q(x_{i+1/2})]$$

Solving Hydrodynamics II – van Leer advection

- If we would like to use our advection skills from last week, we can reformulate the system of equations, such that it looks like an advection problem (see 6.3.4)

$$\partial_t \rho + \nabla \cdot [\rho v] = 0$$
$$\partial_t \rho v + \nabla \cdot [\rho v \otimes v] = -\nabla P$$
$$\partial_t E + \nabla \cdot [E v] = -\nabla \cdot [P v]$$

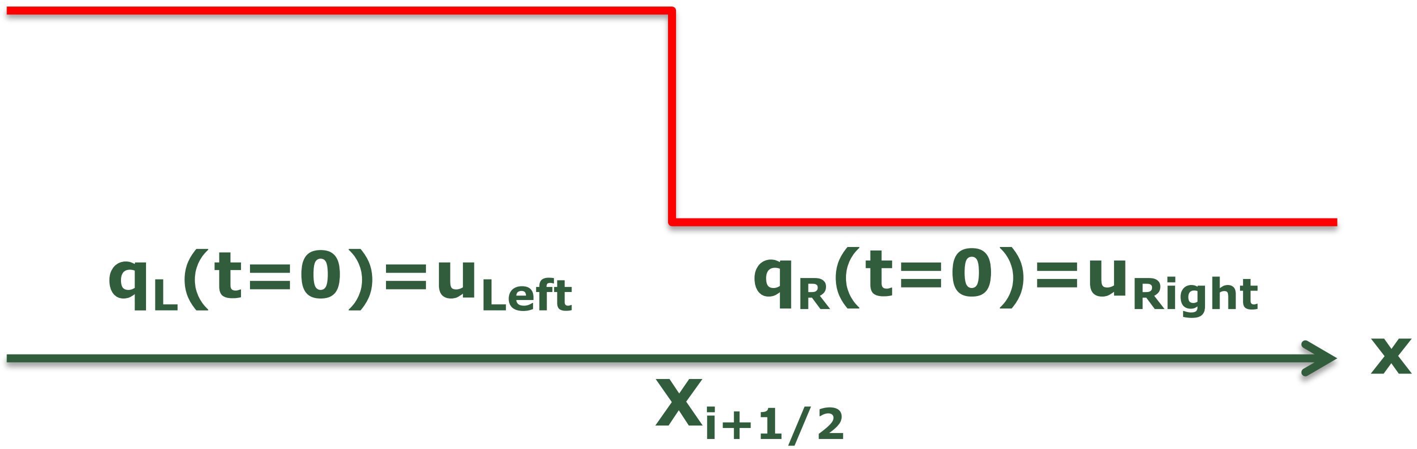
- The pressure term is now a source for momentum and energy.
  → limits precision if there is a detailed balance f.x. soundwave

- We can solve the equation if we remember our flux formulation including sources:

$$u_i^{n+1} - u_i^n = -\frac{\Delta t}{\Delta x}\left(\Delta x\, \tilde{S}_i^{n+1/2} + \tilde{F}_{i+1/2}^{n+1/2} - \tilde{F}_{i-1/2}^{n+1/2}\right)$$

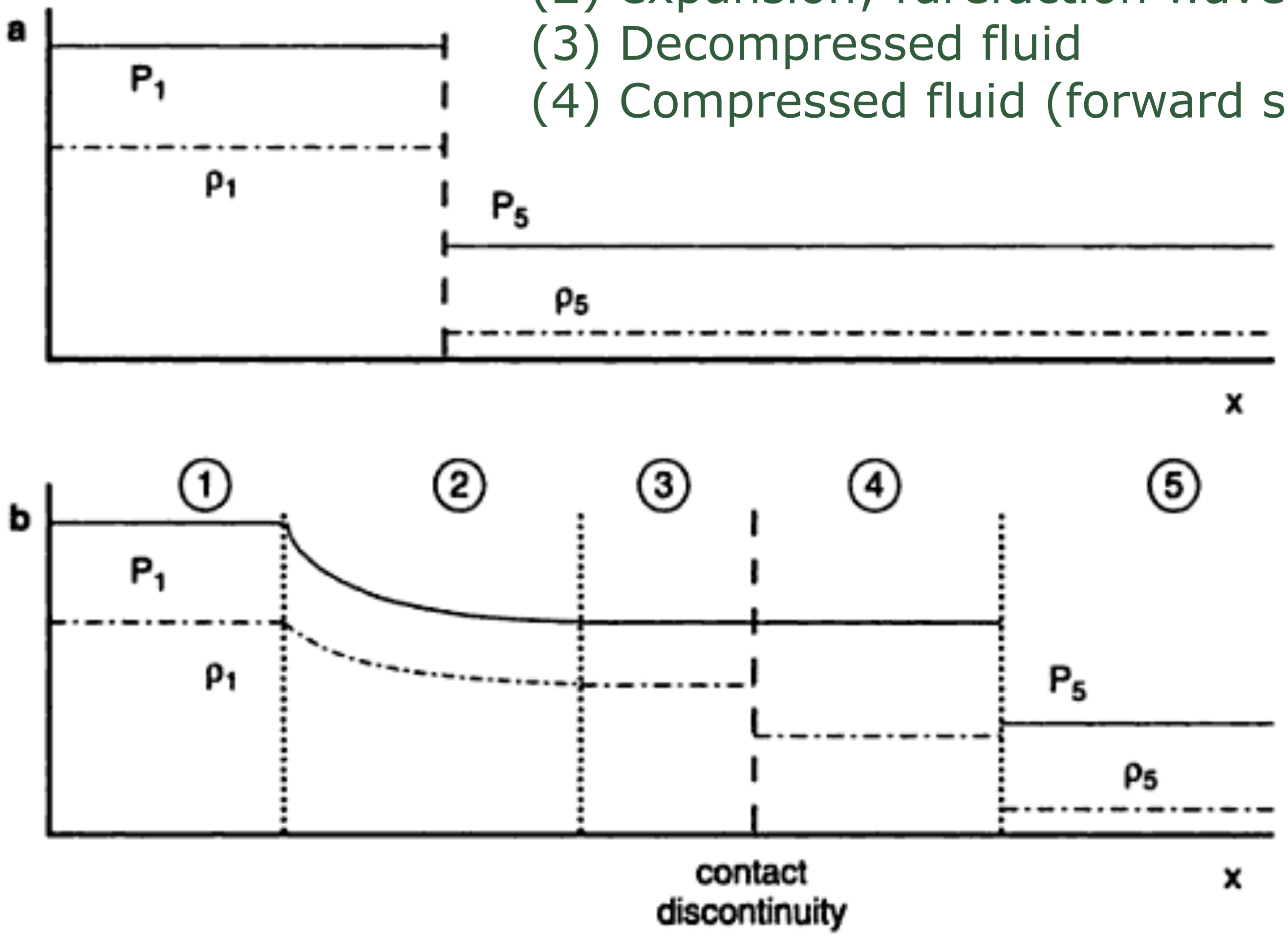
Solving Hydrodynamics III – Godunov method

- Modern methods use instead an approach based on exploiting the wave nature and full conservation properties of the equations
- These are know as Godunov methods (due to Sergei Godunov) and rely on solving the *Riemann problem* at each cell interface.
- This is related to finding the flux through a cell interface



HD shock structure in general (between two cells)

(1)+(5) undisturbed fluid
(2) expansion; rarefaction wave (reverse shock)
(3) Decompressed fluid
(4) Compressed fluid (forward shock)



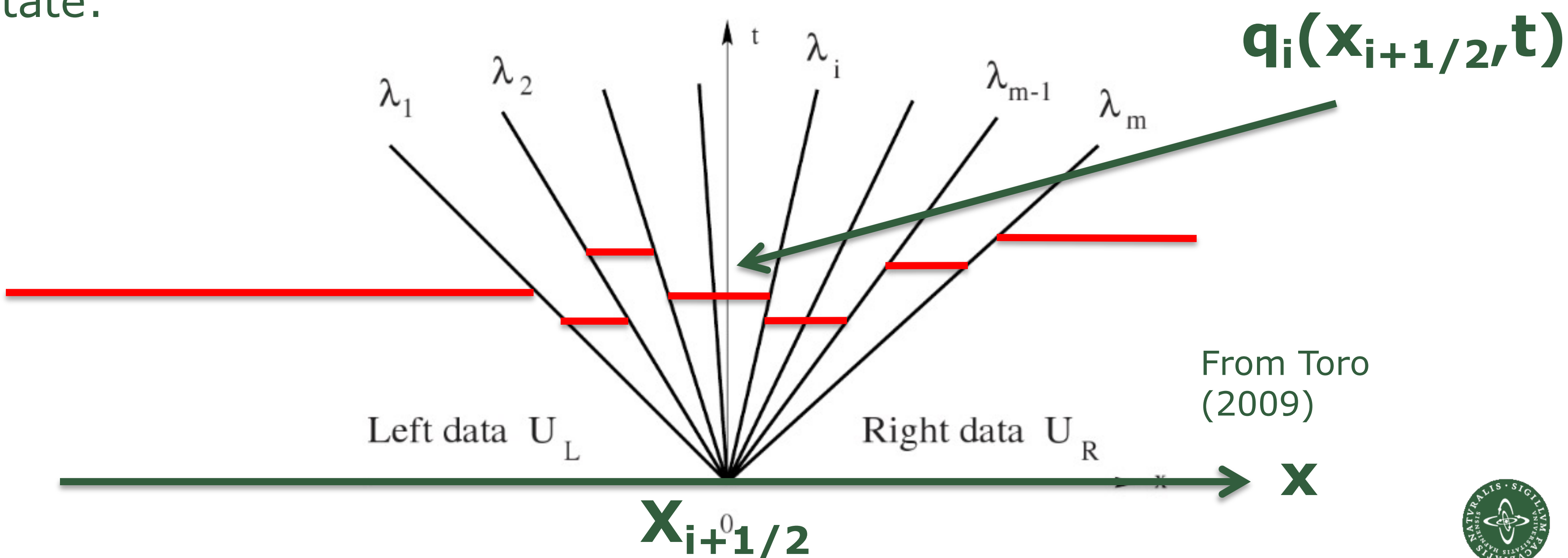
(see also fig. 6.5 in the book)

The Riemann Problem

- The problem is to find an expression for the time averaged fluxes

$$\tilde{F}^{n+1/2}_{i+1/2} = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt\, F_{i+1/2}[q(x_{i+1/2})]$$

- For a general system of equations there will be several wave speeds apart from advection (HD 3, MHD 7). Compute to find interface state:



From Toro (2009)

The Riemann problem

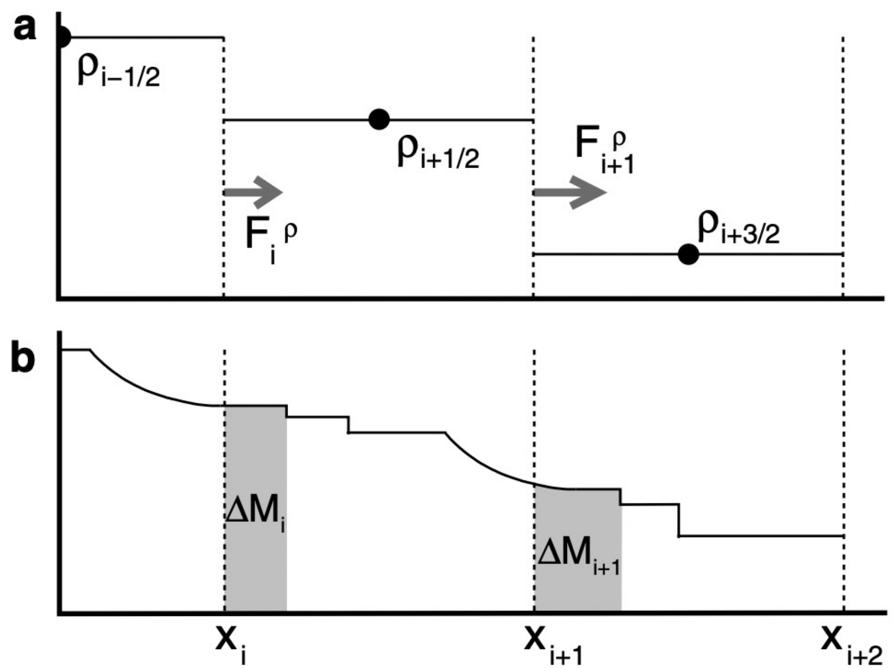


FIGURE 6.5 Advection of mass in Godunov schemes. Based on initial states (a), the Riemann problem is solved at each x_i (b). Once the solution is obtained, the amounts of mass ΔM_i and ΔM_{i+1} can be found that, within the time step, have flowed, respectively, into and out of the grid cell located between x_i and x_{i+1}. Alternatively, fluxes $\mathcal{F}_i^\rho$ and $\mathcal{F}_{i+1}^\rho$ can be calculated, where $\Delta M_i = \Delta t\, \mathcal{F}_i^\rho$. The same scheme is applied to each advected quantity.

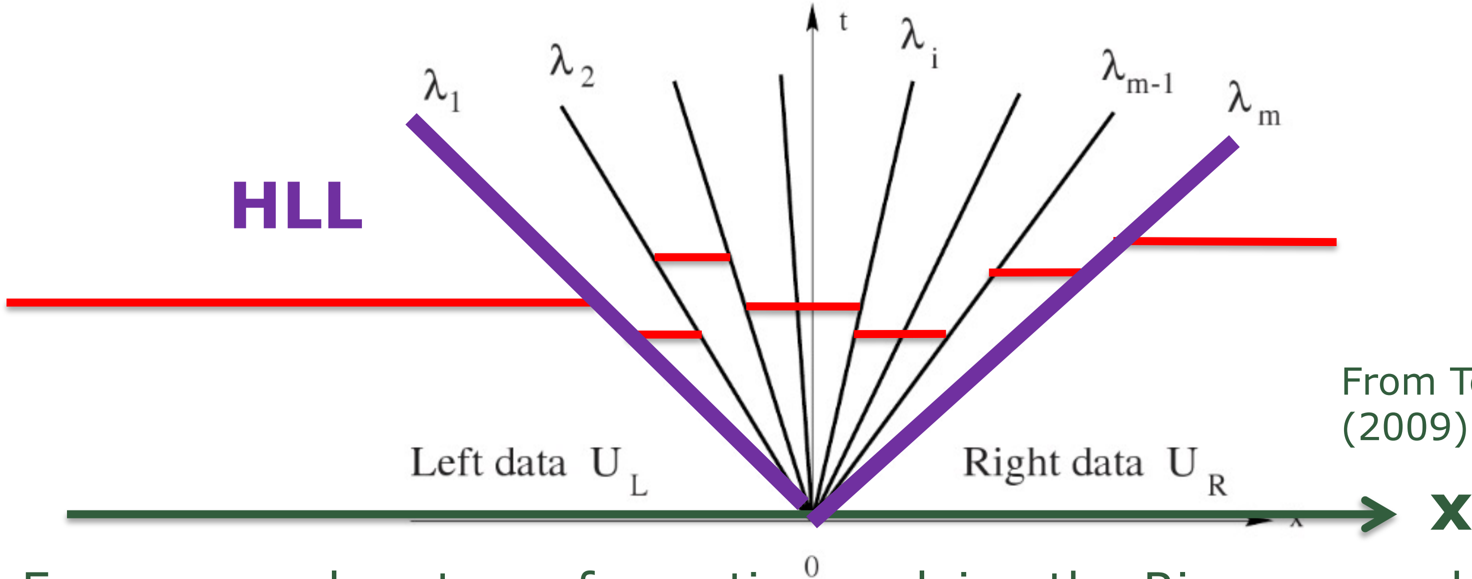
Solving Hydrodynamics III – Godunov method

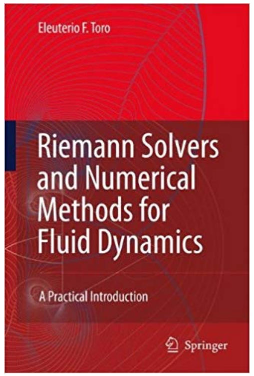
- While this method works, it has to be complemented by using slope reconstruction, otherwise it will be very diffusive

- For the time evolution a popular choice is the so-called MUSCL scheme:
  1. Given an initial state $U = (\rho, \rho v, E_{tot})$ use slope reconstruction to find interface values – this can be done in U or it can be done using *primitive variables* $q = (\rho, v, P)$
  2. To make the integral time centered advance the *cell-centered* solution half a timestep using a simple predictor. Either in *q* or in *u*.
  3. Now solve the Riemann problem and compute fluxes at the interfaces

- You will explore the MUSCL method in the exercise

The Riemann Problem II





From Toro (2009)

- For a general system of equations solving the Riemann problem is not easy, and for many systems the exact solution is either unknown, or results in transcendental equations requiring iterative solvers
- This has to be done for each and every interface!!
- As an alternative to this, what are called approximate solvers have been developed. They are fast but smears out the states to keep them stable.
- Two examples are included in the exercise notebook: LLF, HLL. See the *Riemann bible* by Toro (2009) for more complicated schemes
- On Thursday we will talk more about the HLL scheme

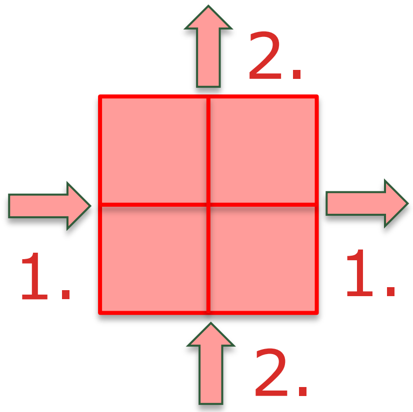
Higher dimensionalities

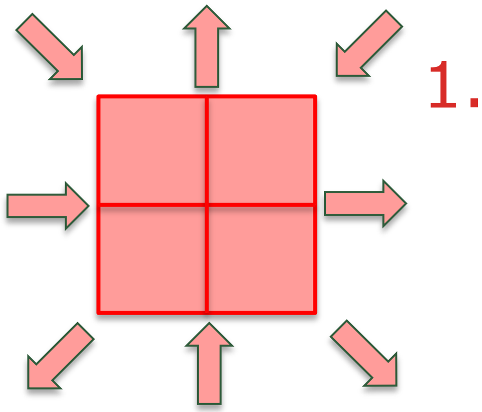
<u>(non)-split fluxes:</u>

- Compute fluxes sequentially thorugh each edge (face) in 2D (3D).
- Problem is overadvection where density or pressure becomes negative
- Solution also not manifestly isotropic
- Can induce so-called *carbuncle* instabilities

<u>Higher dimensionality fluxes:</u>

- Compute fluxes thorugh each edge and corner (face, edge) in 2D (3D).
- This is much more isotropic
- Super complicated to solve Riemann problem.
- Done only by few groups. Implementation papers are small books.





Assignment 2a:

- Using an approximate Riemann solver you will play around with a 1D soundwave, and implement the energy equation yourself in to the solver

- This is a warmup for Thursday where we will continue with hydrodynamical shock waves