Phase correlation - Mathematical Background

1. **Fourier Transfom**
   Given two images:

   - $I_\alpha(x, y)$ (reference image)
   - $I_\beta(x, y)$ (reference image)

   Their 2D Discrete Fourier Transforms (DFT) are:

   $$F_\alpha(u, v) = \mathcal{F}[I_\alpha(x, y)]$$
   $$F_\beta(u, v) = \mathcal{F}[I_\beta(x, y)]$$

2. **Fourier Shift Theorem**
   If $I_\beta(x, y)$ is a shifted version of $I_\alpha(x, y)$ by $(\Delta x, \Delta y)$, then in the frequency domain:

   $$F_\beta(u, v) = F_\alpha(u, v)e^{-2\pi j(u\Delta x + v\Delta y)}$$

   This introduces a phase difference but preserves the magnitude.

3. **Cross Power Spectrum**
   The normalized cross power spectrum is computed as:

   $$C(u, v) = \frac{F_\alpha(u, v) \cdot F_\beta^\star(u, v)}{|F_\alpha(u, v) \cdot F_\beta^\star(u, v)|},$$

   where $F_\beta^\star(u, v)$ is the complex conjugate of $F_\beta(u, v)$.

4. **Inverse Fourier Transform**
   The inverse Fourier transform of $C(u, v)$ gives a peak at the displacement $(\Delta x, \Delta y)$:

   $$c(x, y) = \mathcal{F}^{-1}[C(u, v)].$$

   The coordinates of the maximum valie in $c(x, y)$ correspond to the shift $(\Delta x, \Delta y)$.

Phase correlation - Python implementation

1. **Compute the Fourier Transfroms of the Two Images**

   - Let G_α and G_β be the Fourier transforms of the reference image I_α and the shot image I_β:

   $$G_\alpha = \mathcal{F}\{I_\alpha\},\ G_\beta = \mathcal{F}\{I_\beta\}$$

2. **Compute the Cross-Power Spectrum**

   - The cross-power spectrum is given by:

   $$R = \frac{G_\alpha \cdot G_\beta^\star}{|G_\alpha \cdot G_\beta^\star|}$$

   - $G_\beta^\star$ is the complex conjugate of G_α.
   - The denominator $|G_\alpha \cdot G_\beta^\star|$ ensures normalization, making the result purely phase-based.

3. **Inverse Fourier Transform to Get the Phase Correlation Map**

   - The inverse Fourier transform is applied to obtain the phase correlation function $r(x, y)$:

   $$r = \mathcal{F}^{-1}\{R\}$$

   - $r(x, y)$ is a real-valued function, even though R is complex.

4. **Find the Peak Location**

   - The shift $(\Delta x, \Delta y)$ is dound by detecting the peak of $r(x, y)$:

   $$(\Delta x, \Delta y) = argmax\{r(x, y)\}$$