We are given a set of n coin denominations $A = \{a_1, a_2, \cdots, a_n\}$, such that $a_1 > a_2 > \cdots > a_n = 1$. We are asked to make a change for S "cents". (Note that if the smallest coin is not 1, then it's impossible to make change for 1 cent. On the other hand, if we have a 1-cent coin, then it is possible to make change for S cents using S 1-cent coins. The problem is to use the minimum number of coins.) Let x_t denote the number of coins of type a_t used in a solution, that is, $S = \sum_{t=1}^{n} a_t x_t$. The number of coins used in such a solution is $\sum_{t=1}^{n} x_t$.

We study the following simple greedy algorithm in class (Lecture Slide 6). It takes as many of the largest coins as possible, then as many of the next largest coins as possible, etc. The following is an implementation of it and clearly takes $O(n)$ time, independent of S.

Algorithm *GreedyCoinChange* (A, S)
  $CoinCount \leftarrow 0$; $U \leftarrow S$
  for $t \leftarrow 1$ **to** n **do**
    $x_t \leftarrow U$ div a_t (* i.e., $x_t \leftarrow \lfloor U/a_t \rfloor$ *)
    $U \leftarrow U$ mod a_t (* i.e., $U \leftarrow U - a_t x_t$ *)
    $CoinCount \leftarrow CoinCount + x_t$
  return $((x_1, x_2, \cdots, x_n); CoinCount)$
end

Let $Opt(S)$ (respectively, $G(S)$) denote the number of coins used in an optimal (respectively, the greedy) solution with respect to the coin denomination set A. We say a coin denomination set A is *Regular* if $Opt(S) = G(S)$ with respect to A, for all S, i.e., when given the coin denomination set A, the greedy algorithm always produces an optimum solution for any change amount S. We want to find a polynomial-time checkable necessary and sufficient condition for regularity of a coin denomination. To this end, let us define $m_t = \lceil a_{t-1}/a_t \rceil$ and $S_t = m_t a_t$. In what follows, we will give a proof of the following fact as stated in Lecture Slide 6. Note that the stated condition can be checked in $O(n^2)$ time, since we essentially need to run the $O(n)$-time greedy algorithm for $O(n)$ "critical" change values.

> **Fact 1: [The Regularity Theorem of Magazine-Nemhauser-Trotter, 1975]**
> Suppose we have the Pre-Condition that $S_t < a_{t-2}$ for all $t = 3..n$. Then
>
> $Opt(S) = G(S)$ for all S if and only if $G(S_t) \le m_t$ for all $t = 2..n$.

Proof:

[Necessity:]

$G(S) = Opt(S)$ for all S. In particular, $G(S_t) = Opt(S_t)$ for all t. Clearly m_t a_t-coins is a feasible solution for $S_t = m_t a_t$. So, $Opt(S_t) \le m_t$ for all t. Therefore, we must have $G(S_t) \le m_t$ for all t.

[Sufficiency:]

Consider the restricted set of smallest coins $A_t = \{a_t, a_{t+1}, \cdots, a_n\}$. A solution for S with respect to A_t is a solution which uses only coins from the restricted set A_t (i.e., we are not allowed to use coins $a_1, \cdots, a_{t-1}$). Let $Opt_t(S)$ (respectively, $G_t(S)$) denote the number of coins used in the optimal (respectively, greedy) solution w.r.t. A_t. Note that $Opt(S) = Opt_1(S)$ and $G(S) = G_1(S)$.

An instance is a pair (A, S), where we want to make change for S "cents" using the permitted coin denomination set A. Suppose to the contrary that there is a counter-example (A, S), i.e., $G(S) > Opt(S)$ with respect to A. If so, then there must be a lexicographically minimal counter-example (A, S), i.e., $|A|$ is minimal, and with respect to that A, S is minimal. We will see below that these assumptions will lead to the contradictory conclusion that $G(S) \le Opt(S)$. This will refute our assumption that there is a counter-example.

Consider our minimal counter-example (A, S). Let $a_t \in A$ be the largest coin such that $S \ge a_t$. So, we have $\infty = a_0 > \cdots > a_{t-1} > S \ge a_t > a_{t+1} > \cdots > a_n = 1$. (We have introduced the fictitious $a_0 = \infty$ in case $t = 1$, in which case $a_{t-1} = a_0$.) Note that $t < n$, for if $t = n$, then clearly $Opt(S) = G(S) = S$. So, $a_{t+1} \in A$. Below, we give an alphabetically labeled list of observations that will lead to the conclusion of the proof.

(a) $a_{t-1} > S \ge a_t > a_{t+1}$.

(b) For all t and $S' < a_t$ we have $Opt(S') = Opt_{t+1}(S')$ and $G(S') = G_{t+1}(S')$ (since larger coins cannot be used).

(c) $Opt_t(S') \le Opt_{t+1}(S')$ for all t and S' (since further restriction of coin utilization cannot improve the optimum solution).

(d) $Opt_t(S' + S'') \le Opt_t(S') + Opt_t(S'')$ (since the left hand side is the optimum solution for $S' + S''$ w.r.t. A_t, while the right hand side is a feasible solution for the same).

(e) $G_t(a_t x + U) = x + G_t(U)$ for all t and $U \ge 0$ and $a_t x + U \ge 0$ and integers x (even if x is negative!).

(f) $G_{t'}(S') = Opt_{t'}(S')$ if $t' > t$, or $t' = t$ but $S' < S$ (since $(A_{t'}, S')$ is lexicographically smaller than the minimal counter-example (A, S)).

(g) Since $a_{t-1} > S \ge a_t$, then $G(S)$ will use the coin a_t. By minimality of (A, S) we conclude that $Opt(S)$ will NOT use coin a_t. For otherwise, we would have $0 > Opt(S) - G(S) = [1 + Opt(S - a_t)] - [1 + G(S - a_t)] = Opt(S - a_t) - G(S - a_t)$. Hence, $Opt(S - a_t) < G(S - a_t)$, i.e., the instance $(A, S - a_t)$ is a lexicographically smaller counter-example than (A, S); a contradiction.

(h) Let us consider the optimum solution $Opt(S)$.
Define $x = \lfloor S/a_{t+1} \rfloor$ and $U = S - a_{t+1} x$. So, $x \ge 0$ and $0 \le U < a_{t+1}$. Now we have

$$
\begin{aligned}
Opt(S) &= Opt_t(S) && \text{[by (a) and (b)]} \\
&= Opt_{t+1}(S) && \text{[by (g)]} \\
&= G_{t+1}(S) && \text{[by (f)]} \\
&= G_{t+1}(a_{t+1} x + U) && \text{[by the definition of U]}
\end{aligned}
$$

$$= x + G_{t+1}(U) \qquad \text{[by (e)]}$$

In summary, we conclude

$$Opt(S) = x + G_{t+1}(U) \tag{1}$$

(i) Now let us consider the Pre-Condition stated in the Theorem.
We have $m_{t+1} = \lceil a_t / a_{t+1} \rceil$ and $S_{t+1} = a_{t+1} m_{t+1}$. Define $V = S_{t+1} - a_t$. We have $0 \le V < a_{t+1}$. Therefore

$$\begin{aligned}
m_{t+1} &\ge G(S_{t+1}) && \text{[by Theorem's Condition]} \\
&= G_t(S_{t+1}) && \text{[by (b) and the Pre-Condition } S_{t+1} < a_{t-1}\text{]} \\
&= G_t(a_t + V) && \text{[by definition of } V\text{]} \\
&= 1 + G_t(V) && \text{[by (e)]} \\
&= 1 + G(V) && \text{[by (b) and (i): } V < a_{t+1}\text{]}
\end{aligned}$$

In summary, we conclude

$$G(V) - m_{t+1} + 1 \le 0 \tag{2}$$

(j) Now let us consider the greedy solution $G(S)$.

$$\begin{aligned}
G(S) &= G_t(S) && \text{[by (a) and (b)]} \\
&= 1 + G_t(S - a_t) && \text{[by (a) and (e)]} \\
&= 1 + Opt_t(S - a_t) && \text{[by (f)]} \\
&\le 1 + Opt_{t+1}(S - a_t) && \text{[by (c)]} \\
&= 1 + G_{t+1}(S - a_t) && \text{[by (f)]} \\
&= 1 + G_{t+1}(\, a_{t+1}(x - m_{t+1}) + U + V \,) && \text{[by def. of U in (h) \& V in (i)]} \\
&= 1 + (x - m_{t+1}) + G_{t+1}(U + V) && \text{[by (e)]} \\
&= 1 + x - m_{t+1} + Opt_{t+1}(U + V) && \text{[by (f)]} \\
&\le 1 + x - m_{t+1} + Opt_{t+1}(U) + Opt_{t+1}(V) && \text{[by (d)]} \\
&= 1 + x - m_{t+1} + G_{t+1}(U) + G_{t+1}(V) && \text{[by (f)]} \\
&= Opt(S) + 1 - m_{t+1} + G_{t+1}(V) && \text{[by (1)]} \\
&= Opt(S) + 1 - m_{t+1} + G(V) && \text{[by (b) and (i): } V < a_{t+1}\text{]}
\end{aligned}$$

$$\leq Opt(S) \qquad\qquad [\text{by } (2)]$$

In summary, we conclude

$$G(S) \leq Opt(S) \tag{3}$$

This contradicts our assumption that (A, S) is a counter-example and completes the proof. □

Remark: We notice that the theorem's Pre-Condition is needed only in the proof of (2) which is subsequently needed to complete the proof of (3). So, we can simplify the theorem by removing its Pre-Condition and instead replace its necessary and sufficient condition $G(S_{t+1}) \leq m_{t+1}$ for all t, by the sufficient condition (2), namely, $G(S_{t+1} - a_t) \leq m_{t+1} - 1$ for all t. The simplified theorem is

Theorem: $Opt(S) = G(S)$ for all S if $G(S_t - a_{t-1}) \leq m_t - 1$ for all $t = 2..n$.

The sufficient condition stated in this theorem can also be checked in $O(n^2)$ time, since we essentially need to run the $O(n)$-time greedy algorithm for only O(n) critical change values. □

Exercise: Is the sufficient condition stated in the above theorem also necessary (without any pre-condition)? Explain your answer. □

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