Revision Course – Asset Management
Portfolio Construction and CAPM
Solutions

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1 Portfolio Construction

Exercise 2
Consider three assets, A, B, and C, with variance-covariance matrix $\mathbf{\Sigma}$ and expected rates of return r_A, r_B, and r_C:

$$\mathbf{\Sigma} = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \end{array} \qquad \text{and} \qquad \mathbf{R} = \begin{pmatrix} r_A \\ r_B \\ r_C \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.8 \end{pmatrix}$$

1. Find the minimum variance portfolio (MVP).
2. Find a second efficient portfolio. (Hint: tangency portfolio (TP) with $r_f = 0$)
3. If the risk free rate is $r_f = 0.2$, find an efficient portfolio of risky assets (TP) and compute the expected return and variance of this portfolio.
4. Calculate the expected return and the variance for an efficient portfolio which has a 50% weight in the MVP and 50% in the TP.

Solution: Exercise 2

*I am grateful to Maximilian Bredendiek for graciously sharing his notes.

1. Investors choose the minimum variance portfolio by solving the following optimization problem for K assets:

$$\min_{\mathbf{w}\in\mathbb{R}^K} \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w},$$

subject to

$$\mathbf{w}'\mathbf{1} = 1,$$

where $\mathbf{w}$ is a $K \times 1$ vector of portfolio weights, $\boldsymbol{\Sigma}$ is the $K \times K$ variance-covariance matrix, and $\mathbf{1}$ is a $K \times 1$ vector of ones. Solving this optimization problem yields the weights of the minimum variance portfolio,

$$\mathbf{w}_{MVP} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

The solution above can be found by solving the first-order conditions of the Lagrangian:

$$\begin{aligned}
Z(\mathbf{w},\lambda) &:= \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \lambda\left(\mathbf{w}'\mathbf{1} - 1\right)\\
\frac{\partial Z}{\partial \mathbf{w}} &= \boldsymbol{\Sigma}\mathbf{w} - \lambda\iota = 0\\
\frac{\partial Z}{\partial \lambda} &= \mathbf{w}'\mathbf{1} = 1\\
\Rightarrow \mathbf{w} &= \lambda\boldsymbol{\Sigma}^{-1}\iota\\
\Rightarrow 1 &= \lambda\iota'\boldsymbol{\Sigma}^{-1}\iota
\end{aligned}$$

To compute the weights, we need the inverse of the variance-covariance matrix (e.g. Gauss–Jordan elimination), which is

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix}.$$

Detailed computations are as follows:

$$(A|I) = \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$(A|I) = \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$(A|I) = \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 2 & \frac{1}{2} & -1 & \frac{3}{2} \end{array}\right)$$

$$(A|I) = \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right)$$

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right)$$

Thus, the minimum variance weights are

$$\mathbf{w}_{MVP} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \frac{1}{4} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \frac{1}{4} \begin{pmatrix} 3-2+1 \\ -2+4-2 \\ 1-2+3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Using the vector of portfolio weights, the vector of expected returns, and the variance-covariance matrix, the expected return and the variance of the minimum variance portfolio are

$$\begin{aligned} \mathbb{E}\left(r_{MVP}\right) &= \frac{1}{2} \times 0.4 + \frac{1}{2} \times 0.8 = 0.6 \qquad \text{and} \\ \sigma^2_{MVP} &= \left(\frac{1}{2}\right)^2 \times 2 + \left(\frac{1}{2}\right)^2 \times 2 + 2 \times \frac{1}{2} \times \frac{1}{2} \times 0 = 1. \end{aligned}$$

2. A second efficient portfolio is the tangency portfolio (TP) for any risk-free rate r_f. Investors choose a portfolio by solving the following optimization problem

$$\min_{\mathbf{w}\in\mathbb{R}^K} \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$

subject to

$$\mathbf{w}'\mathbf{R} + (1 - \mathbf{w}'\mathbf{1})\, r_f = \mu$$

Solving this optimization problem gives the weights of the tangency portfolio,

$$\mathbf{w}_{TP} = \frac{\mathbf{\Sigma}^{-1}(\mathbf{R} - \mathbf{1}r_f)}{(\mathbf{R} - \mathbf{1}r_f)' \, \mathbf{\Sigma}^{-1}\mathbf{1}}.$$

We can compute

$$\mathbf{\Sigma}^{-1}(\mathbf{R} - \mathbf{1}r_f) = \frac{1}{4}\begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 0.4 - r_f \\ 0.4 - r_f \\ 0.8 - r_f \end{pmatrix} \overset{\text{for } r_f = 0}{=} \begin{pmatrix} 0.3 \\ -0.2 \\ 0.5 \end{pmatrix}$$

As $\mathbf{1}'\left(\mathbf{\Sigma}^{-1}\mathbf{R}\right) = 0.3 - 0.2 + 0.5 = 0.6$, the weights of the tangency portfolio for $r_f = 0$ are then given by

$$\mathbf{w}_{TP, r_f = 0} = \frac{1}{0.6}\begin{pmatrix} 0.3 \\ -0.2 \\ 0.5 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}.$$

3. Similar, the weights of the tangency portfolio with $r_f = 0.2$ can be computed equivalently to the case with $r_f = 0$:

$$\frac{1}{4}\begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 0.4 - 0.2 \\ 0.4 - 0.2 \\ 0.8 - 0.2 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -0.2 \\ 0.4 \end{pmatrix}$$

Realizing that $\mathbf{1}'\left(\mathbf{\Sigma}^{-1}(R - r_f)\right) = 0.2 - 0.2 + 0.4 = 0.4$ yields the weights of the tangency portfolio with $r_f = 0.2$:

$$\mathbf{w}_{TP} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

The expected return and the variance of the tangency portfolio with $r_f = 0.2$ are

$$\mathbb{E}\left(r_{TP}\right) = \frac{1}{2} \times 0.4 - \frac{1}{2} \times 0.4 + 1 \times 0.8 = 0.8 \qquad \text{and}$$

$$\sigma_{TP}^2 = \left(\frac{1}{2}\right)^2 \times 2 + \left(-\frac{1}{2}\right)^2 \times 2 + 1^2 \times 2 + \left(2 \times \frac{1}{2} \times \left(-\frac{1}{2}\right) \times 1\right) + \left(2 \times \left(-\frac{1}{2}\right) \times 1 \times 1\right) + \left(2 \times \frac{1}{2} \times 1 \times 0\right) = 1.5.$$

4. The expected return of portfolio E, made up of 50% MVP and 50% TP ($r_f = 0.2$), is

$$\mathbb{E}\left(r_E\right) = \frac{1}{2} \times \mathbb{E}\left(r_{MVP}\right) + \frac{1}{2} \times \mathbb{E}\left(r_{TP}\right) = \frac{1}{2} \times 0.6 + \frac{1}{2} \times 0.8 = 0.7.$$

The variance of the portfolio is

$$\sigma_E^2 = \left(\frac{1}{2}\right)^2 \times \sigma_{MVP}^2 + \left(\frac{1}{2}\right)^2 \times \sigma_{TP}^2 + 2 \times \frac{1}{2} \times \frac{1}{2} \times Cov\left(r_{MVP}, r_{TP}\right)$$

with

$$Cov\left(r_{MVP}, r_{TP}\right) = \mathbf{w}_{MVP}' \boldsymbol{\Sigma} \mathbf{w}_{TP} = 1.$$

Therefore, $\sigma_E^2 = 1.125$.

An alternative way is specifying weights of portfolio E by averaging the weights of the minimum variance and the tangency portfolio, which gives

$$\mathbf{w}_E = \frac{1}{2} \times \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \times \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

One can use this new vector of weights, the vector of returns, and the variance-covariance matrix to compute the expected return and the variance of portfolio E.

2 CAPM

Exercise 3 (based on BKM, Chapter 9, Exercise 21)

Suppose the rate of return on short-term government securities (perceived to be risk-free) is about 5%. Suppose also that the market risk premium is 7%. According to the capital asset pricing model:

1. What is the expected rate of return on the market portfolio?
2. What would be the expected rate of return on a risk neutral stock?
3. Suppose you consider buying a share of stock at \$40. The stock is expected to pay \$3 dividends next year and you expect it to sell then for \$41. The stock risk has been evaluated at $\beta = -0.5$. Is the stock overpriced or underpriced?

Solution: Exercise 3

1. The market risk premium is the expected market return less the risk-free interest rate. The expected market return is thus the market risk premium plus the risk-free interest rate, $r_m = 7\% + 5\% = 12\%$.
2. A risk neutral stock has a beta of 0. Thus, the stock is not affected by systematic risk, which is the only risk priced under the CAPM. Hence, the stock's expected rate of return in market equilibrium is the risk-free rate of 5%.
3. The expected return of the stock is calculated as

$$\mathbb{E}\left(r_E\right) = \frac{\mathbb{E}\left(P_{t+1}\right)}{P_t} - 1 = \frac{41+3}{40} - 1 = 0.1 = 10\%.$$

   Given the risk-free rate of 5% and a market return of 12%, the required expected return of a stock with $\beta = -0.5$ is

$$\mathbb{E}\left(r\right) = r_f + \beta \times \left(\mathbb{E}\left(r_m\right) - r_f\right) = 5\% + (-0.5) \times (12\% - 5\%) = 1.5\%.$$

   As the expected return of the stock is much higher than predicted by the SML, we can say that the stock is underpriced.

Exercise 4 (BKM, Chapter 9, Exercise 9)
Consider the next figure which gives a security analysts expected return on two stocks for two particular market returns:

Market Return	Aggressive Stock	Defensive Stock
5%	-2%	6%
25%	38%	12%

1. What are the betas of the two stocks?
2. What is the expected rate of return on each stock if the market return is equally likely to be 5% or 25%?
3. If the T-Bill rate is 6% and the market return is equally likely to be 5% or 25%, draw the SML for this economy.
4. Plot the two securities on the SML graph. What are the alphas of each?

Solution: Exercise 4

$$\mathbb{E}\left(r\right) = r_f + \beta \times \left(\mathbb{E}\left(r_m\right) - r_f\right)$$

1. The betas for both stocks are:

$$\beta_{Aggressive} = \beta_A = \frac{\mathbb{E}\left(r\right)}{\mathbb{E}\left(r_m\right)} = \frac{-2\% - r_f}{5\% - r_f} = \frac{38\% - r_f}{25\% - r_f}$$
$$\beta_{Defensive} = \beta_D = \frac{\mathbb{E}\left(r\right)}{\mathbb{E}\left(r_m\right)} = \frac{6\% - r_f}{5\% - r_f} = \frac{12\% - r_f}{25\% - r_f}$$

Solving the equations for beta gives

$$\beta_A = 2 \qquad \text{and} \qquad \beta_D = 0.3.$$

This can be computed, for instance, as follows,

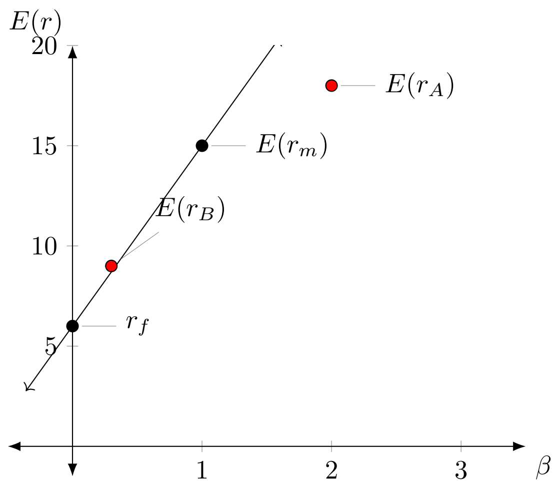
$$\begin{aligned} -2\% =& r_f + \beta_A\left(5\% - r_f\right) \\ 38\% =& r_f + \beta_A\left(25\% - r_f\right) \\ \Leftrightarrow 40\% =& \beta_A\left(25\% - r_f - 5\% + r_f\right) \\ \text{and } 6\% =& r_f + \beta_D\left(5\% - r_f\right) \\ 12\% =& r_f + \beta_D\left(25\% - r_f\right) \\ \Leftrightarrow 6\% =& \beta_D\left(25\% - r_f - 5\% + r_f\right) \end{aligned}$$

2. With the two scenarios equally likely, the expected return is an average of the two possible outcomes:

$$\begin{aligned} \mathbb{E}\left(r_A\right) &= \frac{1}{2} \times (-2\%) + \frac{1}{2} \times 38\% = 18\% \\ \mathbb{E}\left(r_D\right) &= \frac{1}{2} \times 6\% + \frac{1}{2} \times 12\% = 9\% \end{aligned} \tag{1}$$

3. The security market line (SML) is determined by the expected return of the market $(0.5 \times 25\% + 0.5 \times 5\%) = 15\%$, the risk-free rate $r_f = 6\%$, and the market beta. The equation for the SML is

$$\mathbb{E}\left(r\right) = r_f + \beta \times \left(\mathbb{E}\left(r_m\right) - r_f\right) = 6\% + \beta \times 9\%.$$



4. Based on its riskiness, the aggressive stock has a required expected return of

$$\mathbb{E}\left(r_A\right) = 0.06 + 2 \times 0.09 = 0.24 = 24\%.$$

The analyst's forecast of expected return is only 18%. The stock's alpha is the difference between the expected return and the required return given the stock's risk:

$$\alpha_A = 18\% - 24\% = -6\%.$$

Thus, CAPM indicates that the aggressive stock underperforms based on the analysts forecasts. Doing the same for the defensive stock, we get $\alpha_D = 0.3\%$.

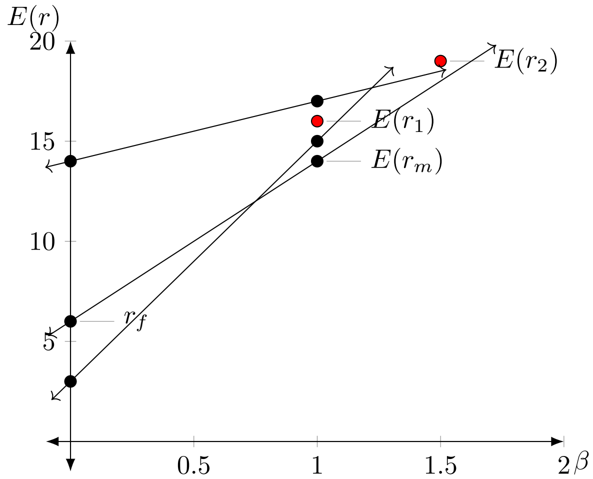
Exercise 5 (BKM, Chapter 9, Exercise 20)

Two investment advisers are comparing performance. One averaged a 19% rate of return and the other a 16% rate of return. However, the beta of the first investor was 1.5, whereas that of the second was 1.

1. Can you tell which investor was a better selector of individual stocks (aside from the issue of general movements in the market)?
2. If the T-bill rate were 6% and the market return during the period were 14%, which investor would be the superior stock selector?
3. What if the T-bill rate were 3% and the market return were 15%?

Solution: Exercise 5

1. To determine which investor was a better selector of individual stocks we look at abnormal return, which is the ex-post alpha; that is, the abnormal return is the difference between the actual return and that predicted by the SML. Without information about the parameters of this equation (risk-free rate and market rate of return) we cannot determine which investor was more accurate.



2. We first compute required returns given the beta estimates of the two investors as

$$
\begin{aligned}
\mathbb{E}(r_1) &= r_f + \beta \times (\mathbb{E}(r_m) - r_f) = 6\% + 1.5 \times (14\% - 6\%) = 18\% \\
\mathbb{E}(r_2) &= r_f + \beta \times (\mathbb{E}(r_m) - r_f) = 6\% + 1 \times (14\% - 6\%) = 14\%.
\end{aligned}
$$

We calculate the alphas of both investors, given their performance and the required return from above as

$$\alpha_1 = 19\% - 18\% = 1\%$$
$$\alpha_2 = 16\% - 14\% = 2\%.$$

Both investors have positive alphas, which shows that their investments performed better than required by their riskiness. However, the second investor's alpha is higher by 1%, which means that he was a better selector of stocks, given the market conditions.

3. We do the same computations as before, with $r_f = 3\%$ and $r_m = 15\%$. We get

$$\mathbb{E}(r_1) = 21\%$$
$$\mathbb{E}(r_2) = 15\%$$

and

$$\alpha_1 = 19\% - 21\% = -2\%$$
$$\alpha_2 = 16\% - 15\% = 1\%$$

We can see that the second investor is still the better stock picker. However, the alpha drops to 1%. The first investor's alpha even becomes negative, which means that his investments underperformed on a risk-adjusted basis (value destroying). (Try the same with e.g. a market return of 6% and a risk-free rate of 2% to see that the first investor can be better than the second one.)

Exercise 6 (BKM, Chapter 9, Exercise 23)

1. A mutual fund with $\beta = 0.8$ has an expected rate of return of 14%. If $r_f = 5\%$, and you expect the rate of return on the market portfolio to be 15%, should you invest in this fund? What is the fund's alpha?

2. What passive portfolio comprised of a market-index portfolio and a money market account would have the same beta as the fund? Show that the difference between the expected rate of return on this passive portfolio and that of the fund equals the alpha from part (a).

Solution: Exercise 6

1. We first calculate the required return given by the SML, which is

$$\mathbb{E}(r) = r_f + \beta \times (\mathbb{E}(r_m) - r_f) = 5\% + 0.8 \times (15\% - 5\%) = 13\%.$$

The fund's alpha is the difference between expected and required return, which is

$$\alpha = 14\% - 13\% = 1\%.$$

As the fund has a positive alpha, it is underpriced in the market, which makes it a valuable investment.

2. The beta of a portfolio is the portfolio-weighted average of individual assets.

   To see that the beta of a portfolio is the portfolio-weighted average of individual assets, consider the portfolio returns r^P of investing a fraction w in the market portfolio and $1 - w$ in the risk-free asset:

$$r^P = (1 - w)r_f + wr_m$$

   Then, we have

$$\mathbb{E}(r^P) = (1 - w)\mathbb{E}(r_f) + w\mathbb{E}(r_m)$$

   Recall that the CAPM equation implies that $\beta_P = \frac{\mathrm{Cov}(r^P, r_m)}{\mathrm{Var}(r_m)}$. Therefore, we have

$$\begin{aligned}\beta_P = \mathrm{Cov}(r^P, r_m) &= \mathbb{E}\left(\left(r^P - \mathbb{E}(r^P)\right)(r_m - \mathbb{E}(r_m))\right)/\mathrm{Var}(r_m) \\ &= \mathbb{E}\left(((1 - w)(r_f - \mathbb{E}(r_f)) + w(r_m - \mathbb{E}(r_m)))(r_m - \mathbb{E}(r_m))\right)/\mathrm{Var}(r_m) \\ &= w\mathbb{E}\left((r_m - \mathbb{E}(r_m))^2\right)/\mathrm{Var}(r_m) = w\end{aligned}$$

   We know that the market-index portfolio has a beta of 1 and that the money market account has a beta of 0. Our portfolio should have the same beta as the fund, which is 0.8. We thus solve for the individual weights

$$\beta_P = w \times \beta_m + (1 - w) \times \beta_f = w \times 1 + (1 - w) \times 0 = w = 0.8.$$

   Thus, we know that our portfolio consists of 0.8 times the market-index portfolio and $(1 - 0.8) = 0.2$ times the money market account. The expected rate of return of this portfolio is

$$\mathbb{E}\left(r_P\right) = 0.8 \times \mathbb{E}\left(r_m\right) + 0.2 \times r_f = 0.8 \times 15\% + 0.2 \times 5\% = 13\%.$$

The difference between this passive strategy and the fund is exactly the fund's alpha from before, $\alpha = 14\% - 13\% = 1\%$.