

Spatial autocorrelation and why it matters
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Measuring SAC
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Gaussian processes, correlation function
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Variogram
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Spatial autocorrelation

Vojtěch Barták

LS 2022

- 1 Spatial autocorrelation and why it matters
- 2 Measuring SAC
- 3 Gaussian processes, correlation function
- 4 Variogram

Spatial autocorrelation and why it matters



Measuring SAC



Gaussian processes, correlation function



Variogram



Section 1

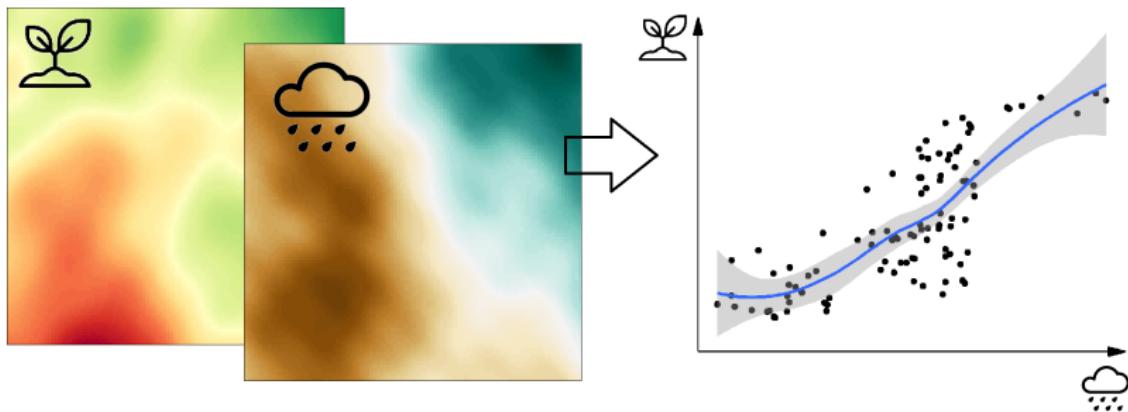
Spatial autocorrelation and why it matters

Types of data in spatial statistics

- Geostatistical data
- Point pattern data
- Areal (lattice) data

Problem with spatially autocorrelated data

Spatial or structural relationship?



Possible explanations:

- ① A direct causal effect
- ② An indirect causal effect
- ③ An accidental spatial association

Chapman 2010

Global Ecology and Biogeography, (Global Ecol. Biogeogr.) (2010) 19, 831–841



Weak climatic associations among British plant distributions



Data:

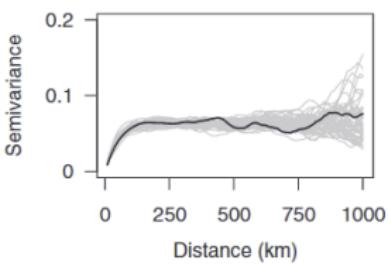
- 100 UK plant species at 10x10 km grid
- 23 climatic variables
- simulated variables with same properties

Models:

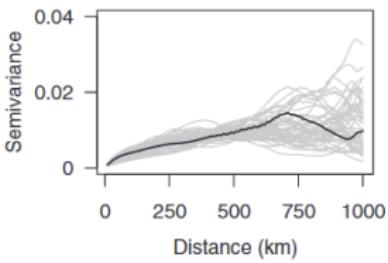
- GLM, random forests
- non-spatial models, trend surfaces, autoregression
- Single predictors, PCA

Chapman 2010

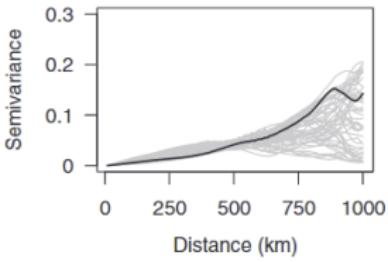
(a)



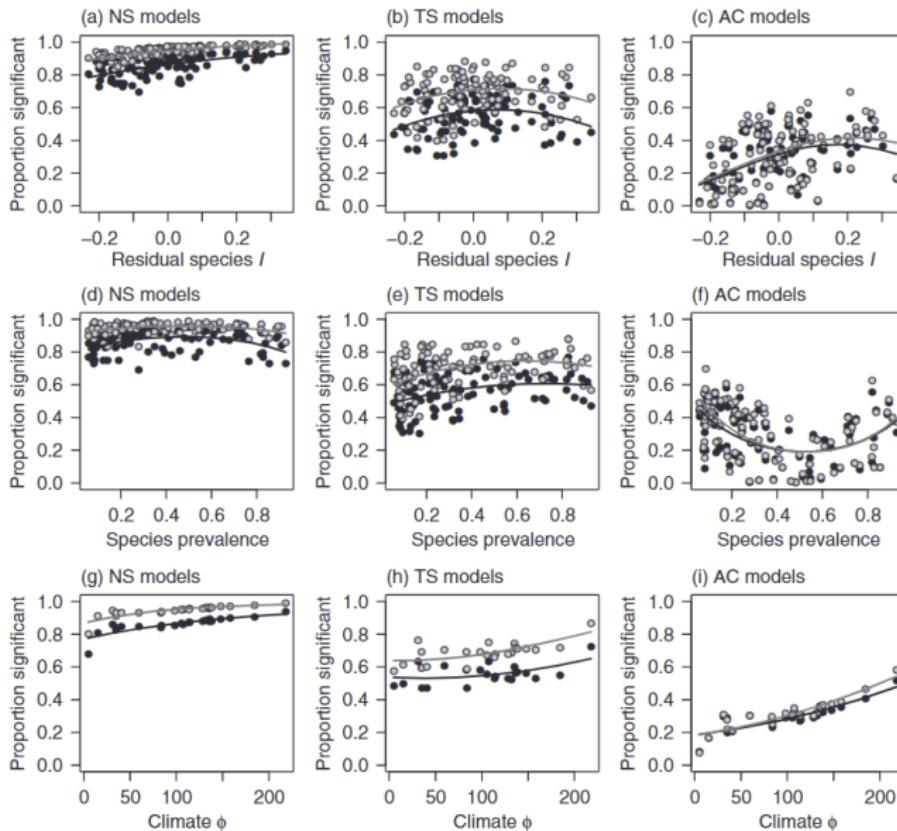
(b)



(c)



Chapman 2010



Fourcade, Besnard, Secondi 2018

MACROECOLOGICAL METHOD

WILEY

Global Ecology
and Biogeography

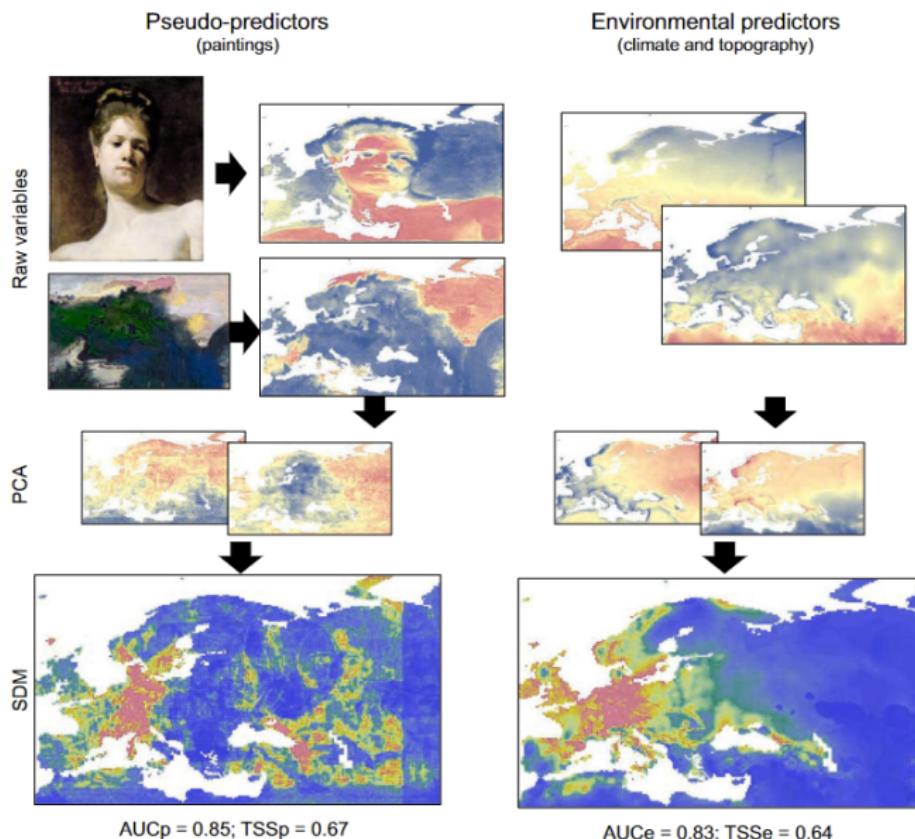
A Journal of
Macroecology

Paintings predict the distribution of species, or the challenge of selecting environmental predictors and evaluation statistics

- 497 species from European Red List (GBIF)
- real environmental predictors vs paintings



Fourcade, Besnard, Secondi 2018



Spatial autocorrelation and why it matters
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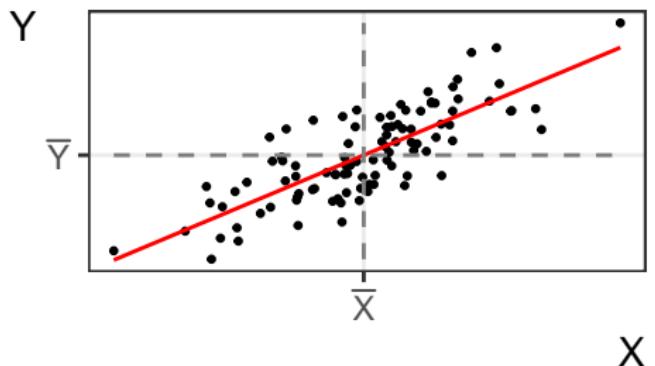
Gaussian processes, correlation function
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Variogram
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Section 2

Measuring SAC

Correlation of two variables



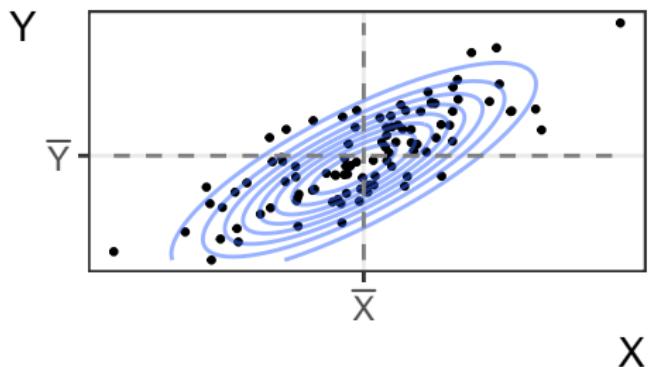
Sample covariance: $S_{XY} = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})$

Correlation coefficient: $r = \frac{S_{XY}}{S_X S_Y}, -1 \leq r \leq 1$

$S_X, S_Y \dots$ sample standard deviations of X, Y

Correlation of two variables

Mathematical model: $(X, Y) \sim MVN(\mu, \Sigma)$



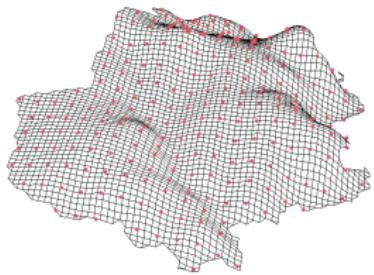
Variance matrix: $\Sigma = \begin{pmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{pmatrix}$

Covariance: $Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$

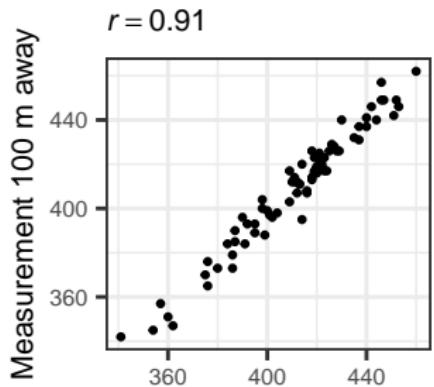
Correlation coefficient: $\rho = \frac{Cov(X, Y)}{\sigma(X)\sigma(Y)}$

Spatial autocorrelation

Point observations in space:

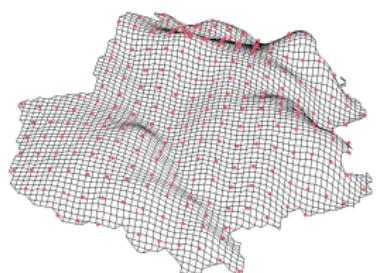


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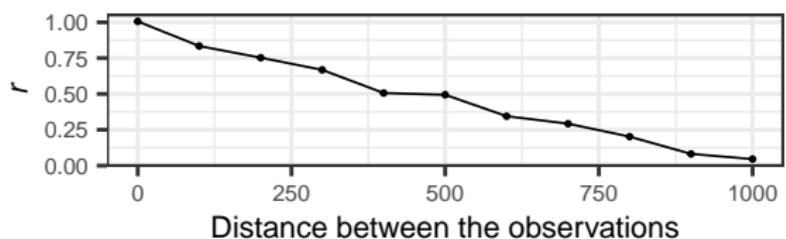


Spatial autocorrelation

Point observations in space:



Correlation as a function of distance (correlogram):



- Assumption: the underlying spatial process is **stationary** and **isotropic**!

Moran's I

$$I = \frac{n}{\sum_{i=1}^n \sum_{j=1}^n w_{ij}} \frac{\sum_{i=1}^n \sum_{j=1}^n w_{ij}(y_i - \bar{y})(y_j - \bar{y})}{\sum_{i=1}^n (y_i - \bar{y})^2}$$

- w_{ij} measures the magnitude of spatial interactions between sites i and j
- originally designed for lattice data!
- Standard metric used for correlograms
- Under the null hypothesis of spatial independence:
 $E(I) = -1/(n - 1)$
- Alternative measure: Geary's C

Local Moran's I

The (global) Moran's I can be decomposed into components associated with the individual sites:

$$I_i = \frac{(y_i - \bar{y}) \sum_{j=1}^n w_{ij}(y_j - \bar{y})}{\frac{\sum_{j=1}^n (y_j - \bar{y})^2}{n}}$$

$$\sum_{i=1}^n I_i = I$$

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Section 3

Gaussian processes, correlation function

Gaussian (spatial random) process

Gaussian process: $\{S(x) : x \in \mathbb{R}^2\}$

For any $x_1, x_2, \dots, x_n \rightarrow \{S(x_1), S(x_2), \dots, S(x_n)\} \sim MVN(\mu, \Sigma)$

- $\mu(x) = E[S(x)] \dots$ mean function
- $C(x, x') = Cov(S(x), S(x')) \dots$ covariance function

Stationary, isotropic process: $\mu(x) = \mu$, $C(x, x') = C(u)$

We can then define:

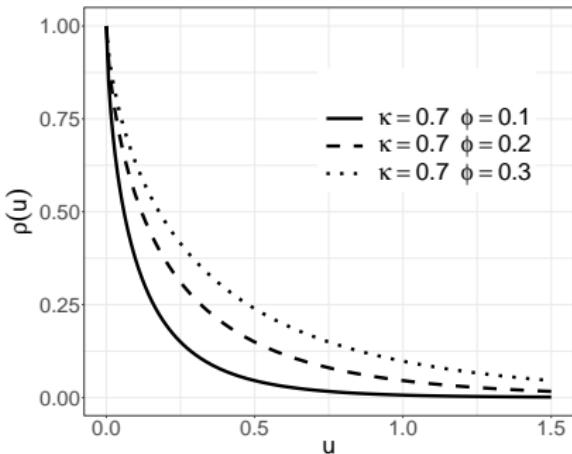
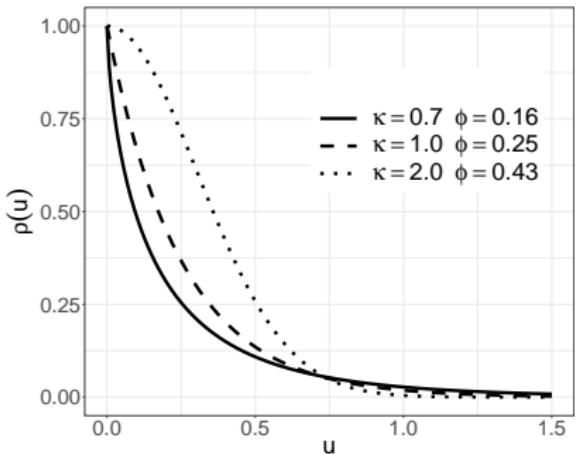
- $\sigma^2 = C(0) \dots$ (constant) variance of the process
- $\rho(u) = C(u)/\sigma^2 \dots$ **correlation function** of the process

Power-Exponential family

$$\rho(u) = \exp \left\{ - \left(\frac{u}{\phi} \right)^\kappa \right\}$$

ϕ ... scale, $\phi > 0$

κ ... shape, $0 < \kappa \leq 2$

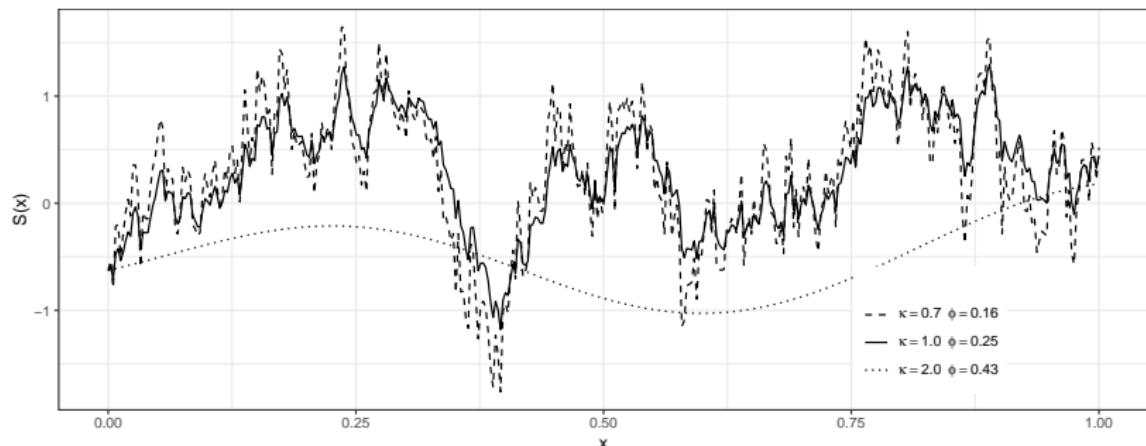


Power-Exponential family

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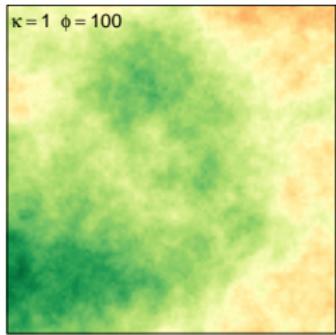
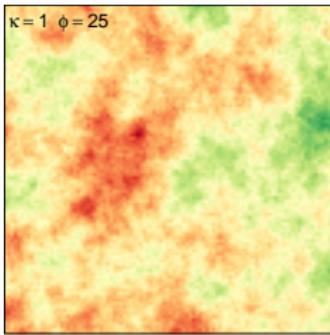
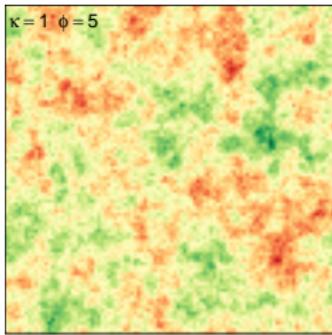
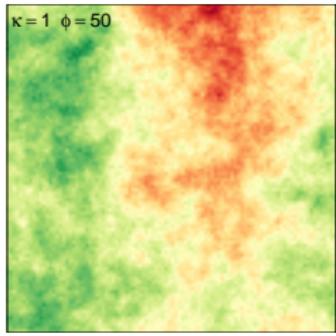
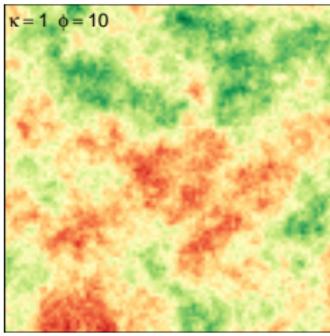
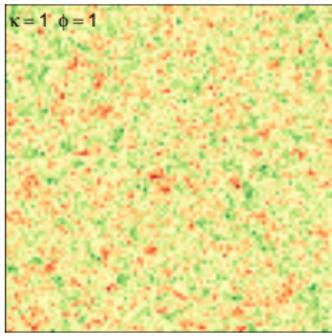
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Gaussian processes, correlation function
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Variogram
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Power-Exponential family



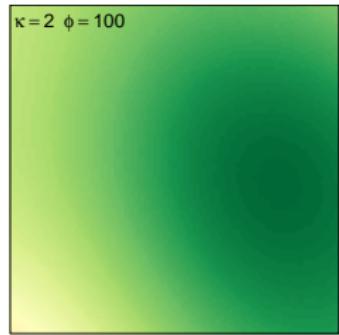
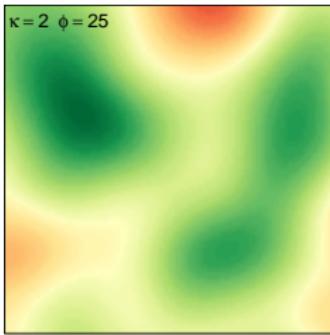
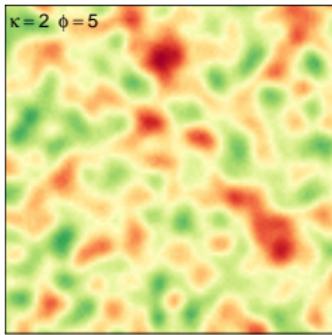
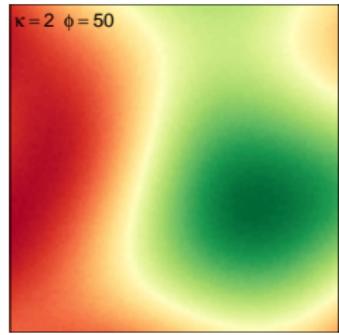
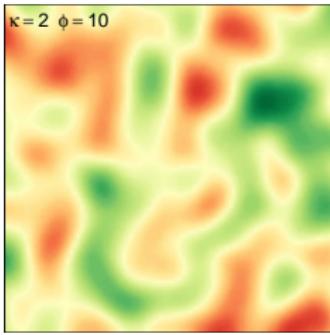
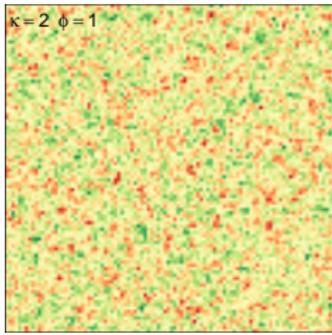
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Power-Exponential family



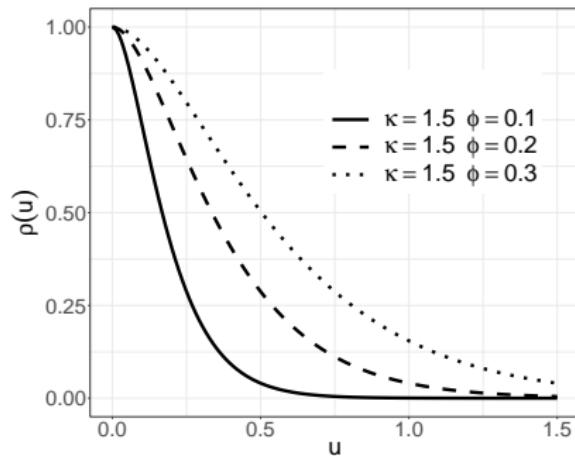
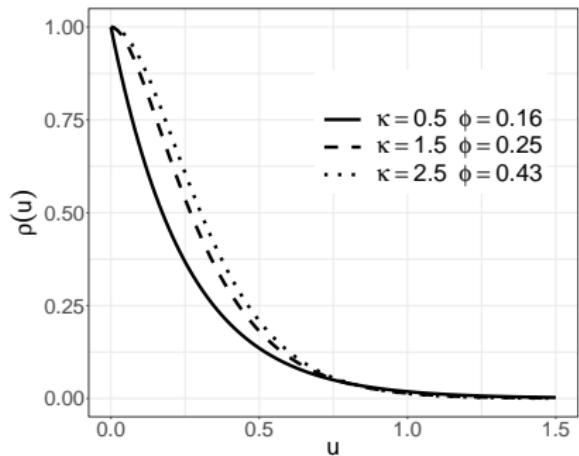
Matérn family

$$\rho(u) = \{2^{\kappa-1}\Gamma(\kappa)\}^{-1} \left(\frac{u}{\phi}\right)^{\kappa} K_{\kappa} \left(\frac{u}{\phi}\right)$$

ϕ ... scale, $\phi > 0$

κ ... order, $\kappa > 0$

K_{κ} ... modifikovaná Besselova funkce druhého typu řádu κ



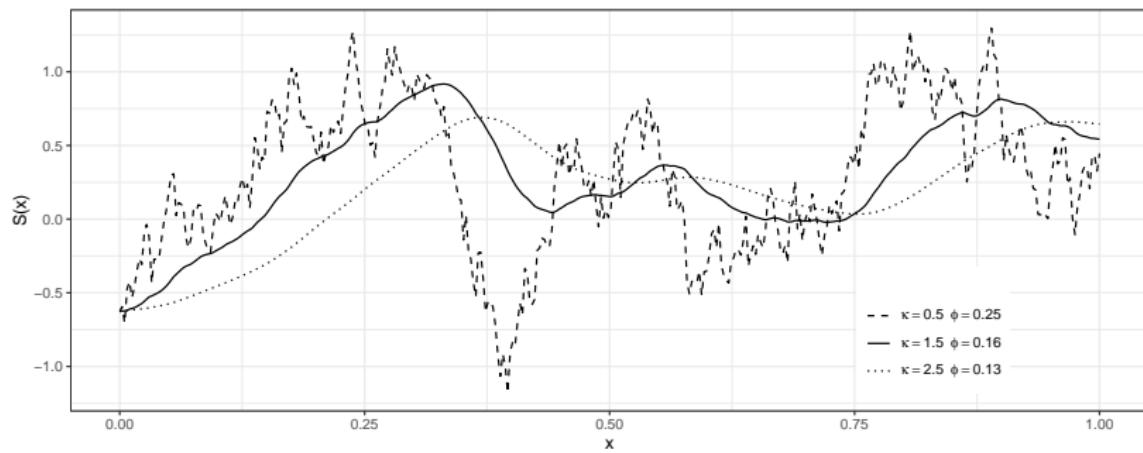
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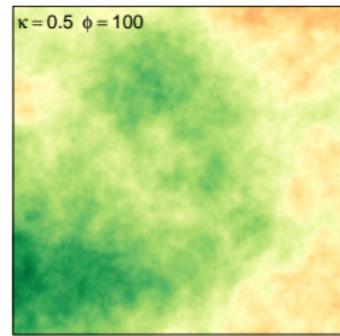
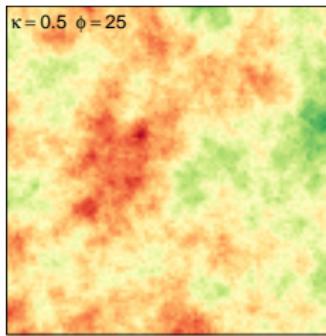
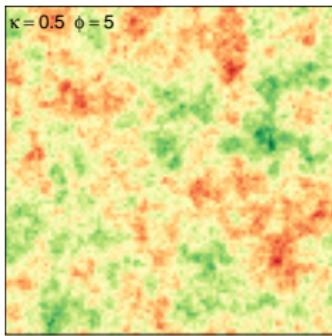
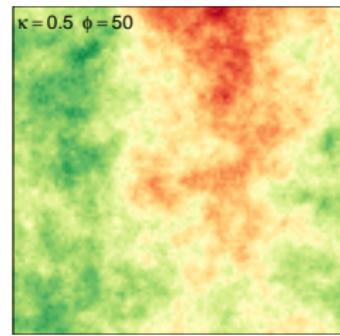
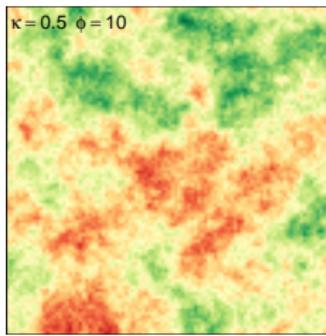
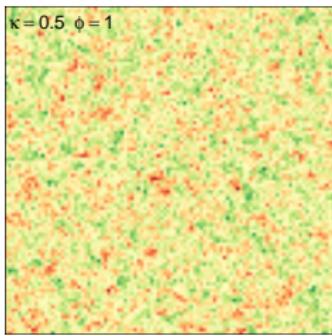
Spatial autocorrelation and why it matters
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Gaussian processes, correlation function
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Variogram
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Matérn family



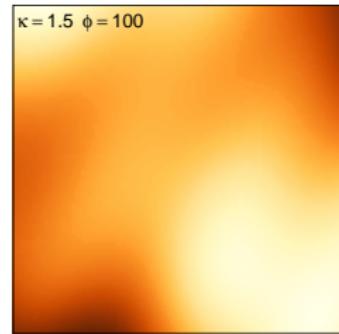
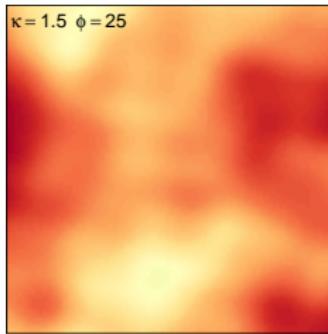
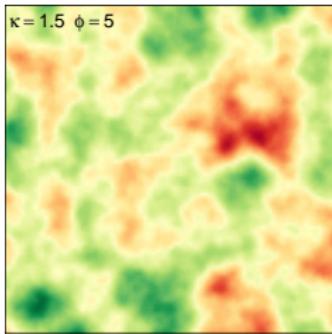
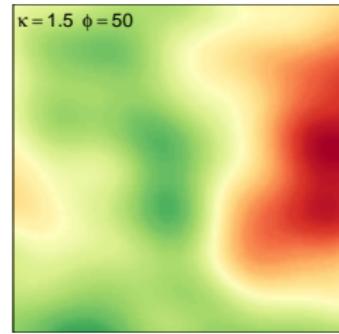
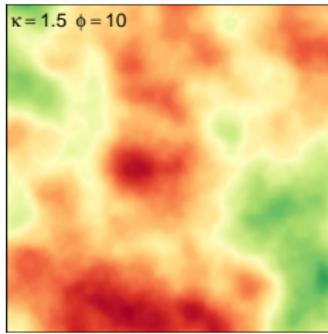
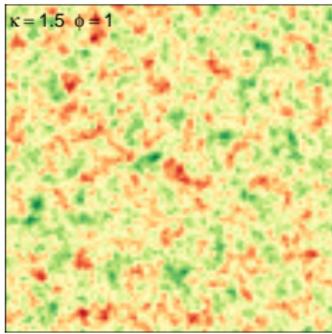
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Matérn family



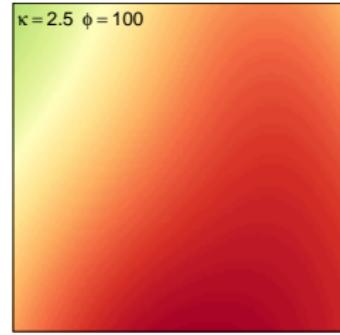
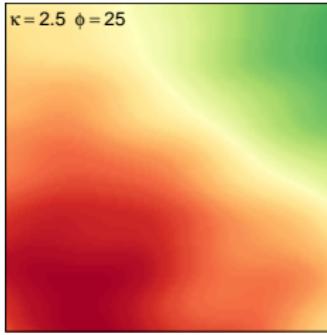
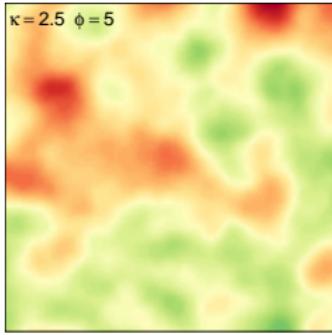
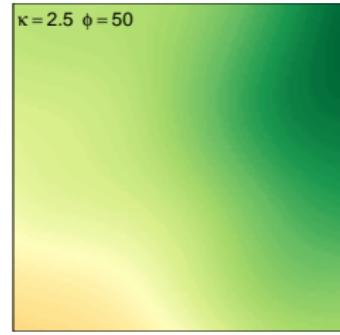
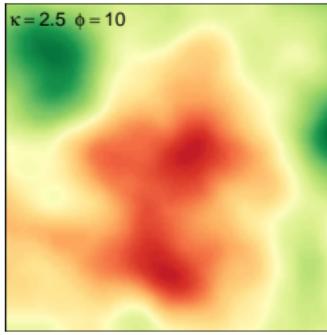
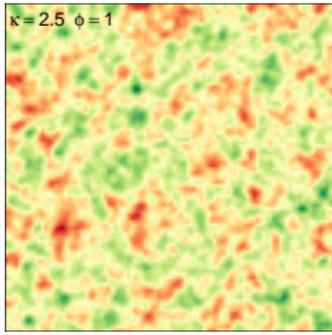
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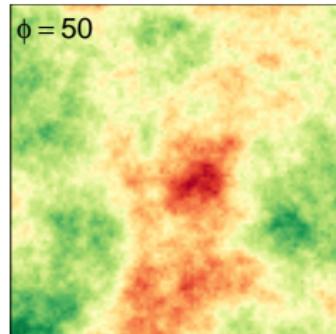
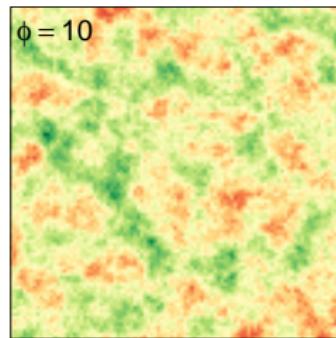
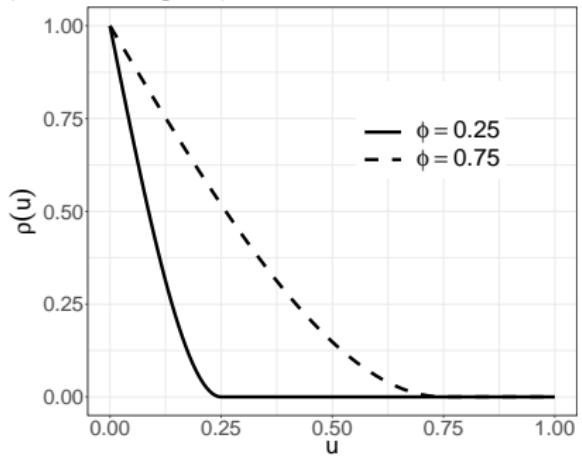
Matérn family



Spherical family

$$\rho(u) = \begin{cases} 1 - \frac{3}{2} \left(\frac{u}{\phi} \right) + \frac{1}{2} \left(\frac{u}{\phi} \right)^3 & 0 \leq u \leq \phi \\ 0 & u > \phi \end{cases}$$

ϕ ... range, $\phi > 0$



Spatial autocorrelation and why it matters
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Section 4

Variogram

Variogram

(Semi)Variogram . . . alternative description of the autocorrelation structure:

$$V(x, x') = \frac{1}{2} \text{Var}(S(x) - S(x'))$$

For stationary isotropic process, it is easy to show that:

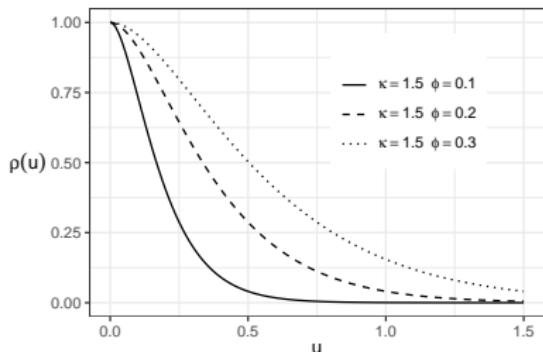
$$V(u) = \sigma^2 - C(u)$$

$$V(u) = \sigma^2 \{1 - \rho(u)\}$$

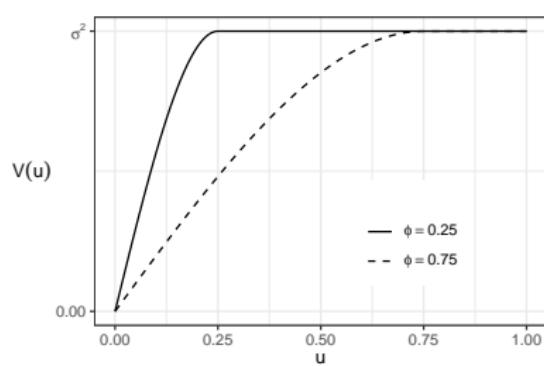
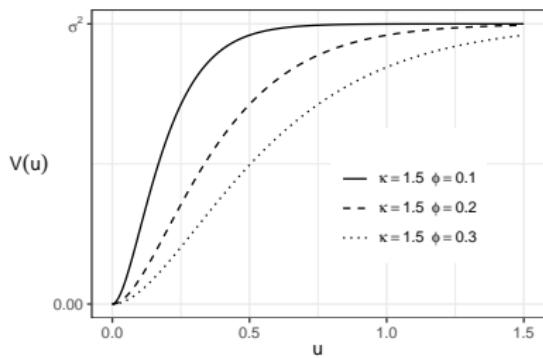
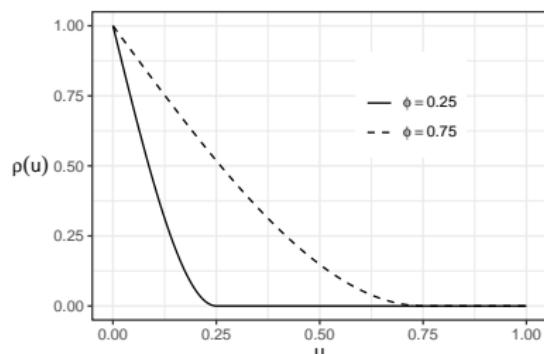
-> Variogram has an *inverse shape* compared to the correlation function.

Variogram

Matérn:



Spherical:



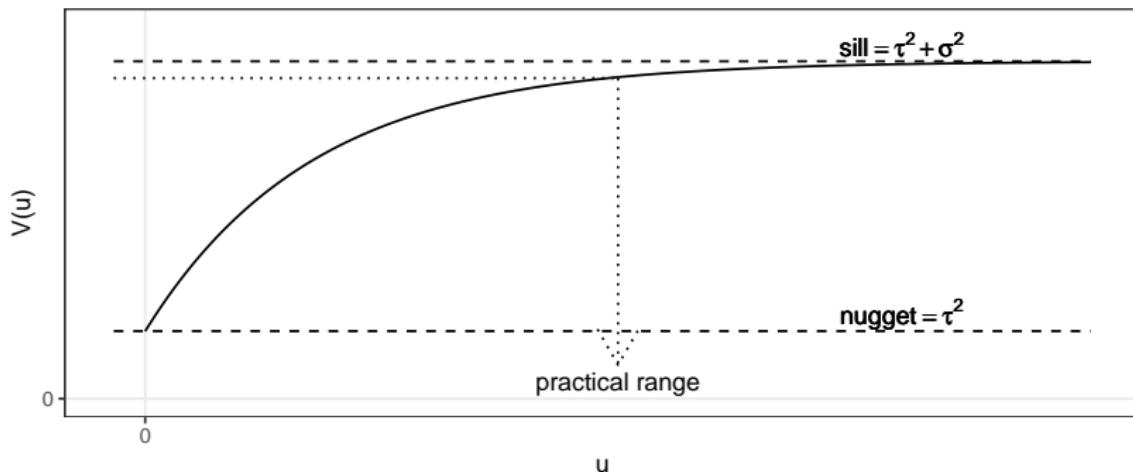
Nugget effect

Realistic assumption: data (x_i, y_i) are generated by a stationary “observational” process

$$Y_i = S(x_i) + \epsilon_i, \text{ where } z_i \sim iid, E(z_i) = 0, Var(z_i) = \tau^2$$

Variogram is then:

$$V_Y = \tau^2 + \sigma^2 \{1 - \rho(u)\}$$



Nugget effect

First approach:

- we decide to *model the nugget effect*
- τ^2 is estimated from data
- two-fold interpretation:
 - ① measurement error
 - ② variability of $S(x)$ in the scale smaller than the scale of the measurement
- cannot be separated, unless we have repeated measurements at the same site
- interpolation does not fit the data points precisely

Second approach:

- we decide to *set the $\tau^2 = 0$ fixly* (no measurement error)
- interpolation exactly fits the data points

Variogram estimation

- ① Compute $v_{ij} = \frac{1}{2} (Y_i - Y_j)^2$
- ② Plot v_{ij} against $u \rightarrow$ **variogram cloud**
- ③ Smooth out the variogram cloud \rightarrow **empirical variogram**

