Models for autocorrelated data

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Section 1

General overview

General model for spatial data

$$\boldsymbol{Y}(\boldsymbol{s}) = \boldsymbol{\mu}(\boldsymbol{s}) + \boldsymbol{e}(\boldsymbol{s})$$

$\boldsymbol{Y}$ ... vector of response observations
$\boldsymbol{s}$ ... vector of spatial coordinates
$\boldsymbol{\mu}$ ... deterministic mean function
$\boldsymbol{e}$ ... random "error" component

The spatial structure observed in $\boldsymbol{Y}$ can be modelled:

- in the **mean component**
  - Spatially structured predictors (induced spatial structure)
  - Autocovariate models
  - Trend surface models
  - Moran's eigenvector mapping
- in the **random component**
  - Geostatistical models
  - Autoregressive models
- both

Modeling spatial structure in the mean component

Linear models:

$$\boldsymbol{\mu}(\boldsymbol{s}) = \boldsymbol{X}(\boldsymbol{s})\beta, \ \boldsymbol{e}(\boldsymbol{s}) \sim MVN(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$$

$\boldsymbol{X}(\boldsymbol{s})$ ... matrix of fixed predictors, including spatially structured ones
β ... vector of unknown parameters (fixed effects)

- estimation by *ordinary least squares* (OLS)

Generalized linear models:

1. Y_i are mutually independent, following a common distributional family (Gaussian, Poisson, Binomial, ...)
2. $\boldsymbol{\mu}(\boldsymbol{s}) = u[\boldsymbol{X}(\boldsymbol{s})\beta]$, u ... link function

- estimation by maximum likelihood (ML)

Modeling spatial structure in the random component

$$\boldsymbol{e} \sim (\mathbf{0}, \boldsymbol{\Sigma}(\boldsymbol{\theta}))$$

- $\boldsymbol{\Sigma}$ is a positive definite matrix with at least some non-zero off-diagonal elements
- $\boldsymbol{\theta}$ is a vector o parameters describing the spatial dependence
- Trying to capture the nature of spatial dependence - the real spatial autocorrelation
- Relies on **stationarity** assumption
- Fixed effects can be estimated by *generalized least squares* (GLS)
- Can be viewed as **mixed models** (estimation by ML/REML)

Section 2

Models with correlated errors

Geostatistical linear model

$$\boldsymbol{\mu}(\boldsymbol{s}) = \boldsymbol{X}(\boldsymbol{s})\beta$$

$$\boldsymbol{e}(\boldsymbol{s}) = \boldsymbol{S}(\boldsymbol{s}) + \epsilon(\boldsymbol{s})$$

$\boldsymbol{S}(\boldsymbol{s}) \sim MVN(\mathbf{0}, \boldsymbol{C})$ ... Gaussian process
$\boldsymbol{C} = (\sigma_{i,j})$, $\sigma_{i,j} = Cov(S_i, S_j) = C(u)$ ... Covariance function
$\epsilon(\boldsymbol{s}) \sim MVN(\mathbf{0}, \tau^2\boldsymbol{I})$ ... Nugget effect
$\boldsymbol{\Sigma} = \boldsymbol{C} + \tau^2\boldsymbol{I}$

Covariance function estimated from data:

- by fitting the curve to the sample variogram (*classical geostatistics*)
- by ML/REML techniques together with other parameters (*model-based approach*)

Geostatistical linear model

Spatial prediction/interpolation: **kriging**

$$\hat{Y}(\boldsymbol{s}_0) = \boldsymbol{x}'\hat{\beta}_{GLS} + \boldsymbol{c}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \boldsymbol{X}\hat{\beta}_{GLS})$$

$\boldsymbol{c} = (C(\boldsymbol{s}_0, \boldsymbol{s}_1), C(\boldsymbol{s}_0, \boldsymbol{s}_2), ..., C(\boldsymbol{s}_0, \boldsymbol{s}_n))'$
$\boldsymbol{c}'\boldsymbol{\Sigma}^{-1}$ ... *kriging weights*
$var(\hat{Y}(\boldsymbol{s}_0))$ ... *kriging variance*

- Simple kriging ... known constant mean
- Ordinary kriging ... unknown constant mean
- Universal kriging ... unknown mean depending on covariates

Geostatistical GLM

Generalized linear model (GLM):

1. Y_i ... mutually independent Gaussian/Poisson/Binomial variables
2. $\boldsymbol{\mu}(\boldsymbol{s}) = u[\boldsymbol{X}(\boldsymbol{s})\beta]$, u ... link function

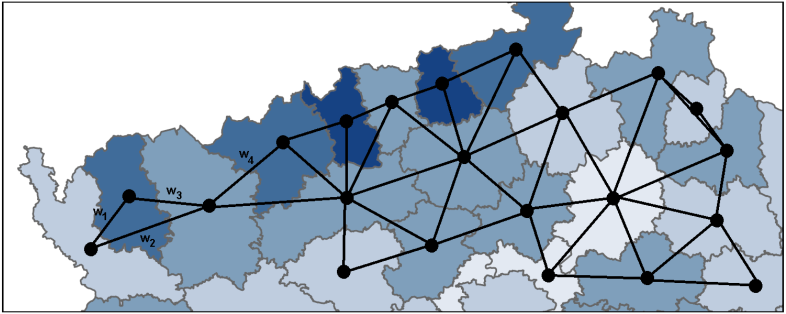
Generalized linear geostatistical model (GLGM):

1. Y_i ... mutually independent Gaussian/Poisson/Binomial variables
2. $\boldsymbol{\mu}(\boldsymbol{s}) = u[\boldsymbol{X}(\boldsymbol{s})\beta + \boldsymbol{S}(\boldsymbol{s})]$, u ... link function
3. $\boldsymbol{S}(\boldsymbol{s})$ ... Gaussian process with zero mean and some covariance function

- Special case of GLMM

Autoregressive models

- Based on discrete locations with a **neighborhood structure**
- Magnitude of spatial interactions between neighbors -> **spatial weights**



- Not necessarily areal data...

Simultanuous autoregressive model

$$\boldsymbol{Y}(\boldsymbol{s}) = \boldsymbol{X}(\boldsymbol{s})\beta + \boldsymbol{e}(\boldsymbol{s})$$
$$\boldsymbol{e}(\boldsymbol{s}) = \boldsymbol{B}\boldsymbol{e}(\boldsymbol{s}) + \epsilon(\boldsymbol{s})$$

$\boldsymbol{B}$ ... matrix of spatial dependence parameters, $b_{i,i} = 0$
$\epsilon(\boldsymbol{s}) \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$
$\boldsymbol{\Sigma}_{SAR} = (\boldsymbol{I} - \boldsymbol{B})^{-1}\sigma^2 \boldsymbol{I}(\boldsymbol{I} - \boldsymbol{B}')^{-1}$

Usually:

$$\boldsymbol{B} = \rho \boldsymbol{W}$$

ρ ... single correlation parameter
$\boldsymbol{W}$ ... matrix of spatial weights

Conditional autoregressive model

Assumption: Spatial process is *Markov random field*

$$E[Y(\boldsymbol{s}_i)|\boldsymbol{Y}(\boldsymbol{s})_{-i}] = \boldsymbol{x}(\boldsymbol{s}_i)'\beta + \sum_{j=1}^{n} b_{i,j}[Y(\boldsymbol{s}_j) - \boldsymbol{x}(\boldsymbol{s}_j)'\beta]$$

$$Var[Y(\boldsymbol{s}_i)|\boldsymbol{Y}(\boldsymbol{s})_{-i}] = \sigma^2$$

- $\boldsymbol{\Sigma}_{CAR} = (\boldsymbol{I} - \boldsymbol{B})^{-1}\sigma^2$
- Again, usually $\boldsymbol{B} = \rho\boldsymbol{W}$
- In these constant-variance versions, every SAR model can be expressed as a CAR model, and vice versa

Section 3

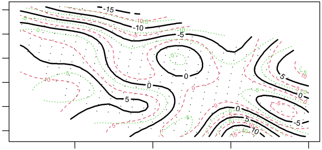
Models with uncorrelated errors

Autocovariate models

- Each observation is modeled as depending on a summary of neighboring observations
- This *autocovariate* is prepared first a added as a fixed predictor
- A "naive" approach to autoregression
- Typically a weighted average of the neighboring values
- Can be used in any type of model (G)L(M)M, GAM, Random Forests, . . .)
- Assumes stationarity

Trend surface models

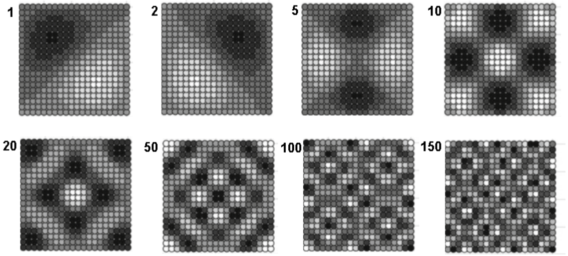
- Originally: a simple (linear, quadratic) spatial trend is added as fixed predictor
- Combined with geostatistical models
- Extended to complex, nonlinear smooth surfaces, describing the unexplained spatial structure
- Typically a smooth term `s(x, y)` in a **generalized additive model** (GAM)



- Description, but not explanation!
- **Does not assume stationarity!**

Moran's eigenvectors mapping

- PCA applied to the distance or weight matrix
- The resulting variables (called Moran's eigenvectors) used as fixed predictors
- Only those associated with positive eigenvalues (positive spatial dependence)



- Similar to trend surface models and Fourier transform
- Significant eigenvectors represent spatial dependence at different scales