Obsah

Matrix Operations

Systems of Linear Equations

Vector Spaces

Linear Transformations

Bilinear and Quadratic Forms

Quadratic form

Definition: $\mathcal{Q}_B$ is a quadratic form, if it takes a vector and outputs a real number. $\mathcal{Q}_B$: $\mathcal{V}\rightarrow \mathbb{R}$.

$$\mathcal{Q}_B(v)= \mathcal{B}(v,v) = \mathcal{B}_S(v,v)$$

Example: $\mathcal{Q}:\mathbb{R}^3\rightarrow \mathbb{R};\mathcal{Q}(x) = x^T x = x_1^2 + x_2^2 + x_3^2$, always positive or zero

Example: $\mathcal{Q}:\mathbb{R}^2\rightarrow \mathbb{R};\mathcal{Q}(x) = x^T Ax = a_{11}x_1^2 + (a_{12}+a_{21})\, x_1x_2 + a_{22}x_2^2$, contains zero

Example: $\mathcal{Q}: \mathcal{F} \rightarrow\mathbb{R};\mathcal{Q}(f) = f^2(1)$

$\mathcal{Q}_B(\alpha x) = \alpha^2 \mathcal{Q}(x)$, (prove by $B(\alpha x, \alpha x)$)

Property: $1 \in R\,(\mathcal{Q}) \Rightarrow \mathcal{Q}(x) = 1; \mathcal{Q}(\alpha x) = \alpha^2 \mathcal{Q}(x) = \alpha^2$: if 1 is in the range, all positive numbers and zero is in the range; similarly if -1 is in the range, all negative numbers and zero is in the range.

Property: $\mathcal{Q}(0)= \mathcal{Q}(0 \bullet 0) = 0 \bullet \mathcal{Q}(0) = 0$

A bilinear form generating a quadratic form is not unique.
By adding any antisymmetric bilinear form, I'd get a new one.

Example: Find a symmetric bilinear form $\mathcal{B}_S$ for quadratic form $\mathcal{Q}_B$.

$$\begin{aligned}\mathcal{B}_S(x+y,x+y)&= \mathcal{B}(x,x)+\mathcal{B}(x,y)+\mathcal{B}(y,x)+\mathcal{B}(y,y)\\ \mathcal{Q}(x)&= \mathcal{Q}(x) + 2\mathcal{B}_S(x,y)+\mathcal{Q}(y)\\ \mathcal{B}_S(x,y) &= \tfrac{1}{2}\left(\mathcal{Q}(x+y)-\mathcal{Q}(x)-\mathcal{Q}(y)\right)\end{aligned}$$

A symmetric bilinear form generating a quadratic is unique and given by $\mathcal{B}_S(x,y,)=\frac{1}{2}\left(\mathcal{Q}(x+y)-\mathcal{Q}(x)-\mathcal{Q}(y)\right)$.

Example: $\mathcal{B}(x,y)=2x_1y_1-3x_1y_2+5x_2y_1-x_2y_2; \varepsilon=\{[1,0],[0,1]\}$

$$B=\begin{bmatrix}2&-3\\5&-1\end{bmatrix}; B_S=\frac{1}{2}\left(B+B^T\right)=\begin{bmatrix}2&1\\1&-1\end{bmatrix}; Q=B_S=\begin{bmatrix}2&1\\1&-1\end{bmatrix}$$

A matrix of quadratic form is equal to a matrix of corresponding bilinear form.

Classification

Definition: A positive definite form: $\mathcal{Q}:\mathcal{V}\to\mathbb{R}:\forall x\in\mathcal{V}, x\neq o:\mathcal{Q}(x)>0$.

Definition: A negative definite form: $\mathcal{Q}:\mathcal{V}\to\mathbb{R}:\forall x\in\mathcal{V}, x\neq o:\mathcal{Q}(x)<0$.

Definition: A positive semi definite form: $\mathcal{Q}:\mathcal{V}\to\mathbb{R}:\forall x\in\mathcal{V}, x\neq o:\mathcal{Q}(x)\geq 0$.

Definition: A negative semi definite form: $\mathcal{Q}:\mathcal{V}\to\mathbb{R}:\forall x\in\mathcal{V}, x\neq o:\mathcal{Q}(x)\leq 0$.

Definition: A indefinite form: $\mathcal{Q}:\mathcal{V}\to\mathbb{R}:\exists x\in\mathcal{V}, x\neq o:\mathcal{Q}(x)<0\wedge\exists y\in\mathcal{V}, y\neq o:\mathcal{Q}(x)>0$.

Example: Classify $\mathcal{Q}(x)=-2x_1^2+2x_1x_2-3x_2^2$.

$\mathcal{Q}(x)=-2x_1^2+2x_1x_2-3x_2^2=-2\left(x_1^2-x_1x_2+\frac{1}{4}x^2\right)-\frac{5}{2}x_2^2=-2\left(x_1-\frac{1}{2}x_2\right)^2-\frac{5}{2}x_2^2 \xrightarrow{\begin{array}{l}s_1=x_1-\frac{1}{2}x_2\\ s_2=x_2\end{array}} -2s_1^2-\frac{5}{2}s_2^2$, aka it's negative definite form.

$[Q]_N=\begin{bmatrix}-2&1\\1&-3\end{bmatrix}; s=\left(S=\begin{bmatrix}1&-\frac{1}{2}\\0&1\end{bmatrix}\right)x; [Q]_S=\begin{bmatrix}-2&0\\0&-\frac{5}{2}\end{bmatrix}$; We diagnolised our matrix.

Congruent operations

Perform a row operation and at the same time the corresponding column operation.

$[Q]_N=\begin{bmatrix}-2&1\\1&-3\end{bmatrix}\sim\begin{bmatrix}-2&1\\0&-\frac{5}{2}\end{bmatrix}\sim\begin{bmatrix}-2&0\\0&-\frac{5}{2}\end{bmatrix}$

For all symmetric matrices $[A|I]\sim[D|\widetilde{L}]$ is called LD decomposition.

Example: Classify $\mathcal{Q}(x):Q=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$

$\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\sim\begin{bmatrix}2&0&0\\0&6&-2\\0&-2&2\end{bmatrix}\sim\begin{bmatrix}2&0&0\\0&6&0\\0&0&\frac{4}{3}\end{bmatrix}$, it is a positive definite form.

Inner Product

Definition: Inner product on the real vector space $\mathcal{V}$ is a symmetric bilinear form on $\mathcal{V}$, which corresponding quadratic form is a positive definite one.

$$\begin{aligned}(u+v,w)&=(u,w)+(v+w)\\(\alpha u,v)&=\alpha(u,v)\\(u,v)&=(v,u)\\(u,u)&>0; u\neq 0\end{aligned}$$

Example: Examine, whether $(f,g) = f(0)g(0) + f(1)g(1)$ is an inner product. (HOWTO: Examine all 4 conditions.)

Example: Examine, whether $(x,y)_A = x^T Ay$ is an inner product. (It is.)

Norms

Definition: The norm of the vector on the real vector space $\mathcal{V}$ is a positive definite one quadratic form.

$\|x+y\| \leq \|x\| + \|y\|$ (triangular inequality)

$$\|\alpha x\| = \alpha \|x\| \|x\| = 0 \Leftrightarrow x = 0$$

Example: $\|x\|_2 = \sqrt{x^2+y^2}$ (Euclidian norm); $\|x\|_0 = \sum_i^n |x_i|$ (zero norm); $\|x\|_\infty = \max_i |x_i|$ (maximum norm)

Example: Energy norms: $(x,y)_A = x^T Ay$; $\|x\|_A = \sqrt{(x,y)_A}$

Cauchy–Bunyakovsky–Schwarz inequality: $(x,y)^2 \leq (x,x) \bullet (y,y)$

Orthonormality

Definition: An orthonormal means that the norm of all vectors is equal to one and is orthogonal.
Definition: An orthogonal means, that the inner product of any two vector is equal to zero.

Gram-Schmidt Algorithm: Make a set of vectors to orthonormal set.

1. Take the first vector from the set.
2. Orthogonalize the vector. We know $f'_n = e_n - \alpha_{n-1} f_{n-1} - \ldots$, where $\alpha_y = (f_y, e_n), \ldots$.
3. Orthonormalize the vector – divide it by its norm.

Example: Find an orthonormal basis: $(x,y)_A = x^T Ay;\ A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$; $\|x\|_A = \sqrt{x^T Ax}$; $\mathcal{V} = \mathbb{R}^3$

$$\varepsilon = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}; f = \left\{ \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\| = \sqrt{2}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{bmatrix}, \frac{\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\| = \frac{\sqrt{6}}{2}} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \bullet \frac{\sqrt{6}}{2}, \frac{\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \right\| = \frac{2\sqrt{3}}{3}} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \bullet \frac{\sqrt{3}}{2} \right\}$$

Example: Find an orthonormal basis of $\mathcal{P}^2$: $(p,q) = \int_0^1 p(x)q(x)dx$; $\|p\|_A = \sqrt{\int_0^1 p(x)^2 dx}$

$$\varepsilon = \{1; x; x^2\}; f = \left\{ \frac{x^2}{\|x^2\| = \frac{\sqrt{5}}{5}} = \sqrt{5}x^2, \frac{x - \frac{5}{4}x^2}{\|x - \frac{5}{4}x^2\| = \frac{\sqrt{12}}{12}} = -5\sqrt{3}x^2 + 4\sqrt{3}x, \ldots \right\}$$

Variation Principle

Theorem: The matrix A is symmetric and positive definite, $(x,y)_A = x^T Ay$, $b(x) = b^t x$, $q(x) = \frac{1}{2}(x,x)_A - b(x)$, vectors b and $\overline{x}$.

$$A\overline{x} = b \Leftrightarrow (x,y)_A = b(y), \forall y \in \mathbb{R}^n \Leftrightarrow ! \exists \overline{x} : q(\overline{x}) \leq q(x)$$

Notes: $q(x)$ is an error function. The solution $\overline{x}$ has the smallest error possible.

Example: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, f = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \bullet \frac{\sqrt{6}}{2}, \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \bullet \frac{\sqrt{3}}{2} \right\}$. Solve $Ax = b$.

Let's use the middle equation $(x, y)_A = b(y), y \in f$. We can then count only $\begin{matrix} \alpha_1 = b(f_1) \\ \cdots \\ \alpha_n = b(f_n) \end{matrix}$.

Least Square Method

$r(x) = Ax - b$ is called the <u>residual vector</u>. It changes the problem to minimalize the residuum.

Theorem: The minimal residuum is achieved at $A^T Ax = A^T b$

Example: $\begin{matrix} x_1 + x_2 = 0 \\ 2x_1 + x_2 = 4 \\ 3x_1 + x_2 = 4 \end{matrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}; A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}; b = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}; A^T A = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}; A^T b = \begin{bmatrix} 20 \\ 8 \end{bmatrix}; x = \begin{bmatrix} 2 \\ -\frac{4}{3} \end{bmatrix}$$

Eigenvalues and Eigenvectors

Spectral Decomposition

Spectral decomposition, a.k.a. diagonalization, is concerned with a linear transformation $A_{F,F} = T \bullet A_{E,E} \bullet T^{-1}$, where the matrix $A_{E,E} := \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$. Assume, that for some A it holds true, $A \bullet e_i = \lambda_i \bullet e_i$ and $e_1, \ldots, e_n$ are linearly independent. That implies $A \bullet (T := [e_1, \ldots, e_n]) = [\lambda_1 \bullet e_1, \ldots, \lambda_n \bullet e_n]$, and so $A \bullet T = T \bullet \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$.

Conclusion 1: For any matrix $A \in C^{n \times n}$: $A = T \bullet D \bullet T^{-1}$.

Assume that A is real and symmetric. We know $A \bullet e = \lambda \bullet e$. We can multiple by $e^* \bullet$ to get $(e^*)^T \bullet A \bullet e = \lambda \bullet \|e\|^2$. Now we know $e^T \bullet A^T \bullet e^* = e^T \bullet A \bullet e^* = e^T \bullet (A \bullet e)^* = e^T \bullet \lambda^* \bullet e^* = \lambda^* \bullet \|e\|^2$, so $\lambda \in \mathbb{R}$.

Conclusion 2: For any real symmetric matrix $A \in C^{n \times n}$: $\forall \lambda_i : \lambda_i \in \mathbb{R}$

Assume that A is real and symmetric. We know $A \bullet e = \lambda \bullet e$. We can multiply by $e_j \bullet$, where $\lambda_i \neq \lambda_j$. Now $e_j^T \bullet A \bullet e_i = \lambda_i \bullet e_j^T \bullet e_i$, so $e_j^T \bullet A \bullet e_i = e_i^T \bullet A^T \bullet e_j = e_i^T \bullet (A \bullet e_j) = \lambda_j \bullet e_i^T \bullet e_j$. Now because $(\lambda_i - \lambda_j) \bullet (e_i \bullet e_j) = 0$.

Conclusion 3: For any real symmetric matrix $A \in C^{n \times n}$: $\forall (e_i, e_j) : e_i \perp e_j, \lambda_i \neq \lambda_j$

$A = T \bullet (D \in R^{n \times n}) \bullet T^{-1} = Q \bullet D \bullet Q^T$, where Q is orthogonal matrix

Example: Compute the spectral decomposition of $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$.

$$|A-\lambda I|=\begin{vmatrix}3-\lambda & 2\\ 2 & 1-\lambda\end{vmatrix}=\lambda^2-4\lambda-1;\lambda_1=2+\sqrt{5},\lambda_2=2-\sqrt{5}$$

$$\lambda_1=2+\sqrt{5}:e_1'=t\bullet\begin{bmatrix}-2\\ 1-\sqrt{5}\end{bmatrix},\ t^2\bullet\alpha=1\Rightarrow e_1=\begin{bmatrix}-\frac{2}{\sqrt{\alpha}}\\ \frac{1-\sqrt{5}}{\sqrt{\alpha}}\end{bmatrix}$$

$$\lambda_2:e_2=s\bullet\begin{bmatrix}-2\\ 1+\sqrt{5}\end{bmatrix},s=\frac{1}{\sqrt{\beta}},\beta=4+\left(1+\sqrt{5}\right)^2$$

$$Q=[e_1,e_2];D=\begin{bmatrix}2+\sqrt{5} & 0\\ 0 & 2-\sqrt{5}\end{bmatrix};A=Q\bullet D\bullet Q^T$$