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Datum aktualizace originálu: **26. 12. 2023 v 9:16:21**

Datum vygenerování PDF: **2. 7. 2025 v 21:51:14**

Obsah

Matrix Operations

Systems of Linear Equations

Vector Spaces

Linear Transformations

Bilinear and Quadratic Forms

Quadratic form

Definition: $\mathcal{Q}_B$ is a quadratic form, if it takes a vector and outputs a real number. $\mathcal{Q}_B$: $\mathcal{V}\rightarrow \mathbb{R}$.

$$\mathcal{Q}_B(v)= \mathcal{B}(v,v) = \mathcal{B}_S(v,v)$$

Example: $\mathcal{Q} :\mathbb{R}^3\rightarrow \mathbb{R};\mathcal{Q}(x) = x^Tx = x_1^2 + x_2^2 + x_3^2$, always positive or zero

Example: $\mathcal{Q} :\mathbb{R}^2\rightarrow \mathbb{R};\mathcal{Q}(x) = x^TAx = a_{11}x_1^2 + (a_{12} + a_{21})\, x_1x_2 + a_{22}x_2^2$, contains zero

Example: $\mathcal{Q} : \mathcal{F} \rightarrow\mathbb{R};\mathcal{Q}(f) = f^2(1)$

$\mathcal{Q}_B(\alpha x) = \alpha^2\mathcal{Q}(x)$, (prove by $B(\alpha x, \alpha x)$)

Property: $1 \in R\,(\mathcal{Q}) \Rightarrow \mathcal{Q}(x) = 1;\, \mathcal{Q}(\alpha x) = \alpha^2\mathcal{Q}(x) = \alpha^2$: if 1 is in the range, all positive numbers and zero is in the range; similarly if -1 is in the range, all negative numbers and zero is in the range.

Property: $\mathcal{Q}(0)= \mathcal{Q}(0 \bullet 0) = 0 \bullet \mathcal{Q}(0) = 0$

A bilinear form generating a quadratic form is not unique.
By adding any antisymmetric bilinear form, I'd get a new one.

Example: Find a symmetric bilinear form $\mathcal{B}_S$ for quadratic form $\mathcal{Q}_B$.

$$\begin{aligned}\mathcal{B}_S(x+y,x+y)&= \mathcal{B}(x,x)+\mathcal{B}(x,y)+\mathcal{B}(y,x)+\mathcal{B}(y,y)\\ \mathcal{Q}(x)&= \mathcal{Q}(x) + 2\mathcal{B}_S(x,y)+\mathcal{Q}(y)\\ \mathcal{B}_S(x,y) &= \tfrac{1}{2}\,(\mathcal{Q}(x+y)-\mathcal{Q}(x)-\mathcal{Q}(y))\end{aligned}$$

A symmetric bilinear form generating a quadratic is unique
and given by $\mathcal{B}_S(x,y,) = \frac{1}{2}\,(\mathcal{Q}(x+y)-\mathcal{Q}(x)-\mathcal{Q}(y))$.

Example: $\mathcal{B}(x,y) = 2x_1y_1 - 3x_1y_2 + 5x_2y_1 - x_2y_2; \varepsilon = \{[1,0],[0,1]\}$

$$B = \begin{bmatrix} 2 & -3 \\ 5 & -1 \end{bmatrix}; B_S = \frac{1}{2}\left(B + B^T\right) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}; Q = B_S = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

A matrix of quadratic form is equal to a matrix of corresponding bilinear form.

Classification

Definition: A positive definite form: $\mathcal{Q} : \mathcal{V} \to \mathbb{R} : \forall x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) > 0$.

Definition: A negative definite form: $\mathcal{Q} : \mathcal{V} \to \mathbb{R} : \forall x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) < 0$.

Definition: A positive semi definite form: $\mathcal{Q} : \mathcal{V} \to \mathbb{R} : \forall x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) \geq 0$.

Definition: A negative semi definite form: $\mathcal{Q} : \mathcal{V} \to \mathbb{R} : \forall x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) \leq 0$.

Definition: A indefinite form: $\mathcal{Q} : \mathcal{V} \to \mathbb{R} : \exists x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) < 0 \land \exists y \in \mathcal{V}, y \neq o : \mathcal{Q}(x) > 0$.

Example: Classify $\mathcal{Q}(x) = -2x_1^2 + 2x_1x_2 - 3x_2^2$.

$\mathcal{Q}(x) = -2x_1^2 + 2x_1x_2 - 3x_2^2 = -2\left(x_1^2 - x_1x_2 + \frac{1}{4}x^2\right) - \frac{5}{2}x_2^2 = -2\left(x_1 - \frac{1}{2}x_2\right)^2 - \frac{5}{2}x_2^2 \xrightarrow{\substack{s_1 = x_1 - \frac{1}{2}x_2 \\ s_2 = x_2}} -2s_1^2 - \frac{5}{2}s_2^2$, aka it's negative definite form.

$[Q]_N = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}; s = \left(S = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}\right) x; [Q]_S = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix}$; We diagnolised our matrix.

Congruent operations

Perform a row operation and at the same time the corresponding column operation.

$$[Q]_N = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & -\frac{5}{2} \end{bmatrix} \sim \begin{bmatrix} -2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix}$$

For all symmetric matrices $[A|I] \sim [D|\widetilde{L}]$ is called LD decomposition.

Example: Classify $\mathcal{Q}(x) : Q = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$, it is a positive definite form.

Inner Product

Definition: Inner product on the real vector space $\mathcal{V}$ is a symmetric bilinear form on $\mathcal{V}$, which corresponding quadratic form is a positive definite one.

$$\begin{aligned} (u+v, w) &= (u, w) + (v + w) \\ (\alpha u, v) &= \alpha(u, v) \\ (u, v) &= (v, u) \\ (u, u) &> 0; u \neq 0 \end{aligned}$$

Example: Examine, whether $(f, g) = f(0)g(0) + f(1)g(1)$ is an inner product. (HOWTO: Examine all 4 conditions.)

Example: Examine, whether $(x, y)_A = x^T Ay$ is an inner product. (It is.)

Norms

Definition: The norm of the vector on the real vector space $\mathcal{V}$ is a positive definite one quadratic form.

$\|x+y\| \leq \|x\|+\|y\|$ (triangular inequality)

$$\|\alpha x\| = \alpha\,\|x\|\|x\| = 0 \Leftrightarrow x = 0$$

Example: $\|x\|_2 = \sqrt{x^2+y^2}$ (Euclidian norm); $\|x\|_0 = \sum_i^n |x_i|$ (zero norm); $\|x\|_\infty = \max_i |x_i|$ (maximum norm)

Example: Energy norms: $(x,y)_A = x^TAy$; $\|x\|_A = \sqrt{(x,y)_A}$

Cauchy–Bunyakovsky–Schwarz inequality: $(x,y)^2 \leq (x,x)\bullet(y,y)$

Orthonormality

Definition: An orthonormal means that the norm of all vectors is equal to one and is orthogonal.
Definition: An orthogonal means, that the inner product of any two vector is equal to zero.

Gram-Schmidt Algorithm: Make a set of vectors to orthonormal set.

1. Take the first vector from the set.
2. Orthogonalize the vector. We know $f'_n = e_n - \alpha_{n-1}f_{n-1} - \ldots$, where $\alpha_y = (f_y, e_n), \ldots$.
3. Orthonormalize the vector – divide it by its norm.

Example: Find an orthonormal basis: $(x,y)_A = x^TAy;\ A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$; $\|x\|_A = \sqrt{x^TAx}; \mathcal{V} = \mathbb{R}^3$

$$\varepsilon = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}; f = \left\{\frac{\begin{bmatrix}1\\0\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\0\end{bmatrix}\right\| = \sqrt{2}} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\0\\0\end{bmatrix}, \frac{\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}}{\left\|\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}\right\| = \frac{\sqrt{6}}{2}} = \begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}\bullet\frac{\sqrt{6}}{2}, \frac{\begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\1\end{bmatrix}}{\left\|\begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\1\end{bmatrix}\right\| = \frac{2\sqrt{3}}{3}} = \begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\1\end{bmatrix}\bullet\frac{\sqrt{3}}{2}\right\}$$

Example: Find an orthonormal basis of $\mathcal{P}^2$: $(p,q) = \int_0^1 p(x)q(x)dx; \|p\|_A = \sqrt{\int_0^1 p(x)^2dx}$

$$\varepsilon = \left\{1;x;x^2\right\}; f = \left\{\frac{x^2}{\|x^2\| = \frac{\sqrt{5}}{5}} = \sqrt{5}x^2, \frac{x-\frac{5}{4}x^2}{\left\|x-\frac{5}{4}x^2\right\| = \frac{\sqrt{12}}{12}} = -5\sqrt{3}x^2+4\sqrt{3}x, \ldots\right\}$$

Variation Principle

Theorem: The matrix A is symmetric and positive definite, $(x,y)_A = x^TAy$, $b(x) = b^tx$, $q(x) = \frac{1}{2}(x,x)_A - b(x)$, vectors b and $\overline{x}$.

$$A\overline{x} = b \Leftrightarrow (x,y)_A = b(y), \forall y \in \mathbb{R}^n \Leftrightarrow !\exists\overline{x} : q(\overline{x}) \leq q(x)$$

Notes: $q(x)$ is an error function. The solution $\overline{x}$ has the smallest error possible.

Example: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, b = \begin{bmatrix}1\\1\\1\end{bmatrix}, f = \left\{\begin{bmatrix}\frac{\sqrt{2}}{2}\\0\\0\end{bmatrix},\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}\bullet\frac{\sqrt{6}}{2},\begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\1\end{bmatrix}\bullet\frac{\sqrt{3}}{2}\right\}$. Solve $Ax=b$.

Let's use the middle equation $(x,y)_A = b(y), y \in f$. We can then count only $\begin{matrix}\alpha_1 = b(f_1)\\ \cdots \\ \alpha_n = b(f_n)\end{matrix}$.

Least Square Method

$r(x) = Ax - b$ is called the residual vector. It changes the problem to minimalize the residuum.

Theorem: The minimal residuum is achieved at $A^TAx = A^Tb$

Example: $\begin{aligned} x_1+x_2&=0 \\ 2x_1+x_2&=4 \\ 3x_1+x_2&=4 \end{aligned}$

$$A=\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix};A^T=\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix};b=\begin{bmatrix}0\\4\\4\end{bmatrix};A^TA=\begin{bmatrix}14&6\\6&3\end{bmatrix};A^Tb=\begin{bmatrix}20\\8\end{bmatrix};x=\begin{bmatrix}2\\-\frac{4}{3}\end{bmatrix}$$

Eigenvalues and Eigenvectors

Spectral Decomposition

Spectral decomposition, a.k.a. diagonalization, is concerned with a linear transformation $A_{F,F}=T\bullet A_{E,E}\bullet T^{-1}$, where the matrix $A_{E,E}:=\begin{bmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\ddots&0\\0&0&0&\lambda_n\end{bmatrix}$. Assume, that for some A it holds true, $A\bullet e_i=\lambda_i\bullet e_i$ and $e_1,\ldots,e_n$ are linearly independent. That implies $A\bullet(T:=[e_1,\ldots,e_n])=[\lambda_1\bullet e_1,\ldots,\lambda_n\bullet e_n]$, and so $A\bullet T=T\bullet\begin{bmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\ddots&0\\0&0&0&\lambda_n\end{bmatrix}$.

Conclusion 1: For any matrix $A\in C^{n\times n}$: $A=T\bullet D\bullet T^{-1}$.

Assume that A is real and symmetric. We know $A\bullet e=\lambda\bullet e$. We can multiple by $e^*\bullet$ to get $(e^*)^T\bullet A\bullet e=\lambda\bullet\|e\|^2$. Now we know $e^T\bullet A^T\bullet e^*=e^T\bullet A\bullet e^*=e^T\bullet(A\bullet e)^*=e^T\bullet\lambda^*\bullet e^*=\lambda^*\bullet\|e\|^2$, so $\lambda\in\mathbb{R}$.

Conclusion 2: For any real symmetric matrix $A\in C^{n\times n}$: $\forall\lambda_i:\lambda_i\in\mathbb{R}$

Assume that A is real and symmetric. We know $A\bullet e=\lambda\bullet e$. We can multiply by $e_j\bullet$, where $\lambda_i\neq\lambda_j$. Now $e_j^T\bullet A\bullet e_i=\lambda_i\bullet e_j^T\bullet e_i$, so $e_j^T\bullet A\bullet e_i=e_i^T\bullet A^T\bullet e_j=e_i^T\bullet(A\bullet e_j)=\lambda_j\bullet e_i^T\bullet e_j$. Now because $(\lambda_i-\lambda_j)\bullet(e_i\bullet e_j)=0$.

Conclusion 3: For any real symmetric matrix $A\in C^{n\times n}$: $\forall(e_i,e_j):e_i\perp e_j,\lambda_i\neq\lambda_j$

$A=T\bullet(D\in R^{n\times n})\bullet T^{-1}=Q\bullet D\bullet Q^T$, where Q is orthogonal matrix

Example: Compute the spectral decomposition of $A=\begin{bmatrix}3&2\\2&1\end{bmatrix}$.

$$|A-\lambda I|=\begin{vmatrix}3-\lambda&2\\2&1-\lambda\end{vmatrix}=\lambda^2-4\lambda-1;\lambda_1=2+\sqrt{5},\lambda_2=2-\sqrt{5}$$

$$\lambda_1=2+\sqrt{5}:e_1'=t\bullet\begin{bmatrix}-2\\1-\sqrt{5}\end{bmatrix},\ t^2\bullet\alpha=1\Rightarrow e_1=\begin{bmatrix}-\frac{2}{\sqrt{\alpha}}\\\frac{1-\sqrt{5}}{\sqrt{\alpha}}\end{bmatrix}$$

$$\lambda_2:e_2=s\bullet\begin{bmatrix}-2\\1+\sqrt{5}\end{bmatrix},s=\frac{1}{\sqrt{\beta}},\beta=4+\left(1+\sqrt{5}\right)^2$$

$$Q=[e_1,e_2];D=\begin{bmatrix}2+\sqrt{5}&0\\0&2-\sqrt{5}\end{bmatrix};A=Q\bullet D\bullet Q^T$$

Závěrečné informace

Veškeré materiály v tomto dokumentu jsou osobními poznámkami autora, vytvořenými na základě univerzitních přednášek. Jsou poskytovány bez záruky a slouží výhradně ke studijním účelům.

Datum aktualizace originálu: **26. 12. 2023 v 9:16:21**

Datum vygenerování PDF: **2. 7. 2025 v 21:51:14**