

# Obsah

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- Matrix Operations
- Systems of Linear Equations
- Vector Spaces
- Linear Transformations
- Bilinear and Quadratic Forms
  - Quadratic form
    - Classification
    - Congruent operations
  - Inner Product
    - Norms
    - Orthonormality
  - Variation Principle
    - Least Square Method
- Eigenvalues and Eigenvectors
  - Spectral Decomposition

# Matrix Operations

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# Systems of Linear Equations

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# Vector Spaces

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# Linear Transformations

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# Bilinear and Quadratic Forms

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## Quadratic form

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**Definition:**  $Q_B$  is a quadratic form, if it takes a vector and outputs a real number.  $Q_B: \mathcal{V} \rightarrow \mathbb{R}$ .

$$Q_B(v) = \mathcal{B}(v, v) = \mathcal{B}_S(v, v)$$

**Example:**  $Q: \mathbb{R}^3 \rightarrow \mathbb{R}; Q(x) = x^T x = x_1^2 + x_2^2 + x_3^2$ , always positive or zero

**Example:**  $Q: \mathbb{R}^2 \rightarrow \mathbb{R}; Q(x) = x^T A x = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$ , contains zero

**Example:**  $Q: \mathcal{F} \rightarrow \mathbb{R}; Q(f) = f^2(1)$

$$Q_B(\alpha x) = \alpha^2 Q(x), \text{ (prove by } \mathcal{B}(\alpha x, \alpha x))$$

**Property:**  $1 \in R(Q) \Rightarrow Q(x) = 1; Q(\alpha x) = \alpha^2 Q(x) = \alpha^2$ : if 1 is in the range, all positive numbers and zero is in the range; similarly if -1 is in the range, all negative numbers and zero is in the range.

**Property:**  $Q(0) = Q(0 \bullet 0) = 0 \bullet Q(0) = 0$

A bilinear form generating a quadratic form is not unique.  
By adding any antisymmetric bilinear form, I'd get a new one.

**Example:** Find a symmetric bilinear form  $\mathcal{B}_S$  for quadratic form  $Q_B$ .

$$\begin{aligned} \mathcal{B}_S(x + y, x + y) &= \mathcal{B}(x, x) + \mathcal{B}(x, y) + \mathcal{B}(y, x) + \mathcal{B}(y, y) \\ Q(x) &= Q(x) + 2\mathcal{B}_S(x, y) + Q(y) \\ \mathcal{B}_S(x, y) &= \frac{1}{2} (Q(x + y) - Q(x) - Q(y)) \end{aligned}$$

A symmetric bilinear form generating a quadratic is unique and given by  $\mathcal{B}_S(x, y) = \frac{1}{2} (\mathcal{Q}(x+y) - \mathcal{Q}(x) - \mathcal{Q}(y))$ .

**Example:**  $\mathcal{B}(x, y) = 2x_1y_1 - 3x_1y_2 + 5x_2y_1 - x_2y_2; \varepsilon = \{[1, 0], [0, 1]\}$

$$B = \begin{bmatrix} 2 & -3 \\ 5 & -1 \end{bmatrix}; B_S = \frac{1}{2} (B + B^T) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}; Q = B_S = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

A matrix of quadratic form is equal to a matrix of corresponding bilinear form.

## Classification

**Definition:** A positive definite form:  $\mathcal{Q} : \mathcal{V} \rightarrow \mathbb{R} : \forall x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) > 0$ .

**Definition:** A negative definite form:  $\mathcal{Q} : \mathcal{V} \rightarrow \mathbb{R} : \forall x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) < 0$ .

**Definition:** A positive semi definite form:  $\mathcal{Q} : \mathcal{V} \rightarrow \mathbb{R} : \forall x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) \geq 0$ .

**Definition:** A negative semi definite form:  $\mathcal{Q} : \mathcal{V} \rightarrow \mathbb{R} : \forall x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) \leq 0$ .

**Definition:** A indefinite form:  $\mathcal{Q} : \mathcal{V} \rightarrow \mathbb{R} : \exists x \in \mathcal{V}, x \neq o : \mathcal{Q}(x) < 0 \wedge \exists y \in \mathcal{V}, y \neq o : \mathcal{Q}(y) > 0$ .

**Example:** Classify  $\mathcal{Q}(x) = -2x_1^2 + 2x_1x_2 - 3x_2^2$ .

$$\begin{aligned} \mathcal{Q}(x) &= -2x_1^2 + 2x_1x_2 - 3x_2^2 = -2 \left( x_1^2 - x_1x_2 + \frac{1}{4}x_2^2 \right) - \frac{5}{2}x_2^2 = -2 \left( x_1 - \frac{1}{2}x_2 \right)^2 - \frac{5}{2}x_2^2 \\ &\xrightarrow{s_1 = x_1 - \frac{1}{2}x_2, s_2 = x_2} -2s_1^2 - \frac{5}{2}s_2^2, \end{aligned}$$

aka it's negative definite form.

$$[Q]_N = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}; s = \left( S = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \right) x; [Q]_S = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix}; \text{We diagonalised our matrix.}$$

## Congruent operations

Perform a row operation and at the same time the corresponding column operation.

$$[Q]_N = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & -\frac{5}{2} \end{bmatrix} \sim \begin{bmatrix} -2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix}$$

For all symmetric matrices  $[A|I] \sim [D|\tilde{L}]$  is called LD decomposition.

**Example:** Classify  $\mathcal{Q}(x) : Q = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}, \text{ it is a positive definite form.}$$

## Inner Product

**Definition:** Inner product on the real vector space  $\mathcal{V}$  is a symmetric bilinear form on  $\mathcal{V}$ , which corresponding quadratic form is a positive definite one.

$$(u + v, w) = (u, w) + (v, w)$$

$$(\alpha u, v) = \alpha(u, v)$$

$$(u, v) = (v, u)$$

$$(u, u) > 0; u \neq 0$$

**Example:** Examine, whether  $(f, g) = f(0)g(0) + f(1)g(1)$  is an inner product. (HOWTO: Examine all 4 conditions.)

**Example:** Examine, whether  $(x, y)_A = x^T A y$  is an inner product. (It is.)

## Norms

**Definition:** The norm of the vector on the real vector space  $\mathcal{V}$  is a positive definite one quadratic form.

$$\|x + y\| \leq \|x\| + \|y\| \text{ (triangular inequality)}$$

$$\|\alpha x\| = \alpha \|x\| \quad \|x\| = 0 \Leftrightarrow x = 0$$

**Example:**  $\|x\|_2 = \sqrt{x^2 + y^2}$  (Euclidian norm);  $\|x\|_0 = \sum_i^n |x_i|$  (zero norm);  $\|x\|_\infty = \max_i |x_i|$  (maximum norm)

**Example:** Energy norms:  $(x, y)_A = x^T A y$ ;  $\|x\|_A = \sqrt{(x, y)_A}$

**Cauchy-Bunyakovsky-Schwarz inequality:**  $(x, y)^2 \leq (x, x) \bullet (y, y)$

## Orthonormality

**Definition:** An orthonormal means that the norm of all vectors is equal to one and is orthogonal.

**Definition:** An orthogonal means, that the inner product of any two vector is equal to zero.

**Gram-Schmidt Algorithm:** Make a set of vectors to orthonormal set.

1. Take the first vector from the set.
2. Orthogonalize the vector. We know  $f'_n = e_n - \alpha_{n-1}f_{n-1} - \dots$ , where  $\alpha_y = (f_y, e_n), \dots$
3. Orthonormalize the vector – divide it by its norm.

**Example:** Find an orthonormal basis:  $(x, y)_A = x^T A y$ ;  $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ ;  $\|x\|_A = \sqrt{x^T A x}$ ;  $\mathcal{V} = \mathbb{R}^3$

$$\varepsilon = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}; f = \left\{ \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\| = \sqrt{2}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{bmatrix}, \frac{\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\| = \frac{\sqrt{6}}{2}} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \cdot \frac{\sqrt{6}}{2}, \frac{\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \right\| = \frac{2\sqrt{3}}{3}} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \cdot \frac{\sqrt{3}}{2} \right\}$$

**Example:** Find an orthonormal basis of  $\mathcal{P}^2$ :  $(p, q) = \int_0^1 p(x)q(x)dx$ ;  $\|p\|_A = \sqrt{\int_0^1 p(x)^2 dx}$

$$\varepsilon = \{1; x; x^2\}; f = \left\{ \frac{x^2}{\|x^2\| = \frac{\sqrt{5}}{5}} = \sqrt{5}x^2, \frac{x - \frac{5}{4}x^2}{\|x - \frac{5}{4}x^2\| = \frac{\sqrt{12}}{12}} = -5\sqrt{3}x^2 + 4\sqrt{3}x, \dots \right\}$$

## Variation Principle

**Theorem:** The matrix  $A$  is symmetric and positive definite,  $(x, y)_A = x^T A y$ ,  $b(x) = b^t x$ ,  $q(x) = \frac{1}{2}(x, x)_A - b(x)$ , vectors  $b$  and  $\bar{x}$ .

$$A\bar{x} = b \Leftrightarrow (x, y)_A = b(y), \forall y \in \mathbb{R}^n \Leftrightarrow \exists \bar{x} : q(\bar{x}) \leq q(x)$$

**Notes:**  $q(x)$  is an error function. The solution  $\bar{x}$  has the smallest error possible.

**Example:**  $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, f = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \bullet \frac{\sqrt{6}}{2}, \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \bullet \frac{\sqrt{3}}{2} \right\}$ . Solve  $Ax = b$ .

Let's use the middle equation  $(x, y)_A = b(y), y \in f$ . We can then count only

$$\begin{aligned} \alpha_1 &= b(f_1) \\ &\vdots \\ \alpha_n &= b(f_n) \end{aligned}$$

## Least Square Method

$r(x) = Ax - b$  is called the residual vector. It changes the problem to minimize the residuum.

**Theorem:** The minimal residuum is achieved at  $A^T Ax = A^T b$

$$x_1 + x_2 = 0$$

**Example:**  $2x_1 + x_2 = 4$

$$3x_1 + x_2 = 4$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}; A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}; b = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}; A^T A = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}; A^T b = \begin{bmatrix} 20 \\ 8 \end{bmatrix}; x = \begin{bmatrix} 2 \\ -\frac{4}{3} \end{bmatrix}$$

# Eigenvalues and Eigenvectors

## Spectral Decomposition

Spectral decomposition, a.k.a. diagonalization, is concerned with a linear transformation  $A_{F,F} = T \bullet A_{E,E} \bullet T^{-1}$ , where the

matrix  $A_{E,E} := \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$ . Assume, that for some  $A$  it holds true,  $A \bullet e_i = \lambda_i \bullet e_i$  and  $e_1, \dots, e_n$  are linearly

independent. That implies  $A \bullet (T := [e_1, \dots, e_n]) = [\lambda_1 \bullet e_1, \dots, \lambda_n \bullet e_n]$ , and so  $A \bullet T = T \bullet \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$ .

**Conclusion 1:** For any matrix  $A \in C^{n \times n}$ :  $A = T \bullet D \bullet T^{-1}$ .

Assume that  $A$  is real and symmetric. We know  $A \bullet e = \lambda \bullet e$ . We can multiply by  $e^* \bullet$  to get  $(e^*)^T \bullet A \bullet e = \lambda \bullet \|e\|^2$ . Now we know  $e^T \bullet A^T \bullet e^* = e^T \bullet A \bullet e^* = e^T \bullet (A \bullet e)^* = e^T \bullet \lambda^* \bullet e^* = \lambda^* \bullet \|e\|^2$ , so  $\lambda \in \mathbb{R}$ .

**Conclusion 2:** For any real symmetric matrix  $A \in C^{n \times n}$ :  $\forall \lambda_i : \lambda_i \in \mathbb{R}$

Assume that  $A$  is real and symmetric. We know  $A \bullet e = \lambda \bullet e$ . We can multiply by  $e_j \bullet$ , where  $\lambda_i \neq \lambda_j$ . Now  $e_j^T \bullet A \bullet e_i = \lambda_i \bullet e_j^T \bullet e_i$ , so  $e_j^T \bullet A \bullet e_i = e_j^T \bullet A^T \bullet e_j = e_j^T \bullet (A \bullet e_j) = \lambda_j \bullet e_j^T \bullet e_j$ . Now because  $(\lambda_i - \lambda_j) \bullet (e_i \bullet e_j) = 0$ .

**Conclusion 3:** For any real symmetric matrix  $A \in C^{n \times n}$ :  $\forall (e_i, e_j) : e_i \perp e_j, \lambda_i \neq \lambda_j$

$$A = T \bullet (D \in R^{n \times n}) \bullet T^{-1} = Q \bullet D \bullet Q^T, \text{ where } Q \text{ is orthogonal matrix}$$

**Example:** Compute the spectral decomposition of  $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ .

$$|A-\lambda I|=\begin{vmatrix}3-\lambda&2\\2&1-\lambda\end{vmatrix}=\lambda^2-4\lambda-1;\lambda_1=2+\sqrt{5},\lambda_2=2-\sqrt{5}$$

$$\lambda_1=2+\sqrt{5}:e'_1=t\bullet\begin{bmatrix}-2\\1-\sqrt{5}\end{bmatrix},\,t^2\bullet\alpha=1\Rightarrow e_1=\begin{bmatrix}-\frac{2}{\sqrt{\alpha}}\\\frac{1-\sqrt{5}}{\sqrt{\alpha}}\end{bmatrix}$$

$$\lambda_2:e_2=s\bullet\begin{bmatrix}-2\\1+\sqrt{5}\end{bmatrix},s=\frac{1}{\sqrt{\beta}},\beta=4+\left(1+\sqrt{5}\right)^2$$

$$Q=[e_1,e_2];D=\begin{bmatrix}2+\sqrt{5}&0\\0&2-\sqrt{5}\end{bmatrix};A=Q\bullet D\bullet Q^T$$