# Matrix Operations

# Systems of Linear Equations

# Vector Spaces

# Linear Transformations

# Bilinear and Quadratic Forms

## Quadratic form

**Definition**: is a quadratic form, if it takes a vector and outputs a real number. .

**Example**: , always positive or zero

**Example:** , contains zero

**Example:**

, (prove by )

**Property:**  : if 1 is in the range, all positive numbers and zero is in the range; similarly if -1 is in the range, all negative numbers and zero is in the range.

**Property:**

A bilinear form generating a quadratic form is not unique.  
By adding any antisymmetric bilinear form, I’d get a new one.

**Example:** Find a symmetric bilinear form for quadratic form .

A symmetric bilinear form generating a quadratic is unique  
and given by .

**Example:**

A matrix of quadratic form is equal to a matrix of corresponding bilinear form.

### Classification

**Definition**: A positive definite form: .

**Definition**: A negative definite form: .

**Definition**: A positive semi definite form: .

**Definition**: A negative semi definite form: .

**Definition**: A indefinite form: .

**Example:** Classify .

, aka it’s negative definite form.

We diagnolised our matrix.

### Congruent operations

Perform a row operation and at the same time the corresponding column operation.

For all symmetric matrices is called LD decomposition.

**Example:** Classify

, it is a positive definite form.

## Inner Product

**Definition:** Inner product on the real vector space is a symmetric bilinear form on , which corresponding quadratic form is a positive definite one.

**Example**: Examine, whether is an inner product. (HOWTO: Examine all 4 conditions.)

**Example**: Examine, whether is an inner product. (It is.)

### Norms

**Definition**: The norm of the vector on the real vector space is a positive definite one quadratic form.

(triangular inequality)

**Example:** (Euclidian norm); (zero norm); (maximum norm)

**Example:** Energy norms:

**Cauchy–Bunyakovsky–Schwarz inequality**:

### Orthonormality

**Definition**: An orthonormal means that the norm of all vectors is equal to one and is orthogonal.  
**Definition:** An orthogonal means, that the inner product of any two vector is equal to zero.

**Gram-Schmidt Algorithm:** Make a set of vectors to orthonormal set.

1. Take the first vector from the set.
2. Orthogonalize the vector. We know , where .
3. Orthonormalize the vector – divide it by its norm.

**Example:** Find an orthonormal basis:

**Example:** Find an orthonormal basis of :

## Variation Principle

**Theorem:** The matrix is symmetric and positive definite, , , , vectors and .

**Notes:** is an error function. The solution has the smallest error possible.

**Example**: . Solve .

Let’s use the middle equation . We can then count only .

### Least Square Method

is called the residual vector. It changes the problem to minimalize the residuum.

**Theorem:** The minimal residuum is achieved at

**Example:**

# Eigenvalues and Eigenvectors

## Spectral Decomposition

Spectral decomposition, a.k.a. diagonalization, is concerned with a linear transformation , where the matrix . Assume, that for some it holds true, and are linearly independent. That implies , and so .

**Conclusion 1**: For any matrix : .

Assume that is real and symmetric. We know . We can multiple by to get . Now we know , so .

**Conclusion 2:** For any real symmetric matrix :

Assume that is real and symmetric. We know . We can multiply by , where . Now , so . Now because .

**Conclusion 3:** For any real symmetric matrix :

, where is orthogonal matrix

**Example:** Compute the spectral decomposition of