

ZERMELO-FRAENKEL'S SET THEORY

NOTES FOR THE “REAL MATHS”: PART 04

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Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können.

No one shall be able to expel us from the paradise, that Cantor created for us.

David Hilbert, Münster, 4 June 1925

A brief synopsis. In Chapter 1 we recall the original definition of a set and we recall why this “naïve” approach leads to a contradiction. This is why Set Theory is build in the axiomatic way and we will present the (nowadays commonly agreed upon) axiomatics of *Zermelo* and *Fraenkel* in Chapter 2. For many constructions in mathematics one often needs a further axiom — *The Axiom of Choice* (AC). We say what this axiom is in Chapter 3 and we try to give the motivation for this axiom. In Chapter 4 we show how and where AC is typically used in mathematics. We also say why the axiom has certain controversial aspects. Finally, in Chapter 5, we give a brief overview of the literature and biographies of *Ernst Zermelo* and *Abraham Fraenkel*.

1. RUSSELL'S PARADOX ONCE MORE

Unter einer “Menge” verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die “Elemente” von M genannt werden) zu einem Ganzen.

By an “aggregate” (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects m of our intuition or our thought. These objects are called the “elements” of M .

Georg Cantor, [5]

The naïve notion of a set like a collection of distinguishable elements brings a substantial difficulty. In fact, this difficulty is so grave that one has to abandon the naïve idea altogether.

1.1. Example (Russell's paradox). Suppose we consider the collection

$$R = \{x \mid x \text{ is a set and } x \notin x\}$$

Then R is certainly an “aggregate” of definite and separate objects. Therefore R must be a set according to Cantor's definition. Moreover, one of the following two cases must hold:

(1) $R \in R$.

In this case, R has to have the property that all elements of R have: $R \notin R$. This is a contradiction.

(2) $R \notin R$.

In this case, R has the property of all elements of R . Thus, $R \in R$ must hold. This is a contradiction.

Since both cases led to contradiction, and since there is no third option, we have to conclude that R *cannot be a set*.

1.2. Remark. Russell discovered the above paradox in May or June 1901 (he informed Gottlob Frege about it in a letter) and published it in 1903 in [23]. According to [35], the same paradox had been discovered in 1899 by Ernst Zermelo in Göttingen, but he did not publish it. The paradox thus was known only to members of the University of Göttingen, which definitely included David Hilbert and Edmund Husserl. At the end of the 1890's Cantor himself had already realised that his definition of a set would lead to a contradiction. Indeed, Cantor proved that $\text{card}(X) < \text{card}(P(X))$ has to hold for any set X , see, e.g., [32]. Thus, if we instantiate the set U of all sets for X , we would have $\text{card}(U) < \text{card}(P(U))$, which is a contradiction with the fact that $\text{card}(U)$ should be the maximal possible. Cantor told David Hilbert and Richard Dedekind about this difficulty in a letter.

An excellent account of the above paradox and other logical antinomies can be found in the book [12].

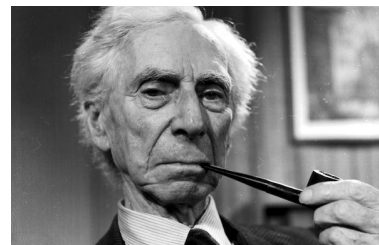
So: is the notion of a set as of a collection of elements doomed forever? Can one not get away from the horrors of Russell's paradox?

There is a way out of these difficulties, of course. In the next chapter we will start building sets in the *axiomatic* way. But there is another danger lurking at us: if we formulate axioms and deduce all the theorems purely from the axioms, will the resulting theory resemble what we wanted to do naïvely?

In many cases, when building an axiomatic theory, one need not bother with defending the “truth” or at least “plausibility” of the axioms. However, Set Theory aspires to be the *foundational* theory of mathematics. Ideally, any mathematical proposition could be translated into the set-theoretical language and its validity should be derivable from the axioms. Thus, we should better defend the “plausibility” of the axioms so that mathematics does not become an abstract formal game only. We will try to achieve such a defense by saying that a new set can be formed only after the existence of its elements has been established. Thus, we will introduce an informal concept of a “stage” at which some set is defined. Also, after introducing each axiom, we will try to defend it by showing which sets the axiom guarantees to exist.

A brief biography of Bertrand Russell. *Bertrand Arthur William Russell*, (3rd Earl Russell) was born on 8 May 1872 in Ravenscroft, Trellech, Monmouthshire, Wales, into an influential and liberal family of the British aristocracy.

Although Russell is nowadays considered mainly as one of the founders of *analytic philosophy* (along with his predecessor Gottlob Frege and Russell's PhD student Ludwig Wittgenstein), he was also a prominent logician. With his colleague *Alfred North Whitehead* he wrote a very influential book *Principia Mathematica* [24], where Russell and Whitehead attempted to give a logical basis to all of mathematics.



In 1910 Russell became a lecturer at Trinity College at the University of Cambridge. Russell was an outspoken pacifist during WWI and his activities led him first to be fired from Trinity and, in 1918, he was even imprisoned¹ for six months when he opposed the US to become the ally of the UK in war efforts.

In August 1920, Russell became a member of an official delegation to Soviet Russia.² He wrote several articles about the visit. Unlike his then-mistress Dora Black, Russell was not enthusiastic about the Bolshevik Revolution. Russell even met Vladimir Ilyich Lenin and had an hour-long conversation with him. Russell wrote in his autobiography that he found Lenin disappointing, sensing an “impish cruelty” in him. The events of the visit ended Russell's support for the revolution.

For most of his life, Russell was an ardent pacifist, engaging publicly in various anti-war movements and often even addressing world leaders. After WWII he was particularly active in the movement for nuclear disarmament, he opposed the imperialism of Europe during the Suez Crisis, and he was against waging the Vietnam War. Russell also proposed that Europe should change: Germany should become one state and the middle of Europe (consisting of Germany, Poland, Hungary and Czechoslovakia) should become a neutral zone.

Although Russell described himself as a moderate socialist, he saw no attraction in the dialectic materialism teachings of Marx. Russell wrote several books on the issues of politics, ethics and education.

Russell was married four times and he had fathered four children. Russell married his first wife, Allys Pearsal Smith, in 1894. The marriage became to fall apart already in 1901. In 1921 the couple got divorced and Russell married writer and feminist activist Dora Black. His second marriage had lasted for fifteen years. In 1936, Russell married his third wife, Patricia Spence, a governess of his children. Russell married his fourth wife, Edith Finch, soon after the divorce of Spence in 1952.

Bertrand Russell died of influenza on 2 February 1970, aged 98, at his home in Penrhyndeudraeth, Wales.

¹“I found prison in many ways quite agreeable. I had no engagements, no difficult decisions to make, no fear of callers, no interruptions to my work. I read enormously; I wrote a book, *Introduction to Mathematical Philosophy* ... and began the work for *Analysis of Mind*”, wrote Russell in his autobiography later. In September 1961, after taking part in an anti-nuclear demonstration in London, Russell was jailed again for seven days in Brixton Prison. Russell was 89 then. The magistrate offered him a pardon if he pledged himself to “good behaviour”, to which Russell replied: “No, I won’t”.

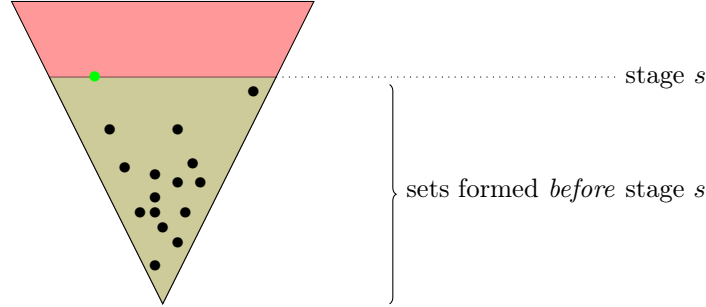
²See the fascinating book [25] on the relationship of Soviet Russia and the West in the early 1920’s.

2. CONSTRUCTING NEW SETS FROM OLD — THE AXIOMS OF ZERMELO & FRAENKEL

Thirty years ago, when I was a Privatdozent in Göttingen, I came under the influence of David Hilbert, to whom I am surely the most indebted for my mathematical development. As a result I began to do research on the foundations of mathematics, especially on the fundamental problems of Cantorian set theory, whose true significance I learned to appreciate through the fruitful collaboration of the mathematicians at Göttingen.

Ernst Zermelo, Zermelo's Nachlass

The idea we will pursue in this chapter is the following: we will assume that to each set we will assign the *stage* at which it is formed and we will allow new sets to be formed only of elements that have been introduced at *earlier* stages. Thus, the sets will eventually fill a “cone” as in the following picture, where the “green set” at stage s could only consist of “black dots” as elements.



We do not aspire to give any axiomatics of what stages are. In fact, we might run into difficulties very soon: what would the collection of some stages be? We will use the above picture only to give an *intuitive plausibility* of the real axioms for sets that we will introduce.

Will the above idea of stages prevent us from Russell's Paradox? Let us have a look at how Russell's paradoxical set R would have to be interpreted: the set R would be formed at some stage s and it would have to contain as elements only sets that are formed at some earlier stage. Thus:

$$R = \{x \mid x \text{ is a set formed at stage } < s \text{ and } x \notin x\}$$

Therefore we cannot assert $R \in R$ or $R \notin R$, for that would mean that R was also formed at a stage earlier than s . Thus, we cannot perform the analysis by cases of Example 1.1 that leads to Russell's Paradox!

The language of set theory. Since we will present axioms, we have to ask first in *what language* the axioms will be written. The answer is: we will work in the first-order language with the *identity* symbol $=$, with one *binary predicate symbol* \in , having *variables* x, y, z, \dots . The rest of the symbols are standard connectives and quantifiers: $\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg, \forall, \exists$. To make the language less cumbersome, we also introduce *shorthands*:

- (1) The quantifier $\exists!$ for there *exists exactly one*, with

$$\exists!x \varphi(x)$$

being a shorthand for the formula

$$\exists x \left(\varphi(x) \wedge \forall y (\varphi(y) \Rightarrow x = y) \right)$$

- (2) The *restricted quantifiers*: formulas

$$\left(\exists x \in y \right) \varphi(x) \quad \text{and} \quad \left(\forall x \in y \right) \varphi(x)$$

are shorthands for

$$\exists x \left(x \in y \wedge \varphi(x) \right) \quad \text{and} \quad \forall x \left(x \in y \Rightarrow \varphi(x) \right)$$

respectively.

- (3) Notations

$$x \notin y \quad \text{and} \quad x \neq y$$

are shorthands for formulas

$$\neg(x \in y) \quad \text{and} \quad \neg(x = y)$$

respectively.

(4) The formula

$$x \subseteq y$$

is a shorthand for the formula

$$\forall z (z \in x \Rightarrow z \in y)$$

We read $x \subseteq y$ as *the set x is a subset of the set y* .

The axioms of set theory ZF. The axioms we present now are the axioms of the so-called *Zermelo-Fraenkel Set Theory*, abbreviated as ZF. The following system of axioms is (almost) universally agreed upon nowadays. Essentially all axioms below were given by Ernst Zermelo [36] in 1908 with the following exceptions: Axiom of replacement is due to Abraham Fraenkel [11] from 1922 (the same axiom was independently proposed by Thoralf Skolem [26] in 1923), and Axiom of regularity is due to John von Neumann [21] from 1925.

We do not exhibit the most efficient axiomatics. For example, the following Axiom 2.3 is not needed, since the existence of at least one set follows from the Axiom of infinity 2.21 below. And once a set exists, one can *define* the empty set.

All the axioms of ZF are summarised in Remark 2.24 below.

2.1. Axiom (Axiom of extensionality). $\forall x \forall y \left(\left(\forall z (z \in x \Leftrightarrow z \in y) \right) \Rightarrow x = y \right)$.

2.2. Commentary. Any set should be completely determined by its elements: we want to keep this idea of Georg Cantor. And Axiom 2.1 says precisely that. Observe that the axiom does *not* postulate the existence of any set. It only says that *if* we are given sets x and y such that the elements of x are precisely the elements of y , *then* the sets x and y are the same.

Why the name “extensionality”? The opposite of “extensional” is “intensional”, in this context. Axiom 2.1 expresses the fact that sets are determined by their *extension* (i.e., by their elements) not by their *intension* (i.e., how the sets are defined). For example, the sets of all non-negative reals and all squares of reals are *extensionally* the same but they need not be *intensionally* the same.

We start modestly: we postulate the existence of a set having no elements whatsoever.

2.3. Axiom (Axiom of an empty set). $\exists x \forall y \left(y \notin x \right)$.

2.4. Commentary. Axiom 2.3 postulates that there exists at least one set. We denote this set by

$$\emptyset$$

and call it an *empty set*.

We can start building sets now. One way of building a set should be “collecting” finitely many sets as elements into another set. Of course, it suffices to “collect” two existing sets.

2.5. Axiom (Axiom of pairing). $\forall a \forall b \exists x \forall z \left(z \in x \Leftrightarrow (z = a \vee z = b) \right)$.

2.6. Commentary. Axiom 2.5 says the following: for all sets a, b there exists a set x that contains as elements *precisely* a or b . This new set, guaranteed by the axiom, is denoted by

$$\{a, b\}$$

Observe that by Axiom 2.1, the set $\{a, b\}$ is determined by a and b uniquely.

2.7. Remark (Singletons and ordered pairs). Since we can form pairs $\{a, b\}$, we can also form *singleton* sets. Namely, we put

$$\{a\} = \{a, a\}$$

Furthermore, we can define *ordered pairs*

$$(a, b) = \{\{a\}, \{a, b\}\}$$

See, e.g., [32] that this definition makes sense.

Analogously we can form *ordered triples*

$$(a, b, c) = (a, (b, c))$$

ordered quadruples, etc.

Another way of forming a set should be “collecting” all the sets having a certain property into another set.

2.8. Axiom (Axiom of separation). $\forall x \exists y \forall z (z \in y \Leftrightarrow (z \in x \wedge \varphi(z)))$.

2.9. Commentary. Let us see what Axiom 2.8 says first and then we will comment on its name. The axiom says: for every set x one can form a set y such that elements of y are precisely those elements z of x such that $\varphi(z)$ holds. In other words: we can form the set y , usually denoted by

$$\{z \in x \mid \varphi(z)\}$$

Thus, the axiom allows us to *separate* the elements of a set x into two collections: those elements z that satisfy $\varphi(z)$ and those elements z that do not satisfy $\varphi(z)$.

Notice that Axiom 2.8 is not a single axiom. It is, in fact, a single axiom for every formula $\varphi(z)$. Such axioms are often called *axiom-schemata* and we will see another such scheme as Axiom of replacement 2.17 later.

2.10. Remark. Observe that Axiom 2.8 does *not* allow us to form $\{z \mid z \notin z\}$ that leads to Russell's Paradox. Indeed, Axiom 2.8 cleverly states that one must have a set x *first* from which one can separate z 's having the property $z \notin z$. And the set

$$y = \{z \in x \mid z \notin z\}$$

poses no difficulty.

2.11. Axiom (Axiom of union). $\forall x \exists y \forall z (z \in y \Leftrightarrow \exists w (z \in w \wedge w \in x))$.

2.12. Commentary. Axiom 2.11 allows us to form a *union* y of any set x . We denote this set y by

$$\bigcup x$$

By the axiom, $z \in \bigcup x$ if and only if $z \in w$ for some $w \in x$.

Let us give an explanation that this is indeed how a union should behave: suppose $x = \{a, b\}$ and think of x as of a system of two sets a, b . Then the union $\bigcup \{a, b\}$ should consist of all elements z for which there is $w \in x$ (i.e., w is one of a, b) such that $z \in w$. We will write

$$a \cup b$$

instead of $\bigcup \{a, b\}$.

2.13. Remark (Intersection and difference of sets). Axioms 2.11 and 2.8 allow us to define sets

$$x \cap y = \{z \in x \cup y \mid z \in x \wedge z \in y\} \quad \text{and} \quad x \setminus y = \{z \in x \cup y \mid z \in x \wedge z \notin y\}$$

called the *intersection* and *difference* of sets x and y .

More generally, we can define an *intersection* $\bigcap x$ of a set x as follows:

$$\bigcap x = \{z \in \bigcup x \mid \forall y (y \in x \Rightarrow z \in y)\}$$

We are now ready to state and prove the usual basic properties of union, intersection and difference. We will not do it, we leave it as an (easy) exercise.

2.14. Axiom (Axiom of powerset). $\forall x \exists y \forall z (z \in y \Leftrightarrow z \subseteq x)$.

2.15. **Commentary.** Axiom 2.14 allows us to form the *powerset*

$$\{z \mid z \subseteq x\}$$

of any set x . We will denote the powerset of x by $P(x)$.

2.16. **Remark (Cartesian product $x \times y$ of two sets x and y).** The set $x \times y$ as the set of all ordered pairs (a, b) , where $a \in x$ and $b \in y$ is defined as follows:

- (1) Recall from Remark 2.7 that any ordered pair (a, b) is the set $\{\{a\}, \{a, b\}\}$. Hence $(a, b) \in P(P(x \cup y))$.
- (2) The set $x \times y$ must therefore be a subset of $P(P(x \cup y))$ and we will use Axiom of separation 2.8 to define it.
- (a) Denote by $\varphi(z)$ the following formula

$$\exists a \exists b \left((a \in x) \wedge (b \in y) \wedge (z = \{\{a\}, \{a, b\}\}) \right)$$

- (b) Define

$$x \times y = \{z \in P(P(x \cup y)) \mid \varphi(z)\}$$

Summarised: $x \times y$ exists due to Axiom of union 2.11, Axiom of powerset 2.14 and Axiom of separation 2.8.

2.17. **Axiom (Axiom of replacement).** Let $\varphi(s, t)$ be a formula such that $\forall s \exists! t \varphi(s, t)$ holds. Then $\forall x \exists y \forall y' \left(y' \in y \Leftrightarrow \exists x' (x' \in x \wedge \varphi(x', y')) \right)$ holds.

2.18. **Commentary.** The formula $\varphi(s, t)$ should be thought of as a function: for every set s , it delivers a unique t such that $\varphi(s, t)$ holds. Axiom 2.17 then states that for any set x , all “function values” y' for all $x' \in x$ (i.e., all y' such that $\varphi(x', y')$ holds) form a set again. In other words, an *image* under a function of a set is a set.

What is Axiom of replacement good for? For example, suppose we would want to construct, given sets $x_i, i \in I$, the set y containing *precisely* all the sets $x_i, i \in I$, as its elements. In other words, we want to form the set

$$\{x_i \mid i \in I\}$$

To that end, we define a formula $\varphi(s, t)$ as follows:

$$(s \in x \Rightarrow t = x_s) \vee (s \notin x \Rightarrow t = \emptyset)$$

where we put $x = I$, to conform with the notation of Axiom 2.17. It is easy to see that $\forall s \exists! t \varphi(s, t)$ holds. Axiom of replacement then provides us with the set y having the property

$$y = \{y' \mid \exists x' (x' \in x \wedge \varphi(x', y'))\}$$

By renaming the bounded variables, we see that

$$y = \{x_i \mid i \in I\}$$

holds.

2.19. **Axiom (Axiom of regularity).** $\forall x (x \neq \emptyset \Rightarrow \exists y (y \in x \wedge (y \cap x = \emptyset)))$.

2.20. **Commentary.** The wording of Axiom 2.19 seems quite mysterious. Let us therefore decypher, what regularity means. The axiom *prohibits* the existence of certain sets. For example, there cannot be a set x such that $x \in x$ holds. Why? Suppose x is a set such that $x \in x$ holds. Then $\{x\}$ is a set by Axiom 2.5. By Axiom 2.19 there exists an element $y \in \{x\}$ such that $y \cap \{x\} = \emptyset$. But we can choose only $y = x$, since $\{x\}$ has no other elements. Thus we proved $x \cap \{x\} = \emptyset$, which means $x \notin x$; a contradiction.

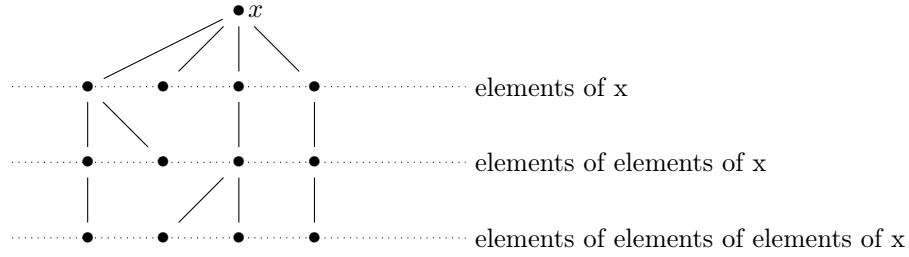
Similar argument shows that there cannot be a set x and an infinite chain

$$\cdots \in x_{i+1} \in x_i \in \cdots \in x_0 \in x$$

of its elements. Or there cannot be a “loop” of the form

$$x \in x_{n-1} \in x_{n-2} \in \cdots \in x_0 \in x$$

Thus, Axiom 2.19 states that sets should be *well-founded*.³ The latter means that if we form the \in -structure of any non empty set x :



then it is a (rooted) tree with each branch finite.

Especially in Computer Science, however, one often needs to work with sets that are *not* well-founded. This happens, for example, when one needs to deal with sets defined by *recursive equations*. Within such a realm, for example, a set satisfying the *recursive equation*

$$x = \{x\}$$

exists. A perfect account of set theory *without* Axiom of regularity can be found in the book [4]. The description of \in -structures, arising from non-wellfounded sets is given in [1].

2.21. Axiom (Axiom of infinity). $\exists x (\emptyset \in x \wedge \forall z (z \in x \Rightarrow z \cup \{z\} \in x))$.

2.22. Commentary. Strictly speaking, Axiom 2.21 does not postulate the existence of an infinite set x . Instead, the axiom says that there exists a set x , that is *inductive*, i.e., the set x satisfies the following two conditions:

- (1) The empty set \emptyset is an element of x .
- (2) If z is an element of x , then the successor $z \cup \{z\}$ of z is also an element of x .

The name “inductive” for the set x as above is used, since the above two conditions resemble the induction as we know it from natural numbers. The set \mathbb{N} of all natural numbers is precisely the set that satisfies the following two conditions:

- (1) The number 0 is an element of \mathbb{N} .
- (2) If z is an element of \mathbb{N} , then the successor $z + 1$ of z is also an element of \mathbb{N} .

The set \mathbb{N} of all natural numbers is the *smallest* such that the above two conditions hold. However, Axiom 2.21 does *not* assert that the inductive set x should be smallest in any sense.

In what sense is the set x infinite then? Observe that $z \cup \{z\} \neq z$ holds. For if $z \cup \{z\} = z$ were true, then $z \in z$ would hold and that would violate Axiom of regularity 2.19. Thus, x contains at least the infinite series

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \quad \dots$$

of mutually distinct sets.

2.23. Remark (The set \mathbb{N} of all natural numbers). All the machinery we have allows us to *define* the set \mathbb{N} of all natural numbers as follows:

- (1) Let x be the inductive set provided by Axiom 2.21.
- (2) Use Axiom 2.14 to form the powerset $P(x)$ of x .
- (3) That z is an inductive set can be written as a formula

$$\emptyset \in z \wedge \forall z' (z' \in z \Rightarrow z' \cup \{z'\} \in z)$$

Denote the above formula by $\iota(z)$ and use Axiom 2.8 to form the set

$$y = \{z \in P(x) \mid \iota(z)\}$$

³This is why the Axiom of regularity is also called *Axiom of foundation*. In fact, Ernst Zermelo named this axiom *Axiom der Fundierung*: every non-empty set reaches its *foundation* after finitely many steps in every \in -chain of its elements. See [37].


(4) Form the set

$$\bigcap y$$

and call it \mathbb{N} .

We have defined \mathbb{N} as the *smallest* inductive set.

2.24. Remark (Summary of the axioms of ZF). Let us summarise the all the axioms of Zermelo-Fraenkel Set Theory. We have observed that there are essentially *three types* of axioms. In the summary we will therefore distinguish among these types.

 **Logical axioms.** There two axioms in this group.

Axiom of extensionality 2.1 brings together the symbol \in and the equality symbol $=$. Two sets are considered as equal if and only if they have the same elements.

Axiom of regularity 2.19 *prescribes* the behaviour of the elementhood relation \in . Namely, a non empty set x must not contain either an infinite chain of the form

$$\cdots \in x_{i+1} \in x_i \in \cdots \in x_0 \in x$$




or a loop of the form

$$x \in x_{n-1} \in x_{n-2} \in \cdots \in x_0 \in x$$

  **Postulation axioms** postulate the *existence* of certain sets. There are two such axioms.

Axiom of an empty set 2.3 postulates the existence of an empty set. The empty set is denoted by \emptyset .

Axiom of infinity 2.21 postulates the existence of an infinite set.

   **Formation axioms** *infer* the existence of certain sets from the sets that *have already been formed*. There are five axioms in this group.

Axiom of pairing 2.5. Given that sets a, b have been formed, we can form the set

$$\{a, b\}$$

Axiom of separation 2.8. Given that a set x has been formed, we can form the set

$$\{z \in x \mid \varphi(z)\}$$

where $\varphi(z)$ is any formula.

Axiom of union 2.11. Given that a set x has been formed, we can form the set

$$\bigcup x$$

called the union of x .

Axiom of powerset 2.14. Given that a set x has been formed, we can form the set

$$P(x) = \{z \mid z \subseteq x\}$$

called the powerset of x .

Axiom of replacement 2.17. Given that f is a function on sets and given that a set x has been formed, we can form the set

$$f[x] = \{f(z) \mid z \in x\}$$

called the image of x under f .

Observe that all the formation axioms have the form “if sets ... have been formed, then a set ... is formed”. This is in accordance with our intuitive notion of “stages” at which sets are formed: assuming that sets have been formed at stage $< s$, then we can form a new set at stage s . The “first” stage is the stage when the empty set and the infinite set have been formed.

2.25. Remark (ZF and Gödel’s theorems). Since ZF is a first order theory and since first order arithmetics can be developed in ZF, Gödel’s Incompleteness Theorems apply to ZF (see, e.g., [33]). Thus, we know the following two facts:

- (1) ZF is either inconsistent or Gödel incomplete. That means: either we can derive a contradiction from the axioms of ZF, or there are statements in the language of ZF that are neither provable nor refutable from the axioms of ZF. This is the contents of the *First Incompleteness Theorem*, when applied to ZF.
- (2) ZF is not strong enough to prove its own consistency. This is the contents of the *Second Incompleteness Theorem*, when applied to ZF.

Gödel's work also shows us that we cannot do any better. Any first order theory (capable of first order arithmetics) must have the above two deficiencies.

3. CHOOSING SHOES AIN'T CHOOSING SOCKS — THE AXIOM OF CHOICE

Just as liberal feminists are frequently content with a minimal agenda of legal and social equality for women and 'pro-choice', so liberal (and even some socialist) mathematicians are often content to work within the hegemonic Zermelo-Fraenkel framework (which, reflecting its nineteenth century origins already incorporates the axiom of equality) supplemented only by the axiom of choice. But this framework is grossly insufficient for liberal mathematics, as was proven long ago by Cohen.

Alan Sokal, in [28] aka *Sokal hoax*

In this chapter we formulate *another* axiom of Set Theory, the so called *Axiom of choice*. By adding this axiom to the axioms of ZF we end up with the Set Theory denoted by

ZFC

and called *Zermelo-Fraenkel Set Theory with Choice*.

We start motivating the new axiom by introducing the following problem.⁴

3.1. Example (Choosing shoes and choosing socks). Suppose that we are facing the following two tasks:

(Shoes) We have a collection of pairs of shoes. Can we find a procedure that *chooses* a shoe out of any pair?

(Socks) We have a collection of pairs of socks. Can we find a procedure that *chooses* a sock out of any pair?

First of all, we have to acknowledge that tasks **(Shoes)** and **(Socks)** are different: a pair of shoes contains a *left* shoe and a *right* shoe, whereas a pair of socks contains *two* socks, and that is it.

Thus, we can answer the problem **(Shoes)** by the following procedure: choose (say) the *left* shoe out of any pair and we are done.

Task **(Socks)** is much more problematic: we can certainly choose one sock in every pair, if we have a *finite* collection of pairs. Why cannot we do the same trick when the collection of pairs of socks is *infinite*? The trouble lies in what we want to allow as a proof of the existence of a procedure. Most certainly, we should be able to give a *finite* argument that persuades us about the existence of such procedure. Therefore we cannot go on choosing one sock from a pair “for ever”. But *how* do we choose a sock from a pair simultaneously for all pairs out of an infinite collection?

The above example was no doubt a bit funny, but is it real mathematics? Let us see a “proper” mathematical example that will uncover the same phenomena we saw in Example 3.1.

3.2. Example (Splitting of surjections). Again, we are facing two tasks:⁵

- (S1)** A surjection $e : \mathbb{N} \longrightarrow \mathbb{N}$ is given. Can we find a function $m : \mathbb{N} \longrightarrow \mathbb{N}$ such that $e \cdot m = id$? That is, can we find $m : \mathbb{N} \longrightarrow \mathbb{N}$ such that the triangle

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{m} & \mathbb{N} \\ & \searrow id & \downarrow e \\ & & \mathbb{N} \end{array}$$

commutes?

⁴The wording of the problem is by Bertrand Russell.

⁵Let us comment on the terminology: a surjection $e : X \longrightarrow Y$ is said to *split*, if there exists $m : Y \longrightarrow X$ such that $e \cdot m = id$ holds. The mapping m is called a *splitting* of e .

(S2) A surjection $e : \mathbb{R} \longrightarrow \mathbb{N}$ is given. Can we find a function $m : \mathbb{N} \longrightarrow \mathbb{R}$ such that $e \cdot m = id$? That is, can we find $m : \mathbb{N} \longrightarrow \mathbb{R}$ such that the triangle

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{m} & \mathbb{R} \\ & \searrow id & \downarrow e \\ & & \mathbb{N} \end{array}$$

commutes?

Now, first of all, the above looks very much like mathematical questions. The tricky part seems to be that we know *nothing* about the mapping e in both (S1) and (S2), except that e is *surjective*.

Recall from [32] that general mappings of the form $f : B \longrightarrow C$ can be thought of as *colourings*: the set C is the set of *colours*, and $b \in B$ has colour $c \in C$, if $f(b) = c$. Moreover, the set

$$f^{-1}(c) = \{b \in B \mid f(b) = c\}$$

yields the set of all elements of B that have the *same* colour c . For a general mapping f , however, it can happen that there exists a colour $c \in C$ with an empty set $f^{-1}(c)$; no element of B has colour c . But such a thing *cannot happen*, if f is a *surjection*. Thus, if $f : B \longrightarrow C$ is a surjection, *every* colour $c \in C$ must be “painted on” at least one element $b \in B$. This is a fancy way of saying that, for a surjection $f : B \longrightarrow C$, the set

$$f^{-1}(c) = \{b \in B \mid f(b) = c\}$$

is *non empty*, for every $c \in C$. Moreover, for $c_1 \neq c_2$ in C , we have

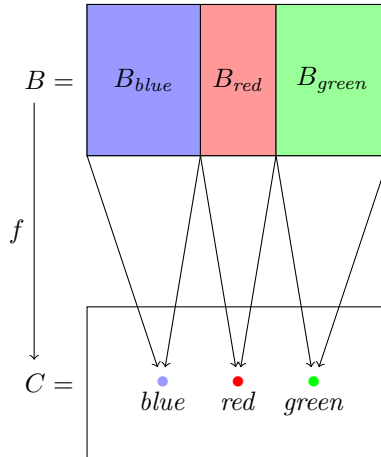
$$f^{-1}(c_1) \cap f^{-1}(c_2) = \emptyset$$

by the very fact that f is a function.

Hence, to give a surjection $f : B \longrightarrow C$, is to give a *partition*

$$\{B_c \mid c \in C\}, \quad B_c = \{b \in B \mid f(b) = c\}$$

of B into sets by grouping elements of colour c into the set B_c . See the following picture for $C = \{\text{blue}, \text{red}, \text{green}\}$:



Let us go back to tasks (S1) and (S2). In both cases, the given function e is surjective. Hence we can use the above colouring idea.

(S1) Since $e : \mathbb{N} \longrightarrow \mathbb{N}$ is a surjection, the above reasoning provides us with a partition

$$N_k, \quad k \in \mathbb{N}$$

of the set \mathbb{N} , where

$$N_k = \{x \in \mathbb{N} \mid e(x) = k\}$$

Every set N_k is non empty, since e is a surjection.

The question about $m : \mathbb{N} \longrightarrow \mathbb{R}$ with $e \cdot m = id$ is equivalent to the question: can we describe a procedure that *chooses* an element $m(k)$ in every N_k ? Is there any *distinguished* element in a general non empty set of natural numbers?

Yes, *there is*:⁶ every element of N_k must have a least element. Hurrah: the least element of N_k plays the rôle of the left shoe in a pair of shoes. We define

$$m(k) = \text{the least element of } N_k$$

and we are done: the equality $e \cdot m = id$ clearly holds.

(S2) Since $e : \mathbb{R} \rightarrow \mathbb{N}$ is a surjection, the above reasoning provides us with a partition

$$R_k, \quad k \in \mathbb{N}$$

of the set \mathbb{R} , where

$$R_k = \{x \in \mathbb{R} \mid e(x) = k\}$$

Every set R_k is non empty, since e is a surjection.

The question about $m : \mathbb{N} \rightarrow \mathbb{R}$ with $e \cdot m = id$ is equivalent to the question: can we describe a procedure that *chooses* an element $m(k)$ in every R_k ? Is there any *distinguished* element in a general non empty set of real numbers?

Alas, there is not. We are stuck. We would need someone to *point out a canonical choice* in every R_k . And this is exactly what *Axiom of choice* does.

3.3. Axiom (Axiom of choice). Every surjection $e : X \rightarrow Y$ splits, i.e., there exists $m : Y \rightarrow X$ such that $e \cdot m = id$.

3.4. Commentary. The above phrasing of Axiom of choice may come as a shock to any reader well-versed in reading books on Set Theory. We chose, in our opinion, the easiest formulation of the axiom and we explain now why it is equivalent to the “traditional” formulation.

The considerations in Example 3.2 show that the following pieces of information are equivalent:

- (1) A surjection $e : X \rightarrow Y$ with a splitting $m : Y \rightarrow X$.
- (2) A partition

$$X_y, \quad y \in Y$$

of the set X , together with an element

$$m(y) \in X_y$$

for every $y \in Y$.

Now we can see that our phrasing of AC is equivalent to the “classical” one:

Classical formulation of AC. Suppose $\{S_i \mid i \in I\}$ ⁷ is any family of non empty sets. Then there exists a *choice function* for $\{S_i \mid i \in I\}$, i.e., a function

$$c : \{S_i \mid i \in I\} \rightarrow \bigcup_{i \in I} S_i$$

such that

$$c(S_i) \in S_i$$

holds for all $i \in I$.

Indeed, given a family $\{S_i \mid i \in I\}$ of pairwise disjoint non empty sets, define

$$X = \bigcup_{i \in I} S_i, \quad Y = I, \quad e : X \rightarrow Y, \quad e(x) = i \text{ iff } x \in S_i$$

Then e is a surjection. Any splitting $m : Y \rightarrow X$ of e yields a choice function

$$c : \{S_i \mid i \in I\} \rightarrow \bigcup_{i \in I} S_i, \quad c(S_i) = m(i)$$

Hence our formulation of AC implies the classical one.

Conversely, given a surjection $e : X \rightarrow Y$, define

$$\{S_y \mid y \in Y\}, \text{ where } S_y = \{x \in X \mid e(x) = y\}$$

⁶Recall that *every* non empty subset of \mathbb{N} has a least element. We will prove this fact formally in a later set on notes.

⁷Observe that, without loss of generality, the sets S_i can be considered to be pairwise disjoint. If they are not, simply pass to a new family $\{\{i\} \times S_i \mid i \in I\}$, whose members *are* pairwise disjoint. It is easy to see that $\{S_i \mid i \in I\}$ has a choice function if and only if $\{\{i\} \times S_i \mid i \in I\}$ has a choice function.

The classical formulation of AC provides us with a choice function

$$c : \{S_y \mid y \in Y\} \longrightarrow \bigcup_{y \in Y} S_y, \quad c(S_y) \in S_y \text{ for all } y \in Y$$

This allows us to define a splitting $m : Y \longrightarrow X$ of e by putting

$$m(y) = c(S_y)$$

We proved that the classical formulation of AC is equivalent to our formulation.

3.5. Remark (Relationship of AC to the axioms of ZF). What is the relationship of AC to the axioms ZF? It is reminiscent of the relationship of the *parallel's postulate* to the rest of the axioms of planar geometry. More in detail:

- (1) In 1940 Kurt Gödel showed that if ZF is consistent, so is ZFC. See [13]. Recall, however, that the consistency of ZF cannot be proved within ZF by *Gödel's Incompleteness Theorems*, see Remark 2.25 and, e.g., [33].
- (2) In 1963 Paul Cohen proved that AC was independent of ZF, so it could not be proved from ZF. See [7] and [8] for the summary, or the book [9] for the full account.

3.6. Remark (The continuum hypothesis CH). There are very many statements that are undecidable in ZF. Mostly they are quite specialised statements of interest to “pure” Set Theory. One of the statements can be encountered quite often in “mainstream” mathematics, however. It is, in fact, the statement that has initiated set theory. Namely, the following seemingly quite innocent question was asked by Georg Cantor already in the 1870's:

How many real numbers are there?

In our notation, the question can be rephrased as follows:

Which cardinal is $\text{card}(\mathbb{R})$?

Cantor's diagonal slash (see, e.g., [33]) shows that

$$\text{card}(\mathbb{R}) > \text{card}(\mathbb{N}) = \aleph_0$$

holds. This means that the set \mathbb{R} is *uncountable*. The definition of reals by Dedekind cuts (see, e.g., [31]) can be used to show even that

$$2^{\aleph_0} = \text{card}(P(\mathbb{N})) = \text{card}(\mathbb{R}) > \text{card}(\mathbb{N}) = \aleph_0$$

holds. Unfortunately, the inequality $2^{\aleph_0} > \aleph_0$ is (almost) the only thing one can prove about $\text{card}(\mathbb{R})$ in ZF (or, even in ZFC).

The equality

$$2^{\aleph_0} = \aleph_1 \quad (\text{where } \aleph_1 \text{ is the smallest cardinal larger than } \aleph_0)$$

is *independent* of ZF. The above equality is called the *Continuum Hypothesis*⁸ (abbreviated to CH). Accordingly, the cardinal 2^{\aleph_0} is often denoted by the *Fraktur* letter \mathfrak{c} :

$$\mathfrak{c}$$

and thus one often sees CH written as the equality

$$\mathfrak{c} = \aleph_1$$

The *consistency* of CH with axioms of ZF was proved by Kurt Gödel in [13]. That CH is *independent* of the axioms of ZF was proved by Paul Cohen in [7] and [8].

There is also the *Generalised Continuum Hypothesis* (abbreviated to GCH), stating that the equality

$$2^\kappa = \kappa^+ \quad (\text{where } \kappa^+ \text{ is the smallest cardinal larger than } \kappa)$$

holds for every infinite cardinal κ .

It was proved in 1947 by Waław Sierpiński [27] that, surprisingly, GCH implies AC.

⁸The name is derived from calling the real line *continuum*.

4. WHY AC IS REALLY USEFUL IN MATHS AND WHY IT ALSO BRINGS NIGHTMARES

The axiom of choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?

Jerry Bona, quoted in [18]

The axiom of choice can be stated in very many equivalent ways. In fact, the book [22] lists heaps of statements that are equivalent to AC in ZF. We will list only few such statements. We divide them into “set theoretical” and “practical for the working mathematician”.

4.1. Theorem (The set theoretical equivalents of AC). *In ZF, the following statements are equivalent:*

- (1) **Axiom of choice.**
- (2) **The well-ordering principle.** *On every set x there exists a linear order $<$, such that every non empty subset of x has the least element w.r.t. $<$.*
- (3) **Zorn's lemma.** *Suppose (X, \sqsubseteq) is a poset with the following property:
Every linearly ordered subset of (X, \sqsubseteq) has an upper bound.
Then (X, \sqsubseteq) has a maximal element.*
- (4) **Tarski's square theorem.** *For every infinite cardinal κ , the equality $\kappa \cdot \kappa = \kappa$ holds.*

4.2. Theorem (Equivalents of AC for the working mathematician). *In ZF, the following statements are equivalent:*

- (1) **Axiom of choice.**
- (2) **The cartesian product theorem.** *The cartesian product of any system of non empty sets is non empty.*
- (3) **Finiteness of sets.** *A set X is finite iff it either has at most one element or has smaller cardinality than $X \times X$.*
- (4) **The trichotomy theorem.** *For any pair X, Y of sets, one of $\text{card}(X) < \text{card}(Y)$, $\text{card}(X) = \text{card}(Y)$, $\text{card}(X) > \text{card}(Y)$ holds.*
- (5) **The vector space basis theorem.** *Any vector space over any field has a basis.*
- (6) **Tykhonoff theorem.** *A product of any family of compact spaces is a compact space.*
- (7) **Binary relations contain functions.** *Suppose $R \subseteq A \times B$ is a binary relation, and put*

$$A_0 = \{a \in A \mid \text{there exists } b \in B \text{ such that } (a, b) \in R\}$$

Then there exists $f \subseteq R$ such that $f : A_0 \longrightarrow B$.

In the rest of this chapter we indicate why AC stirs quite a lot of discussions even today. The reason is: AC can be used to show things we would rather not know about. On the other hand, the world with the negation of AC is also quite disturbing.

Horrifying results holding under AC. The above results show that AC is equivalent to “nice” theorems in ZF. However, AC can also bring results that are highly unintuitive. We present two such results: a non-measurable subset of reals and Banach-Tarski Paradox.

4.3. Theorem (Giuseppe Vitali [34] (in ZFC)). *There cannot exist a function*

$$\mu : B(\mathbb{R}) \longrightarrow [0; +\infty)$$

where $B(\mathbb{R})$ denotes the set of all bounded subsets of \mathbb{R} with the following properties:

- (1) **Nontriviality:** *there is X in $B(\mathbb{R})$ such that $\mu(X) > 0$.*
- (2) **Translation-invariance:** *for every translation $t : \mathbb{R} \longrightarrow \mathbb{R}$ and every X in $B(\mathbb{R})$ the equality $\mu(X) = \mu(t[X])$ holds.*
- (3) **σ -additivity** *For every countable family $(X_i)_{i=0}^{+\infty}$ of pairwise disjoint bounded subsets of \mathbb{R} the equality $\mu(\bigcup_{i=0}^{+\infty} X_i) = \sum_{i=0}^{+\infty} \mu(X_i)$ holds.*

4.4. Commentary. Let us decypher Vitali's result. What would the existence of a function

$$\mu : B(\mathbb{R}) \longrightarrow [0; +\infty)$$

satisfying (1)–(3) really mean? Well, nothing else than a “well-behaved” notion of “volume” of bounded subsets of \mathbb{R} . Thus, for a bounded set $X \subseteq \mathbb{R}$, the non-negative real number

$$\mu(X)$$

is the “volume” of the set X . In this light, conditions (1)–(3) mean the following:

- (1) The “volume” cannot be zero for all bounded sets X .
- (2) If we take a bounded set X and “shift” it by a translation t to a set $t[X]$, then the “volumes” of X and $t[X]$ are the same.
- (3) If we have a countable family of pairwise disjoint sets, then the “volume” of its union is simply the sum of “volumes” of individual sets.

Guiseppe Vitali showed that no such notion of a “volume” can exist. Thus, there must exist a bounded set $X_0 \subseteq \mathbb{R}$ that cannot be assigned a “volume”.

In mathematical language, it is much more customary to use the word *measure* than “volume”, and the bounded set X_0 above is called *non-measurable*. In fact, although this is a highly nonintuitive result, it is at the core of various problems concerning the *theory of integration* or *probability theory*. See, e.g., [17] and [6].

4.5. Theorem (Banach-Tarski [2] (in ZFC)). *A unit ball in \mathbb{R}^3 can be cut into finitely many pieces that can be rearranged to form two disjoint unit balls in \mathbb{R}^3 .*

4.6. Commentary. Having absorbed the fact about non-measurable subsets of \mathbb{R} , we suspect that the above result would be of similar sort. This is true: some of the finitely many pieces of the sphere are non-measurable, this time in \mathbb{R}^3 . Hence if we rearrange them differently, we can “duplicate” the unit sphere. The above result by Stefan Banach and Alfred Tarski is therefore *no* paradox: its name is only historical.

Horrifying results holding under the negation of AC. We present just two quite counterintuitive results that happen when we add the negation of AC to the axioms of ZF.

4.7. Theorem. *There is a model of ZF and $\neg AC$, where \mathbb{R} is a union of countably many countable sets.*

4.8. Commentary. The above result *does not* say that \mathbb{R} is a countable set! It only states that a certain uncountable set (namely, the set \mathbb{R} of all reals) can be expressed as follows:

$$\mathbb{R} = \bigcup_{i=0}^{+\infty} S_i \quad \text{where every } S_i \subseteq \mathbb{R} \text{ is a countable set}$$

4.9. Theorem. *There is a model of ZF and $\neg AC$, where there exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with the following property*

$$\lim_{n \rightarrow +\infty} f(x_n) = f(a), \text{ whenever } \lim_{n \rightarrow +\infty} x_n = a$$

but the function f is not continuous at a .

4.10. Commentary. First recall that *continuity* of $f : \mathbb{R} \longrightarrow \mathbb{R}$ at $a \in \mathbb{R}$ is the validity of the formula

$$\forall \varepsilon > 0 \exists \delta > 0 \left(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon \right)$$

If f is continuous at a and AC holds, then we can indeed replace the above formula by saying that $\lim_{n \rightarrow +\infty} f(x_n) = f(a)$, whenever $\lim_{n \rightarrow +\infty} x_n = a$ holds. This is called *Heine Theorem* in calculus. The theorem is no longer true when AC does not hold.

5. FINAL REMARKS

A brief biography of Ernst Zermelo. *Ernst Friedrich Ferdinand Zermelo* was born on 27 July 1871 in Berlin, Germany. Zermelo's father was a college professor and he encouraged young Ernst academic interests.



After the studies at Gymnasium in Berlin, Zermelo started studying mathematics, physics and philosophy at the universities of Berlin, Halle and Freiburg. He received his PhD in 1894 at the University of Berlin with a thesis in theory of variations. He remained in Berlin, began the work on the habilitation, and became an assistant to Max Planck.

In 1897 Zermelo went to Göttingen, finished his habilitation (in hydrodynamics!) and was appointed a lecturer. Zermelo began to work on problems of set theory under Hilbert's influence. In 1902 Zermelo published his first work concerning the addition of transfinite cardinals.

Zermelo began to axiomatise set theory in 1905; in 1908, he published his results in [35].

In 1910, Zermelo left Göttingen for Zürich University. His health worsened but he was helped by the award of a prize of 5000 marks for his contributions to set theory.⁹ Zermelo had to resign from his Zürich position for health reasons in 1916. After having stayed in Schwarzwald for ten years, Zermelo was appointed to an honorary position at the University of Freiburg in 1926. He resigned from this position in 1935, this time

for political reasons: Zermelo disapproved of Adolf Hitler's regime. Zermelo was reinstated to his honorary position in Freiburg after the end of WWII.

Ernst Zermelo died on 21 May 1953 in Freiburg im Breisgau, Germany.

A brief biography of Abraham Fraenkel. *Abraham Halevi (Adolf) Fraenkel* was born on 17 February 1891 in München, Germany. He was known as Adolf until later in his life when he started call himself Abraham (after leaving Germany for Mandatory Palestine in 1933).

Fraenkel studied at universities of München, Marburg, Berlin and Breslau. He received his PhD in 1914 for his thesis on divisors of zero and decomposition of rings. Fraenkel immediately started on his habilitation, but he received a call to the army. Fraenkel spent WWI serving mostly in medical corps. Even when in the army, Fraenkel collected enough material to write a habilitation thesis on the problem concerning simple extensions of decomposable rings in 1915. Due to war, he was not able to go to Germany to defend the thesis and to give the inaugural lecture. Fraenkel fell ill in 1916, while serving in Serbia, and he managed to defend his thesis and become a Privatdozent at the University of Marburg.

After WWI, Fraenkel started teaching at the University of Marburg and published the book [10] on Set Theory.¹⁰ He also got married at that time. Since the beginning of 1920's Fraenkel started to work in Set Theory and its axiomatics rather seriously. Fraenkel was promoted to full professor in Marburg in 1922.

Fraenkel chose to accept a position at the Hebrew University of Jerusalem in 1929. There he spent the rest of his career.

Abraham Fraenkel died on 15 October 1965 in Jerusalem, Israel.



⁹The prize was awarded by the initiative of David Hilbert to help Zermelo improve his health. Yet another instance of Hilbert's nobility.

¹⁰In the book's introduction, Fraenkel says that the only prerequisites for reading the book are interest and patience. He also says that this is what he required from his fellow soldiers when he kept introducing them to Set Theory during the war.

Remarks on the literature. The book [14] by Derk Goldrei is an absolutely fantastic book to learn Set Theory by yourself.¹¹ The book contains loads of exercises with detailed hints.

- (1) A good overview of the history of Set Theory (and of approaches different from ZF) is in the book [12] by Abraham A. Fraenkel, Yehoshua Bar-Hillel and Azriel Levy. An almost standard textbook of Set Theory is the 750-pages long treatise [16] by Thomas Jech. In Czech language, the book [30] by Petr Štěpánek is available.
- (2) An account of the history and mathematics of AC is given by Gregory H. Moore in the book [20]. A detailed study of AC is also given by Thomas Jech in [15].
- (3) A thorough review of themes relevant to Set Theory (Model Theory, Recursion Theory and Proof Theory), along with certain themes of Set Theory, is given in a thick volume [3] edited by Jon Barwise.
- (4) If you want to learn *how* independence of axioms is proved, then you are looking for the book [19] by Kenneth Kunen.
- (5) We mentioned that, sometimes, ZF *without* Axiom of regularity is a useful theory. The book [4] by Jon Barwise and Larry Moss is an excellent introduction into the intricacies of non-wellfounded Set Theory.
- (6) The abuse of modern mathematics (set theory, logic, topology, etc.) and modern physics (quantum theory, relativity theory, etc.) in the work of some postmodern authors is criticised in the very good book by Alan Sokal and Jean Bricmont [29].¹² In fact, the book appeared as a reaction to the fact that a nonsensical paper by Alan Sokal, peppered with quotations of the abuses, was accepted and published [28].

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¹¹Beware: Axiom of regularity contains a typo!

¹²One would want to say that, in some cases, not only that the Emperor has no clothes but that he also has a bad rash.

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