DEDEKIND'S CONSTRUCTION OF THE REALS

NOTES FOR THE "REAL MATHS": PART 01

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> Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.
> *Leopold Kronecker* (7 December 1823 —- 29 December 1891)

A brief synopsis. In Chapter 1 we introduce the *historical background* of Dedekind's construction. What the system of real numbers *should intuitively look like* is summarised in Chapter 2. We start with a recollection of various algebraic notions: *fields* and *ordered fields* Chapter 3. We *define* the reals as a *Dedekind complete field* in Chapter 4. In Chapter 5 we construct the field of *rationals* from the integers. This is an instance of a more general construction called a *field of fractions for an integral domain*. We develop the *completion by cuts* for any poset in Chapter 6 and in Chapter 7 we apply the construction to the rational numbers to obtain the reals. Chapter 8 is devoted to the study of *sequential completeness* for fields. Finally, in Chapter 9, we mention some other constructions of the reals and their equivalence. As a result, given a set of integers, we will be able to say in an *essentially unique* way what the reals are.

We will thus fulfill the Leopold Kronecker motto above: the integers were made by Our Lord, the rest is the work of Man.

1. Let us meet Prof Richard Dedekind

> Für mich war damals das Gefühl der Unbefriedigung ein so überwältigendes, dass ich den festen Entschluss fasste, so lange nachzudenken, bis ich eine rein arithmetische und völlig strenge Begründung der Principien der Infinitesimalanalysis gefunden haben würde... Dies gelang mir am 24. November 1858,... aber zu einer eigentlichen Publikation konnte ich mich nicht recht entschliessen, weil erstens die Darstellung nicht ganz leicht, und weil ausserdem die Sache so wenig fruchtbar ist.

> At that point, my sense of dissatisfaction was so strong that I firmly resolved to start thinking until I should find a purely arithmetic and absolutely rigorous foundation of the principles of infinitesimal analysis... I achieved this goal on 24 November 1858,... but I could not really decide upon a proper publication, because, firstly, the subject is not easy to present, and, secondly, the material is not very fruitful.

Richard Dedekind in [6]

Julius Wilhelm Richard Dedekind was born in the city of Braunschweig in Lower Saxony, Germany, on 6 October 1831. He started his university studies in Braunschweig but in 1850 he moved to the famous University of Götingen, where he became the last student of Karl Friedrich Gauss.[1] He graduated from Götingen in 1852 with a thesis *Über die Theorie der Eulerschen Integral*. Then Dedekind went to the University of Berlin to habilitate himself.

After habilitation, in 1854, Dedekind returned to Götingen as a Privatdozent. In 1858 he began to teach calculus at the Polytechnik in Zürich.[2] This is the time when he developed the *construction of the reals* by the technique that is called *Dedekind cuts* nowadays. He published the construction in [6], after his return to Braunschweig Technical University in 1872. Later, in 1888, Dedekind published [7] where he, very exactly and in the axiomatic way, had introduced the *natural numbers*.

Dedekind was rather mathematically active mostly in abstract algebra. He proposed the words *der Ring* (the ring), *der Körper* (the field) and *die Einheit* (the unit) for use in mathematics.

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[1] In Götingen, Dedekind befriended a fellow student (and also a student of Gauss), named Georg Friedrich Bernhard Riemann. Does the name ring a bell?

[2] Now ETH Zürich.

Dedekind met *Georg Ferdinand Ludwig Phillip Cantor* in 1872 and became quite enthusiastic about Cantor's *Set Theory*. Dedekind used sets in his work, intuitively, before Cantor. Dedekind and Cantor kept exchanging letters for the most of the 1870's and in these letters they discussed and proved important results about *finite* and *infinite* sets.

Richard Dedekind died in Braunschweig on 12 February 1916.

2. What properties should the system of real numbers have?

In forming domains for "computing", the sequence

$$\mathbb{N} \rightsquigarrow \mathbb{Z} \rightsquigarrow \mathbb{Q} \rightsquigarrow \mathbb{R}$$

of "bigger and bigger sets of numbers" seems to be agreed upon universally. One starts with an ordered set $\mathbb{N}$ of natural numbers (together with addition and multiplication that respects the order) and proceeds to richer and richer structures: the set $\mathbb{Z}$ of all integers, the set $\mathbb{Q}$ of all rationals, up to the set $\mathbb{R}$ of all real numbers.

At each step of the sequence, we expect to enrich the algebraic structure and the properties of the order structure of $\mathbb{N}$.

The passage from $\mathbb{N}$ to $\mathbb{Z}$ is caused by the fact that we cannot solve certain problems posed in $\mathbb{N}$. For example, one can prove that there is no element x in $\mathbb{N}$, solving the problem

$$7 + x = 3$$

Thus, we can produce $\mathbb{Z}$ as the set where problems like above will have a solution. Hence

$$\mathbb{Z} = \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\}$$

where, for $n \in \mathbb{N}$, the integer $-n$ is called the *opposite* of n, with the property $n + (-n) = 0 = (-n) + n$. Of course, one has to extend the algebraic and order properties from $\mathbb{N}$ to $\mathbb{Z}$ in a *conservative* way. The latter means that, when restricted to naturals, the algebraic and order properties of integers are the same as they were on $\mathbb{N}$.

Analogously, the passage from $\mathbb{Z}$ to $\mathbb{Q}$ can be motivated by the impossibility to solve problem like

$$2x = -3$$

in $\mathbb{Z}$. This time it is less clear how the definition of $\mathbb{Q}$ should look like. For example, the equality

$$\frac{2}{4} = \frac{1}{2}$$

should hold in $\mathbb{Q}$. Thus, we cannot simply say that $\mathbb{Q}$ consists of fractions, we have to be more careful: there is a certain identification going on. This identification can possibly muddle the extension of the order relation as well. We will address these problems later.

When passing from $\mathbb{Q}$ to $\mathbb{R}$, there seem to be further problems:

(1) There should be "no gaps" in the set of real numbers. This clearly concerns the order. But what does that mean, really? The requirement seems to be supported by the intuitive fact that real numbers should correspond to points on a line.

Could "no gaps" mean that between any two distinct reals x and y there is another one? In that sense, there are "no gaps" between any two distinct rationals x and y. Indeed, for

$$x < y$$

define the rational number

$$z = \frac{x + y}{2}$$

Then $x < z < y$ holds.

(2) Clearly, "no gaps" must mean something different than the above. There is an ancient *Method of Exhaustion* that will give us a clue. The method originates in Ancient Greece in 500 BC and it was made rather rigorous by Eudoxus of Cnidus.[3] Euclid,[4] in Proposition XII.2 of his book *Elements* [10], uses the method to prove, for example, that the area A of a circle is proporitonal to the square of its

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[3]Eudoxus of Cnidus (cca 390 BC – cca 337 BC) was an ancient Greek mathematician. All of his work is lost, some was preserved and made rigorous by Euclid in Book V of *Elements* [10]. See, for example, 9.2 below.

[4]Euclid lived around 300 BC, for most of his life in Alexandria, presumably. His book *Elements* [10] gave rise to the axiomatic method in mathematics.

diameter d (the constant of proportion is, of course, $\pi/4$ since $A = \pi/4 \cdot d^2$). The method is used as follows: given a circle C, insribe a sequence P_n of polygons into it in such a way that the inequalities

$$\text{area}(P_1) < \text{area}(P_2) < \ldots < \text{area}(P_n) < \ldots$$

hold. The sequence $(\text{area}(P_n))_n$ of areas is bounded from above by $\text{area}(C)$. The Method of Exhaustion says then that $\text{area}(C)$ is a "limit" of the sequence $(\text{area}(P_n))_n$.

Ideas like the above lie at the very core of calculus. Limits of sequences were starting to be understood well in Dedekind's times. And it is precisely the Method of Exhaustion that will be the crucial part of Dedekind's construction of the reals.

We will formulate precise definitions of the notions above in Section 3. We will use the mathematical language of modern algebra. Apart from the terminology, we will follow the ideas of Richard Dedekind very closely.

We will skip the initial passage

$$\mathbb{N} \rightsquigarrow \mathbb{Z}$$

from the naturals to the integers. That is, we will *assume* that we know all the necessary properties of $\mathbb{N}$ and $\mathbb{Z}$ and we will focus on the passages

$$\mathbb{Z} \rightsquigarrow \mathbb{Q} \rightsquigarrow \mathbb{R}$$

in this text.

3. Fields and ordered fields

In this chapter we collect the terminology that we will find necessary later, when constructing the reals. The terminology is a standard part of modern algebra and it can be found in any textbook, see, for example, [4].

We will start with a precise formulation of a set of "numbers" equipped with addition and multiplication that "behave as expected". The resulting structure is called a *commutative ring with a unit*.

3.1. Definition (Commutative ring with a unit). A *commutative ring with a unit* is given by a set $\mathbb{K}$ (a typical member of $\mathbb{K}$ will be denoted by r and thought of as a "number") that is equipped with two functions

$$+ : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K} \quad (\text{read: } \textit{the addition}), \quad \cdot : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K} \quad (\text{read: } \textit{the multiplication}),$$

that are subject to the following two axioms:

(1) Axioms for addition.
  (a) The existence of zero.
      There is an element 0 in $\mathbb{K}$ such that, for all r in $\mathbb{K}$, such that $r + 0 = 0 + r = r$ holds for all r in $\mathbb{K}$. The element 0 is called *zero*.
  (b) The commutativity of addition.
      For all r, s in $\mathbb{K}$, the equality $r + s = s + r$ holds.
  (c) The associativity of addition.
      For all r, s, t in $\mathbb{K}$, the equality $r + (s + t) = (r + s) + t$ holds.
  (d) The existence of an additive inverse.
      For every r in $\mathbb{K}$ there is a unique s such that the equality $r + s = s + r = 0$ holds. The uniquely determined s is called the *additive inverse* to r and it is denoted by $-r$.
(2) Axioms for multiplication.
  (a) The existence of unit.
      There is an element 1 such that the equalities $1 \cdot r = r = r \cdot 1$ holds for all r in $\mathbb{K}$. The element 1 is called *unit*.
  (b) The commutativity of multiplication.
      For all r, s in $\mathbb{K}$, the equality $r \cdot s = s \cdot r$ holds.
  (c) The associativity of multiplication.
      For all r, s, t in $\mathbb{K}$ the equality $r \cdot (s \cdot t) = (r \cdot s) \cdot t$ holds.
(3) The distributive law.
      For all r, s, t the equality $r \cdot (s + t) = (r \cdot s) + (r \cdot t)$ holds.

3.2. Remark. We will make the usual *notational conventions* that will make our life easier:

(1) We assume that multiplication *binds stronger* than addition. Thus, we write, for example

$$r \cdot s + t$$

instead of the cumbersome $(r \cdot s) + t$.

(2) We will often omit the dot notation of the multiplication and write, for example

$$rs$$

instead of $r \cdot s$.

(3) We use associativity of addition and multiplication to omit parentheses. Thus, we write, for example

$$r_1 + r_2 + r_3 + r_4 + r_5 \quad \text{or} \quad rst$$

instead of properly parenthesised expressions.

3.3. Task for you (easy). *Assume without proof that the set $\mathbb{Z}$, with the usual multiplication and addition, is a commutative ring with a unit. Prove that, for any natural $m \geq 2$, the set*

$$\mathbb{Z}_m = \{0, 1, 2, \ldots, m - 1\}$$

together with the addition modulo m and multiplication modulo m is a commutative ring with a unit.

HINTS FOR THE PROOF. You will make use of the theorem of division with a remainder for integers.   ∎

A *field* is an abstraction of the *algebraic* properties that the collection of all real numbers should have. The notion of a field builds upon the notion of a commutative ring with a unit. Observe that *any field is automatically a commutative ring with a unit*. The precise definition is as follows:

3.4. Definition (Field). A *field* is a commutative ring with a unit, where the following condition

(IT) The invertibility test.

For every r in $\mathbb{F}$, the inequality $r \neq 0$ holds if and only if there is a unique r^{-1} such that the equalities $r \cdot r^{-1} = 1 = r^{-1} \cdot r$ holds. The uniquely determined r^{-1} is called the (multiplicative) *inverse* of r.

holds.

3.5. Task for you (easy). *Prove that the examples below are examples of fields:*
(1) $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ *with the usual definitions of addition modulo 5 and multiplication modulo 5.*
(2) *The set $\mathbb{Z}_p = \{0, 1, 2, 3, 4, \ldots, p - 1\}$ with the usual definitions of addition modulo p and multiplication modulo p, where p is a prime number.*

HINTS FOR THE PROOF. Pretty straightforward. If in doubt, consult Chapter 1 in [21].   ∎

3.6. Task for you (easy). *Prove that, in every field $\mathbb{F}$, the following properties hold:*
(1) $0 \neq 1$.
(2) $r \cdot 0 = 0$.
(3) $(-r) \cdot s = -(r \cdot s)$.
(4) $r \cdot s = 0$ *if and only if $r = 0$ or $s = 0$.*
(5) *Every equation $a + x = b$, where a, b are in $\mathbb{F}$, has a unique solution.*
(6) *Every equation $a \cdot x = b$, where $a \neq 0$, b are in $\mathbb{F}$, has a unique solution.*

HINTS FOR THE PROOF. Use Definition 3.4. If in doubt, consult Chapter 1 in [21].   ∎

3.7. Definition (Ordered commutative ring with a unit). A commutative ring with a unit $\mathbb{K}$ is called *ordered*, if there is a subset $\mathbb{K}_+$ in $\mathbb{K}$ (the so-called *positive cone of $\mathbb{K}$*) with the following properties:

(1) For every r in $\mathbb{K}$ we have either $-r \in \mathbb{K}_+$, or $r = 0$, or $r \in \mathbb{K}_+$.
(2) If $r \in \mathbb{K}_+$ and $s \in \mathbb{K}_+$, then $r + s \in \mathbb{K}_+$ and $rs \in \mathbb{K}_+$ hold.

In every ordered commutative ring with a unit we denote by

$$r < s$$

the fact that $s - r \in \mathbb{K}_+$.

We will also use the notation

$$r \leq s$$

to denote that $r < s$ or $r = s$ holds.

3.8. Remark (An ordered field). Since every field is a commutative ring with a unit, we can speak of *ordered fields*. We retain the notation $\mathbb{F}_+$ for the positive cone of an ordered field $\mathbb{F}$. We also use the symbols $<, \leq$, etc. for ordered fields.

3.9. Task for you (easy). *Prove that, in every ordered commutative ring with a unit $\mathbb{K}$, the following properties hold:*

(1) $r < r$ *holds for no $r \in \mathbb{K}$.*
(2) *If $r < s$ and $s < t$, then $r < t$.*
(3) *Either $0 = 1$ or $0 < 1$ holds.*
(4) *If $r < s$ and $r' < s'$, then $r + r' < s + s'$.*
(5) *If $r < s$ and $0 < a$, then $ar < as$.*
(6) *If $r < s$, then $-s < -r$.*
(7) *If $r \neq 0$, then $0 < r^2$.*
(8) *If $0 < r$ and r^{-1} exists, then $0 < r^{-1}$.*
(9) *If $r < s$, $0 < r$, $0 < s$ and both r^{-1} and s^{-1} exist, then $s^{-1} < r^{-1}$.*

HINTS FOR THE PROOF. Use Definition 3.7. If in doubt, consult Chapter 1 in [21] and adapt the proofs therein for rings. ■

We may be shocked by the fact that $0 = 1$ can hold in a ring. In fact, the equality $0 = 1$ holds iff the ring is *trivial*, as the next result shows. We will often exclude trivial rings from our considerations.

3.10. Task for you (easy). *Prove that if $0 = 1$ holds in a commutative ring with a unit $\mathbb{K}$, then $0 = r$ for all r in $\mathbb{K}$.*

HINTS FOR THE PROOF. First prove that $0 \cdot r = 0$ for every r in $\mathbb{K}$. Since also $1 \cdot r = r$ holds for all r in $\mathbb{K}$, we can deduce $0 = r$ from the equality $0 = 1$ by multiplying it by r. ■

4. THE DEFINITION OF REALS

We come to the last notion that we need to introduce: that of a *Dedekind complete field*. The definition is motivated by the *Method of Exhaustion* mentioned in Chapter 2.

4.1. Definition (Dedekind completeness). Let $\mathbb{F}$ be an ordered field. We say that $\mathbb{F}$ is *Dedekind complete*, if the *supremum* $\sup S$ of S exists for every nonempty subset S that is bounded from above. More in detail:

(1) A set $S \subseteq \mathbb{F}$ is *bounded from above*, if there exists r in $\mathbb{F}$ such that $x \leq r$ for all $x \in S$. Such an r is also called an *upper bound* of S.
(2) We say that s is a *supremum* of S and we denote this fact by $s = \sup S$, if the following conditions hold:[5]
    (a) s is an upper bound of S.
    (b) If r is any upper bound of S, then $s \leq r$.

We can now define the reals "synthetically". That is, we say that the set of reals constitutes a field, satisfying certain properties.

4.2. Definition (The reals). We say that a field $\mathbb{F}$ is a *field of reals*, provided that it is a Dedekind complete field.

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[5]The conditions say that s is the *least upper bound* of S.

4.3. **Remark.** There is, of course, a problem (well, two problems):

(1) Does there exist an ordered, Dedekind complete field?
(2) If a field of reals exists, is it in a certain sense unique?

These two questions will be addressed in the rest of this text. Namely, in Chapter 7 we provide a construction of the reals via *Dedekind cuts* on the ordered field $\mathbb{Q}$ of rational numberes and in Chapter 9 we show that the reals are determined *essentially uniquely* by Definition 4.2.

5. The field of rationals as a field of fractions

In the current chapter we introduce the set $\mathbb{Q}$ of *rational numbers* as arising from the *integral domain* $\mathbb{Z}$. We prove that $\mathbb{Q}$ is an *ordered field* w.r.t. "naturally defined" operations of addition and multiplication. The passage

$$\mathbb{Z} \rightsquigarrow \mathbb{Q}$$

from $\mathbb{Z}$ to $\mathbb{Q}$ is an instance of a general procedure of constructing a *field* $\mathrm{Frac}(\mathbb{K})$ *of fractions* out of *an integral domain* $\mathbb{K}$. What are integral domains? They are simply "fields where axiom of invertibility is replaced by something slightly weaker". See the *integral domain axiom* in the following definition.

5.1. **Definition** (**Integral domain**). An *integral domain* is a commutative ring with a unit $\mathbb{K}$, where the property

(ID) The integral domain axiom.

If $rs = 0$, then $r = 0$ or $s = 0$.

holds.

5.2. **Task for you (easy).** *Prove that:*
(1) *The set $\mathbb{Z}$ of all integers is an integral domain that is not a field.*
(2) *Any field is an integral domain.*
(3) *$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, with the operations of addition modulo 6 and multiplication modulo 6, is not an integral domain.*

Hints for the proof. The proof is easy.   ∎

5.3. **Example** (**Fractions a bit differently**). In giving the set $\mathbb{Q}$ of rational numbers one usually stumbles upon the fact that fractions are not determined uniquely. A common way to get over it is to say that in a fraction

$$\frac{n}{d}$$

the numerator n is an *integer number*, the denominator d is a *positive integer*, and n and d are *relatively prime*. The latter means that the greatest common divisor of n and d is number 1.

Sometimes one even speaks about the fraction with the above properties as about the rational number in a *canonical form*. Thus, for example, we have

$$\frac{-3}{9} = \frac{3}{-9} = \frac{1}{-3} = \frac{-1}{3}$$

where only the last fraction is in the canonical form.

In the rest of this section we will approach fractions rather differently:

A fraction will be a *set* of pairs (n, d) that give the same value when evaluated as n divided by d.

Thus, for example, we will have

$$\frac{1}{-3} = \{\dots, (-3, 9), (3, -9), (-1, 3), \dots\}$$

Obviously, the set on the right is meant to be the set of *all* pairs that give the same value as a fraction. Thus, we may say that the expression

$$\frac{1}{-3}$$

is given by choosing the pair $(1, -3)$ as a *representative*. We could have chosen a different representative of the same fraction, of course. For example, we have

$$\frac{-3}{9} = \{\ldots, (-3, 9), (3, -9), (-1, 3), \ldots\}$$

We will now make the idea of Example 5.3 more precise. Fractions will be *equivalence classes* of a certain equivalence relation on pairs of elements of an integral domain.

5.4. **Remark** (**A step aside: Equivalence relations, etc.**). Recall that a relation E on a set X is called an *equivalence relation*, if the following three properties hold:

(1) The relation E is *reflexive*: for every x in X we have $x \, E \, x$.
(2) The relation E is *symmetric*: for every x, y in X, if $x \, E \, y$ holds, then so does $y \, E \, x$.
(3) The relation E is *transitive*: for every x, y, z in X, if $x \, E \, y$ and $y \, E \, z$ hold, then so does $x \, E \, z$.

An archetype equivalence relation is the *identity relation* Δ_X on every set X: $x \, \Delta_X \, y$ holds iff $x = y$. It is easy to see that Δ_X is reflexive, symmetric and transitive.

A good slogan for an equivalence relation E on X is that the relation E allows us to see two elements x and y as "the same", whenever $x \, E \, y$ holds. Thus, an equivalence relation E on X allows us to "gather together objects that are the same". More formally, for every element x of X we form a set

$$[x]_E = \{y \in X \mid x \, E \, y\}$$

of all elements y that are "the same" as x. The set $[x]_E$ is called an *equivalence class, represented by x*. It is then easy to prove the following facts:

(1) Every $[x]_E$ is a nonempty subset of X.
(2) The following holds:

$$[x]_E \cap [x']_E = \begin{cases} \emptyset, & \text{if } x \, E \, x' \text{ does not hold,} \\ [x]_E = [x']_E, & \text{if } x \, E \, x' \text{ does hold.} \end{cases}$$

(3) The equality

$$X = \bigcup_{x \in X} [x]_E$$

holds.

A family of subsets of X with the above three properties are called a *partition* of X. For the record: a family $\{S_i \mid i \in I\}$ of subsets of X is called a partition of X, if the following three conditions hold:

(1) The set S_i is nonempty for every $i \in I$.
(2) The following holds:

$$S_i \cap S_j = \begin{cases} \emptyset, & \text{if } i \neq j, \\ S_i = S_j, & \text{if } i = j. \end{cases}$$

(3) The equality

$$X = \bigcup_{i \in I} S_i$$

holds.

The above considerations showed that every equivalence relation E on X yields a partition of X (consisting of equivalence classes w.r.t. E).

Conversely, every partition $\{S_i \mid i \in I\}$ of X gives rise to an equivalence relation E on X, if we put

$$x \, E \, y \quad \text{iff} \quad x \text{ and } y \text{ are both in } S_i \text{ for some } i \in I$$

Equivalence classes are used mainly to construct the *quotient set*. More in detail, given an equivalence relation E on X, we can define the quotient set X/E by putting

$$X/E = \{[x]_E \mid x \in X\}$$

There are various possible interpretations of a quotient set. Here is our favourite one: an equivalence relation E on X "blurs" our vision — we can recognise as different only different equivalence classes, we see all elements of an equivalence class as the same. The set X "blurred" by E is the quotient set X/E.

5.5. Task for you (easy) (A field of fractions). *Let $\mathbb{K}$ be an integral domain. Consider all pairs (p, q) of elements of $\mathbb{K}$, where $q \neq 0$, and define a relation $\sim$ as follows*

$$(p, q) \sim (p', q') \quad \text{iff} \quad \text{the equality } pq' = p'q \text{ holds in } \mathbb{K}$$

Prove that:

(1) *The relation $\sim$ is an equivalence relation on $\mathbb{K} \times (\mathbb{K} \setminus \{0\})$, i.e., it is reflexive, symmetric and transitive.*

(2) *Denote by*

$$\frac{p}{q}$$

the $\sim$-equivalence class, represented by (p, q). Define

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \quad \text{and} \quad \frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2}$$

Show that these two operations on the $\sim$-equivalence classes turn the quotient set $\mathbb{K}/\sim$ into a field, denoted by $\mathrm{Frac}(\mathbb{K})$ and called the field of fractions on $\mathbb{K}$.

(3) *Prove that the mapping*

$$i : \mathbb{K} \longrightarrow \mathrm{Frac}(\mathbb{K}), \quad r \mapsto \frac{r}{1}$$

respects zero, unit, addition and multiplication. That is prove that:

(a) *$i(0)$ is a zero element in $\mathrm{Frac}\,\mathbb{K}$.*

(b) *$i(1)$ is a unit element in $\mathrm{Frac}\,\mathbb{K}$.*

(c) *The equality $i(r_1 + r_2) = i(r_1) + i(r_2)$ holds in $\mathrm{Frac}\,\mathbb{K}$.*

(d) *The equality $i(r_1 \cdot r_2) = i(r_1) \cdot i(r_2)$ holds in $\mathrm{Frac}\,\mathbb{K}$.*

HINTS FOR THE PROOF. The proof is easy.   ∎

5.6. Definition (Field of rationals). The field of fractions $\mathrm{Frac}(\mathbb{Z})$ of the integral domain $\mathbb{Z}$ is called the *field of rational numbers* and denoted by $\mathbb{Q}$.

5.7. Task for you (easy). *Prove that there is no positive element r in $\mathbb{Q}$ such that $r^2 = 2$ holds.[a]*

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[a]In plain English we are saying here that $\sqrt{2}$ is not a rational number. This fact has been first noted by Hippasus of Metapontum (roughly 500 BC). The proof you are asked to give is based on Hippasus thoughts.

HINTS FOR THE PROOF. Proceed by contradiction: denote by $r = \frac{a}{b}$ the positive rational number with the property $r^2 = 2$. Assume (and argue why you can do it) that a, b are positive relatively prime natural numbers. Deduce that $a^2 = 2b^2$ holds. Deduce further that it follows that both a and b are even numbers (hence not relatively prime).[6] Conclude a contradiction, proving the assertion.   ∎

5.8. Task for you (easy). *Prove that the field $\mathbb{Q}$ is not Dedekind complete.*

HINTS FOR THE PROOF. Consider the set

$$S = \{r \in \mathbb{Q} \mid r \text{ is positive and } r^2 < 2 \text{ holds}\}$$

and show that it is nonempty and bounded from above. Were $\mathbb{Q}$ Dedekind complete, the set S would have a supremum in $\mathbb{Q}$, say, s. Prove that then $s^2 = 2$ would hold, a contradiction with Task 5.7.   ∎

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[6]You will have to prove for every natural number n that if n^2 is even, so is n. The easiest strategy is the proof by contraposition.

We conclude this chapter by showing that ordered fields of fractions have a quite rigid structure. For example: by Task 3.9 we know that $0 < 1$ and therefore the inequality

$$0 < \underbrace{1 + 1 + \ldots + 1}_{n\text{-times}}$$

holds for all positive natural numbers n. Fields (or rings) where

$$0 \neq \underbrace{1 + 1 + \ldots + 1}_{n\text{-times}}$$

holds for all positive natural numbers n are said to have *characteristic* 0 and one can show the following.

5.9. Task for you (easy). *Prove the following:*

(1) *Any nontrivial (i.e., one, where where $0 \neq 1$ holds) ordered commutative ring with unit $\mathbb{K}$ is an integral domain of characteristic 0.*

(2) *The ordering from $\mathbb{K}$ can be extended to the ordering on $\mathrm{Frac}(K)$ in just one way:*

$$\frac{p}{q} > 0 \quad \textit{iff} \quad pq > 0 \ \textit{in } \mathbb{K}$$

(3) *Any ordered field $\mathbb{F}$ is necessarily infinite.*

(4) *The field $\mathbb{Q}$ can be ordered in a unique way, extending the order of $\mathbb{Z}$.*

HINTS FOR THE PROOF.

(1) Proceed by contraposition: start with $r \neq 0$ and $s \neq 0$ and prove that $rs \neq 0$. Start with observing $rs > 0$, if both $r > 0$ and $s > 0$ (see Definition 3.7). Then do the case analysis of the remaining three cases. The assertion about characteristic 0 has been showed already.

(2) Since $\mathbb{K}$ is an integral domain, the formation of $\mathrm{Frac}(\mathbb{K})$ makes sense. Secondly, if $\frac{p}{q} > 0$, then $pq = \frac{p}{q} \cdot q^2 > 0$ must hold. Thus we only need to prove that the definition of positive fractions defines an order on $\mathrm{Frac}(\mathbb{K})$.

  (a) Suppose $\frac{p}{q} = \frac{p'}{q'}$. Then $pq' = p'q$, hence $pqq'^2 = p'q^2q'$. Thus $pq > 0$ iff $pqq'^2 > 0$ iff $p'q^2q' > 0$ iff $p'q' > 0$. Thus, the condition does not depend on the representative of the fraction.

  (b) For any fraction $\frac{p}{q}$ only one of the conditions $\frac{p}{q} > 0$, $\frac{p}{q} = 0$, $\frac{p}{q} < 0$ holds.

  (c) Consider $\frac{p_1}{q_1} > 0$ and $\frac{p_2}{q_2} > 0$, i.e., $p_1q_1 > 0$ and $p_2q_2 > 0$. Then $(p_1q_2 + p_2q_1) \cdot q_1q_2 = p_1q_1q_2^2 + p_2q_2q_1^2 > 0$, proving $\frac{p_1}{q_1} + \frac{p_2}{q_2} > 0$. Further, $p_1p_2q_1q_2 > 0$, proving $\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} > 0$.

(3) Since $\mathbb{K}$ has characteristic 0, it must contain an infinite number of distinct elements

$$\underbrace{1 + 1 + \ldots + 1}_{n\text{-times}}$$

indexed by natural numbers.

(4) Show that $\mathbb{Z}$ can be ordered in a unique way. The inequality $0 < 1$ must hold, hence

$$0 < \underbrace{1 + 1 + \ldots + 1}_{n\text{-times}}$$

holds for any positive integer n. Hence

$$0 > \underbrace{-1 - 1 - \ldots - 1}_{n\text{-times}}$$

holds for any positive integer n. Then use part (2).

$\blacksquare$

The above considerations will allow us to "embed" the structures of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ into any ordered field.

5.10. Definition ($\mathbb{F}$-naturals, $\mathbb{F}$-integers and $\mathbb{F}$-rationals). Let $\mathbb{F}$ be an ordered field.

(1) Every n in $\mathbb{N}$ is considered as an $\mathbb{F}$-*natural* (or, a *natural number in* $\mathbb{F}$), i.e., as an element
$$n \cdot 1 = \underbrace{1 + \ldots + 1}_{n\text{-times}}$$
of $\mathbb{F}$.

(2) Every z in $\mathbb{Z}$ is considered as an $\mathbb{F}$-*integer* (or, an *integer in* $\mathbb{F}$), i.e., as an element
$$z \cdot 1 = \begin{cases} z \cdot 1, & \text{if } z > 0 \\ 0, & \text{if } z = 0 \\ -((-z) \cdot 1), & \text{if } z < 0 \end{cases}$$
of $\mathbb{F}$.

(3) Every $\frac{p}{q}$ in $\mathbb{Q}$ is considered as an $\mathbb{F}$-*rational* (or, a *rational in* $\mathbb{F}$), i.e., as an element
$$\frac{p}{q} \cdot 1 = (p \cdot 1) \cdot (q \cdot 1)^{-1}$$
of $\mathbb{F}$.

We will make the notation lighter and frequently we will write just
$$n \quad z \quad \frac{p}{q}$$
to denote the appropriate elements on $\mathbb{F}$.

5.11. Task for you (easy). *Let $\mathbb{F}$ be an ordered field. Prove that the assignment*
$$j : r \mapsto r \cdot 1$$
from $\mathbb{Q}$ to $\mathbb{F}$ is indeed a function and prove that j is injective, it respects the addition, the multiplication, the zero and and the unit. That is, prove that the following hold for all rational numbers r, s:

(1) $j(r + s) = j(r) + j(s)$.
(2) $j(r \cdot s) = j(r) \cdot j(s)$.
(3) $j(0) = 0$.
(4) $j(1) = 1$.

HINTS FOR THE PROOF. This proof is pretty straightforward. ∎

6. Posets and their Dedekind completions

The simple and ingenious Dedekind's ideas were as follows:

(1) If the reals correspond to points on a line, then observe that every point p of a line *cuts* the line into two parts: the points "to the left of" p and "the points to the right of" p. Thus, *a real number should correspond uniquely to the pair of collections of these points*.
(2) To avoid circularity of the argument, one could restrict each of the two collections above to *consist only of rational numbers*.

Dedekind called the pair as above a *cut* (*der Schnitt* in German) on the rationals. This allows one to *define a real number to be a cut on $\mathbb{Q}$*. In fact, as we show later, the above ideas constitute a certain *completion process*, see Definition 6.18 below. When applied to $\mathbb{Q}$, the process yields a Dedekind complete field, see Chapter 7.

The notion of a cut makes sense for any ordered field. By Definition 3.7, for every pair r, s of an every ordered field $\mathbb{F}$, it holds that
$$r \leq s \quad \text{or} \quad s \leq r$$

The above condition is a special property of an *abstract* notion of an order, see Definition 6.21 below. In practice, a binary relation can have the intuitive meaning of "ordering" without the above property. For example, given a set X, we might say that $A \subseteq X$ is "at most as big as" $B \subseteq X$, if $A \subseteq B$ holds. It is then

easy to find an example of X, $A \subseteq X$, $B \subseteq X$, such that

$$\text{neither} \quad A \subseteq B \quad \text{nor} \quad B \subseteq A$$

holds. See Example 6.3 below.

In this chapter we recall the very basics of the theory of *partial orders* (the above "order" by being a subset is an example).

The completion by Dedekind cuts can be performed for every *partially ordered set*. Since this is an important construction on its own and since this construction in partially ordered sets is rather illuminating, we present it in this generality.

6.1. **Definition** (**Poset**). A set X, together with a binary relation $\sqsubseteq$, is called a *poset*[7], provided that the following three conditions hold:

(1) Reflexivity of $\sqsubseteq$: for every x in X, $x \sqsubseteq x$ holds.
(2) Transitivity of $\sqsubseteq$: for all x, y, z in X if $x \sqsubseteq y$ and $y \sqsubseteq z$ hold, then $x \sqsubseteq z$ holds.
(3) Antisymmetry of $\sqsubseteq$: for all x, y in X if $x \sqsubseteq y$ and $y \sqsubseteq x$ hold, then $x = y$ holds.
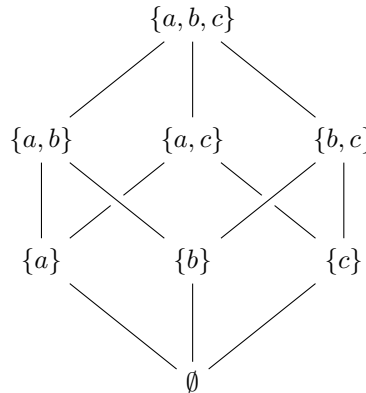
We will denote a poset by $(X, \sqsubseteq)$.

> 6.2. **Task for you (easy)** (**Opposite poset**). *Prove that for any poset $(X, \sqsubseteq)$, the pair $(X, \sqsubseteq^{op})$ is a poset, where*
>
> $$x \sqsubseteq^{op} y \quad \text{iff} \quad y \sqsubseteq x$$
>
> *The poset $(X, \sqsubseteq^{op})$ is called the* opposite *of $(X, \sqsubseteq)$.*

HINTS FOR THE PROOF. Prove that the relation $\sqsubseteq^{op}$ is reflexive, transitive and antisymmetric. ∎

6.3. **Example** (**Hasse diagram**). Posets can be conveniently drawn by the use of *Hasse diagrams*.[8] In Hasse diagrams one draws only the necessary information about the *neighbours* in the partial order. See the following picture

of a poset on the powerset

$$P(\{a, b, c\}) = \Big\{ S \mid S \subseteq \{a, b, c\} \Big\}$$

of all subsets of $\{a, b, c\}$, ordered by inclusion.

What we drew above is not the relation $\subseteq$ on $P(\{a, b, c\})$ but the least relation R such that $\subseteq$ is a *reflexive and transitive hull* of R. In other words, to obtain the full grip of the poset $(P(\{a, b, c\}), \subseteq)$, one would have to add paths of all finite lenghts (including loops).

6.4. **Definition** (**Suprema and infima in a poset**). Let $(X, \sqsubseteq)$ be a poset, let $S \subseteq X$. We say that

(1) An element s is an *upper bound* of S, if $x \sqsubseteq s$ holds for all $x \in S$.[9]

---

[7]It is an acronym of *partially ordered set*. The word *poset* is in common usage in mathematics.
[8]Named after the German mathematician Helmut Hasse (25 August 1898 — 26 December 1979).
[9]If a set S has an upper bound, we also say that S is *bounded from above*. See also Definition 4.1.

(2) An element s is a *supremum* of S (notation[10] $s = \sup_{\sqsubseteq} S$), if the following two conditions are satisfied:[11]
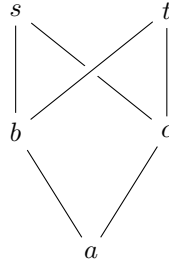
      (a) s is an upper bound of S.

      (b) If s' is an upper bound of S, then $s \sqsubseteq s'$.

An element i is a *lower bound* of S in $(X, \sqsubseteq)$, if i is an upper bound of S in the opposite poset $(X, \sqsubseteq^{op})$. An element i is an *infimum* S in $(X, \sqsubseteq)$ (denoted by $i = \inf S$), if i is a supremum of S in the opposite poset $(X, \sqsubseteq^{op})$.

6.5. Remark (Maxima and minima in a poset). Suprema and infima of subsets of a poset are a generalisation of the better-known concepts of *maxima* and *minima*. Recall that m is a *maximum* of S in a poset $(X, \sqsubseteq)$ (denoted by $m = \max S$), if $m = \sup S$ and $m \in S$. Equivalently, $m = \max S$, if $m \in S$ and m is an upper bound of S.

    An element m is a *minimum* of S in $(X, \sqsubseteq)$ (denoted by $m = \min S$), if $m = \max S$ in $(X, \sqsubseteq^{op})$.

6.6. Example (Suprema need not exist). Consider the following Hasse diagram

of a poset on the set $X = \{a, b, c, s, t\}$. Put $S = \{b, c\}$. Then both s and t are upper bounds of S, but neither of them is a supremum of S.

    The previous example showed that suprema need not exist. However, if suprema do exist, they are determined uniquely. Thus, the notation $s = \sup S$ is unambiguous.

> **6.7. Task for you (easy) (Suprema are unique, if they exist).** *Suppose that S is a subset of a poset $(X, \sqsubseteq)$. Prove that if s and s' are suprema of S, then $s = s'$.*
>   *Extend the result to prove that also infima, minima and maxima are unique, provided they exist.*

HINTS FOR THE PROOF. Use Definition 6.4 to conclude that both $s \sqsubseteq s'$ and $s' \sqsubseteq s$ hold. Then use antisymmetry of $\sqsubseteq$. ■

    In what follows we will work with the sets of all lower bounds and the sets of all upper bounds of a subset.

6.8. Definition (Sets of all lower bounds and all upper bounds). Let $(X \sqsubseteq)$ be a poset. For every $A \subseteq X$ we define

$$\blacktriangledown A = \{x \in X \mid x \sqsubseteq a \text{ for all } a \in A\} \quad \text{and} \quad \blacktriangle A = \{x \in X \mid a \sqsubseteq x \text{ for all } a \in A\}$$

and call them the *set of all lower bounds* and the *set of all upper bounds* of the set A, respectively. Instead of $\blacktriangledown\{a\}$ and $\blacktriangle\{a\}$, for $a \in X$, we write simply $\blacktriangledown a$ and $\blacktriangle a$.

---

[10]We will mostly omit the subscript $\sqsubseteq$ and write $\sup S$ instead of $\sup_{\sqsubseteq} S$.

[11]Due to these conditions, a supremum is sometimes caled a *least upper bound* and denoted by $\operatorname{lub} S$. Analogously, an infimum is called a *greatest lower bound* and denoted by $\operatorname{glb} S$. We will not use this notation.

6.9. **Task for you (easy).** *Prove that the following properties of subsets A, B of a poset $(X, \sqsubseteq)$ hold:*
   (1) *If $A \subseteq B$, then $\blacktriangledown B \subseteq \blacktriangledown A$.*
   (2) *If $A \subseteq B$, then $\blacktriangle B \subseteq \blacktriangle A$.*
   (3) *$A \subseteq \blacktriangledown\blacktriangle A$.*
   (4) *$A \subseteq \blacktriangle\blacktriangledown A$.*
   (5) *$\blacktriangledown A = \blacktriangledown\blacktriangle\blacktriangledown A$.*
   (6) *$\blacktriangle A = \blacktriangle\blacktriangledown\blacktriangle A$.*

HINTS FOR THE PROOF. We give hints for proving (1), (3), (5).
   (1) Consider $x \in \blacktriangledown B$. Then $x \sqsubseteq b$ for every $b \in B$. Since $A \subseteq B$, this implies that $x \sqsubseteq a$ for every $a \in A$. Thus $x \in \blacktriangledown A$ holds.
   (3) Consider $a \in A$. We need to prove that $a \in \blacktriangledown\blacktriangle A$. In other words, we need to prove $a \sqsubseteq x$ for every upper bound x of A. But this is trivial, since x is an upper bound of A.
   (5) The inclusion $\blacktriangledown\blacktriangle\blacktriangledown A \subseteq \blacktriangledown A$ follows from (1), applied to $A \subseteq \blacktriangle\blacktriangledown A$ (which holds by (4)).

   Thus we only need to prove that $\blacktriangledown A \subseteq \blacktriangledown\blacktriangle\blacktriangledown A$ holds. To this end, take any $x \in \blacktriangledown A$. Thus, $x \sqsubseteq a$ holds for all $a \in A$. We need to prove that $x \sqsubseteq b$ for every b in $\blacktriangle\blacktriangledown A$. That is, we need to prove that $x \sqsubseteq b$ for every upper bound b of the set $\blacktriangledown A$. But this is trivial: $x \sqsubseteq b$ holds, since x is a lower bound of A.

The rest can be proved analogously or by passing to the opposite poset $(X, \sqsubseteq^{op})$ ∎

The sets of all lower bounds and all upper bounds of subsets of a poset allow us to define *cuts* on a poset. The notion of a cut will be at the core of forming a *completion* of a poset.

6.10. **Definition** (**Cut on a poset**). Let $(X, \sqsubseteq)$ be a poset. A pair $c = (L, U)$ of subsets $L \subseteq X$ and $U \subseteq X$ is called a *cut* on $(X, \sqsubseteq)$, if the following three conditions hold:
   (1) Neither L nor U is empty.
   (2) L is the set of all lower bounds of U, i.e., $L = \blacktriangledown U$ holds.
   (3) U is the set of all upper bounds of L, i.e., $U = \blacktriangle L$ holds.
The set L will be called a *lower part* and U an *upper part* of the cut $c = (L, U)$.

6.11. **Task for you (easy).** *Suppose that $A \subseteq X$ is nonempty and bounded from above, $B \subseteq X$ is nonempty and bounded from below. Then the pairs*
$$(\blacktriangledown\blacktriangle A, \blacktriangle A) \quad and \quad (\blacktriangledown B, \blacktriangle\blacktriangledown B)$$
are both cuts on the poset $(X, \sqsubseteq)$.

HINTS FOR THE PROOF. We give a hint for the proof that the pair $(\blacktriangledown\blacktriangle A, \blacktriangle A)$ is a cut.
   (1) Since A is nonempty and $A \subseteq \blacktriangledown\blacktriangle A$ holds by Task 6.9, the set $\blacktriangledown\blacktriangle A$ is nonempty. Since A is bounded from above, the set $\blacktriangle A$ is nonempty.
   (2) Using Task 6.9, the equality $\blacktriangle\blacktriangledown\blacktriangle A = \blacktriangle A$ holds.
   (3) The equality $\blacktriangledown\blacktriangle A = \blacktriangledown\blacktriangle A$ holds trivially.
That the pair $(\blacktriangledown B, \blacktriangle\blacktriangledown B)$ is a cut is proved analogously or by passing to the opposite poset $(X, \sqsubseteq^{op})$. ∎

6.12. **Task for you (easy).** *Suppose (L, U) is a cut on a poset $(X, \sqsubseteq)$. Prove that there exists a nonempty subset $A \subseteq X$ such that $L = \blacktriangledown A$ and $U = \blacktriangle\blacktriangledown A$. In particular, the following conditions hold:*
   (1) *The set L is a* lowerset, *i.e., if $x \in L$ and $y \sqsubseteq x$, then $y \in L$.*
   (2) *The set U is an* upperset, *i.e., if $x \in U$ and $x \sqsubseteq y$, then $y \in U$.*

HINTS FOR THE PROOF. Define $A = U$. Then A is nonempty and $L = \blacktriangledown A$ holds by Definition 6.10. Then $U = \blacktriangle\blacktriangledown A$ holds by Definition 6.10. For (1) and (2) use that $L = \blacktriangledown U$ and $U = \blacktriangle L$. ∎

6.13. Task for you (easy). *Suppose (L_1, U_1) and (L_2, U_2) are cuts on a poset $(X, \sqsubseteq)$. Prove that $L_1 \subseteq L_2$ holds iff $U_2 \subseteq U_1$ does.*

HINTS FOR THE PROOF. Just use the definition of lower/upper bounds.   ∎

6.14. Task for you (easy). *Suppose (L, U) is a cut on $(X, \sqsubseteq)$. Then $L \cap U$ is either empty or it contains precisely one element.*

HINTS FOR THE PROOF. Consider elements a and a' that are in $L \cap U$. Prove that $a = a'$ as follows: since a is a lower bound of U and since a is an element of U, we have $a = \min U$. Analogously, $a' = \min U$.[12] Conclude the proof by the fact that a minimum in a poset is determined uniquely.   ∎

6.15. Definition (Gap). A cut (L, U) on $(X, \sqsubseteq)$ is called a *gap*, if $L \cap U = \emptyset$.

By Definition 6.15 and by Task 6.14, every cut (L, U) on $(X, \sqsubseteq)$ is either a gap, or it can be identified with a unique element of X. Moreover, the notion of a gap is closely related to the existence of suprema of nonempty sets bounded from above.

6.16. Task for you (easy). *Prove that, in a poset $(X, \sqsubseteq)$, the following conditions are equivalent:*
(1) *Every nonempty subset bounded from above has a supremum in $(X, \sqsubseteq)$.*
(2) *No cut on $(X, \sqsubseteq)$ is a gap.*

HINTS FOR THE PROOF. Hints for (1) implies (2). Let (L, U) be a cut on $(X, \sqsubseteq)$. Since $\blacktriangle L = U$, which is a nonempty set, the set L is nonempty and bounded from above. Thus $s = \sup L$ exists. In particular $s \in U = \blacktriangle L$ and $s = \min U$. Thus, s is a lower bound of U. In particular, $s \in \blacktriangledown U = L$. Hence

$$L \cap U = \{s\}$$

and the cut (L, U) is not a gap.

Hints for (2) implies (1). Consider a nonempty subset S with an upper bound. Thus $\blacktriangle S$ is nonempty. Since $S \subseteq \blacktriangledown \blacktriangle S$ holds by Task 6.9, the set $\blacktriangledown \blacktriangle S$ is nonempty. Therefore $(\blacktriangledown \blacktriangle S, \blacktriangle S)$ is a cut by Task 6.11. Since $(\blacktriangledown \blacktriangle S, \blacktriangle S)$ is not a gap, we have

$$\blacktriangledown \blacktriangle S \cap \blacktriangle S = \{s\}$$

by Task 6.14. But the above states that $s = \min \blacktriangle S$, again by Task 6.14. In other words: $s = \sup S$.   ∎

6.17. Definition (Dedekind complete poset). A poset $(X, \sqsubseteq)$ is called *Dedekind complete*, if every nonempty subset bounded from above has a supremum in $(X, \sqsubseteq)$.

We are now ready to define the desired completion process.

---

[12]By the dual reasoning, $a = a' = \max L$.

6.18. Definition (Dedekind-MacNeille completion of a poset). Let $(X, \sqsubseteq)$ be a poset. Define a poset

$$\mathrm{Compl}(X, \sqsubseteq)$$

to be the poset of all cuts (L, U) on $(X, \sqsubseteq)$, ordered as follows:

$$(L_1, U_1) \preceq (L_2, U_2) \quad \text{iff} \quad L_1 \subseteq L_2$$

is called the *Dedekind-MacNeille completion*[a] of $(X, \sqsubseteq)$.

_____

[a]Dedekind's construction was generalised in the PhD Thesis of Holbrook Mann MacNeille (11 May 1907 – 30 September 1973) in 1935.

6.19. Task for you (easy). *The Dedekind-MacNeille completion* $\mathrm{Compl}(X, \sqsubseteq)$ *of* $(X, \sqsubseteq)$ *is a* Dedekind complete *poset. That is: every nonempty subset of* $\mathrm{Compl}(X, \sqsubseteq)$ *of* $(X, \sqsubseteq)$, *that is bounded from above, has a supremum.*

Moreover, the map

$$j : a \mapsto (\blacktriangledown a, \blacktriangle\blacktriangledown a)$$

from $(X, \sqsubseteq)$ *to* $\mathrm{Compl}(X, \sqsubseteq)$ *is a* monotone embedding, *i.e.,* $a \sqsubseteq a'$ *holds iff* $j(a) \preceq j(a')$ *does.*

HINTS FOR THE PROOF. Choose a nonempty subset of $\mathrm{Compl}(X, \sqsubseteq)$ that is bounded from above. More in detail, consider cuts $c_i = (L_i, U_i)$ on $(X, \sqsubseteq)$, where $I \neq \emptyset$, such that there exists a cut $c = (L, U)$ with $c_i \preceq c$ for all $i \in I$. We need to prove that the supremum $\sup_{\preceq}\{c_i \mid i \in I\}$ exists.

Define the sets

$$L^* = \blacktriangledown\blacktriangle\left(\bigcup_{i \in I} L_i\right)$$

and

$$U^* = \blacktriangle L^* = \blacktriangle\left(\bigcup_{i \in I} L_i\right)$$

By Task 6.9, the inclusion

$$\bigcup_{i \in I} L_i \subseteq \blacktriangledown\blacktriangle\left(\bigcup_{i \in I} L_i\right) = L^*$$

holds. Since $\bigcup_{i \in I} L_i$ is nonempty (I is nonempty and every L_i is nonempty), the set L^* is nonempty.

Since $(L_i, U_i) \preceq (L, U)$ for all $i \in I$, we have $L_i \subseteq L$ for all $i \in I$. Thus $\bigcup_{i \in I} L_i \subseteq L$ holds. By Task 6.9, the inclusion

$$U = \blacktriangle L \subseteq \blacktriangle\left(\bigcup_{i \in I} L_i\right) = U^*$$

holds. Since U is nonempty, so is U^*. Thus the pair $s = (L^*, U^*)$ is a cut by Task 6.12.

We claim that $s = \sup_{\preceq}\{c_i \mid i \in I\}$.

(1) s is an upper bound of $\{c_i \mid i \in I\}$.

By the definition of $\preceq$, it suffices to prove $L_i \subseteq L^*$ for all $i \in I$. The inclusion $L_i \subseteq \bigcup_{i \in I} L_i$ holds for all $i \in I$. Thus, by Task 6.9, the inclusion

$$L_i \subseteq \blacktriangledown\blacktriangle L_i \subseteq \blacktriangledown\blacktriangle\left(\bigcup_{i \in I} L_i\right) = L^*$$

holds for all $i \in I$.

(2) s is a least upper bound of $\{c_i \mid i \in I\}$.

Consider any upper bound (M, V) of (L_i, U_i), $i \in I$. This means that $L_i \subseteq M$ holds for all $i \in I$. In particular, $\bigcup_{i \in I} L_i \subseteq M$ holds. Then

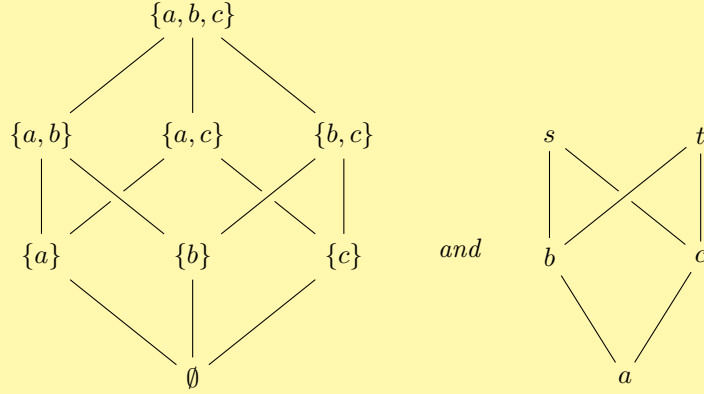$$L^* = \blacktriangledown\blacktriangle\bigcup_{i \in I} L_i \subseteq \blacktriangledown\blacktriangle M = M$$

holds, which we were supposed to prove.

The proof is concluded by observing that $(\blacktriangledown a, \blacktriangle\blacktriangledown a)$ is a cut on $(X, \sqsubseteq)$ and that

$$a \sqsubseteq a' \text{ holds} \quad \text{iff} \quad \blacktriangledown a \subseteq \blacktriangledown a' \text{ does}$$

∎

6.20. **Task for you (easy).** *Compute the Dedekind-MacNeille completions of the posets*

of Examples 6.3 and 6.6, respectively.

HINTS FOR THE PROOF. Use Task 6.14 to realise which cuts are gaps. ∎

The order-theoretic properties of Dedekind-MacNeille completions are quite lucid when considering *linearly* ordered sets. Notice that all ordered rings are ordered linearly, see 3.7.

6.21. **Definition (Linear order).** A poset $(X, \sqsubseteq)$ is called a *linear* order, if for every x, y in X, $x \sqsubseteq y$ or $y \sqsubseteq x$ holds.

6.22. **Task for you (easy).**
   (1) *Let $(X, \sqsubseteq)$ be a linear order. Then, for any subsets A, B of X, it holds that $\blacktriangledown A \subseteq \blacktriangledown B$ or $\blacktriangledown B \subseteq \blacktriangledown A$.*
   (2) *Let (L, U) be a cut on a linear order $(X, \sqsubseteq)$. Then the equality $L \cup U = X$ holds.*
   (3) *Find an example of a poset $(X, \sqsubseteq)$ and a cut (L, U) on it, such that $L \cup U \neq X$.*

HINTS FOR THE PROOF.
   (1) Suppose $\blacktriangledown A \not\subseteq \blacktriangledown B$ holds. Fix $x \in \blacktriangledown A$ such that $x \notin \blacktriangledown B$.
       Now consider any $y \in \blacktriangledown B$. Since $\sqsubseteq$ is linear, $x \sqsubseteq y$ or $y \sqsubseteq x$ holds. But $x \sqsubseteq y$ cannot hold, since then $x \in \blacktriangledown B$. Therefore $y \sqsubseteq x$ holds and this means that $y \in \blacktriangledown A$.
   (2) Take any $x \in X$ such that $x \notin L$. Then x is not a lower bound of U. Thus there exists $y \in U$ such that $x \not\sqsubseteq y$. Since $\sqsubseteq$ is assumed to be linear, this means that $y \sqsubseteq x$. Then $x \in U$, since U is an upperset. Analogously: assume $x \notin U$ and prove that $x \in L$.
   (3) The order must not be linear by (2).

∎

6.23. **Task for you (easy).** *The Dedekind-MacNeille Completion of a linear order is a linear order.*

HINTS FOR THE PROOF. Use (1) of Task 6.22 and Definition 6.18. ∎

We conclude this chapter by proving that Dedekind completeness could have been stated equivalently in terms of infima of subsets bounded from below.

6.24. **Task for you (easy).** *Prove that, for a poset $(X, \sqsubseteq)$, the following conditions are equivalent:*

(1) *Every nonempty set S bounded from above has a supremum.*
(2) *Every nonempty set R bounded from below has an infimum.*

HINTS FOR THE PROOF. We give a hint for (1) implies (2). Let R be a nonempty subset of X bounded from below. Thus, $\blacktriangledown R$ is a nonempty set. Since $R \subseteq \blacktriangle\blacktriangledown R$ holds by Task 6.9, and since R is nonempty, the set $\blacktriangle\blacktriangledown R$ is nonempty.

Thus $(\blacktriangledown R, \blacktriangle\blacktriangledown R)$ is a cut on $(X, \sqsubseteq)$ and it is not a gap by Task 6.16. Therefore, by Task 6.14, there is a unique element s such that

$$\blacktriangledown R \cap \blacktriangle\blacktriangledown R = \{s\}$$

holds. This means that s is an infimum of R.

The implication (2) implies (1) is proved analogously. ∎

7. DEDEKIND CUTS ON RATIONALS

We will now apply the theory of Chapter 6 to the *ordered field* $\mathbb{Q}$ of rationals. A very readable treatment of Dedekind's construction is given in [23]. Let us recapitulate Definition 4.2 of the reals:

7.1. **Definition (The reals).** We say that a field $\mathbb{F}$ is a *field of reals*, provided that it is a Dedekind complete field.

Our strategy will be as follows:

(1) We start with the *ordered field* $\mathbb{Q}$ of rationals, see Definition 5.6. By Task 5.8 we know that $\mathbb{Q}$ is *not* Dedekind complete.
(2) We apply the construction of *Dedekind-MacNeille completion* of Definition 6.18 to $\mathbb{Q}$. We know that the result will be a Dedekind complete *poset* that we will denote by $\mathbb{R}$.
(3) We impose *algebraic* operations on $\mathbb{R}$ in such a way that $\mathbb{R}$ becomes an ordered field that extends $\mathbb{Q}$.

We first need to understand the ordered field $\mathbb{Q}$ a bit better.

7.2. **Task for you (easy).** *Prove that for every positive r in $\mathbb{Q}$ and for every s in $\mathbb{Q}$, there exists a positive integer n such that $s < nr = \underbrace{r + \ldots + r}_{n\text{-times}}$.*

HINTS FOR THE PROOF. The proof is easy. ∎

The above property of the order on $\mathbb{Q}$ is important and it has a name.

7.3. **Definition (The Archimedean property).** An ordered field $\mathbb{F}$ is called *Archimedean*,[a] if the following holds: for every r in $\mathbb{F}_+$ and for every s in $\mathbb{F}$, there exists a positive integer n such that $s < nr = \underbrace{r + \ldots + r}_{n\text{-times}}$.

[a] Obviously, the property is named after the Greek mathematician Archimedes (cca 287 BC — cca 212 BC). Archimedes is probably best known in pop culture for running around Syracuse and shouting *Heureka*! But Archimedes was most likely the best mathematician of the Ancient Greece. He formulated the above property as Axiom V in his book *On the sphere and cylinder*.

7.4. **Remark (The intuition behind the Archimedean property).** The Archimedean property of an ordered field $\mathbb{F}$ states that "$\mathbb{F}$ contains no infinitely large elements".

More in detail, suppose that $\mathbb{F}$ does *not* have the Archimedean property. Then there is r in $\mathbb{F}_+$ and s in $\mathbb{F}$ such that the inequality

$$s > nr = \underbrace{r + \ldots + r}_{n\text{-times}}$$

holds for all positive integers n. Intuitively, such an s could safely be called an "infinitely large" element of $\mathbb{F}$, since no integer multiple nr of the positive element r can exceed s.

Not all ordered fields are Archimedean, see the following example. Notice that the example is a field of fractions for a certain integral domain. Hence fields of fractions are not necessarily Archimedean.

7.5. **Example** (**A non-Archimedean field**). It is easy to observe that the set $\mathbb{Q}[x]$ of all polynomials in indeterminate x with rational coefficients is an integral domain.

Consider the field

$$\mathbb{Q}(x) = \mathrm{Frac}(\mathbb{Q}[x])$$

of fractions of the integral domain $\mathbb{Q}[x]$.

That means, we consider the following set

$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} \mid p(x) \in \mathbb{Q}[x], q(x) \in \mathbb{Q}[x], q(x) \neq 0 \right\}$$

of fractions of polynomials with coefficients in $\mathbb{Q}$, with addition and multiplication defined in the usual way.

We define the subset $\mathbb{Q}(x)_+$ of $\mathbb{Q}(x)$ to consist of fractions $\frac{p(x)}{q(x)}$ where the leading coefficients[13] of both $p(x)$ and $q(x)$ have the same sign. It is then easy to see that this definition makes $\mathbb{Q}(x)$ into an ordered field.

The field $\mathbb{Q}(x)$ is not Archimedean, however. Indeed: consider the positive element

$$r(x) = 1$$

Were $\mathbb{Q}(x)$ Archimedean, then for $s(x) = x$ there would exist an integer n such that

$$x = s(x) < nr(x) = n$$

holds, see Definition 7.3. But $x - n$ is positive for every integer n. Thus

$$x = s(x) > nr(x) = n$$

holds; a contradiction.

Recall from Definition 5.10 and Task 5.11 that we can consider $\mathbb{Q}$ to be embedded into any ordered field. This allows us to state the following result: the Archimedean property of a field is equivalent to the statement that between any two elements there is a rational number.[14]

7.6. **Task for you (easy)** (**Characterisation of Archimedean fields**). *Prove that, for an ordered field $\mathbb{F}$, the following conditions are equivalent:*

(1) *$\mathbb{F}$ is Archimedean.*
(2) *The field $\mathbb{Q}$ of rationals is* order-dense *in $\mathbb{F}$: for every pair a, b in $\mathbb{F}$ such that $a < b$ there exists a rational r such that $a < r < b$ holds.*

HINTS FOR THE PROOF. (1) implies (2). We first consider the case $0 < a < b$. Then, since $\mathbb{F}$ is Archimedean, there exist rationals p, q such that $0 < p < a$ and $0 < q < b - a$. Then $0 < p + q < b$. Thus, the set

$$S = \{n \in \mathbb{N} \mid p + nq < b\}$$

is nonempty. Since $\mathbb{F}$ is Archimedean, the set S must be finite. Therefore the set S has a maximal element, denote it by N. Thus $p + Nq < b$ holds. We claim that $a < p + Nq$ holds. Suppose not. Then $p + Nq \leq a$ and $p + (N + 1)q = (p + Nq) + q < a + q < b$. Thus N could not have been the maximal element of S. Therefore we have

$$a < p + Nq < b$$

and since $p + Nq$ is clearly rational, we are done.

The other two cases: $a < 0 < b$ and $a < b < 0$ can be proved similarly.

---

[13] The leading coefficient of $p(x)$ is the coefficient by the largest power of x in $p(x)$.

[14] Recall Chapter 2: the latter property was one of our guesses for what $\mathbb{R}$ should fulfill. We see now that this property amounts precisely to saying that there are no "infinitely large" real numbers.

(2) implies (1). Suppose that $\mathbb{F}$ is non-Archimedean. Choose r such that $n < r$ for all natural n. Then there is no rational between r and $r + 1$. ∎

We are now able to state the basic relationship between Dedekind complete and Archimedean fields. A characterisation of Dedekind complete fields as certain Archimedean fields is given in Task 8.9 below.

> 7.7. **Task for you (easy).** *Show that the following hold:*
>   (1) *Every Dedekind complete field is Archimedean.*
>   (2) *There exists an Archimedean field that is not Dedekind complete.*

HINTS FOR THE PROOF.

(1) Proceed by contradiction: suppose that $\mathbb{F}$ is a non-Archimedean Dedekind complete field. In the notation of Remark 7.4, consider the set

$$S = \{nr \mid n \text{ is a positive integer}\}$$

where $r > 0$. Then S is nonempty and s is the upper bound of S. Since $\mathbb{F}$ is Dedekind complete, the set S has a supremum, say, $a = \sup S$. Then, for all positive integers n, we have

$$(n + 1)r \leq a$$

hence $nr \leq a - r$ holds for all positive integers n. This means that $a - r < a$ and $a - r$ is an upper bound of S; a contradiction with the definition of a.

(2) The field $\mathbb{Q}$ of rationals is Archimedean by Task 7.2 and it is not Dedekind complete by Task 5.8.

∎

We *define* $\mathbb{R}$ as the Dedekind-MacNeille completion of the ordered field $\mathbb{Q}$:

$$\mathbb{R} = \mathrm{Compl}(\mathbb{Q}, \leq)$$

By Task 6.19 we know that $\mathbb{R}$ is a Dedekind complete poset. We also have a monotone embedding

$$j : \mathbb{Q} \longrightarrow \mathbb{R}, \quad j(r) = (\blacktriangledown r, \blacktriangle\blacktriangledown r)$$

that we will simply treat as inclusion.

What do we have to do further? We need to impose an *algebraic* structure of an ordered field on the poset $\mathbb{R}$. We divide the task to the definition of addition and multiplication. The definitions are quite technical. We believe, however, that various examples inserted into the text will help. To make the construction as clear as possible, we break it up into the following steps:

(1) We first recall the *order* on the reals. This will help us to work with *positive* reals first.
(2) We define *addition* for all cuts and *multiplication* for positive cuts. We prove the basic properties of addition and multiplication and then extend multiplication to all reals.
(3) We prove the properties that make the reals into a field.

For the sake of clarity, we will denote the "new" operations on the set $\mathbb{R}$ by

$$+_{\mathbb{R}} \quad \cdot_{\mathbb{R}} \quad -_{\mathbb{R}} \quad 0_{\mathbb{R}} \quad 1_{\mathbb{R}}$$

The order on reals — the positive and the negative cuts. Recall from Definition 6.18 that the order on cuts is defined as follows:

$$(L_1, U_1) \preceq (L_2, U_2) \quad \text{iff} \quad L_1 \subseteq L_2 \ (\text{or, equivalently, iff } U_2 \subseteq U_1)$$

We will use the usual notation

$$(L_1, U_1) \prec (L_2, U_2) \quad \text{iff} \quad L_1 \subsetneq L_2 \ (\text{or, equivalently, iff } U_2 \subsetneq U_1)$$

for strict inequality.

The zero element 0 in $\mathbb{Q}$ is represented by the cut

$$0_{\mathbb{R}} = (\blacktriangledown 0, \blacktriangle\blacktriangledown 0)$$

in $\mathbb{R} = \mathrm{Compl}(\mathbb{Q}, \leq)$, see Task 6.19.

7.8. Task for you (easy). *The order $\preceq$ on $\mathbb{R}$ is linear. That is: for any cuts (L_1, U_1), (L_2, U_2), it holds that $(L_1, U_1) \preceq (L_2, U_2)$ or $(L_2, U_2) \preceq (L_1, U_1)$.*

HINTS FOR THE PROOF. Use Task 6.23. ∎

The following definition is sort of obvious:

7.9. Definition (Positive and negative cuts). Let us say that a cut (L, U) is *positive*, if

$$\blacktriangledown 0 \subsetneqq L$$

or, equivalently, if L contains at least one positive rational number. We denote this fact by $0_{\mathbb{R}} \prec (L, U)$.

A cut (L, U) is *negative*, if

$$\blacktriangle \blacktriangledown 0 \subsetneqq U$$

or, equivalently, if U contains at least one negative rational number. We denote this fact by $(L, U) \prec 0_{\mathbb{R}}$.

7.10. Definition (Additive inverse). For a cut (L, U), define its *additive inverse*

$$-_{\mathbb{R}}(L, U) = (\ominus U, \ominus L)$$

where $\ominus X = \{-x \mid x \in X\}$ for any subset $X \subseteq \mathbb{Q}$.

7.11. Task for you (easy). *Prove that:*
  (1) *If (L, U) is a cut, so is $-_{\mathbb{R}}(L, U)$.*
  (2) *If (L, U) is a positive cut, then $-_{\mathbb{R}}(L, U)$ is negative.*
  (3) *If (L, U) is a negative cut, then $-_{\mathbb{R}}(L, U)$ is positive.*
  (4) *The equality $-_{\mathbb{R}}(L, U) = (L, U)$ holds iff $(L, U) = 0_{\mathbb{R}}$.*

HINTS FOR THE PROOF. Straightforward. ∎

The addition of reals. The addition on cuts is defined as follows.

7.12. Definition (Addition of cuts). Given two cuts (L_1, U_1), (L_2, U_2) on $\mathbb{Q}$, define

$$(L_1, U_1) +_{\mathbb{R}} (L_2, U_2) = (\blacktriangledown(U_1 \oplus U_2), \blacktriangle\blacktriangledown(U_1 \oplus U_2))$$

where $U_1 \oplus U_2 = \{x_1 + x_2 \mid x_1 \in U_1, \ x_2 \in U_2\}$.

7.13. Task for you (easy). *Prove the following:*
  (1) *$(L_1, U_1) +_{\mathbb{R}} (L_2, U_2)$ is a Dedekind cut, whenever (L_1, U_1) and (L_2, U_2) are cuts.*
  (2) *The operation $+_{\mathbb{R}}$ is associative, commutative and it has the cut $0_{\mathbb{R}}$ as a neutral element.*
  (3) *The additive inverse to (L, U) is the cut $-_{\mathbb{R}}(L, U)$.*

HINTS FOR THE PROOF. The proofs follow from the definitions. ∎

The multiplication of reals. By Task 6.19 the multiplicative unit 1 in $\mathbb{Q}$ is represented by the cut

$$1_{\mathbb{R}} = (\blacktriangledown 1, \blacktriangle\blacktriangledown 1)$$

in $\mathbb{R} = \mathrm{Compl}(\mathbb{Q}, \leq)$.

Notice that the following definition performs a careful case analysis, so that the multiplication will obey the usual sign laws.

7.14. Definition (Multiplication of cuts). Given two cuts $c_1 = (L_1, U_1)$, $c_2 = (L_2, U_2)$, we define their *multiplication* as follows:

$$c_1 \cdot_{\mathbb{R}} c_2 = \begin{cases} (\blacktriangledown(U_1 \odot U_2), \blacktriangle\blacktriangledown(U_1 \odot U_2)), & \text{if } 0_{\mathbb{R}} \prec c_1 \text{ and } 0_{\mathbb{R}} \prec c_2 \\[2mm] -_{\mathbb{R}}(-_{\mathbb{R}}c_1) \cdot_{\mathbb{R}} (-_{\mathbb{R}}c_2), & \text{if } c_1 \prec 0_{\mathbb{R}} \text{ and } c_2 \prec 0_{\mathbb{R}} \\[2mm] -_{\mathbb{R}}\left(c_1 \cdot_{\mathbb{R}} (-_{\mathbb{R}}c_2)\right), & \text{if } 0_{\mathbb{R}} \prec c_1 \text{ and } c_2 \prec 0_{\mathbb{R}} \\[2mm] -_{\mathbb{R}}\left((-_{\mathbb{R}}c_1) \cdot_{\mathbb{R}} c_2\right), & \text{if } c_1 \prec 0_{\mathbb{R}} \text{ and } 0_{\mathbb{R}} \prec c_2 \\[2mm] 0_{\mathbb{R}}, & \text{if } c_1 = 0_{\mathbb{R}} \text{ or } c_2 = 0_{\mathbb{R}} \end{cases}$$

where $U_1 \odot U_2 = \{x_1 \cdot x_2 \mid x_1 \in U_1, \ x_2 \in U_2\}$.

7.15. Definition (Multiplicative inverse). For a nonzero cut $c = (L, U)$ we define its *multiplicative inverse* by

$$c^{-1} = \begin{cases} (\blacktriangledown L^{-1}, L^{-1}), & \text{if } 0_{\mathbb{R}} \prec c \\[2mm] -(-c)^{-1}, & \text{if } c \prec 0_{\mathbb{R}} \end{cases}$$

where $L^{-1} = \{x^{-1} \mid x \in L \text{ and } x > 0\}$.

7.16. Task for you (easy). *Prove the following:*
(1) *$(L_1, U_1) \cdot_{\mathbb{R}} (L_2, U_2)$ is a Dedekind cut, whenever (L_1, U_1) and (L_2, U_2) are positive cuts.*
(2) *The operation $\cdot_{\mathbb{R}}$ is associative, commutative and it has the cut $1_{\mathbb{R}}$ as a neutral element.*
(3) *The multiplicative inverse to (L, U) is the cut $(L, U)^{-1}$.*
(4) *The distributive laws for $+_{\mathbb{R}}$ and $\cdot_{\mathbb{R}}$ hold.*

HINTS FOR THE PROOF. The proofs follow from the definitions. ∎

The results of this chapter can be summed up as follows:

7.17. Corollary. *The Dedekind-MacNeille completion $\mathrm{Compl}(\mathbb{Q}, \leq)$ of $(\mathbb{Q}, \leq)$ is a Dedekind complete field.*

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8. SEQUENTIAL COMPLETENESS OF ORDERED FIELDS

At the very beginning of calculus one studies *limits* of sequences of real numbers. It is a remarkable property of real numbers that every *Cauchy* sequence of real numbers has a limit. In this chapter we will study this phenomenon in general, i.e., we will study the behaviour of sequences in ordered fields and *sequentially complete* ordered fields.

We start by the definition of an *absolute value* in an ordered commutative ring with a unit.

8.1. Task for you (easy). *In every ordered commutative ring with a unit $\mathbb{K}$, define*

$$|r| = \begin{cases} r, & \text{if } r > 0, \\ 0, & \text{if } r = 0, \\ -r, & \text{if } r < 0. \end{cases}$$

and call it an absolute value *of r. Prove the following properties of absolute values:*
(1) *$|r| \geq 0$ for every r in $\mathbb{K}$.*
(2) *$|rs| = |r| \cdot |s|$ for every r, s in $\mathbb{K}$.*
(3) *$|r + s| \leq |r| + |s|$ for every r, s in $\mathbb{K}$.*

HINTS FOR THE PROOF. Proceed by analysing various cases (as you do when you are proving these facts in the field $\mathbb{R}$). ∎

The definition of absolute value allows us to introduce the usual cornucopia of notions for sequences of elements in an ordered field.

8.2. Definition. Let (s_n) be a sequence of elements of an ordered field $\mathbb{F}$.
- (1) We say that (s_n) is *bounded*, if there is an element M of $\mathbb{F}$ such that $|s_n| \leq M$ holds for all n.
- (2) We say that (s_n) is *monotone*, if either of the following holds:[15]
  - (a) (s_n) is *increasing*, if $s_0 < s_1 < s_2 < \ldots$ holds.
  - (b) (s_n) is *decreasing*, if $s_0 > s_1 > s_2 > \ldots$ holds.
  - (c) (s_n) is *non-decreasing*, if $s_0 \leq s_1 \leq s_2 \leq \ldots$ holds.
  - (d) (s_n) is *non-increasing*, if $s_0 \geq s_1 \geq s_2 \geq \ldots$ holds.

Having defined the absolute value, we can proceed in defining *Cauchy* and *convergent* sequences. The definitions are formulated in the usual way. In fact, observe that most of the following notions and results have the same phrasing as in any "Calculus 101" material, see, for example [18], or [19].

8.3. Definition (Sequentially complete ordered field). Let $\mathbb{F}$ be an ordered field. Let (s_n) be a sequence of elements of $\mathbb{F}$.
- (1) The sequence (s_n) is a *Cauchy sequence*, if the following property holds:[a]
  For every $\varepsilon > 0$ there exists n_0 such that $|s_n - s_m| < \varepsilon$ holds for all $m \geq n_0$, $n \geq n_0$.
- (2) The sequence (s_n) is a *convergent sequence* with a limit s in $\mathbb{F}$ (we denote it by $s = \lim s_n$), if the following condition holds:
  For every $\varepsilon > 0$ there exists n_0 such that $|s_n - s| < \varepsilon$ holds for all $n \geq n_0$.

We say that $\mathbb{F}$ is *sequentially complete*, if every Cauchy sequence in $\mathbb{F}$ is convergent.

---
[a]Warning: when we write $\varepsilon > 0$, we mean the order in $\mathbb{F}$, whereas $n \geq n_0$, etc., refers to the order on $\mathbb{N}$. This is a usual abuse of notation, but one has to be careful.

8.4. Task for you (easy). *Let $\mathbb{F}$ be an ordered field. Prove that every convergent sequence in $\mathbb{F}$ is Cauchy.*

HINTS FOR THE PROOF. Let $\lim s_n = s$. Choose $\varepsilon > 0$ and find n_0 such that[16] $|s_n - s| < 2^{-1}\varepsilon$ holds for all $n \geq n_0$. Then, using Task 8.1, observe that for all $m \geq n_0$ and $n \geq n_0$ it holds that

$$|s_m - s_n| = |(s_m - s) + (s - s_n)| \leq |s_m - s| + |s - s_n| < 2^{-1}\varepsilon + 2^{-1}\varepsilon = 2 \cdot 2^{-1}\varepsilon = \varepsilon$$

Thus, (s_n) is a Cauchy sequence.   ■

Limits of monotone sequences behave in a rather nice way. Let us show their behaviour for non-decreasing sequences.

8.5. Task for you (easy). *Let (s_n) be a non-decreasing sequence in an ordered field. Then the following conditions hold:*
- (1) *If $s = \lim s_n$, then $s = \sup\{s_n \mid n \geq 0\}$.*
- (2) *If $s = \sup\{s_n \mid n \geq 0\}$, then $s = \lim s_n$.*
- (3) *Suppose $s = \lim s_{n_k}$ for some subsequence (s_{n_k}) of (s_n). Then $s = \lim s_n$.*

HINTS FOR THE PROOF.
- (1) Show first that s is an upper bound of $S = \{s_n \mid n \geq 0\}$. If not, there exists n_0 such that $s < s_n$ for all $n \geq n_0$. Define $\varepsilon = s_{n_0} - s > 0$. Then $s_n - s = |s_n - s| > \varepsilon$ for all $n \geq n_0$. This contradicts $s = \lim s_n$. Hence s is an upper bound of $S = \{s_n \mid n \geq 0\}$.

---
[15]Connoisseurs of calculus know that in all of the definitions one could replace "the property $(*)$ of (s_n) holds for all $n \geq 0$" by "there is n_0 such that the property $(*)$ of s_n holds for all $n \geq n_0$. The reader is invited to think this through.
[16]Recall that, in any ordered field $\mathbb{F}$, we *define* $2 = 1 + 1$, see Definition 5.10. Since $\mathbb{F}$ has characteristic 0 by Task 5.9, we know that $1 + 1 \neq 0$, hence 2 is invertible.

Show that s is the least upper bound of $S = \{s_n \mid n \geq 0\}$. If not, there exists $s' < s$ such that $s_n \leq s'$ for all $n \geq 0$. Define $\varepsilon = s - s' > 0$. Then $|s_n - s| < \varepsilon$ holds for no n, a contradiction.

(2) Choose any $\varepsilon > 0$. We need to prove that $-\varepsilon < s - s_n < \varepsilon$ holds for all but finitely many n. Since s is an upper bound of $\{s_n \mid n \geq 0\}$, we have $s_n \leq s$ for all n. Thus, $-\varepsilon < 0 \leq s - s_n$ holds for all n. Thus we only need to show that $s - s_n < \varepsilon$ holds for all but finitely many n. Suppose this is not the case: let there be infinitely many terms s_n such that $s - s_n \geq \varepsilon$. Since (s_n) is non-decreasing, we have $s - s_n \geq \varepsilon$ for all n. This means that $s - \varepsilon$ is an upper bound of $\{s_n \mid n \geq 0\}$. Since $s - \varepsilon < s$, this contradicts $s = \sup\{s_n \mid n \geq 0\}$.

(3) The equality $s = \sup\{s_{n_k} \mid k \geq 0\}$ holds by (1). Since (s_n) is non-decreasing, $s = \sup\{s_n \mid n \geq 0\}$. Now use (2): $s = \lim s_n$.

■

Next we show that the *Method of Exhaustion* that we stated informally in Chapter 2 characterises Dedekind complete fields. The proof is taken from [3].

> 8.6. **Task for you (easy)** (**Monotone Sequence Property**). *For an ordered field $\mathbb{F}$, the following conditions are equivalent:*
>
> (1) $\mathbb{F}$ *is Dedekind complete.*
> (2) *Every non-decreasing sequence (s_n) in $\mathbb{F}$ that is bounded from above is convergent. Moreover, the limit of (s_n) is the supremum $\sup s_n$.*
> (3) *Every non-increasing sequence (s_n) in $\mathbb{F}$ that is bounded from below is convergent. Moreover, the limit of (s_n) is the infimum $\inf s_n$.*

HINTS FOR THE PROOF. (1) implies (2). Let (s_n) be a non-decreasing sequence bounded from above. Then the set $S = \{s_n \mid n \geq 0\}$ is nonempty and bounded from above. Let $s = \sup S$. Then $s = \lim s_n$ by Task 8.5.

(2) implies (3). Given a non-increasing sequence (s_n) bounded from below by M, observe that $(-s_n)$ is a non-decreasing sequence bounded from above by $-M$. By (2), $s = \lim -s_n$ exists. Then $-s = \lim s_n$.

(3) implies (1). Observe first that any field $\mathbb{F}$ satisfying (3) must necessarily be Archimedean. Indeed: were $\mathbb{F}$ non-Archimedean, then the sequence $s_n = -n$ would be non-increasing and bounded from below. Therefore $s = \lim s_n$ would exist and $s = \inf\{-n \mid n \geq 0\}$. This would be a contradiction.

Let us start with the proof of (3) implies (1). Consider a nonempty set S with an upper bound M. We first prove the following claim:

(∗) For every $n \geq 1$ there exists $y_n \in S$ such that

$$x \leq y_n + \frac{1}{n}$$

holds for any $x \in S$.

Let us fix $n \geq 1$. Choose any $z_1 \in S$. Then one of the following cases hold:

(a) For all $x \in S$, the inequality $x \leq z_1 + \frac{1}{n}$ holds. Then define $y_n = z_1$.
(b) There exists $z_2 \in S$, such that the inequality $z_2 \geq z_1 + \frac{1}{n}$ holds. If $x \leq z_2 + \frac{1}{n}$ holds for all $x \in S$, define $y_n = z_2$. If not, then there exists $z_3 \in S$, such that the inequality $z_3 \geq z_1 + \frac{1}{n}$ holds. And so on. Either we have defined y_n, or there is an infinite sequence (z_k) such that

$$z_k > z_1 + \frac{k-1}{n}$$

holds for all $k \geq 1$. But this cannot happen, since $\mathbb{F}$ is Archimedean: we would have $z_k > M$ for some k. Therefore the process must terminate and we can define y_n.

This concludes the proof of (∗).

Construct a sequence (M_n) as follows:

$$M_0 = M, \quad M_{n+1} = \min\{M_n, y_{n+1} + \frac{1}{n+1}\}$$

Then (M_n) is a non-increasing sequence bounded from below by any element of S. Thus, by (3), we have

$$L = \lim M_n = \inf\{M_n \mid n \geq 0\}$$

We claim that $L = \sup S$. Consider any $L' < L$. Then there exists n such that

$$L' + \frac{1}{n} < L \leq M_n \leq y_n + \frac{1}{n}$$

since $\mathbb{F}$ is Archimedean. Therefore $L' < y_n$ and L' cannot be an upper bound of S. ∎

We aim at proving that Dedekind complete fields are *precisely* the sequentially complete Archimedean fields, see Task 8.9 below. We start with a technical lemma from [20] that will allow us to derive the characterisation rather quickly.

> **8.7. Task for you (easy)** (**Rising Sun Lemma**). *Every infinite sequence in an ordered field admits a monotone subsequence.*

HINTS FOR THE PROOF. Consider an infinite sequence (s_n). Say that a natural number N is a *peak* of (s_n) if $s_n < s_N$ holds for all $n > N$. Consider two mutually exclusive cases:

(1) The sequence (s_n) has infinitely many peaks $N_0 < N_1 < \ldots$.

In this case we have $s_{N_1} > s_{N_2} > \ldots$ and we have found a monotone subsequence.
(2) The sequence (s_n) has finitely many peaks $N_0 < N_1 < \ldots < N_k$.

In this case, denote by N the natural number such that there is no peak of (s_n) after N. Thus, for $n_1 = N$ there exists $n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Now n_2 is not a peak, so there exists $n_3 > n_2$ such that $s_{n_3} \geq s_{n_2}$. Proceed like this to construct a monotone sequence $s_{n_1} \leq s_{n_2} \leq s_{n_3} \ldots$

The proof is finished. ∎

> **8.8. Task for you (easy)** (**Bolzano-Weierstrass Property**). *For an ordered field $\mathbb{F}$, the following conditions are equivalent:*
>
> (1) *$\mathbb{F}$ is Dedekind complete.*
> (2) *$\mathbb{F}$ has the* Bolzano-Weierstrass Property:[a] *every bounded sequence in $\mathbb{F}$ admits a convergent subsequence.*
>
> ――――――――――
> [a]Bernardus Placidus Johann Nepomuk Gonzal Bolzano (5 October 1781 – 18 December 1848) was a Prague-based theologian and mathematician. He tried to understand the notion of continuity and he gave the first proof of the *Intermediate Value Theorem* in 1817. Karl Theodor Wilhelm Weierstrass (31 October 1815 – 19 February 1897) was a German mathematician. He made significant contributions to the theory of continuous functions and the foundations of modern analysis.

HINTS FOR THE PROOF. To prove that (1) implies (2), consider a bounded sequence (s_n) in $\mathbb{F}$. By Task 8.7 there is a monotone subsequence (s_{n_k}) that is, of course, bounded. Now use the Monotone Sequence Property of $\mathbb{F}$ (see Task 8.6).

To prove (2) implies (1), proceed by contraposition, using Task 8.6. Let (s_n) be a non-decreasing sequence bounded from above that does not converge in $\mathbb{F}$. Then (s_n) cannot admit any convergent subsequence by Task 8.5. ∎

We are now ready to clarify the connection between Archimedean and Dedekind complete fields. See also Task 7.7.

> **8.9. Task for you (not so easy).** *For an ordered field $\mathbb{F}$, the following are equivalent:*
>
> (1) *$\mathbb{F}$ is Dedekind complete.*
> (2) *$\mathbb{F}$ is sequentially complete and Archimedean.*

HINTS FOR THE PROOF. (1) implies (2). We know that every Dedekind complete field is Archimedean by Task 7.7. It therefore suffices to prove that $\mathbb{F}$ is sequentially complete.

Let (s_n) be a Cauchy sequence in $\mathbb{F}$. Use the following hints:

(a) (s_n) is bounded.

Indeed: choose $\varepsilon = 1$. Then there exists n_0 such that $|s_n - s_m| < 1$ for all $m \geq n_0$ and all $n \geq n_0$. Put $m = n_0$. Then $|s_n - s_{n_0}| < 1$ for all $n \geq n_0$. Thus $|s_n| < |s_{n_0}| + 1$ holds for all $n \geq n_0$. Define

$$M = \max\{|s_0|, \ldots, |s_{n_0-1}|, |s_{n_0}| + 1\}$$

Then $|s_n| \leq M$ for all $n \geq 0$.

(b) By Task 8.8, (s_n) contains a convergent subsequence s_{n_k}, since $\mathbb{F}$ is Dedekind complete.

Indeed: (s_n) is bounded by part (a) above.

(c) Prove that any Cauchy sequence that admits a convergent subsequence is itself convergent.

Let (x_n) be a Cauchy sequence, with (x_{n_k}) a convergent subsequence. Let $L = \lim x_{n_k}$. We claim that $L = \lim x_n$. To that aim, choose $\varepsilon > 0$. Find N_1 such that $|x_n - x_m| < 2^{-1}\varepsilon$ for all $n \geq N_1$ and all $m \geq N_1$. Find N_2 such that $|x_{n_k} - L| < 2^{-1}\varepsilon$ for all $k \geq N_2$. Put $N = \max\{N_1, N_1\}$. Then

$$|x_n - L| \leq |x_n - x_{n_N}| + |x_{n_N} - L| < 2^{-1}\varepsilon + 2^{-1}\varepsilon = \varepsilon$$

holds for all $n \geq N$.

Combining (a), (b), (c) we finish the proof of (1) implies (2).

(2) implies (1).[17] Let S be a nonempty set bounded from above. Recall the notation

$$\blacktriangle S = \{r \in \mathbb{F} \mid s \leq r \text{ for all } s \in S\}$$

for the set of all upper bounds of S. We will construct a non-increasing Cauchy sequence (r_n) of elements of $\blacktriangle S$ and we will prove that

$$L = \lim r_n = \sup S$$

holds.

For any $s \in S$ and any $r \in \blacktriangle S$ we choose a negative integer m and a positive integer M such that

$$m < s \leq r < M$$

holds. We can do it, since $\mathbb{F}$ is Archimedean.

Define

$$S_n = \{k \in \mathbb{Z} \mid \frac{k}{2^n} \in \blacktriangle S \text{ and } \frac{k}{2^n} < M\}$$

for every $n \geq 0$.

Every set S_n is finite and nonempty. Indeed: $2^n \cdot M$ is an element of S_n and every element of S_n is in between $2^n \cdot m$ and $2^n \cdot M$. Put

$$k_n = \min S_n, \quad r_n = \frac{k_n}{2^n}$$

Then $\frac{2k_n}{2^{n+1}} = \frac{k_n}{2^n} \in \blacktriangle S$ and $\frac{2k_n - 2}{2^{n+1}} = \frac{k_n - 1}{2^n} \notin \blacktriangle S$. Hence

$$\text{either } k_{n+1} = 2k_n \text{ or } k_{n+1} = 2k_n - 1$$

and therefore

$$\text{either } r_{n+1} = 2r_n \text{ or } r_{n+1} = r_n - \frac{1}{2^{n+1}}$$

Therefore (r_n) is a non-increasing sequence.

(a) The sequence (r_n) is Cauchy.

For all $1 \leq m < n$ we have the inequalities

$$0 \leq r_m - r_n = (r_m - r_{m+1}) + (r_{m_1} - r_{m+2}) + \ldots + (r_{n-1} - r_n) \leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \ldots \frac{1}{2^n} = \frac{1}{2^m}$$

Thus (r_n) is a Cauchy sequence.

(b) Denote by $L = \lim r_n$. We prove that $L = \sup S$.

Suppose L is not an upper bound of S. Then there exists $s \in S$ such that $L < s$. Therefore there is n such that $r_n - L = |r_n - L| < s - L$. Hence $r_n < s$, which contradicts $r_n \in \blacktriangle S$. Thus, L is an upper bound of S.

Choose an upper bound L' of S with $L' < L$. Then there is $n \geq 1$ such that $\frac{1}{2^n} < L - L'$. Therefore $r_n - \frac{1}{2^n} \geq L - \frac{1}{2^n} > L'$. Thus $r_n - \frac{1}{2^n} = \frac{k_n - 1}{2^n} \in \blacktriangle S$, which contradicts the definition of k_n. Hence L is a least upper bound of S.

---

[17]The proof follows closely [3] and [17].

The proof is finished.   ∎

8.10. **Example** (**Sequentially complete non-Archimedean ordered field**). Examples of sequentially complete non-Archimedean ordered fields exist. They are, however, a bit complicated to describe. We will present perhaps the simplest example with no full proof of its properties.

The example is the field

$$\mathbb{R}(\!(x)\!)$$

of *formal Laurent series* in indeterminate x and with real coefficients. Such a series is an expression of the form

$$s(x) = \sum_{n=-\infty}^{+\infty} a_n x^n$$

where there exists N, such that $a_n = 0$ for all $n < N$. The operations are defined as follows

$$\left(\sum_{n=-\infty}^{+\infty} a_n x^n \right) + \left(\sum_{n=-\infty}^{+\infty} b_n x^n \right) = \sum_{n=-\infty}^{+\infty} (a_n + b_n) x^n$$

$$\left(\sum_{n=-\infty}^{+\infty} a_n x^n \right) \cdot \left(\sum_{n=-\infty}^{+\infty} b_n x^n \right) = \sum_{n=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} (a_{n-k} \cdot b_k) \right) \cdot x^n$$

having the obvious zero and unit. It is easy to prove that $\mathbb{R}(\!(x)\!)$, together with the above operations, is a field.

If, for a Laurent series $s(x)$ as above, we denote by

$$\nu(s(x)) = \text{ the smallest } N \text{ such that } a_N \neq 0$$

We declare $s(x)$ to be *positive*, if $a_{\nu(s(x))}$ is a positive real number.

Then it is easy to show that $\mathbb{R}(\!(x)\!)$ is an *ordered field*. Since the series

$$s(x) = \frac{1}{x}$$

satisfies the property

$$n < s(x)$$

for all natural numbers n, the field $\mathbb{R}(\!(x)\!)$ is *non-Archimedean*.

The technical bit now requires to show the following characterisation of *convergent* sequences in $\mathbb{R}(\!(x)\!)$. A sequence

$$s_k(x) = \sum_{n=-\infty}^{+\infty} a_n^{(k)} x^n$$

is convergent in $\mathbb{R}(\!(x)\!)$ iff the following two conditions

(1) There is an integer N such that for all $k \geq 0$ and all $n < N$ the equality $a_n^{(k)} = 0$ holds.
(2) For each integer n there is k_0 such that $a_n^{(k)}$ is constant for all $k \geq k_0$.

are satisfied.

The above characterisation allows one to prove that every Cauchy sequence in $\mathbb{R}(\!(x)\!)$ is convergent. Hence the field $\mathbb{R}(\!(x)\!)$ is a sequentially complete non-Archimedean ordered field. For more details we refer to [13].

The results of this chapter can be summed up as follows:

8.11. **Corollary.** *For a field $\mathbb{F}$, the following conditions are equivalent:*

(1) *$\mathbb{F}$ is* Dedekind complete, *i.e., every nonempty subset bounded from above has a supremum.*
(2) *Every nonempty subset of $\mathbb{F}$ bounded from below has an infimum.*
(3) *Every non-decreasing sequence (s_n) in $\mathbb{F}$ that is bounded from above is convergent. Moreover, the limit of (s_n) is the supremum $\sup s_n$.*
(4) *Every non-increasing sequence (s_n) in $\mathbb{F}$ that is bounded from below is convergent. Moreover, the limit of (s_n) is the infimum $\inf s_n$.*
(5) *$\mathbb{F}$ has the* Bolzano-Weierstrass Property*: every bounded sequence in $\mathbb{F}$ admits a convergent subsequence.*
(6) *$\mathbb{F}$ is* sequentially complete *and* Archimedean.

8.12. **Remark.** There is a vast amount of completeness properties of ordered fields. A very nice and readable survey is in [13]. The notions usually pick a theorem in calculus and turn it into a definition. For example, an ordered field $\mathbb{F}$ is called *Cantor complete*, if every sequence

$$\{[a_n; b_n] \mid n \geq 0\} \quad \text{with } [a_0; b_0] \supseteq [a_1; b_1] \supseteq [a_2; b_2] \supseteq \ldots$$

of closed bounded intervals[18] in $\mathbb{F}$ has a nonempty intersection.

The above definition is clearly based on *Cantor's Intersection Theorem*, well-known from any basic calculus course. See, e.g., [17] or [19].

9. Final remarks

Of course, Dedekind's construction of the reals is not the only possible one. A survey of other possible approaches to constructing reals is given in [22]. We comment on some of the constructions in this final chapter.

9.1. **Cantor's construction of the reals via Cauchy sequences.** Probably the best-known construction of the reals out of rationals is given by considering certain *sequences* of rational numbers. Although this construction is often dubbed as "Cauchy[19] reals", the construction was presented by Georg Cantor in 1873 in [2]. Recall from Definition 8.3 that a sequence (s_n) of rational numbers is a *Cauchy sequence*, if the following condition holds:

For every $\varepsilon > 0$ there exists n_0 such that $|s_n - s_m| < \varepsilon$ holds for all $m \geq n_0$, $n \geq n_0$.

If we denote by C the class of all Cauchy sequences of rationals, we can define an equivalence relation $\sim$ on C by putting

$$(s_n) \sim (s'_n) \quad \text{iff} \quad \lim_n |s_n - s'_n| = 0$$

Then the set of reals is the quotient set

$$C/\sim$$

The operations on $C/\sim$ are then defined as follows:

$$[(s_n)]_\sim + [(t_n)]_\sim = [(s_n + t_n)]_\sim \qquad [(s_n)]_\sim \cdot [(t_n)]_\sim = [(s_n \cdot t_n)]_\sim$$

Zero and unit are defined by equivalence classes of sequences constantly 0 and 1, respectively.

It takes lengthy computations to prove that the above definitions yield a Dedekind complete field. See, e.g., [17].

9.2. **Eudoxus' construction of the reals.** A remarkable construction of the reals can be derived from Eudoxus' *Theory of Proportions*. We do not have original Eudoxus' papers but Book V of Euclid's *Elements* [10] is devoted to Euclid's treatment of proportions. We sketch the idea of what a proportion is based on the explanation by *Thomas Little Heath*.[20]

The famous *Definition 5* of Book V of *Elements* reads in modern English as follows:

---

[18]A *closed bounded interval* in an ordered field $\mathbb{F}$ is the set $[a; b] = \{x \in \mathbb{F} \mid a \leq x \leq b\}$.

[19]Augustin-Louis Cauchy (21 August 1789 – 23 May 1857) was a prominent French mathematician. He was probably the first bringing rigour to calculus. There are many concepts in mathematics bearing Cauchy's name, Cauchy sequences being one of them.

[20]Sir Thomas Little Heath (5 October 1861 – 16 March 1940) was an English mathematician and a historian of classical Greek mathematics. He translated *Elements* into English in 1920 and gave vast commentaries to the text.
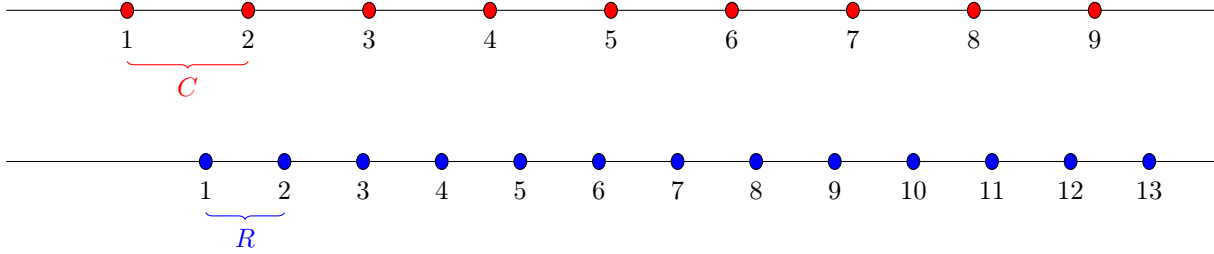
> Magnitudes are said to be in the same ratio, the first to the second, and the third to the
> fourth, when equal multiples of the first and the third either both exceed, are both equal to,
> or are both less than, equal multiples of the second and the fourth, respectively, being taken
> in corresponding order, according to any kind of multiplication whatever.

<div align="right">Euclid, [10]</div>

Using the notation $a \div b$ for the *ratio* of a, b, the above Euclid quote reads as follows:

> Ratios $a \div b$ and $c \div d$ are *equal*, if for all positive integers m and n, the statements $ma > nb$
> and $mc > nd$ are either both true or both false, and analogously for $ma = nb$ and $ma < nb$.

Heath attributes the following interpretation of this definition to *Augustus de Morgan*.[21] Imagine a fence
with equally spaced railings in front of a colonnade of equally spaced columns as in the following picture:

If the distance between two consecutive railings is denoted by R and the distance between two consecutive
columns by C, we can see, for example, that the fifth railing is in between the fourth and the fifth columns.
This means that $4C < 5R < 5C$ holds. Thus

$$\frac{4}{5} < R \div C < \frac{1}{1}$$

We can get a higher degree of accuracy by noting that, for example $7C < 10R < 8C$, hence

$$\frac{7}{10} < R \div C < \frac{8}{10}$$

Assuming that both the collonade and the railing extend infinitely we can, by proceeding as above, say that
the proportion $R \div C$ is a representation of a real number. The details of this construction of the reals can
be found, e.g., in [1].

9.3. Task for you (easy). *Suppose that $\mathbb{F}$ is a Dedekind complete field. Then $\mathbb{F}$ is isomorphic to the
field $\mathbb{R}$ of reals, introduced in Chapter 7.*

HINTS FOR THE PROOF. We will construct a bijection

$$I : \mathbb{F} \longrightarrow \mathbb{R}$$

and show that the properties

$$I(r + s) = I(r) + I(s) \qquad I(r \cdot s) = I(r) \cdot I(s) \qquad \text{if } r > 0, \text{ then } I(r) > 0$$

hold.

Recall from Task 5.11 that $\mathbb{F}$ contains $\mathbb{Q}$ as a subfield. Define

$$I(r) = (L_r, U_r), \quad \text{where } L_r = \{x \in \mathbb{Q} \mid x \le r\} \text{ and } U_r = \blacktriangle L_r$$

Verify that the mapping $I : \mathbb{F} \longrightarrow \mathbb{R}$ has the desired properties.   ■

9.4. Task for you (easy) (The extended reals). *Prove that one could define the* extended reals
$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$ by considering cuts (L, U) on $\mathbb{Q}$ and allowing either L or U to be empty.

---

[21]Augustus De Morgan (27 June 1806 – 18 March 1871) was a British mathematician and logician. He made *mathematical
induction* rigorous. He is also famous for laws of classical logic, nowadays called *de Morgan's Laws*.

HINTS FOR THE PROOF. Modify the Dedekind-MacNeille completion accordingly and denote the resulting poset by $\mathrm{Compl}^*(X, \sqsubseteq)$. Show that the cut $(X, \emptyset)$ corresponds to the largest element of $\mathrm{Compl}^*(X, \sqsubseteq)$ and that the cut $(\emptyset, X)$ corresponds to the least element of $\mathrm{Compl}^*(X, \sqsubseteq)$. Thus, in the modified $\mathrm{Compl}^*(X, \sqsubseteq)$, *every* subset has a supremum and an infimum. Then go through Chapter 7 and work it through with the definition $\mathbb{R}^* = \mathrm{Compl}^*(\mathbb{Q}, \leq)$. ∎

9.5. Remark. In the literature, the above "modifed" Dedekind-MacNeille completion is often used instead of our Definition 6.18. The modified definition is then called *MacNeille completion*, see, e.g., [5].

9.6. Remark (Hyperreals — A way to nonstandard analysis). There is another construction that might be very appealing during a "coffee-conversation" about differential and integral calculus: How about *extending* the "usual reals" $\mathbb{R}$ to an ordered field $^\star\mathbb{R}$ that contains *infinitely large* and *infinitely small* real numbers?

The "classical" definition of continuity of $f : \mathbb{R} \longrightarrow \mathbb{R}$ at point a, i.e., the statement

$$\lim_{x \to a} f(x) = f(a)$$

would then have the form $f(a) - f(x)$ *is infinitely small, whenever* $a - x$ *is infinitely small*.

In constructing $^\star\mathbb{R}$ one usually proceeds as follows:

(1) Form $\mathbb{R}^\mathbb{N}$, the set of all sequences of real numbers.
(2) Equip $\mathbb{R}^\mathbb{N}$ with operations $+$ and $\cdot$

$$(a_n) + (b_n) = (a_n + b_n), \quad (a_n) \cdot (b_n) = (a_n \cdot b_n)$$

   that turn $\mathbb{R}^\mathbb{N}$ into a commutative ring with a unit.
(3) Define an equivalence relation $\sim$ on $\mathbb{R}^\mathbb{N}$ that allows you to treat sequences (a_n) and (b_n) as the same, whenever the set $\{n \mid a_n = b_n\}$ is "large enough". Then form the quotient set

$$^\star\mathbb{R} = \mathbb{R}/\!\!\sim$$

   and prove that it has the required properties.

There is a catch, of course: we have not said what it means that a subset of $\mathbb{N}$ is "large enough". As it turns out, the proper notion is that of an *ultrafilter* $\mathscr{U}$ on $\mathbb{N}$. What is an ultrafilter, then? We say that a collection $\mathscr{U}$ of subsets of $\mathbb{N}$ is an *ultrafilter*, if the following three conditions are satisfied:

(a) If $A \in \mathscr{U}$ and $A \subseteq B$, then $B \in \mathscr{U}$.
(b) If $A \in \mathscr{U}$ and $B \in U$, then $A \cap B \in \mathscr{U}$.
(c) For every $A \subseteq \mathbb{N}$, either A or $\mathbb{N} \setminus A \in \mathscr{U}$.

The above three properties then justify the claim "A is large enough iff $A \in \mathscr{U}$":

(a) A superset B of a "large enough A" is "large enough".
(b) The intersection $A \cap B$ of "large enough A and B" is "large enough".
(c) For every subset $A \subseteq \mathbb{N}$ either A or its complement $\mathbb{N} \setminus A$ is "large enough".

We will postpone the construction of $^\star\mathbb{R}$ for further notes, since to understand ultrafilters requires to understand a certain axiom of Set Theory, called *The Axiom of Choice*. Namely, the proof of the existence of at least one "nice" ultrafilter on $\mathbb{N}$ uses Axiom of Choice. Moreover, there are quite a lot of "nice" ultrafilters on $\mathbb{N}$ and we may wonder which one to choose. That it does not matter which one we choose, follows from another axiom of Set Theory, called *The Continuum Hypothesis*.

See [11] for a brief intro to $^\star\mathbb{R}$ or the beautiful book [12] that contains all the details. Or simply wait, we're gonna get there

The literature. Here we comment on where to find more details about the notions from this text:

(1) It is certainly worthwhile to read the original papers [6] and [7] by Richard Dedekind. Or their translation into English [8].
(2) Rings, fields, integral domains, etc. and their ordered variants are covered in any textbook of algebra. We recommend the books [4] by Paul Moritz Cohn.

(3) Dedekind-MacNeille completion is described in full detail (as part of an even more general construction) in an easy-going, reader friendly book [5] by Brian Davey and Hillary Priestley.

(4) The details of Dedekind's construction of reals can be found, e.g., in the freely available book [23] by Elias Zakon.

(5) The structure of the set $\mathbb{R}$ is far richer than we have indicated here. For a thorough information about problems concerning $\mathbb{R}$ we recommend the book [9] by Oliver Deiser.

(6) A very interesting and relevant book is [14], edited by Stephen Hawking. The first edition contains translations and commentaries of works of great mathematicians throughout the history: Euclid, Archimedes, Diophantus, René Descartes, Isaac Newton, Pierre Simon de Laplace, Jean Baptiste Fourier, Karl Friedrich Gauss, Augustin-Louis Cauchy, George Boole, Georg Friedrich Bernhard Riemann, Karl Weierstrass, Richard Julius Wilhelm Dedekind, Georg Cantor, Henri Lebesgue, Kurt Gödel and Alan Mathison Turing. The new edition [15] of this book adds works by Nikolai Ivanovich Lobachevsky, János Bolyai and Évariste Galois.

This amazing book that has also a *physics counterpart* [16]. There you can find the works of Nicolaus Copernicus, Galileo Galilei, Johannes Kepler, Isaac Newton and Albert Einstein.

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