A 2D Bin-packing and Knapsack problem

Algorithmic Methods for Mathematical Models

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Intro

Problem description

Dimensions	$x \times y$ (in millimeters)
Capacity	c (in grams)
Products	n

Price	p_i (in euros)
Weight	w_i (in grams)
Side	s_i (in millimeters)

Methodology

Methodology

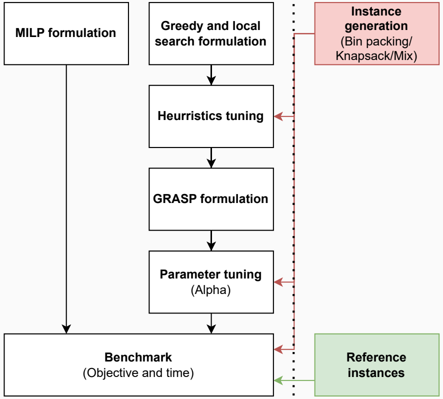
MILP formulation
Greedy and local search formulation
Instance generation (Bin packing/ Knapsack/Mix)
Heurristics tuning
GRASP formulation
Parameter tuning (Alpha)
Benchmark (Objective and time)
Reference instances

Figure 1: Proposed methdology

Instance Generation

We consider three types of generated instances

- Bin packing (designed with optimal objective $2n^2$)
- Knapsack
- Mix (mimics reference instances)

Mixed Integer Linear Programming Approaches

Plan-of-attack

Two ideas on how to express the non-overlapping constraint

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Two ideas on how to express the non-overlapping constraint

- If we select products $i, j \in P$ then, product i is to the left or below product j – or vice verse.

Plan-of-attack

Two ideas on how to express the non-overlapping constraint

- If we select products $i, j \in P$ then, product i is to the left or below product j – or vice verse.
- Or, each suitcase cell can at most be occupied by one product

MILP I - Plan-of-attack

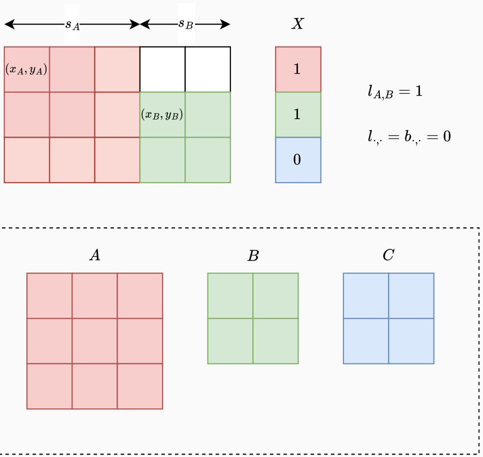

Figure 2: Non-overlapping constraint using a geometric interpretation

MILP I - Formulation

- X_i, binary, is one if product $i \in P$ is chosen, otherwise zero.
- (x_i, y_i), integers, coordinate of the bottom left corner of product $i \in P$

MILP I - Formulation (continued)

$$
\begin{aligned}
\max \quad & \sum_{k \in P} p_k \cdot X_k && && (1)\\
\text{s.t.} \quad & \sum_{(i,j)\in S,\ k \in P} w_k \cdot X_i \leq c && && (2)\\
& x_k \geq 1 && \forall k \in P && (3)\\
& y_k \geq 1 && \forall k \in P && (4)\\
& x_k + s_k \leq x && \forall k \in P && (5)\\
& y_k + s_k \leq y && \forall k \in P && (6)\\
X_i = X_j = 1 \implies & (x_i + s_i \leq x_j) && \forall i \in P, \forall j \in P : i < j && (7)\\
& \vee (x_j + s_j \leq x_i) && && \\
& \vee (y_i + s_i \leq y_j) && && \\
& \vee (y_j + s_j \leq y_i) && &&
\end{aligned}
$$

MILP I - Linearization

$$
\begin{aligned}
x_i + s_i &\leq x_j + M \cdot (1 - l_{i,j}) && \forall i \in P, \forall j \in P : i < j \\
x_j + s_j &\leq x_i + M \cdot (1 - l_{j,i}) && \forall i \in P, \forall j \in P : i < j \\
y_i + s_i &\leq y_j + M \cdot (1 - b_{i,j}) && \forall i \in P, \forall j \in P : i < j \\
y_j + s_j &\leq y_i + M \cdot (1 - b_{j,i}) && \forall i \in P, \forall j \in P : i < j \\
l_{i,j} + b_{i,j} + l_{j,i} + b_{j,i} &\geq X_i + X_j - 1 && \forall i \in P, \forall j \in P : i < j
\end{aligned}
$$

MILP II - Plan-of-attack

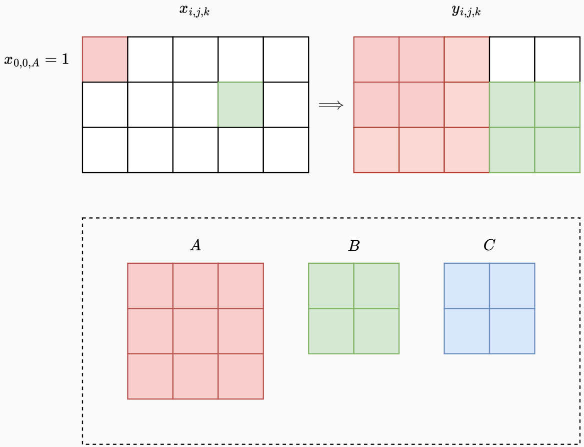

Figure 3: Non-overlapping constraint using an assignment interpretation

MILP II - Formulation

- $x_{i,j,k}$, binary, is one if product $k \in P$ has the bottom left corner in position (i, j), zero otherwise.
- $y_{i,j,k}$, binary, is one if product $k \in P$ occupies position (i, j).

MILP II - Formulation (continued)

$$
\begin{aligned}
\max \quad & \sum_{(i,j)\in S, k\in P} p_k \cdot x_{i,j,k} && && (8)\\
\text{s.t.} \quad & \sum_{(i,j)\in S,\ k\in P} w_k \cdot x_{i,j,k} \leq c && && (9)\\
& \sum_{k\in P} x_{i,j,k} \leq 1 && \forall (i,j) \in S && (10)\\
& \sum_{k\in P} y_{i,j,k} \leq 1 && \forall (i,j) \in S && (11)\\
& \sum_{(i,j)\in S} x_{i,j,k} \leq 1 && \forall k \in P && (12)\\
& x_{i,j,k} = 0 && \forall k \in P, \forall (i,j) \in O_k && (13)\\
& x_{i,j,k} \leq y_{i',j',k} && \forall (i,j) \in S, \forall k \in P && (14)\\
& && \forall (i',j') \in S'_{i,j,k} &&
\end{aligned}
$$

MILP II - Formulation (continued)

- $S = \{1, \dots, x\} \times \{1, \dots, y\}$ are the suitcase cell indices
- $S_{i,j,k} = S \cap \{i, .., i + s_k - 1\} \times \{j, \dots, j + s_k - 1\}$ are the cells occupied by product $k \in P$, when placed at (i, j).
- $O_k = \{(i, j) \mid i + s_k - 1 > x \vee j + s_k - 1 > y\}$ are the corner placements for product $k \in P$, such that product k would be out-of-bounds.

Metaheuristics

Greedy search

Selecting the greedy function

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🚫 Generate all product and placement pairs. Select the one that maximizes q (depends on position).

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Selecting the greedy function

- 🚫 Generate all product and placement pairs. Select the one that maximizes q (depends on position).
- ✅ Select product that maximizes q (independent of position). Place it according to `MaxRect` algorithm[2].

Greedy Search

```
function CandidateSet($products$, $suitcase$)
    $candidates \leftarrow \{\}$
    for $product \in products$ do
        if $product \notin suitcase$ then
            if $suitcase.weight + product.weight \leq c$ then
                if CanFit($product$, $suitcase$) then          ▷ Try to add square w.
MaxRect algo.
                    $candidate \leftarrow$ AddProduct($suitcase$, $product$)
                    $candidates \leftarrow candidates \cup \{candidate\}$
                end if
            end if
        end if
    end for
    return $candidates$
end function
```

Greedy Search

```
function GreedySearch($products$, $q$)
    $suitcase \leftarrow$ EmptySuitcase($x, y$)
    for $product$ in Sort($products, q$) do                          ▷ Sort by $q(\cdot, \cdot)$
        if $suitcase.weight + product.weight \leq c$ then       ▷ Check weight limit
            if CanFit($product$, $suitcase$) then                    ▷ Try MaxRect algo.
                $suitcase \leftarrow$ AddProduct($suitcase, product$)
            end if
        end if
    end for
    return $suitcase$
end function
```

Local Search - Plan of Attack

1. We remove a product from the suitcase
2. Optionally, we repack the suitcase
3. We consider new suitcase, with each product that is not in the suitcase added.

Local Search - Pseudocode

```
function LocalSearch($products, suitcase, repack$)
    $improved \leftarrow$ false
    while $\neg improved$ do
        $improved \leftarrow$ false
        for $neighbour \in$ NeighborhoodSolutions($products, suitcase$) do
            if $neighbour.value > suitcase.value$ then
                $suitcase \leftarrow neighbour$
                $improved \leftarrow$ true
                break
            end if
        end for
    end while
    return $suitcase$
end function
```

Local Search - Pseudocode (continued)

```
function NeighborhoodSolutions(*products*, *suitcase*, *repack*)
    $N \leftarrow \{\}$
    for *productToRemove* $\in$ *products* do

*neighborhoodSuitcase* $\leftarrow$ RemoveProduct(*suitcase*, *productToRemove*)
        if *repack* then
            *neighborhoodSuitcase* $\leftarrow$ Repack(*neighborhoodSuitcase*)
        end if
        $N \leftarrow N \cup$ CandidateSet(*products*, *neighborhoodSuitcase*)
    end for
    return $N$
end function
```

GRASP

```
function GRASP(products, q, alpha)
    $suitcase \leftarrow \textsc{EmptySuitcase}(x, y)$
    while true do
        $candidates \leftarrow \textsc{CandidateSet}(products, suitcase)$
        if $\textsc{Empty}(candidates)$ then
            break
        end if
        $q_{min} \leftarrow \min\{q(p, suitcase) \mid p \in candidates\}$
        $q_{max} \leftarrow \max\{q(p, suitcase) \mid p \in candidates\}$
        $RCL_{max} \leftarrow \{p \in candidates \mid q(p, suitcase) \geq q_{max} - \alpha(q_{max} - q_{min})\}$
        $product \leftarrow \textsc{ChooseRandom}(RCL_{max})$
        $suitcase \leftarrow \textsc{AddSquare}(suitcase, square)$
    end while
    $suitcase \leftarrow \textsc{LocalSearch}(products, suitcase)$
    return $suitcase$
end function
```

[1]

Results

Heuristics tuning

- For bin-packing, s_i was optimal
- For knapsack, $\frac{p_i}{w_i}$ was optimal
- For mix, p_i was optimal. (On the reference instances, $\frac{p_i}{s_i}$ was optimal)

GRASP tuning

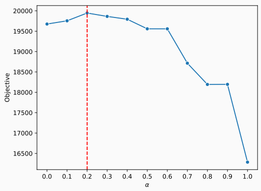

Figure 4: Bin packing

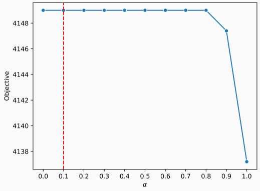

Figure 5: Knapsack

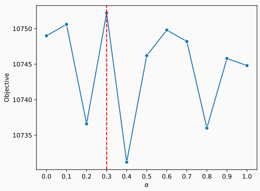

Figure 6: Mix

- Mean performance of various α values, using GRASP across generated instances ($n = 50$).

Comparisons of Algorithms - Bin Packing i

Table 1: Bin Packing: Success and Failure

	50	100	200
Greedy	✅	✅	✅
Greedy w. local search	✅	✅	✅
GRASP	✅	✅	✅
MILP I	⌛	⌛	⌛
MILP II	⌛	☠️	☠️

Comparisons of Algorithms - Bin Packing ii

Table 2: Bin Packing: Best Performance

	50	100	200
Greedy	🥈	🥈	🥈
Greedy w. local search	🥈	🥈	🥈
GRASP	🥇	🥇	🥇
MILP I	4️⃣	4️⃣	4️⃣
MILP II	5️⃣	5️⃣	5️⃣

- The bin packing problem was designed so that the objective value of the optimal solution was $2n^2$. The objective value of the GRASP solutions is within 99.36 %, 99.40% and 99.60% of the optimal solution for bin packing with $n = 50$, $n = 100$ and $n = 200$ respectively.

Comparisons of Algorithms - Knapsack

Table 3: Knapsack: Success and Failure

	50	100	200
Greedy	✅	✅	✅
Greedy w. local search	✅	✅	✅
GRASP	✅	✅	✅
MILP I	✅	✅	✅
MILP II	✅	✅	✅

Comparisons of Algorithms - Knapsack (continued)

Table 4: Knapsack: Best Performance

	50	100	200
Greedy	🥉	4️⃣	4️⃣
Greedy w. local search	🥉	4️⃣	4️⃣
GRASP	🥉	🥇	🥇
MILP I	🥇	🥇	🥇
MILP II	🥇	🥇	🥇

Comparisons of Algorithms - Mix

Table 5: Mix: Success and Failure

	50	100	200
Greedy	✅	✅	✅
Greedy w. local search	✅	✅	✅
GRASP	✅	✅	✅
MILP I	⌛	⌛	⌛
MILP II	☠️	☠️	☠️

Comparisons of Algorithms - Mix (continued)

Table 6: Mix: Best Performance

	50	100	200
Greedy	🥉	🥇	🥇
Greedy w. local search	🥉	🥇	🥇
GRASP	🥈	🥇	🥇
MILP I	🥇	🥇	4️⃣
MILP II	5️⃣	5️⃣	5️⃣

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Discussion and Conclusion

- The MILP formulations exhibited different performance.
- Metaheuristic techniques proved efficient.
- It is essential to tune metaheuristic techniques to the specific problem.
- Faster GRASP implementation had better performance than clever heuristics.

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