Time Series Analysis - Assignment 3

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April 2023

1 Univariate ARIMA

1.1 Presenting the data

This report aims to forecast the average house prices for several regions of Denmark and interpret the forecasting model.

The dataset used for forecasting is derived from Danmarks Statistik.[2] It includes the quarterly interest rate and inflation rate from 1992Q2 to 2023Q1 and the quarterly average house price in Greater Copenhagen, Zealand, Central Jutland and rural areas (specifically, Southern Denmark and Northern Jutland) from 1992Q2 to 2022Q3.

The quarterly interest rate, inflation rate and average house sales price (in thousands) throughout Denmark are given in fig. 1.

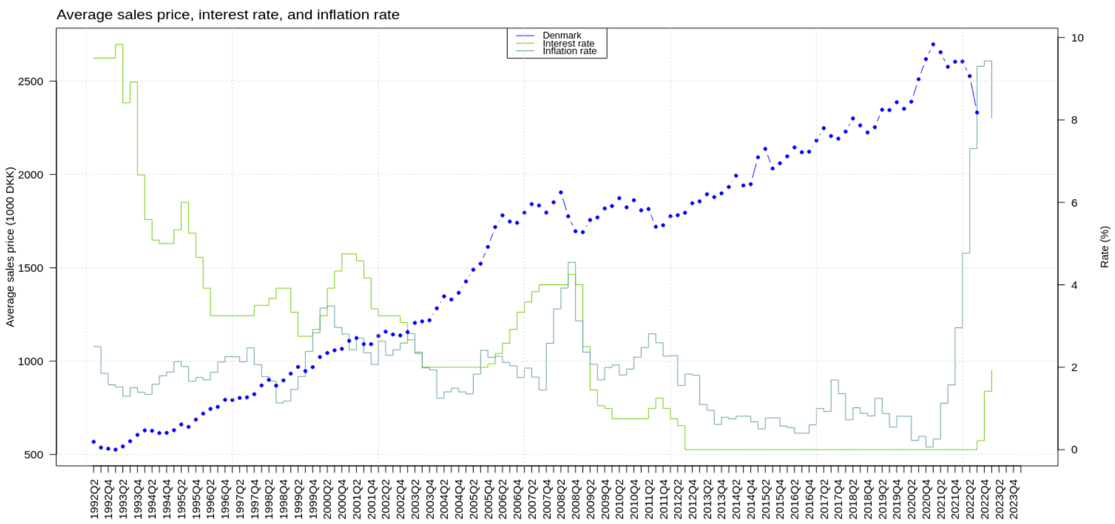

Figure 1: Danish quarterly, average, sales prices, with interest and inflation rates.

There is a clear positive linear trend in the average house price, and a deviation from the linear trend, from 2005 to 2011, with drops in 2007-2008 (likely due to the global financial crisis) and 2010-2011 (likely due to the European debt crisis). Further, a peak occurred in 2020.

Basic macroeconomics would suggest that interest rates and inflation are negatively correlated. Higher interest rates should also result in lower house prices through more defaults and fewer purchases being financed.

Overall, the average house price is non-stationary, and a differencing and, optionally, a variance-stabilizing transformation is required.

1.2 ACF and PACF

In this section we will consider the ACF and PACF for the average house price, to determine an appropriate variance-stabilizing transformation and order of differencing, to ensure stationarity, which is an assumption for the ARMA models. The cross-correlation function for the four regions is handled in section 2, since it is not relevant for the univariate model. We will use the same approach as in [1, Section 6.7] to determine the appropriate transformations.

From the linear trend in fig. 1, we would expect an ACF that does not decrease sufficiently fast. Further, it would not be surprising to see a yearly seasonality.

The ACF and PACF of the quarterly average house price is given in respectively fig. 2 and fig. 3.

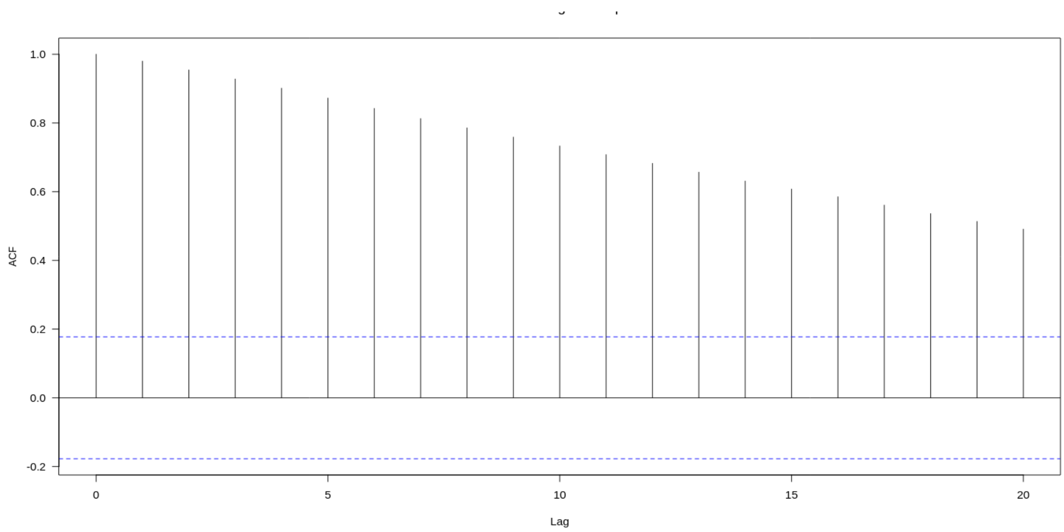

Figure 2: ACF for Danish quarterly average house price in Denmark.

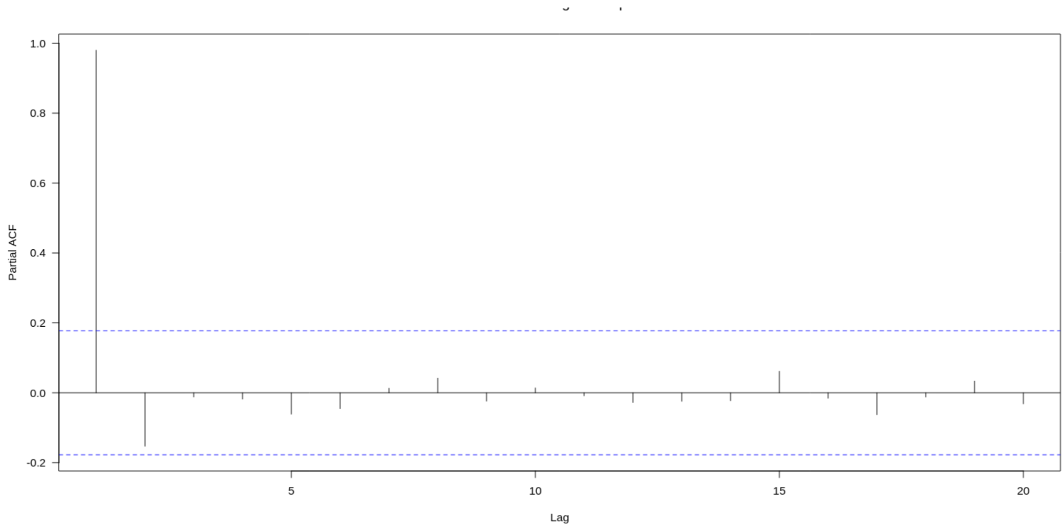

Figure 3: PACF for Danish quarterly average house price in Denmark.

We have determined that the log transform[1] is an appropriate variance-stabilizing transformation from a mean-variance plot, as in [1, Section 6.7], from which a linear trend is clear, indicating that the original data potentially has multiplicative error. Multiplicative error makes intuitive sense as well, as it means the size of the fluctuations of prices are proportional to the prices. The log-transformed data and the associated ACF and PACF is given fig. 4.

[1]Another possible variance-stabilizing transform is the square-root transform. However, in terms of interpretability, the log-transform is favorable. Further, from an optimization perspective, the log transform utilizes the whole domain for the dependent variable.

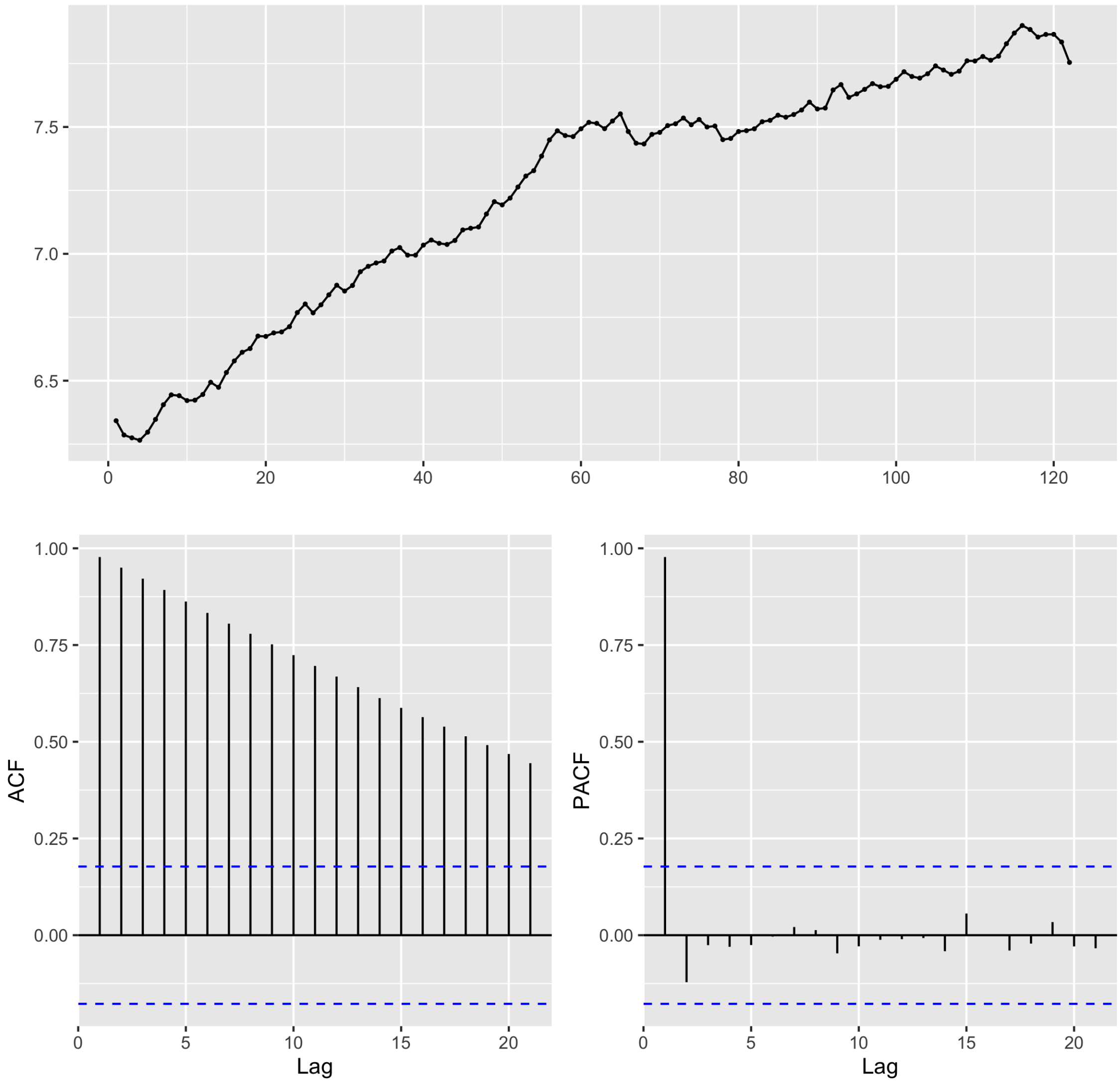

Figure 4: Log-transformed quarterly average sales prices, $\ln Y_t$, (top) and associated ACF (bottom left) and PACF (bottom right).

From fig. 4, we see a linear trend in the log-transformed average house price and further, it is clear that the ACF does not decrease sufficiently fast. We will therefore difference the series once, for which the resulting is shown in fig. 5.

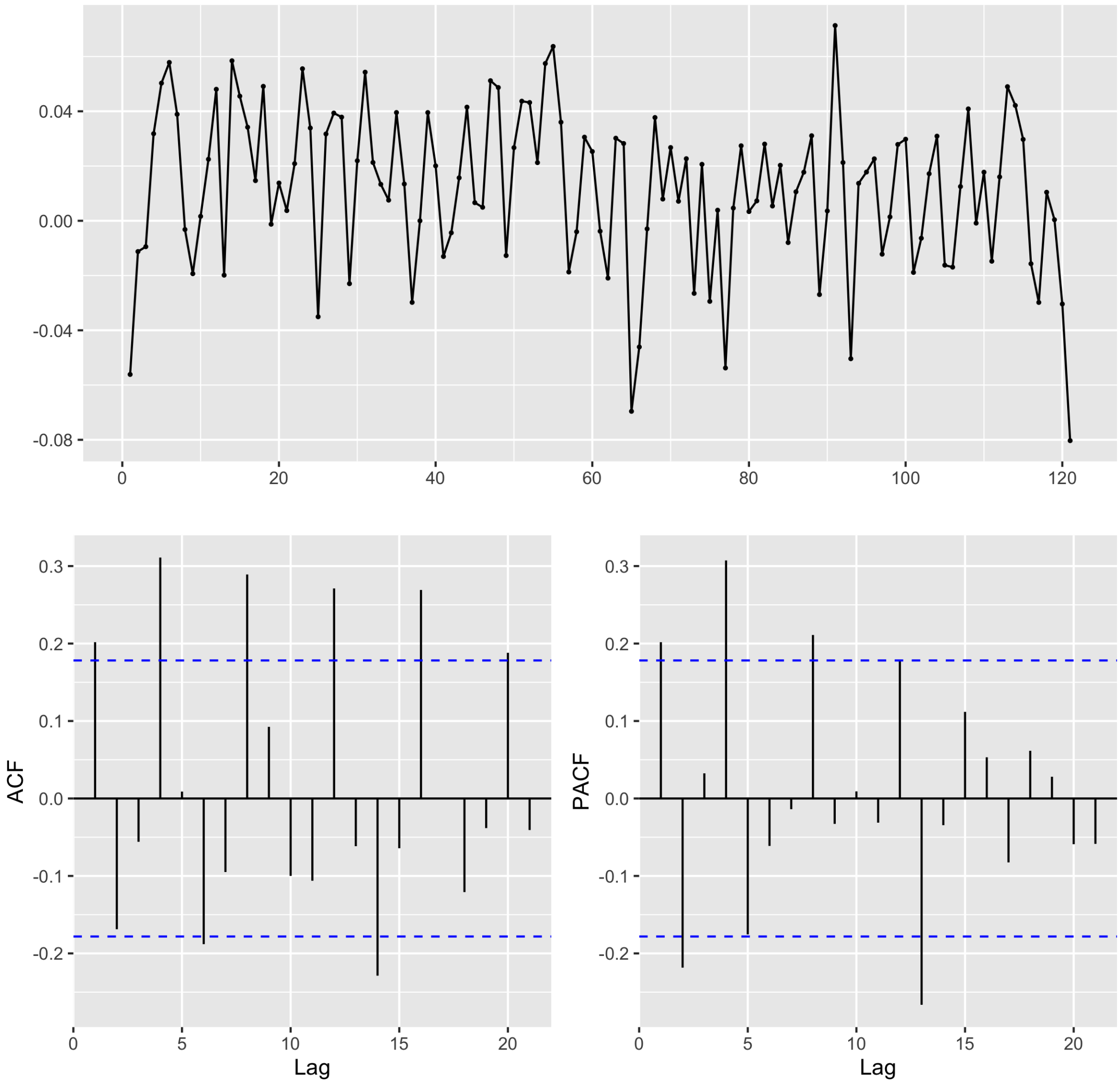

Figure 5: Log-transformed quarterly average sales prices differenced with lag 1 $\nabla \ln Y_t$ (top) and associated ACF (bottom left) and PACF (bottom right).

From fig. 5, we see slowly decreasing auto-correlations at lag 4, 8, 12, and 16. We choose to account for that with a seasonal differencing with period 4, for which the result is shown in fig. 6.

We use differencing as the peaks decrease slowly towards zero and furthermore, one might argue there is a vague curvature. Choosing a period of 4 is further supported by the fact the average house price is quarterly data, so a period of 4 corresponds to a year. [2]

[2] One could also choose to model the seasonal lags with the autoregressive model of appropriate order in the seasonal part of SARIMA. This is elaborated later.

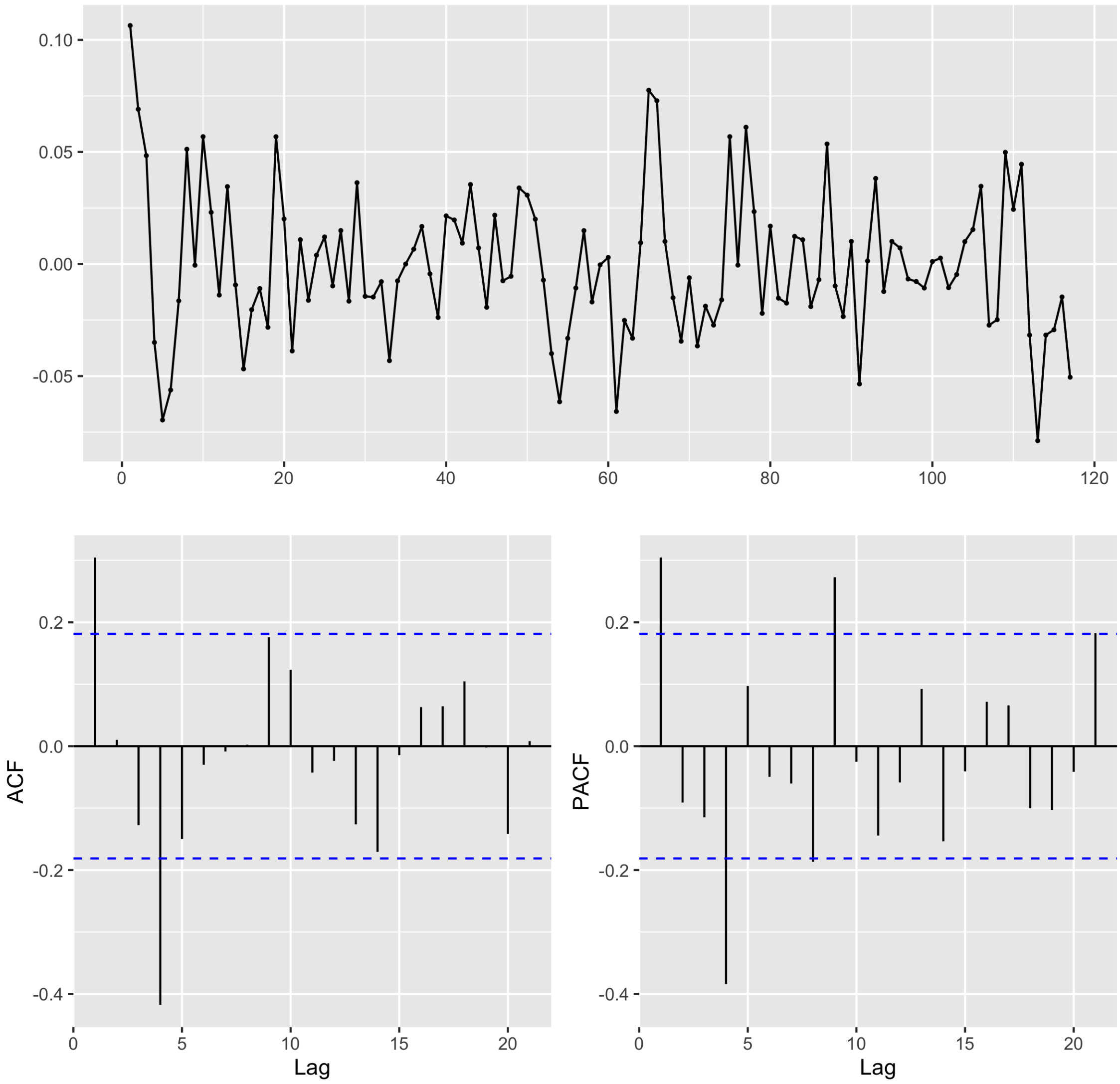

Figure 6: Log-transformed quarterly average sales prices differenced with lag 1 and 4, i.e. $\nabla\nabla_4 \ln Y_t$ (top), and associated ACF (bottom left) and PACF (bottom right).

Finally, using the Kwiatkowski–Phillips–Schmidt–Shin (KPSS) tests, we cannot reject the null hypothesis that the series is stationary at significance level $\alpha = 0.05$.

Hence, the stationarity assumption for ARMA is likely fulfilled, and we can consider the ACF and PACF to determine the model order. From the damped sine in the ACF and PACF, the ARMA is neither a pure MA nor AR process. We see that at least an autoregressive part of order one should be appropriate.

One could argue further for the model order. Instead, we will consider the most parsimonious model in this family, namely ARIMA$(1, 1, 0) \times (0, 1, 1)_4$ of the log-transformed data and expand the model iteratively and determine the final model using information criterion and likelihood-ratio tests.

1.3 Univariate model selection

In the following section, we will determine the model order.

We have iteratively expanded the initial model, namely ARIMA$(0,1,0) \times (0,1,0)_4$, considering the ACF and PACF of the residuals at each step. We arrived at model 3, namely ARIMA$(1,1,0) \times (0,1,1)_4$ for which there were no notable significant lags in the ACF and PACF of the residuals, see fig. 7. However, we expanded the model further, with the intention to test via the maximum-likelihood ratio test, whether higher-order AR or MA parts were justified, see table 2, which was not the case. We refer to section 6.5.2.1 for the Likelihood ratio test and to section 6.5.3 for the model reduction based on the information criteria. The results from the table 1.3 indicate that the model 3 has the lowest information criteria. Since the models in the table are nested in one another, we perform the Likelihood ratio tests with model 3. Table 2 presents the results in terms of the p-values. We test the null hypothesis that the reduction to the simpler model is possible. We see that the null hypothesis is rejected for models 1 and 2, which indicates that the parameters $p = 1, Q = 1$ are justified. The results for the test for models 4-7 indicate that the reduction to model 3 can be performed. Hence, we conclude that model 3 is appropriate for modelling the Y_t out of the models we have considered.

Model number	**p**	**d**	**q**	**P**	**D**	**Q**	**Frequency**	**AIC**	**BIC**
1	0	1	0	0	1	0	4	-468.6657	-465.9035
2	1	1	0	0	1	0	4	-479.4874	-473.9631
3	1	1	0	0	1	1	4	**-534.5516**	**-526.2651**
4	1	1	0	1	1	1	4	-532.9580	-521.9093
5	1	1	1	0	1	1	4	-534.4460	-523.3973
6	2	1	0	0	1	1	4	-534.5133	-523.4646
7	1	1	1	1	1	1	4	-532.7329	-518.9221

Table 1: Model identification based on information criteria

	Model 1	Model 2	Model 4	Model 5	Model 6	Model 7
Model 3	6.68e-16	4.22e-14	0.5238	0.1687	0.1613	0.336

Table 2: Maximum-likelihood ratio test with the model 3

We arrived at the following model. We note that the model does not include an intercept since the differencing was applied.

$$(1 + \phi_1 B)\nabla\nabla_4 \log Y_t = \left(1 + \Theta_1 B^4\right) \epsilon_t \tag{1}$$

The estimated parameters with the uncertainty are presented with the following table:

Coefficients	**Estimate**	**Standard Error**
ϕ_1	0.3931	0.0899
Θ_1	-1.0000	0.3696
σ^2	0.0005235	

Table 3: Estimated model statistics, coefficients and associated uncertainty

1.4 Residual diagnostics

In this section, we perform model checking, to check if the residuals for the selected model can be considered white noise and normally distributed.

Firstly, the residuals are plotted along with the ACF and PACF in fig. 7.

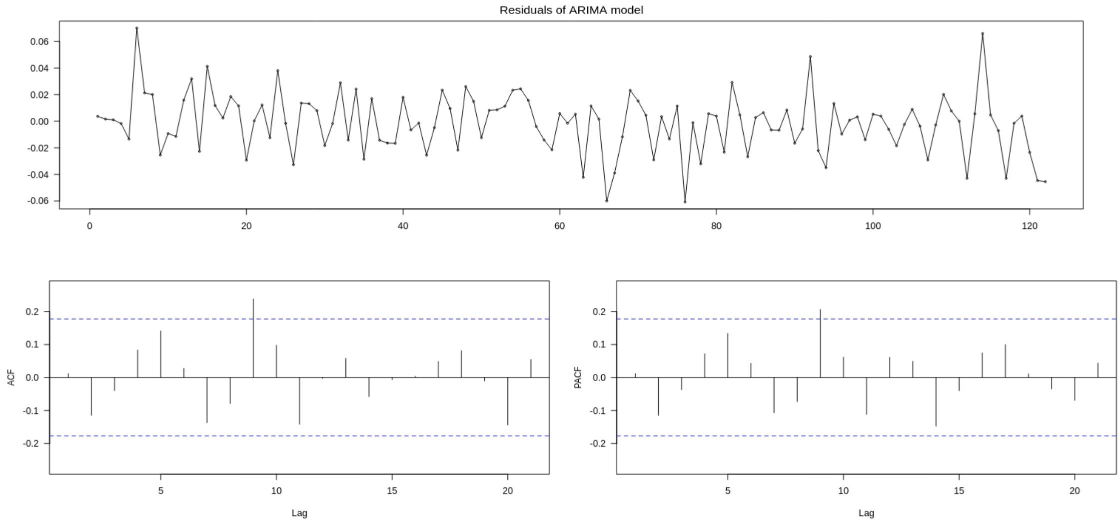

Figure 7: Residual analysis of the selected model

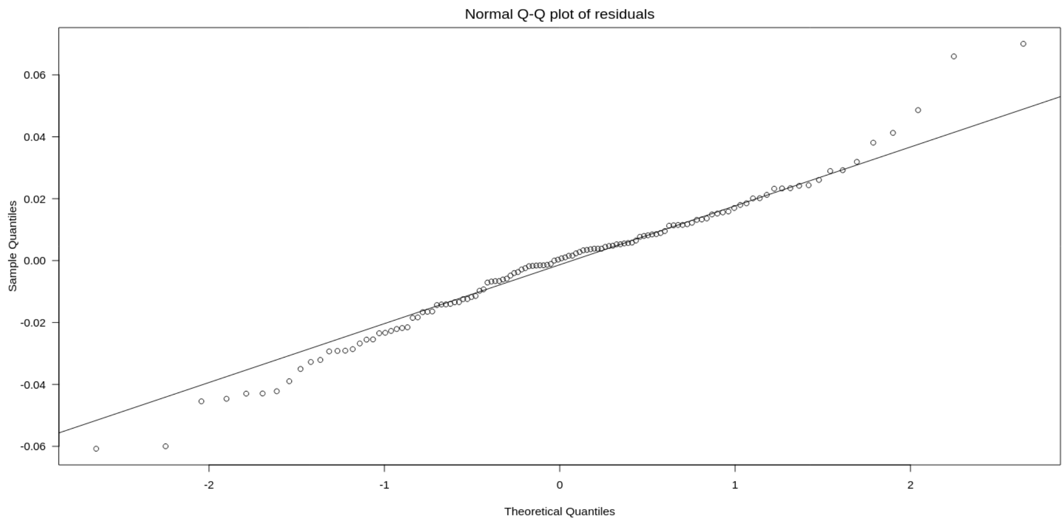

Figure 8: QQ-normal plot of residuals

The sign test yields a p-value of 0.7861, so we cannot reject the null-hypothesis of sign changes being binomially distributed with probability 0.5. We use the t-test to test if the mean is significantly different from 0. We get a p-value of 0.4485, so the difference is not significant either. Finally, when looking at the ACF and PACF plots, we see that they each have one peak outside the 95% CF, which is expected since we have roughly 20 lags. Also, just looking at the plots of the residuals, it looks a lot like white noise

Thus, it is plausible to assume the residuals are white noise, so the model is adequate.

1.5 Forecasting the future house prices - I

Figure 9 presents the plot of the predictions 6-time steps beyond the training data. Table 4 presents the numerical results along with the confidence interval. We note that the confidence interval becomes wide indicating high uncertainty of the predictions. We further note that the predictions capture the linear trend in the training data, indicating the robustness of the model.

	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6
Mean	2302.057	2336.149	2382.349	2345.723	2375.841	2435.621
Lower (95%)	2198.879	2158.766	2141.162	2058.894	2041.003	2052.143
Upper (95%)	2408.774	2524.089	2643.069	2661.099	2749.692	2869.719

Table 4: Summary of predictions

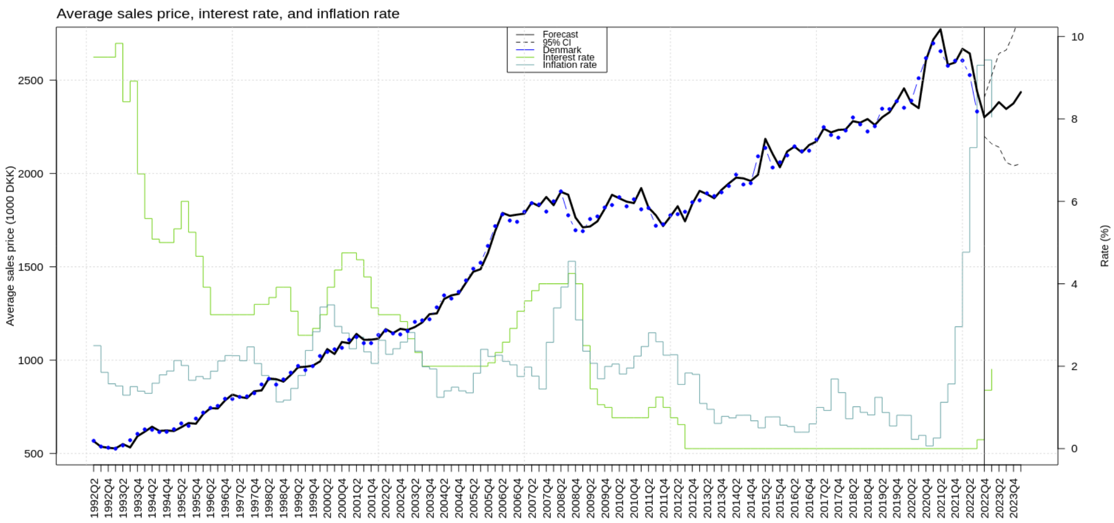

Figure 9: ARIMA forecast of Danish, quarterly, average, sales prices, with interest and inflation rates.

1.6 External inputs

We have two external regressors at our disposal, namely Inflation Rate and Interest Rate. Therefore we deal with a multiple-input, single-output (MISO) system, as described in section (8.5). Remark (8.3) states that the pre-whitning approach for determining the impulse responses (detecting what lags of the external inputs are significant for the prediction of Y) is not appropriate. We initially investigated the linear regression method described in section (8.5.3), however since no procedure for concluding which impulse responses are significant is provided and we couldn't use the p-values from the t-test in linear regression because of the autocorrelated residuals, we decided to take a different approach.

We decided to estimate an extended ARIMAX model with external regressors of Inflation Rate and Interest Rate of lags from 0 up to 5 and then successively reduce the model by the Likelihood ratio tests. Therefore our initial extended model had 12 external regressors. We keep the ARIMA structure from the previous section, namely from the model 1. We iteratively reduced the model until no further reductioins can be performed, using the significance level of 0.05. Table 5 presents the external regressors that were not reduced during the procedure.

Thereby we arrive at the following model:

$$(1 - \phi_1 B)\nabla\nabla_4 \ln Y_t = (\omega_{T,1} B + \omega_{T,4} B^4)T_t + (\omega_{F,1} B + \omega_{F,4} B^4)F_t + (1 + \Theta_1 B^4)\epsilon_t \quad (2)$$

	InterestRate lag1	InterestRate lag4	InflationRate lag1	InflationRate lag4
p-vlaue	0.02568	0.01827	0.00002	0.00015

Table 5: Summary of Likelihood ratio test

where Y_t is the house price as usual, T_t is the standardized interest rate and F_t is the standardized inflation rate. Note the minus in front of ϕ_1. This is to follow the sign convention from the arima implementation in R. This way, the parameter presented in table 9, is the actual parameter outputted by the model. Also note that the inflation rate and interest rate are standardized in order for optimization to work better as suggested by the documentation of the "arima" function.

1.7 Forecasting the future house prices - II

Figure 10 presents the plot of the predictions 6-time steps beyond the training data for the model 2. Note, that as indicated in the assignment description, we assume that the external regressors will stay at the same level as 2023Q1 beyond the training data. Table 7 presents the numerical results along with the confidence intervals. For comparison, we also estimate the ARIMAX model where external regressors are included without any lags, which has the form:

$$(1 - \phi_1 B)\nabla\nabla_4 \ln Y_t = \tilde{\omega}_T T_t + \tilde{\omega}_F F_t + (1 + \Theta_1 B^4)\epsilon_t \tag{3}$$

Table 1.7 presents the numerical results along with the confidence intervals for the model 3. The estimated parameters of the model 3 are presented in the table 6.

Variable	Value	Standard Error
ϕ_1	0.3368	0.0904
Θ_1	-1.0000	0.0604
$\tilde{\omega}_T$	1.7568	1.0707
$\tilde{\omega}_F$	-2.6665	1.2114
σ^2	0.0005	

Table 6: The estimated parameters for the model presented in eq. (2)

The comparison between models shows the importance of including the external regressors lagged appropriately. We note that the confidence intervals for the model 3 are much wider than for the model 2.

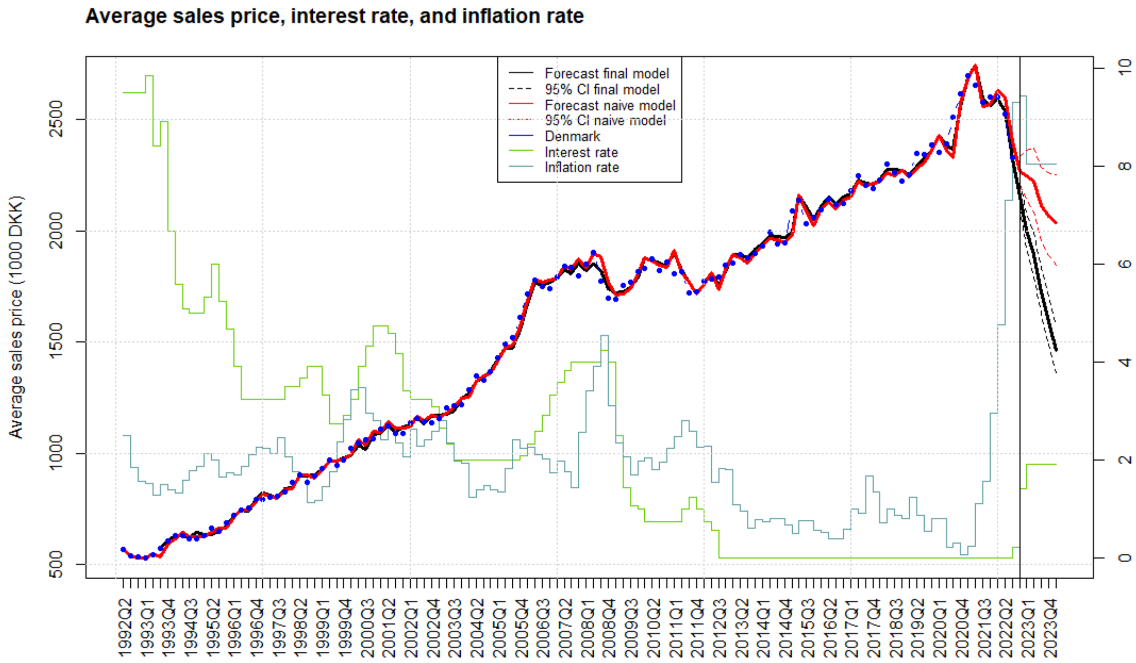

Figure 10: ARIMA forecast of Danish, quarterly, average, sales prices, with interest and inflation rates.

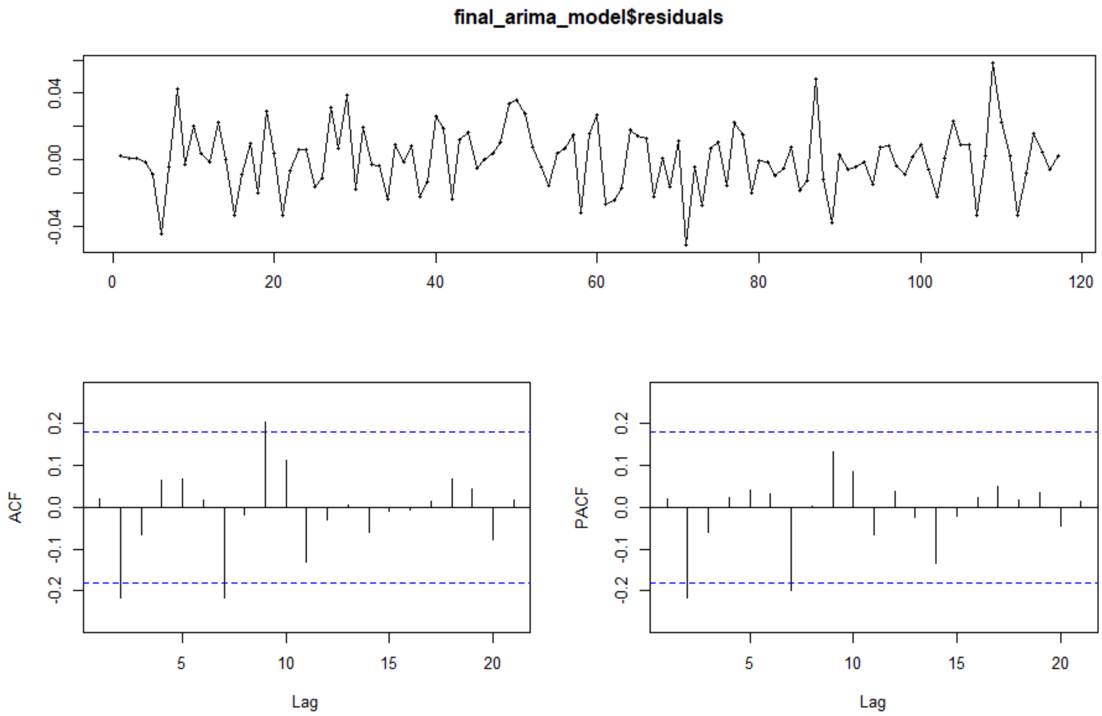

Figure 11: Residual analysis of the final ARIMAX model 2

	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6
Mean	2156.831	2005.753	1895.454	1726.391	1592.817	1463.968
Lower (95%)	2244.717	2130.869	2045.414	1887.317	1762.201	1637.151
Upper (95%)	2072.385	1887.983	1756.487	1579.186	1439.714	1309.105

Table 7: Summary of predictions for the final model 2

	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6
Mean	2268.389	2249.222	2225.774	2116.143	2066.555	2039.357
Lower (95%)	2371.988	2423.319	2456.794	2383.607	2370.690	2378.216
Upper (95%)	2169.316	2087.633	2016.477	1878.692	1801.437	1748.780

Table 8: Summary of predictions for the model 3

1.8 Conclusions - I

When including the external regressors in the model, we see that the predictions are quite different compared to the case where we don't include external regressors. We compare the models 2 and 1 with the Likelihood ratio test since they are nested. The p-value was calculated to be 0.00001338, indicatding that the 4 external regressors can not be reduced. Hence, the inflation rate and the interest rate impact the forecasting.

Variable	Value	Standard Error
ϕ_1	0.1382	0.0954
Θ_1	-1.0000	0.0653
$\omega_{T,1}$	-11.9672	5.2803
$\omega_{T,4}$	12.8722	5.3519
$\omega_{F,1}$	18.2332	4.5524
$\omega_{F,4}$	-21.3102	4.6443
σ^2	0.0004014	

Table 9: The estimated parameters for the model presented in eq. (2)

Let's take a look at the parameter estimates in table 9. Quite interestingly we see that a high interest rate in one quarter results in a lower house price the quarter immediately after. At the same time however, the house price 4 quarters later (so a year later) are impacted positively. And the opposite holds for inflation rate.

When looking at the plots of the external regressors combined with the predictions, these actually agree quite nicely with this interpretation.

Clearly, the model has found a relation between the external regressors and the house prices. So can the model be trusted? The variances of the parameter estimates are actually very high, which indicates that we should be cautious using this model. Hence, the relatively low confidence intervals of the predictions can not be fully trusted. Assuming the model can be trusted, we should definitely not buy a house now, because the price will keep dropping the next 6 quarters.

2 Multivariate ARIMAX (MARIMA)

In this section we will model the quarterly average house price for the regions Greater Copenhagen, Zealand, Central Jutland and rural areas (specifically, Southern Denmark and Northern Jutland) and their interactions with each other.

2.1 Presenting the data

The quarterly interest rate, inflation rate and average house price for each region is given in fig. 12.

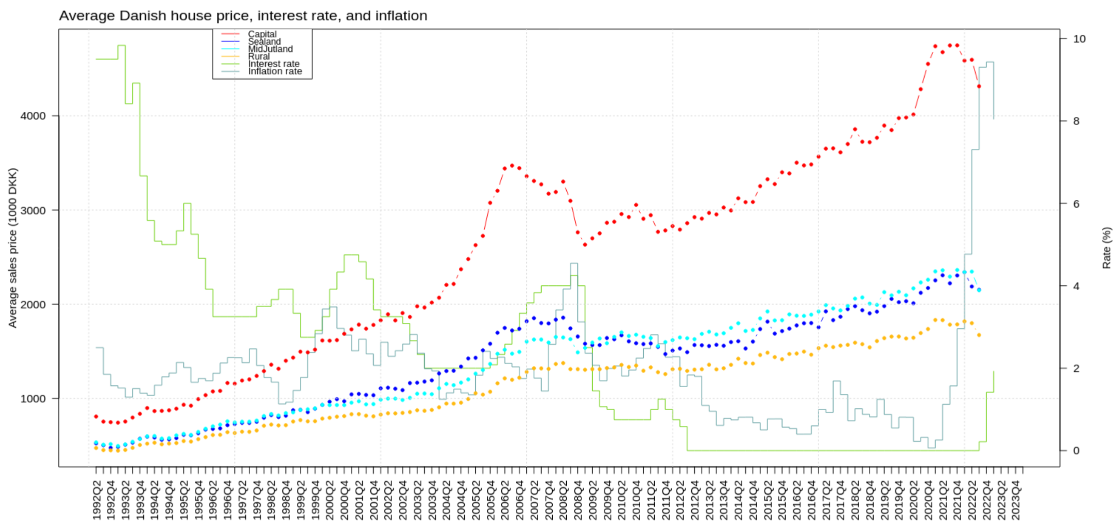

Figure 12: Quarterly average, sales prices, with interest and inflation rates.

In general we see a trend similar to fig. 1 for each region, but Copenhagen displays a magnified or proportionately larger version of this trend, characterized by more significant peaks and valleys, along with a more positive linear trend.

The similarity to the national price is a given, since it is merely a weighted mean of these four regions. Each of the regions also appear to be highly correlated. This suggests that the signals likely contain information about each other which could be utilized for predictions. A MARIMAX model may therefore be a relevant choice.

Like before, the data probably needs similar differencing and a variance-stabilizing log-transformation. This is because the national data is a weighted mean of the four regions.

2.2 ACF and PACF

In this section, we will present our transformations[3] on the data determined via the ACF and PACF for the quarterly average house price for the regions of Denmark. This is to determine the model structure for the associated MARIMA model.

Using the same approach as in section 1, the same variance-stabilizing as in section 1.3 was determined to be appropriate.

[3]For interpretability and parsimony, using the same variance-stabilizing transformation and differencing for all regions is preferable

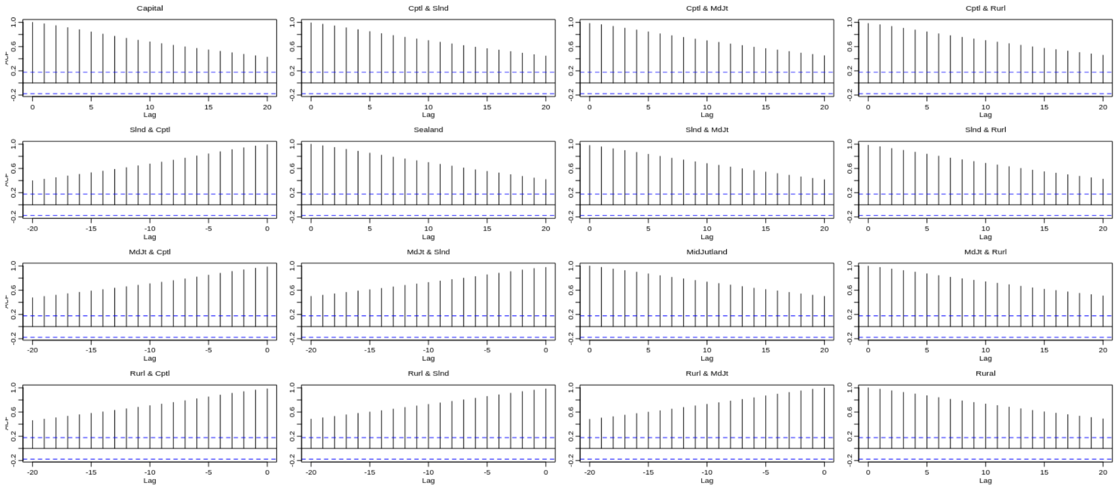

Figure 13: ACF for regional, quarterly, average, sales prices.

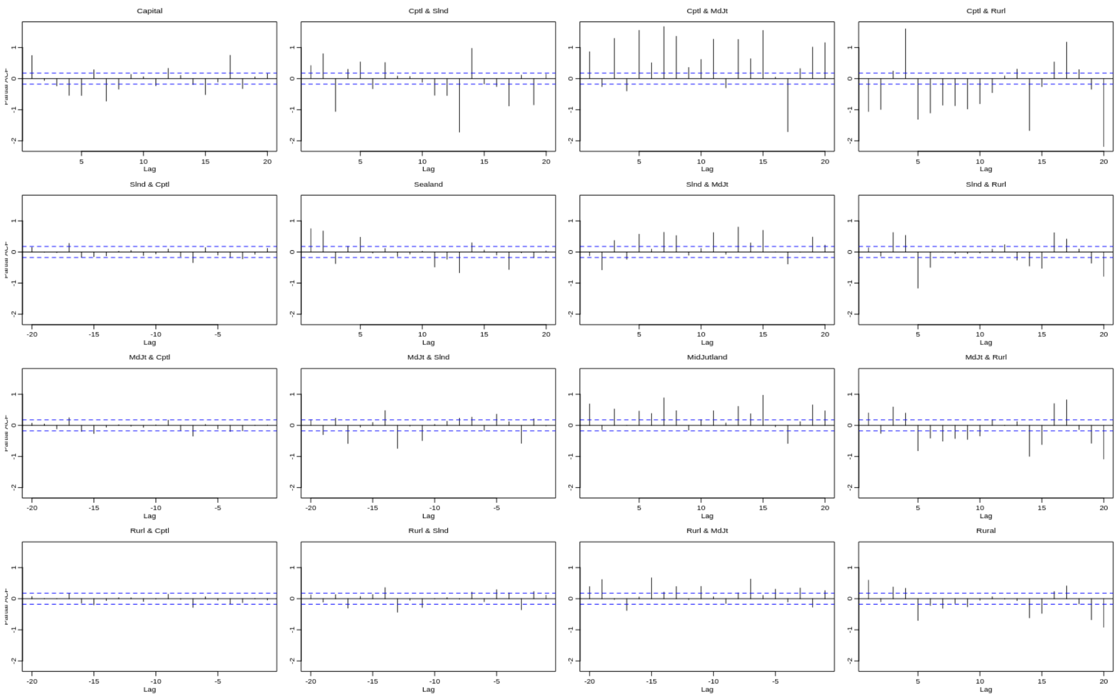

Figure 14: PACF for regional, quarterly, average, sales prices.

The ACF on fig. 13 is very similar to the univariate case. As before, after differencing all the series with lag 1, the resulting ACF (not shown) indicates that an annual differencing may be appropriate - this is done.

The PACF on fig. 14, however, looks a lot different and has a lot more spikes, especially on the cross-correlating plots. After doing the above differencing, many spikes are still present all over the place as seen on fig. 16. This is indicative of the series likely being correlated at various lags, but the model order selection cannot be determined unambiguously based on these plots.

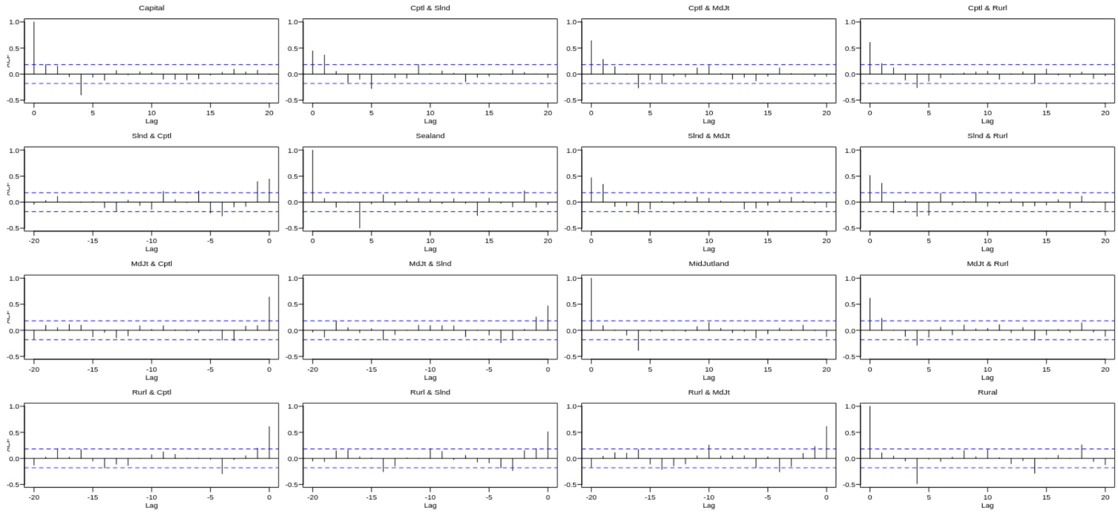

Figure 15: ACF for differenced and transformed regional quarterly average sales prices.

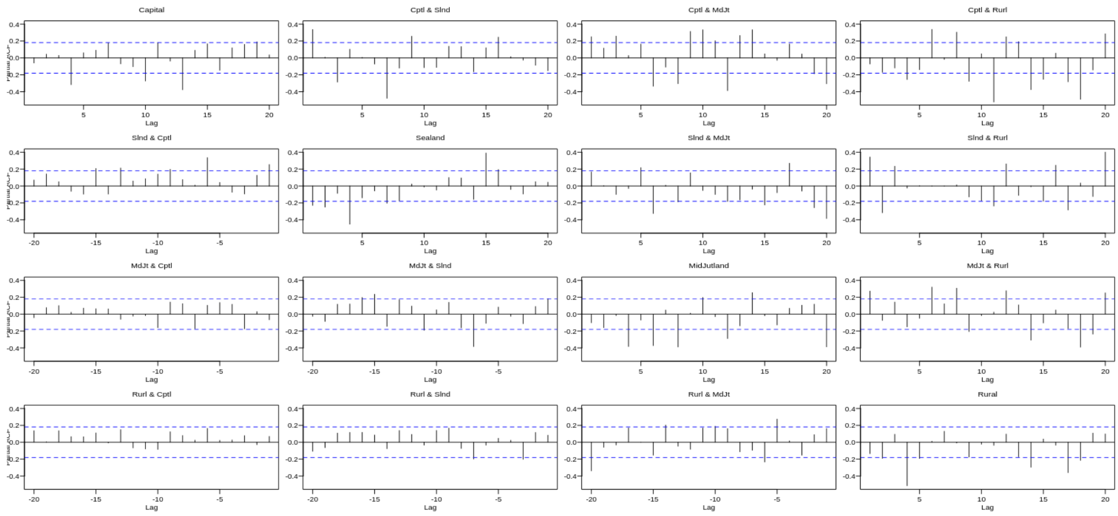

Figure 16: PACF for differenced and transformed regional quarterly average sales prices.

The diagonal of fig. 15 has much of the same story as in the univariate case. This suggest the same MA order as before, but again, this choice cannot be unambiguously made because of the cross-correlations.

Generally, from fig. 15, we see significant symmetric peaks for lag 0 and 1 in the CCF (the off-diagonal plots) for lag 0 and 1 for all pairs of regions. That is, significant mutual interactions exist between the quarterly average house prices between the regions in the first couple of lags. Exceptionally, we have that the pairs Copenhagen, Central Jutland and Zealand, Rural Denmark, are not symmetric. Specifically for each pair, significant lags at 0 and 1 suggest that the former may influence the latter. In contrast, the lack of a significant lag at -1 indicates that the latter may not influence the former or at least not to the same extent.

Most spikes in fig. 16 appear insignificant. Nevertheless, Copenhagen especially has many correlations with all the other regions - likely leading. The same could be said for Rural Denmark; however, they are close to being insignificant, so they may just be spurious and used sparingly for deciding the model structure.

2.3 Multivariate model selection

As evident from the previous section 2.2 the same differencing and log-transformation can be used. The interest rates and inflation rates are also included at lags 1 and 4, as the univariate sections showed they were significant for modelling the national sales price and because of the macroeconomic connections. They are both included as regression variables as both are quantities mostly determined arbitrarily by human decision-making processes.[4]. They are therefore harder to model, so the most parsimonious choice is to include them directly as regression variables to also incorporate their non-stationarity. The model orders, however, are harder to determine. For these, we use the following procedure:

1. Start with all orders till 10 and MARIMAX fitting penalty of 5.
2. Loop till the tests are deemed adequate
   (a) Fit a MARIMAX model.
   (b) Transform residuals according to [1, Chapter 9.8]
   (c) Investigate CCF and PCCF for transformed residuals
   (d) Run Shapiro-Wilk normality test on each transformed residual dimension.
   (e) Check forecasts for poor fits.
   (f) Remove model orders dependent upon amount of insignificant AR and MA p-values.
   (g) If none of the tests look good, reduce the penalty slightly to reduce biasing parameters towards 0.
3. The fitted model is chosen

[4]Inflation is calculated via weighting of different goods, but goods can be substituted or reweighted. The interest rate has a lower bound set by the national bank.

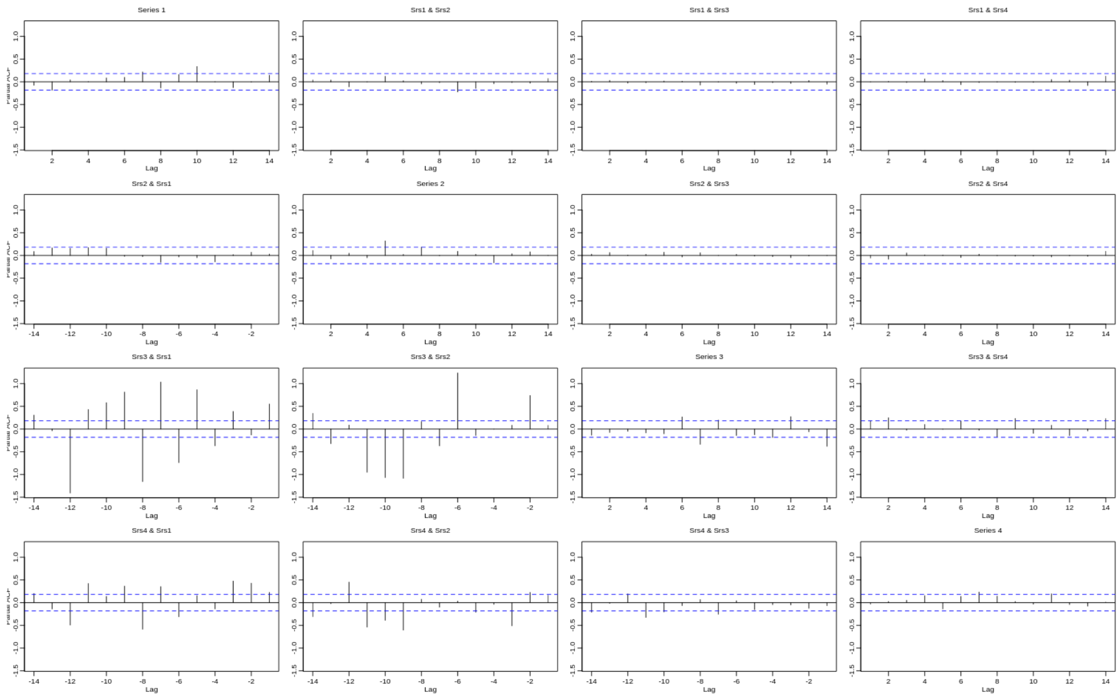

Figure 17: ACF for model residuals after residual transformation

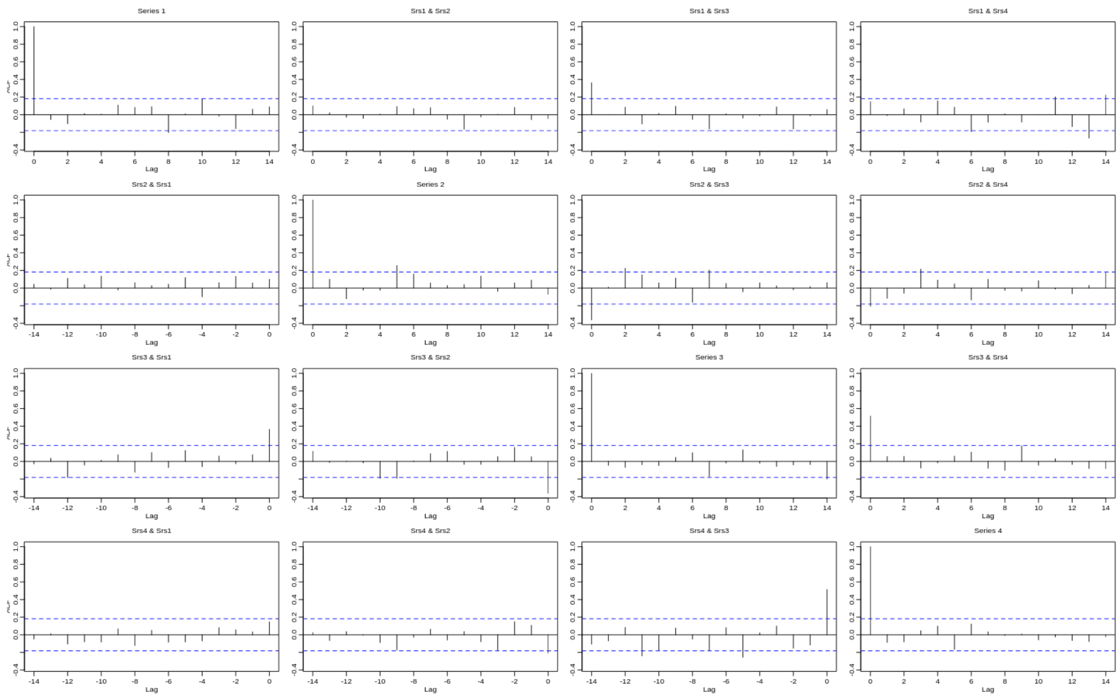

Figure 18: PACF for model residuals after residual transformation

The final chosen MARIMAX model has an AR(1) and an MA(1) component fitted at penalty 2 with the marimax package. Recall that the residuals have been transformed according to [1, Chapter 9.8]. As seen on fig. 17 the final model manages to model the data quite well, but there are still a few significant correlations present. These are not modelled because of a bug in the marimax package that causes it to not be able to add individual correlations for lags greater than the highest order AR or MA, since the package does not instantiate higher order matrices afterwards. A higher order model is not chosen to mitigate this as it would not be parsimonious. Likewise, the same story can be seen for the PCCF on fig. 18, where most of the plots look insignificant for most lags.

The transformed residual dimensions also are unable to be rejected by the Shapiro-Wilk tests, so they may be normal.

Shapiro-Wilk test p-value	0.264	0.242	0.976	0.142

2.4 Multivariate explanation

The MARIMAX model is a multivariate generalization of the ARIMAX model. The MARIMAX model joint ARIMAX processes and allows for modelling dependencies and feedback between them. Informally, it uses a similar approach as ARIMAX but with additional features to model the correlations between multiple variables, by allowing a variable to depend on previous values of the other variables and their errors. MARIMAX extends ARIMAX, so it can handle multiple time series in which variables depend on each other. The reader is referred to [1, Chapter 9] for a more theoretical introduction.

We will now interpret, in general, the estimated model coefficients for the selected MARIMAX model and point out notable cases. You can find the model parameters at section 2.7.

First, consider the regional sales price dependence on interest and inflation rates. Generally, the AR, seasonal AR and seasonal MA have zero coefficients wrt. the interest rate or the 4-lagged interest rate.[5] The regional sales prices do not depend **directly** on the interest rate. However, regional sales prices generally depend on inflation, which in turn from a macroeconomic perspective depends on the interest rate. Specifically, the AR part for all regions wrt. the 4-lagged inflation rate has positive coefficients. That is, a high inflation is positively correlated with the regional sales price four quarters or a year later.

From the coefficient in the AR part, we can interpret the following:

- Greater Copenhagen is negatively correlated with the previous quarter for Zealand, but notably not autocorrelated.
- Zealand is autocorrelated and is negatively correlated with the previous quarters for Central Jutland and Rural Denmark
- Central Jutland is positively correlated with Copenhagen, negatively correlated with the previous quarters for Zealand and Rural Denmark, but notably not autocorrelated.
- Rural Denmark is negatively correlated with the previous quaryer for Central Jutland but notably not autocorrelated.

Finally, we will interpret the MA components.

For the MA part, notably we have positive coefficients wrt. Copenhagen for Zealand and Central Jutland. That is, for these, a sudden spike in Copenhagen in one quarter is positively correlated with the sales price in the next quarter.

The seasonal MA part is a diagonal matrix with negative coefficients. That is, for a region, a sudden spike in one quarter is negatively correlated with the sales price 4 quarters or a year later for the same region, but not across regions.

[5] Apart from Central Jutland and Rural Denmark with have a small positive coefficient in the AR part

2.5 Forecasting the future house prices - III

Capital	Mean	Lower	Upper
Step 1	4219.505	3967.577	4487.431
Step 2	3959.051	3587.083	4369.592
Step 3	3729.301	3269.203	4254.151
Step 4	3407.811	2908.990	3992.168
Step 5	3045.176	2519.636	3680.332
Step 6	2628.023	2107.862	3276.544
Sealand	Mean	Lower	Upper
Step 1	2058.369	1937.767	2186.476
Step 2	1980.642	1801.886	2177.132
Step 3	1823.842	1614.301	2060.582
Step 4	1701.860	1473.446	1965.682
Step 5	1570.120	1320.568	1866.829
Step 6	1404.630	1145.223	1722.797
MidJutland	Mean	Lower	Upper
Step 1	2133.916	2028.344	2244.983
Step 2	2029.814	1883.417	2187.592
Step 3	1956.066	1781.592	2147.626
Step 4	1790.158	1606.712	1994.549
Step 5	1655.695	1454.346	1884.921
Step 6	1485.418	1275.862	1729.394
Rural	Mean	Lower	Upper
Step 1	1583.8927	1518.2632	1652.359
Step 2	1515.2651	1419.7285	1617.230
Step 3	1417.0754	1303.6984	1540.312
Step 4	1273.6009	1154.0951	1405.481
Step 5	1120.6597	996.2343	1260.625
Step 6	998.4396	869.2976	1146.767

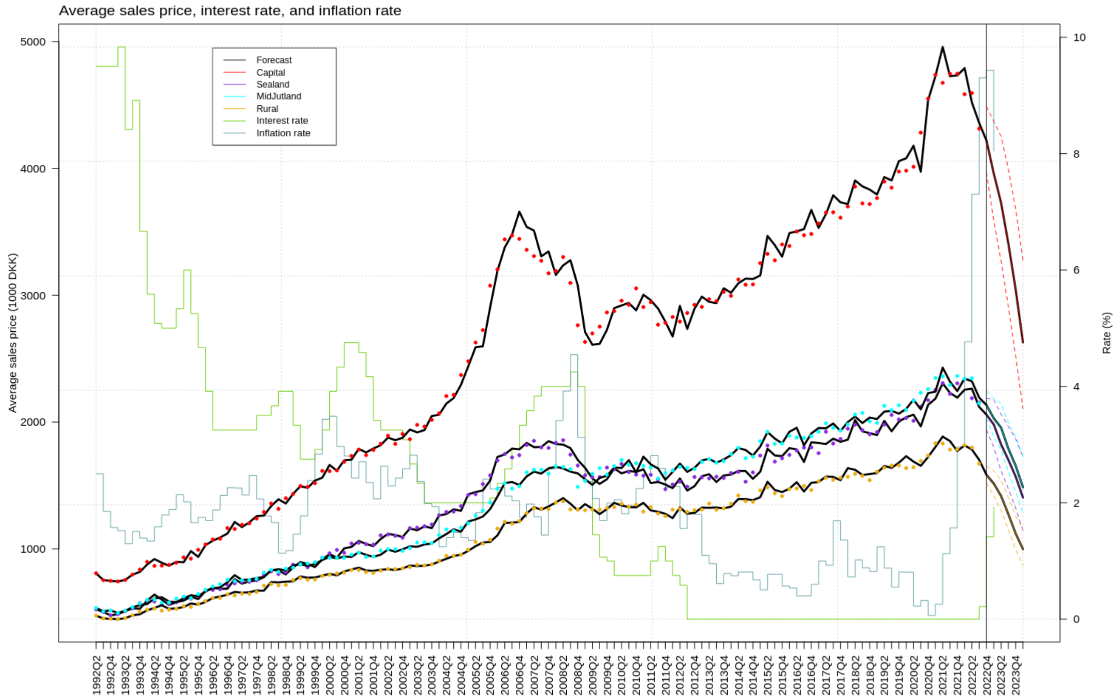

Figure 19: Forecast of regional, quarterly, average, sales prices, with interest and inflation rates.

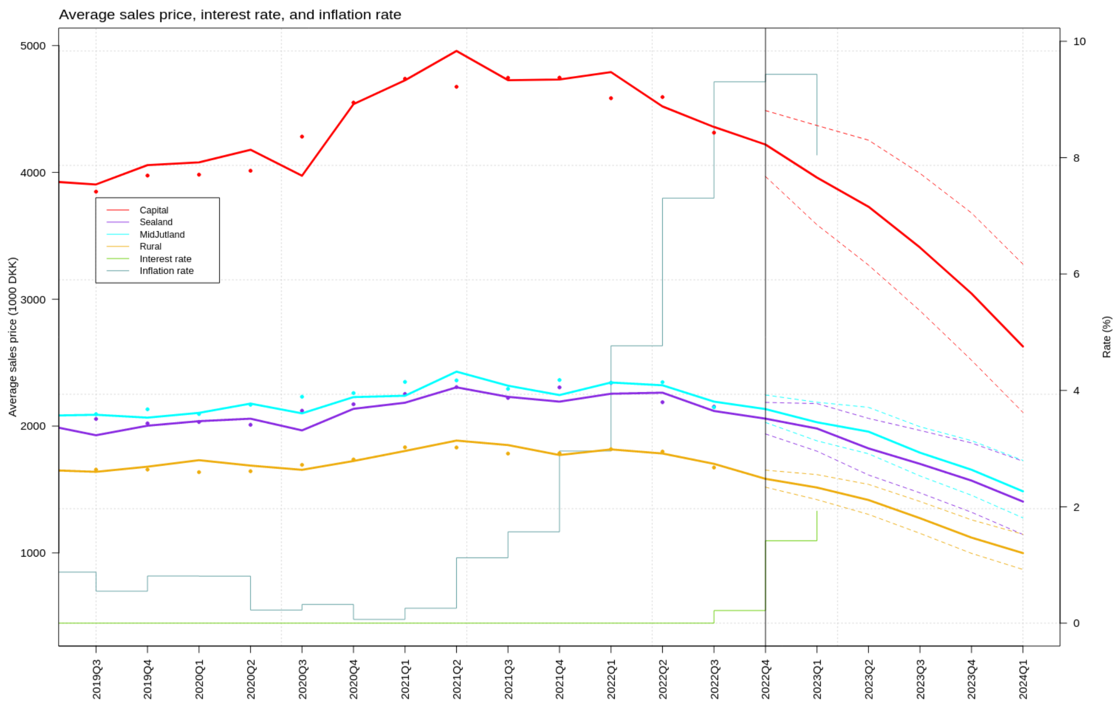

Figure 20: Zoomed forecast of regional, quarterly, average, sales prices, with interest and inflation rates.

2.6 Conclusions - II

In general we see that the univariate and multivariate models give similar results when taking external regressors into account.

From fig. 19, we see a significant drop in quarterly average sales prices for the prediction horizon. This is likely due to the behavior of inflation, which is particularly high close the the prediction horizon, when compared the the rest of the series. From a macroeconomic perspective the interest rates are also starting to be raised significantly to deal with the rampant inflation. This naturally leads to lower house prices.

Based on the model analysis and the forecast plots, the model exhibits favourable performance in modelling consistent trends. However, with substantial fluctuations, like in times of crisis, the model tends to demonstrate a bias towards excessive optimism initially, followed by undue pessimism, suggesting that while the model is reliable for general trend modelling, it may be inappropriate to rely on it during times of crisis.

The forecast results indicate that the Greater Copenhagen region is expected to experience the most significant decrease in sales prices. While this trend may not be advantageous for individuals with a short-term investment horizon, it can be particularly favorable for those with a larger investment horizon, if they purchased the asset during a low point in the market. The model does, however, not appear trustworthy in times of crisis, but the old saying: "time in the market, beats timing the market" may, be very applicable for this model.

References

[1] Henrik Madsen. *Time Series Analysis*. Oct. 2008. ISBN: 978-1-4200-5967-0. DOI: `10.1201/9781420059687`.

[2] Danmarks Statistik. *Økonomi*. da. URL: `https://www.dst.dk/da/Statistik/emner/oekonomi` (visited on 04/11/2023).

Appendix

2.7 MARIMA estimates

```
Model dimension = kvar =  8  N =  117
Averages for all variables:
 -0.0002378131  -3.770962e-05  -0.0001987403  -0.0001663835  -0.1170391  -0.1747774
Covariance(all data):
         y1       y2       y3       y4       y5       y6       y7       y8
y1   0.0018   0.0008   0.0009   0.0007  -0.0018  -0.0032  -0.0062  -0.0097
y2   0.0008   0.0018   0.0006   0.0006  -0.0009  -0.0025  -0.0044  -0.0060
y3   0.0009   0.0006   0.0011   0.0005  -0.0008  -0.0022  -0.0052  -0.0055
y4   0.0007   0.0006   0.0005   0.0008  -0.0003  -0.0013  -0.0048  -0.0059
y5  -0.0018  -0.0009  -0.0008  -0.0003   0.6239   0.5334   0.1467   0.0684
y6  -0.0032  -0.0025  -0.0022  -0.0013   0.5334   0.5528   0.1819   0.1511
y7  -0.0062  -0.0044  -0.0052  -0.0048   0.1467   0.1819   0.4701   0.4148
y8  -0.0097  -0.0060  -0.0055  -0.0059   0.0684   0.1511   0.4148   0.8138
Covariance(residuals):
         u1       u2       u3     u4       u5       u6       u7       u8
u1   0.0010   0.0004   0.0005  3e-04   0.0002   0.0013  -0.0002  -0.0005
u2   0.0004   0.0009   0.0004  3e-04  -0.0039  -0.0040  -0.0014   0.0007
u3   0.0005   0.0004   0.0007  3e-04  -0.0005   0.0000  -0.0012   0.0009
u4   0.0003   0.0003   0.0003  5e-04  -0.0002   0.0001  -0.0004  -0.0007
u5   0.0002  -0.0039  -0.0005 -2e-04   0.6239   0.5334   0.1467   0.0684
u6   0.0013  -0.0040   0.0000  1e-04   0.5334   0.5528   0.1819   0.1511
u7  -0.0002  -0.0014  -0.0012 -4e-04   0.1467   0.1819   0.4701   0.4148
u8  -0.0005   0.0007   0.0009 -7e-04   0.0684   0.1511   0.4148   0.8138
Random variables are 1 2 3 4

AR definition:
 , , Lag=1

   x1=y1 x2=y2 x3=y3 x4=y4 x5=y5 x6=y6 x7=y7 x8=y8
y1     1     1     1     1     1     1     1     1
y2     1     1     1     1     1     1     1     1
y3     1     1     1     1     1     1     1     1
y4     1     1     1     1     1     1     1     1
y5     0     0     0     0     0     0     0     0
y6     0     0     0     0     0     0     0     0
y7     0     0     0     0     0     0     0     0
y8     0     0     0     0     0     0     0     0

MA definition:
 , , Lag=1

   x1=y1 x2=y2 x3=y3 x4=y4 x5=y5 x6=y6 x7=y7 x8=y8
y1     1     1     1     1     0     0     0     0
y2     1     1     1     1     0     0     0     0
y3     1     1     1     1     0     0     0     0
y4     1     1     1     1     0     0     0     0
```

```
y5      0     0     0     0     0     0     0     0
y6      0     0     0     0     0     0     0     0
y7      0     0     0     0     0     0     0     0
y8      0     0     0     0     0     0     0     0

, , Lag=4

   x1=y1 x2=y2 x3=y3 x4=y4 x5=y5 x6=y6 x7=y7 x8=y8
y1      1     1     1     1     0     0     0     0
y2      1     1     1     1     0     0     0     0
y3      1     1     1     1     0     0     0     0
y4      1     1     1     1     0     0     0     0
y5      0     0     0     0     0     0     0     0
y6      0     0     0     0     0     0     0     0
y7      0     0     0     0     0     0     0     0
y8      0     0     0     0     0     0     0     0

AR estimates:
, , Lag=1

    x1=y1   x2=y2   x3=y3   x4=y4   x5=y5  x6=y6  x7=y7  x8=y8
y1 0.0000 -0.3995  0.0000  0.0000  0.0000 0.0000 0.0000 0.0085
y2 0.0000  0.1911 -0.2665 -0.3838  0.0000 0.0000 0.0000 0.0040
y3 0.2864 -0.2160  0.0000 -0.2300 -0.0097 0.0117 0.0000 0.0043
y4 0.0000 -0.1152  0.0000  0.0000 -0.0039 0.0000 0.0084 0.0047
y5 0.0000  0.0000  0.0000  0.0000  0.0000 0.0000 0.0000 0.0000
y6 0.0000  0.0000  0.0000  0.0000  0.0000 0.0000 0.0000 0.0000
y7 0.0000  0.0000  0.0000  0.0000  0.0000 0.0000 0.0000 0.0000
y8 0.0000  0.0000  0.0000  0.0000  0.0000 0.0000 0.0000 0.0000

AR f-values (squared t-values):
, , Lag=1

    x1=y1   x2=y2  x3=y3  x4=y4  x5=y5  x6=y6  x7=y7  x8=y8
y1 0.0000 31.4881 0.0000 0.0000 0.0000 0.0000 0.0000 5.7087
y2 0.0000  5.0620 2.9646 7.7620 0.0000 0.0000 0.0000 1.1664
y3 7.3541  8.7149 0.0000 3.5594 1.5686 1.7979 0.0000 1.7020
y4 0.0000  5.2056 0.0000 0.0000 1.9958 0.0000 2.7129 2.2318
y5 0.0000  0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
y6 0.0000  0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
y7 0.0000  0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000
y8 0.0000  0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

MA estimates:
, , Lag=1

    x1=y1 x2=y2   x3=y3 x4=y4 x5=y5 x6=y6 x7=y7 x8=y8
y1 0.0000     0  0.0000     0     0     0     0     0
y2 0.4119     0 -0.3187     0     0     0     0     0
```

```
y3  0.3896       0  -0.2859       0       0       0       0       0
y4  0.0000       0   0.0000       0       0       0       0       0
y5  0.0000       0   0.0000       0       0       0       0       0
y6  0.0000       0   0.0000       0       0       0       0       0
y7  0.0000       0   0.0000       0       0       0       0       0
y8  0.0000       0   0.0000       0       0       0       0       0

 , , Lag=4

      x1=y1    x2=y2    x3=y3    x4=y4 x5=y5 x6=y6 x7=y7 x8=y8
y1  -0.7311   0.0000   0.0000   0.0000     0     0     0     0
y2   0.0000  -0.6603   0.0000   0.0000     0     0     0     0
y3   0.0000   0.0000  -0.6395   0.0000     0     0     0     0
y4   0.0000   0.0000   0.0000  -0.7038     0     0     0     0
y5   0.0000   0.0000   0.0000   0.0000     0     0     0     0
y6   0.0000   0.0000   0.0000   0.0000     0     0     0     0
y7   0.0000   0.0000   0.0000   0.0000     0     0     0     0
y8   0.0000   0.0000   0.0000   0.0000     0     0     0     0

MA f-values (squared t-values):
 , , Lag=1

    x1=y1 x2=y2 x3=y3 x4=y4 x5=y5 x6=y6 x7=y7 x8=y8
y1   0.00     0  0.00     0     0     0     0     0
y2  12.80     0  2.69     0     0     0     0     0
y3   7.96     0  5.06     0     0     0     0     0
y4   0.00     0  0.00     0     0     0     0     0
y5   0.00     0  0.00     0     0     0     0     0
y6   0.00     0  0.00     0     0     0     0     0
y7   0.00     0  0.00     0     0     0     0     0
y8   0.00     0  0.00     0     0     0     0     0

 , , Lag=4

    x1=y1 x2=y2 x3=y3 x4=y4 x5=y5 x6=y6 x7=y7 x8=y8
y1   58.8   0.0  0.00   0.0     0     0     0     0
y2    0.0  42.9  0.00   0.0     0     0     0     0
y3    0.0   0.0 40.03   0.0     0     0     0     0
y4    0.0   0.0  0.00  54.3     0     0     0     0
y5    0.0   0.0  0.00   0.0     0     0     0     0
y6    0.0   0.0  0.00   0.0     0     0     0     0
y7    0.0   0.0  0.00   0.0     0     0     0     0
y8    0.0   0.0  0.00   0.0     0     0     0     0

r$> mulx_model$ar.pvalues
 , , 1

      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]     1    0    0    0    0    0    0    0
```

```
[2,]    0    1    0    0    0    0    0    0
[3,]    0    0    1    0    0    0    0    0
[4,]    0    0    0    1    0    0    0    0
[5,]    0    0    0    0    1    0    0    0
[6,]    0    0    0    0    0    1    0    0
[7,]    0    0    0    0    0    0    1    0
[8,]    0    0    0    0    0    0    0    1

, , 2

            [,1]         [,2]       [,3]        [,4]      [,5]
[,6]      [,7]      [,8]
[1,] 0.000000000 1.448958e-07 0.00000000 0.000000000 0.0000000 0.0000000 0.000
[2,] 0.000000000 2.646460e-02 0.08794247 0.006294701 0.0000000 0.0000000 0.000
[3,] 0.007797587 3.879520e-03 0.00000000 0.061920728 0.2131464 0.1828076 0.000
[4,] 0.000000000 2.442119e-02 0.00000000 0.000000000 0.1605344 0.0000000 0.102
[5,] 0.000000000 0.000000e+00 0.00000000 0.000000000 0.0000000 0.0000000 0.000
[6,] 0.000000000 0.000000e+00 0.00000000 0.000000000 0.0000000 0.0000000 0.000
[7,] 0.000000000 0.000000e+00 0.00000000 0.000000000 0.0000000 0.0000000 0.000
[8,] 0.000000000 0.000000e+00 0.00000000 0.000000000 0.0000000 0.0000000 0.000


r$> mulx_model$ma.pvalues
, , 1

     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    1    0    0    0    0    0    0    0
[2,]    0    1    0    0    0    0    0    0
[3,]    0    0    1    0    0    0    0    0
[4,]    0    0    0    1    0    0    0    0
[5,]    0    0    0    0    1    0    0    0
[6,]    0    0    0    0    0    1    0    0
[7,]    0    0    0    0    0    0    1    0
[8,]    0    0    0    0    0    0    0    1

, , 2

             [,1] [,2]       [,3] [,4] [,5] [,6] [,7] [,8]
[1,] 0.0000000000    0 0.00000000    0    0    0    0    0
[2,] 0.0005178247    0 0.10403975    0    0    0    0    0
[3,] 0.0056965230    0 0.02656682    0    0    0    0    0
[4,] 0.0000000000    0 0.00000000    0    0    0    0    0
[5,] 0.0000000000    0 0.00000000    0    0    0    0    0
[6,] 0.0000000000    0 0.00000000    0    0    0    0    0
[7,] 0.0000000000    0 0.00000000    0    0    0    0    0
[8,] 0.0000000000    0 0.00000000    0    0    0    0    0

, , 3
```

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    0    0    0    0    0    0    0    0
[2,]    0    0    0    0    0    0    0    0
[3,]    0    0    0    0    0    0    0    0
[4,]    0    0    0    0    0    0    0    0
[5,]    0    0    0    0    0    0    0    0
[6,]    0    0    0    0    0    0    0    0
[7,]    0    0    0    0    0    0    0    0
[8,]    0    0    0    0    0    0    0    0

, , 4

     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
[1,]    0    0    0    0    0    0    0    0
[2,]    0    0    0    0    0    0    0    0
[3,]    0    0    0    0    0    0    0    0
[4,]    0    0    0    0    0    0    0    0
[5,]    0    0    0    0    0    0    0    0
[6,]    0    0    0    0    0    0    0    0
[7,]    0    0    0    0    0    0    0    0
[8,]    0    0    0    0    0    0    0    0

, , 5

             [,1]         [,2]        [,3]         [,4] [,5] [,6] [,7] [,8]
[1,] 6.619145e-12 0.000000e+00 0.00000e+00 0.000000e+00    0    0
0    0
[2,] 0.000000e+00 1.957458e-09 0.00000e+00 0.000000e+00    0    0
0    0
[3,] 0.000000e+00 0.000000e+00 5.95953e-09 0.000000e+00    0    0
0    0
[4,] 0.000000e+00 0.000000e+00 0.00000e+00 3.268207e-11    0    0
0    0
[5,] 0.000000e+00 0.000000e+00 0.00000e+00 0.000000e+00    0    0
0    0
[6,] 0.000000e+00 0.000000e+00 0.00000e+00 0.000000e+00    0    0
0    0
[7,] 0.000000e+00 0.000000e+00 0.00000e+00 0.000000e+00    0    0
0    0
[8,] 0.000000e+00 0.000000e+00 0.00000e+00 0.000000e+00    0    0
0    0
```