MARKOV DECISION PROCESSES

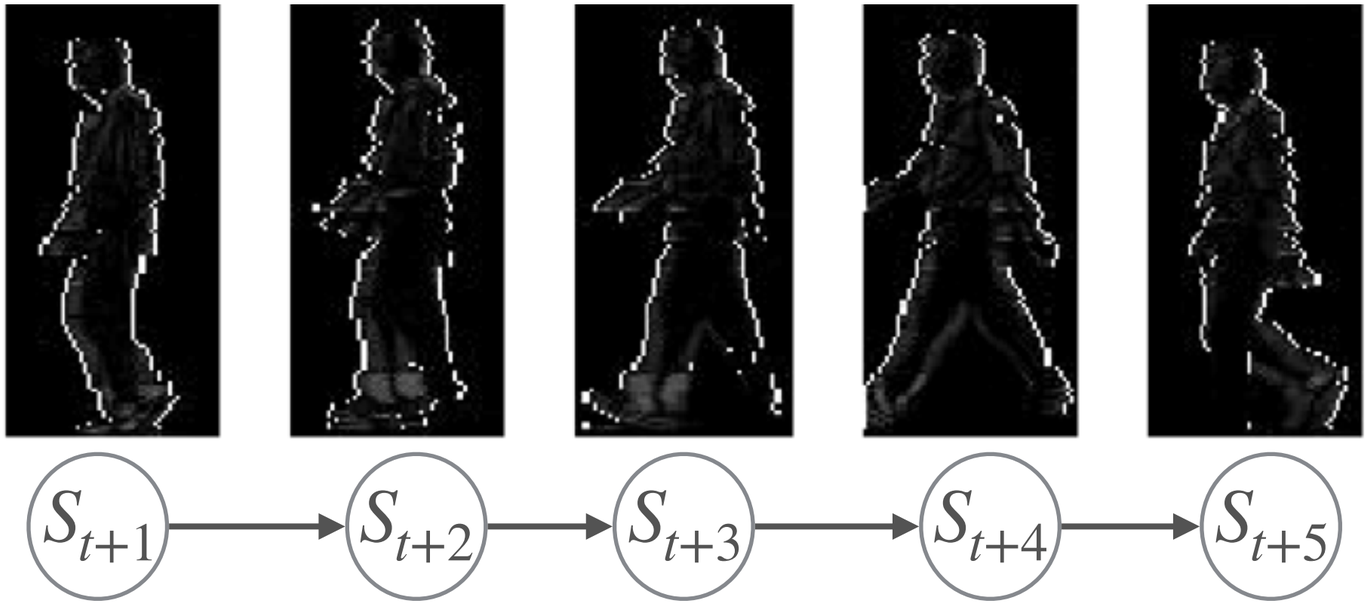
Volker Krüger

Slides partially borrowed from a presentation by David Silver

OVERVIEW

- Markov Processes (MP)
- Markov Reward Processes (MRP)
- Markov Decision Processes (MDP)
  - Formal Model of Reinforcement Learning Tasks
  - Value Functions
  - Bellman Equation
  - Optimal Value Function: solving the MDP
- Policy Iteration

MARKOV PROCESSES



- *The next state is dependent only on the present state*
- A random state S_t is *Markov* if and only if

$$P(S_{t+1} \mid S_t) = P(S_{t+1} \mid S_1, \ldots, S_t)$$

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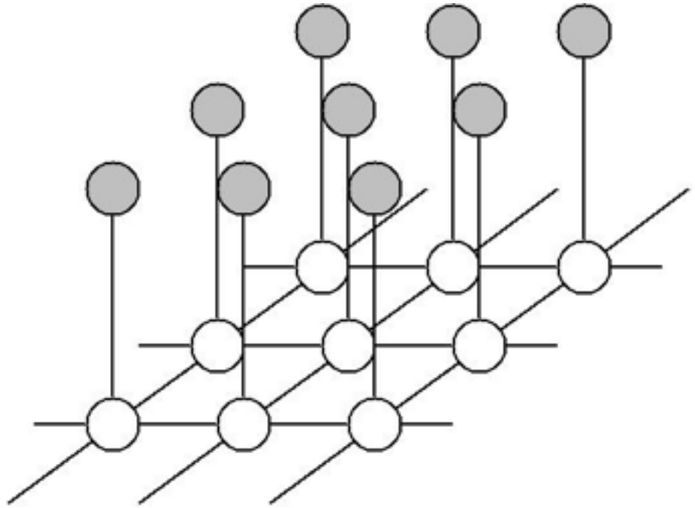
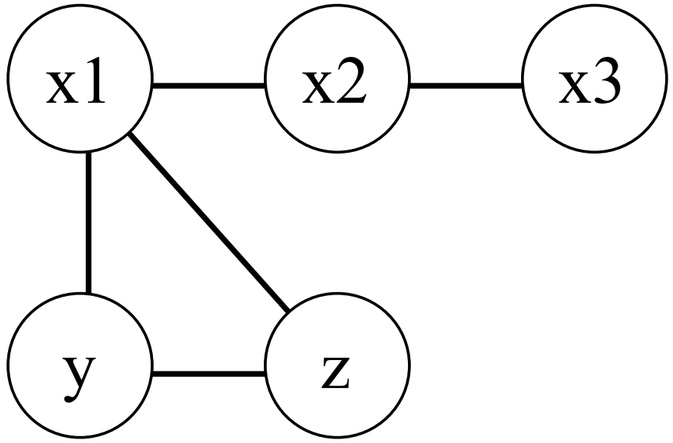
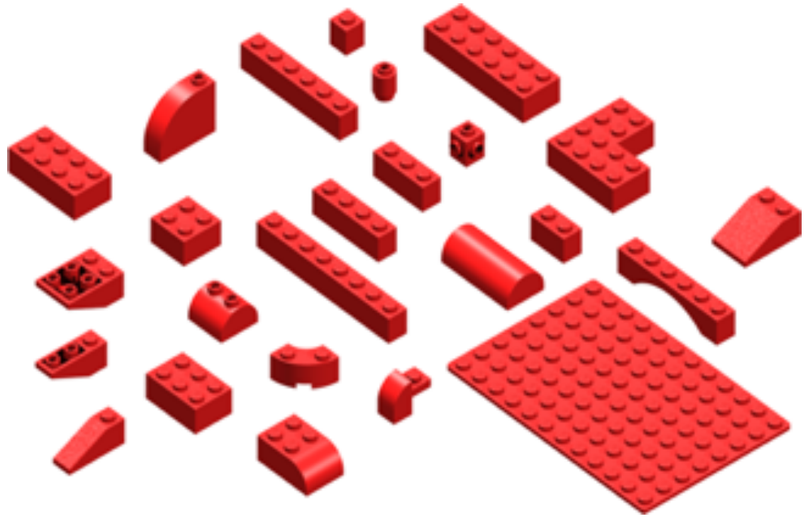
MARKOV RANDOM FIELDS



GRAPHICAL MODELS
TINKER TOYS TO BUILD COMPLEX PROBABILITY DISTRIBUTIONS

- Circles represent random variables.
- Lines represent statistical dependencies.
- There is a corresponding equation that gives $P(x_1, x_2, x_3, y, z)$ but often it's easier to understand things from the picture.
- These tinker toys for probabilities let you build up, from simple, easy-to-understand pieces, complicated probability distributions involving many variables.





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STATE TRANSITION MATRIX

- For a Markov state s and a successor state s', the *state transition probability* is defined by

$$P_{ss'} = P(S_{t+1} = s' \mid S_t = s)$$

- A state transition probability matrix summaries the possible state transitions

$$\mathbf{P} = \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & & \\ P_{n1} & \cdots & P_{nn} \end{bmatrix}$$

- Note that each row of $\mathbf{P}$ sums to 1.

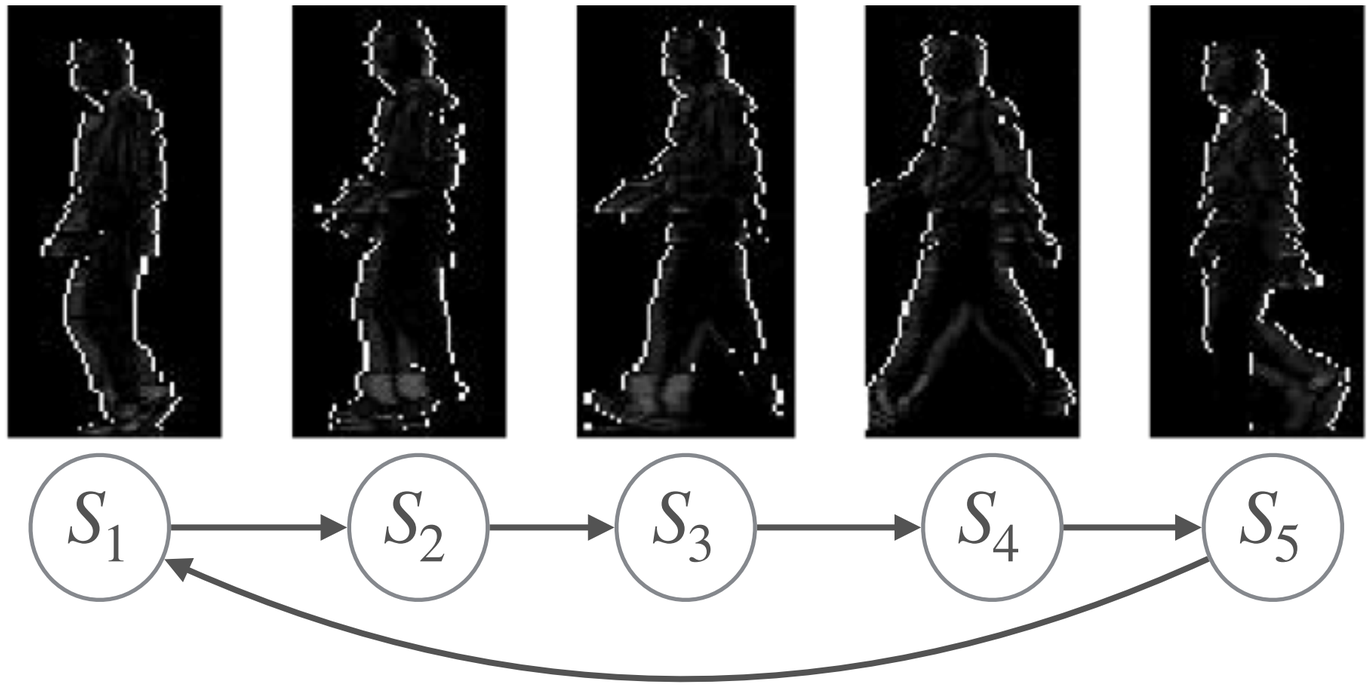
MARKOV PROCESS

- A Markov process is a sequence $S_1, S_2, \ldots$ of Markov states, i.e., a sequence of random states that have the Markov property

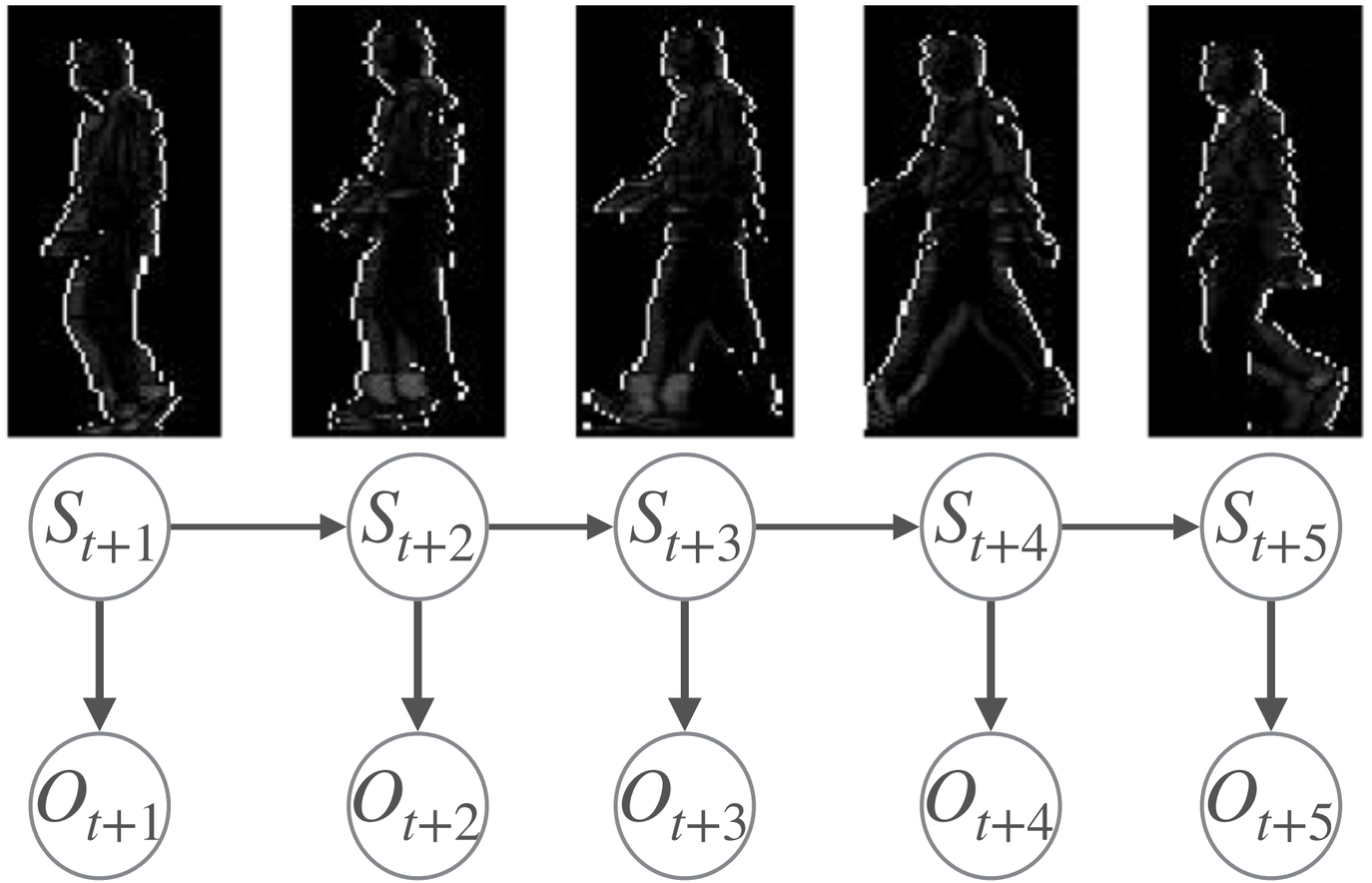
- **Definition:** A Markov process (or Markov Chain) is a tuple $(\mathbf{S}, \mathbf{P})$
  - $\mathbf{S}$ is a (finite) set of states
  - $\mathbf{P}$ is a state transition probability matrix, with

$$P_{ss'} = P(S_{t+1} = s' \mid S_t = s)$$

EXAMPLE GAIT MARKOV CHAIN

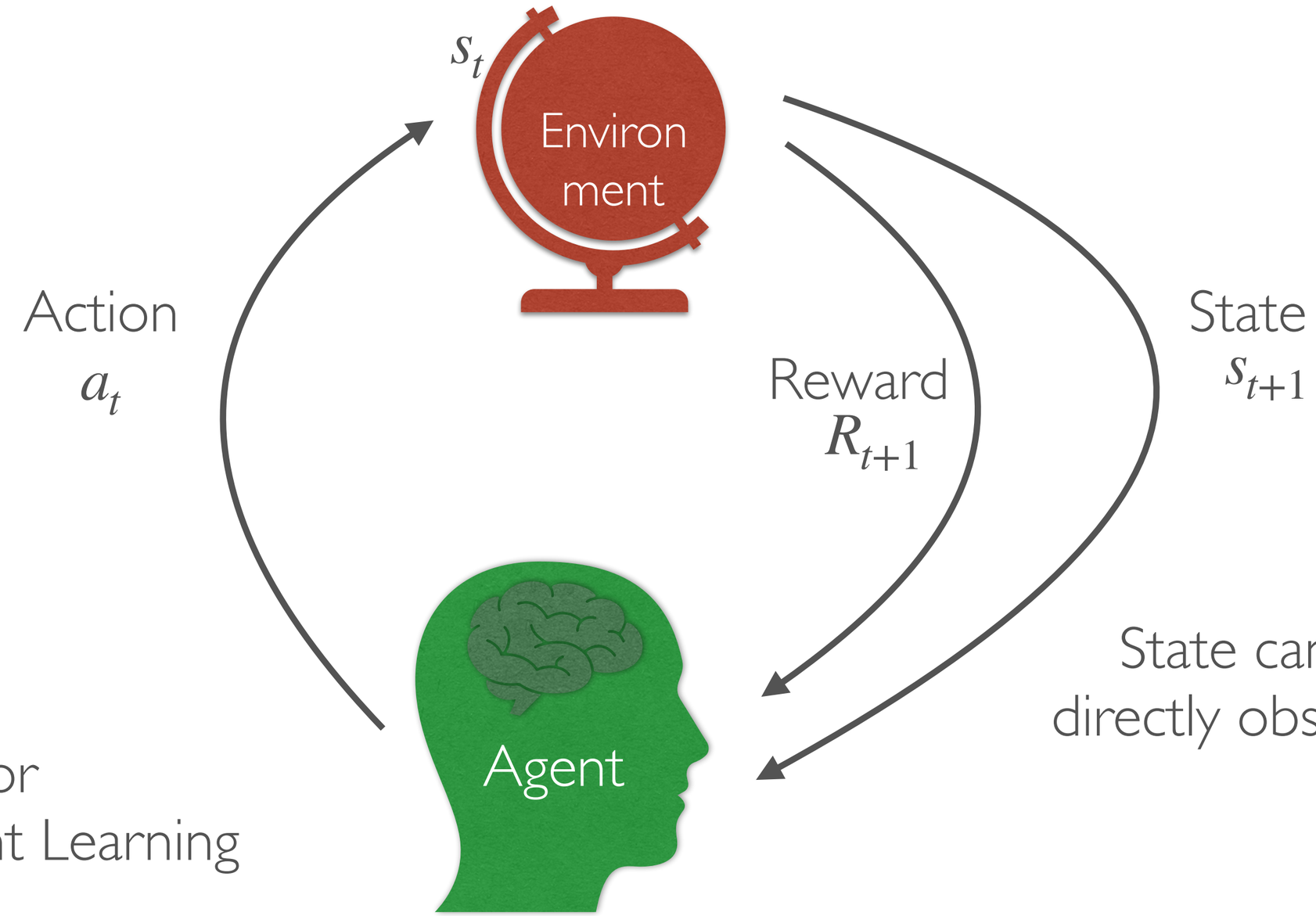


HIDDEN MARKOV MODELS



- Extension: Hidden Markov Model
  - From last lecture: Likelihood of an observation o_t, given that we are in state S_t

MARKOV DECISION PROCESS



State can be directly observed!!

Model used for Reinforcement Learning

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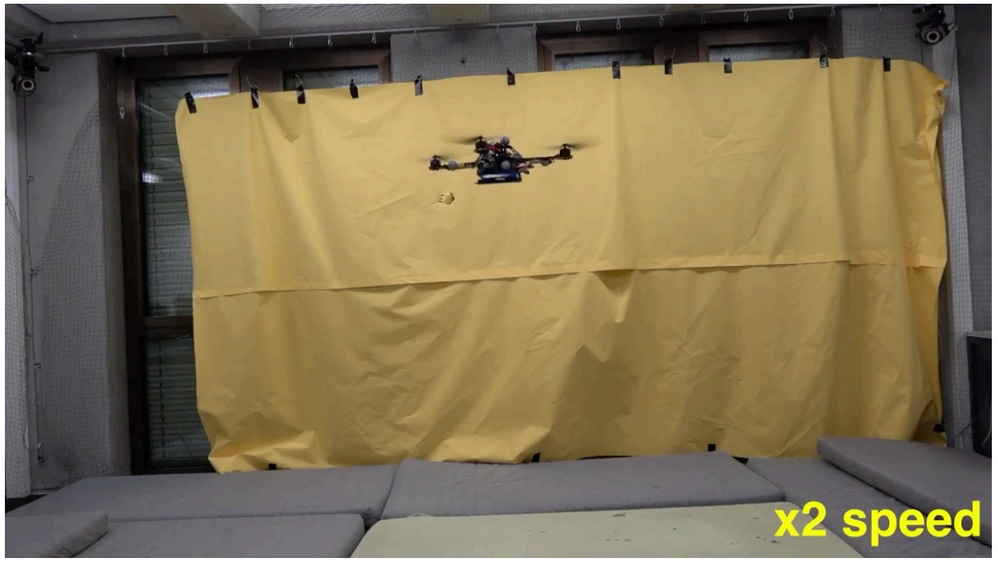
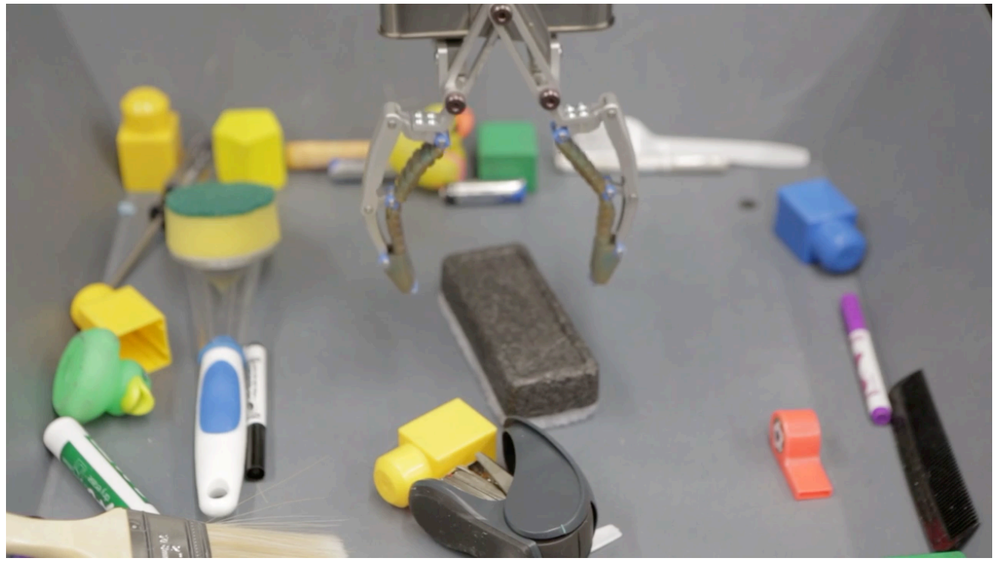
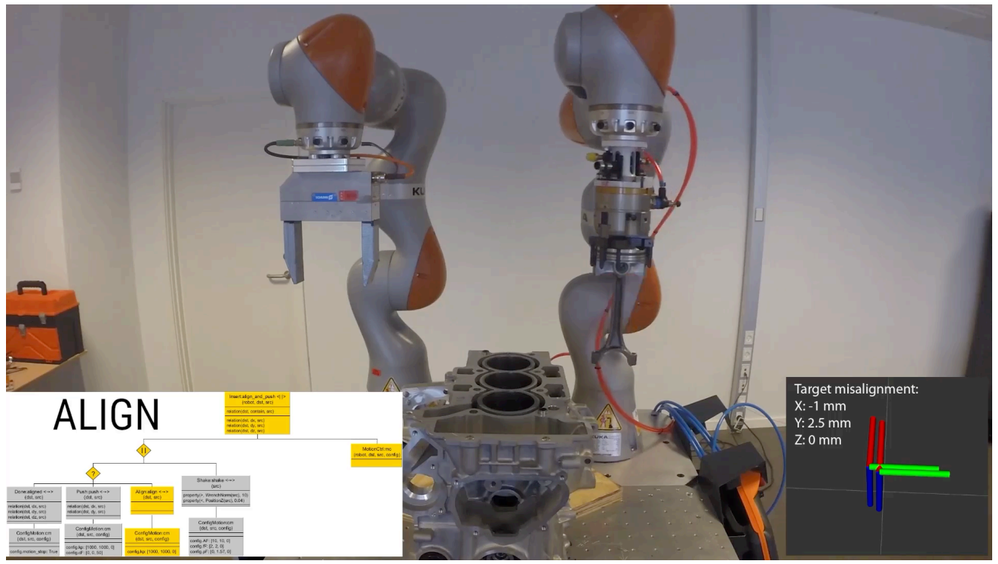
REINFORCEMENT LEARNING





- Value function + Policy modelled using DNN and CNN.
- 14 robots needed 3000h (800.000 grasp attempts) before something intelligent happened



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EXAMPLE: PICKING MARKOV CHAIN

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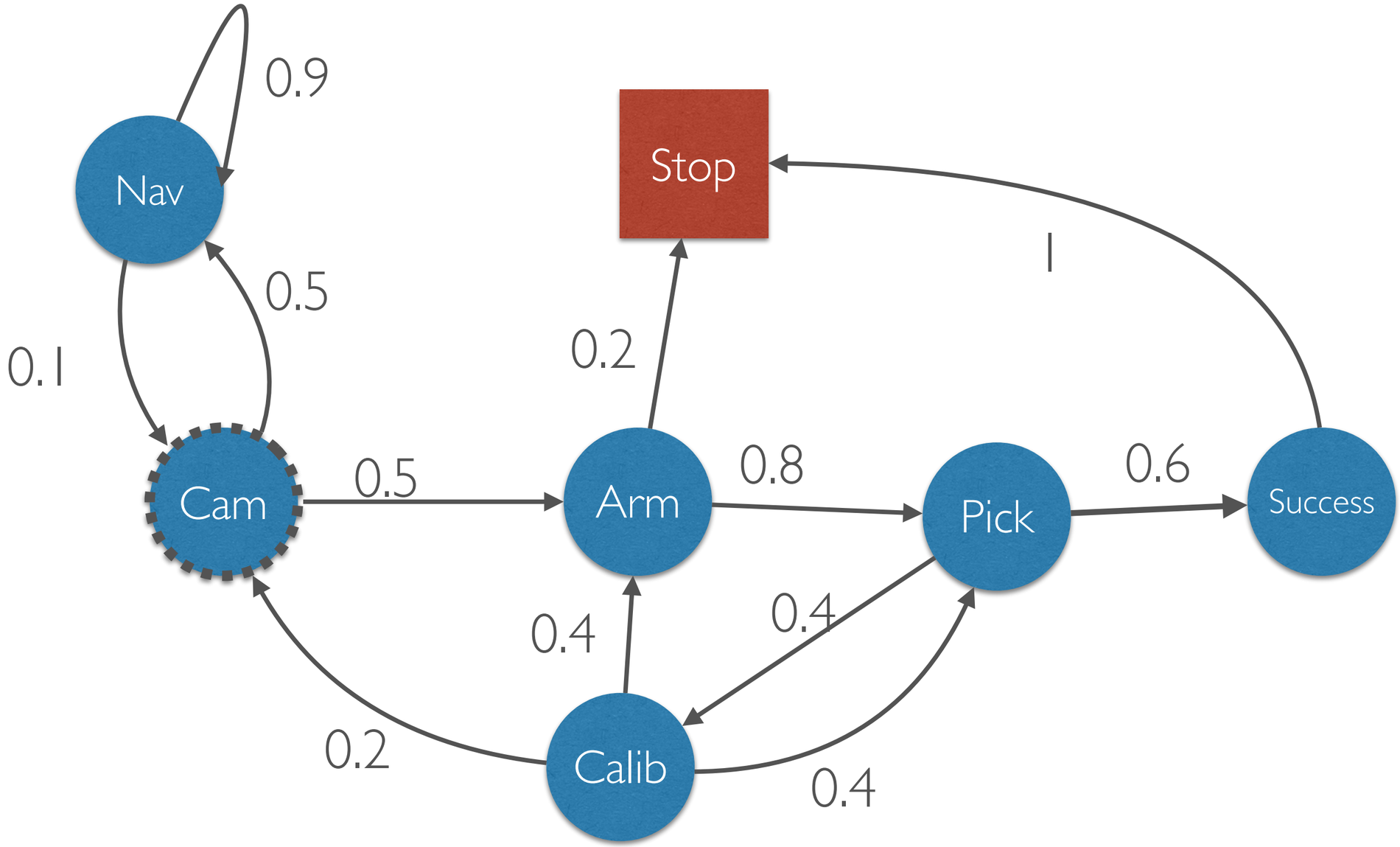
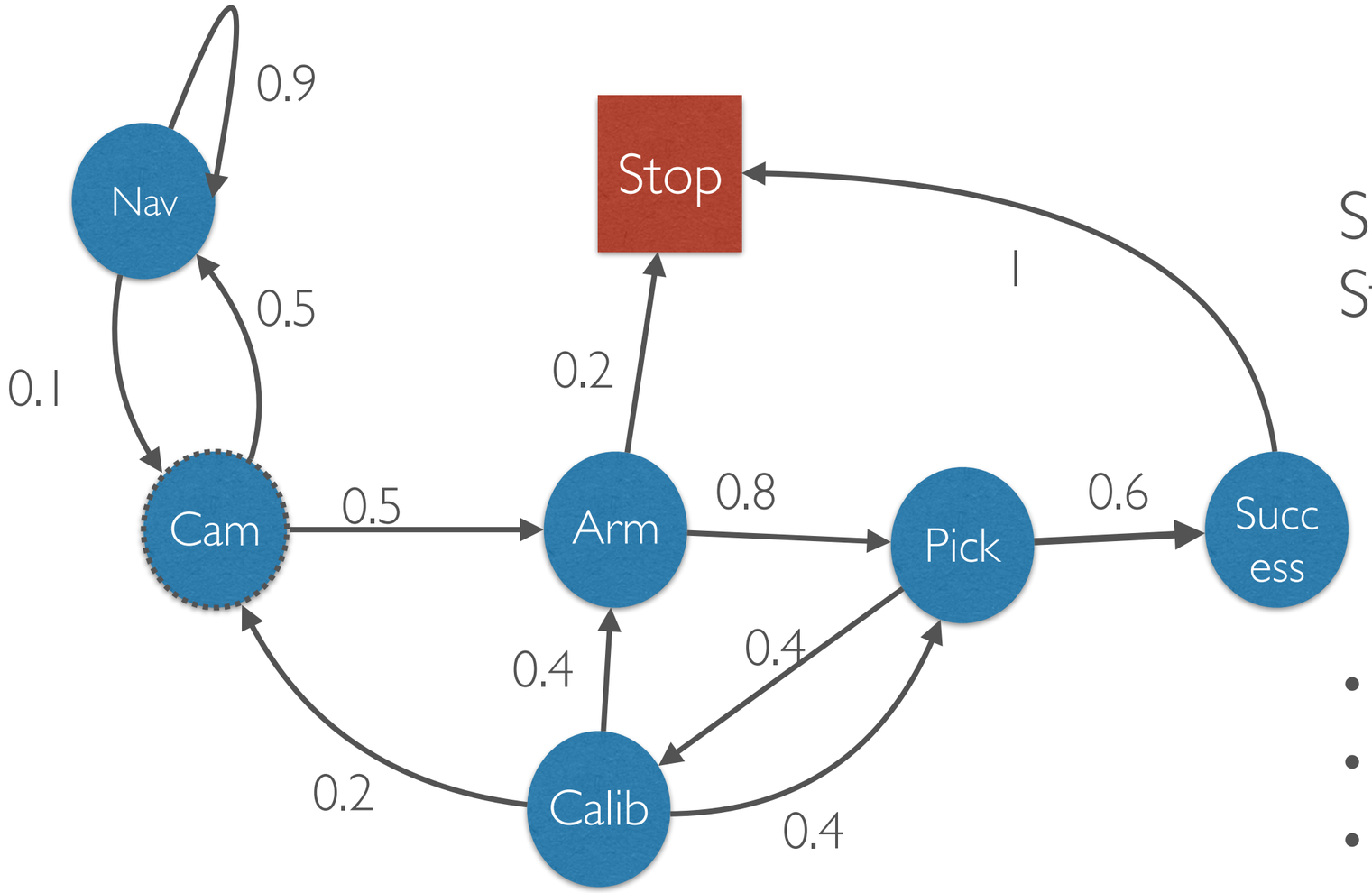


EXAMPLE: PICKING MARKOV CHAIN

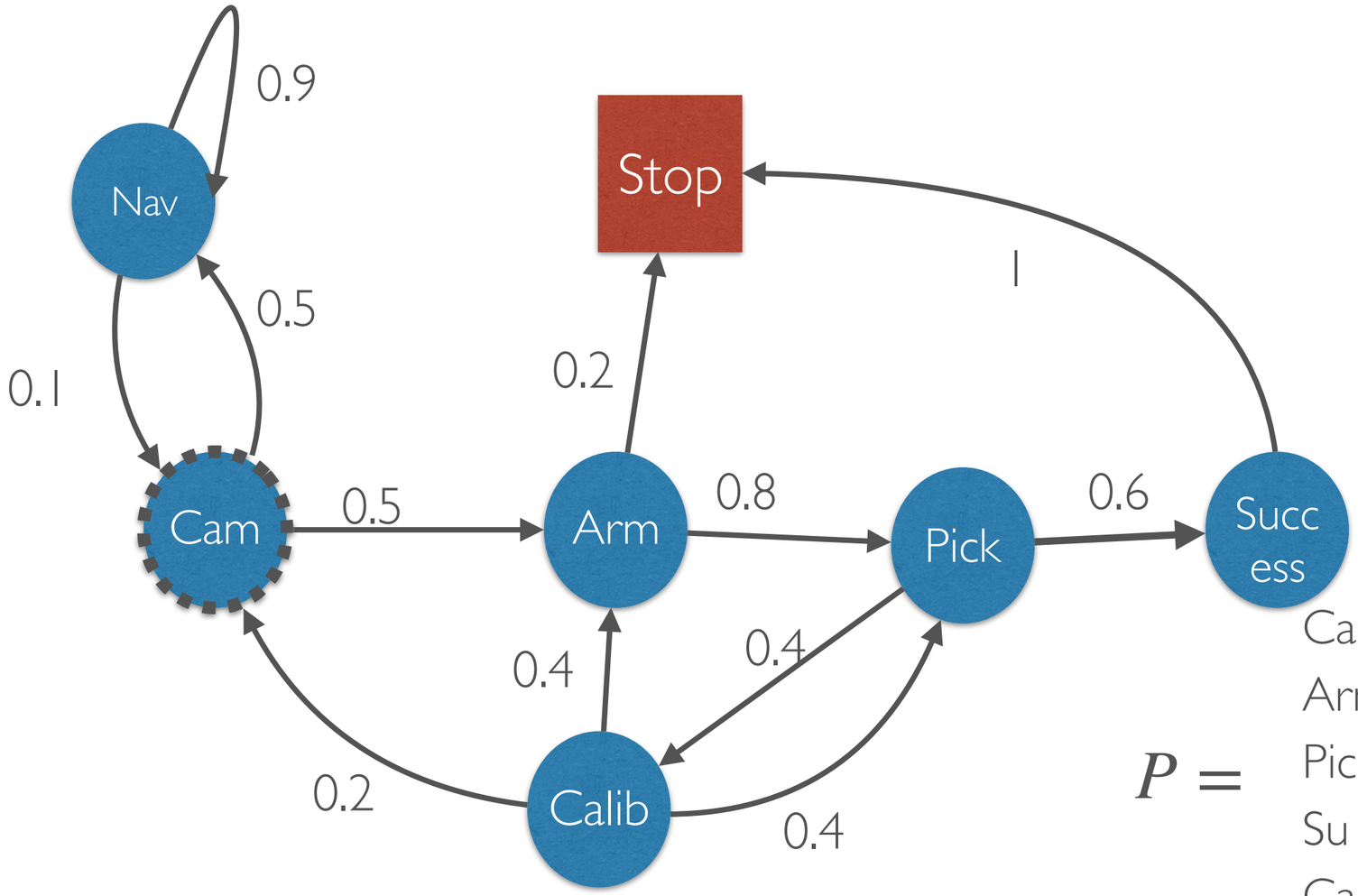


Sample **episodes**:
Start state $S_1 = \text{Cam}$

$$S_1, S_2, \ldots, S_T$$

- Cam, Nav, Cam, Arm, Pick, Su, S
- Cam, Arm, Pick,Cal,Pick, Su,S
- Can, Nav, Cam, Arm, S

EXAMPLE: PICKING MARKOV CHAIN



$$P = \begin{array}{c} \\ \text{Cam} \\ \text{Arm} \\ \text{Pick} \\ \text{Su} \\ \text{Cal} \\ \text{Nav} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{Cam} & \text{Arm} & \text{Pick} & \text{Su} & \text{Cal} & \text{Nav} & \text{S} \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.2 & 0.4 & 0.4 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

MARKOV REWARD PROCESSES

MARKOV REWARD PROCESS

- **Definition**: A Markov Reward Process is a tupel $(\mathbf{S}, \mathbf{P}, \mathbf{R}, \gamma)$
  - $\mathbf{S}$ is a finite set of states
  - $\mathbf{P}$ is a state transition probability matrix $P_{ss'} = P\left(S_{t+1} = s' \,|\, S_t = s\right)$
  - $\mathbf{R}$ is a reward function $R_s = E\left(R_{t+1} \,|\, S_t = s\right)$
  - γ is a discount factor $\gamma \in [0,1]$

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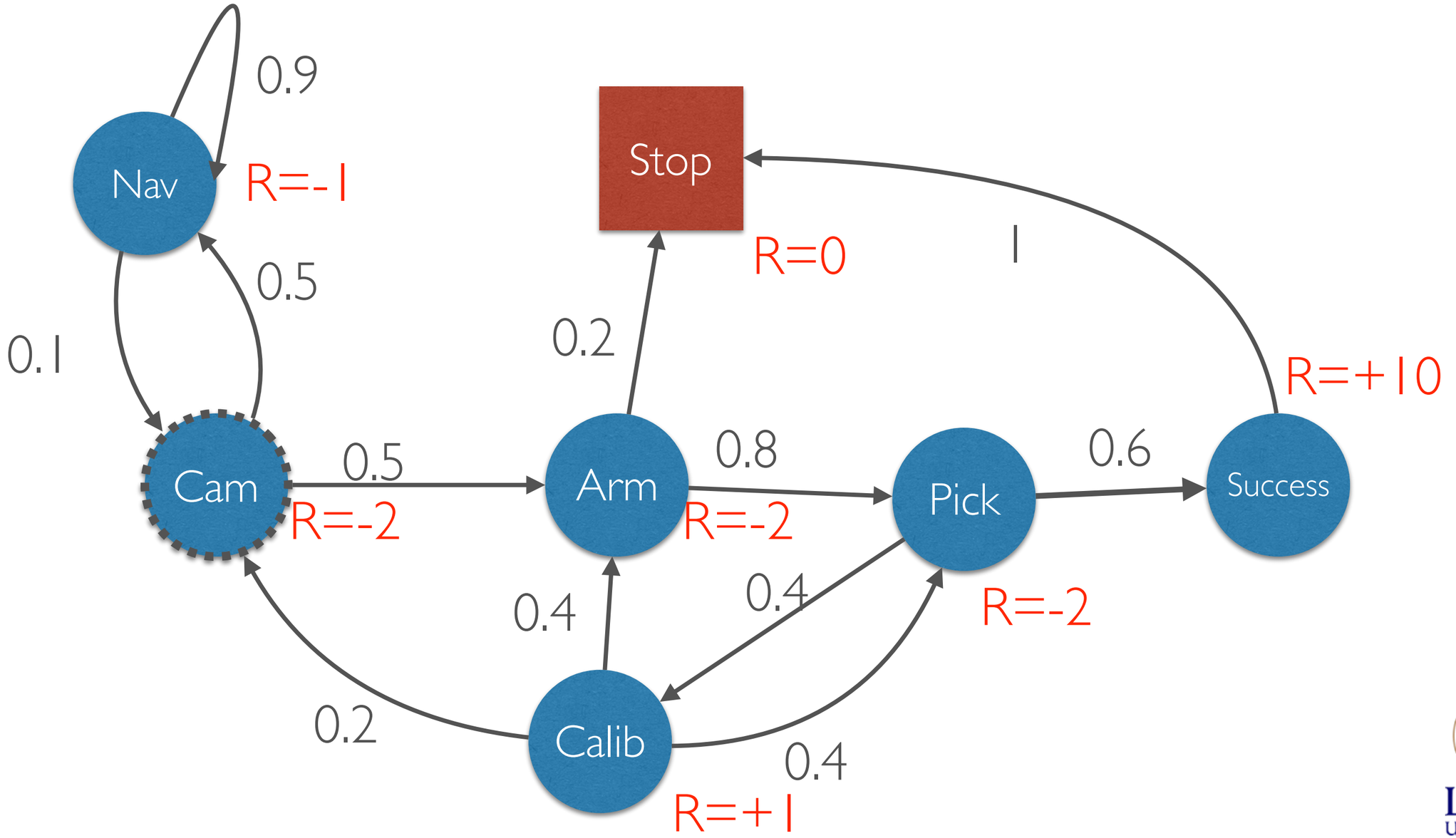
EXAMPLE: PICKING MARKOV CHAIN



RETURN

Definition: The ***return*** G_t is the total discounted reward from time-step t .

$$G_t = R_{t+1} + \gamma R_{t+2} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The *discount* $\gamma \in [0,1]$ weights the future rewards.
- The value of a reward R that is *k+1* steps into the future is $\gamma^k R$
  - Short-sighted: γ close to 0
  - Far-sighted: γ close to 1

ROLE OF THE DISCOUNT FACTOR

- Mathematically convenient as γ^k vanishes with k for $\gamma < 1$
- Assures finite returns even in cyclic Markov processes
- Represents the uncertainty about the future rewards
- Animal/human behaviour shows preference for immediate reward
- Possible to use *undiscounted* Markov reward process ($\gamma = 1$) if all sequences terminate (assures finite return)

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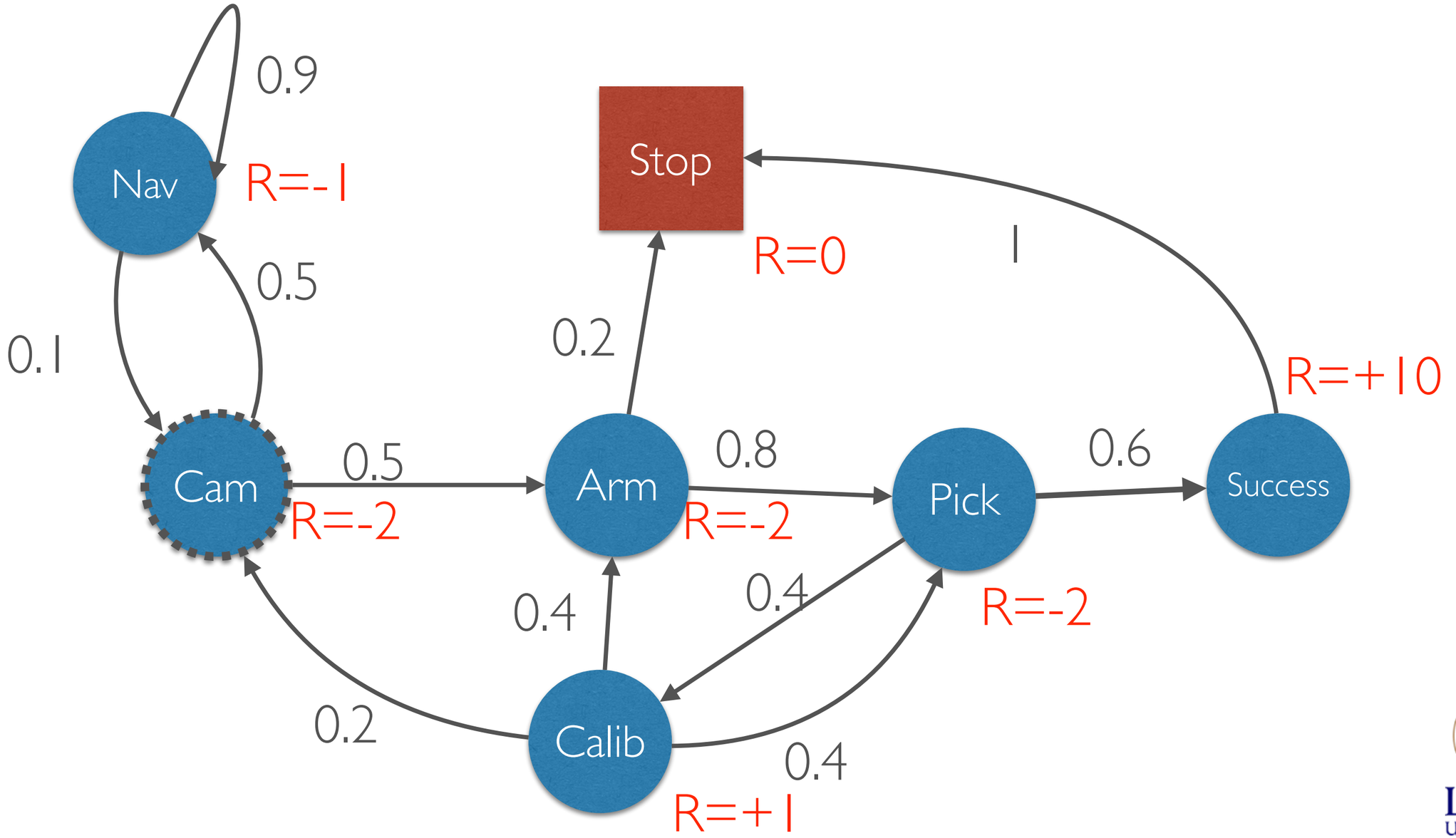
EXAMPLE: PICKING MARKOV CHAIN



If we get rewards, how good is it to be in a certain state

STATE VALUE FUNCTION

- The state value function $v(s)$ gives the long-term value of a state s

Definition: The ***state value function*** $v(s)$ of a MRP is the expected return starting from state s

$$v(s) = E\left\{G_t \,|\, S_t = s\right\}$$

REWARD OF THE PICKING NETWORK

Sample **returns**: $S_1 = \text{Cam with } \gamma = \frac{1}{2}$

$$G_1 = R_2 + \gamma R_3 + \ldots + \gamma^{T-2} R_T$$

Sample **episodes**:

- Cam, Nav, Cam, Arm, Pick, Su, S
- Cam, Arm, Pick,Cal,Pick, Su, S
- Can, Nav, Cam, Arm, S

$$v_1 = -2 + \frac{1}{2}(-1) + \frac{1}{4}(-2) + \frac{1}{8}(-2) + \frac{1}{16}(-2) + \frac{1}{32}10 = -3.28$$

$$v_1 = -2 + \frac{1}{2}(-2) + \frac{1}{4}(-2) + \frac{1}{8}(1) + \frac{1}{16}(-2) + \frac{1}{32}10 = -3.18$$

$$v_1 = -2 + \frac{1}{2}(-2) + \frac{1}{4}(-1) + \frac{1}{8}(-2) + \frac{1}{16}(-2) = -3.62$$

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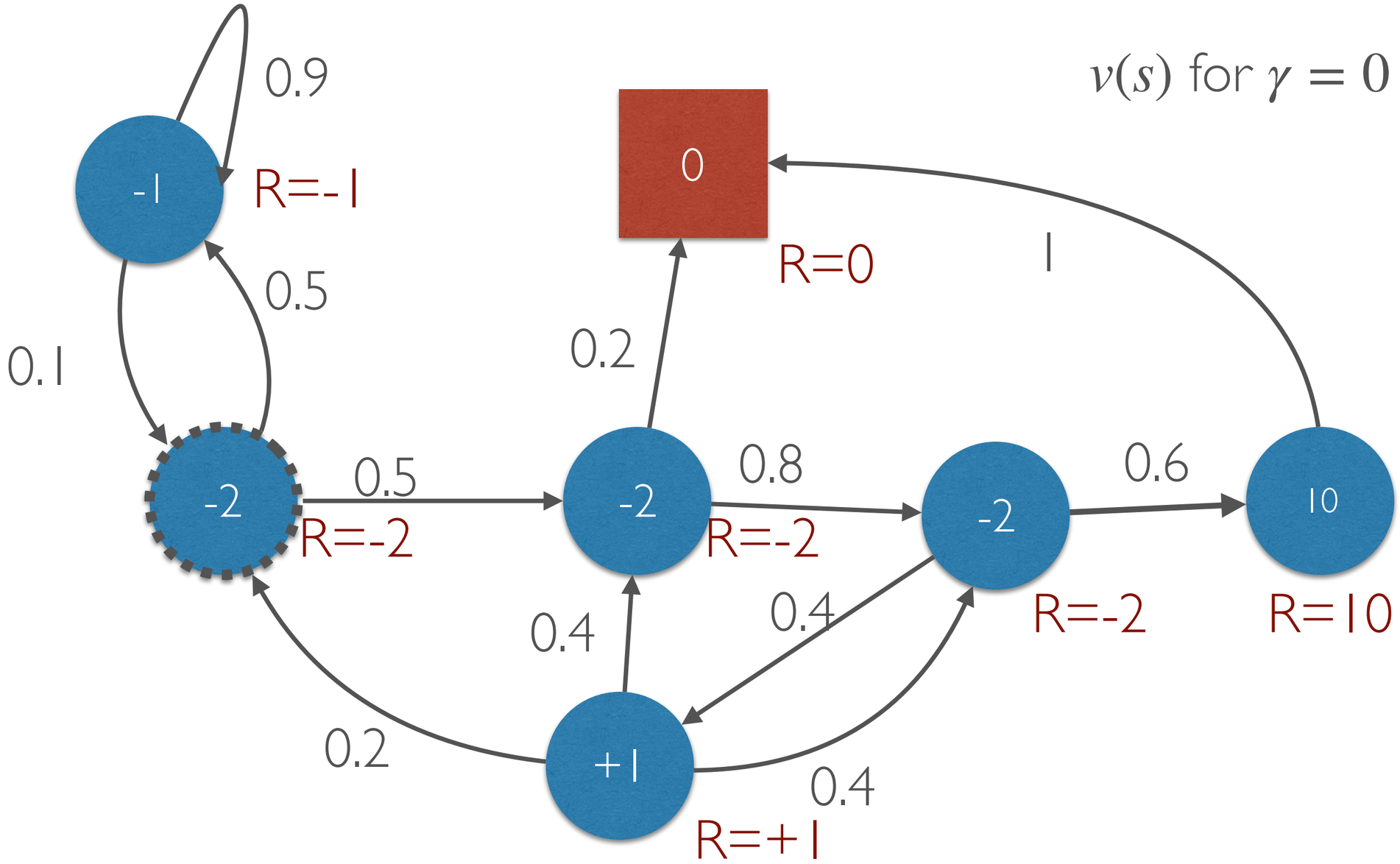

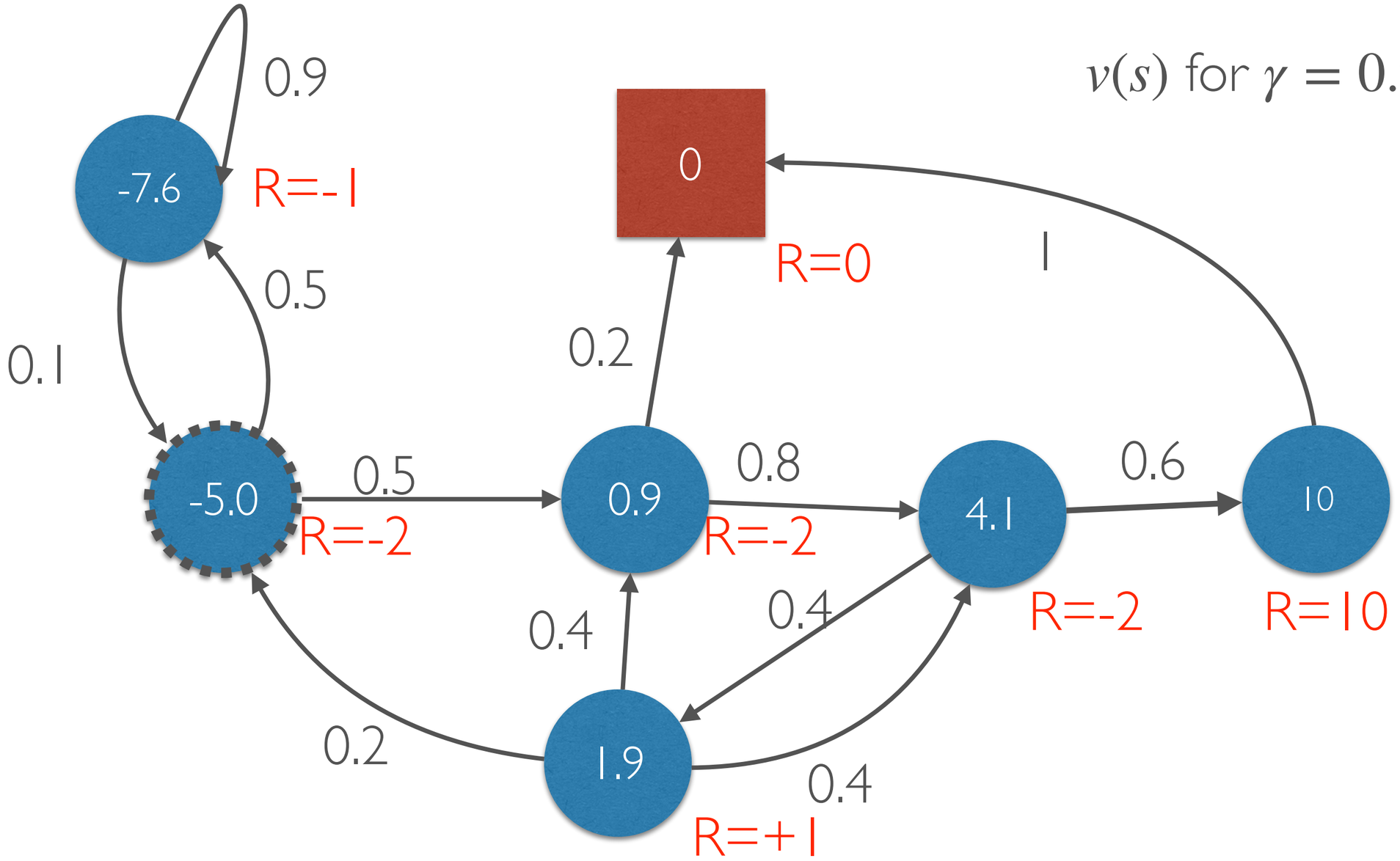

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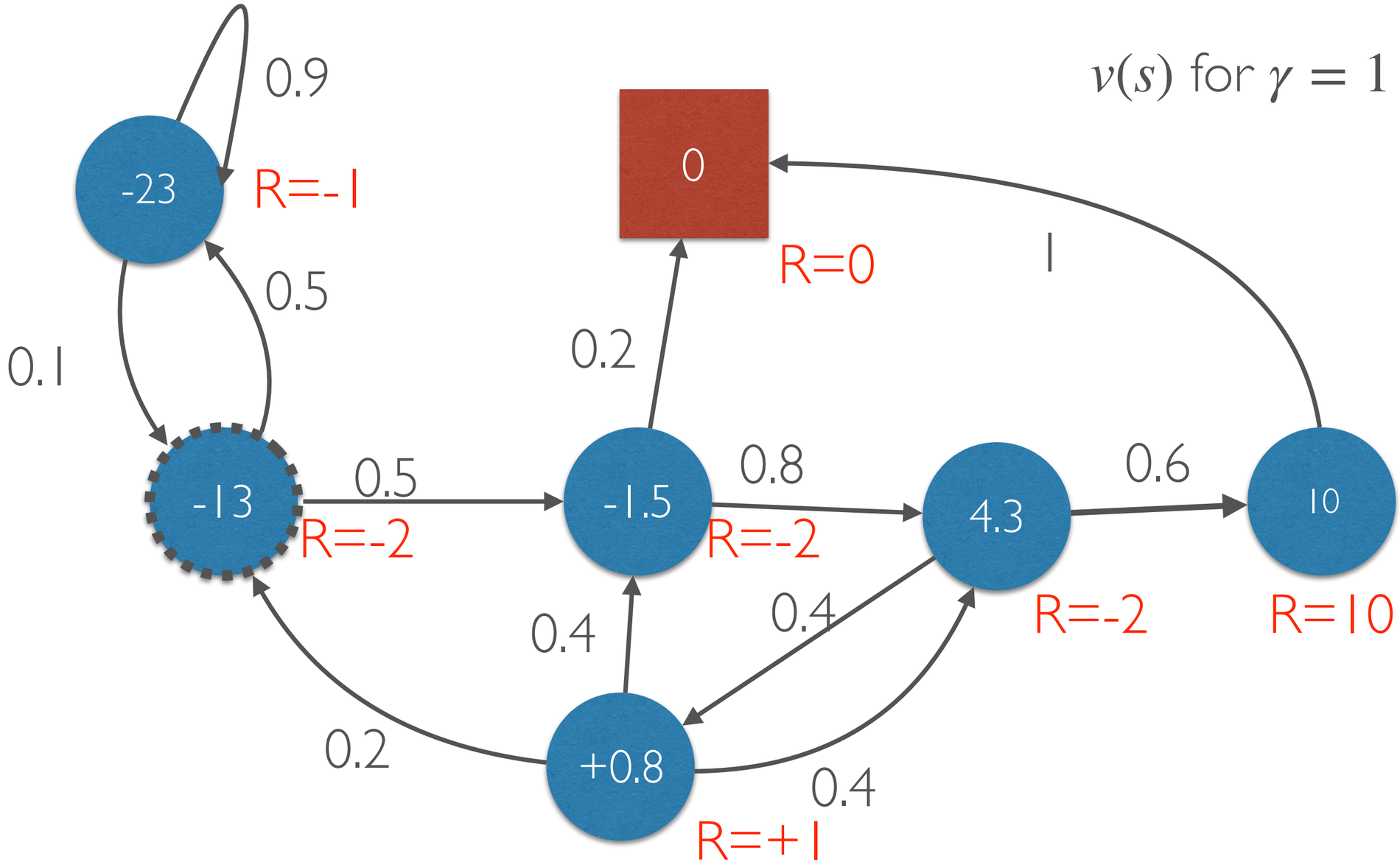
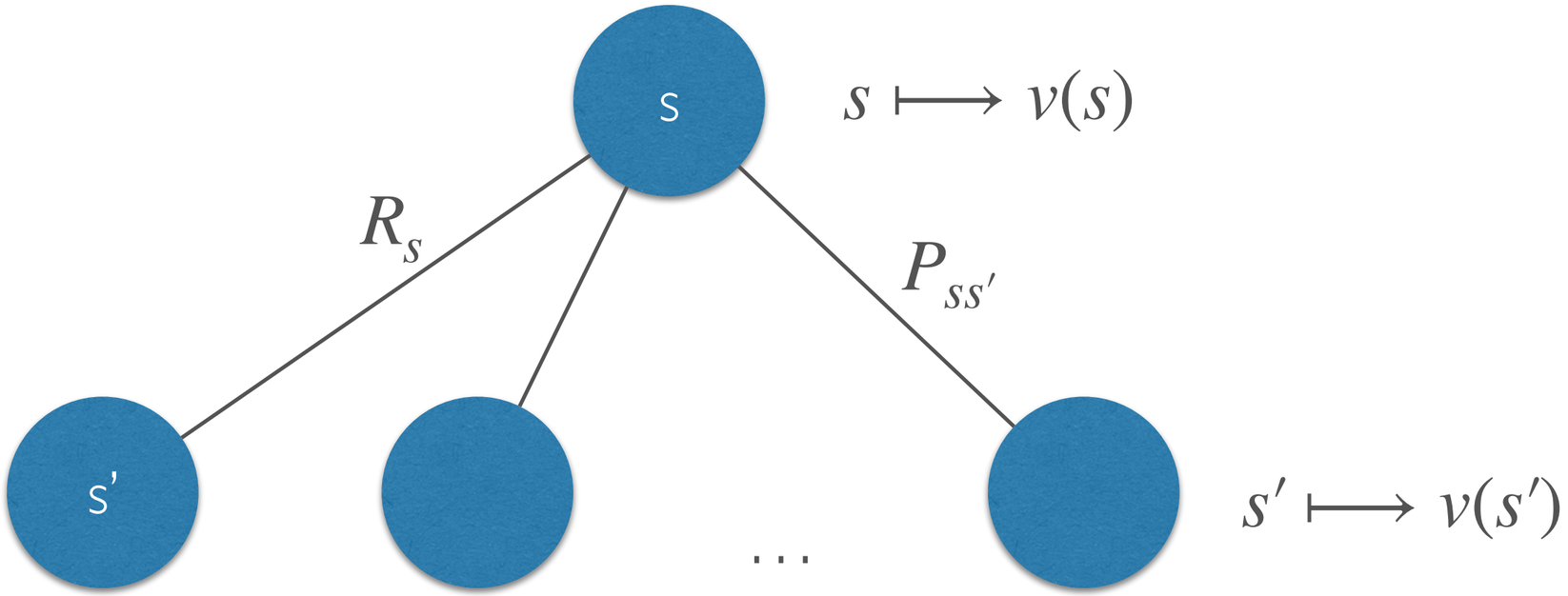

$v(s)$ for $\gamma = 1$
0.9
-23
R=-1
0.5
0.1
-13
0.5
R=-2
0
R=0
0.2
1
-1.5
0.8
R=-2
4.3
0.6
10
R=-2
R=10
0.4
0.4
0.2
+0.8
0.4
R=+1

BELLMAN EQUATION FOR MRP

- The value function can be compute recursively
  - The immediate reward R_{t+1}
  - The discounted future rewards $\gamma v(S_{t+1})$

$$
\begin{aligned}
v(s) &= E\left\{G_t \,|\, S_t = s\right\} \\
&= E\left\{R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \,|\, S_t = s\right\} \\
&= E\left\{R_{t+1} + \gamma\left(R_{t+2} + \gamma R_{t+3} + \ldots\right) \,|\, S_t = s\right\} \\
&= E\left\{R_{t+1} + \gamma\left(G_{t+1}\right) \,|\, S_t = s\right\} \\
&= E\left\{R_{t+1} + \gamma v\left(S_{t+1}\right) \,|\, S_t = s\right\}
\end{aligned}
$$

$$v(s) = E\left\{ R_{t+1} + \gamma v\left(S_{t+1}\right) \mid S_t = s \right\}$$



Marginalizing over the next possible states:

$$v(s) = R_s + \gamma \sum_{s' \in S} P_{ss'} v(s')$$

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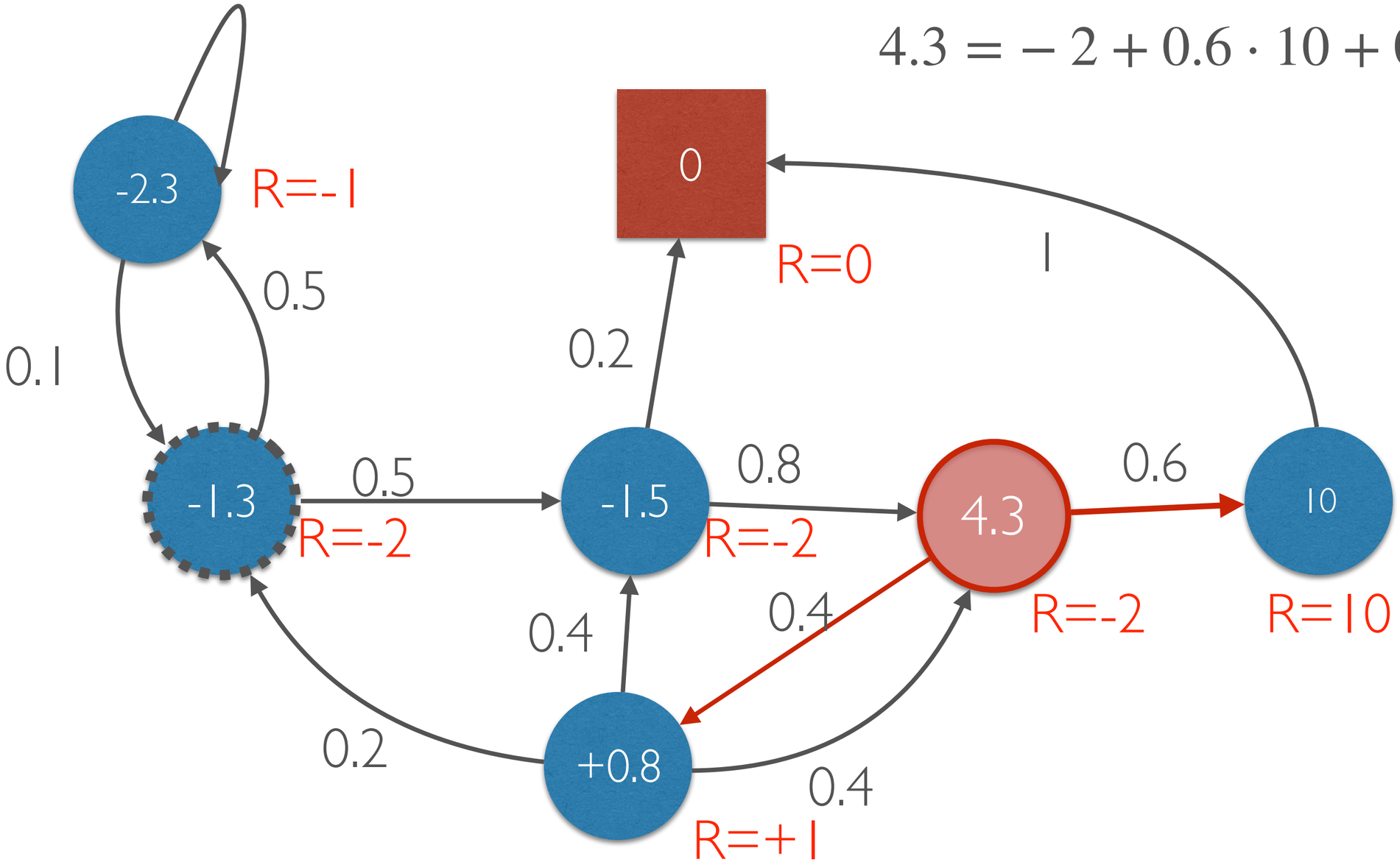
$$v(s) = R_s + \gamma \sum_{s' \in S} P_{ss'} v(s') \quad \gamma = 1$$

$$4.3 = -2 + 0.6 \cdot 10 + 0.4 \cdot 0.8$$



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The Bellman equation can be expressed using matrices

$$v = R + \gamma P v$$

where v is a column vector with one entry for each state

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix} + \gamma \begin{bmatrix} P_{11} & \dots & P_{1n} \\ \vdots & & \\ P_{n1} & \dots & P_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

- The Bellman equation is linear. It can be solved directly

$$
\begin{aligned}
v &= R + \gamma P v \\
\left(I - \gamma R\right) v &= R \\
v &= \left(I - \gamma P\right)^{-1} R
\end{aligned}
$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs
- Many iterative methods for solving large MRPs

MARKOV DECISION PROCESSES

MARKOV DECISION PROCESS

- A Markov Decision Process is a Markov Reward Process with decisions on actions

Definition: A **Markov Decision Process** is a tupel $(\mathbf{S}, \mathbf{A}, \mathbf{P}, \mathbf{R}, \gamma)$

- $\mathbf{S}$ is a finite set of states
- $\mathbf{A}$ is a finite set of actions
- $\mathbf{P}$ is a state transition probability matrix $P^{a}_{ss'} = P\left(S_{t+1} = s' \,|\, S_t = s, A_t = a\right)$

- $\mathbf{R}$ is a reward function $R^{a}_{s} = E\left(R_{t+1} \,|\, S_t = s, A_t = a\right)$
- γ is a discount factor $\gamma \in [0,1]$

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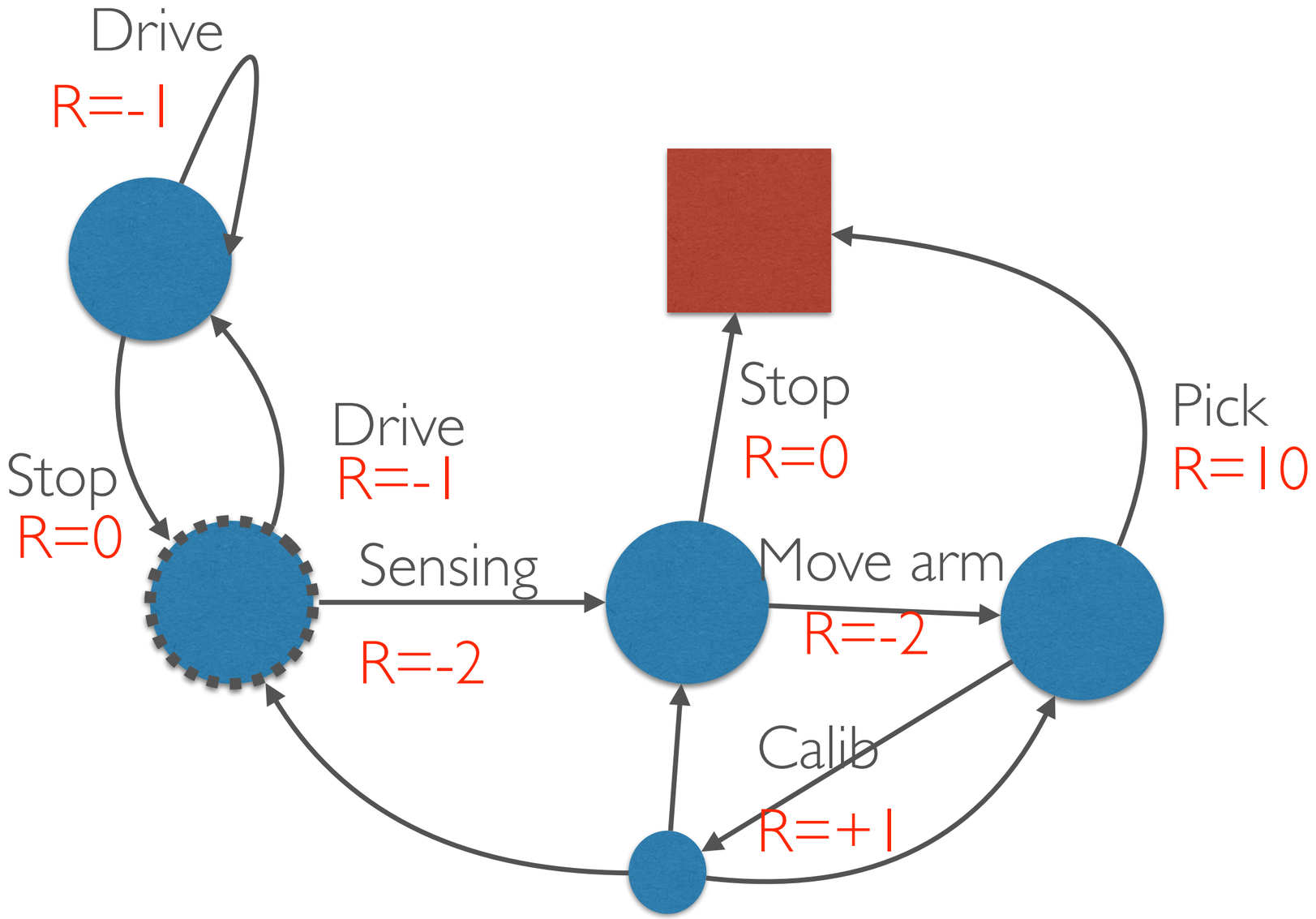

POLICIES: WHAT DO TO WHEN

Definition: A ***policy*** π *is a distribution over actions given states,*

$$\pi(a \mid s) = P(A_t = a \mid S_t = s)$$

- *Given that I am in state* s *what action* a *should I take?*
- Policies define the behaviour of the agent
- Policies are *stationary,* i.e., they do not depend on the time

$$\sum_{a \in A} \pi(a \mid s) = 1$$

POLICIES

- We can reduce a MDP to a MRP.
- Let $\mathbf{M} = (\mathbf{S}, \mathbf{A}, \mathbf{P}, \mathbf{R}, \gamma)$ be an MDP with a policy π
- If we marginalise out (average over) the actions

$$\hat{P}_{s,s'} = \sum_{a \in A} \pi(a \mid s) P^a_{ss'}$$

$$\hat{R}_s = \sum_{a \in A} \pi(a \mid s) R^a_s$$

- then $(\mathbf{S}, \hat{\mathbf{P}}, \hat{\mathbf{R}}, \gamma)$ is a Markov Reward Process.

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VALUE FUNCTIONS

Definition: The *state-value function* $v_\pi(s)$ of an MDP is the expected return starting from state s and following policy π

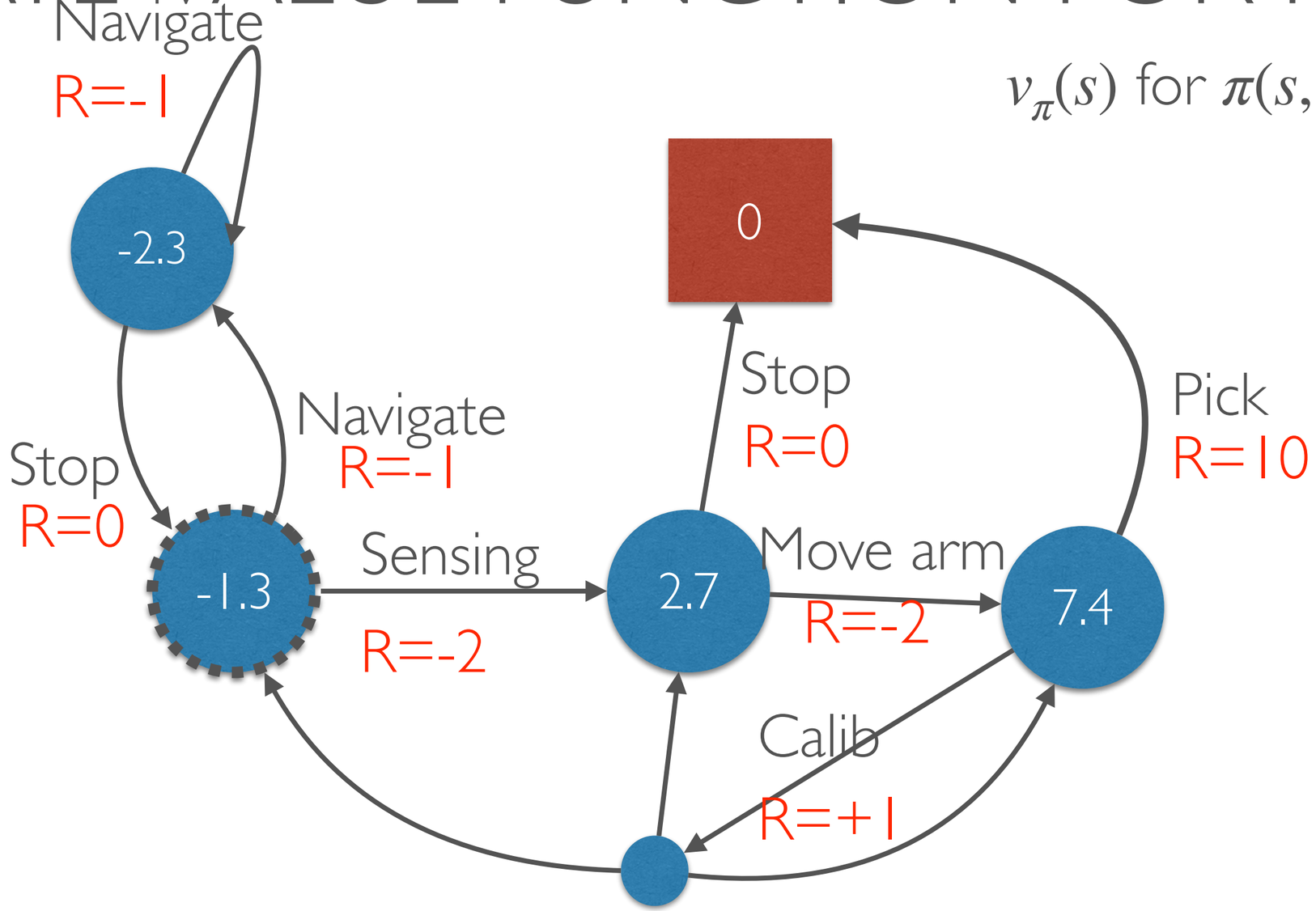
$$v_\pi(s) = E_\pi \left\{ G_t \,|\, S_t = s \right\}$$

Definition: The *action-value function* $q_\pi(s, a)$ of an MDP is the expected return starting from state s by taking action a and following policy π

$$q_\pi(s, a) = E_\pi \left\{ G_t \,|\, S_t = s, A_t = a \right\}$$

STATE-VALUE FUNCTION FOR PICKING

$v_\pi(s)$ for $\pi(s,a) = 0.5, \gamma = 1$



BELLMAN EXPECTATION EQUATION

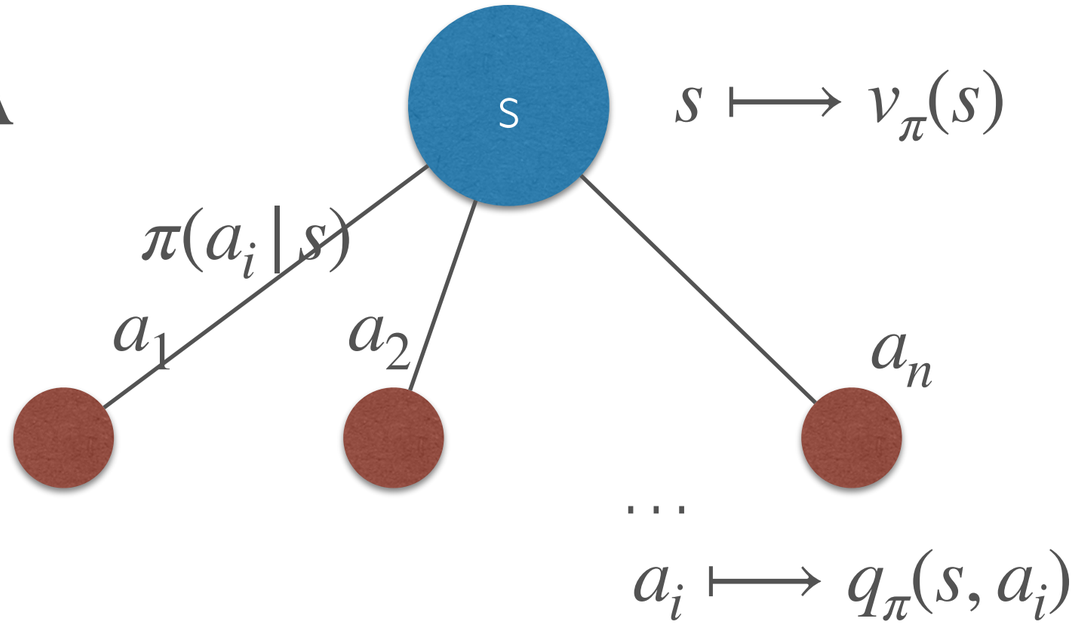
- The state-value function and the action value function can again be recursively computed

$$v_\pi(s) = E_\pi \left\{ R_{t+1} + \gamma v_\pi(S_{t+1}) \,|\, S_t = s \right\}$$

$$q_\pi(s, a) = E_\pi \left\{ R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \,|\, S_t = s, A_t = a \right\}$$

BELLMAN EXPECTATION EQUATION

- From state s, we can take different actions $a_i \in \mathbf{A}$ chosen according to $\pi(a \mid s)$



Averaging / marginalising over a

$$v_\pi(s) = \sum_{a \in \mathbf{A}} \pi(a \mid s) q_\pi(s, a)$$

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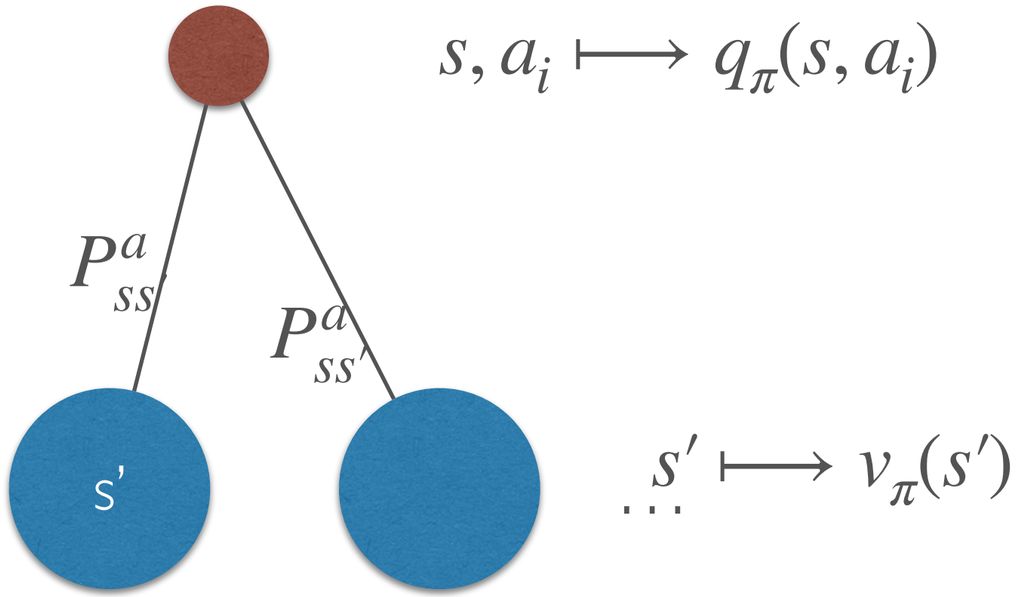
BELLMAN EXPECTATION EQUATION

- Based on s and a we get a reward R_s^a

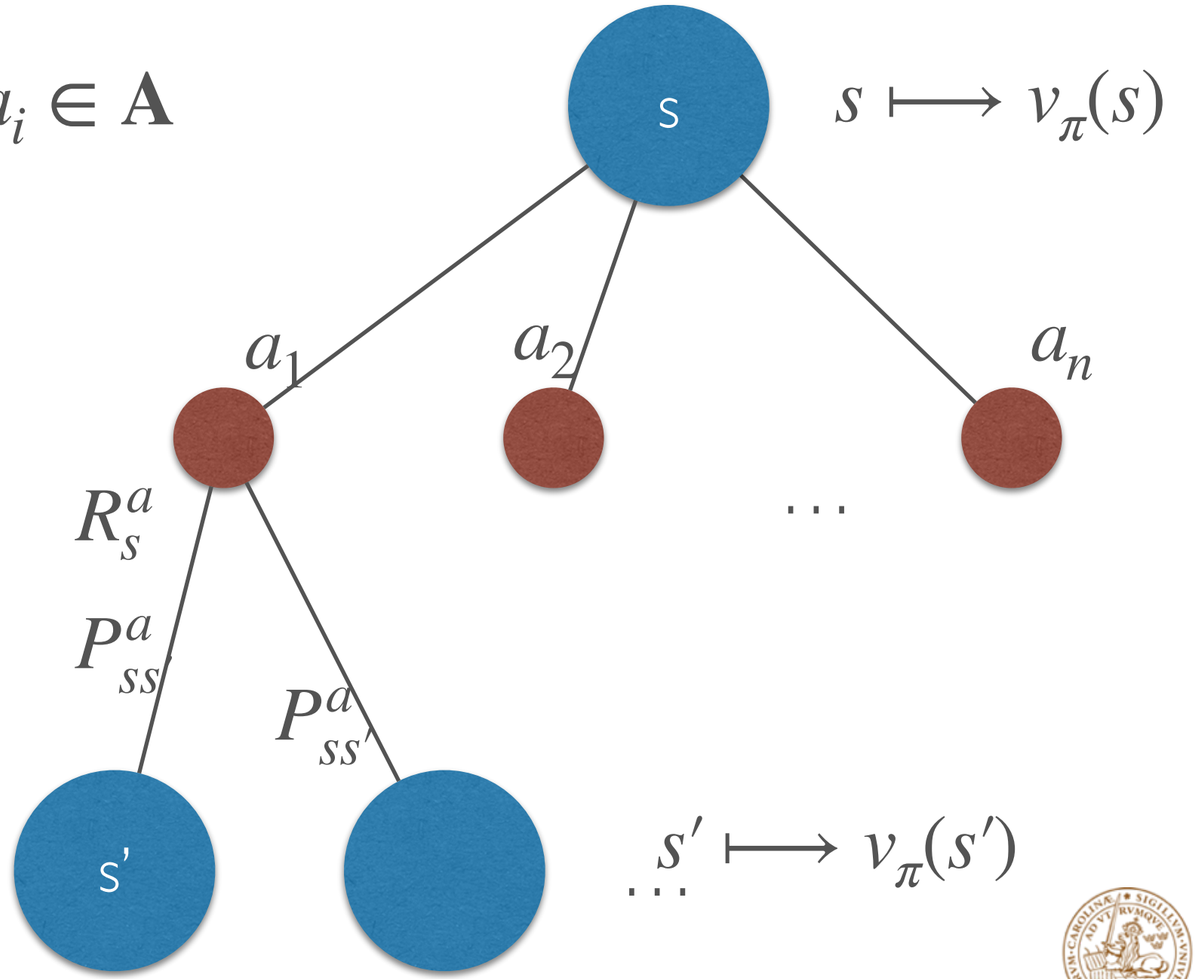
$$q_\pi(s,a) = R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a v_\pi(s')$$



BELLMAN EXPECTATION EQUATION

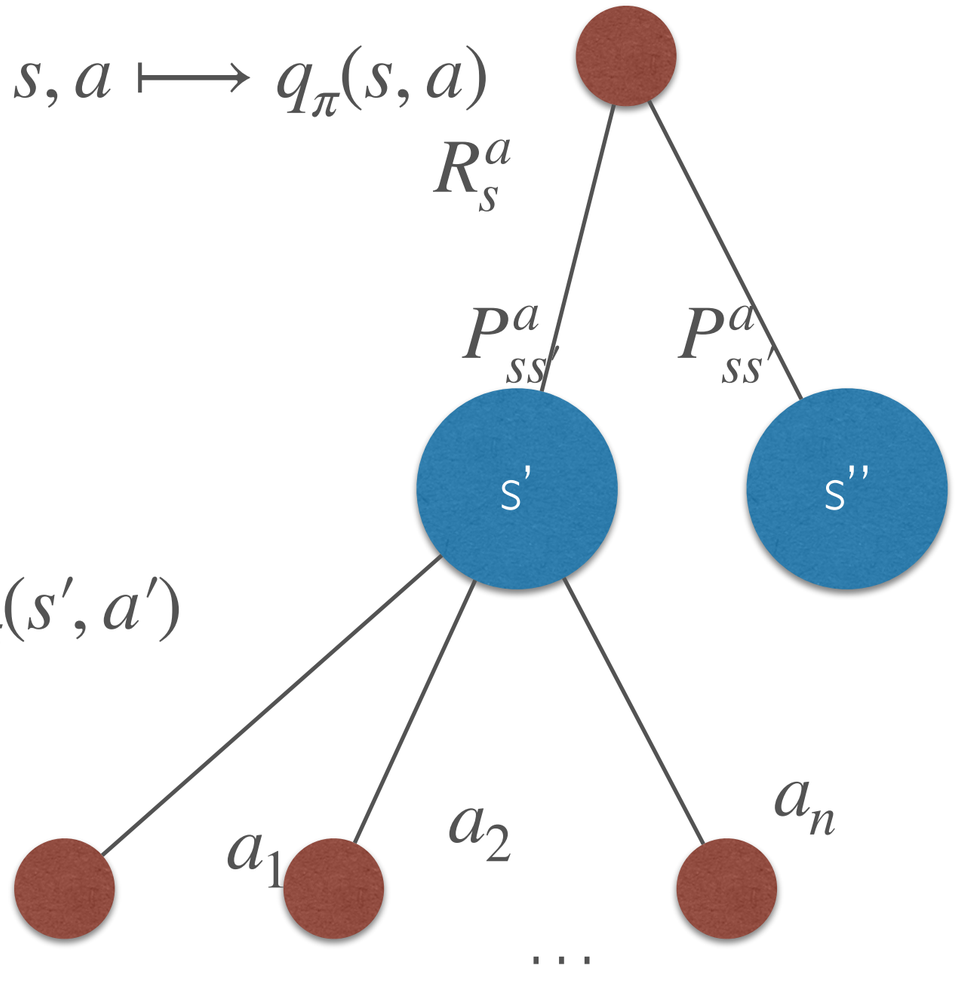
- From state s, we can take different actions $a_i \in \mathbf{A}$ chosen according to $\pi(a \mid s)$
- Based on s and a we get a reward R_s^a

$$v_\pi(s) = \sum_{a \in \mathbf{A}} \pi(a \mid s) \left(R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a v_\pi(s') \right)$$

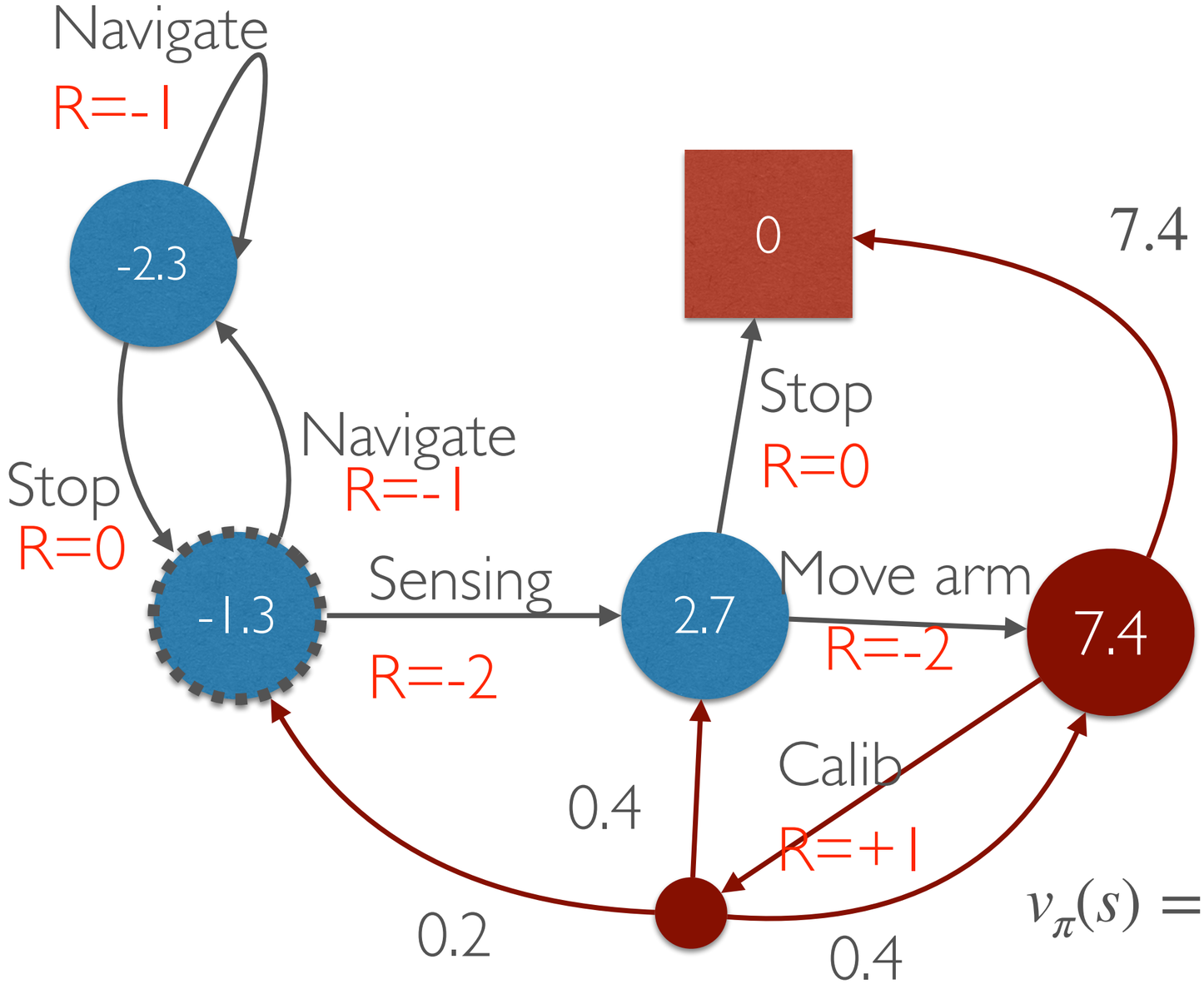


BELLMAN EXPECTATION EQUATION

$$q_\pi(s,a) = R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a \sum_{a' \in \mathbf{A}} \pi(a' \mid s') q_\pi(s', a')$$



BELLMAN EXPECTATION EQUATION



$v_\pi(s)$ for $\pi(s,a) = 0.5,\ \gamma = 1$

$$7.4 = 0.5 \cdot \big(1 + 0.2 \cdot (-1.3) + 0.4 \cdot 2.7 + 0.4 \cdot 7.4\big) + 0.5 \cdot 10$$

$$v_\pi(s) = \sum_{a \in \mathbf{A}} \pi(a \mid s) \left(R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a v_\pi(s') \right)$$

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OPTIMAL VALUE FUNCTION

- So far, we have asked: Given a policy π what are the state values.
- however, we can also aim to optimise the policy π to maximise the state value function

$$v^*(s) = \max_\pi v_\pi(s)$$

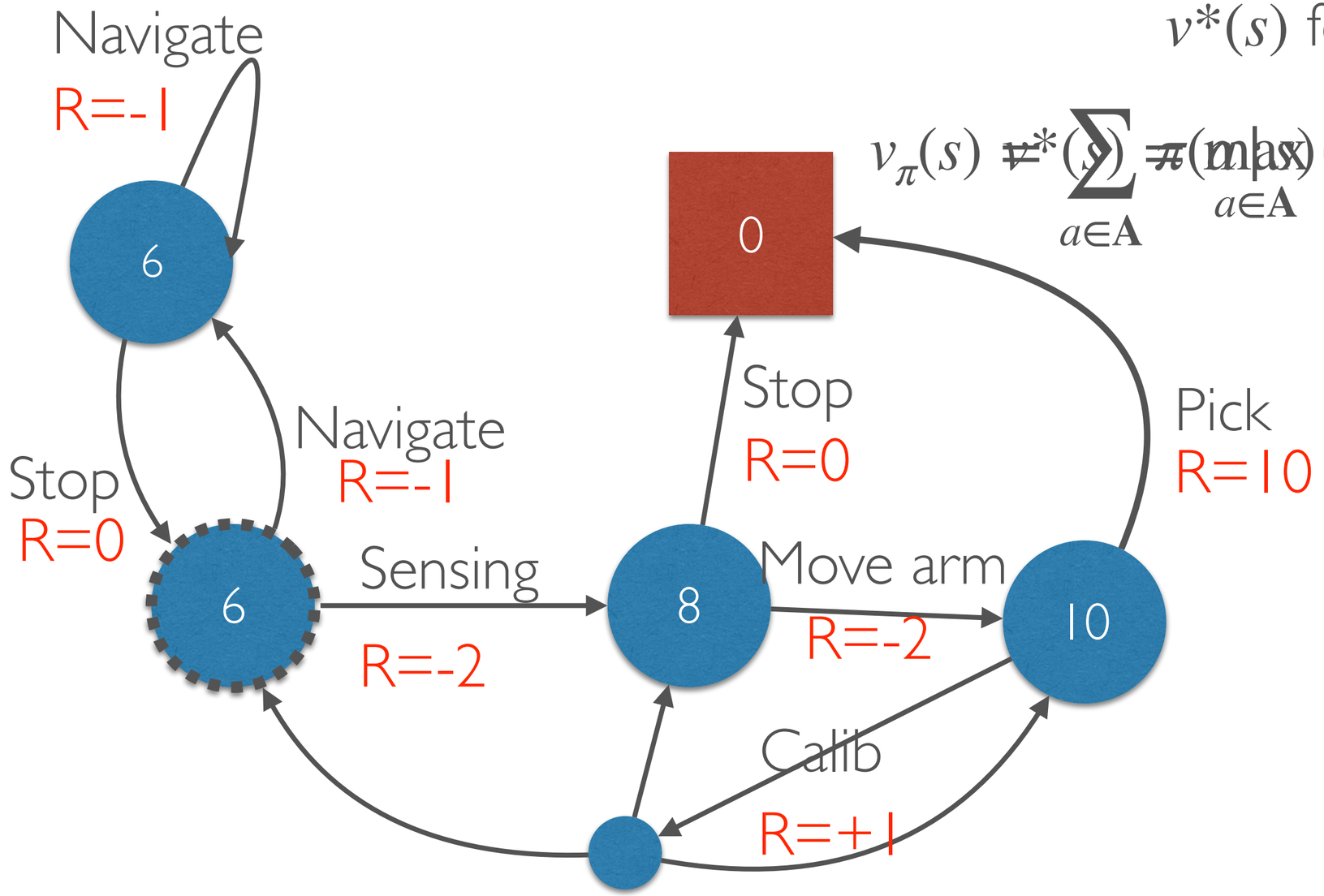
Definition: $v^*(s)$ is called the ***optimal state-value*** *function*

- Likewise, we can find the π that optimises the action-value function

$$q^*(s, a) = \max_\pi q_\pi(s, a)$$

Definition: $q^*(s, a)$ is called the ***optimal action-value function***

EXAMPLE: OPTIMAL STATE-VALUE FUNCTION FOR PICKING



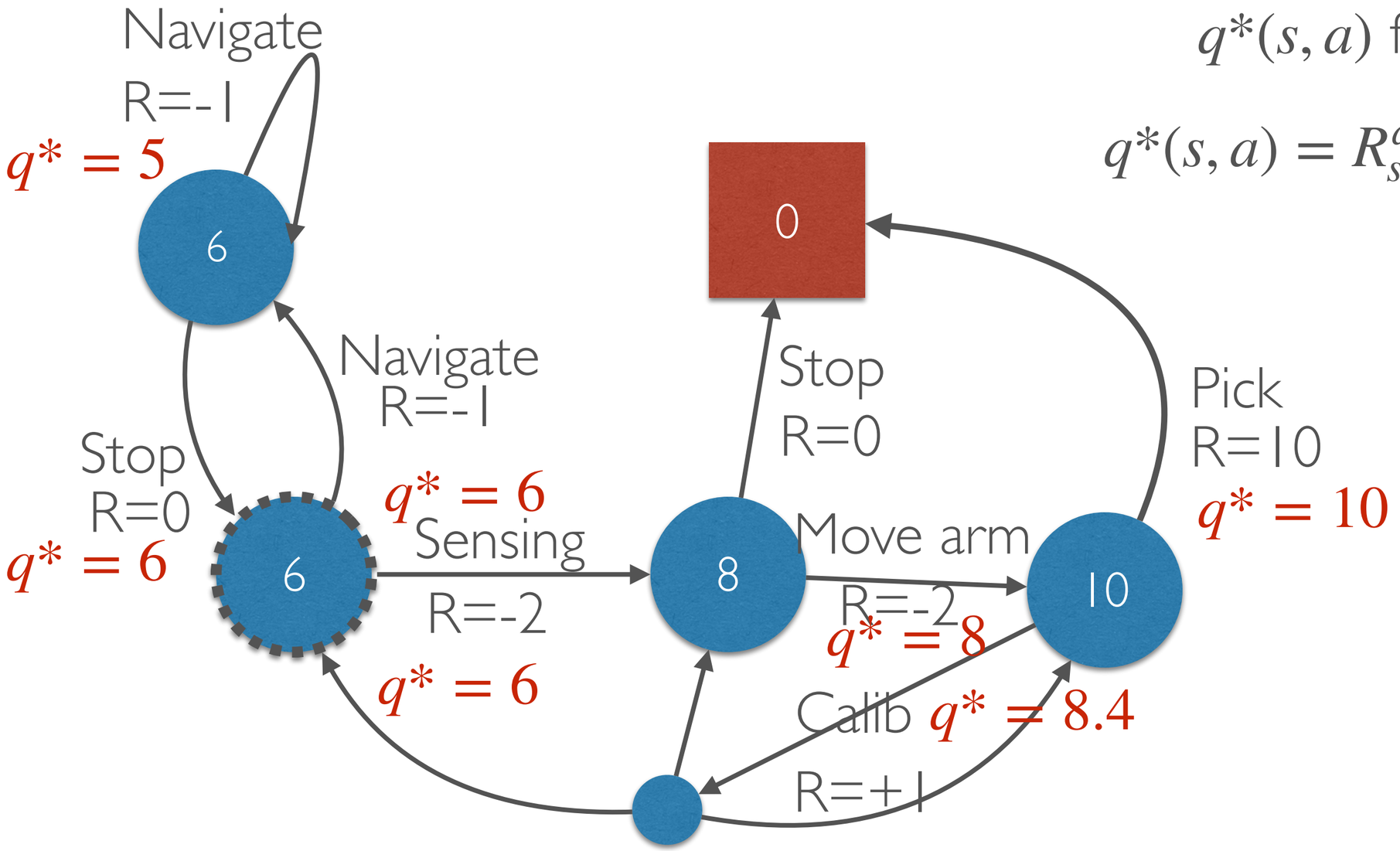
$$v^*(s) \text{ for } \gamma = 1$$

$$v_\pi(s) = \max_{a \in \mathbf{A}} \pi(a|s) \left(R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a v_\pi(s') \right)$$

OPTIMAL ACTION-VALUE FUNCTION FOR PICKING

$q^*(s,a)$ for $\gamma = 1$

$$q^*(s,a) = R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a v^*(s')$$



OPTIMAL POLICY

Define a partial ordering over policies:

$$\pi \geq \pi' \text{ if } v_\pi(s) \geq v_{\pi'}(s) \forall s$$

- **Theorem**: For any Markov Decision Process
- There exists an optimal policy π^*that is better than or equal to all other policies

$$\pi^* \geq \pi \forall \pi$$

- All optimal policies achieve the optimal value function $v_{\pi^*}(s) = v^*(s)$

- All optimal policies achieve the optimal action-value function $q_{\pi^*}(s,a) = q^*(s,a)$

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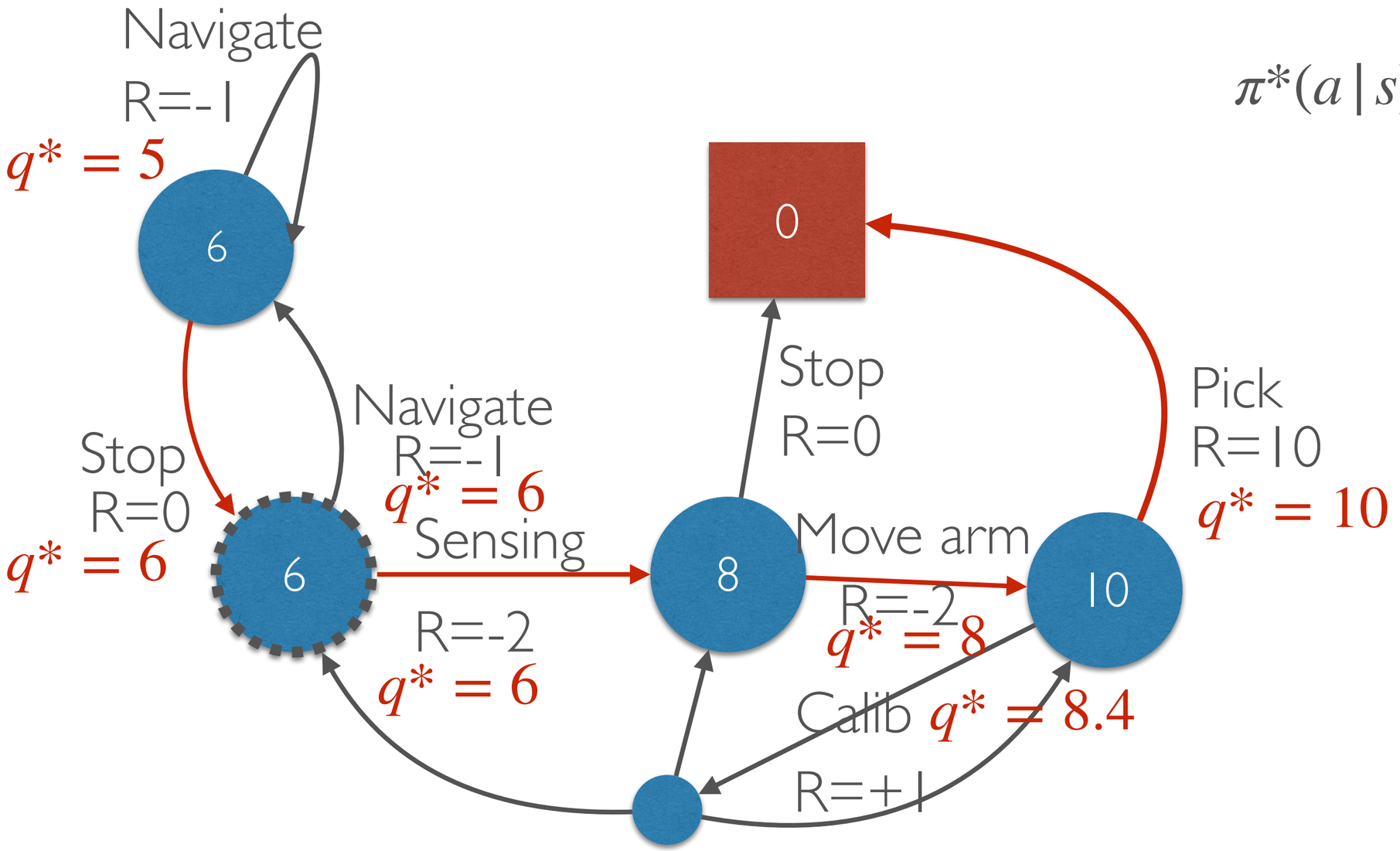
FINDING AN OPTIMAL POLICY

- An optimal policy can be found by maximising over $q^*(s, a)$

$$\pi^*(a \mid s) = \begin{cases} 1 & \text{if } a = \arg\max_{a \in \mathbf{A}} q^*(s, a) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

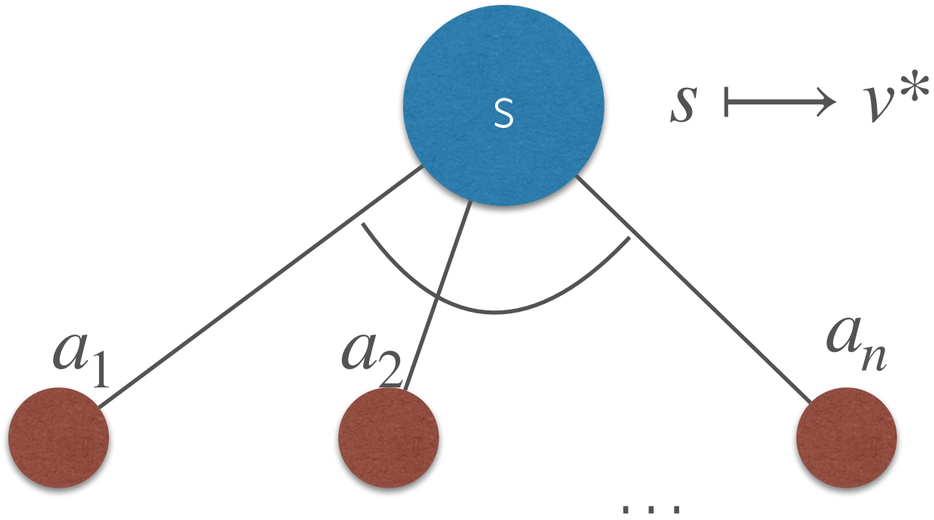
- There is always a deterministic optimal policy for any MDP
- If we know $q^*(s, a)$, we immediately have the optimal policy.



$\pi^*(a \mid s)$ for $\gamma = 1$

BELLMAN OPTIMALITY EQUATION

- From state s, we can action $a_i \in \mathbf{A}$ that has maximal value $q(s, a)$



$$a_i \longmapsto q^*(s, a_i)$$

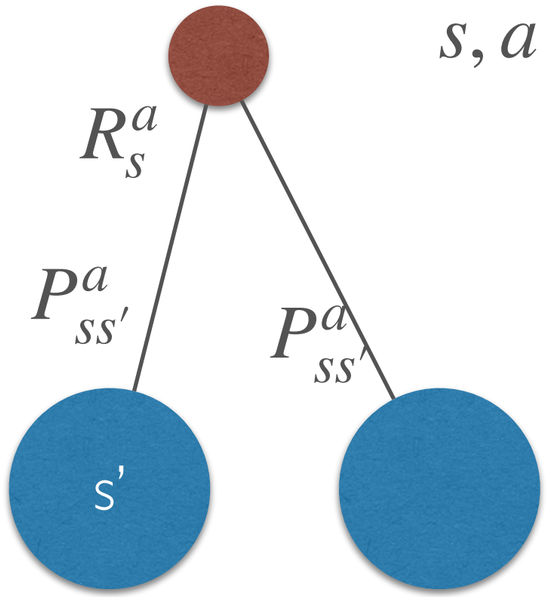
$$v^*(s) = \max_a q^*(s, a)$$

Maximizing over a

BELLMAN OPTIMALITY EQUATION

- Based on s and a we get a reward R_s^a

$$q^*(s,a) = R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a v^*(s')$$



$$s, a \longmapsto q^*(s,a)$$

$$s' \longmapsto v^*(s')$$
$$\dots$$

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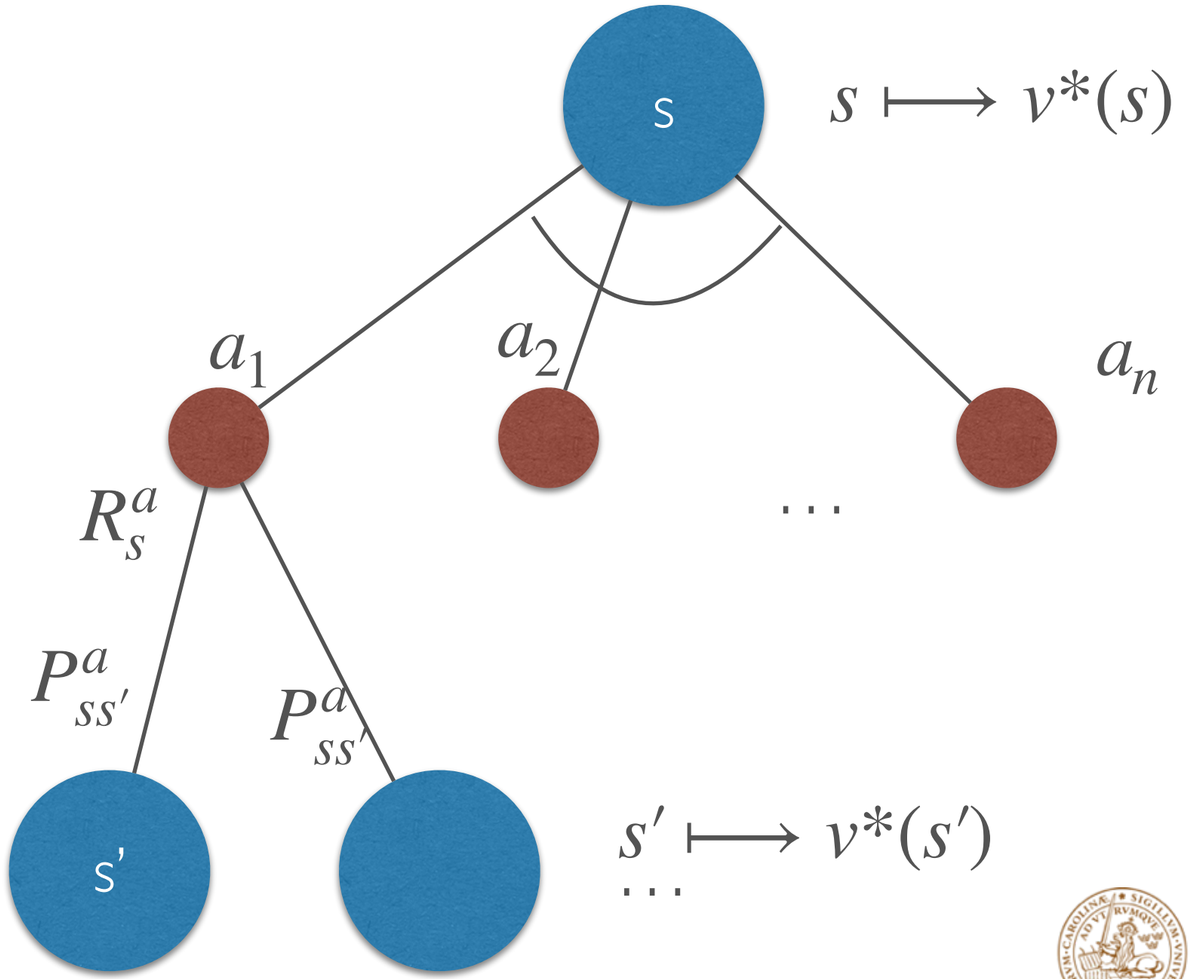
BELLMAN OPTIMALITY EQUATION

- From state s, we can action $a_i \in \mathbf{A}$ that has maximal reward $q(s, a)$
- Based on s and a we get a reward R_s^a

$$v_*(s) = \max_{a \in \mathbf{A}} R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a v^*(s')$$



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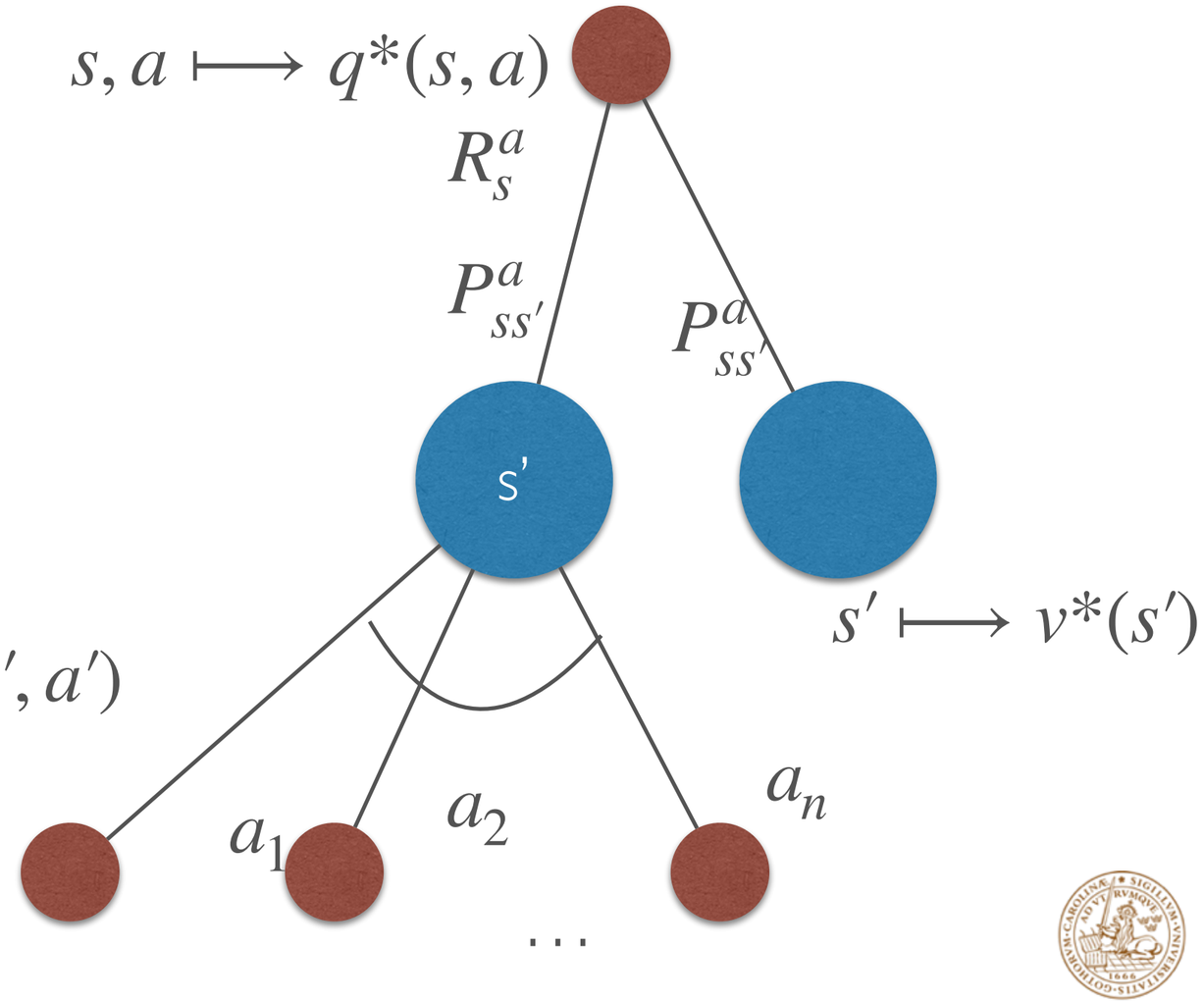
BELLMAN OPTIMALITY EQUATION

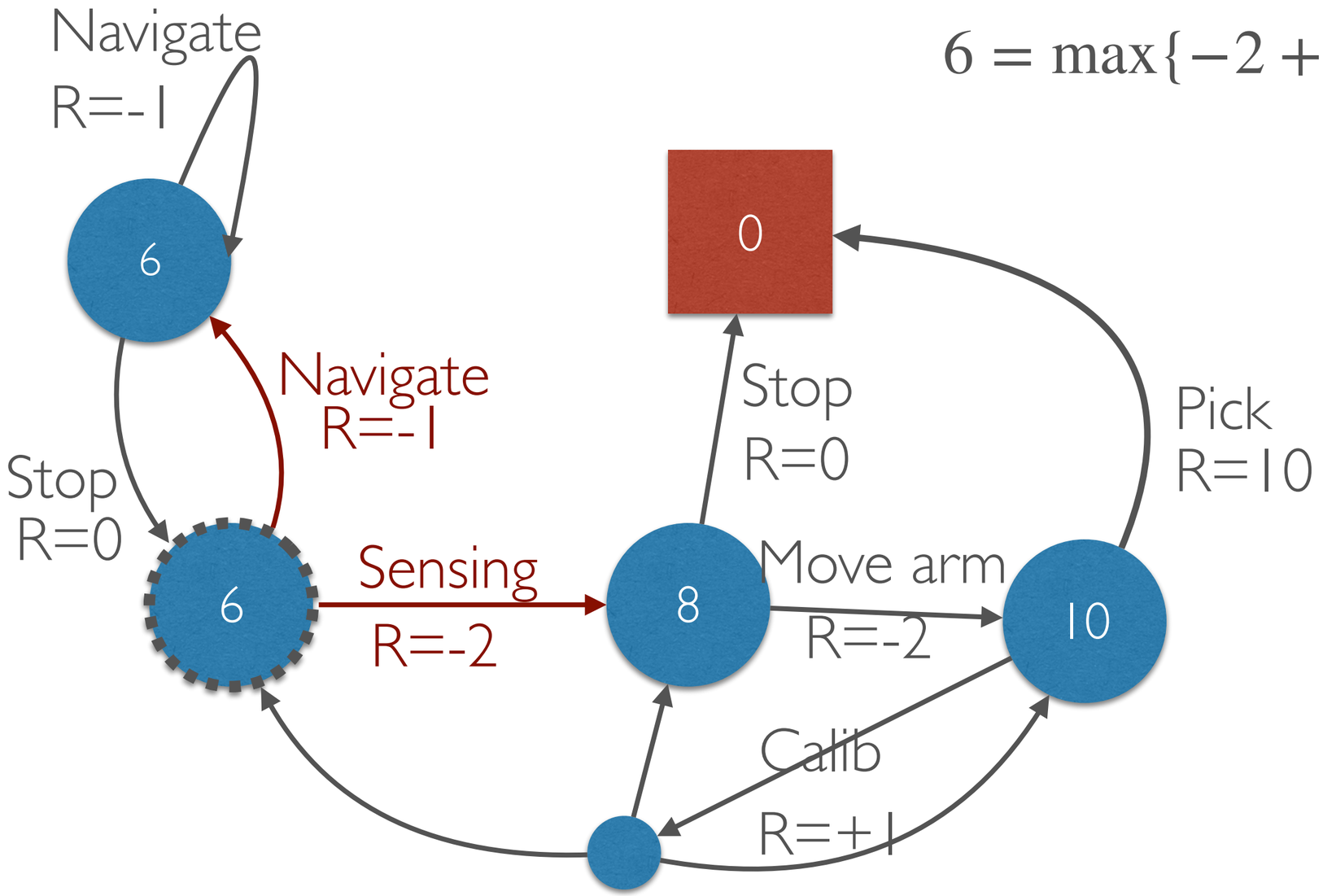
$$q^*(s,a) = R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a \max_{a' \in \mathbf{A}} q^*(s',a')$$



$$s', a' \longmapsto q^*(s',a')$$

BELLMAN OPTIMALITY EQUATION

$$6 = \max\{-2 + 8, -1 + 6\}$$



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SOLVING THE BELLMAN OPTIMALITY EQUATION

- Bellman Optimality Equation is non-linear
- No closed form solution (in general)
- Many iterative solution methods
  - Dynamic Programming: Value Iteration, Policy Iteration
  - Q-learning
  - Sarsa

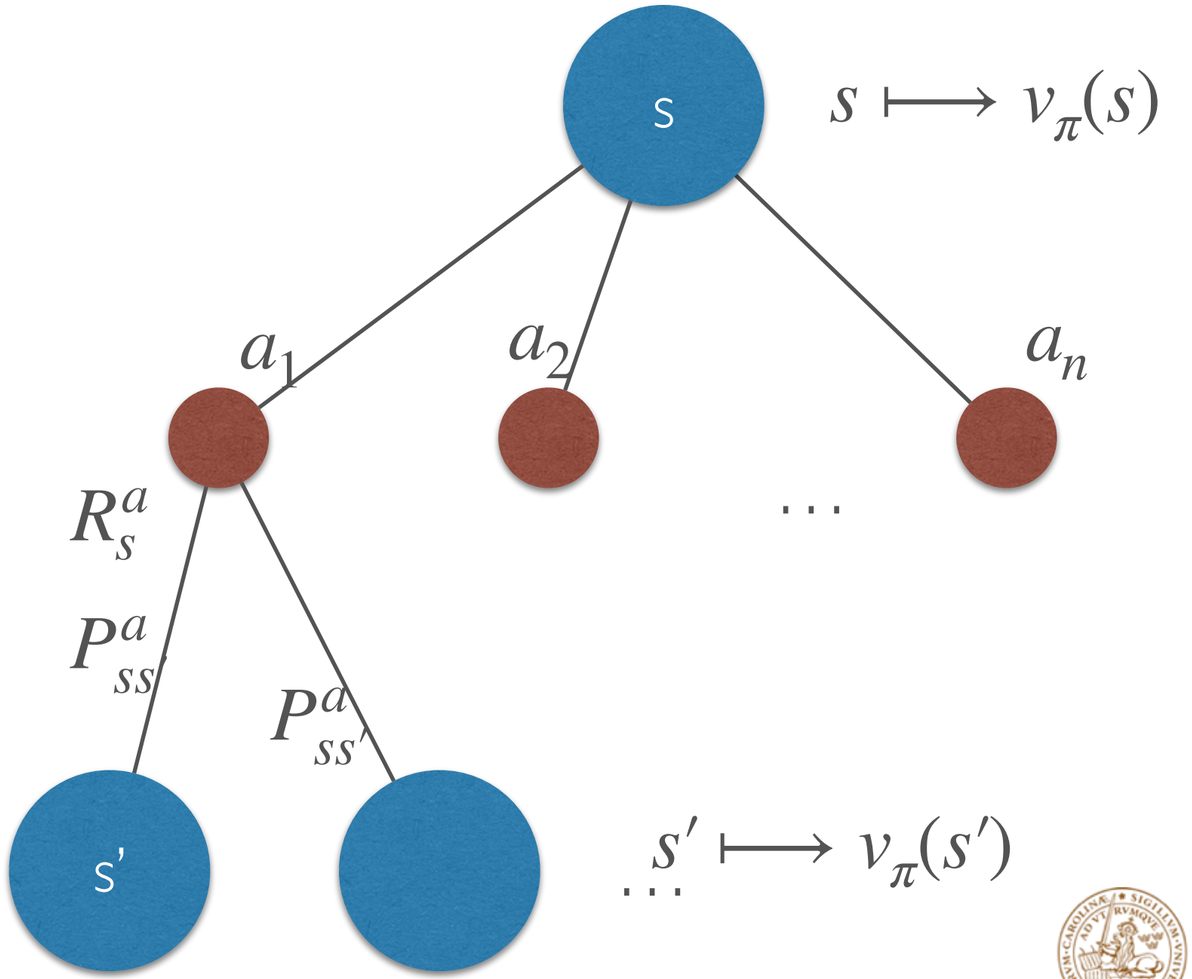
POLICY ITERATION

ITERATIVE POLICY EVALUATION

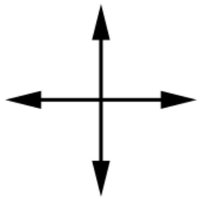
- Goal: evaluate a given policy π
- Solution: Iterative application of the Bellman expectation equation
- $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_\pi$
- At each iteration $k+1$
  - For all states $s \in S$
  - Update $v_{k+1}(s)$ from $v_k(s')$ where s is a successor state of s'

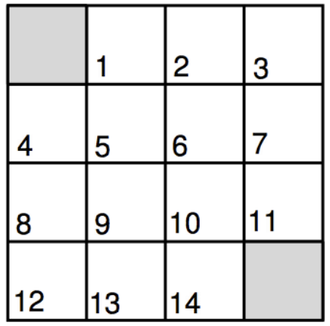
$$v_\pi(s) = \sum_{a \in \mathbf{A}} \pi(a \mid s) \left(R_s^a + \gamma \sum_{s' \in \mathbf{S}} P_{ss'}^a v_\pi(s') \right)$$

$$\mathbf{v}_{k+1} = \mathbf{R}^\pi + \gamma \mathbf{P}^\pi \mathbf{v}_k$$



EXAMPLE: RANDOM POLICY IN THE SMALL GRIDWORLD





$r = -1$ for all transition

$\gamma = 1$

- Non-terminal states 1…14
- terminal state in grey
- Actions leading out of the grid leave state unchanged
- Reward is -1 until the terminal state is reached
- Agent follows uniform random policy

$$\pi(n, \cdot) = \pi(e, \cdot) = \pi(s, \cdot) = \pi(w, \cdot)$$

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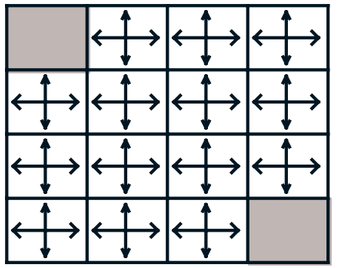
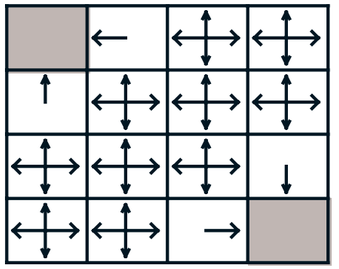
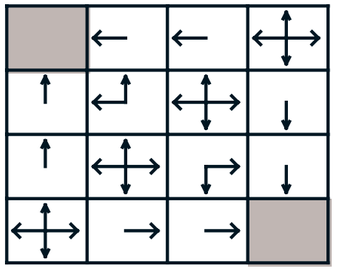
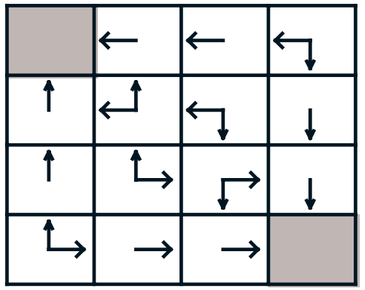
EXAMPLE: RANDOM POLICY IN THE SMALL GRIDWORLD

	v_k for the Random Policy	Greedy Policy w.r.t. v_k
$k = 0$	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0	 ← random policy
$k = 1$	0.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 0.0	
$k = 2$	0.0 -1.7 -2.0 -2.0 -1.7 -2.0 -2.0 -2.0 -2.0 -2.0 -2.0 -1.7 -2.0 -2.0 -1.7 0.0	

EXAMPLE: RANDOM POLICY IN THE SMALL GRIDWORLD

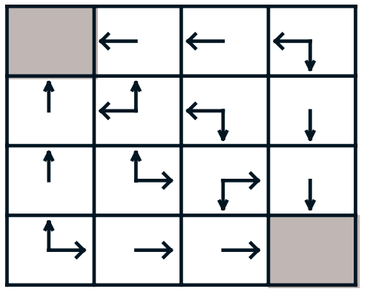
$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0



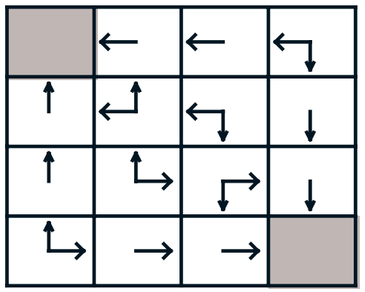
$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0



$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



optimal policy

HOW TO IMPROVE THE POLICY

- Given a policy.
  - Evaluate the policy

$$v_\pi = E\left\{R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right\}$$

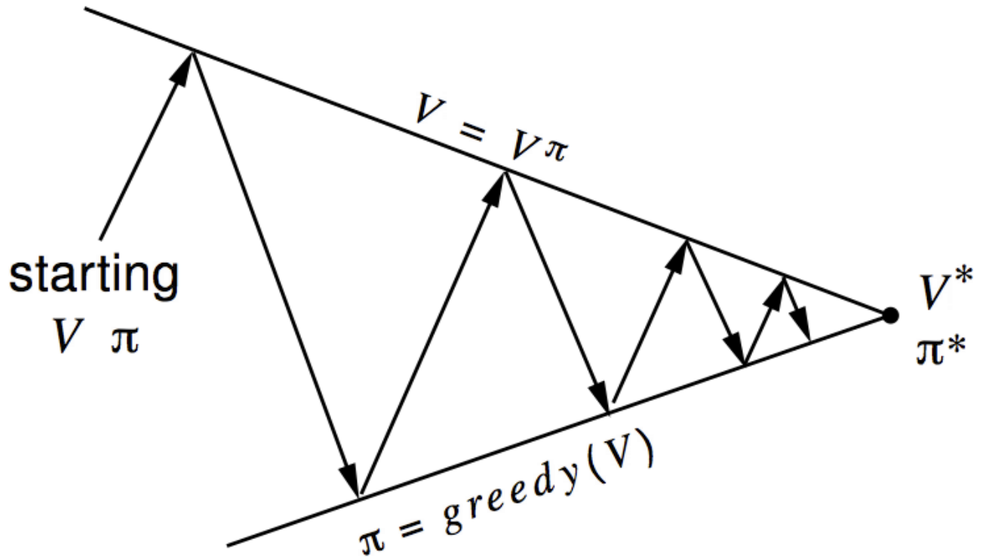
  - Improve the policy by acting greedily with respect to

$$\pi' = \text{greedy}(v_\pi)$$

- In the Gridworld the improved policy was optimal $\pi' = \pi^*$
- In general, more iterations are needed
- But this process of **policy iteration** always converges to π^*

POLICY ITERATION

- Policy Evaluation: Estimate v_π
  - Iterative policy evaluation
- Policy improvement: Generate $\pi' \geq \pi$
  - Greedy policy improvement



JACK'S CAR RENTAL

- States: Two locations, maximum 20 cars at each location
- Actions: Move up to 5 cars between locations overnight
- Reward: 10€ for each car rented if available
- Transitions: Cars returned and requested randomly
  - Poisson distribution, *n* returns/requests with probability $\frac{\lambda^n}{n!}e^{-\lambda}$
  - 1st location: average requests=3, average returns=3
  - 2nd location: average requests=4, average returns =2

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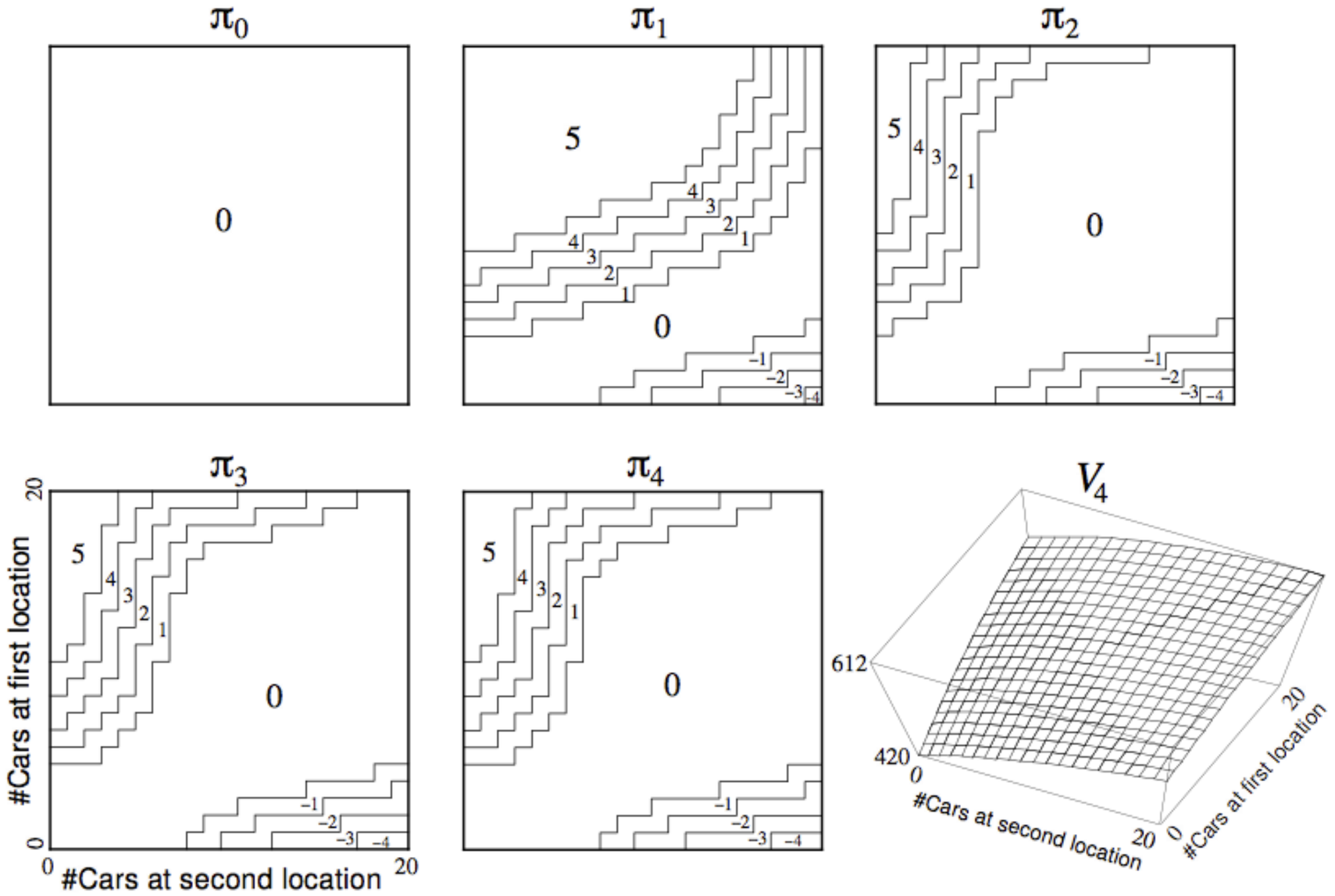
POLICY ITERATION IN JACK'S CAR RENTAL



SUMMARY

- Markov Model, Markov Chain
- Markov Reward Process, Bellman Equation for the state value function
- Markov Decision Process, Bellman Eq. for state value and action value functions
- Identify the optimal policy, state value function and action value function
- Policy Iteration for finding the optimal policy.
  - Very simple approach, but still finds π^*
  - the MDP needs to be known completely

VOLUNTARY EXERCISES

- Gridworld and Jack's car rental are exercises in the Book by Sutton and Barto, Sect 3 and beginning of Sect. 4
  - Correct results are in these slides.