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Traveltime tomography in anisotropic media—I. Theory

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SUMMARY

In principle, crosshole, traveltime tomography is ideal for directly detecting and measuring seismic anisotropy. The traveltimes of multiple rays with wide angular coverage will be sensitive both to inhomogeneities and to anisotropy. In practice, the traveltimes will depend only on a limited number of the anisotropic velocity parameters, and the data may not be adequate even to determine these parameters uniquely. In addition, trade-offs may exist between anisotropy and inhomogeneities. In this paper, we use the linear perturbation theory for traveltimes in general, weakly anisotropic media to discuss the dependence of traveltimes in 2-D crosshole tomographic experiments on the anisotropic parameters. In a companion paper, we apply the results to synthetic and real data examples. We show that when measurements are restricted to a 2-D plane, the qP and qS traveltimes depend on subsets of the complete set of 21 anisotropic velocity parameters. Formulae are developed for the differential coefficients of the traveltimes with respect to these parameters in piecewise homogeneous and in linearly interpolated models. It is shown how in a generally oriented model element, the local parameters are related to the same parameters in the global model. The parameters that can be determined from 2-D tomographic data do not in general determine the full nature of the anisotropy. Rather, these parameters serve only to describe the intersection of the slowness sheet with the 2-D plane. Since many models may fit this description, additional information on symmetry properties and orientations is required. For example, if *a priori* information suggests that the anisotropy is transversely isotropic (TI), then we can determine some of the TI parameters and some information on the orientation of the axis of symmetry. Formulae are given relating the general parameters to those of a TI system with general orientation of the symmetry axis. The general formulae for qS traveltimes are intrinsically more complicated than those for qP. In the qS case, the traveltime perturbation depends on the polarization, which in turn depends on the perturbation. This makes the general problem non-linear even for small perturbations. However, the mean qS traveltime and the traveltime dependence on various subsets of parameters are linear. Although linear perturbation theory is invalid for qS rays, degenerate perturbation theory is valid for the calculation of the traveltimes and could be used in a non-linear inversion scheme.

Key words: anisotropy, traveltime tomography.

1 INTRODUCTION

In recent years, tomographic reconstruction techniques have been applied to crosshole, traveltime data with some success. With multiple sources and receivers in two approximately parallel boreholes, considerable angular coverage of rays is possible. The first-arrival times are interpreted in terms of 2-D, inhomogeneous structure between the holes. Exact solutions for traveltime tomography are only known for straight rays, e.g. filtered back projection. In most geological structures, straight rays are an unrealistic, poor approximation. It is necessary to use perturbation theory to iteratively correct the velocities and

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ray paths to fit the traveltime data. Various iterative algorithms for inverting the linearized system with or without curved ray tracing have been developed. The existence of unresolved structures, e.g. in regions of poor ray coverage, or for vertical structures between vertical boreholes, etc., and artifacts in the image, e.g. elongation of structure along ray paths, etc., are now well understood. Many examples of studies with synthetic and real data sets have appeared in the literature, e.g. Bois *et al.* (1972), Dines & Lytle (1979), Mason (1981), Wong, Hurley & West (1983, 1984, 1985), Wong *et al.* (1987), Fehler & Pearson (1984), Ivansson (1985), Peterson, Paulsson & McEvilly (1985) and Bregman, Bailey & Chapman (1989a,b). Some applications allow the use of surface sources and receivers, i.e. VSP geometry; this improves the resolution by increasing the angular coverage, but introduces the extra uncertainties of a slow, attenuating, variable surface layer. While more work is probably needed on different regularization methods, and more than the first-arrival times might be used, in general terms the curved ray, iterative inversion techniques are successful and await more and better data sets!

Crosshole traveltime tomographic experiments are ideal for determining seismic anisotropy between the boreholes, except that we might expect a trade-off between inhomogeneities and anisotropy. The angular coverage makes the seismic traveltimes directly sensitive to any anisotropy. Crosshole experiments are often done in locations where anisotropy might be anticipated, e.g. due to fine layering or cracking. In fact, given that the purpose of borehole experiments is often to investigate fine structure and fracturing between the boreholes, it would be surprising if anisotropy did not exist. Nevertheless, most data sets collected to date have been interpreted assuming isotropy. In this paper we develop the theory necessary to include anisotropy in crosshole tomography experiments. In a companion paper (Pratt & Chapman 1992), we analyse the resultant linear systems and we apply the results to synthetic and real data examples. In general, anisotropy and inhomogeneity will coexist, so we have presented the theory for inhomogeneous media. In view of the past success assuming isotropy, we have assumed that the anisotropy is weak. In some circumstances this assumption may fail, but in general we believe it will be adequate. Assuming weak anisotropy enables the perturbation theory developed by Červený (1982), Červený & Jech (1982) and Jech & Pšenčík (1989) to be used to calculate the traveltimes in anisotropic media, and to calculate the differentials needed for tomographic interpretation. They have reported good results for up to about 10 per cent anisotropy.

It is extremely convenient to use isotropic unperturbed media, and to consider anisotropic effects to be due to weak perturbations of isotropic systems. This assumption allows one to proceed without developing new methods for ray tracing in anisotropic media—all rays are traced in unperturbed isotropic media. Iterative inversions yield estimates of the corrections to the elastic tensor, some of which are required due to errors in the isotropic velocities, and some of which are required due to ignoring the anisotropy.

Anisotropy has been included or reported in relatively few traveltime tomography experiments. McCann *et al.* (1989) have reported inversions for P and S velocities between boreholes, and boreholes to the surface in Oxford clays. Using straight ray tomography, the initial image showed little relationship to the known stratigraphy between the boreholes. However, it was known that the clay is anisotropic, and a second tomographic inversion assuming 10 per cent anisotropy was performed. This showed an improved agreement with the known stratigraphy. No attempt was made to use curved ray tomography, nor to invert directly for the anisotropy (although no improvement was obtained with 20 per cent anisotropy). Winterstein & Paulsson (1990) have interpreted crosshole and VSP traveltime data with a transversely isotropic model. Anisotropy was identified by the existence of shear wave splitting and the angular variation of one shear wave velocity. The only inhomogeneity modelled was a uniform vertical gradient. In this simple situation, it was relatively easy to interpret the velocity anisotropy which was consistent with previous measurements. Stewart (1988) has suggested an algorithm for inverting for the weak, transverse isotropic parameters using Thomsen's (1986) formula for the angular dependence of the velocity. We are unaware of any attempt to invert simultaneously for general inhomogeneity and anisotropy, or to invert for anisotropy without initial assumptions about the nature of the anisotropy.

In common with other crosshole tomographic studies, we assume that the structure is 2-D. Structure perpendicular to the plane of the boreholes will cause ray paths to deviate from the plane, but to first order this can be ignored, i.e., by Fermat's principle the effect on the traveltimes is second order in the deviation. In anisotropic media, the ray paths may deviate from the plane even if the structure is 2-D, but again to first order we can ignore it. We consider two methods of parametrizing the model. First, for simplicity we consider parametrizing the model with small, homogeneous elements. An example where this method has been used for tomographic reconstruction is Dyer & Worthington (1988). This method is straightforward to program and is a good first approximation when iterative, curved ray inversion is not used. It is used for the examples in the companion paper (Pratt & Chapman 1992). Small homogeneous elements are unsuitable for curved ray tracing, as instabilities arise from the many discontinuities in the model which may cause total reflections. We also consider a model parametrization which allows for continuous 2-D inhomogeneities. Various parametrization methods are possible (Červený 1987). The different methods have different numerical advantages and disadvantages. Provided the model is parametrized with sufficient detail (and the overparametrization handled by regularizing the solution), the differences between the solutions should not be physically significant. It is important that rays can be traced through the numerically specified model accurately and efficiently. Linear interpolation is one such method, because the ray paths are known to be arcs of circles and, as we shall see, the various expressions needed for anisotropic ray tracing and traveltime perturbations are also known analytically. We follow the procedures developed in Bregman *et al.* (1989a). The 2-D model is parametrized by a grid of points which need not be regularly distributed (Fig. 1). The grid is divided into triangular model elements. Within each model element, parameters

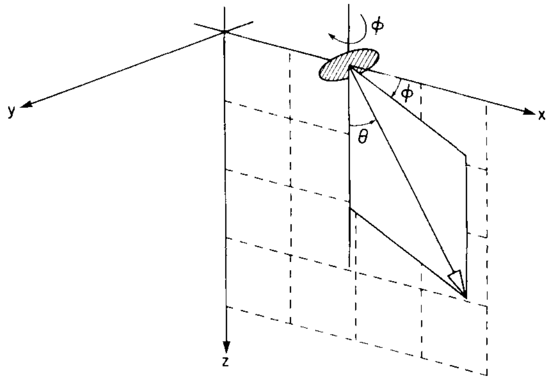

Figure 1. The global coordinate system is defined in the plane of the tomographic experiment and the model is divided into a grid of triangles. At each model point, the anisotropy may be interpreted as TI with an arbitrary symmetry axis. The axes are arranged in the usual geographical orientation with the z-axis positive downwards, the x-axis positive to the right, and the y-axis positive out of the paper (to form a RH set of axes). The (Euler) angles θ and ϕ define the orientation of the TI symmetry axis with respect to the global coordinates (see also Fig. E1). Note that θ is a positive rotation about the y-axis, and ϕ is a positive rotation about the z-axis (in a RH sense).

are linearly interpolated. Normally this results in models which are continuous between model elements but contain second-order discontinuities. In each model element, the constant gradient may be arbitrarily oriented (Fig. 2). The solution in each model element is simplified by considering a coordinate system aligned with the gradient vector. We refer to this coordinate system as the 'local' coordinate system in which the gradient is 'locally vertical', i.e. is aligned with the 'local' z-axis, and the coordinate system describing the entire model as the 'global' system in which the z-axis is vertically downwards.

Various approaches to tomographic interpretation in

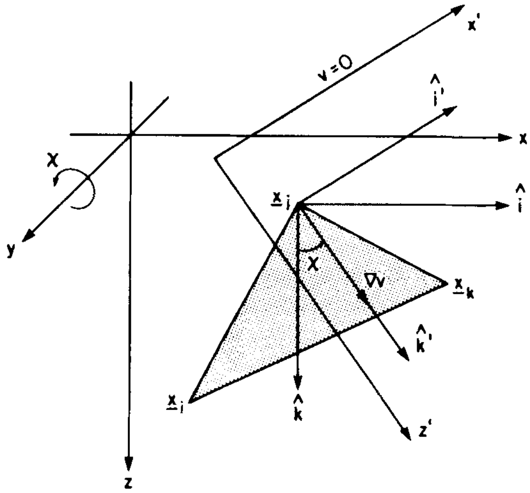

Figure 2. In each triangle, the isotropic velocity defines a local velocity gradient vector. The local coordinate system is defined with this as the vertical, i.e. a rotation through the angle χ. Again as in Fig. 1, the axes are drawn in the usual geographic orientation, and χ is a positive rotation about the y-axis. The origin of the local coordinate system is at the centre of the ray arc.

anisotropic media are possible. Assuming some model for the anisotropy, e.g. some specific crystal symmetry, fine layering, or crack orientation, we might parametrize this model with the natural physical parameters, e.g. crack density and aspect ratio, or the limited velocity parameters needed, and then calculate the differential coefficients of the traveltimes with respect to these parameters. We believe that this approach is mistaken. It does not allow one to test whether the data are in fact compatible with the physical model or whether some other model of anisotropy is equally or more appropriate. Instead we believe that the differential coefficients of the traveltime with respect to a general anisotropic model should be calculated. In general, this would allow for 21 velocity parameters although, we shall see that in the special case of borehole tomography, the traveltimes depend only on small subsets of these parameters. If some anisotropic parameters are undetermined by the experiment, this should be demonstrated by the solutions rather than initial assumptions. Having inverted for the velocity parameters which can be determined from the tomographic experiment (with no assumption restricting the type of anisotropy), the velocity parameters can then be interpreted in terms of specific models of anisotropy. For instance, at each model point we can interpret the velocity parameters in terms of a transversely isotropic (TI) system [five velocity parameters, plus two parameters to specify the symmetry axis (Love 1944, p. 160)]. We follow the terminology suggested by Winterstein (1990), that TI refers to a transversely isotropic medium with an arbitrary axis of symmetry and TIV refers to a TI medium with a vertical axis of symmetry (Fig. 1). Several models of anisotropy may be equally compatible with the velocity parameters determined by the tomographic data set. If *a priori* information is available restricting the type of anisotropy, then this can be included as a regularization in the inversion.

The emphasis in this paper is on 2-D tomography in anisotropic media. Nevertheless, the formulae derived can be used for calculating traveltimes for any 2-D geometry. It is also straightforward to use the same formulae in 3-D models. In 3-D, the model is divided into tetrahedra. Within each tetrahedron, the parameters are linearly interpolated. Within any tetrahedron, the ray path is restricted to a plane defined by the initial ray direction and the velocity gradient in the tetrahedron, and locally the problem is 2-D. In the plane of the ray path within the model element, the formulae derived in this paper for the traveltime perturbations can be used. However, the anisotropic model perturbations must be appropriate to the local plane of the ray. The fourth-order tensor elements of the anisotropic model parameters are transformed between the global, model coordinate system and the local plane of the ray with velocity gradient 'vertical' and another coordinate perpendicular to the plane of the ray, by a rotation transformation.

In the next section of this paper, we review the linearized perturbation theory developed by Červený (1972, 1982), Červený & Jech (1982) and Jech & Pšenčík (1989) for calculating traveltimes in weakly anisotropic media. In the following section, we apply this to the 2-D, tomographic geometry. The important result here is to identify the subsets of velocity parameters which each ray type is sensitive to and might determine in a tomographic experiment. For piecewise homogeneous models, the computations necessary to define a suitable linear system are straightforward. For more generally inhomogeneous models, the results are algebraically more complicated and the details have been relegated to various appendices. Finally we discuss the application of the results to tomographic inversion, and show how results from an inversion might be interpreted in terms of TI media.

2 THEORY

2.1 Traveltime perturbations

The theory for tracing rays in anisotropic, inhomogeneous media is well known (Červený 1972). Červený (1982) and Červený & Jech (1982) have developed a theory for linearized perturbations to the traveltime in anisotropic media. This has been extended to cover the case of degenerate qS rays by Jech & Pšenčík (1989). In this section we summarize the results of those papers and clarify the rôle of the slowness perturbation (equations 13 and 18).

Defining the density normalized elastic parameters as

$$a_{ijkl} = c_{ijkl}/\rho, \tag{1}$$

the slowness as

$$\mathbf{p} = \nabla T, \tag{2}$$

and the Christoffel matrix as

$$\Gamma_{jk} = p_i p_l a_{ijkl}, \tag{3}$$

then

$$G(\mathbf{p}, \mathbf{x}) = 1 \tag{4}$$

defines the slowness surface, where G solves the eigenvalue equation (Červený 1972)

$$(\Gamma_{jk} - \delta_{jk} G)\hat{g}_k = 0, \tag{5}$$

with $\hat{\mathbf{g}}$ the normalized, i.e. unit, polarization vector. From this equation we have

$$G = \Gamma_{jk}\hat{g}_j\hat{g}_k = a_{ijkl}p_i p_l \hat{g}_j \hat{g}_k. \tag{6}$$

The kinematic ray equations are (Červený 1972)

$$\frac{dx_i}{dT} = \frac{1}{2}\frac{\partial G}{\partial p_i} = a_{ijkl}p_l\hat{g}_j\hat{g}_k = V_i, \tag{7}$$

$$\frac{dp_i}{dT} = -\frac{1}{2}\frac{\partial G}{\partial x_i} = -\frac{1}{2}\frac{\partial a_{mjkl}}{\partial x_i} p_m p_l \hat{g}_j \hat{g}_k, \tag{8}$$

where V_i is a component of the group velocity vector, $\mathbf{V} = d\mathbf{x}/dT$.

The fundamental equation giving the traveltime associated with a ray path, $\mathscr{L}$, in a general media is

$$T = \int_{\mathscr{L}} U\, dl \tag{9}$$

where $U = 1/V = 1/|\mathbf{V}|$ is the ray, or group, slowness at each location on the ray path and dl is the incremental length along the ray path. Using Fermat's principle, which allows us to assume that the traveltime perturbation due to ray path perturbations can be ignored to first order, the traveltime perturbation is

$$\delta T = \int_{\mathscr{L}} \delta U\, dl. \tag{10}$$

In isotropic media, these equations, (9) and (10), are straightforward and have been widely used in traveltime tomography, but in anisotropic media the group slownesses are non-linear functions of both the media properties and the direction of propagation, and the application is more complicated.

It is convenient to write the traveltime integral (9) in terms of the phase slowness vector (2) $\mathbf{p}$, which is perpendicular to the wavefront. From the geometry of the wavefronts and rays, which need not be perpendicular, we have

$$\mathbf{p}\cdot\mathbf{V} = p_i \frac{dx_i}{dT} = 1. \tag{11}$$

This result can also be obtained algebraically from equations (4), (6) and (7) (Červený 1972), but is a basic geometrical result for non-dispersive waves. Thus

$$T = \int_{\mathscr{L}} p_i\, dx_i = \int_{\mathscr{L}} \frac{\hat{p}_i\, dx_i}{v} \tag{12}$$

where $\hat{\mathbf{p}} = v\mathbf{p}$ is the normalized, i.e. unit, slowness vector, and v is the phase velocity. The traveltime perturbation is given by

$$\begin{aligned}\delta T &= \int_{\mathscr{L}} \delta p_i\, dx_i = -\int_{\mathscr{L}} \frac{\delta v \hat{p}_i\, dx_i}{v^2} + \int_{\mathscr{L}} \frac{\delta \hat{p}_i\, dx_i}{v} \\ &= -\int_{\mathscr{L}} \frac{\delta v\, dT}{v} + \int_{\mathscr{L}} \frac{\delta \hat{p}_i\, dx_i}{v}\end{aligned} \tag{13}$$

where, by Fermat's principle, the integrals can be taken along the unperturbed ray path. In isotropic media, the last integral in (13) is identically zero but in anisotropic media it must be retained. Fermat's principle does not allow us to use the unperturbed phase direction, only the unperturbed ray path, i.e. the group direction. Later we shall see that the second integral cancels with an extra term from δv in the other integral.

To evaluate (13), we need an expression for the phase velocity perturbation, δv, in terms of the media parameters. We rewrite the eigenvalue equation (5) as

$$(\hat{\Gamma}_{jk} - v^2\,\delta_{jk})\hat{g}_k = 0, \tag{14}$$

where we have defined the normalized Christoffel matrix $\hat{\Gamma}_{jk} = a_{ijkl}\hat{p}_i\hat{p}_l = v^2\Gamma_{jk}$. Perturbing this equation (14) and retaining only first-order terms, we obtain

$$(\hat{\Gamma}_{jk} - v^2\,\delta_{jk})\,\delta\hat{g}_k + (\delta\hat{\Gamma}_{jk} - 2v\,\delta v\,\delta_{jk})\hat{g}_k = 0. \tag{15}$$

Multiplying by $\hat{g}_j$, this equation reduces to

$$\delta\hat{\Gamma}_{jk}\hat{g}_j\hat{g}_k - 2v\,\delta v = 0 \tag{16}$$

as $\hat{g}_j\hat{g}_j = 1$ and $\hat{g}_j\,\delta\hat{g}_j = 0$ since the eigenvector $\hat{\mathbf{g}}$ is normalized. Hence

$$\delta v = \frac{\delta \hat{\Gamma}_{jk} \hat{g}_j \hat{g}_k}{2v} = \frac{(\delta a_{ijkl} \hat{p}_i \hat{p}_l + 2a_{ijkl} \hat{p}_i \, \delta \hat{p}_l) \hat{g}_j \hat{g}_k}{2v} = \frac{1}{2v} \delta a_{ijkl} \hat{p}_i \hat{p}_l \hat{g}_j \hat{g}_k + V_l \, \delta \hat{p}_l, \tag{17}$$

where the simplification of the final term comes from the kinematic ray equation (7). Substituting equation (17) in equation (13), we obtain

$$\delta T = -\frac{1}{2} \int_{\mathscr{L}} \frac{\delta a_{ijkl} \hat{p}_i \hat{p}_l \hat{g}_j \hat{g}_k \, dT}{v^2} - \int_{\mathscr{L}} \frac{V_l \, \delta \hat{p}_l \, dT}{v} + \int_{\mathscr{L}} \frac{\delta \hat{p}_i \, dx_i}{v} = -\frac{1}{2} \int_{\mathscr{L}} \frac{\delta a_{ijkl} \hat{p}_i \hat{p}_l \hat{g}_j \hat{g}_k}{v^2} \, dT, \tag{18}$$

where the terms in $\delta \hat{p}_i$ cancel due to (7).

We have used the normalized slowness, $\hat{\mathbf{p}}$, and perturbed the normalized eigenvector equation (14), including more detail than previous publications, to emphasize the difference from isotropic media (particularly that $\delta \hat{p}_i \, dx_i = \delta \hat{\mathbf{p}} \cdot \mathbf{V} \, dT \neq 0$). Algebraically, it is simpler to perturb the unnormalized equation (5) so equations (15) and (16) are replaced by

$$\delta \Gamma_{jk} \hat{g}_k + (\Gamma_{jk} - \delta_{jk}) \, \delta \hat{g}_k = 0 \tag{19}$$

and

$$\delta \Gamma_{jk} \hat{g}_j \hat{g}_k = 0. \tag{20}$$

Expanding the latter equation (20), we obtain

$$V_l \, \delta p_l = -\tfrac{1}{2} \, \delta a_{ijkl} p_i p_l \hat{g}_j \hat{g}_k. \tag{21}$$

Substituting in the first expression for δT in (13), we obtain (18) again.

We can also calculate the perturbation to the polarization vector. It is now necessary to distinguish the different eigenvectors of the eigenvector equation (5). We add a Greek letter superscript to enumerate the various eigenvalues. As the Christoffel matrix (3) is real and symmetric, the polarization vectors are orthogonal. For a given phase slowness direction, $\hat{\mathbf{p}}$, at a given position, the three eigenvectors, $\hat{\mathbf{g}}^{(\mu)}$, $\mu = 1$ to 3, are solutions of the eigenvalue equation (14). Only one (which we have used up to now without a superscript) corresponds to the actual wave polarization (and has $G^{(\mu)} = 1$), but the other two correspond to polarizations of the other wave types with the same phase slowness direction (and have $G^{(\nu)} = v^{(\nu)2}/v^{(\mu)2}$, etc.). Normally, these will not be the polarizations of the other (independent) waves at the same point which will have different phase slowness directions. As the polarization vectors form an orthonormal set, we can write

$$\delta \hat{\mathbf{g}}^{(\mu)} = c_{\mu\nu} \hat{\mathbf{g}}^{(\nu)} + c_{\mu\lambda} \hat{\mathbf{g}}^{(\lambda)}, \tag{22}$$

where μ, ν and λ are a cyclic set of indices (no summation). Substituting (22) in the perturbation equation (15) and multiplying by $\hat{g}_j^{(\nu)}$, we obtain

$$c_{\mu\nu} (\hat{\Gamma}_{jk} \hat{g}_k^{(\nu)} \hat{g}_j^{(\nu)} - v^{(\mu)2}) + \delta \hat{\Gamma}_{jk} \hat{g}_k^{(\mu)} \hat{g}_j^{(\nu)} = 0, \tag{23}$$

so

$$c_{\mu\nu} = \frac{\delta \hat{\Gamma}_{jk} \hat{g}_k^{(\mu)} \hat{g}_j^{(\nu)}}{v^{(\mu)2} - v^{(\nu)2}} \tag{24}$$

if $\mu \neq \nu$ (and $c_{\mu\mu} = 0$ by definition). The result (24) indicates that unfortunately this perturbation theory breaks down if $v^{(\mu)} = v^{(\nu)}$ with $\mu \neq \nu$. This happens for S-waves in isotropic media because $v^{(2)} = v^{(3)} = \beta$, where β is the shear wave velocity in the isotropic medium. We must use degenerate perturbation theory.

In order to include the degenerate S wave case, we follow the degenerate perturbation theory developed in Jech & Pšenčík (1989) to compute the perturbed velocities. The eigenvalue equation does not define the degenerate eigenvectors uniquely. We define

$$\begin{aligned} \hat{g}_i^{(2)} &= a_{22} \hat{e}_i^{(2)} + a_{23} \hat{e}_i^{(3)}, \\ \hat{g}_i^{(3)} &= a_{32} \hat{e}_i^{(2)} + a_{33} \hat{e}_i^{(3)}, \end{aligned} \tag{25}$$

where $\hat{\mathbf{e}}^{(2)}$ and $\hat{\mathbf{e}}^{(3)}$ are arbitrarily chosen to be mutually perpendicular and perpendicular to $\hat{\mathbf{g}}^{(1)}$ (i.e. a_{ij} form a 2-D rotation matrix). Multiplying the perturbation equation (15) for $\mu = 2$ by $\hat{e}_j^{(2)}$ we obtain

$$a_{22} \, \delta \hat{\Gamma}_{jk} \hat{e}_k^{(2)} \hat{e}_j^{(2)} - 2a_{22} v^{(2)} \, \delta v^{(2)} + a_{23} \, \delta \hat{\Gamma}_{jk} \hat{e}_k^{(3)} \hat{e}_j^{(2)} = 0, \tag{26}$$

and similarly for $\hat{e}_j^{(3)}$. These simultaneous equations can be written

$$\begin{aligned} a_{22}(B_{22} - 2v^{(2)} \, \delta v^{(2)}) + a_{23} B_{32} &= 0, \\ a_{22} B_{23} + a_{23}(B_{33} - 2v^{(2)} \, \delta v^{(2)}) &= 0, \end{aligned} \tag{27}$$

where

$$B_{\mu\nu} = \delta \hat{\Gamma}_{jk} \hat{e}_j^{(\mu)} \hat{e}_k^{(\nu)}. \tag{28}$$

Solving we obtain

$$\delta v^{(\mu)} = \frac{1}{4\beta} (B_{22} + B_{33} \pm B) \tag{29}$$

with the alternative signs giving $\mu = 2$ or 3, where β is the shear wave velocity in the unperturbed medium and

$$B = [(B_{22} - B_{33})^2 + 4B_{23}^2]^{1/2}. \tag{30}$$

The question of how to associate the alternative signs with the two qS-waves in a unique fashion is difficult. If the qS slowness sheets degenerate in the perturbed medium, then the roles of the two solutions may reverse at the degenerate point, and there is no simple method of determining which solution is which.

The perturbed traveltimes due to perturbations of the parameters a_{ijkl} can be calculated by substituting equation (29) in the integral (13). We now proceed to discuss some specific cases relevant to crosshole tomography.

2.2 2-D tomography and weak anisotropy

If non-degenerate perturbation theory is valid (17), as it is for qP rays, then the traveltime perturbation is given by

$$\delta T^{(\mu)} = -\frac{1}{2}\int_{\mathscr{L}} p_i p_l \hat{g}_j^{(\mu)} \hat{g}_k^{(\mu)}\, \delta a_{ijkl}\, dT, \tag{31}$$

where in general the integral may contain 21 independent terms. If the unperturbed model is isotropic and we are considering qP rays, equation (31) is considerably simplified. In isotropic media, the P-wave polarization, the slowness direction and the ray direction are all parallel, so $\hat{\mathbf{g}}^{(1)} = \hat{\mathbf{p}} = \alpha\mathbf{p}$. Thus

$$\delta T^{(1)} = -\frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_i p_j p_k p_l\, \delta a_{ijkl}\, dT. \tag{32}$$

Let us now make the further assumption that data are collected for source/receiver pairs that are all co-planar. Small deviations from the plane can be corrected for by projection onto the plane with appropriate traveltime corrections. With this assumption, we can orient our coordinate system so that the x_2 coordinate is normal to the plane [in this paper we use (x_1, x_2, x_3) and (x, y, z) interchangeably for Cartesian coordinaes]. In isotropic 2-D media rays are confined to the x_1–x_3 plane, and hence p_2 will always be zero. The perturbed rays in the anisotropic media may deviate from the plane, but if the anisotropy is weak the deviation is small. Again, to first order Fermat's principle allows us to ignore this deviation. With these assumptions (weak anisotropic, 2-D media), the P traveltime perturbation reduces to

$$\begin{aligned}\delta T^{(1)} = &-\frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_1^4\, \delta q_1\, dT - \frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_1^3 p_3\, \delta q_2\, dT \\ &-\frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_1^2 p_3^2\, \delta q_3\, dT - \frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_1 p_3^3\, \delta q_4\, dT \\ &-\frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_3^4\, \delta q_5\, dT,\end{aligned} \tag{33}$$

where

$$\begin{aligned} q_1 &= a_{1111} = A_{11}, \\ q_2 &= 4a_{1131} = 4A_{15}, \\ q_3 &= 2a_{1133} + 4a_{3131} = 2A_{13} + 4A_{55}, \\ q_4 &= 4a_{3331} = 4A_{35}, \\ q_5 &= a_{3333} = A_{33}. \end{aligned} \tag{34}$$

As expected the qP ray traveltimes are only sensitive to a limited number of the 21 parameters, a_{ijkl}. Only five independent terms occur in the traveltime perturbation (33). In (34), we have also written the parameters q_j in terms of the elements of the commonly used 6×6 symmetric matrix representation, A_{mn}, of the 21 parameters a_{ijkl}. The reduced indices (m and $n = 1$ to 6) enumerate the Voigt pairs—11, 22, 33, 23, 31, 12—in the usual order (Thomsen 1986; Winterstein 1990). In the matrix $\mathbf{A}$, the qP traveltimes are sensitive to the elements

$$\begin{pmatrix} A_{11} & \cdot & A_{13} & \cdot & A_{15} & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & A_{33} & \cdot & A_{35} & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & A_{55} & \cdot \\ & & & & & \cdot \end{pmatrix}. \tag{35}$$

While it is intuitively obvious why the qP traveltimes will be sensitive only to parameters containing the indices 1 and 3, we are unable to give a simple physical (non-algebraic) reason why the parameters a_{1133} and a_{3131} occur only in combination.

It is often convenient to consider the isotropic velocity as a separate parameter. For instance, in tomographic inversion we may wish to invert for the isotropic velocity model first and then include anisotropic parameters. To do this include an extra parameter $\bar{q}_0 = \alpha$ and write

$$\begin{aligned} &\bar{q}_0 = \alpha, \quad \bar{q}_1 = q_1 - \alpha^2, \quad \bar{q}_2 = q_2, \\ &\bar{q}_3 = q_3 - 2\alpha^2, \quad \bar{q}_4 = q_4, \quad \bar{q}_5 = q_5 - \alpha^2. \end{aligned} \tag{36}$$

For the traveltime perturbation we obtain

$$\begin{aligned}\delta T^{(1)} = &-\int_{\mathscr{L}} \alpha^{-1}\, \delta\bar{q}_0\, dT - \frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_1^4\, \delta\bar{q}_1\, dT \\ &-\frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_1^3 p_3\, \delta\bar{q}_2\, dT - \frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_1^2 p_3^2\, \delta\bar{q}_3\, dT \\ &-\frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_1 p_3^3\, \delta\bar{q}_4\, dT - \frac{1}{2}\int_{\mathscr{L}} \alpha^2 p_3^4\, \delta\bar{q}_5\, dT.\end{aligned} \tag{37}$$

The first term is equivalent to that used in normal isotropic tomography, e.g. Bregman *et al.* (1989a). It is trivial to replace the $\bar{q}_0$ parameter by α^2, $1/\alpha$ or $1/\alpha^2$, etc. if preferred. Here we have used α to make the expressions equivalent to Bregman *et al.* (1989a). The six parameters, $\bar{q}_j$ are no longer independent but this can be handled in the numerical scheme used for inversion.

Several different methods can be used to implement equations (37) numerically. The simplest is to divide the model into piecewise homogeneous elements so the integrals can be discretized. Then equation (37) can be rewritten as

$$\delta T_k^{(1)} = \frac{\partial T_k^{(1)}}{\partial \bar{q}_{ij}}\, \delta\bar{q}_{ij} \tag{38}$$

where $\delta\bar{q}_{ij}$ is the jth parameter $\delta\bar{q}_j$ ($j = 0, 1, \ldots, 5$) in the ith element, and $T_k^{(1)}$ is the traveltime of the kth ray. Then assuming the starting model is isotropic, for $j = 0$

$$\frac{\partial T_k^{(1)}}{\partial \bar{q}_{i0}} = -\frac{\Delta l_{ki}}{\alpha_i^2} \tag{39}$$

and for $j = 1, \ldots, 5$

$$\frac{\partial T_k^{(1)}}{\partial \bar{q}_{ij}} = -\tfrac{1}{2}\alpha_i\, \Delta l_{ki} p_1^{5-j} p_3^{j-1} \tag{40}$$

where Δl_{ki} is the distance travelled by the kth ray in the ith element, α_i is the isotropic compressional velocity in the ith element, and the p_1 and p_3 are evaluated for the kth ray in the ith element. All of these quantities are simple to compute given the ray path in the original isotropic medium and the geometry of the regions.

It is straightforward to trace 'curved' (bent) rays through a model of homogeneous elements using Snell's law at each discontinuity (Dyer & Worthington 1988). Nevertheless, discontinuities introduced into the model by the perturbations may result in total reflections, and further ray tracing will be unstable. In general it will be necessary to model inhomogeneity and anisotropy simultaneously and to trace curved rays through the inhomogeneous model. Another possible parametrization for an inhomogeneous model is to divide the model into triangular elements and interpolate the velocity linearly in each element, e.g. Chapman (1985) and Bregman *et al.* (1989a). Again the traveltime perturbation can be written as equation (38) where the parameters $\bar{q}_{ij}$ are the values at the ith grid point (vertex) in the model. The expressions for the matrix elements $\partial T_k^{(1)}/\partial \bar{q}_{ij}$ are naturally more complicated as the integrands in (37) are variable and each matrix element depends on possible segments in several triangles, and algebraic details are given in the appendices. In Appendices A and B we develop algebraic results for the ray integrals through an element with a constant velocity gradient. In general, this gradient need not be vertical nor in the same direction in each model element. In Appendix C we show how the expressions in Appendices A and B can be accurately computed. In Appendix D we demonstrate how the results in Appendix A, which apply in each model element with local coordinates arranged so that the velocity gradient is 'vertical', can be transformed into global coordinates. In particular we show that the combination of parameters that occurs in $\bar{\mathbf{q}}$ transforms into the same parameters in the global system.

The basic form of the integrals (37) is simple and they can easily be evaluated for other model parametrizations, e.g. Červený (1987). Virieux, Farra & Madariaga (1988) have suggested linear interpolation of the squared slowness. As the terms making up the matrix elements contain many repeated expressions (Appendices A and C), most of the computations reduce to simple arithmetic. We can see no great advantage in any particular model parametrization and for brevity do not give details of other parametrizations.

The application of expression (13) for S rays is intrinsically more complicated than for P rays. Expression (29) is non-linear in the model perturbations due to the square root in equation (30). This complication arises because in order to calculate the perturbation (31) it is necessary to know the polarization of the ray. For P rays this is known uniquely because it is longitudinal for isotropic media and is only slightly perturbed in weakly anisotropic media. For S rays, however, it is ambiguous. In isotropic media any transverse polarization is permitted and the actual polarization depends on the initial conditions at the source, not on the medium. In anisotropic media, the polarizations are determined by the medium and ray, e.g. solutions of equation (14). If an isotropic medium is perturbed to an anisotropic medium then the traveltime perturbation depends on the model perturbation and the polarizations. Since the polarizations are themselves defined by the perturbations [solving (25) and (27)], this leads to the non-linear behaviour (29). If we assume that the polarizations after the perturbation are known, then we can choose $\hat{\mathbf{e}}^{(2)}$ and $\hat{\mathbf{e}}^{(3)}$ to correspond to the polarizations, making $B_{23}=0$. Then

$$\delta v^{(\mu)} = \frac{B_{\mu\mu}}{2\beta} \tag{41}$$

for $\mu = 2$ and 3 [just as for P rays (17)]. But perturbing from an isotropic medium, the polarizations are not known until the anisotropy is known. Thus the S ray traveltime perturbation from an isotropic medium is intrinsically non-linear.

The mean S traveltime is linear, however, as

$$\tfrac{1}{2}(\delta v^{(2)} + \delta v^{(3)}) = \frac{1}{4\beta}(B_{22} + B_{33}), \tag{42}$$

a result previously obtained by Červený & Jech (1982). It is also straightforward, apart from the question of uniquely identifying the solution with each qS ray, to use the formulae (29) to solve the forward problem, i.e. calculating the perturbed qS traveltimes. However, for the inverse problem there is no solution to the intrinsic non-linearity except to introduce further restrictions.

Each term $B_{\mu\nu}$ depends only on a subset of all the elastic parameters [the r's, s's and t's below, see (44), (47) and (51)]. If we assume that parameters only in one subset are perturbed, then the resultant perturbation is linear. To calculate the perturbed S ray results, we can use any suitable polarization vectors. A convenient choice is the SV and SH vectors. For wave propagation in the x_1–x_3 plane, we use

$$\begin{aligned}\hat{\mathbf{e}}^{(1)} &= \beta(p_1, 0, p_3),\\ \hat{\mathbf{e}}^{(2)} &= \beta(-p_3, 0, p_1),\\ \hat{\mathbf{e}}^{(3)} &= (0, 1, 0).\end{aligned} \tag{43}$$

With these vectors, many elements in the perturbation integrals are zero, and many are repeated due to the symmetries. The term B_{22} depends on the three combinations of parameters $a_{1111} + a_{3333} - 2a_{1133}$, $4(a_{3331} - a_{1131})$ and $4a_{3131}$. As with the P results, it is often convenient to separate off the isotropic behaviour. Thus we define

$$\begin{aligned}\bar{r}_0 &= \beta,\\ \bar{r}_1 &= a_{1111} + a_{3333} - 2a_{1133} - 4\beta^2\\ &= A_{11} + A_{33} - 2A_{13} - 4\beta^2,\\ \bar{r}_2 &= 4(a_{3331} - a_{1131}) = 4(A_{35} - A_{15}),\\ \bar{r}_3 &= 4(a_{3131} - \beta^2) = 4(A_{55} - \beta^2).\end{aligned} \tag{44}$$

If only these parameters are perturbed, then the traveltime perturbations are given by

$$\begin{aligned}\delta T^{(2)} &= -\frac{1}{2}\int_{\mathscr{L}} \frac{B_{22}}{v^{(2)2}}\,dT\\ &= -\int_{\mathscr{L}} \beta^{-1}\,\delta\bar{r}_0\,dT - \frac{1}{2}\int_{\mathscr{L}} \beta^2 p_1^2 p_3^2\,\delta\bar{r}_1\,dT\\ &\quad - \frac{1}{2}\int_{\mathscr{L}} \frac{1}{2}\beta^2 p_1 p_3 (p_1^2 - p_3^2)\,\delta\bar{r}_2\,dT\\ &\quad - \frac{1}{2}\int_{\mathscr{L}} \frac{1}{4}\beta^2 (p_1^2 - p_3^2)^2\,\delta\bar{r}_3\,dT,\\ \delta T^{(3)} &= 0.\end{aligned} \tag{45}$$

For these parameters, the traveltime perturbation can be written

$$\delta T_k^{(2)} = \frac{\partial T_k^{(2)}}{\partial \bar{r}_{ij}} \delta \bar{r}_{ij} \tag{46}$$

where the notation is similar to equation (38). We note that the same parameters, e.g. (35), appear in $\mathbf{r}$ as in $\mathbf{q}$ but in different combinations.

Similarly, for B_{33} we define

$$\begin{aligned}
\bar{s}_0 &= \beta, \\
\bar{s}_1 &= a_{1212} - \beta^2 = A_{66} - \beta^2, \\
\bar{s}_2 &= 2a_{1223} = 2A_{46}, \\
\bar{s}_3 &= a_{2323} - \beta^2 = A_{44} - \beta^2
\end{aligned} \tag{47}$$

(for notational simplicity we include $\bar{s}_0 = \bar{r}_0$) and obtain

$$\begin{aligned}
\delta T^{(2)} &= 0, \\
\delta T^{(3)} &= -\frac{1}{2}\int_{\mathscr{L}} \frac{B_{33}}{v^{(3)2}}\, dT \\
&= -\int_{\mathscr{L}} \beta^{-1}\, \delta \bar{s}_0\, dT - \frac{1}{2}\int_{\mathscr{L}} p_1^2\, \delta \bar{s}_1\, dT \\
&\quad -\frac{1}{2}\int_{\mathscr{L}} p_1 p_3\, \delta \bar{s}_2\, dT - \frac{1}{2}\int_{\mathscr{L}} p_3^2\, \delta \bar{s}_3\, dT
\end{aligned} \tag{48}$$

and

$$\delta T_k^{(3)} = \frac{\partial T_k^{(3)}}{\partial \bar{s}_{ij}} \delta \bar{s}_{ij}. \tag{49}$$

In the 6×6 matrix $\mathbf{A}$, the elements of $\mathbf{s}$ depend on

$$\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & \cdot & \cdot & \cdot & \cdot \\
 & & & A_{44} & \cdot & A_{46} \\
 & & & & \cdot & \cdot \\
 & & & & & A_{66}
\end{pmatrix}. \tag{50}$$

Finally, for B_{23} we define

$$\begin{aligned}
\bar{t}_1 &= a_{3312} - a_{1112} = A_{36} - A_{16}, \\
\bar{t}_2 &= a_{3323} - a_{1123} = A_{34} - A_{14}, \\
\bar{t}_3 &= 2a_{3112} = 2A_{56}, \\
\bar{t}_4 &= 2a_{2331} = 2A_{45},
\end{aligned} \tag{51}$$

and obtain

$$\begin{aligned}
\delta T^{(2)} = -\delta T^{(3)} &= -\frac{1}{2}\int_{\mathscr{L}} \frac{B_{23}}{v^{(2)2}}\, dT \\
&= -\frac{1}{2}\int_{\mathscr{L}} \beta p_1^2 p_3\, \delta \bar{t}_1\, dT - \frac{1}{2}\int_{\mathscr{L}} \beta p_1 p_3^2\, \delta \bar{t}_2\, dT \\
&\quad -\frac{1}{2}\int_{\mathscr{L}} \frac{1}{2}\beta p_1 (p_1^2 - p_3^2)\, \delta \bar{t}_3\, dT \\
&\quad -\frac{1}{2}\int_{\mathscr{L}} \frac{1}{2}\beta p_3 (p_1^2 - p_3^2)\, \delta \bar{t}_4\, dT
\end{aligned} \tag{52}$$

and

$$\delta T_k^{(2)} = -\delta T_k^{(3)} = \frac{\partial T_k^{(2)}}{\partial \bar{t}_{ij}} \delta \bar{t}_{ij}. \tag{53}$$

In the 6×6 matrix $\mathbf{A}$, the elements of $\mathbf{t}$ depend on

$$\begin{pmatrix}
\cdot & \cdot & \cdot & A_{14} & \cdot & A_{16} \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & \cdot & A_{34} & \cdot & A_{36} \\
 & & & \cdot & A_{45} & \cdot \\
 & & & & \cdot & A_{56} \\
 & & & & & \cdot
\end{pmatrix}. \tag{54}$$

Note that none of the traveltime perturbations depends of the parameters $a_{22ij} = A_{2k} = A_{k2}$, but all the other elastic parameters appear in one or more traveltime perturbations.

For a piecewise homogeneous model, the partial derivatives in equations (46), (49) and (53) are obtained in a straightforward fashion as in (39) and (40). The results for linearly interpolated models are given in Appendices A, B and C.

3 DISCUSSION

In this paper we have demonstrated how the traveltimes of rays in inhomogeneous, weakly anisotropic media in 2-D tomographic experiments depend on subsets of the full anisotropic velocity parameters. For instance, the qP traveltimes depend on the five parameters, $\mathbf{q}$ (equation 34). One of these parameters (q_3) is a linear combination of the basic velocity parameters. The five parameters determined by a tomographic experiment need not be compatible with a specific model of anisotropy, e.g. TI. We believe that in general it is important to invert for these parameters independently rather than restrict the number of parameters *a priori* by introducing some assumption about the anisotropy.

For computational purposes, it may be convenient to use transformed coordinates in each model element, e.g. so that the coordinates are aligned with the 'local' gradient. The subset of parameters in $\mathbf{q}$, for instance, in the 'local' coordinates are simply related to the same subset in the 'global' coordinates (Appendix D). While it is physically obvious that this result must apply, as it cannot matter in which coordinate system the traveltimes are calculated, it is not intuitively obvious why the same linear combinations of velocity parameters occur in both systems. The partial derivatives can be calculated in 'local' coordinate systems using straightforward analytic expressions (Appendices A, B and C), and then converted using linear transformations to the 'global' system (Appendix D). Although the algebraic details are tedious, it is straightforward to calculate the integrals for the differential coefficients in the 'local' system for a linearly interpolated model because they can be reduced to standard trigonometrical integrals. Although many integrals are necessary, e.g. the 15 elements of $\mathbf{Q}$ for the qP terms, the computations are efficient because the expensive special functions are used repeatedly (Table 1). In addition, differences of the trigonometrical functions are calculated directly from the geometry of the ray (Appendix

Table 1. The integrands and indefinite integrals needed for the partial derivatives in models with linear interpolation. The notation is explained in Appendix A.

Integral	Integrand	Indefinite Integral
Q_{01}	$\sec^2\psi$	$\tan\psi$
Q_{02}	$\sec\psi\,\tan\psi$	$\sec\psi$
Q_{03}	$\sec\psi$	$\tanh^{-1}(\sin\psi)$
Q_{11}	$\cos\psi$	$\sin\psi$
Q_{12}	$\cos\psi\,\sin\psi$	$-(\cos 2\psi)/4$
Q_{13}	$\cos^2\psi$	$(\sin 2\psi + 2\psi)/4$
Q_{21}	$-\sin\psi$	$\cos\psi$
Q_{22}	$-\sin^2\psi$	$(\sin 2\psi - 2\psi)/4$
Q_{23}	$-\sin\psi\,\cos\psi$	$(\cos 2\psi)/4$
Q_{31}	$\sin\psi\,\tan\psi$	$\tanh^{-1}(\sin\psi) - \sin\psi$
Q_{32}	$\sin^2\psi\,\tan\psi$	$\ln(\sec\psi) + (\cos 2\psi)/4$
Q_{33}	$\sin^2\psi$	$(2\psi - \sin 2\psi)/4$
Q_{41}	$-\sin\psi\,\tan^2\psi$	$-\sec\psi - \cos\psi$
Q_{42}	$-\sin^2\psi\,\tan^2\psi$	$3\psi/2 - \tan\psi - (\sin 2\psi)/4$
Q_{43}	$-\sin^2\psi\,\tan\psi$	$-\ln(\sec\psi) - (\cos 2\psi)/4$
Q_{51}	$\sin\psi\,\tan^3\psi$	$(\tan\psi\,\sec\psi - 3\tanh^{-1}(\sin\psi))/2 + \sin\psi$
Q_{52}	$\sin^2\psi\,\tan^3\psi$	$(2\sec^2\psi - \cos 2\psi)/4 + 2\ln(\cos\psi)$
Q_{53}	$\sin^2\psi\,\tan^2\psi$	$\tan\psi + (\sin 2\psi)/4 - 3\psi/2$
R_{1j}	Q_{3j}	
R_{2j}	$(Q_{2j} - Q_{4j})/2$	
R_{3j}	$(Q_{1j} - 2Q_{3j} + Q_{5j})/4$	
S_{11}	$\sec\psi$	$\tanh^{-1}(\sin\psi)$
S_{12}	$\tan\psi$	$\ln(\sec\psi)$
S_{13}	1	ψ
S_{21}	$-\tan\psi\,\sec\psi$	$-\sec\psi$
S_{22}	$-\tan^2\psi$	$\psi - \tan\psi$
S_{23}	$-\tan\psi$	$\ln(\cos\psi)$
S_{31}	$\tan^2\psi\,\sec\psi$	$(\tan\psi\,\sec\psi - \tanh^{-1}(\sin\psi))/2$
S_{32}	$\tan^3\psi$	$(\sec^2\psi)/2 + \ln(\cos\psi)$
S_{33}	$\tan^2\psi$	$\tan\psi - \psi$
T_{11}	$\tan\psi$	$\ln(\sec\psi)$
T_{12}	$\sin\psi\,\tan\psi$	$\tanh^{-1}(\sin\psi) - \sin\psi$
T_{13}	$\sin\psi$	$-\cos\psi$
T_{21}	$-\tan^2\psi$	$\psi - \tan\psi$
T_{22}	$-\tan^2\psi\,\sin\psi$	$-\sec\psi - \cos\psi$
T_{23}	$-\tan\psi\,\sin\psi$	$\sin\psi - \tanh^{-1}(\sin\psi)$
T_{31}	$(\tan^2\psi - 1)/2$	$(\tan\psi - 2\psi)/2$
T_{32}	$\sin\psi(\tan^2\psi - 1)/2$	$(\sec\psi)/2 + \cos\psi$
T_{33}	$\cos\psi(\tan^2\psi - 1)/2$	$(\tanh^{-1}(\sin\psi))/2 - \sin\psi$
T_{41}	$\tan\psi(1 - \tan^2\psi)/2$	$\ln(\sec\psi) - (\sec^2\psi)/4$
T_{42}	$\tan\psi\,\sin\psi(1 - \tan^2\psi)/2$	$(5\tanh^{-1}(\sin\psi) - \tan\psi\,\sin\psi)/4 - \sin\psi$
T_{43}	$\sin\psi(1 - \tan^2\psi)/2$	$-\cos\psi - (\sec\psi)/2$

Table 2. The definite integrals needed for the partial derivatives in models with linear interpolation. The results are expressed in a form which will retain numerical accuracy. The notation is explained in Appendices A and C.

Integral	Definite Integral
Q_{01}	$(\sin\Delta\psi)/\cos\psi_1\cos\psi_2$
Q_{02}	$-(\Delta\cos\psi)/\cos\psi_1\cos\psi_2$
Q_{03}	$\lvert\nabla\alpha\rvert\Delta T$
Q_{11}	$\Delta\sin\psi$
Q_{12}	$-(\Delta\cos\psi)(\cos\psi_1 + \cos\psi_2)/2$
Q_{13}	$(\sin\Delta\psi)(\cos\psi_1\cos\psi_2 - \sin\psi_1\sin\psi_2)/2 + (\Delta\psi)/2$
Q_{21}	$\Delta\cos\psi$
Q_{22}	$Q_{13} - \Delta\psi$
Q_{23}	$-Q_{12}$
Q_{31}	$Q_{03} - Q_{11}$
Q_{32}	$\mathrm{dln}(-(\Delta\cos\psi)/\cos\psi_2) - Q_{12}$ if $\cos\psi_1 > \cos\psi_2$
	$-\mathrm{dln}((\Delta\cos\psi)/\cos\psi_1) - Q_{12}$ if $\cos\psi_1 < \cos\psi_2$
Q_{33}	$-Q_{22}$
Q_{41}	$-Q_{02} - Q_{21}$
Q_{42}	$2\Delta\psi - Q_{01} - Q_{13}$
Q_{43}	$-Q_{32}$
Q_{51}	$(\Delta\sin\psi)(1 + \sin\psi_1\sin\psi_2)/2\cos^2\psi_1\cos^2\psi_2 - 3Q_{03}/2 + Q_{11}$
Q_{52}	$-(\Delta\cos\psi)(\cos\psi_1 + \cos\psi_2)/2\cos^2\psi_1\cos^2\psi_2 - 2Q_{32} - Q_{12}$
Q_{53}	$-Q_{42}$
R_{1j}	Q_{3j}
R_{2j}	$(Q_{2j} - Q_{4j})/2$
R_{3j}	$(Q_{1j} - 2Q_{3j} + Q_{5j})/4$
S_{11}	Q_{03}
S_{12}	$Q_{32} + Q_{12}$
S_{13}	$\Delta\psi$
S_{21}	$-Q_{02}$
S_{22}	$S_{13} - Q_{01}$
S_{23}	$-S_{12}$
S_{31}	$Q_{31} + Q_{51}$
S_{32}	$Q_{32} + Q_{52}$
S_{33}	$Q_{33} + Q_{53}$
T_{11}	S_{12}
T_{12}	Q_{31}
T_{13}	$-Q_{21}$
T_{21}	S_{22}
T_{22}	Q_{41}
T_{23}	$-Q_{31}$
T_{31}	$(Q_{01} - 2S_{13})/2$
T_{32}	$(Q_{21} - Q_{41})/2$
T_{33}	$(Q_{31} - Q_{11})/2$
T_{41}	$(S_{12} - S_{32})/2$
T_{42}	$(Q_{31} - Q_{51})/2$
T_{43}	$(Q_{41} - Q_{21})/2$

D) to retain numerical accuracy in the integrals (Table 2). The same integrals are needed for each parameter and the only differences are the values of the coefficients Y_0, Y_x and Y_z (Appendix A). Thus overall, the many terms in the differential coefficients can be computed efficiently and accurately. Similar systems of analytic integrals would apply for interpolations other than linear, but for brevity we have not presented details.

It was not our purpose in this paper to discuss the techniques available for solving the linearized equations, e.g. equation (38), for the tomographic image. Many methods exist for obtaining the generalize inverse of large, sparse systems of linear equations and they have been extensively discussed in the literature. The LSQR algorithm (Paige & Saunders 1982) used by Bregman *et al.* (1989a) seems particularly well suited especially as its numerical properties are well understood. As the system of equations will usually be (very) underdetermined, particularly with the addition of many anisotropic parameters, it will be essential to regularize the solution, e.g. constrain the inversion to look for the smoothest image. In addition it may be appropriate to constrain the solution to minimize the anisotropic parts, while inverting for the isotropic image. This is easily accomplished using the parameters $\bar{\mathbf{q}}$ (36) rather than $\mathbf{q}$ and weighting the isotropic velocity over the anisotropic, or introducing constraints to minimize the anisotropic part. Having executed several iterations to improve the isotropic image with curved ray tracing, the constraint might be relaxed to allow further improvements in the fit using the anisotropic parameters. After each iteration, it is necessary to re-establish the isotropic velocity values. These may not correspond exactly to the isotropic velocities, e.g. $\bar{q}_0$, obtained in the inversion as there is some ambiguity in separating the isotropic velocity from the anisotropic parameters. Backus (1970) has described a unique procedure for separating the isotropic velocity and we have followed this procedure in Pratt & Chapman (1992). It should be noted that it does not correspond to the velocities α_0 and β_0 used in Appendix E.

Other constraints would be possible between the anisotropic parameters if specific models of anisotropy are appropriate. The equations in Appendix E connecting the general parameters, **q**, **r**, **s** and **t** with specific parameters for TI media can be used to introduce these constraints. Similar equations exist for other models of anisotropy, e.g. orthorhombic symmetry, but for brevity we have not included them here. We do argue, however, that these equations should be added as constraints to a general inverse problem, rather than being built into the original formulation. This procedure is much more flexible and allows the consistency of the data to be tested.

Having determined tomographic images of the anisotropic parameters, e.g. the parameters **q** from qP traveltimes, it will often be appropriate to interpret these in terms of a specific model of anisotropy. In Appendix E we have indicated how the general parameters **q** are related to the five TI parameters, A, C, F, L and N [we use this notation of Love (1944) to avoid confusion with the other sets of elastic parameters used in the local and global coordinate systems]. First we note that the qP traveltimes only allow two of the TI parameters and one combination to be determined, A, C and $F+2L$. For these parameters, the system is overdetermined and can be solved in a least-squares sense (assuming the orientation of the symmetry axis is known). The other parameters, N and some other combination of F and L, are completely undetermined by the qP traveltimes. They can be determined independently from the qS traveltimes, but overall the system of equations for the TI parameters remains overdetermined, e.g. qP and mean qS traveltimes depend on **q**, **r** and **s** which contain eight different velocity parameters from which we require five TI parameters. For other simple anisotropic systems, e.g. orthorhombic symmetry, we would expect other overdetermined systems of equations to apply, while for more general systems the equations may be underdetermined. In discussing the TI system, we have assumed that the orientation of the symmetry axis is known, i.e. the angles θ and ϕ in Appendix E. If the orientation is unknown, the equation for the qP parameters (E4) contains five parameters on each side but is non-linear in θ and ϕ. In general, it may be possible to solve for A, C, $F+2L$, θ and ϕ numerically, but certain ambiguities must be anticipated. Some of these ambiguities are artificial due to the choice of coordinates, e.g. if θ is small, ϕ will be very uncertain (or if $\theta=0$, ϕ is indeterminate) but the axis itself is not uncertain, and some may be real, e.g. the symmetry axis can be shifted through 90°—see Pratt & Chapman (1992) for examples.

Despite our reservations, it is possible to express the linear perturbation directly in terms of the parameters of some specific model of anisotropy. Equations such as (38) can be transformed into equations with A, C and $F+2L$ using the linear equations (E4). If required, equation (E4) can be differentiated with respect to A, C, $F+2L$, θ and ϕ to obtain linearized equations with respect to all five parameters. Similar results apply for the qS results and other anisotropy models. The possibilities are endless depending on one's prejudices—it is partly for this reason that we argue that the initial interpretation should be in terms of **q**, **r**, **s** and **t**.

Perturbation theory for S rays is more complicated than for P rays because in isotropic media it is degenerate. This makes the perturbation in the velocity and traveltime non-linear in the δa_{ijkl}'s. The perturbed velocity contains the factor B (30) which is the square root of a quadratic expression in the δa_{ijkl}'s. If parameters in only one subset **r**, **s** or **t** are perturbed, then the perturbation is linear as the factor B contains only one squared factor. However, there is no reason to expect that only one subset of parameters will be non-zero. Although linear inverse theory will not apply to qS rays starting with an isotropic model, the non-linear perturbation formulae for the traveltime perturbation solve the direct problem, and a non-linear inverse scheme may be successful. Another problem may be whether two qS ray traveltimes can be uniquely identified in tomographic experiments. As noted above, the mean qS traveltime does satisfy linear perturbation theory and it is likely that it can be measured more reliably. It must be emphasized however, that shear wave splitting, i.e. the difference in the qS traveltimes, is diagnostic of anisotropy and this information is ignored if only the mean time is used. If we start with an anisotropic model or continue from one in an iterative sequence, then in theory the qS perturbation theory is non-degenerate and will be linear. However, we do not expect this to be a satisfactory situation. Models will necessarily be only weakly anisotropic, and while this is sufficient to make perturbation theory non-degenerate, the situation will be unstable. Many perturbations will be sufficient to cause significant changes in the polarizations, i.e. take the perturbation outside the range of linearity. A satisfactory solution must await shear wave data sets.

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APPENDIX A: LINEAR INTERPOLATION

In order to trace rays and evaluate the various integrals for the traveltimes and perturbations, it is convenient and efficient to use a simple, analytic interpolation for the model. The model is divided into geometrical elements, e.g. triangles, and within each element a single formula is used for interpolation. Various interpolation methods are possible but a straightforward choice is linear interpolation. It is well known that with a linear interpolation of the velocity, the ray paths are arcs of circles. In each element the gradient can be different in magnitude and direction, but the velocity is continuous between elements (except when an interface is introduced between triangles). Within each model element, it is convenient to use local coordinates with the origin at the centre of the ray arc and a coordinate axis parallel to the gradient. The local coordinates are different for each ray and each model element. In this appendix, all parameters are expressed in these local coordinates. In Appendix C, we relate the model parameters in the local coordinate system to those in the global coordinate system used to describe the complete model. In the local coordinates, positions on the ray are defined by the radius of the circle, R, and the angle with the local vertical, ψ (see Fig. A1). In order to produce robust numerical code, it is necessary to obtain expressions that are valid whichever the direction of propagation. We have to include two cases: case 1 is when the ray is propagating counter-clockwise, i.e. $p_1 > 0$, as in Fig. A1, while case 2 is when the ray is propagating clockwise, i.e. $p_1 < 0$. In both cases it

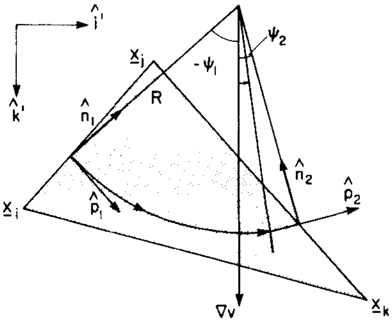

Figure A1. In each triangle, the ray path is an arc of a circle. The ray path is described by the radius of the circle, R, and the angle, ψ, at the centre. The ray path runs from ψ_1 to ψ_2 (in this diagram ψ_1 is negative).

is convenient to measure ψ in the propagation direction. Note that $\hat{\mathbf{k}}$ is always defined as $\hat{\mathbf{k}} = \nabla v/|\nabla v|$ whichever the direction of propagation. This is the opposite of the convention used in Chapman (1985), and we have used v for the velocity to indicate either α or β. The vector $\hat{\mathbf{j}}$ is fixed and in the figures normally points out of the paper. The vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ form a RH set of axes so $\hat{\mathbf{i}}$ is defined by $\hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{k}}$. The position vector on the ray is $\mathbf{x} = x\hat{\mathbf{i}} + z\hat{\mathbf{k}}$, and the slowness vector is $\mathbf{p} = p_1\hat{\mathbf{i}} + p_3\hat{\mathbf{k}}$. If $p_1 > 0$ then x increases along the ray, and if $p_1 < 0$ it decreases. Then

$$x = \operatorname{sgn}(p_1) R \sin \psi, \qquad z = R \cos \psi, \qquad v = |\nabla v|\, z = R\, |\nabla v| \cos \psi,$$
$$p_1 = \operatorname{sgn}(p_1) \cos \psi/v = \operatorname{sgn}(p_1)/|\nabla v|\, R, \qquad p_3 = -\operatorname{sgn}(p_1) \sin \psi/v = -p_1 \tan \psi, \qquad v = \cos \psi/|p_1|, \tag{A1}$$

and for increments along the ray (always positive), we have

$$dT = dl/v = R\, d\psi/v = \sec \psi\, d\psi/|\nabla v|. \tag{A2}$$

Thus the traveltime integral is

$$T = \int dT = \frac{1}{|\nabla v|} \int \sec \psi\, d\psi = \frac{1}{|\nabla v|} \tanh^{-1}(\sin \psi) \Big| \tag{A3}$$

where in this and the following integrals we understand the limits ψ_1 to ψ_2 defined by the ray segment (see Fig. A1). The limits are always in the increasing direction, but the angles can be of either sign depending on the relationship to the turning point. This result is the standard expression for a linear gradient (Gebrande 1976). We shall see that using the variable ψ simplifies the evaluation of the required integrals, and sign problems are accounted for automatically even when a turning point is included.

Within each triangle, various parameters are linearly interpolated, i.e. the parameters $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$. Let us denote such a parameter by γ. We consider a triangle with vertices $\mathbf{x}_i$, $\mathbf{x}_j$ and $\mathbf{x}_k$ and parameter values γ_i, γ_j and γ_k at the vertices. It is worth commenting that the isotropic velocity parameters, e.g. $\bar{q}_0$, and the anisotropic parts of the normalized elastic parameters, e.g. $\bar{q}_1, \bar{q}_2, \ldots$, have different dimensions by definition (equation 36). The latter are squared velocities. We choose to interpolate the isotropic velocity linearly because it gives circular ray arcs. We might consider interpolating anisotropic velocity parameters, e.g. $(a_{ijkl})^{1/2}$, linearly but this introduces a difficulty. The traveltime perturbations are linear in δa_{ijkl} (31). Using $(a_{ijkl})^{1/2}$ as a parameter, we obtain $2(a_{ijkl})^{1/2}\delta(a_{ijkl})^{1/2}$ in the perturbation integral (31). If $a_{ijkl} \neq 0$, then we still obtain a linear perturbation in terms of $(a_{ijkl})^{1/2}$. But if $a_{ijkl} = 0$, then the perturbation is higher order and to first order there is no perturbation. This will occur for some parameters, e.g. q_2, when perturbing from an isotropic model. We therefore choose to interpolate the isotropic part of the velocity linearly [to give circular ray arcs, and to make tomographic inversions compatible with previous work, e.g. Bregman *et al.* (1989a)] and the non-isotropic part of the squared velocity parameters linearly.

For linear interpolation we must have

$$\gamma_i = \gamma_j + \nabla\gamma \cdot (\mathbf{x}_i - \mathbf{x}_j), \qquad \gamma_k = \gamma_j + \nabla\gamma \cdot (\mathbf{x}_k - \mathbf{x}_j), \tag{A4}$$

which we can solve for the gradient

$$\nabla\gamma = \frac{1}{X} \begin{pmatrix} z_k - z_j & -z_i + z_j \\ -x_k + x_j & x_i - x_j \end{pmatrix} \begin{pmatrix} \gamma_i - \gamma_j \\ \gamma_k - \gamma_j \end{pmatrix} \tag{A5}$$

where

$$X = (x_i - x_j)(z_k - z_j) - (x_k - x_j)(z_i - z_j) = x_i(z_k - z_j) + z_i(x_j - x_k) + (z_j x_k - z_k x_j). \tag{A6}$$

In general, the parameter can be written

$$\gamma(\mathbf{x}) = \gamma_j + \nabla\gamma \cdot (\mathbf{x} - \mathbf{x}_j) \tag{A7}$$

which can be differentiated to give ($i \neq j$)

$$\frac{\partial\gamma}{\partial\gamma_i} = \frac{\partial(\nabla\gamma)}{\partial\gamma_i} \cdot (\mathbf{x} - \mathbf{x}_j). \tag{A8}$$

The derivative of the gradient is simply

$$\frac{\partial(\nabla\gamma)}{\partial\gamma_i} = \frac{1}{X} \begin{pmatrix} z_k - z_j \\ -x_k + x_j \end{pmatrix} = \begin{pmatrix} Y_x \\ Y_z \end{pmatrix}, \tag{A9}$$

say, which is in a direction perpendicular to the side of the triangle opposite the perturbed vertex. Substituting and simplifying we find

$$\frac{\partial\gamma}{\partial\gamma_i} = Y_0 + Y_x x + Y_z z = Y_x(x - x_j) + Y_z(z - z_j), \tag{A10}$$

where

$$Y_0 = (z_j x_k - z_k x_j)/X = -Y_x x_j - Y_z z_j, \qquad Y_x = (z_k - z_j)/X, \qquad Y_z = (x_j - x_k)/X. \tag{A11}$$

We note that these expressions are invariant if j and k are interchanged. Overall they depend only on the geometry of the triangle, although intermediate terms depend on the origin of the coordinates.

These expressions can be used for the partial derivatives of any parameter substituted for γ, e.g. for the qP traveltime we need the perturbation of the parameters $\bar{q}_j$ in the triangles:

$$\frac{\partial \bar{q}_j}{\partial \bar{q}_{ij}} = Y_0 + \operatorname{sgn}(p_1) Y_x R \sin\psi + Y_z R \cos\psi, \tag{A12}$$

where $\bar{q}_{ij}$ is the jth parameter at the ith vertex. Remember that the linear interpolation of each parameter is distinct from the linear interpolation of the isotropic velocity and the other parameters, so Y_0, Y_x and Y_z are different for each parameter and vertex, whereas R and ψ are determined by the ray geometry, which depends on the linear interpolation of the isotropic part of the velocity. Depending on the ray path, a parameter perturbation at one vertex may influence the traveltime in several triangles. It is difficult to denote explicitly this summation but trivial to compute—the partial derivative $\partial T_k^{(1)}/\partial \bar{q}_{ij}$ in (38) is a summation over all ray segments on the kth ray influenced by the parameters at the ith vertex. The partial derivatives are for $j = 0$

$$\frac{\partial T^{(1)}}{\partial \bar{q}_{i0}} = -\sum \int \frac{\partial \bar{q}_0}{\partial \bar{q}_{i0}} \frac{dT}{\alpha} \tag{A13}$$

and for $j \neq 0$

$$\frac{\partial T^{(1)}}{\partial \bar{q}_{ij}} = -\sum \frac{1}{2} \int \alpha^2 p_1^{5-j} p_3^{j-1} \frac{\partial \bar{q}_j}{\partial \bar{q}_{ij}}\, dT, \tag{A14}$$

and the summation is over all appropriate ray segments. For brevity we drop the summation sign from the following integrals but it is always understood. Substituting (A12), the partial derivatives can be written

$$\frac{\partial T^{(1)}}{\partial \bar{q}_{i0}} = -\left|\frac{p_1}{\nabla\alpha}\right| \left[Y_0 \int \sec^2\psi\, d\psi + \operatorname{sgn}(p_1) Y_x R \int \sec\psi \tan\psi\, d\psi + Y_z R \int \sec\psi\, d\psi \right]$$

$$= -\left|\frac{p_1}{\nabla\alpha}\right| [Y_0 Q_{01} + \operatorname{sgn}(p_1) Y_x R Q_{02} + Y_z R Q_{03}] \tag{A15}$$

say, and

$$\frac{\partial T^{(1)}}{\partial \bar{q}_{ij}} = -\frac{p_1^2}{2\,|\nabla\alpha|} (-1)^{j-1} \left[Y_0 \int \cos\psi \tan^{j-1}\psi\, d\psi + \operatorname{sgn}(p_1) Y_x R \int \cos\psi \sin\psi \tan^{j-1}\psi\, d\psi + Y_z R \int \cos^2\psi \tan^{j-1}\psi\, d\psi \right]$$

$$= -\frac{p_1^2}{2\,|\nabla\alpha|} [Y_0 Q_{j1} + \operatorname{sgn}(p_1) Y_x R Q_{j2} + Y_z R Q_{j3}], \tag{A16}$$

where the integrals Q_{ij} are listed in Table 1.

Similar expressions apply for the parameters $\mathbf{r}$. Expression (A15) applies for r_0 provided β is substituted for α. For r_j, $j = 1$ to 3 we have

$$\frac{\partial T^{(3)}}{\partial \bar{r}_{ij}} = -\frac{p_1^2}{2\,|\nabla\beta|} [Y_0 R_{j1} + \operatorname{sgn}(p_1) Y_x R R_{j2} + Y_z R R_{j3}]. \tag{A17}$$

For s_j, $j = 1$ to 3 we have

$$\frac{\partial T^{(3)}}{\partial \bar{s}_{ij}} = -\frac{p_1^2}{2\,|\nabla\beta|} [Y_0 S_{j1} + \operatorname{sgn}(p_1) Y_x R S_{j2} + Y_z R S_{j3}], \tag{A18}$$

and finally for t_j, $j = 1$ to 4 we have

$$\frac{\partial T^{(2)}}{\partial \bar{t}_{ij}} = \operatorname{sgn}(p_1) \frac{p_1^2}{2\,|\nabla\beta|} [Y_0 T_{j1} + \operatorname{sgn}(p_1) Y_x R T_{j2} + Y_z R T_{j3}]. \tag{A19}$$

The integrals R_{ij}, S_{ij} and T_{ij} are defined and listed in Table 1.

APPENDIX B: SPECIAL CASES

The results in Appendix A break down if $p_1 = 0$ or $\nabla v = 0$ as in both cases $R \to \infty$. Let us first consider $p_1 = 0$. We can still use

$$\frac{\partial \gamma}{\partial \gamma_i} = Y_0 + Y_x x + Y_z z \tag{B1}$$

where Y_0, Y_x and Y_z are defined for any coordinate system. The simplest choice will be to take $x = 0$ on the ray and measure z from any point, e.g. the point where $v = 0$. The ray may be propagating in either z-direction, i.e. $p_3 > 0$ or $p_3 < 0$. Let

$\Delta z = z_2 - z_1$. Then

$$T = \int \frac{|dz|}{v} = \mathrm{sgn}\,(\Delta z) \frac{1}{|\nabla v|} \ln z \Big| . \tag{B2}$$

The partial derivative for the isotropic P velocity is

$$\frac{\partial T^{(1)}}{\partial \bar{q}_0} = -\int \frac{\partial \alpha}{\partial \alpha_i} \frac{|dz|}{\alpha^2} = -\int (Y_0 + Y_z z) \frac{|dz|}{|\nabla \alpha|^2 z^2} = \mathrm{sgn}\,(\Delta z) \frac{1}{|\nabla \alpha|^2} (Y_0/z - Y_z \ln z) \Big| \tag{B3}$$

and similarly for the partial derivative of the isotropic S velocity, $\partial T^{(2)}/\partial \bar{r}_0$. In each set of anisotropic parameters, only the final partial derivative is non-zero (as $p_1 = 0$). Thus

$$\begin{aligned} \frac{\partial T^{(1)}}{\partial \bar{q}_5} &= -\frac{1}{2} \int \alpha^2 p_3^4 \frac{\partial \bar{q}_5}{\partial \bar{q}_{i5}}\, dT = -\frac{1}{2} \int \frac{1}{|\nabla \alpha|^3 z^3} (Y_0 + Y_z z)\, |dz| = \mathrm{sgn}\,(\Delta z) \frac{1}{2\,|\nabla \alpha|^3} (Y_0/2z^2 + Y_z/z) \Big| , \\ \frac{\partial T^{(2)}}{\partial \bar{r}_3} &= \mathrm{sgn}\,(\Delta z) \frac{1}{8\,|\nabla \beta|^3} (Y_0/2z^2 + Y_z/z), \Big| , \\ \frac{\partial T^{(3)}}{\partial \bar{s}_3} &= \mathrm{sgn}\,(\Delta z) \frac{1}{2\,|\nabla \beta|^3} (Y_0/2z^2 + Y_z/z) \Big| , \\ \frac{\partial T^{(2)}}{\partial \bar{t}_4} &= -\frac{1}{4\,|\nabla \beta|^3} (Y_0/2z^2 + Y_z/z) \Big| . \end{aligned} \tag{B4}$$

The second special case is $\nabla v = 0$. Although $\nabla v = 0$, the expression for

$$\frac{\partial (\nabla \gamma)}{\partial \gamma_i} = \frac{1}{X} \begin{pmatrix} z_k - z_j \\ -x_k + x_j \end{pmatrix} \tag{B5}$$

is still valid. Rays again are straight, so we have complete freedom in the choice of local coordinates. Let us rotate so $p_1 = 0$ and choose the origin so $x = 0$. Hence

$$T = \int \frac{|dz|}{v} = \mathrm{sgn}\,(\Delta z) \frac{z}{v} \Big| , \tag{B6}$$

and

$$\frac{\partial T^{(1)}}{\partial \bar{q}_0} = -\int (Y_0 + Y_z z) \frac{|dz|}{\alpha^2} = -\mathrm{sgn}\,(\Delta z) \frac{1}{\alpha^2} \left(Y_0 z + \frac{1}{2} Y_z z^2 \right) \Big| \tag{B7}$$

for the partial derivative of the isotropic velocity. Again only the partial derivatives of the final components are non-zero and

$$\begin{aligned} \frac{\partial T^{(1)}}{\partial \bar{q}_5} &= -\mathrm{sgn}\,(\Delta z) \frac{1}{2\alpha^3} \left(Y_0 z + \frac{1}{2} Y_z z^2 \right) \Big| , \quad & \frac{\partial T^{(2)}}{\partial \bar{r}_3} &= -\mathrm{sgn}\,(\Delta z) \frac{1}{8\beta^3} \left(Y_0 z + \frac{1}{2} Y_z z^2 \right) \Big| , \\ \frac{\partial T^{(3)}}{\partial \bar{s}_3} &= -\mathrm{sgn}\,(\Delta z) \frac{1}{2\beta^3} \left(Y_0 z + \frac{1}{2} Y_z z^2 \right) \Big| , \quad & \frac{\partial T^{(2)}}{\partial \bar{t}_4} &= \frac{1}{4\beta^3} \left(Y_0 z + \frac{1}{2} Y_z z^2 \right) \Big| . \end{aligned} \tag{B8}$$

APPENDIX C: NUMERICAL INTEGRALS

In Appendix A and Table 1, analytic formulae have been given for the integrals that are required for the traveltime perturbations in model elements with linear gradients. These can be evaluated numerically using a quadrature method or using the analytic forms of the integrals. The advantages of the quadrature method are that integrals can be evaluated directly in the global coordinate system without transformations to and from the local coordinate system, and that it can be used for the forward problem of qS-waves whereas, in general, the analytic method, depending on linearity, fails. The qS problem is non-linear and the analytic formulae apply only for each subset of parameters separately, because the operation of integration and square root do not commute. The quadrature method can be used when several components are combined. The disadvantage of the quadrature method is that results are accurate only if the model elements are small and/or ray radius large, whereas in principle at least, the analytic results are accurate whatever the size of model element. More details are given below about how numerical accuracy can be maintained with the analytic expressions.

Although quadrature results may not always be accurate, they are often useful. Without the transformations and the special functions of the analytic formulae, the computations are much simpler and therefore faster. Being simpler, they provide a useful independent cross-check on the computer code for the analytic formulae. As the analytic forms are available, it is sensible to use quadrature only if a low-order method is accurate. The simplest form of quadrature is to assume that the model elements are homogeneous, and we have already mentioned this approximation in the main text (39 and 40). In inhomogeneous model elements we can use the simple trapezoidal rule. Thus, for instance

$$\frac{\partial T^{(1)}}{\partial \bar{q}_{i1}} = -\frac{1}{2}\int \alpha^2 p_1^4 \frac{\partial \bar{q}_1}{\partial \bar{q}_{i1}}\, dT = -\frac{1}{4}\left[\left(\alpha^2 p_1^4 \frac{\partial \bar{q}_1}{\partial \bar{q}_{i1}}\right)_1 + \left(\alpha^2 p_1^4 \frac{\partial \bar{q}_1}{\partial \bar{q}_{i1}}\right)_2\right]\Delta T, \tag{C1}$$

where ΔT is the time increment across the element, i.e. $\Delta T = T_2 - T_1$ [which is still calculated using the analytic formula (A3) because the traveltime is required accurately]. The terms in the integrand are needed at the entry and exit points. The slowness components and the elastic parameters can be taken in the global coordinates.

If the trapezoidal rule is inaccurate, we can use the analytic formulae given in Appendix A and Table 1. In order to evaluate the definite integrals accurately, it is important to evaluate small quantities directly, rather than numerically subtract almost equal quantities. Here it is necessary to transform variables into the local coordinates defined by the isotropic velocity gradient, and in the following it is understood that these transformations have been made and the notation refers to the transformed variables. Unit vectors defining the local axes and ray direction at the entry and exit points are illustrated in Fig. A1. The local horizontal slowness, p_1, is given by

$$p_1 = \mathbf{p}\cdot\hat{\mathbf{i}} = \frac{\hat{\mathbf{p}}_1\cdot\hat{\mathbf{i}}}{v_1} = \frac{\hat{\mathbf{p}}_2\cdot\hat{\mathbf{i}}}{v_2} \tag{C2}$$

where the subscripts 1 and 2 refer to the entry and exit points (except for the component p_1, which is constant for the ray segment). We note that p_1 is positive or negative depending on the direction of propagation relative to $\hat{\mathbf{i}}$, i.e. whether the ray is propagating in a counterclockwise or clockwise sense of rotation about the y-axis. The radius of the ray path is

$$R = \frac{1}{|p_1 \nabla v|}. \tag{C3}$$

The unit principal normal and binormal are often used in ray tracing. The unit principal normal is defined as

$$\hat{\mathbf{n}} = \frac{d\hat{\mathbf{p}}}{ds}\bigg/\left|\frac{d\hat{\mathbf{p}}}{ds}\right|. \tag{C4}$$

It lies in the osculating plane (the x–z plane for 2-D ray propagation) and points towards the centre of the ray arc. The unit binormal is defined to complete a RH axis set, i.e. $\hat{\mathbf{b}} = \hat{\mathbf{p}}\times\hat{\mathbf{n}}$. Note that if the ray curvature changes sign, i.e. p_1 changes sign, the unit principal normal changes direction relative to $\hat{\mathbf{p}}$. For ray propagation in 2-D, we find $\hat{\mathbf{b}} = \operatorname{sgn}(p_1)\hat{\mathbf{j}}$.

Relative to the centre of the ray arc, the entry and exit points are

$$\mathbf{x}_1 = -R\hat{\mathbf{n}}_1, \qquad \mathbf{x}_2 = -R\hat{\mathbf{n}}_2. \tag{C5}$$

In solving for the exit point, coordinates relative to the entry point are used to retain numerical accuracy, and

$$\Delta\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 = R(\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2) \tag{C6}$$

is calculated directly. This is used to calculate

$$\hat{\mathbf{n}}_2 = \hat{\mathbf{n}}_1 - \Delta\mathbf{x}/R \tag{C7}$$

and then the ray direction, $\hat{\mathbf{p}}_2$, at the exit point (from $\hat{\mathbf{p}} = \hat{\mathbf{n}}\times\hat{\mathbf{b}}$).

In order to evaluate the integrals in Appendix 1 and Table 1, we need trigonometrical functions of the angles ψ_1 and ψ_2. We always measure these angles in the direction of the ray propagation, i.e. counterclockwise in Fig. A1 and clockwise if p_1 is negative. Obviously, $\psi_2 > \psi_1$ but both signs are possible for the angles depending on the relationship to the turning point. From the ray geometry, we have

$$\sin\psi_i = -\hat{\mathbf{p}}_i\cdot\hat{\mathbf{k}} = -\operatorname{sgn}(p_1)\hat{\mathbf{n}}_i\cdot\hat{\mathbf{i}}, \qquad \cos\psi_i = -\hat{\mathbf{n}}_i\cdot\hat{\mathbf{k}} = \operatorname{sgn}(p_1)\hat{\mathbf{p}}_i\cdot\hat{\mathbf{i}}. \tag{C8}$$

Various differences can and must be calculated directly in terms of $\Delta\mathbf{x}$. These are

$$\begin{aligned}
&\sin\Delta\psi = \sin(\psi_2 - \psi_1) = \frac{\hat{\mathbf{p}}_1\cdot\Delta\mathbf{x}}{R} = \frac{\hat{\mathbf{p}}_2\cdot\Delta\mathbf{x}}{R}, \qquad \Delta\sin\psi = \sin\psi_2 - \sin\psi_1 = \operatorname{sgn}(p_1)\frac{\hat{\mathbf{i}}\cdot\Delta\mathbf{x}}{R},\\
&\cos\Delta\psi = \cos(\psi_2 - \psi_1) = 1 - \frac{|\Delta\mathbf{x}|^2}{2R^2}, \qquad \Delta\cos\psi = \cos\psi_2 - \cos\psi_1 = \frac{\hat{\mathbf{k}}\cdot\Delta\mathbf{x}}{R},
\end{aligned} \tag{C9}$$

and $\Delta\psi = \psi_2 - \psi_1$ should be calculated from $\sin\Delta\psi$.

In addition, various integrals contain logarithms. For short ray segments, the arguments of these logarithms are approximately unity. To retain numerical accuracy it is important to calculate the difference from unity directly, and employ this as the argument of a function which uses a power series expansion when the difference is small. Thus we define a new function, dln, which is coded separately:

$$\ln(1+x) = \operatorname{dln}(x) \simeq x \quad \text{(defined for } x > 0\text{)}. \tag{C10}$$

The traveltime integral is then

$$\Delta T = \frac{1}{|\nabla v|}[\tanh^{-1}(\sin\psi_2) - \tanh^{-1}(\sin\psi_1)] = \frac{1}{|\nabla v|}\ln\frac{(1+\sin\psi_2)\cos\psi_1}{(1+\sin\psi_1)\cos\psi_2}$$

$$= \frac{1}{|\nabla v|}\,\mathrm{dln}\frac{\cos\psi_1 - \cos\psi_2 + \sin(\psi_2-\psi_1)}{(1+\sin\psi_1)\cos\psi_2} = \frac{1}{|\nabla v|}\,\mathrm{dln}\frac{(\hat{\mathbf{p}}_1 - \hat{\mathbf{k}})\cdot\mathbf{x}}{R\hat{\mathbf{i}}\cdot\hat{\mathbf{p}}_2[\mathrm{sgn}(p_1) - \hat{\mathbf{i}}\cdot\hat{\mathbf{n}}_1]}. \tag{C11}$$

The expressions used to evaluate the analytic perturbation integrals numerically are given in Table 2.

Expressions for the numerical evaluation of the special cases in Appendix B are straightforward.

APPENDIX D: LOCAL TO GLOBAL TRANSFORMATIONS

For a model in which linear interpolation is used in each model element, the ray integrals for each segment are evaluated in local coordinates defined by the local velocity gradient and ray path (see Appendix A). Model parameters and perturbations must be transformed from the global coordinates to the local coordinates, and vice versa. In this appendix, we use the notation (x, y, z) for the global coordinates and (x', y, z') for the local coordinates. Parameters in the two systems are similarly distinguished by being primed (local) or unprimed (global). The transformation between the global and local coordinates is defined by a rotation through an angle χ about the y-axis (see Fig. 2). Thus

$$\cos\chi = \hat{\mathbf{k}}\cdot\nabla v/|\nabla v| \tag{D1}$$

where $\hat{\mathbf{k}}$ is fixed in the global system. The significant transformation elements are

$$g_{11} = g_{33} = \cos\chi = k, \qquad g_{13} = -g_{31} = -\sin\chi = -K, \tag{D2}$$

which transforms elastic parameters as

$$a'_{i'j'k'l'} = g_{i'i}g_{j'j}g_{k'k}g_{l'l}a_{ijkl}, \tag{D3}$$

where the symbols k and K are used just in this appendix for brevity. Alternatively, the required transformations of the elastic parameters can be written

$$\mathbf{q}' = \mathcal{Q}(\chi)\mathbf{q} \tag{D4}$$

where $\mathbf{q}$ is defined in (34). After some algebra, we find that the matrix is

$$\mathcal{Q} = \begin{pmatrix} k^4 & -k^3K & k^2K^2 & -kK^3 & K^4 \\ 4k^3K & k^2(k^2-3K^2) & -2kK(k^2-K^2) & K^2(3k^2-K^2) & -4kK^3 \\ 6k^2K^2 & 3kK(k^2-K^2) & k^4+K^4-4k^2K^2 & -3kK(k^2-K^2) & 6k^2K^2 \\ 4kK^3 & K^2(3k^2-K^2) & 2kK(k^2-K^2) & k^2(k^2-3K^2) & -4k^3K \\ K^4 & kK^3 & k^2K^2 & k^3K & k^4 \end{pmatrix}. \tag{D5}$$

This transformation is used to convert parameters $\mathbf{q}$ in the global system to $\mathbf{q}'$ in the local system. The same transformation applies for converting the perturbation $\delta\mathbf{q}$ into $\delta\mathbf{q}'$ and the deviations from isotropy $\delta\bar{\mathbf{q}}$ into $\delta\bar{\mathbf{q}}'$ provided we include $\mathcal{Q}_{00} = 1$ in the matrix, i.e. the isotropic velocity is unaltered. These transformations are completely standard but are included here because it is non-trivial that the combination $q_3 = 2a_{1133} + 4a_{3131}$ appears on both sides of equation (D4). It is also much more efficient to evaluate (D4) than (D3). The transformation for the global parameters from the local parameters, $\mathcal{Q}^{-1}$, is easily obtained by substituting $-K$ for K.

For the qS parameters, similar transformations are obtained when the rotation angle χ is defined by the isotropic shear velocity gradient. We use the matrices $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ to transform $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$, respectively. The matrix elements are

$$\mathcal{R} = \begin{pmatrix} (k^2-K^2)^2 & 2kK(k^2-K^2) & 4k^2K^2 \\ -4kK(k^2-K^2) & k^4+K^4 & 4kK(k^2-K^2) \\ 4k^2K^2 & -2kK(k^2-K^2) & (k^2-K^2)^2 \end{pmatrix}, \tag{D6}$$

$$\mathcal{S} = \begin{pmatrix} k^2 & -kK & K^2 \\ kK & k^2-K^2 & -kK \\ K^2 & kK & k^2 \end{pmatrix}, \tag{D7}$$

$$\mathcal{T} = \begin{pmatrix} k(k^2-K^2) & -K(k^2-K^2) & 2k^2K & -2kK^2 \\ K(k^2-K^2) & k(k^2-K^2) & 2kK^2 & 2k^2K \\ -2k^2K & 2kK^2 & k(k^2-K^2) & -K(k^2-K^2) \\ -2kK^2 & -2k^2K & K(k^2-K^2) & k(k^2-K^2) \end{pmatrix}. \tag{D8}$$

The parameter gradients used in Appendix A to evaluate the partial differentials are needed in the local coordinate system. Thus we need

$$\frac{\partial \gamma}{\partial \gamma_i} = Y_0' + Y_x' x' + Y_z' z' \tag{D9}$$

where $(x', z') = [\operatorname{sgn}(p_1')R \sin \psi, R \cos \psi]$ are the local coordinates, relative to the centre of the ray arc, used to evaluate the various integrals.

If ∇v is calculated in the global coordinates and used to evaluate the rotation χ to the local coordinates, it is readily verified that in the local system, $(\nabla v)_{x'} = 0$. The partial derivatives of the gradient can be transformed as

$$\begin{pmatrix} Y_x' \\ Y_z' \end{pmatrix} = \begin{pmatrix} k & -K \\ K & k \end{pmatrix} \begin{pmatrix} Y_x \\ Y_z \end{pmatrix}. \tag{D10}$$

The constant term can be evaluated using

$$Y_0' = -Y_x' x_j' - Y_z' z_j' \tag{D11}$$

where

$$\begin{pmatrix} x_j' \\ z_j' \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(p_1')R \sin \psi \\ R \cos \psi \end{pmatrix} + \begin{pmatrix} k & -K \\ K & k \end{pmatrix} \begin{pmatrix} x_j - x \\ z_j - z \end{pmatrix} \tag{D12}$$

and (x, z) corresponds to the point ψ.

In the special cases (Appendix B) ($p_1' = 0$ or $\nabla v = 0$), the coordinate rotation is defined by

$$\cos \chi = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}. \tag{D13}$$

The gradient terms, Y_x' and Y_z' are calculated as above. For the first special case ($\nabla v \neq 0$), the local coordinate origin is at

$$\mathbf{x}_0 = (v_2 \mathbf{x}_1 - v_1 \mathbf{x})_2 / (v_2 - v_1). \tag{D14}$$

where $v = 0$ and $x' = 0$. Hence

$$\begin{pmatrix} x_j' \\ z_j' \end{pmatrix} = \begin{pmatrix} k & -K \\ K & k \end{pmatrix} \begin{pmatrix} x_j - (v_2 x_1 - v_1 x_2)/(v_2 - v_1) \\ z_j - (v_2 z_1 - v_1 z_2)/(v_2 - v_1) \end{pmatrix}. \tag{D15}$$

In the second case ($\nabla v = 0$), we can choose the origin at $\mathbf{x}_1$ so

$$\begin{pmatrix} x_j' \\ z_j' \end{pmatrix} = \begin{pmatrix} k & -K \\ K & k \end{pmatrix} \begin{pmatrix} x_j - x_1 \\ z_j - z_1 \end{pmatrix} \tag{D16}$$

and $z_1' = 0$ and $z_2' = |\Delta \mathbf{x}|$.

APPENDIX E: TI TO GLOBAL TRANSFORMATION

The basic theory for traveltime perturbations in anisotropic media has been presented for general weak anisotropy. A tomographic experiment allows certain limited combinations of elastic parameters to be determined. The full nature of the anisotropy cannot be determined by tomographic experiments in 2-D. Having determined some of the elastic parameters, it is possible to interpret these in terms of a specific type of anisotropy, e.g. TIV, TI, orthorhombic anisotropy, etc. It is important to remember that this interpretation will include some *a priori* information. A popular model for anisotropy is a TI system because it models anisotropy due to fine layering or aligned cracks. In this appendix we consider the relationship between the elastic parameters in a generally oriented, TI system and the parameters in the traveltime perturbations, e.g. **q**, **r**, **s** and **t**.

In order to define the direction of the symmetry axis in a TI system, we use two Euler angles θ and ϕ (see Fig. E1). Consider an intermediate coordinate system rotated by an angle ϕ about the z-axis (see Figure E1). Vector components in this coordinate system (x_i, y_i, z) and the original system (x, y, z) are related by

$$\mathbf{v} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{v}_i = \begin{pmatrix} p & -P & 0 \\ P & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{v}_i. \tag{E1}$$

This coordinate system is then rotated through an angle θ about the y_i-axis (see Fig. E1). Components in this system are related to the intermediate components by

$$\mathbf{v}_i = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \mathbf{v}_h = \begin{pmatrix} h & 0 & H \\ 0 & 1 & 0 \\ -H & 0 & h \end{pmatrix} \mathbf{v}_h, \tag{E2}$$

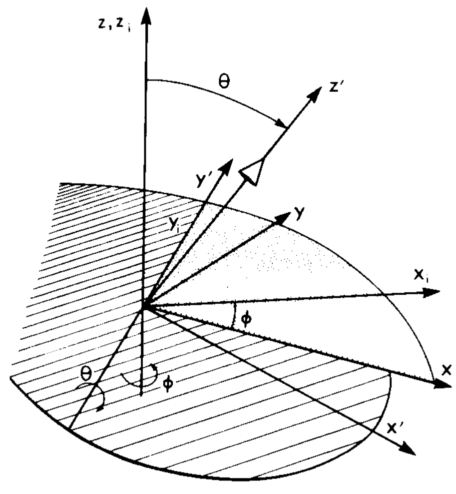

Figure E1. At each model point, the z'-axis is the symmetry axis of the TI system. It is defined by rotating the global x–z plane about the z-axis through an angle ϕ to an intermediate system (x_i, y_i, z). Then the z-axis is rotated through an angle θ about the y_i-axis, to become the z'-axis. Note these axes are drawn inverted (a rotation of π about the x-axis) compared with Fig. 1 to be in the more usual mathematical orientation.

where $\mathbf{v}_h$ is in the TI system (we use the subscript h as a TI and hexagonal system are indistinguishable elastically), and the symbols p, P, h and H have been introduced for brevity. Combining these transformations gives the overall transformation which can also be applied to the elastic parameters. In the TI system, the density normalized elastic parameters are given by

$$A = a_{1111} = a_{2222}, \quad C = a_{3333}, \quad F = a_{1133} = a_{2233}, \quad L = a_{2323} = a_{3131}, \quad N = a_{1212}, \quad A - 2N = a_{1122}, \tag{E3}$$

with the usual symmetries (just here, a_{ijkl} refers to the TI system, i.e. the subscript h is implied), where we have used a notation like Love's (1944, p. 160) (to avoid yet another set of subscripted variables), except that A, C, F, L and N are density normalized, i.e. have dimensions of velocity squared.

Applying the above transformations to the fourth-order elastic parameters, we obtain

$$\mathbf{q} = \begin{pmatrix} (p^2h^2+P^2)^2 & 2p^2H^2(p^2h^2+P^2) & p^4H^4 \\ -phH(p^2h^2+P^2) & phH(P^2+p^2(h^2-H^2)) & p^3hH^3 \\ H^2(3p^2h^2+P^2) & p^2+h^2P^2-6p^2h^2H^2 & 3p^2h^2H^2 \\ -phH^3 & phH(H^2-h^2) & ph^3H \\ H^4 & 2h^2H^2 & h^4 \end{pmatrix} \begin{pmatrix} A \\ F+2L \\ C \end{pmatrix}, \tag{E4}$$

$$\mathbf{r} = \begin{pmatrix} (h^2-p^2h^2)^2 & 4P^2H^2 & 4H^2(p^2+1)(p^2h^2+h^2+P^2) \\ 4phH(h^2-p^2H^2) & 0 & -16phH(h^2-p^2H^2) \\ 4p^2h^2H^2 & 4P^2H^2 & 4(p^2(h^2-H^2)^2+P^2) \end{pmatrix} \begin{pmatrix} A+C-2F \\ N-L \\ L \end{pmatrix}, \tag{E5}$$

$$\mathbf{s} = \begin{pmatrix} p^2P^2H^4 & h^2 & 1-4p^2P^2H^4 \\ 2phP^2H^3 & -2phH & -8phP^2H^3 \\ h^2P^2H^2 & p^2H^2 & p^2+P^2(h^2-H^2)^2 \end{pmatrix} \begin{pmatrix} A+C-2F \\ N-L \\ L \end{pmatrix}, \tag{E6}$$

$$\mathbf{t} = \begin{pmatrix} pPH^2(h^2-p^2H^2) & 2pPH^2 & -4pPH^2(h^2-p^2H^2) \\ hPH(h^2-p^2H^2) & -2hPH & -4hPH(h^2-p^2H^2) \\ 2p^2hPH^3 & -2hPH & -8p^2hPH^3 \\ 2ph^2PH^2 & -2pPH^2 & -8ph^2PH^2 \end{pmatrix} \begin{pmatrix} A+C-2F \\ N-L \\ L \end{pmatrix}. \tag{E7}$$

Note that the matrices depend only on θ and ϕ. Again these transformations are straightforward but it is satisfying that combinations of parameters arise consistently on both sides of the equations.

If the anisotropy is interpreted as TI, then it is attractive to express the anisotropy in the TI system in terms of the weak anisotropy parameters of Thomsen (1986). This is consistent with the assumption throughout this paper that the anisotropy is weak. Thomsen (1986) has shown that the wave velocities in weakly anisotropic TI material can be written

$$v_P(\theta) = \alpha(1 + \delta \sin^2\theta \cos^2\theta + \epsilon \sin^4\theta), \qquad v_{SV}(\theta) = \beta_0\left[1 + \frac{\alpha_0^2}{\beta_0^2}(\epsilon - \delta)\sin^2\theta\cos^2\theta\right], \qquad v_{SH}(\theta) = \beta_0(1 + \gamma\sin^2\theta), \tag{E8}$$

where just here, θ is the angle from the symmetry axis and

$$\alpha_0 = C^{1/2}, \qquad \beta_0 = L^{1/2}, \qquad \epsilon = (A - C)/2C,$$

$$\delta = [(F + L)^2 - (C - L)^2]/2C(C - L) \simeq ((F + 2L) - C)/2C, \qquad \frac{\alpha_0^2}{\beta_0^2}(\epsilon - \delta) \simeq ((A + C - 2F) - 4L)/2L, \qquad \gamma = (N - L)/2L. \tag{E9}$$

The qP-wave velocity depends on α_0, δ and ϵ (E8), which in turn depend on only A, C and $(F - 2L)$ (E9), which depend on only $\mathbf{q}$ (E4), either in global or local coordinates (D4 and D5). While physically the dependence is to be expected, the algebraic relationships are non-trivial. Similarly the qS-wave velocities depend on β_0, γ and $\alpha_0^2(\epsilon - \delta)/\beta_0^2$ (E8), which depend only on $(A + C - 2F)$, $(N - L)$ and L (E9), which depend only on $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$ (E5, E6 and E7) in global or local coordinates (D6, D7 and D8).