Null-Space Smoothing Of Tomographic Images Using TV Norm Minimization

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Abstract

Smoothing is desirable in tomographic imaging when it reduces the effects of noise in the data and is undesirable when it smooths a small feature such as a tumor or a lesion so much that they become undetectable. Linear algebra can be used to identify a significant problem associated with reconstruction from incomplete data set; namely, the rank of the resulting system matrix is less then full. To maximize its benefit and to minimize its harm, when smoothing is used in this case, it seems desirable to give more credence to the row-space component of the reconstruction than the null-space because the tomographic data contains only information about the row-space component of the object. The **objective** of the work presented here is to propose and demonstrate a method, which is called null-space smoothing, for achieving this. **Methodology**. Using computer generated data, ART is used to reconstruct the row-space component of the Shepp and Logan phantom. By solving a convex optimization problem, an image in the null-space was added to the reconstruction so that the resulting image had the minimum TV norm possible; thus, leaving the row-space component unchanged. It is **concluded** that although null-space smoothing can produce smooth images with an unchanged row-space component, more work needs to be done in the future to demonstrate its usefulness with real data.

Introduction

Tomographic reconstruction from incomplete data sets arise when reduced sampling is used in an effort to reduce dose as well as in tomosynthesis and cone-beam tomography. Linear algebra can be used to identify a significant problem associated with reconstruction from incomplete data sets. If an finite-dimensional linear model is used to describe the formation of the tomographic data, the reconstruction process can be described as finding the image $\mathbf{x}$ (represented as a vector rather than a 2 or 3-dimensional array) that satisfies the matrix equation

$$\mathsf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

where A is the "system matrix," which relates the object to be reconstructed to the vector $\mathbf{b}$, which is the tomographic data represented as a vector. In reconstruction from incomplete data sets, the rank of the system will be less then full. As a consequence, the null-space of the matrix will be non-trivial. From linear algebra, it is known that any vector $\mathbf{x}$ can be decomposed into the sum of two vectors as

$$\mathbf{x} = \mathbf{x}_{\text{rowspace}} + \mathbf{x}_{\text{nullspace}} \tag{2}$$

where $\mathbf{x}_{\text{rowspace}}$ is an unique vector in the row-space of the matrix A and $\mathbf{x}_{\text{nullspace}}$ is an unique vector in the null-space of the matrix A.

It can be seen that tomographic data produced by this system matrix does not contain information about the null-space of the object being reconstructed. Let this object be denoted by the vector $\mathbf{x}_{\text{true}}$ and let the vector $\mathbf{b}_{\text{true}}$ be defined by the equation

$$\mathsf{A}\mathbf{x}_{\text{true}} = \mathbf{b}_{\text{true}} \tag{3}$$

Letting $\mathbf{u}$ be a vector in the null-space, we have

$$\mathsf{A}\left(\mathbf{x}_{\text{true}} + \mathbf{u}\right) = \mathsf{A}\left(\mathbf{x}_{\text{true}}\right) = \mathbf{b}_{\text{true}} \tag{4}$$

It is seen in the above equation that adding any image in the null-space of the system matrix to $\mathbf{x}_{\text{true}}$ does not change the measurements, which implies that the measurements do not contain any information about the null-space component of the object being scanned.

Smoothing in tomographic reconstruction is both a blessing and a curse. Smoothing is desirable when it reduces the effects of noise in the data and/or reduces the streaks at result. On the other hand, smoothing could be harmful. A small feature such as a tumor or a lesion, which are being sought after, could be smoothed so much that they are not detectable. To maximize its benefit and to minimize its harm, when smoothing is incorporated in tomographic reconstruction it seems desirable to give more credence to the row-space component of the reconstruction than the null-space because the data contains only information about the row-space component of the object. To achieve this, a new method, which is called null-space smoothing, is proposed here that smoothes only the null-space component of a given reconstructed image while leaving the row-space unchanged.

Methods

It is known that ART will produce, upon convergence, the row-space component of the object when the initial image is the zero vector and the measurement vector is $\mathbf{b}_{\text{true}}$. This is represented symbolically as:

$$\mathbf{x}_{\text{true\_rowspace}} = \text{ART}\Big(\mathbf{x}_{\text{initial}} = \mathbf{0}\,,\ \mathbf{b} = \mathbf{b}_{\text{true}}\Big) \tag{5}$$

Since this image is in the row-space, the null-space component of this image is the zero vector. The Shepp and Logan phantom is used to demonstrate the reconstruction of the row-space of the object. To simplify the displays of the images, the gray scale values of the phantom have been scaled so that the maximum value is one. This 128 x 128 phantom is shown in Figure 3. Using equation 1, MatLab code is used to produced data that is consistent with the phantom. In calculating the tomographic data, 65 parallel views of the objects are taken with 128 projections per view. The nullity of the system matrix is 9106. The value of the relaxation parameter used is 0.75 and 1111 iterations are used. The row-space component reconstructed using this approach is shown in Figure 2a.

It can be argued that ART will produce, upon convergence, the null-space component of the true object when the initial image is the true object and the measurement vector is the zero vector. This is represented symbolically as:

$$\mathbf{x}_{\text{true\_nullspace}} = \text{ART}\Big(\mathbf{x}_{\text{initial}} = \mathbf{x}_{\text{true}}\,,\ \mathbf{b} = \mathbf{0}\Big) \tag{6}$$

The null-space component reconstructed using this approach is shown in Figure 2b.

To add a non-zero null-space component to a reconstruction without changing its row-space component the following convex optimization problem is proposed: Find the vector $\mathbf{u}$ that minimizes the cost function

$$\text{minimize}\ \left\| \mathbf{x}_{\text{art\_rowspace}} + \mathsf{Z}\mathbf{u} \right\|_{\text{TV}} \tag{7}$$

where $\mathbf{x}_{\text{art\_rowspace}}$ is the row-space component of the object calculated in the previous section and the columns of the matrix Z form a basis of the null-space of the system matrix. The image, denoted by $\mathbf{x}_{\text{art\_plus\_tv}}$ here, that has a non-changed row-space component with a null-space component added to it such that the resulting image has a minimum TV norm is calculated as

$$\mathbf{x}_{\text{art\_plus\_tv}} = \mathbf{x}_{\text{art\_rowspace}} + \mathsf{Z}\mathbf{u}^{\dagger} \tag{8}$$

where $\mathbf{u}^{\dagger}$ is the minimizer of the convex problem. This method for adding a null-space component to $\mathbf{x}_{\text{art\_rowspace}}$ is denoted was the ART+TV method here. To solve the previously mentioned optimization problem, CVX, which is a Matlab-based package for specifying and solving convex programs was used. The resulting image is shown in Figure 2c.

The null-space component of the $\mathbf{x}_{\text{art\_plus\_tv}}$ image, which denoted by $\mathbf{x}_{\text{art\_plus\_tv\_nullspace}}$ here, is calculated in an way that is analogous to the one used to calculate the previous null-space image; namely,

$$\mathbf{x}_{\text{art\_plus\_tv\_nullspace}} = \text{ART}\Big(\mathbf{x}_{\text{initial}} = \mathbf{x}_{\text{art\_plus\_tv}}\,,\ \mathbf{b} = \mathbf{0}\Big) \tag{9}$$

This image is shown in Figure 2d.

Results

A casual comparison of the ART+TV image shown in Figure 2c with the phantom reveals that ART+TV image looks much similar to the phantom than the ART Row-Space image does. In particular, the three "tumors" at the bottom of the phantom can be seen in the ART+TV image but cannot be seen in the ART Row-Space image. The black and gray spots seen in the ART Row-Space image in the region that should be totally gray have been eliminated in the ART+TV image except for the top and bottom gray regions. Upon close inspection, however, vertical strips can be seen in the ART+TV image. The ART+TV Null-Space image in Figure 2d shows the null-space component that is in fact added to the ART Row-Space image to produce a smoother image.

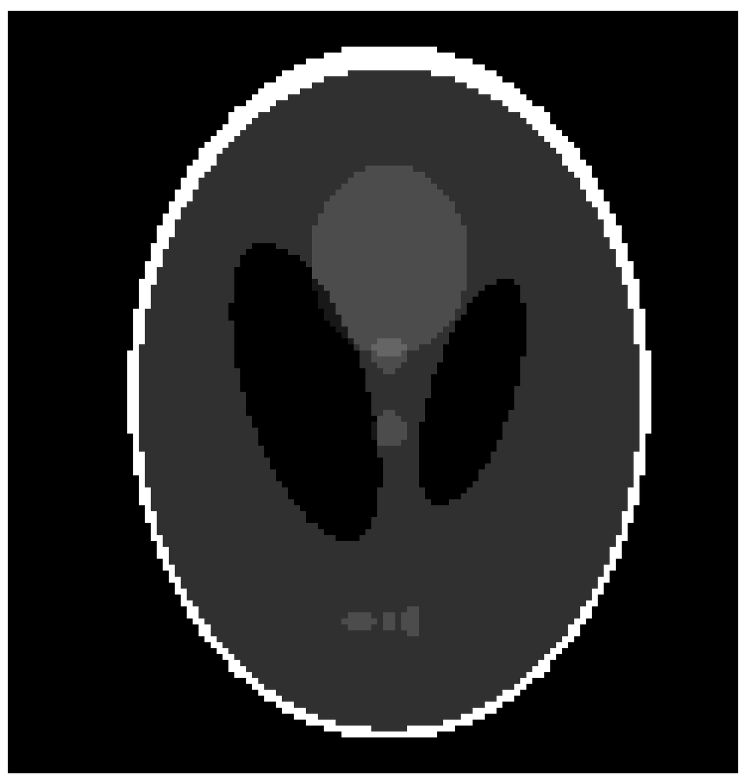
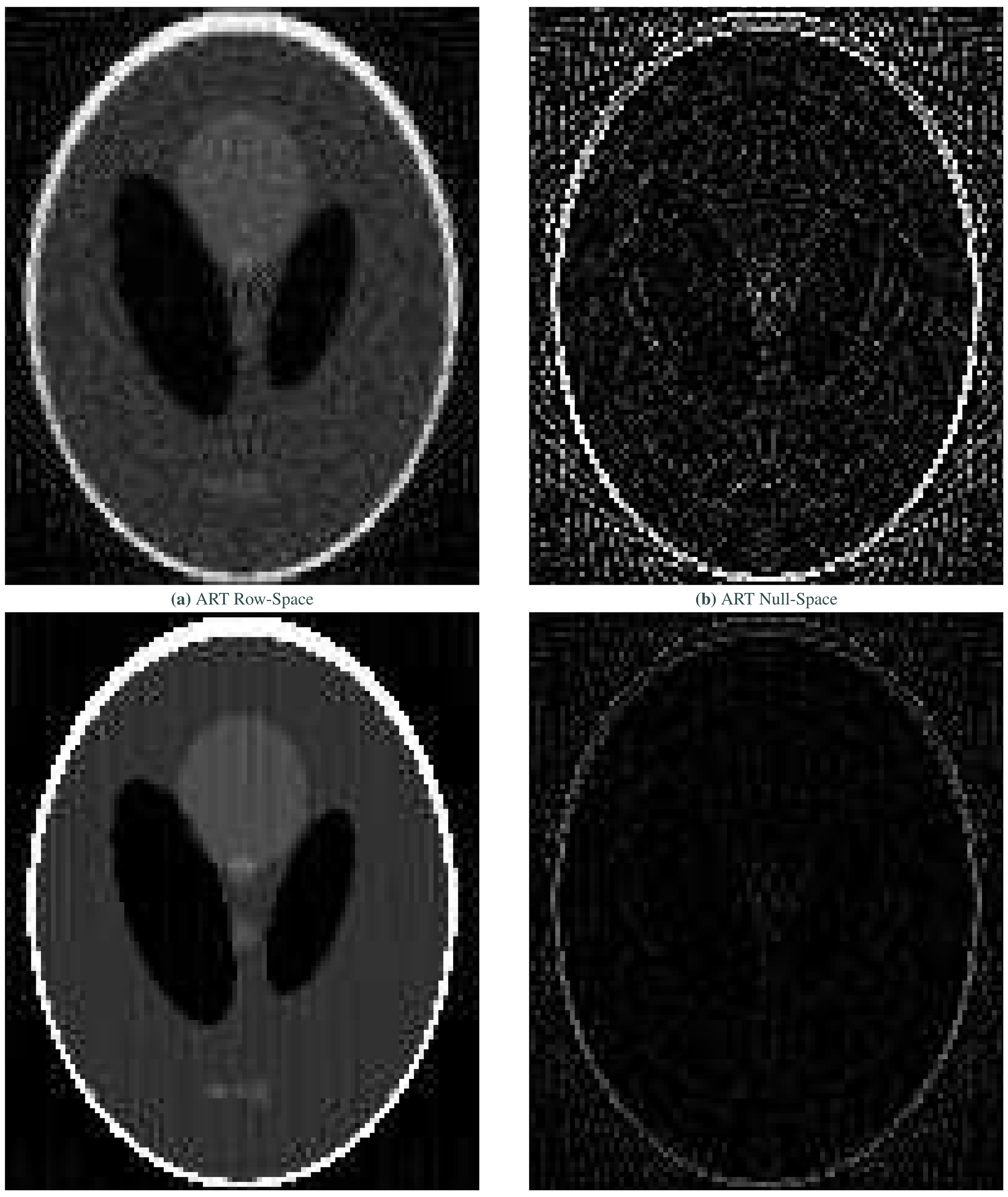


Figure 1: Phantom



(a) ART Row-Space

(b) ART Null-Space

(c) ART+TV

(d) ART+TV Null-Space

Figure 2: ART Row-Space, ART Null-Space, ART+TV and ART+TV Null-Space

Discussion

The images in Figure 2 illustrate the potential of adding a null-space component to a reconstruction to smooth it without changing its row-space component. It is known that he row-space component produced in ART can be controlled by varying the number of iterations and the relaxation parameter used. This combined with null-space smoothing, makes it possible to control both the row-space component and the null-space component of the reconstruction independently.

Conclusions

Using data that was consistent with a normalize Shepp and Logan mathematical phantom, simulations demonstrated that is possible to add a null-space component to a reconstruction to produce a smoother image without changing the row-space component of the reconstruction. Although the results presented here are encouraging, more work needs to be done in the future to demonstrate the usefulness in null-space smoothing with real data.