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Global and Planetary Change 11 (1995) 57–70

GLOBAL AND PLANETARY CHANGE

Resolution of ground temperature histories inverted from borehole temperature data

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Received 14 February 1994; accepted after revision 11 January 1995

Abstract

Inversion methods have been used to determine ground temperature history from borehole temperature data. This paper examines the resolution of an inversion method based on singular value decomposition. The method includes in the constraints the rate of change of subsurface temperature if available. The response of the Earth to a unit pulselike change in surface temperature and the model resolution were calculated to determine the details of ground temperature variations obtainable from geothermal data. The model parameters estimated by the inversion correspond to averages over time intervals of increasing length with time. For a unit impulse temperature change occurring at time t before present, the estimated ground temperature history shows the event "spread" over at least 60% the time of occurrence. In practice, the resolution is not improved by increasing the sampling frequency of the data. Likewise, the simultaneous inversion of the temperature profile and of its time derivative does not significantly improve the resolution. Selection of a particular parameterization may improve resolution of the chosen parameters, but not the actual resolution since the parameters represent longer time averages. Temperature data from the Minchin Lake borehole in western Ontario, which was logged twice over 15 years, have been inverted to illustrate these conclusions.

1. Introduction

It was recognized early in this century that climate related ground temperature changes propagate and are recorded in the subsurface (Lane, 1923). Hotchkiss and Ingersoll (1934) were the first to attempt and infer the timing of the last deglaciation from geothermal data. It is common practice to make an adjustment to heat flow density (HFD) measurements made in northern regions to account for the climatic changes of recent glaciations (e.g. Birch, 1948; Jessop, 1971). During the last decades, several studies have inferred very recent variations of the ground temperature from borehole data (e.g. Beck and Judge, 1969; Cermak, 1971; Beck, 1982; Shen and Beck, 1983; Vasseur et al., 1983). Following Vasseur et al. (1983), several authors have used inversion methods to infer ground temperature histories from borehole data (Shen and Beck, 1991; Wang, 1992; Mareschal and Beltrami, 1992; see also Shen et al., 1992, and Beck et al., 1992).

The interest of the scientific community for bore-

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SSDI 0921-8181(95)00002-X

hole temperature data as a record of climate change was aroused by the paper of Lachenbruch and Marshall (1986) who suggested that the warming inferred from temperature logs in permafrost on Alaska's north coast could be an early sign of the enhanced greenhouse effect predicted by general circulation models. Since the publication of this paper, several studies of borehole temperature data have detected recent (100–500 years) climatic variations in different parts of the world. Several papers and further references can be found in the special volume of Global and Planetary Change devoted to this topic (Lewis, 1992).

Most of the studies present ground temperature histories with long term (100 years) trends that are generally consistent with meteorological and historical records. On the other hand, the high frequency variations are not recovered and the resolution of inversion appears limited. The main objective of this paper is to evaluate and discuss the resolution of model parameters estimated by singular value decomposition (SVD). A secondary objective is to determine the quality and quantity of data required to estimate model parameters. Finally, the paper will discuss whether it is possible to improve the determination of ground temperature histories (GTHs) from geothermal data. The response of the Earth and model resolution for a unit impulsive change in surface temperature are calculated to determine the finest details of ground temperature variations that can be inverted from these data. The study shows that the pulse is spread over an interval about 60% the time of occurrence and that the resolution is not improved by finer sampling of the data. Likewise, an inversion scheme including the rate of change of temperature shows that repeat measurements of borehole temperature does not improve the model resolution. The analysis of data from the Minchin Lake borehole in western Ontario illustrates these conclusions.

2. Theoretical framework

For a homogeneous, isotropic, source-free half space, the temperature perturbation due to a time varying ground surface temperature, is a solution of the heat diffusion equation in one dimension with initial and boundary conditions (Carslaw and Jaeger, 1959):

$$\kappa \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial t} \tag{1}$$

where κ is the thermal diffusivity of the rock, z is depth (positive downwards), and t is time. The use of the one-dimensional equation is valid if long term surface temperature changes have a spatial wavelength much larger than the depth to which they penetrate.

The temperature at depth z, $T(z)$, is the superposition of the equilibrium temperature and of $T_t(z)$ the temperature perturbation arising from the time varying surface temperature. i.e.

$$T(z) = T_0 + q_0 R(z) - HM(z) + T_t(z) \tag{2}$$

where T_0 is the equilibrium ground temperature, q_0 is the equilibrium surface heat flow density, $R(z)$ is the thermal resistance (Bullard, 1939) from the surface to the depth z, H is the rate of heat generation by radioactive decay and M is the moment:

$$R(z) = \int_0^z \frac{dz'}{K(z')} \tag{3a}$$

$$M(z) = \int_0^z \frac{z' dz'}{K(z')} \tag{3b}$$

where $K(z)$ is the thermal conductivity measured in core samples and/or estimated from the lithology. If there is no systematic variation of thermal conductivity and if the rate of heat generation is negligible, Eq. 2 reduces to:

$$T(z) = T_0 + T_t(z) + q_0 R(z) \tag{4}$$

This is the case encountered for the great majority of the borehole temperature logs analyzed by the authors.

The present temperature perturbation $T_t(z)$ in a semi-infinite solid with past surface temperature $T_0(t)$, where t is time before present, is given by (e.g. Vasseur et al., 1983):

$$T_t(z) = \frac{z}{2\sqrt{\pi\kappa}} \int_0^\infty T_0(t) t^{-3/2} \exp\left(-\frac{z^2}{4\kappa t}\right) dt \tag{5}$$

Eq. 5 can be integrated for various functions describing the surface temperature history. For an instantaneous change in ground surface temperature ΔT at time t before present, it yields (Carslaw and Jaeger, 1959)

$$T_t(z) = \Delta T \operatorname{erfc} \frac{z}{2\sqrt{\kappa t}} \tag{6}$$

where erfc is the complementary error function.

Since short period variations are attenuated rapidly with depth (e.g. Carslaw and Jaeger, 1959), the surface temperature history can be approximated by the average surface temperature over a series of time intervals that are often assumed of equal duration:

$$T_0(t) = T_k \quad (k-1)\Delta \le t < k\Delta \tag{7}$$

and $k = 1,..K$. The assumption of equal duration is not necessary; it has been introduced only for the sake of simplicity.

Eq. 2 can then be written as:

$$\Theta_j = A_{jl} X_l \tag{8}$$

where Θ_j are the J values of temperature measured at depth z_j, corrected for heat production between the surface and that depth if necessary. X_l is a vector containing the model parameters, i.e. the reference surface temperature T_0 and heat flow q_0 and the averaged past surface temperatures $\mathbf{X} = \{T_0, q_0, T_1, ..., T_K\}$, and A_{jl} is a matrix containing 1's in the first row, the thermal resistance to depth z_j in the second row, $R(z_j)$, and, in the K following rows, the J elements formed by evaluating the difference between complementary error functions at depth z_j and times t_{k-1} and t_k:

$$A_{jk+2} = \operatorname{erfc}\left\{\frac{z_j}{2\sqrt{\kappa t_{k-1}}}\right\} - \operatorname{erfc}\left\{\frac{z_j}{2\sqrt{\kappa t_k}}\right\} \tag{9}$$

This yields an usually under and overdetermined system of linear equations. Mareschal and Beltrami (1992) have used the singular value decomposition method (Lanczos, 1961) to obtain a generalized solution to this system.

If two temperature-depth logs of the same borehole have been measured at times τ_1 and τ_2, the additional information may be introduced as another constraint in an attempt to improve the resolution of the GTH inferred from the data.

The temperatures at depth z and at time, τ_1, $t_1(z)$, and at time, τ_2, $t_2(z)$, are given by:

$$T_2(z) - T_1(z) \cong \frac{\partial T_t}{\partial t}(\tau_2 - \tau_1) \tag{10a}$$

$$T_2(z) = T_0 + q_0 R(z) + T_t(z) \tag{10b}$$

where the time derivative of T_t is evaluated at time τ_1. It is implicitly assumed that $\tau_1 \gg (\tau_2 - \tau_1)$.

For a unit stepwise change in surface temperature occurring at time t before present, the time derivative of the subsurface temperature perturbation time is given by:

$$\frac{\partial T(z)}{\partial t} = \frac{-z}{2\sqrt{\pi \kappa t^3}} \exp\left\{-\frac{z^2}{4\kappa T}\right\} \tag{11}$$

The time rate of change of temperature can be included in the inversion by substituting by evaluating the time derivatives (Eq. 10a) for the ground temperature history (7). For a GTH model consisting of a series of independent temperature step changes, the resulting system of linear equations can be written as: $\Theta_j = A_{jl} X_l$, where X_l is a vector containing the unknowns $\{T_0, q_0, T_1, ...T_K\}$ Θ_j, is a vector containing the measured temperature perturbations and their time derivatives, and A_{jl} is now a $2J \times (K+2)$ matrix whose J first rows are identical to Eq. 9 and the J following rows contain 0's in the first two columns and in the next K columns the differences between time derivatives at depth z_j and time t_k and t_{k-1}:

$$A_{J+j,2+k} = \frac{z_j}{2\sqrt{\pi\kappa t_k^3}} \exp\left\{-\frac{z_j^2}{4\kappa t_k}\right\} - \frac{z_j}{2\sqrt{\pi\kappa t_{k-1}^3}} \exp\left\{-\frac{z_j^2}{4\kappa t_{k-1}}\right\} \tag{12}$$

This formulation assumes that the rate of change of the temperature perturbation is constant over the time interval between logs. This assumption was verified by evaluation of the second derivative of the temperature perturbation which is very small for the times and depth range considered in most practical applications. It also assumes that changes in the average surface temperature between measurements have not affected the temperature profiles. This assumption is also verified in most applications below

20 m with time between repeat measurements on the order of 10–20 years.

The system of linear equations can formally be written as:

$$AX = \Theta \tag{13}$$

where **X** is a column vector containing the unknown model parameters: $\mathbf{X} = (T_0, q_0, T_1, T_2, ..., T_K)^T$ and Θ is a column vector containing the measured temperature (and its time derivative) at each depth $\Theta = (\Theta_1, \Theta_2 ..., \Theta_J)$ (or $\Theta = (\Theta_1, ..., \Theta_J, \dot{\Theta}_1, ... \dot{\Theta}_J)$) where the dot denotes a time derivative) and **A** is a matrix of dimensions $J \times (K+2)$ or $2J \times (K+2)$.

It has been shown (Lanczos, 1961) that a matrix **A** $(N \times M)$ can be decomposed as:

$$A = U \Lambda V^T \tag{14}$$

where Λ is a $(N \times M)$ diagonal matrix whose elements are the non-zero singular values $\lambda_{(r)} r = 1, ..., R$ (R is the rank of A), **U** is an $(N \times N)$ orthonormal matrix of eigenvectors spanning data space and **V** is an $(M \times M)$ orthogonal matrix of eigenvectors spanning model space, i.e.

$$V = \{\overline{V}_1, ... \overline{V}_M\} \tag{15}$$

and

$$U = \{\overline{U}_1, ... \overline{U}_N\} \tag{16}$$

If the rank of A, $R < M$, the system is underdetermined; if $R < N$, it is also overconstrained. In this case, the system does not have an exact and unique solution. However, a generalized solution to Eq. (13) is given by (Lanczos, 1961):

$$X = V \Lambda^{-1} U^T \Theta \tag{17}$$

where Λ^{-1} is an $M \times N$ diagonal matrix whose elements are $1/\lambda_{(r)}$, for the $R \lambda_{(r)} \neq 0$, and zero if for the nul singular values. If the system is underdetermined, the solution is non unique and any linear combination of the M–R vectors of **V** corresponding to a zero singular value can be added to the solution.

The parameters given by the generalized solution (17) can be written explicitly as:

$$X_i = \Theta_j \left(\frac{v_{ir} u_{rj}}{\lambda_{(r)}} \right) \tag{18}$$

with $r = 1, .. R$, $i = 1, ... M$, and $j = 1, ... N$ and summation is performed over repeated indices.

Eq. 18 shows that the determination of model parameters requires dividing the data by the singular values. Any error in the data will be multiplied by $1/\lambda_{(r)}$ and be amplified for the very small singular values. In order to reduce the impact of errors on the solution, it is necessary to eliminate the singular values which are smaller than a cutoff value. Although this solution is no longer "true", the singular value decomposition selects the linear combination of model parameters that is best constrained by the data. The solution obtained by retaining only a few singular values can reproduce the gross features of a GTH (Menke, 1989).

Eq. 18 also indicates that the largest contribution to the standard error in the estimated model parameters is that of the eigenvector in **V** associated with the smallest retained singular value. The variance of the estimated model parameters can be written as (e.g. Jackson, 1972):

$$\sigma_m^2 = \sum_{r=1}^{R} \frac{V_{rm}^2}{\lambda_{(r)}^2} \tag{19}$$

The variance represents the amplification of the measurement errors in the solution (i.e. it is the standard error on the estimated parameter corresponding to a 1° K standard deviation in the temperature measurements). The order of magnitude of the variance is inversely proportional to the smallest singular value. For a 10^{-2} singular value cutoff, the error on estimated parameters is thus on the order of 100 times the measurement error. The parameters that have a large component in the last eigenvector are more affected by errors than those that are weakly represented in the eigenvectors with low singular values. For a lower singular value cutoff, these parameters (corresponding to larger times) are more strongly represented and better resolved, but the error level becomes unacceptable (i.e. for a cutoff of 10^{-4} or 10^{-6}, the inversion amplifies 10^4 or 10^6 times the measurements errors).

The model resolution matrix **P** and the data resolution matrix **Q** are defined as:

$$\boldsymbol{P} = \boldsymbol{V}\boldsymbol{V}^T \tag{20a}$$

$$\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{U}^T \tag{20b}$$

where only the vectors corresponding to non zero singular values are retained. The matrices **P** and **Q** depend only on the data kernel of **A**, i.e. the experimental geometry and the assumptions applied to the model. They are independent of the actual values of the data. The matrix **P** relates the parameters of an arbitrary model to the parameters that would be obtained by inversion of the corresponding data (e.g. Menke, 1989). $\mathbf{T}^{est} = \mathbf{P}\mathbf{T}^{true}$ The model parameters are perfectly resolved if **P** is an identity matrix; if **P** is not an identity matrix, the estimated model parameters are averages of the true model parameters. Similarly, the matrix **Q** relates the observed data to those that the inverted model parameters would produce $\Theta^{pred} = \mathbf{Q}\Theta^{true}$. If **Q** is not the identity matrix, a particular datum is not uniquely determined but appears as average of neighboring data.

If all the vectors in **V** were retained, the model resolution matrix would be the identity matrix. The resolution is limited because the rank of the system is less than the dimension of model space; it is further reduced by the elimination of the small singular values. The inclusion of more singular values improves the resolution, but it also increases the variance of the estimated model parameters as shown by Eq. 19. This is the common trade off between resolution and stability of a solution. The choice of one at the expense of the other depends on the quality of the data as well as on the nature of the problem.

3. Sampling interval and resolution

In order to determine if the resolution can be improved by closer temperature measurements in a borehole, model resolution matrices were calculated for a fixed set of model parameters and for different sampling intervals of the temperature depth profiles. The model parameters consist of a series of fifty 20-yr temperature averages. The singular value spectrum and the model resolution matrix were calculated for temperature-depth profiles sampled at 1, 5, 10, 15 and 20 m from the surface down to 600 m. The singular value spectrum for these geometries is given in Table 1. The first two eigenvectors corresponding to these singular values represent the reference surface temperature and heat flow. These parameters are extremely well constrained, almost independently of the other parameters, since the singular values are large. Although the singular values increase with increasing sampling rate, the important

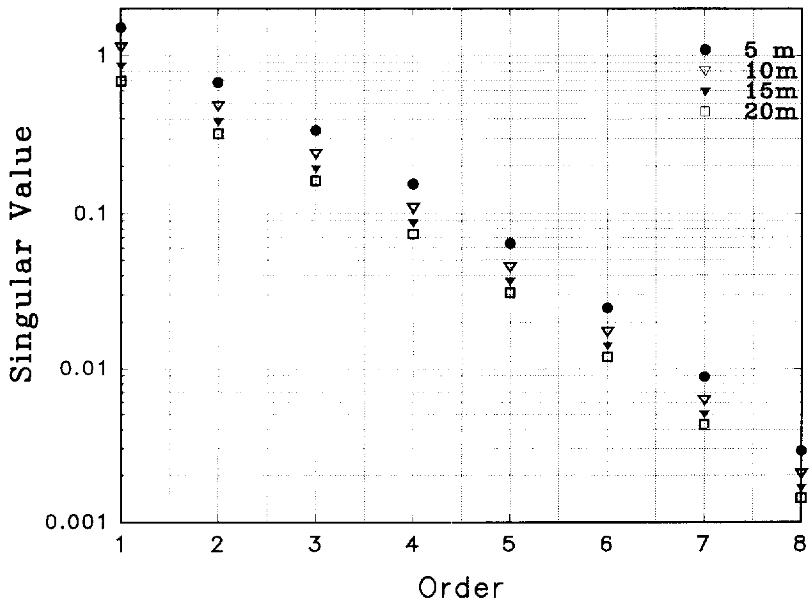

Fig. 1. Singular values from 3 to 10 in semilogarithmic scale for a 600-m borehole sampled at different intervals. The model parameters are the reference surface temperature and heat flow and 50 values of surface temperature averaged over 20 years.

parameter is the ratio of each singular value to the largest one (i.e. the condition number). Fig. 1 shows the singular values from 3 to 10 on semilogarithmic scale. The distance between singular values, i.e. their ratio, remains almost the same regardless of sampling rate. Therefore, increasing the sampling rate does not improve the resolution of the parameters of a GTH since the number of singular values that can be retained is not increased. This point is further demonstrated by Fig. 2 that compares slices of the model resolution matrix at 100 years for a singular value cutoff of 0.025 and different sampling rates with the same slices for a cutoff of 0.001; (for most of the field data, the lowest singular value retained is 0.025). The resolution matrix slices show how one parameter of the model (i.e. the temperature 100 years ago) is spread over neighboring parameters. Perfect resolution would yield 1 for the temperature 100 years ago, and zero for all the other parameters. The parameter appears spread over approximately 50–60 years regardless of sampling rate. This discussion applies for a given set of model parameters. Better resolution can be achieved for the most recent part of the GTH with a higher sampling rate in the shallower part of the borehole. In general, the resolution can be improved only by decreasing the cutoff and including additional singular values in the inversion. This requires the noise to signal ratio to be low.

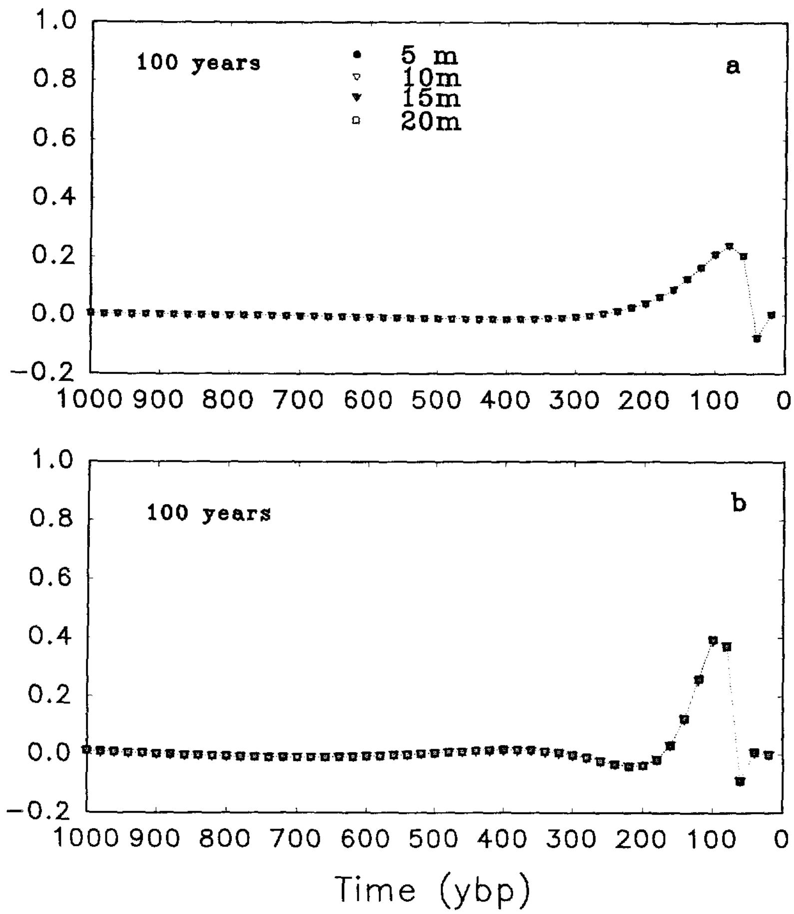

Fig. 2. Slices of the model resolution matrices for four (5-, 10-, 15- and 20-m) sampling intervals. There is no improvement in the resolution for an event which occurred 100 yr B.P. (or any other time) by sampling data more closely. The singular value cutoff was set at 0.025 in (a) and 0.001 in (b).

Table 1
Singular value spectrum for 600-m temperature log sampled at different intervals

1-m sampling	5-m sampling	10-m sampling	20-m sampling
8517	3866	2784	2041
12.54	5.587	3.933	2.76
3.374	1.429	0.938	0.5624
1.051	0.4623	0.3186	0.2094
0.4057	0.1802	0.1258	0.0849
0.1576	0.0705	0.0497	0.0342
0.0583	0.0263	0.0187	0.0131
0.0199	0.00902	0.00646	0.0046
0.0061	0.00278	0.00201	0.00145
0.00165	0.000762	0.000556	0.00041
0.00037	0.000184	0.000136	0.00010

The model parameters consist of reference surface temperature and heat-flow and 50 values of the ground temperature averaged over 20 years. The first two eigenvectors in model space correspond to reference surface temperature and heat flow. These parameters with large singular values are well resolved.

Table 2
Singular value spectrum for inversion of the single synthetic temperature log and two synthetic logs inversion "measured" after a 40 year period

Temperature and time derivative	Temperature log only
2784	2784
3.933	3.933
0.939	0.938
0.319	0.3186
0.126	0.1258
0.0498	0.0497
0.0187	0.0187
0.00649	0.00646
0.00202	0.00201
0.000556	0.00056
0.000137	0.000136

The models consist of 50 20-yr step temperature changes; only first 11 singular values are shown. The joint inversion of both logs takes into account the time rate of change of the temperature perturbation over the 40-yr period.

Mareschal and Beltrami (1992) and Clow (1992) had reached similar conclusions and found that finer sampling does not improve the resolution.

In order to show the resolving power of the data, the model resolution was calculated for 500, 400, 300, 200, and 100 yr B.P., assuming that temperature profiles were sampled at 10-m intervals down to 600 m. The model parameters consist of 50 values of

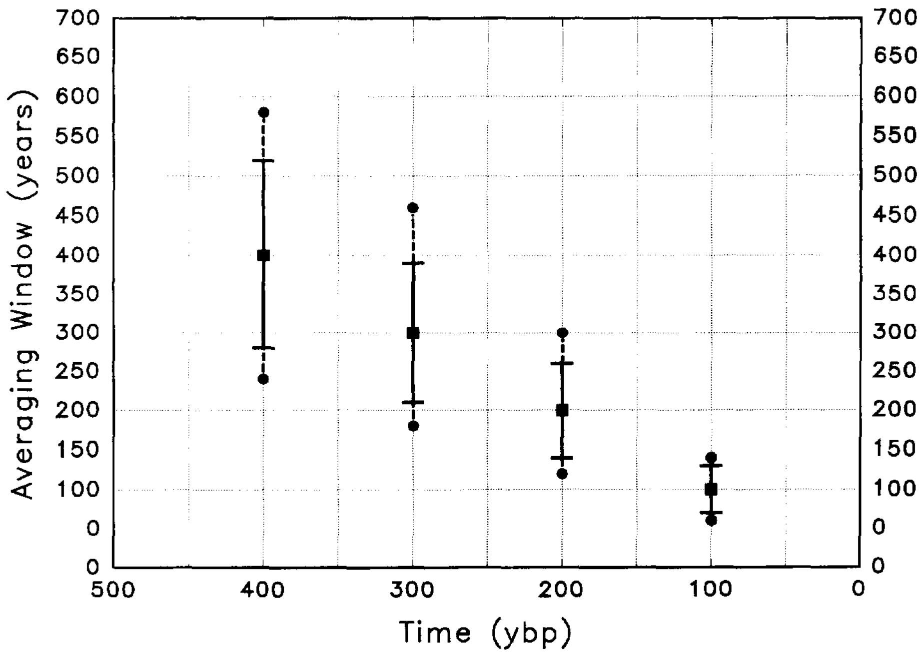

Fig. 3. Response of the Earth to a unit step temperature change. The vertical bars show the "spread" of the signal recovered by inversion at the indicated times. Bars delimited by circles correspond to the inversion with a 0.001 singular value cutoff, and inner bars are for a 0.01 cutoff (see text).

average temperature for 20 years intervals. The model resolution represents the parameters that would be obtained by inversion if the surface temperature history consists only of a single 1° C increase during a 20 years interval. The singular value cutoff set at 0.001 which is lower than the value retained in most practical applications. The estimated temperature history was integrated within limits that were varied until $\int_{t_1}^{t_2} \Delta T \mathrm{d}t \sim 1$. This procedure determines the minimum value for $(t_2 - t_1)$. The limits of integration obtained are summarized in Fig. 3. The inner bars in the vertical axis represent the averaging intervals for the estimated parameters at the given time, that is, the time span over which the recovered signal is spread. The figure shows that the interval is almost 60% of the time of occurrence. This estimate is an upper limit on time resolution since the singular value cutoff is too low to be practical with noisy data. The outer circles represent the analogous situation but for a larger singular value cutoff. Note that the spread increases and the symmetry of the spread breaks.

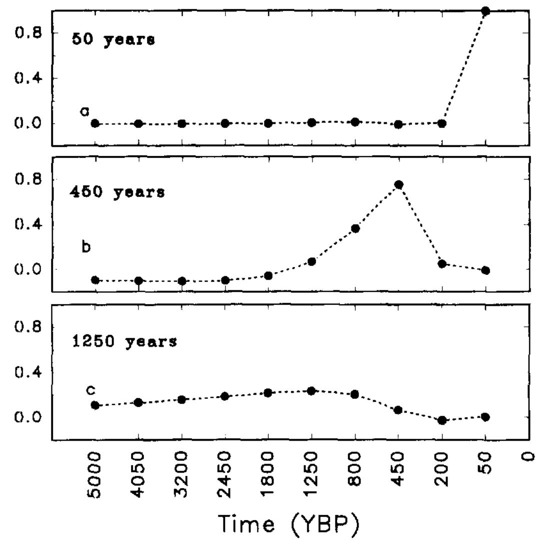

Fig. 5. Slices of the model resolution matrix for a different parameterization of the GTH which is averaged over time intervals set at $50 l^2$ yr (see text). The slices represent the resolution for 50, 450 and 1250 yr B.P. The sizes of the averaging intervals can be inferred from approximately half the width of the peak on the time axis. The singular value cutoff was set at 0.025, and the number of singular values retained is five. The scale on the time axis is varied according to parameter order.

4. Repeat temperature measurements

It has been suggested that the resolution of the ground temperature signal may be improved by measurements of temperature depth profiles from the same borehole repeated over suitable time intervals

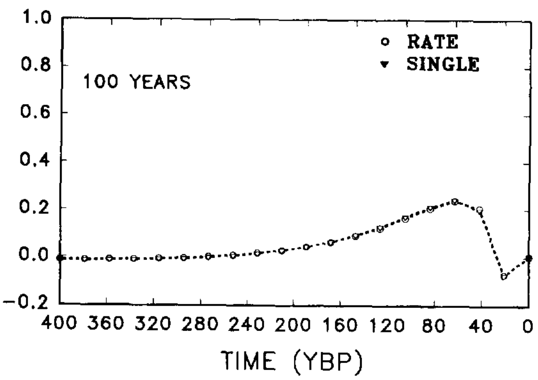

Fig. 4. Slices of resolution matrices for inversion of temperature and its time derivative for an event occurring 100 yr B.P. Circles correspond to the joint inversion and triangles to the single log inversion. There is no improvement of the resolution.

(Mareschal and Beltrami, 1992). Systematic temperature logging of the same boreholes for the purpose of improving the GTH have been reported by Chapman and Harris (1993).

The time rate of change for the subsurface temperature perturbation in Eq. 11 is maximum at depth $z = \sqrt{2\kappa t}$. For a 1° C step increase 100 yr B.P., it gives $z_{\max} =$ 78 m for a thermal diffusivity. After 10 years, the temperature change will be 0.024° C. For the same surface temperature change 50 and 20 yr B.P., the maximum rate of change will be at about 56 and 36 m, with changes of 0.05 and 0.12° C respectively after 10 years. Such changes could be easily detected and included as additional constraints in the inversion scheme. Repeat measurements have been performed and reported by several authors (Nielsen and Beck, 1989; Chapman and Harris, 1993).

The model resolution matrix was calculated for a data set consisting of temperature and its time derivative sampled at 10 m interval over 600 m and for model parameters consisting of 50 values of ground temperature averaged over 20 years. Table 2 compares the singular value spectra of inversion from temperature only with that of inversion from temperature and its time derivative. It came as a major surprise to the authors that the singular values for inversion of temperature and its time derivative are almost identical to those for temperature only. The singular values decrease as rapidly and the same number of singular values will thus be retained in both cases. The introduction of the time derivatives as new constraints does not significantly improve resolution. Fig. 4 compares the model resolution of a temperature pulse that occurred 100 years ago, for temperature only or for temperature and its derivative. With the singular-value cutoff set at 0.025, the number of singular values retained was only 5 for both inversions. Despite the inclusion of the time derivative, the resolution has not been improved and the impulse has been spread over a period of 60 years.

It might be possible to include an additional singular value in the inversion with derivative if repeat measurements eliminate the effects of random noise on the data. However, the errors are usually larger on the derivatives than on the data, and the inclusion of the derivative could further decrease the stability of the solution. Model resolution could more easily be improved by stacking repeat logs of the same borehole over a short period of time. The report by Chapman and Harris (1993) shows a fine structure in borehole temperature data that is stable over the time interval between their measurements. The same consistency can be observed in repeat measurements of the Minchin Lake borehole that will be discussed later. For determining the GTH, these small persistent perturbations are coherent noise that will not be eliminated by stacking measurements from the same borehole.

5. Parameterization

It is well known that the final problem (by opposition to initial problem) for the heat equation is an

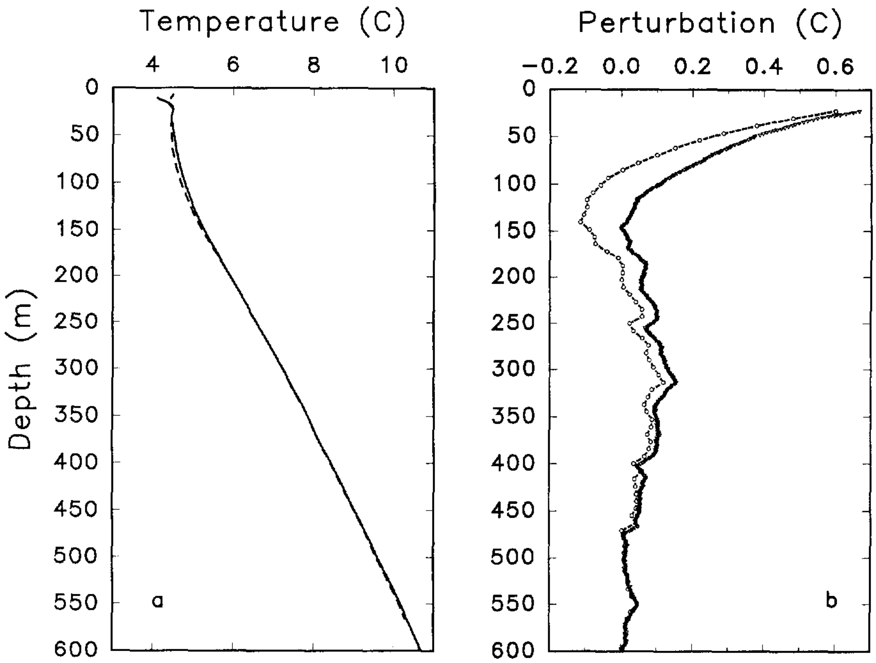

Fig. 6. (a) Temperature depth profiles for Minchin Lake measured in 1970 (dashed line) and in 1985 (solid line). (b) Temperature departures from steady state for the same logs; the 1970 temperature anomaly is represented by circles and the 1985 anomaly by triangles. The reference surface temperature and heat flow density are 3.5° C and 35 mW/m^2 respectively.

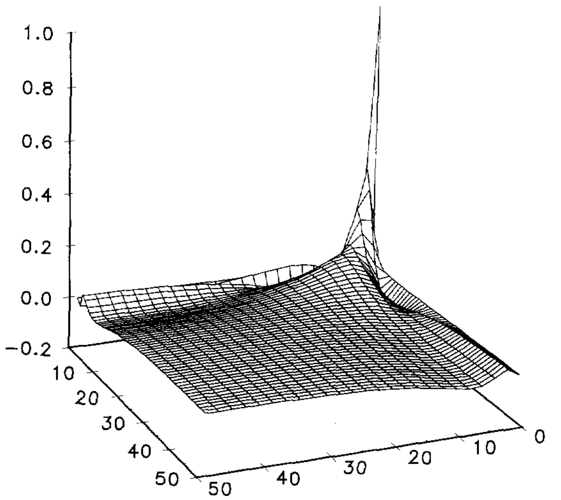

Fig. 7. Model Resolution Matrix for Minchin Lake for a model with 50 values of temperature averaged over 20 years with a singular value cutoff set at 0.025. Departures from a diagonal form show the loss of resolution in time. The labels on X and Y axes refer to the parameters order.

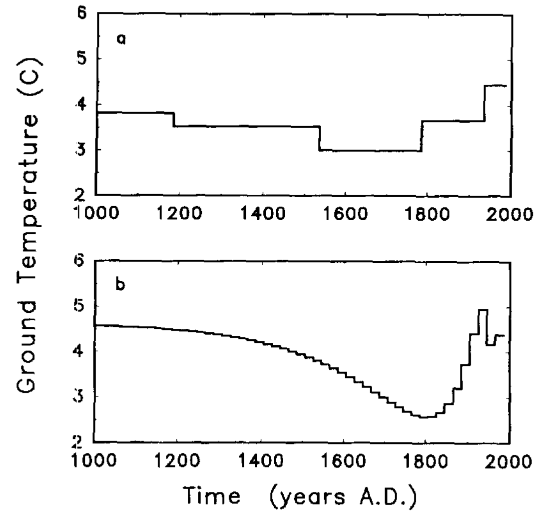

Fig. 9. Solutions for the 1985 Minchin Lake temperature log with fixed and variable interval model. Only the last 1000 years are given for the variable interval. The singular value cutoff ratio was set at 0.025 for both models and the same number of singular values were included in both inversions.

ill-posed problem (i.e. the solution of the heat equation backwards in time is unstable). The loss of resolution discussed above is just another example of this instability and it cannot be recovered. Nevertheless, it is possible to select model parameters that

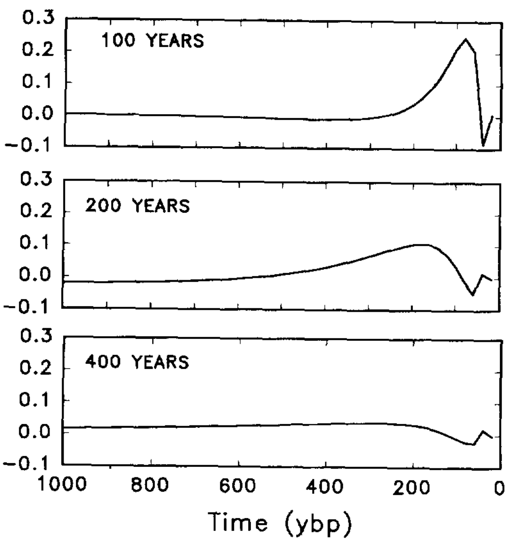

Fig. 8. Slices of the model resolution matrix in Fig. 7 at indicated times. The averaging interval for the estimated model parameters increases with the width of the peak.

would be better constrained by the data. This could be done, for instance, by increasing the duration of the time intervals over which the ground temperature is averaged according to what the data can resolve. To some extent, the singular value decomposition operates this selection as it replaces the parameters chosen initially by a linear combination of these parameters. Mareschal and Vasseur (1992) have used models consisting of a series of temperature values averaged over intervals varying in a logarithmic fashion. The model investigated consists of 10 values of temperature values over time intervals set at $50I^2$ yr, where I is the order of the model parameter. The total time considered is 5000 years.

The slices of the model resolution matrix at 50, 450 and 1250 yr are shown in Fig. 5. The parameters at 50 yr and at 450 yr are obviously better resolved from the data, but these parameters do not give a more detailed GTH than the previous model.

6. Example: the Minchin Lake data

Inversion by singular value decomposition has been applied to temperature–depth profiles in eastern and central Canada (Beltrami et al., 1992; Beltrami and Mareschal, 1992,1993) and in Europe (Mareschal and Vasseur, 1992; Clauser and Mareschal, 1995). The Minchin Lake borehole, located north of Lake Superior in western Ontario (52° 42.7′N, 90° 28.8′W), has been logged for temperature in 1970 and 1985 (for details see Nielsen and Beck, 1989). The 1970 measurements were taken at about 8 m (25 feet) interval; the 1985 log is continuous. These logs provide the type of data required for inverting temperature with improved sampling and inclusion of the time derivative. In addition, these data were analyzed by several groups (Shen et al., 1992; Beck et al., 1992) and present a standard benchmark of inversion procedures. Fig. 6a shows the two temperature profiles, one measured in 1970, the other measured in 1985. The reference surface temperature and heat flow density found by inversion are 3.5° C and 35 mW/m^2. Details of the temperature logs appear more clearly on Fig. 6b that shows the temperature perturbations calculated as deviation from steady state. The small oscillations appear consistent between these temperature logs obtained at 15 yr interval.

The model parameters for the inversion consist of 50 temperature values averaged over 20-yr. For the inversion, the continuous 1985 log was sampled at 1.1 m interval. Five singular values were retained in the inversion with a cutoff at 0.025. Inclusion of more singular values improves the resolution but increases the variance. The full model resolution matrix shown in Fig. 7 indicates that the resolution drops rapidly with time, and that the estimated parameters are averaged over time intervals increasing with time. These averaging intervals are better seen on slices of the resolution matrix at each time. Fig. 8 shows three such slices taken at 100, 200 and 400 yr; the spread at 100 yr is about 80 yr, at 200 yr is on the order of 140 yr, and at 400 yr the resolution is

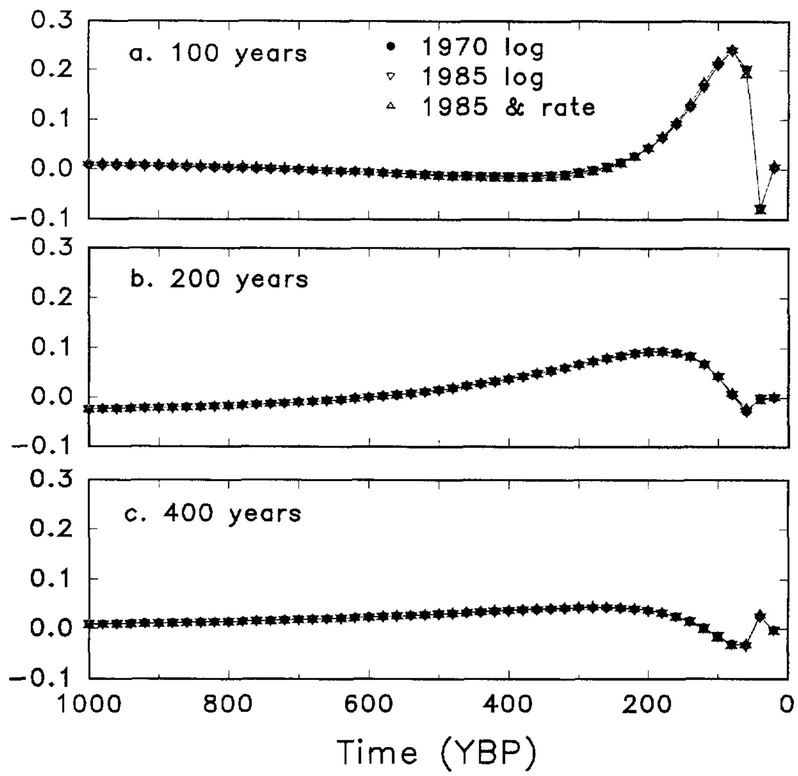

Fig. 10. Slices of the model resolution matrices for the two Minchin Lake logs. All models consist of fifty 20-yr temperature step changes with the same singular value cutoff and same number of singular values retained. Slices are given at 100, 200 and 400 yr B.P. There is no improvement of the resolution.

practically non existent. The decrease in resolution is an effect of heat diffusion but the resolution is further deteriorated because of noise in the data. Were the data free from noise or had a lower noise level, the resolution could be improved by the inclusion of more singular values in the inversion.

The inversion was done with two model parameterization: fifty 20-yr step and averaging intervals that increase with time as described above. The solution for both parameterizations are shown in Fig. 9.

The 1970 and 1985 temperature logs for Minchin Lake provide an example of inversion of real temperature-depth data with different sampling intervals. The singular value spectra, shown in Table 3, and the slices of the resolution matrix for 100, 200 and 400 years before 1985 A.D., shown in Fig. 10, confirm that the sampling density will not significantly improve the resolution.

The two temperature logs from Minchin Lake were measured in 1970 and 1985 can be inverted simultaneously. The results of inversion including the rate of change of the subsurface temperature perturbation can be compared with the inversion of a single log. The singular value spectrum for inversion with time derivative is also given in Table 3. As expected, it is no different than for the single 1985 log. The slices of the resolution matrix, also shown also in Fig. 10, are nearly identical to the ones from the single logs inversion.

Finally, Fig. 11 shows the GTHs obtained from the inversion of each log separately (Fig. 11a,b) and with the rate of change (Fig. 11c). Only the first 600 yr of the GTHs are shown here. The standard errors on the estimated model parameters are small and are included in the solid curves. The GTHs shown in this figure are remarkably consistent with respect to the cool period around 1800 and to the recent warming during this century. The only important difference appears to be in the very recent past where the rate of change of the subsurface temperature perturbation further restricts the solution.

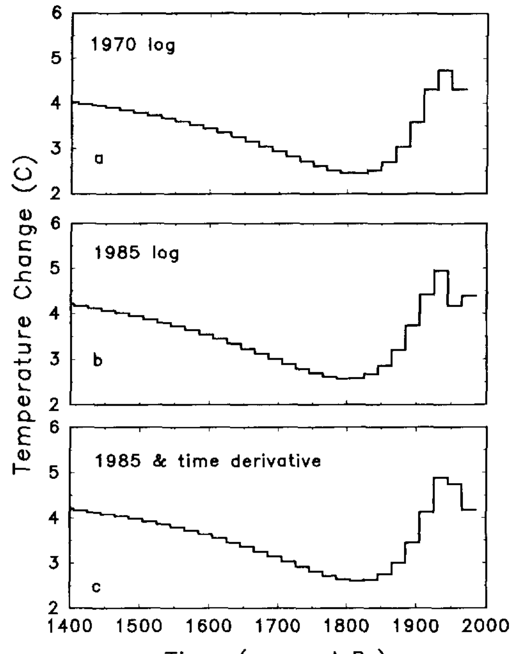

Fig. 11. Ground temperature histories for Minchin Lake: (a) 1970 log, (b) 1985 log, (c) 1970 and 1985 logs taking into account the time rate of change of the subsurface temperature perturbation. The standard error of the estimated parameters are small and they are not shown here.

Table 3
Singular value spectra for the Minchin Lake data sets as indicated

1970 log	1985 log	1985 + 1970
2797	7875	7875
4.306	11.2	11.2
1.807	2.376	2.488
0.355	0.878	1.025
0.136	0.3498	0.4611
0.052	0.138	0.1955
0.019	0.0516	0.0777
0.0062	0.0177	0.0278
0.0005	0.0054	0.0090

Because of denser sampling, the singular values are larger for the 1985 log. However, the ratio of the singular values to the largest, are comparable and the resolution is the same. The spectra for the 1985 log and the joint inversion are very close.

7. Conclusions

Geothermal data are useful to recover long-term climatic trends. However the resolution of the signal is poor and decreases with time. The estimated parameters of the ground temperature history appear as averages over intervals of increasing length as one attempts to resolve events farther back in time. As a result, short-period variations are lost. A different parameterization with a model averaging temperature changes over increasing time intervals for earlier times permits a better determination of the individual parameters, but it does not improve the resolution. Inverting the response of the Earth to a unit impulse change of surface temperature shows that the estimated temperature history is spread over a time spanning 60% of the time of occurrence. These "averaging intervals" were found to be independent of the model used in the inversion and provide a gross estimate of the resolution of geothermal data.

Measurement of temperature in the borehole with a finer sampling interval does not significantly improve the resolution unless it increases the signal to noise ratio. Inversion of the temperature and its time derivative from repeat measurements of the same borehole does not improve the resolution. The small scale perturbations in the temperature log that are stable over long period of times and geologic noise cannot be eliminated either by repeat measurements or finer sampling. Repeat measurements are useful because they offer a test of consistency of the interpretation. The signal to noise ratio could be improved by stacking temperature logs from several boreholes with uncorrelated geologic noise that have experienced consistent surface temperature history.

Although the climatic variations inferred from geothermal data are rather poorly resolved, the integrated estimates of the temperature variations provide long term trends that appear to be robust. Attempts to combine borehole temperature with other climatic indicators are being reported (Beltrami and Taylor, 1994). Geothermal data, which provide a quantitative estimate of long temperature trends, could be used to calibrate high resolution proxy climatic indicators (tree-rings, oxygen isotope data, etc.).

Acknowledgements

H.B. Acknowledges support from a postdoctoral fellowship from the Natural Sciences and Engineering Research Council of Canada (NSERC), O. Jensen (McGill) and from David S. Chapman NSF (University of Utah) grant number EAR-9205031. JCM thanks NSERC for generous and continuing support. We thank Dave Chapman, Rob Harris, and Scott Putnam for discussions while this work was carried out. The comments of two anonymous reviewers are gratefully aknowledged.

References

Beck, A.E., 1982. Precision logging of temperature gradients and the extraction of past climate. Tectonophysics, 83: 1–11.

Beck, A.E. and Judge, A., 1969. Analysis of heat flow data: detailed observation in a single borehole. Geophys. J. R. Astron. Soc., 18: 145–158.

Beck, A.E., Shen, P.Y., Beltrami, H., Mareschal, J.C., Safanda, J., Sebagenzi, S., Vasseur, G. and Wang, K., 1992. A comparison of 5 different analyses for interpreting 5 datasets. Palaeogeogr. Palaeoclimatol. Palaeoecol. (Global Planet. Change Sect.), 98: 101–112.

Beltrami, H., Jessop, A. and Mareschal, J.C., 1992. Ground temperature histories in eastern and central Canada from geothermal measurements. Palaeogeogr. Palaeoclimatol. Palaeoecol. (Global Planet. Change Sect.), 98: 167–184.

Beltrami, H. and Mareschal, J.C., 1992. Ground temperature histories for central and eastern Canada from geothermal measurements: Little Ice Age signature. Geophys. Res. Lett., 19: 698–692.

Beltrami, H. and Mareschal, J.C., 1993. Determination of past climate change from geothermal data: Regional ground temperature histories, correlation with atmospheric CO_2 and comparison with proxy data. Terra Nova, 5: 21–28.

Beltrami, H. and Taylor, A., 1994. Reconstruction of high resolution ground temperature histories from the combination of geothermal and oxygen isotope data in the Canadian Arctic. Submitted.

Birch, F., 1948. The effects of Pleistocene climatic variations upon geothermal gradients. Am. J. Sci., 246: 729–760.

Bullard, E.C., 1939. Heat flow in South Africa. Proc. R. Soc. London, A173: 428–502.

Carslaw, H.S. and Jaeger, J.C., 1959. Conduction of Heat in Solids. Oxford Univ. Press, 2nd ed., 510 pp.

Cermak, V., 1971. Underground temperature and inferred climatic temperature of the past millennium. Palaeogeogr. Palaeoclimatol. Palaeoecol., 10: 1–19.

Chapman, D.S. and Harris, R.N., 1993. Repeat temperature measurements in borehole GC-1, northwestern Utah: Towards isolating a climate–change signal in borehole temperature profiles. Geophys. Res. Lett., 20: 1891–1894.

Clauser, C. and Mareschal, J.C., 1995. Ground temperature history in central Europe from borehole temperature data. Geophys. J. Int., submitted.

Clow, G.D., 1992. The extent of temporal smearing in subsurface temperature histories derived from borehole temperature measurements. Palaeogeogr. Palaeoclimatol. Palaeoecol. (Global Planet. Change), 98: 81–86.

Hotchkiss, W.O. and Ingersoll, L.R., 1934. Post-glacial time calculations from recent geothermal measurements in the Calumet copper mines. J. Geol., 42: 113–142.

Jackson, D.D., 1972. Interpretation of inaccurate, insufficient, and inconsistent data. Geophys. J. R. Astron. Soc., 28: 97–110.

Jessop, A.M., 1971. The distribution of glacial perturbation of heat flow in Canada. Can. J. Earth Sci., 5: 61–68.

Lachenbruch, A. and Marshall, B.V., 1986. Changing climate: geothermal evidence from permafrost in the Alaskan Arctic. Science, 234: 689–696.

Lanczos, C., 1961. Linear Differential Operators. Van Nostrand, New York, 564 pp.

Lane, E.C., 1923. Geotherms of the Lake Superior Copper country. J. Geol., 42: 113–122.

Lewis, T. (Editor), 1992. Climatic Change Inferred from Underground Temperatures. Palaeogeogr. Palaeoclimatol. Palaeoecol. (Global Planet. Change Sect.), 98: 71–282.

Mareschal, J.C. and Beltrami, H., 1992. Evidence for recent warming from perturbed geothermal gradients: examples from eastern Canada. Clim. Dyn., 6: 135–143.

Mareschal, J.C. and Vasseur, G., 1992. Ground temperature history from two deep boreholes in central France. Palaeogeogr. Palaeoclimatol. Palaeoecol. (Global Planet. Change Sect.), 98: 185–192.

Menke, W., 1989. Geophysical Data Analysis: Discrete Inverse Theory (Int. Geophys. Ser., 45). Academic Press, San Diego, CA, 289 pp.

Nielsen, S.B. and Beck, A.E., 1989. Heat flow density values and paleoclimate determined from stochastic inversion of four temperature–depth profiles from the Superior Province of the Canadian Shield. Tectonophysics, 164: 345–359.

Shen, P.Y. and Beck, A.E., 1983. Determination of surface temperature history from borehole temperature gradients. J. Geophys. Res., 88: 7485–7493.

Shen, P.Y. and Beck, A.E., 1991. Least square inversion of borehole temperature measurements in functional space. J. Geophys. Res., 96: 19,965–19,979.

Shen, P.Y., Wang, K., Beltrami, H. and Mareschal, J.C., 1992. A comparative study of inverse methods for estimating climatic histories from borehole temperature data. Palaeogeogr. Palaeoclimatol. Palaeoecol. (Global Planet. Change Sect.), 98: 113–127.

Vasseur, G., Bernard, P., Van de Meulebrouck, J., Kast, Y. and Jolivet, J., 1983. Holocene paleotemperatures deduced from geothermal measurements. Palaeogeogr. Palaeoclimatol. Palaeoecol., 43: 237–259.

Wang, K., 1992. Estimation of ground surface temperature from borehole temperature data. J. Geophys. Res., 97: 2095–2106.