NUMERICAL INVESTIGATION OF CONVECTIVE MOTION IN A CLOSED CAVITY

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The investigation of thermal convection in a closed cavity is of considerable interest in connection with the problem of heat transfer. The problem may be solved comparatively simply in the case of small characteristic temperature difference with heating from the side, when equilibrium is not possible and when slow movement is initiated for an arbitrarily small horizontal temperature gradient. In this case the motion may be studied using the small parameter method, based on expanding the velocity, temperature, and pressure in series in powers of the Grashof number—the dimensionless parameter which characterizes the intensity of the convection [1–4]. In the problems considered it has been possible to find only two or three terms of these series. The solutions obtained in this approximation describe only weak nonlinear effects and the region of their applicability is limited, naturally, to small values of the Grashof number (no larger than 10^3).

With increase of the temperature difference the nature of the motion gradually changes—at the boundaries of the cavity a convective boundary layer is formed, in which the primary temperature and velocity gradients are concentrated; the remaining portion of the liquid forms the flow core. On the basis of an analysis of the equations of motion for the plane case, Batchelor [4] suggested that the core is isothermal and rotates with constant and uniform vorticity. The value of the vorticity in the core must be determined as the eigenvalue of the problem of a closed boundary layer. A closed convective boundary layer in a horizontal cylinder and in a plane vertical stratum was considered in [5, 6] using the Batchelor scheme. The boundary layer parameters and the vorticity in the core were determined with the aid of an integral method. An attempt to solve the boundary layer equations analytically for a horizontal cylinder using the Oseen linearization method was made in [7].

However, the results of experiments in which a study was made of the structure of the convective motion of various liquids and gases in closed cavities of different shapes [8–13] definitely contradict the Batchelor hypothesis. The measurements show that the core is not isothermal; on the contrary, there is a constant vertical temperature gradient directed upward in the core. Further, the core is practically motionless. In the core there are found retrograde motions with velocities much smaller than the velocities in the boundary layer.

The use of numerical methods may be of assistance in clarifying the laws governing the convective motion in a closed cavity with large temperature differences. In [14] the two-dimensional problem of steady air convection in a square cavity was solved by expansion in orthogonal polynomials. The author was able to progress in the calculation only to a value of the Grashof number $G = 10^4$. At these values of the Grashof number G the formation of the boundary layer and the core has really only started, therefore the author's conclusion on the agreement of the numerical results with the Batchelor hypothesis is not justified. In addition, the bifurcation of the central isotherm (Fig. 3 of [14]), on the basis of which the conclusion was drawn concerning the formation of the isothermal core, is apparently the result of a misunderstanding, since an isotherm of this form obviously contradicts the symmetry of the solution.

In [5] the method of finite differences is used to obtain the solution of the problem of strong convection of a gas in a horizontal cylinder whose lateral sides have different temperatures. According to the results of the calculation and in accordance with the experimental data [9], in the cavity there is a practically stationary core. However, since the authors started from the convection equations in the boundary layer approximation they did not obtain any detailed information on the core structure, in particular on the distribution of the temperature in the core.

In the following we present the results of a finite difference solution of the complete nonlinear problem of plane convective motion in a square cavity. The vertical boundaries of the cavity are held at constant temperatures; the temperature varies linearly on the horizontal boundaries. The velocity and temperature distributions are obtained for values of the Grashof number in the range $0 < G \le 4 \cdot 10^5$ and for a value of the Prandtl number $P = 1$. The results of the calculation permit following the formation of the closed boundary layer and the very slowly moving core with a constant vertical temperature gradient. The heat flux through the cavity is found as a function of the Grashof number.

1. Let us consider the plane convective motion of a viscous incompressible fluid in a long horizontal cylinder of square cross-section. We choose the x and y axes in the plane of the section; we direct the x axis horizontally, the y axis vertically upward. We take the temperature of the fluid at the vertical boundary $x = 0$ of the region as the origin, the temperature at the boundary $x = a$ is equal to Θ; along the horizontal boundaries $y = 0$ and $y = a$ the temperature varies linearly. The fluid equations of motion have the form

$$\frac{\partial \triangle\psi}{\partial t} + \left(\frac{\partial\psi}{\partial y}\frac{\partial\triangle\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\triangle\psi}{\partial y}\right) = \\ = \triangle\triangle\psi - G\frac{\partial T}{\partial x} \quad \left(G = \frac{g\beta\Theta a^3}{\nu^2}\right), \qquad (1.1)$$

$$\frac{\partial T}{\partial t} + \left(\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right) = \frac{1}{P}\triangle T \quad \left(P = \frac{\nu}{\chi}\right). \qquad (1.2)$$

Equations (1.1) and (1.2) for the stream function $\psi(x, y, t)$ and the temperature $T(x, y, t)$ are written in dimensionless form; the following units are used: distance is the side a of the square, time is a^2/ν, temperature is Θ, the stream function is ν. The dimensionless parameters G, the Grashof number, and P, the Prandtl number, appear in the equations.

The dimensionless velocity of the fluid is connected with the stream function by the relations $\nu_x = \partial\psi/\partial y$, $\nu_y = -\partial\psi/\partial x$; the projection of the vorticity on the z axis is

$$(\mathrm{rot}\,\mathbf{v})_z = -\triangle\psi \equiv \varphi \qquad \left(\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \qquad (1.3)$$

At the boundaries of the region the velocity is zero and the temperature is specified,

$$\psi = \frac{\partial\psi}{\partial x} = 0,\ T = 0 \text{ for } x = 0;\quad \psi = \frac{\partial\psi}{\partial x} = 0,$$
$$T = 1 \quad \text{for } x = 1,$$
$$\psi = \frac{\partial\psi}{\partial y} = 0, \quad T = x \quad \text{for } y = 0,\ y = 1. \qquad (1.4)$$

We write Eqs. (1.1)–(1.3) in finite difference form, using the central differences for the spatial derivatives

$$\varphi_{i,k}^{n+1} = \varphi_{i,k}^{n} + \Big\{\triangle\varphi_{i,k}^{n} + \frac{G}{2h}\left(T_{i+1,k}^{n} - T_{i-1,k}^{n}\right) -$$
$$- \frac{1}{4h^2}\left[\left(\psi_{i,k+1}^{n} - \psi_{i,k-1}^{n}\right)\left(\varphi_{i+1,k}^{n} - \varphi_{i-1,k}^{n}\right) - \qquad (1.5)$$

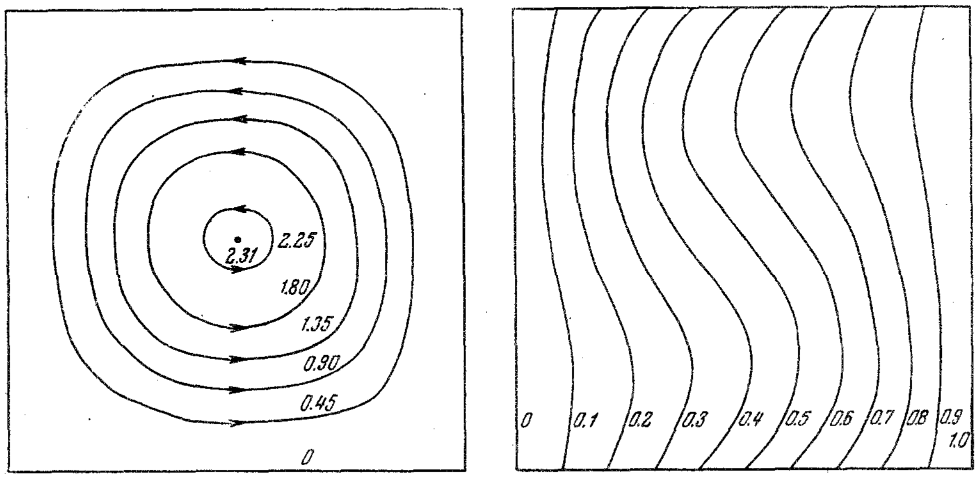

Fig. 1 $G = 2 \cdot 10^3$

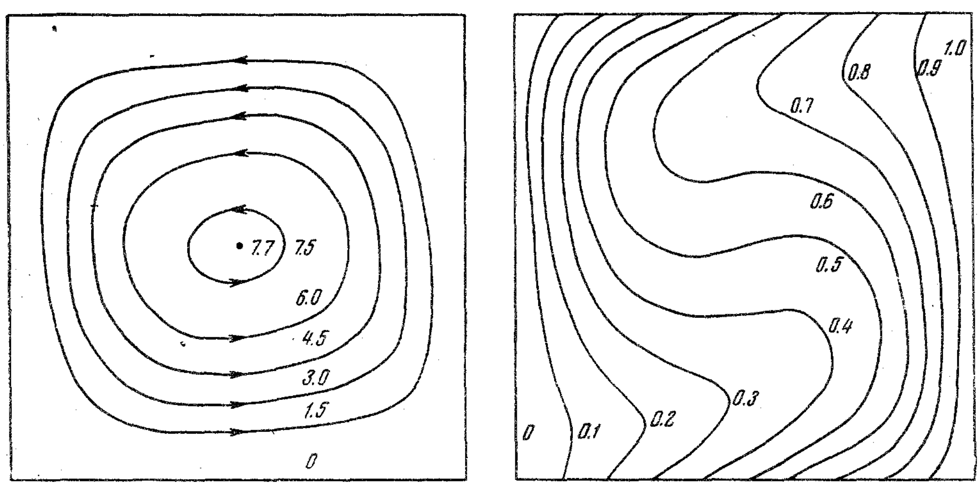

Fig. 2 $G = 14 \cdot 10^3$

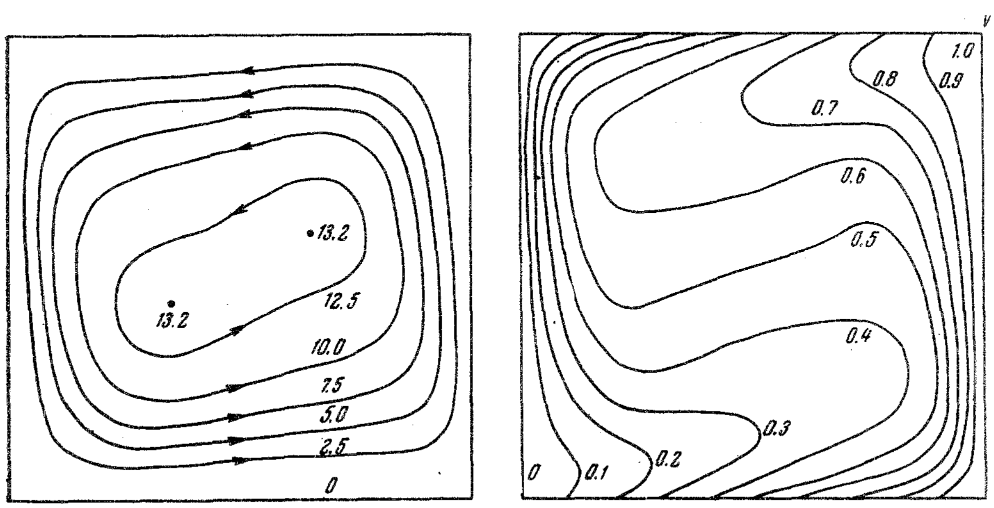

Fig. 3 $G = 80 \cdot 10^3$

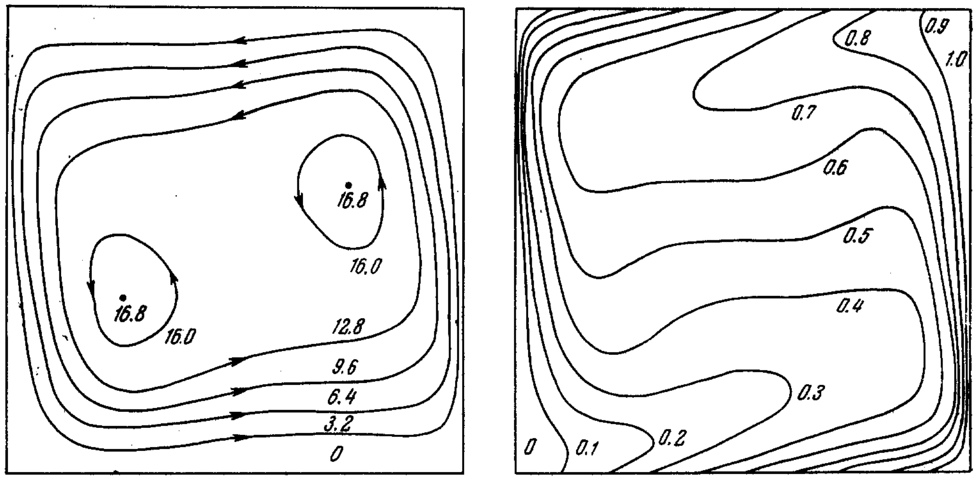

Fig. 4 $G=200\cdot 10^3$

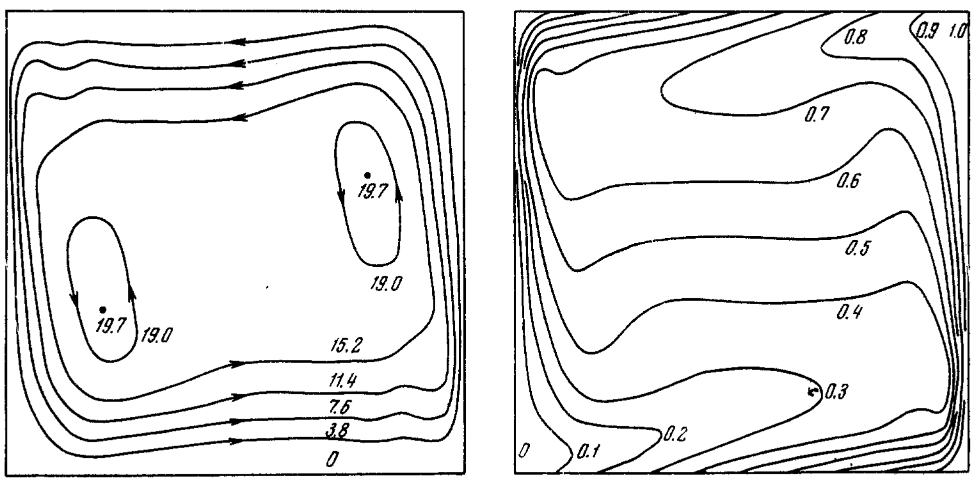

Fig. 5 $G=400\cdot 10^3$

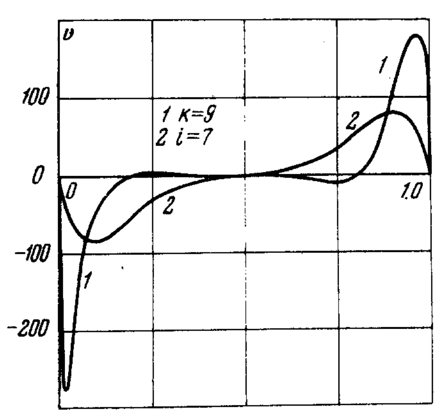

Fig. 6

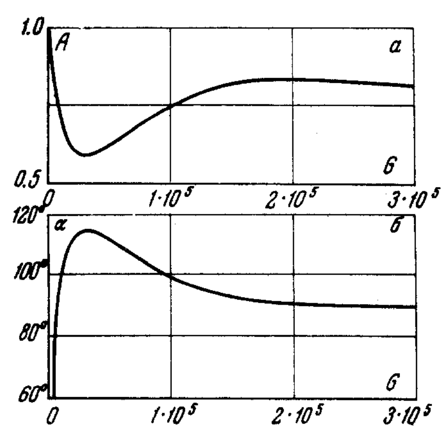

Fig. 7

$$-(\psi_{i+1,k}^{n}-\psi_{i-1,k}^{n})(\varphi_{i,k+1}^{n}-\varphi_{i,k-1}^{n})]\Big\}\tau, \qquad (1.5)\ (\text{cont'd})$$

$$T_{i,k}^{n+1}=T_{i,k}^{n}+\Big\{\frac{1}{P}\Delta T_{i,k}^{n}-\frac{1}{4h^2}[(\psi_{i,k+1}^{n+1}-\psi_{i,k-1}^{n+1})\times$$
$$\times(T_{i+1,k}^{n}-T_{i-1,k}^{n})-(\psi_{i+1,k}^{n+1}-$$
$$-\psi_{i-1,k}^{n+1})(T_{i,k+1}^{n}-T_{i,k-1}^{n})]\Big\}\tau, \qquad (1.6)$$

$$\Delta\psi_{i,k}^{n+1}=-\varphi_{i,k}^{n+1} \qquad (1.7)$$

In Eqs. (1.5)–(1.7) the Laplacians $\Delta\varphi_{i,k}$, $\Delta T_{i,k}$ and $\Delta\psi_{i,k}$ are defined by the equation

$$\Delta f_{i,k}=\frac{1}{h^2}(f_{i+1,k}+f_{i-1,k}+f_{i,k+1}+f_{i,k-1}-4f_{i,k}),$$

and we have introduced the notation

$$f_{i,k}^{n}\equiv f(x_i,y_k,t_n),\quad t_n=n\tau\quad(n=0,1,2,\ldots),$$
$$x_i=ih\quad(i=0,1,2\ldots I),\quad y_k=kh\quad(k=0,1,2\ldots K).$$

The boundary conditions (1.4) in difference form are [16, 17]

$$\psi_{0,k}^{n}=\psi_{I,k}^{n}=\psi_{i,0}^{n}=\psi_{i,K}^{n}=0,\quad \varphi_{0,k}^{n}=\frac{2}{h^2}\psi_{1,k}^{n},$$
$$\varphi_{I,K}^{n}=\frac{2}{h^2}\psi_{I-1,k}^{n},\quad \varphi_{i,0}^{n}=\frac{2}{h^2}\psi_{i,1}^{n},\quad \varphi_{i,K}^{n}=\frac{2}{h^2}\psi_{i,K-1}^{n}$$
$$T_{0,k}^{n}=0,\quad T_{I,k}^{n}=1,\quad T_{i,0}^{n}=T_{i,K}^{n}=x_i. \qquad (1.8)$$

The difference system (1.5)–(1.7) with the boundary conditions (1.8) is solved as follows. The quantities $\psi_{i,k}^{n}$ and $T_{i,k}^{n}$ are assumed known at the time t_n for all x_i, y_k. Then from Eq. (1.5) we can find the values of the vorticity $\varphi_{i,k}^{n+1}$ at all the internal nodes of the spatial grid at the succeeding instant of time. Then, solving the Poisson equation (1.7), we find $\psi_{i,k}^{n+1}$ and from (1.6) we determine $T_{i,k}^{n+1}$. After this we use Eqs. (1.8) to determine the new boundary values for φ^{n+1} and the entire procedure is repeated for the succeeding moment of time. At each step in time the Poisson equation is solved by the method of Liebmann iterations.

The calculations were made with a square grid h = 1/15. Some check calculations were also made with a smaller grid h = 1/25. The time step τ was varied depending on the computation stability; the program also provided for automatic alteration of τ in the process of the computations. The steady-state solution was obtained at the end of the stabilization process and did not depend on the initial distribution of the stream function and temperature. The steady-state solutions were obtained for 25 different values of the Grashof number in the range $0 < G \leq 4\cdot 10^5$ for P = 1. All the calculations were made on the Aragats digital computer at the Computer Center of Perm University.

2. Figures 1–5 show the stream function (a) and the isotherms (b) for several values of the Grashof number. For weak convection the trajectories of the fluid particles in the central portion of the cavity are nearly circular, while the isotherms are curved by the slow convective motion (Fig. 1; $G = 2\cdot 10^3$). With increase of G we note a tendency toward formation of a boundary layer and the isotherms take on an S-shaped form. In Fig. 2b ($G = 14\cdot 10^3$) we see clearly the genesis of a central region with relatively small temperature gradient.

Figures 3–5 (G respectively $80\cdot 10^3$; $200\cdot 10^3$; $400\cdot 10^3$) permit following the further formation of the boundary layer and the flow core with increase of the Grashof number. The velocities in the core are much lower than in the boundary layer and a pair of vortices is formed with slow retrograde motion. The resulting picture of the motion is very close to the experimental results of [13], which presents the trajectories of fluid particles in a square cavity. Figure 6 shows the longitudinal velocity distribution along the horizontal (k = 9) and the vertical (i = 7) sections for the case of the developed boundary layer ($G = 4\cdot 10^5$). We see that the boundary layer is particularly marked near the vertical boundaries of the cavity.

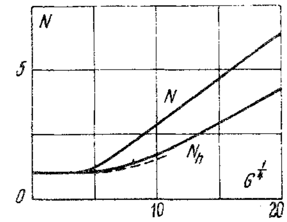

Fig. 8

In Figs. 3b–5b we see that with increase of G a region with constant vertical (directed upward) temperature gradient is formed and increases in size. This temperature gradient in the core is far smaller than the gradient in the boundary layer, however, the core cannot be considered isothermal, since the magnitude of the temperature gradient there is of order Θ/a.

The absolute magnitude of the temperature gradient A in the center of the cavity and the angle α formed by the vector A with the horizontal x axis are shown in Figs. 7a, b. We see from these figures that with increase of G the gradient A rotates and becomes vertical, and for $G > 2\cdot 10^4$ its magnitude essentially stabilizes about the value 0.8 (in units of Θ/a).

We note that the formation of a core with a uniform vertical temperature gradient is closely associated with the absence of marked motions in the core. Actually, it is easy to see that the Navier-Stokes and heat conduction equations are satisfied for $\mathbf{v} = 0$, $T = A\gamma$ (A = const, γ is a unit vertical vector).

The results of the calculation of the temperature field show that for all values of the Grashof number G the heat flux is directed toward the fluid on the heated vertical boundary $x = a$ and on the lower horizontal boundary $y = 0$; through the other two segments of the boundary the heat is transferred from the fluid to the surroundings. The total heat flux through the cavity (per unit length along the z axis) is

$$Q=-\varkappa\int_{\Gamma}\left(\frac{\partial T}{\partial n}\right)_{\Gamma}dl. \qquad (2.1)$$

Here T is dimensional temperature, n is the normal to the boundary. The integration in (2.1) is performed over the two segments $x = a$ and $y = 0$ of the boundary. The heat flux Q may be characterized by the dimensionless Nusselt number $N = Q/\varkappa\Theta$. Also of interest is the heat flux Q_h through the vertical boundary; the corresponding Nusselt number is

$$N_h=\frac{Q_h}{\varkappa\Theta}=-\frac{1}{\Theta}\int_0^a\left(\frac{\partial T}{\partial x}\right)_{y=a}dy. \qquad (2.2)$$

The variation of N and N_h with G is shown in Fig. 8; also shown here by the dashed curve are the values of N_h obtained in [14] for a somewhat different value of the Prandtl number (P = 0.73) by expanding in orthogonal polynomials.

For small values of the Grashof number ($G < 10^3$) the following relations hold

$$N=1+3.25\cdot 10^{-4}G,\quad N_h=1+5.15\cdot 10^{-8}G^2. \qquad (2.3)$$

In Eqs. (2.3) the term which is linear in G in the expression for N, and quadratic in G in the expression for N_h, describes the convective addition to the molecular heat flux for weak convection.

For large values of G there is a linear variation of the dimensionless heat transfer coefficients N and N_h with $G^{1/4}$ which is characteristic for the laminar boundary layer, the slopes of the limiting lines are respectively 0.37 and 0.28.

The steady-state results presented above apply to the range of values of the Grashof number $0 < G \leq 4 \cdot 10^5$. For $G > 4 \cdot 10^5$ the numerical calculations do not lead to steady-state solutions; after the stage of the transient regime, steady-state oscillations are established for which the stream function and the temperature, and also all the solution parameters—the temperature gradient in the core, the Nusselt number, etc. —oscillate about some average values, and the frequency of these oscillations increases with increase of G. It is possible that the appearance of these oscillations is due to the occurrence of small-scale motions which are not resolved by the grid. However, it is also possible that these oscillations are associated with physical causes—the formation of traveling waves in the boundary layer for large G. Such waves have been observed experimentally in [12, 18, 19].

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