Métodos Estabilizados para Elasticidade Não-linear Incompressível

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Defesa do Projeto Final

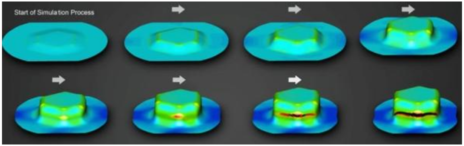
Curso GB-167 - Fenômenos de Transporte Computacional

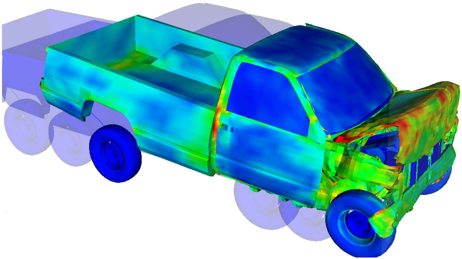
Laboratório Nacional de Computação Científica

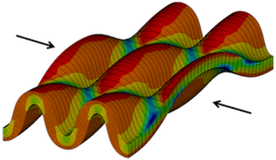
14 de dezembro de 2016

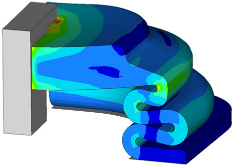
Outline

Motivação

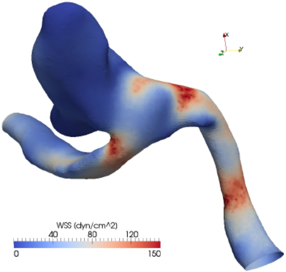


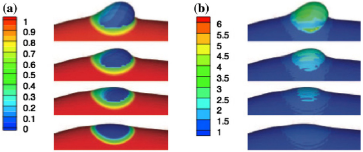


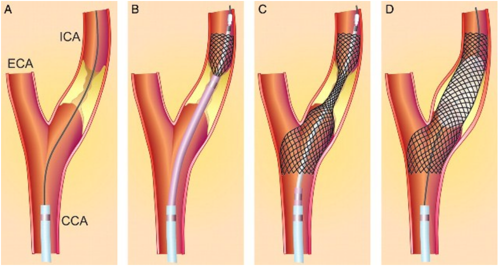


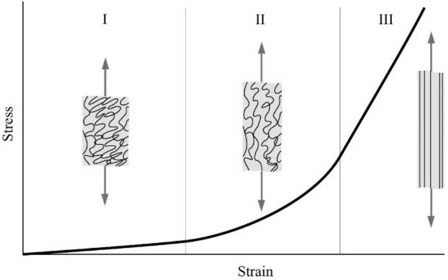


Motivação

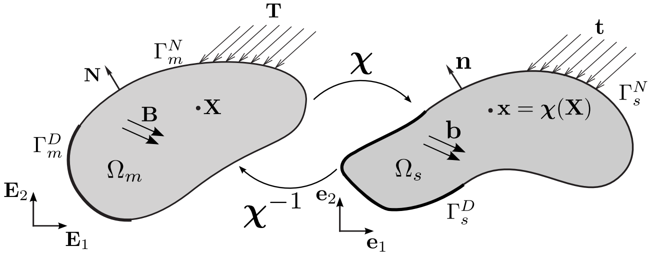

WSS (dyn/cm^2)
40
80
120
0
150


(a)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
(b)
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1

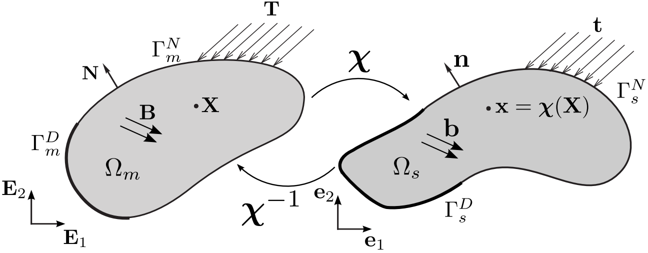

A
B
C
D
ICA
ECA
CCA


I
II
III
Stress
Strain

Cinemática de Grandes Deformações

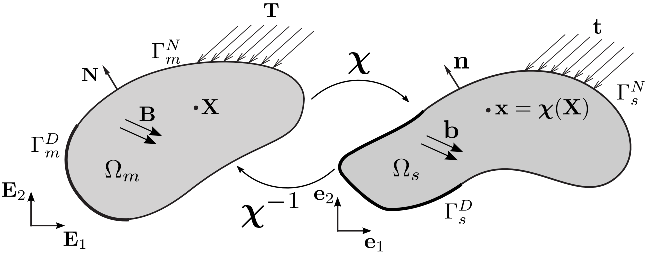


Cinemática de Grandes Deformações



$$
\begin{aligned}
\mathbf{F} &= \nabla \chi \\
\mathrm{d}\mathbf{x} &= \mathbf{F}\mathrm{d}\mathbf{X} \\
\mathrm{d}\Omega_s &= \det \mathbf{F} \mathrm{d}\Omega_m \\
J &= \det \mathbf{F} > 0 \\
\mathbf{n}\mathrm{d}\Gamma_s &= J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}\Gamma_m
\end{aligned}
$$

Cinemática de Grandes Deformações



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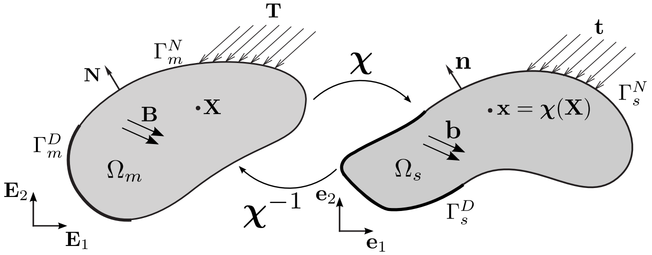
J &= \det\mathbf{F} > 0 \\
\mathbf{n}\mathrm{d}\Gamma_s &= J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}\Gamma_m
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u}(\mathbf{X}) &= \boldsymbol{\chi}(\mathbf{X}) - \mathbf{X} \\
\mathbf{u}_s(\mathbf{x}) &= \mathbf{u}(\boldsymbol{\chi}^{-1}(\mathbf{x})) \\
\mathbf{F}(\mathbf{X}) &= \mathbf{I} + \nabla\mathbf{u}(\mathbf{X}) \\
(\mathrm{grad}\mathbf{u}_s)_m &= \mathbf{F}^{-1}\nabla\mathbf{u} \\
(\mathrm{grad}\theta_s)_m &= \mathbf{F}^{-T}\nabla\theta
\end{aligned}
$$

Cinemática de Grandes Deformações



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(\mathrm{grad}\theta_s)_m &= \mathbf{F}^{-T}\nabla \theta
\end{aligned}
$$

$$
\begin{aligned}
\text{(Cauchy-Green)}\ \mathbf{C} &= \mathbf{F}^T\mathbf{F} \\
\text{(Green-Lagrange)}\ \mathbf{E} &= \frac{1}{2}(\mathbf{C} - \mathbf{I}) \\
&= \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T + (\nabla \mathbf{u})^T \nabla \mathbf{u})
\end{aligned}
$$

Princípio de Potências Virtuais

Forma Espacial

Dado $(\mathbf{b}, \mathbf{t})$ admissíveis encontrar $\mathbf{u}_s \in \mathscr{U}_s$ tal que:

$$\int_{\Omega_s} \boldsymbol{\sigma}(\mathbf{u}_s) \cdot \operatorname{grad}^s \hat{\mathbf{u}}_s d\Omega_s = \int_{\Gamma_s^N} \mathbf{b} \cdot \hat{\mathbf{u}}_s d\Omega_s + \int_{\Gamma_s^N} \mathbf{t} \cdot \hat{\mathbf{u}}_s d\Gamma_s^N \qquad \forall \hat{\mathbf{u}}_s \in \mathscr{V}_s$$

onde:

$$\mathscr{U}_s = \{\mathbf{u}_s : \Omega_s \to \mathbb{R}^{\mathrm{nd}}; \text{suf. reg.}, \mathbf{u}_s|_{\Gamma_s^D} = \bar{\mathbf{u}}_s\}$$
$$\mathscr{V}_s = \{\hat{\mathbf{u}}_s : \Omega_s \to \mathbb{R}^{\mathrm{nd}}; \text{suf. reg.}, \hat{\mathbf{u}}_s|_{\Gamma_s^D} = \mathbf{0}\}.$$

e $\boldsymbol{\sigma} = \mathcal{F}(\mathbf{u}_s)$ é o tensor de tensões de Cauchy.

Forma Forte

$$\begin{cases} \operatorname{div}_s \boldsymbol{\sigma} + \mathbf{b} = 0 & \text{em } \Omega_s \\ \boldsymbol{\sigma}\mathbf{n} = \mathbf{t} & \text{sobre } \Gamma_s^N \\ \mathbf{u}_s = \bar{\mathbf{u}}_s & \text{sobre } \Gamma_s^D \end{cases}$$

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Princípio de Potências Virtuais

Forma Material

Dado $(\mathbf{B},\mathbf{T})$ admissíveis encontrar $\mathbf{u}\in\mathscr{U}$ tal que:

$$\int_{\Omega_m}\mathbf{P}(\mathbf{u})\cdot\nabla\hat{\mathbf{u}}\,\mathrm{d}\Omega_m=\int_{\Gamma_m^N}\mathbf{B}\cdot\hat{\mathbf{u}}\,\mathrm{d}\Omega_m+\int_{\Gamma_s^N}\mathbf{T}\cdot\hat{\mathbf{u}}\,\mathrm{d}\Gamma_s^N\qquad\forall\hat{\mathbf{u}}\in\mathscr{V}$$

onde:

$$\mathscr{U}=\{\mathbf{u}\in H^1(\Omega_m)^{\mathrm{nd}},\mathbf{u}|_{\Gamma_m^D}=\bar{\mathbf{u}}\}$$
$$\mathscr{V}=\{\hat{\mathbf{u}}\in H^1(\Omega_m)^{\mathrm{nd}};\hat{\mathbf{u}}|_{\Gamma_m^D}=\mathbf{0}\}.$$

e $\mathbf{P}=\mathcal{F}(\mathbf{u})$ é o tensor primeiro tensor de Piola-Kirchhoff.

Forma Forte

$$\begin{cases}\operatorname{div}\mathbf{P}+\mathbf{B}=0 & \text{em } \Omega_m\\ \mathbf{PN}=\mathbf{T} & \text{sobre } \Gamma_m^N\\ \mathbf{u}=\bar{\mathbf{u}} & \text{sobre } \Gamma_m^D\end{cases}$$

Princípio de Potências Virtuais

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Dado $(\mathbf{B}, \mathbf{T})$ admissíveis encontrar $\mathbf{u} \in \mathscr{U}$ tal que:

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onde:

$$\mathscr{U} = \{\mathbf{u} \in H^1(\Omega_m)^{\mathrm{nd}}, \mathbf{u}|_{\Gamma_m^D} = \bar{\mathbf{u}}\}$$
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e $\mathbf{P} = \mathcal{F}(\mathbf{u})$ é o tensor primeiro tensor de Piola-Kirchhoff.

Forma Forte

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Princípio de Potências Virtuais

Forma Material

Dado $(\mathbf{B}, \mathbf{T})$ admissíveis encontrar $\mathbf{u} \in \mathscr{U}$ tal que:

$$\int_{\Omega_m} \mathbf{S}(\mathbf{u}) \cdot (\mathbf{F}^T \nabla \hat{\mathbf{u}})^s \, \mathrm{d}\Omega_m = \int_{\Gamma_m^N} \mathbf{B} \cdot \hat{\mathbf{u}} \, \mathrm{d}\Omega_m + \int_{\Gamma_s^N} \mathbf{T} \cdot \hat{\mathbf{u}} \, \mathrm{d}\Gamma_s^N \qquad \forall \hat{\mathbf{u}} \in \mathscr{V}$$

Onde $\mathbf{S}$ é o segundo tensor de Piola-Kirchhoff (simétrico). Temos as equivalencias:

$$\mathbf{P} = \mathbf{F}\mathbf{S}$$

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{P}\mathbf{F}^T$$

$$\delta\mathbf{E}(\mathbf{u}, \hat{\mathbf{u}}) = (\mathbf{F}^T \nabla \hat{\mathbf{u}})^s$$

Forma Forte

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Princípio de Potências Virtuais

Forma Material

Dado $(\mathbf{B}, \mathbf{T})$ admissíveis encontrar $\mathbf{u} \in \mathscr{U}$ tal que:

$$\int_{\Omega_m} \mathbf{P}(\mathbf{u}) \cdot \nabla \hat{\mathbf{u}} \, d\Omega_m - \Pi^{ext}(\hat{\mathbf{u}}) = 0 \qquad \forall \hat{\mathbf{u}} \in \mathscr{V}$$

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Princípio de Potências Virtuais

Forma Material

Dado $(\mathbf{B},\mathbf{T})$ admissíveis encontrar $\mathbf{u} \in \mathscr{U}$ tal que:

$$\mathcal{R}(\mathbf{u},\hat{\mathbf{u}}) = 0 \qquad \forall \hat{\mathbf{u}} \in \mathscr{V}$$

Onde $\mathbf{S}$ é o segundo tensor de Piola-Kirchhoff (simétrico). Temos as equivalencias:

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Modelos Constitutivos

Modelo Hiperelástico

Seja $\Psi : \mathbb{R}^{\mathrm{nd}\times\mathrm{nd}} \to \mathbb{R}^+$ uma energia de deformação então:

$$\mathbf{P} = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}}$$

Onde em geral é admitido o desacoplamento:

$$\Psi(\mathbf{F}) = \Psi^{iso}(\bar{\mathbf{F}}) + \kappa U(J)$$

onde $\bar{\mathbf{F}} = J^{-1/3}\mathbf{F}$, $U(1) = 0, U(J) \to \infty$ quando $J \to 0$ ou $J \to \infty$.

Exemplos

$$\Psi(\mathbf{F}) = \hat{\Psi}(\mathbf{C}) = \frac{\mu}{2}(\mathrm{tr}(\bar{\mathbf{C}}) - 3) + \frac{\kappa}{2}(J-1)^2 \quad \text{(Neo-Hookean)}$$

$$\Psi(\mathbf{F}) \approx \hat{\Psi}(\varepsilon) = \frac{1}{2}\mathbb{C}\varepsilon \cdot \varepsilon \quad \text{(Elasticidade Linear-Infinitesimal)}$$

Obs: $\boldsymbol{\sigma} = \dfrac{\partial \Psi}{\partial \varepsilon} = \mathbb{C}\varepsilon$

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Obs: $\boldsymbol{\sigma} = \dfrac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = \mathbb{C}\boldsymbol{\varepsilon}$

Linearização do Problema

Método de Newton-Raphson

Seja $f : \mathbb{R} \to \mathbb{R}$, queremos achar $x \in \mathbb{R}$ tal que:

$$f(x) = 0$$

para um dado chute $x^0 \in \mathbb{R}$ realizar os passos

$$\begin{aligned} f'(x^k)\delta x^k &= -f(x^k) \\ x^{k+1} &= x^k + \delta x^k \end{aligned}$$

para $k = 0, 1, 2, \ldots$ até obter uma dada convergência.

Linearização do Problema

Método de Newton-Raphson

Queremos achar $\mathbf{u} \in \mathscr{U}$ tal que:

$$\mathcal{R}(\mathbf{u}, \hat{\mathbf{u}}) = 0 \quad \forall \hat{\mathbf{u}} \in \mathscr{V}$$

para um dado chute $\mathbf{u}^0 \in \mathscr{U}$ incrementamos

$$\delta\mathcal{R}(\mathbf{u}^k, \hat{\mathbf{u}}; \delta\mathbf{u}^k) = -\mathcal{R}(\mathbf{u}^k, \hat{\mathbf{u}}) \quad \forall \hat{\mathbf{u}} \in \mathscr{V}$$
$$\mathbf{u}^{k+1} = \mathbf{u}^k + \delta\mathbf{u}^k$$

para $k = 0, 1, 2, \ldots$ até obter uma dada convergência.

Formulação de elementos finitos

Dado $\mathbf{u} \in \mathscr{U}$ defina

$$a_{(\mathbf{u})}(\delta\mathbf{u}, \hat{\mathbf{u}}) = \delta\mathcal{R}(\mathbf{u}, \hat{\mathbf{u}}; \delta\mathbf{u}) = \int_{\Omega_\mu} \mathbb{A}(\mathbf{u})\nabla\delta\mathbf{u} \cdot \nabla\hat{\mathbf{u}}$$

$$f_{(\mathbf{u})}(\hat{\mathbf{u}}) = -\mathcal{R}(\mathbf{u}, \hat{\mathbf{u}}) = -\int_{\Omega_\mu} \mathbf{P}(\mathbf{u})\nabla\delta u \cdot \nabla\hat{\mathbf{u}} \mathrm{d}\Omega_m + \Pi^{ext}(\hat{\mathbf{u}})$$

onde $\mathbb{A} = \dfrac{\partial\mathbf{P}}{\partial\mathbf{F}}$ é chamado de tensor tangente.

Problema Discreto

Tome $V_h \subset \mathscr{V}$, dado $\mathbf{u}_h \in V_h$, achar $\delta\mathbf{u}_h$ tal que:

$$a_{(\mathbf{u}_h)}(\delta\mathbf{u}_h, \hat{\mathbf{u}}_h) = f_{(\mathbf{u}_h)}(\hat{\mathbf{u}}_h) \quad \forall\hat{\mathbf{u}}_h \in V_h$$

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Formulação Mista

- Derivando a energia de deformação temos:

$$\mathbf{P}(\mathbf{F}) = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} = \frac{\partial \Psi_{iso}(\overline{\mathbf{F}})}{\partial \mathbf{F}} + \kappa \frac{\partial U(J)}{\partial \mathbf{F}}$$

- mas:

$$\kappa \frac{\partial U(J)}{\partial \mathbf{F}} = U'(J) \frac{\partial J}{\partial F} = J \kappa U'(J) \mathbf{F}^{-T}$$

definindo $p = \kappa U'(J)$, temos:

$$\mathbf{P} = \mathbf{P}_{iso} + Jp\mathbf{F}^{-T} \quad (\boldsymbol{\sigma} = \boldsymbol{\sigma}_{iso} + p\mathbf{I})$$

- Temos assim, definindo $\mathscr{Q} = H^1(\Omega_m) \subset L^2(\Omega_m)$:

$$\int_{\Omega_m} \mathbf{P}_{iso} \cdot \nabla \hat{\mathbf{u}} + \int_{\Omega_m} Jp\mathbf{F}^{-T} \cdot \nabla \hat{\mathbf{u}} = \Pi_{ext}(\hat{\mathbf{u}}) \quad \forall \hat{\mathbf{u}} \in \mathscr{V}$$

$$\int_{\Omega_m} \left(U'(J) - \frac{1}{\kappa} p \right) \hat{p} = 0 \quad \forall \hat{p} \in \mathscr{Q}$$

Obs: Note que $U'(J) = (J - 1)$ e que se $\kappa \to \infty$, temos $J = 1 \in \Omega_\mu$ (no sentido das distribuiçoes).

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Obs: Note que $U'(J) = (J-1)$ e que se $\kappa \to \infty$, temos $J = 1 \in \Omega_\mu$ (no sentido das distribuiçoes).

Formulação Mista

- Derivando a energia de deformação temos:

$$\mathbf{P}(\mathbf{F}) = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} = \frac{\partial \Psi_{iso}(\overline{\mathbf{F}})}{\partial \mathbf{F}} + \kappa \frac{\partial U(J)}{\partial \mathbf{F}}$$

- mas:

$$\kappa \frac{\partial U(J)}{\partial \mathbf{F}} = U'(J) \frac{\partial J}{\partial F} = J \kappa U'(J) \mathbf{F}^{-T}$$

definindo $p = \kappa U'(J)$, temos:

$$\mathbf{P} = \mathbf{P}_{iso} + Jp\mathbf{F}^{-T} \quad (\boldsymbol{\sigma} = \boldsymbol{\sigma}_{iso} + p\mathbf{I})$$

- Temos assim, definindo $\mathscr{Q} = H^1(\Omega_m) \subset L^2(\Omega_m)$:

$$\int_{\Omega_m} \mathbf{P}_{iso} \cdot \nabla \hat{\mathbf{u}} + \int_{\Omega_m} Jp\mathbf{F}^{-T} \cdot \nabla \hat{\mathbf{u}} = \Pi_{ext}(\hat{\mathbf{u}}) \quad \forall \hat{\mathbf{u}} \in \mathscr{V}$$

$$\int_{\Omega_m} \left(U'(J) - \frac{1}{\kappa} p \right) \hat{p} = 0 \quad \forall \hat{p} \in \mathscr{Q}$$

Obs: Note que $U'(J) = (J-1)$ e que se $\kappa \to \infty$, temos $J = 1 \in \Omega_\mu$ (no sentido das distribuiçoes).

Formulação Mista Estabilizada

- Dado uma partição $\mathcal{T}_h$, em termos da formulação espacial propoe-se adicionar na segunda equação o seguinte termo variacionalmente consistente, resíduo da equação diferencial [Klass et al, 1999]:

$$\sum_{K_s \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\mu} \int_{K_s} \Big(-\operatorname{div}(\boldsymbol{\sigma}_{iso} + p\mathbf{I}) - \mathbf{b} \Big) \cdot \operatorname{grad}\hat{p} \, \mathrm{d}\Omega_s$$

- Simplificando para sem forças de corpo, interpolaçoes lineares e mudando para a configuração de referencia ficamos:

$$\int_{\Omega_m} \mathbf{P}_{iso} \cdot \nabla\hat{\mathbf{u}} \, \mathrm{d}\Omega_m + \int_{\Omega_m} Jp\mathbf{F}^{-T} \cdot \nabla\hat{\mathbf{u}} \, \mathrm{d}\Omega_m - \Pi_{ext}(\hat{\mathbf{u}}) = 0 \quad \forall \hat{\mathbf{u}} \in \mathscr{V}$$

$$\int_{\Omega_m} \left(U'(J) - \frac{1}{\kappa} p \right) \hat{p} \, \mathrm{d}\Omega_m - \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\mu} \int_K J\mathbf{F}^{-T} \nabla p \cdot \mathbf{F}^{-T} \nabla \hat{p} \mathrm{d}\Omega_m = 0 \quad \forall \hat{p} \in \mathscr{Q}$$

- Linearizando, temos o problema de dado um par $(\mathbf{u}, p) \in \mathscr{U} \times \mathscr{Q}$, encontrar $(\delta\mathbf{u}, \delta p) \in \mathscr{V} \times \mathscr{Q}$ tal que:

$$a_{(\mathbf{u},p)}(\delta\mathbf{u}, \hat{\mathbf{u}}) + b_{(\mathbf{u},p)}(\delta p, \hat{\mathbf{u}}) = f_{(\mathbf{u},p)}(\hat{\mathbf{u}}) \quad \forall \hat{\mathbf{u}} \in \mathscr{V}$$
$$c_{(\mathbf{u},p)}(\delta\mathbf{u}, \hat{p}) + d_{(\mathbf{u},p)}(\delta p, \hat{p}) = g_{(\mathbf{u},p)}(\hat{p}) \quad \forall \hat{p} \in \mathscr{Q}$$

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Onde:

$$
\begin{aligned}
a_{(\mathbf{u},p)}(\mathbf{w},\mathbf{v}) &= \int_{\Omega_m} (\mathbb{A}\nabla\mathbf{w})\cdot\nabla\mathbf{v} \\
b_{(\mathbf{u},p)}(q,\mathbf{w}) &= \int_{\Omega_m} Jq\mathbf{F}^{-T}\cdot\nabla\mathbf{w} \\
c_{(\mathbf{u},p)}(\mathbf{w},q) &= \int_{\Omega_m} U''(J)Jq\mathbf{F}^{-T}\cdot\nabla\mathbf{w} + \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\mu}\int_K (\mathbb{E}\nabla\mathbf{w})\cdot\nabla\hat{q} \\
d_{(\mathbf{u},p)}(q,\hat{q}) &= -\int_{\Omega_m}\frac{1}{\kappa}q\hat{q} + \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\mu}\int_K J\mathbf{F}^{-T}\nabla q\cdot\mathbf{F}^{-T}\nabla\hat{q} \\
f_{(\mathbf{u},p)}(\mathbf{w}) &= -\int_{\Omega_m}\mathbf{P}_{iso}\cdot\nabla\mathbf{w} - \int_{\Omega_m} Jp\mathbf{F}^{-T}\cdot\nabla\mathbf{w} + \Pi_{ext}(\mathbf{w}) \\
g_{(\mathbf{u},p)}(q) &= -\int_{\Omega_m}\left(U'(J)-\frac{1}{\kappa}p\right)q - \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\mu}\int_K J\mathbf{F}^{-T}\nabla p\cdot\mathbf{F}^{-T}\nabla q
\end{aligned}
$$

Onde:

$$\mathbb{A} = \mathbb{A}_{iso} + J\mathbf{F}^{-T} \otimes \mathbf{F}^{-T} - J\mathbf{F}^{-T} \odot \mathbf{F}^{-1}$$
$$\mathbb{B} = J\left(\mathbf{C}^{-1} \otimes \mathbf{F}^{-T} - \mathbf{F}^{-1} \circledcirc \mathbf{C}^{-1} - \mathbf{C}^{-1} \odot \mathbf{F}^{-1}\right)$$
$$\mathbb{E} = \mathbb{B}_{ijkl}(\nabla p)_j \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

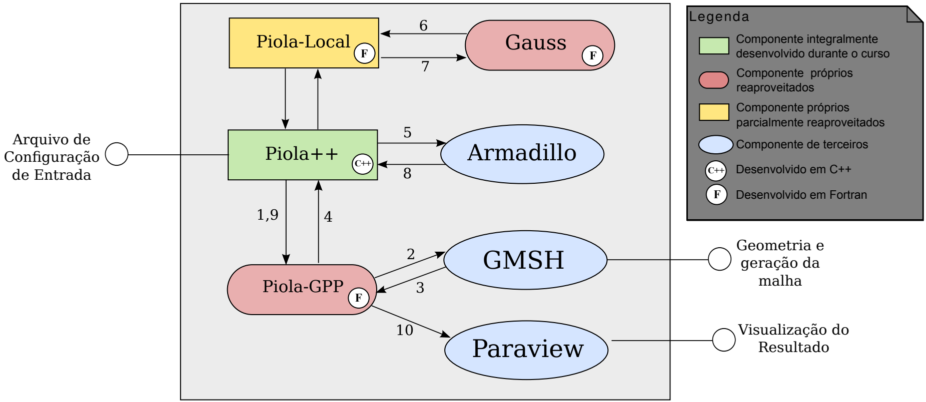
usando as seguintes definiçoes não usuais tensores de segunda ordem arbitrarios $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

$(\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_{ij}B_{kl}$ ou seja $\quad (\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$

$(\mathbf{A} \odot \mathbf{B})_{ijkl} = A_{il}B_{jk}$ ou seja $\quad (\mathbf{A} \odot \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}^T\mathbf{B}^T$

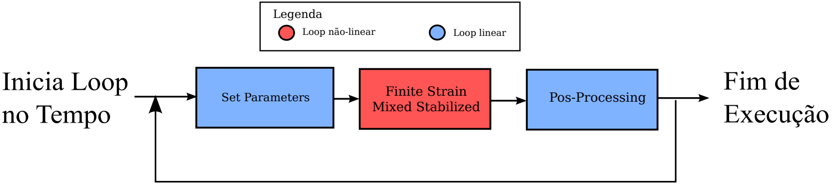
$(\mathbf{A} \circledcirc \mathbf{B})_{ijkl} = A_{ik}B_{jl}$ ou seja $\quad (\mathbf{A} \circledcirc \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^T$

Visão de Execução



Piola++ : Arquivo de Configuração de Loops

```
*TimeStep ( DelT, Tini, Tmax )
1.0 1.0 20.1

#(NSubsteps, MaxElemLib,MaxLCommonPar,MaxNodEl)
*ElementLibraryControl
3 3 23 3

# set material param (isNL)
*SubStep
0
100 0
100 0
901 -1

23*0.0
23*0.0
3 1 2 0.400969 0.0 400942.0 17*0.0

# finite strain mixed stabilized
# (isNL, if(isNL) tol maxit)
*SubStep
1
0.00000001 50
798 -2
539 -3
100 0

0 3 1 2 5 1 0 1 0 0.001 13*0.0
0 3 5 1 0.0625 0.0 1.0 16*0.0
23*0.0

# pos processing
*SubStep
0
797 -4
100 0
100 0
0 1 2 1 0 1 0 4 3 1 4 2 5 3 7 4 7*0.0
23*0.0
```



Piola++ : Interface com Fortran

```fortran
module interfaceFortran

    implicit none

    public executerElementC, executerSymbolicC
    private fortranMatrix2Carray, fortranMatrix2CarrayI

    contains

    subroutine executerElementC(id_Elem_Family, AE, BE, Ma
                                Sol0, Sol1, CommonPar, Par
```

```c
extern "C" {
    void __interfacefortran_MOD_executerelementc(int* ,
                     double*, double*, int*, double*, d
}
```

Piola-Local

- Dados: $\mathbf{u}^e \in \mathbb{R}^{n_u}, \mathbf{p}^e \in \mathbb{R}^{n_p}$, Geometria, Propriedades Materiais.
- Para cada ponto de Gauss avaliar as funções de forma e suas derivadas (em coord. globais) e defina as matrizes (caso particular $n_p = 3, n_u = 3, n_{sd} = 2$):

$$\mathbf{B}_u^e = \begin{bmatrix} \varphi_{1,x} & 0 & \varphi_{2,x} & 0 & \varphi_{3,x} & 0 \\ \varphi_{1,y} & 0 & \varphi_{2,y} & 0 & \varphi_{3,y} & 0 \\ 0 & \varphi_{1,x} & 0 & \varphi_{2,x} & 0 & \varphi_{3,x} \\ 0 & \varphi_{1,y} & 0 & \varphi_{2,y} & 0 & \varphi_{3,y} \end{bmatrix} \in \mathbb{R}^{n_{sd}^2 \times n_u}$$

$$\mathbf{B}_p^e = \begin{bmatrix} \varphi_{1,x} & \varphi_{2,x} & \varphi_{3,x} \\ \varphi_{1,y} & \varphi_{2,y} & \varphi_{3,y} \end{bmatrix} \in \mathbb{R}^{n_{sd} \times n_p}$$

$$\mathbf{N}_p^e = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix}^T \in \mathbb{R}^{n_p}$$

tais que $(\nabla \mathbf{u})_v = \mathbf{B}_u^e \mathbf{u}^e, \nabla p = \mathbf{B}_p^e \mathbf{p}^e, p = \mathbf{N}_p^e \cdot \mathbf{p}$.

- Dado que $\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}$, podemos calcular $J, \mathbf{F}^{-T}, \mathbf{C}, \mathbf{C}^{-1}, \mathbb{B}, \mathbb{E}$ e ainda:

$$(\mathbf{P}_{iso})_{ij} = \frac{\partial \Psi_{iso}}{\partial (\mathbf{F})_{ij}}(\mathbf{F}) = \frac{\Psi_{iso}(\mathbf{F} + \tau \mathbf{e}_i \otimes \mathbf{e}_j) - \Psi_{iso}(\mathbf{F})}{\tau}$$

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Piola-Local:Continuação

- Definir os vetores auxiliares:

$$\mathbf{d}^e = J(\mathbf{B}_u^e)^T(\mathbf{F}^{-T})_v \quad (\in \mathbb{R}^{n_u})$$
$$\mathbf{C}^e = \mathbf{d}^e \otimes \mathbf{N}_p^e \quad (\in \mathbb{R}^{n_u \times n_p})$$
$$\mathbf{D}^e = \mathbf{N}_p^e \otimes \mathbf{N}_p^e \quad (\in \mathbb{R}^{n_p \times n_p})$$

- Computar os blocos das matrizes tangentes elementares:

$$\mathbf{K}_{uu}^e = \int_{K^e} (\mathbf{B}_u^e)^T(\mathbb{A})_m \mathbf{B}_u^e \quad (\in \mathbb{R}^{n_u \times n_u})$$
$$\mathbf{K}_{up}^e = \int_{K^e} \mathbf{C}^e \quad (\in \mathbb{R}^{n_u \times n_p})$$
$$\mathbf{K}_{pu}^e = \int_{K^e} U''(J)(\mathbf{C}^e)^T - \frac{\alpha h_K^2}{2\mu}\int_{\Omega_m} (\mathbf{B}_p^e)^T(\mathbb{E})_m \mathbf{B}_u^e \quad (\in \mathbb{R}^{n_p \times n_u})$$
$$\mathbf{K}_{pp}^e = \int_{\Omega_m} -\frac{1}{\kappa}\mathbf{D}^e - \frac{\alpha h_K^2}{2\mu}\int_{\Omega_m} J(\mathbf{B}_p^e)^T \mathrm{C}^{-1}\mathbf{B}_p^e \quad (\in \mathbb{R}^{n_p \times n_p})$$

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$$\mathbf{K}_{pu}^e = \int_{K^e} U''(J)(\mathbf{C}^e)^T - \frac{\alpha h_K^2}{2\mu} \int_{\Omega_m} (\mathbf{B}_p^e)^T (\mathbb{E})_m \mathbf{B}_u^e \quad (\in \mathbb{R}^{n_p \times n_u})$$
$$\mathbf{K}_{pp}^e = \int_{\Omega_m} -\frac{1}{\kappa}\mathbf{D}^e - \frac{\alpha h_K^2}{2\mu} \int_{\Omega_m} J(\mathbf{B}_p^e)^T \mathbf{C}^{-1} \mathbf{B}_p^e \quad (\in \mathbb{R}^{n_p \times n_p})$$

Piola-Local:Continuação

- Definir os vetores auxiliares:

$$\begin{aligned} \mathbf{d}^e &= J(\mathbf{B}_u^e)^T(\mathbf{F}^{-T})_v \quad (\in \mathbb{R}^{n_u}) \\ \mathbf{C}^e &= \mathbf{d}^e \otimes \mathbf{N}_p^e \quad (\in \mathbb{R}^{n_u \times n_p}) \\ \mathbf{D}^e &= \mathbf{N}_p^e \otimes \mathbf{N}_p^e \quad (\in \mathbb{R}^{n_p \times n_p}) \end{aligned}$$

- Computar os blocos das matrizes tangentes elementares:

$$\begin{aligned} \mathbf{K}_{uu}^e &= \int_{K^e} (\mathbf{B}_u^e)^T(\mathbb{A})_m \mathbf{B}_u^e \quad (\in \mathbb{R}^{n_u \times n_u}) \\ \mathbf{K}_{up}^e &= \int_{K^e} \mathbf{C}^e \quad (\in \mathbb{R}^{n_u \times n_p}) \\ \mathbf{K}_{pu}^e &= \int_{K^e} U''(J)(\mathbf{C}^e)^T - \frac{\alpha h_K^2}{2\mu} \int_{\Omega_m} (\mathbf{B}_p^e)^T(\mathbb{E})_m \mathbf{B}_u^e \quad (\in \mathbb{R}^{n_p \times n_u}) \\ \mathbf{K}_{pp}^e &= \int_{\Omega_m} -\frac{1}{\kappa}\mathbf{D}^e - \frac{\alpha h_K^2}{2\mu} \int_{\Omega_m} J(\mathbf{B}_p^e)^T \mathbf{C}^{-1} \mathbf{B}_p^e \quad (\in \mathbb{R}^{n_p \times n_p}) \end{aligned}$$

Piola-Local:Continuação

- Computar os blocos dos vetores resíduos negativos:

$$\mathbf{F}_u^e = -\int_{K^e} \left[(\mathbf{B}_u^e)^T (\mathbf{P}_{iso})_v + p\mathbf{d}^e \right] + [\Pi_{ext}]_v \quad (\in \mathbb{R}^{n_u})$$

$$\mathbf{F}_p^e = -\int_{K^e} \left[U'(J) - \frac{1}{\kappa} \right] \mathbf{N}_p^e + \frac{\alpha h_K^2}{2\mu} \int_{K^e} J(\mathbf{B}_p^e)^T \mathbf{C}^{-1} \nabla p \quad (\in \mathbb{R}^{n_p})$$

- Retornar as matrizes e vetores globais para Piola++:

$$\mathbf{K}^e = \begin{bmatrix} \mathbf{K}_{uu}^e & \mathbf{K}_{up}^e \\ \mathbf{K}_{pu}^e & \mathbf{K}_{pp}^e \end{bmatrix}$$

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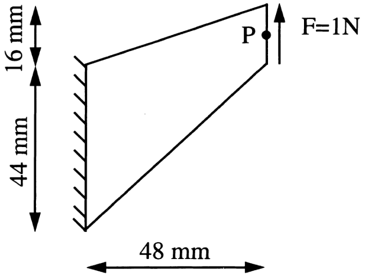
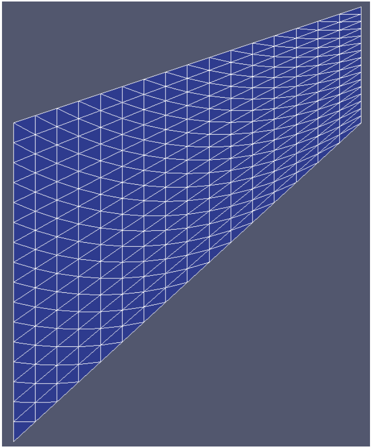
Membrana de Cook:



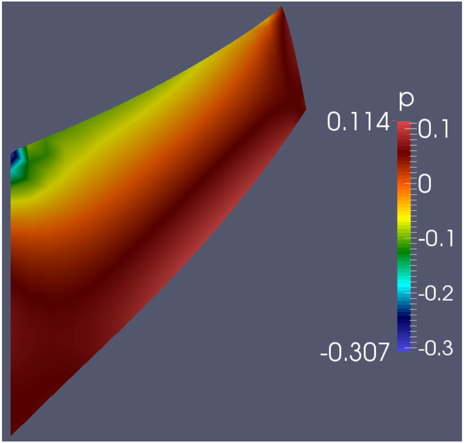
- Modelo Neo-Hookeano com $\mu = 0.801938 N/mm^2$ e $\kappa = 800942.0$ $\left(\frac{\kappa}{\mu} \sim 10^6, \nu \approx 0.49995\right)$.
- Deformação plana, i.e $\mathbf{F} = \begin{bmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- $h_e =$ max comprimento de aresta.
- O carregado é distribuido de forma homogênea na superficie.
- Convergencia com somente 1 passo de carga. Para o caso com locking, foram 50 !!!!.

Teste 1 : Escolha da parâmetro de estabilização

- Variar $\alpha \in \{1.0e-12, 0.0001, 0.001, 0.01, 0.05, 0.1\}$ e ver como se comporta a solução.
- Malha com 16 elementos por lado. Aproximaçao P2P1 e P1P1 estabilizada.



Pressão: P2P1 x P1P1stab $\alpha = 1e - 12$



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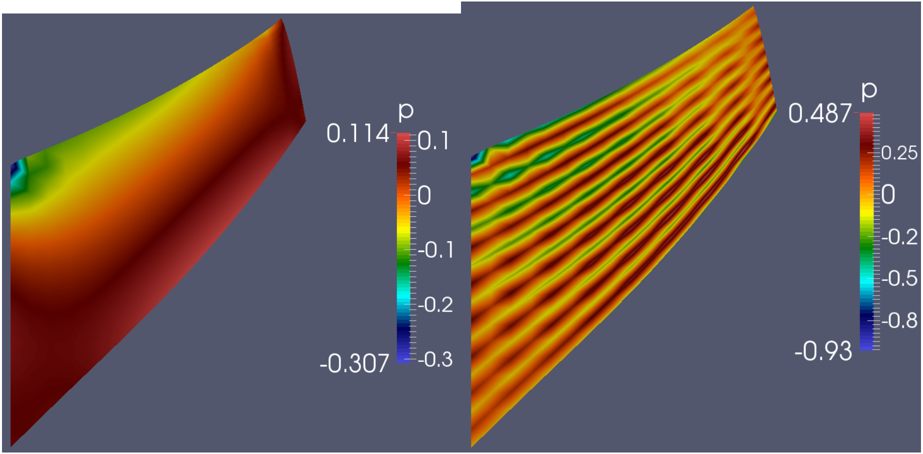
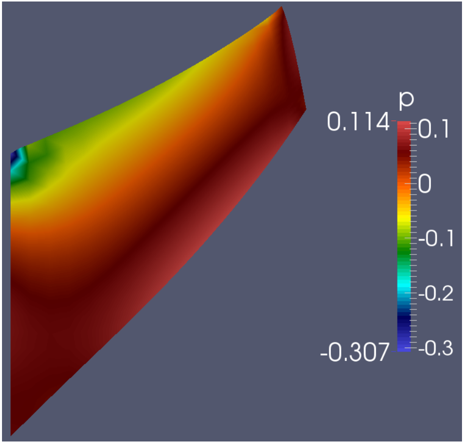
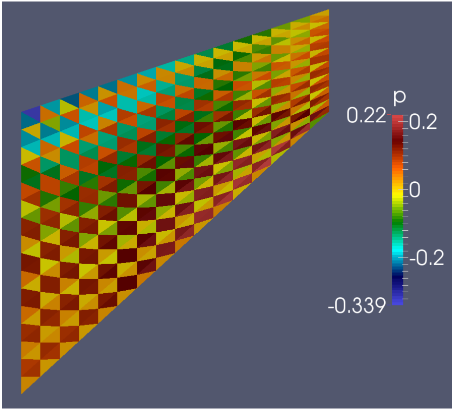
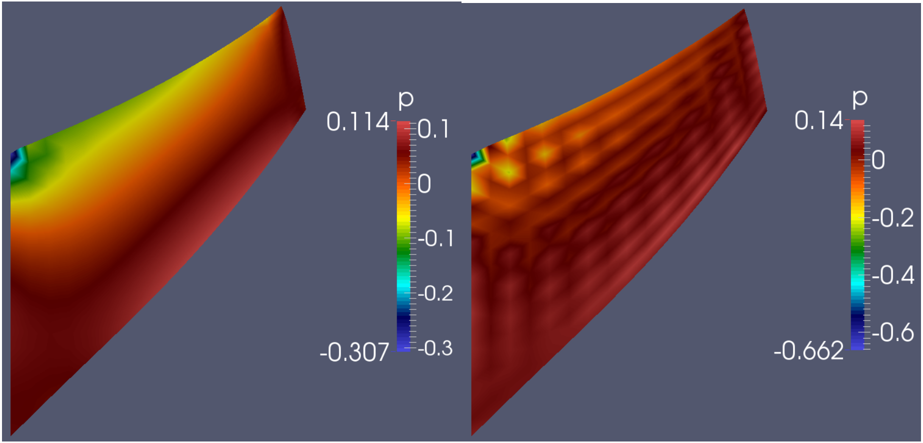


Pressão: P2P1 x P1P1stab $\alpha = 1e - 12$

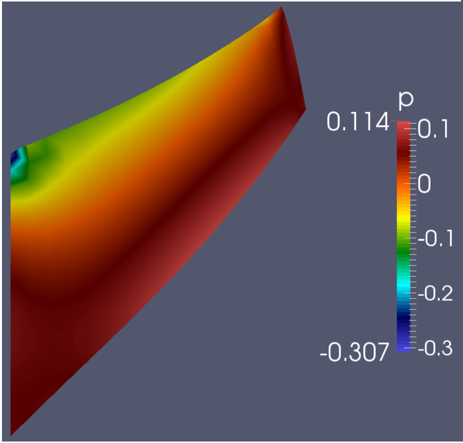


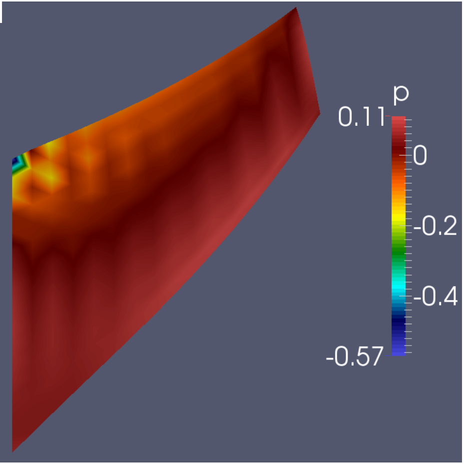


Pressão: P2P1 x P1P1stab $\alpha = 0.0001$

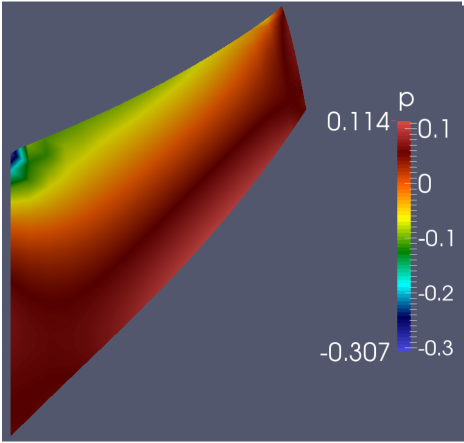


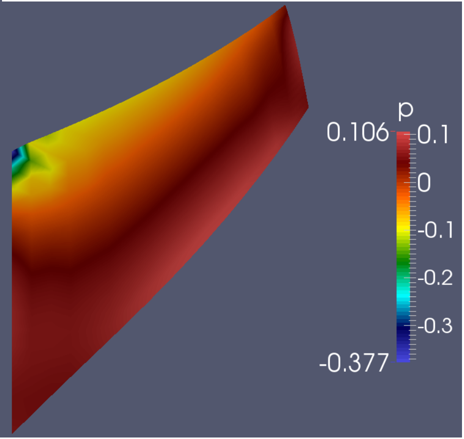
Pressão: P2P1 x P1P1stab $\alpha = 0.001$



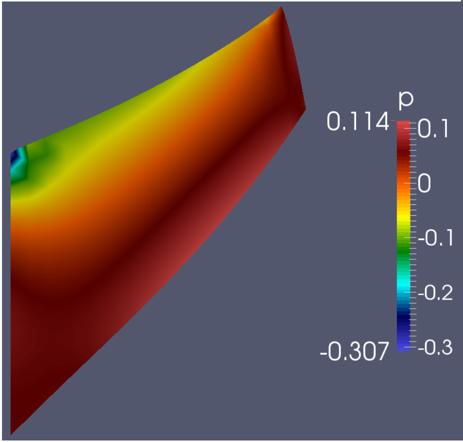


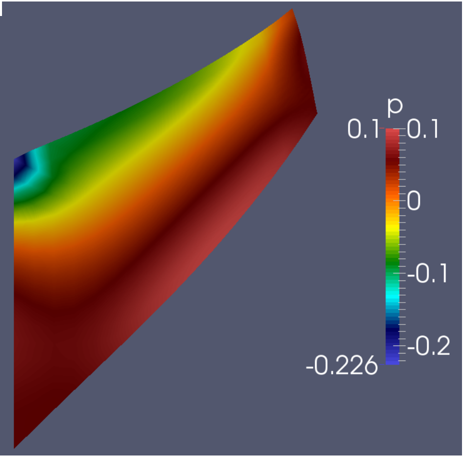
Pressão: P2P1 x P1P1stab $\alpha = 0.01$



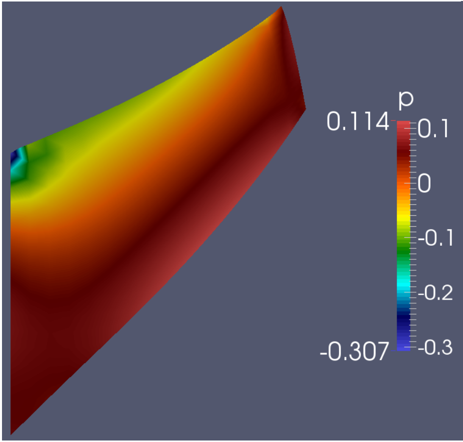


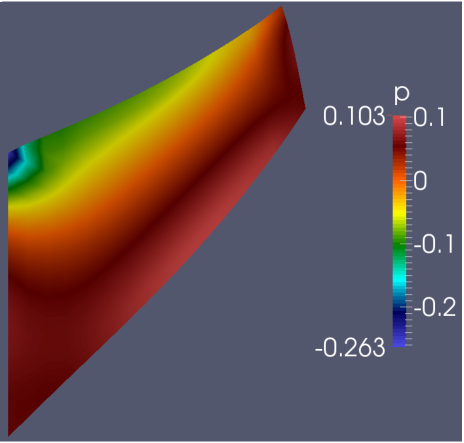
Pressão: P2P1 x P1P1stab $\alpha = 0.1$



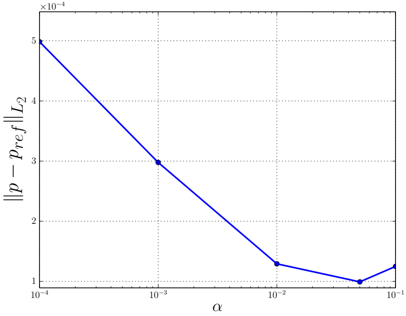


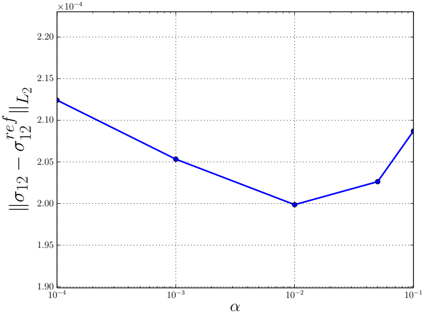
Pressão: P2P1 x P1P1stab $\alpha = 0.05$



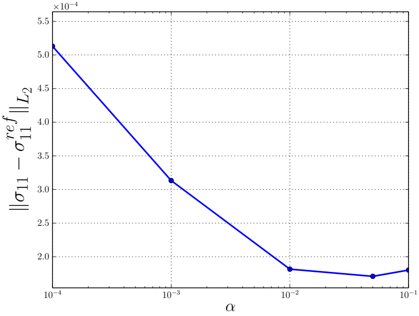


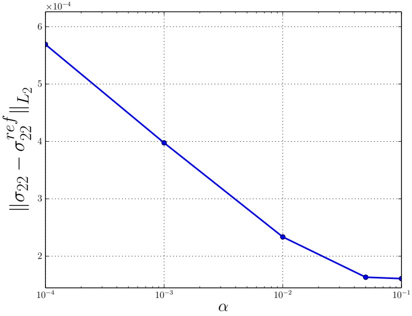
Normas: Pressão e σ_{12}



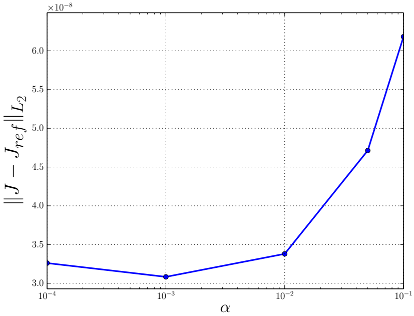


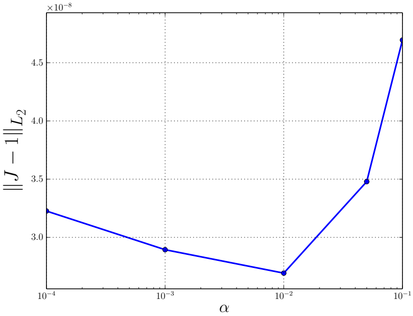
Normas: σ_{11} e σ_{22}





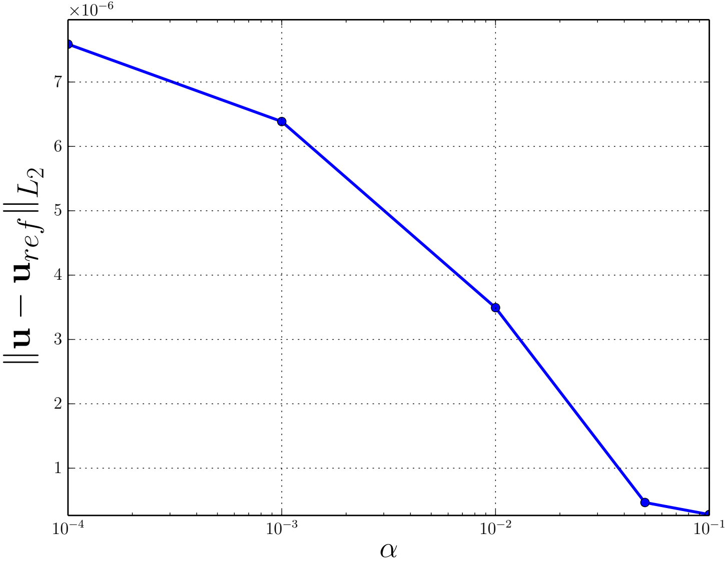
Normas: J (vs P2P1) e J (vs 1)





- Obs: $\|J_{P2P1} - 1\|_{L2} = 4.706e - 09$

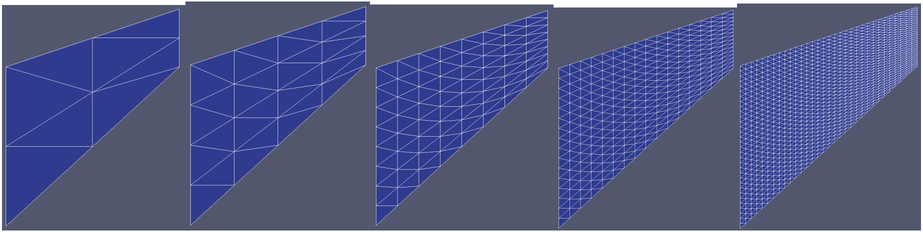
Norma: U



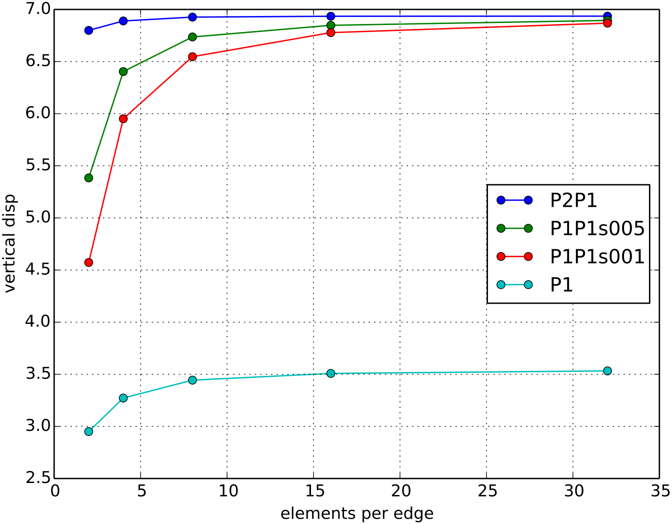
$\times 10^{-6}$
7
6
5
4
3
2
1
10^{-4}
10^{-3}
10^{-2}
10^{-1}
α
$\|\mathbf{u} - \mathbf{u}_{ref}\|_{L_2}$

Teste 2: Refinamento da malha

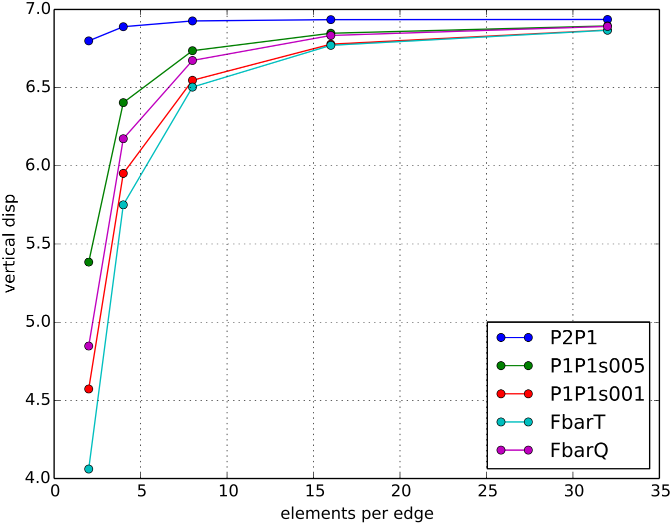
- Malha com número de elementos por lado $\in \{2, 4, 8, 16, 32\}$.
- Aproximação de um campo P1 e mista P2P1 e P1P1 estabilizada com $\alpha = 0.01$ e 0.05.



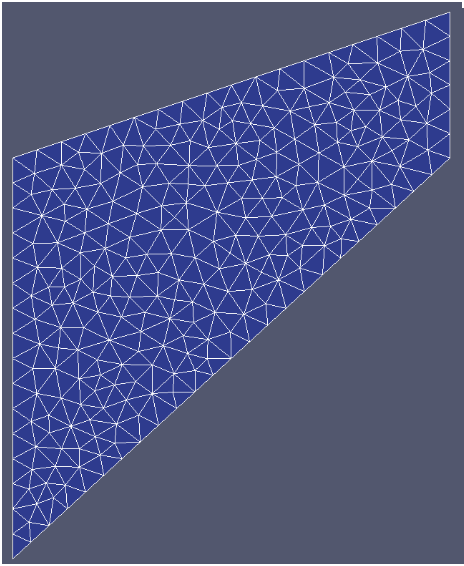
Convegência malha deslocamento topo direito

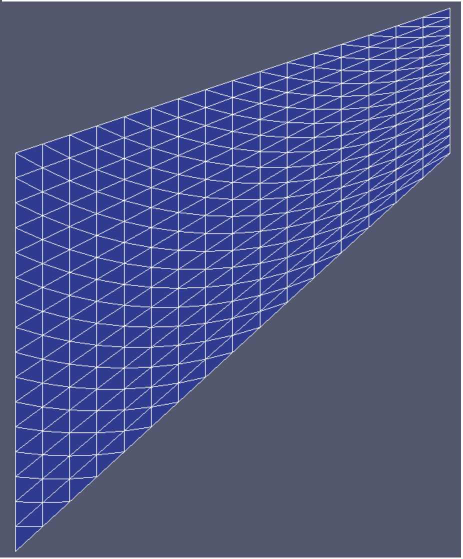


Comparação com a literatura [de Souza Neto et al,2005]



Teste 3: Malha não-Estruturada $\alpha = 0.05$ (506 vs. 512 triangulos)





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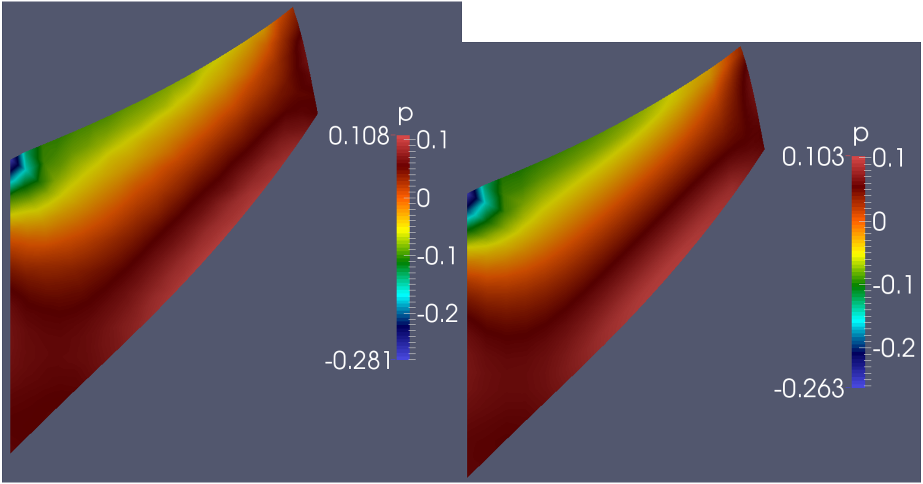
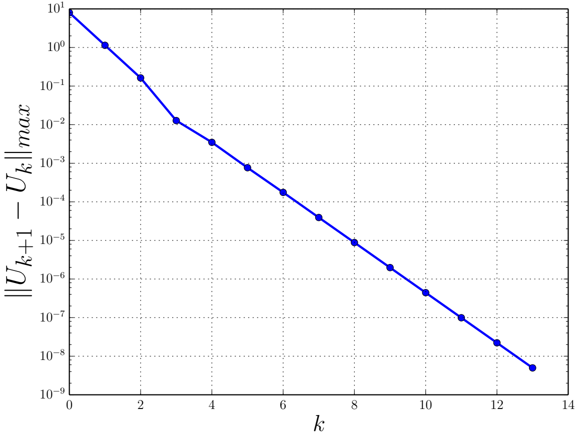


Deslocamento: 6.86908150 vs. 6.84781483

Convergência Newton-Raphson



Para
$e_k = \|\mathbf{U}_{k+1} - \mathbf{U}_k\|_{max}$

$$\frac{\log(e_{k+2}/e_{k+1})}{\log(e_{k+1}/e_k)} \approx 1!!??$$

Considerações Finais

- É resolvido a condição de Locking.
- A estabilização é altamente dependente do parametro de estabilização.
- Pode ser vantajoso dependendo do compromisso entre desempenho e precisão.

Futuras investigações

- Aproximações discontínuas para pressão.
- Outras alternativas para estabilização.

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Further Reading

E. A. de Souza Neto, F. M. Andrade Pires and D. R. J. Owen
F-bar-based linear triangles and tetrahedra for finite strain analysis of nearly incompressible solids. Part I: formulation and benchmarking
INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING 2005.

Ottmar Klaas , Antoinette Maniatty, Mark S. Shephard
A stabilized mixed Finite element method for Finite elasticity. Formulation for linear displacement and pressure interpolation.
Comput. Methods Appli. Mech. Engrg. 1999.

P.J. Sanchez , V.E. Sonzogni, A.E. Huespe
Study of a stabilized mixed finite element with emphasis in its numerical performance for strain localization problems.
COMMUNICATIONS IN NUMERICAL METHODS IN ENGINEERING 2007.

Questions ?