# Lec 01: Prerequisite - Scalar and Vector

10 January 2022 09:05

Scalar $\longrightarrow$ magnitude (eg - Temp, mass, ...)

Vector $\longrightarrow$ magnitude & direction. (eg - force, velocity, acceleration)

$\vec{A}$ & $\vec{B}$

$\vec{R}$

$O \longrightarrow P$

$\vec{A} = \vec{B}$

vector of unit mag $\longrightarrow$ unit vector

vector $\vec{A}$ can be written as

$$\boxed{\vec{A} = A \cdot \vec{a}_A}$$

$\vec{a}_A$ $\rightarrow$ unit vector in the same direction

$A$ $\rightarrow$ magnitude of $\vec{A}$

$$\vec{a}_A = \frac{\vec{A}}{A}$$

# Lec 02: Chapter 01- Introduction to Static fields

10 January 2022 09:05

**Textbooks:**

1. Electromagnetic Waves and Radiating Systems- Jordan and Balmain, PHI, 2nd edition
2. Principles of Electromagnetics Engineering- Matthew N. O.Sadiku , S.V.Kulkarni, Oxford university press, 6th edition
3. Antenna Theory: Analysis and Design, Costantine A. Balanis, John Wiley Publication, 4th edition
4. Antenna and wave Propagation, John D Kraus, A S Khan, McGraw Hill, 4th edition
5. Antenna Theory and Design. Stutzman, Theile, John Wiley and Sons, 3rd edition

**Reference Books:**

1. Engineering Electromagnetics, William H Hayt and John A Buck, Tata McGraw-Hill Publishing Company Limited, 7th edition
2. Antennas and Radio Wave Propagation, R. E. Collin, McGraw Hill, International Student Edition

## Introduction to Electrostatic Field

- Source:

Stationary Charge

- What is Charge?

Excess or deficiency of Electron in an atomic structure

- Example

Rub the surface of the balloon with the cloth and hold the balloon a short distance above your head and watch your hair stick to it!

- The balloon gains electrons from the cloth and becomes negatively-charged, so it attracts your hair, which is positively-charged

- Coulomb's Law
- Electric Field Intensity,
- Electric Field due to
    - Point charge
    - Line charge and
    - Surface charge distributions

# * Coulomb's Law.

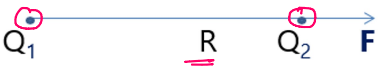
$$F = \frac{Q_1 Q_2}{4\pi\varepsilon R^2}$$

$\varepsilon = \varepsilon_0 \cdot \varepsilon_r$

$\varepsilon_r = 1$

$\varepsilon = \varepsilon_0$

- Experimental law deals with force between two point charges
- The law states that the force F between two point charges $Q_1$ and $Q_2$ is-
  - directly proportional to the products ($Q_1 \times Q_2$) of the charges
  - Inversely proportional to square of distance (R) between them
  - along the line joining them.
  - depends upon medium
  - depends upon the nature of charges (+ or – ).




- The force is expressed as –
- $F = \frac{Q_1 Q_2}{4\pi\epsilon R^2}$
- where ε is permittivity of the medium. For free space $\varepsilon = \varepsilon_0\varepsilon_r = \varepsilon_0 = 8.854 \times 10^{-12}$ F/m i.e. $\varepsilon_r = 1$

## Vector Form:

$\varepsilon = \varepsilon_0 \varepsilon_r$

$\varepsilon_r = 1$

$\varepsilon = \varepsilon_0$

- The force $\mathbf{F_2}$ on $Q_2$ due to $Q_1$ is given by-

$$\therefore \bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

Where $\bar{a}_{12}$ is unit vector in the direction of $\bar{R}_{12}$

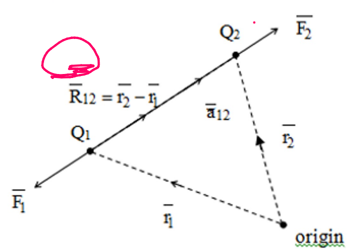
$$\bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}$$

Similarly, force $\bar{F}_1$ on $Q_1$ due $Q_2$ is given by

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}\bar{a}_{21}$$

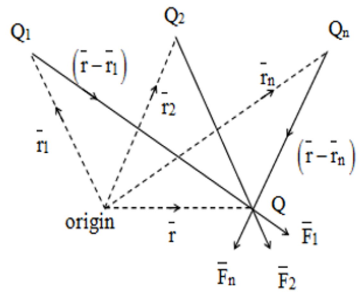
$$= \boxed{\frac{-Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}}\bar{a}_{12}$$

$$\bar{F}_1 = -\bar{F}_2$$



# Principle of Superposition:

- Consider system of n point charges. The total force is given as-




$$\bar{F} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \ldots + \bar{F}_n$$

$$= \frac{QQ_1(\bar{r}-\bar{r}_1)}{4\pi\varepsilon_0(|\bar{r}-\bar{r}_1|)^3} + \frac{QQ_2(\bar{r}-\bar{r}_2)}{4\pi\varepsilon_0(|\bar{r}-\bar{r}_2|)^3} + \ldots + \frac{QQ_n(\bar{r}-\bar{r}_n)}{4\pi\varepsilon_0(|\bar{r}-\bar{r}_n|)^3}$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{Q_k(\bar{r}-\bar{r}_k)}{|\bar{r}-\bar{r}_k|^3}$$

Q Consider charge $Q_1 = 3\times10^{-4}$ C at M(1,2,3) & $Q_2 = -10^{-4}$ C at N(2,0,5) → in a vacuum

we desire to find the force exerted on $Q_2$ by $Q_1$

⇒ The force exerted on $Q_2$ by $Q_1$ is given by

$$\rightarrow F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 \cdot R_{12}^2} \cdot \bar{a}_{12} = F_{12}$$

$$R_{12} = r_2 - r_1 = (2-1)a_x + (0-2)a_y + (5-3)a_z$$

$$R_{12} = a_x - 2a_y + 2a_z$$

Find $|R_{12}|$ & the unit vector

$$|R_{12}| = 3$$

$$\rightarrow a_{12} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|} = \frac{1}{3}(a_x - 2a_y + 2a_z)$$

$$\therefore F_2 = \frac{3\times10^{-4} \times (-10^{-4})}{4\pi(1/36\pi)\times10^{-9} \times (3)^2} \times \frac{1}{3}(a_x - 2a_y + 2a_z)$$

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$$F_2 = -30\left(\frac{a_x - 2a_y + 2a_z}{3}\right) \text{ N}$$

$$\boxed{F_2 = -10\,a_x + 20\,a_y - 20\,a_z \quad \text{N}}$$

P1. Three point charges $Q_1$=1 mC, $Q_2$=2 mC and $Q_3$=-3mC are respectively located at (0,0,4), (-2,6,1) and (3,-4,-8). Calculate force on $Q_1$.

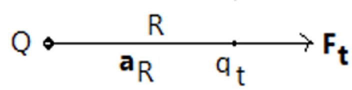
# Electric Field Intensity(E):

- Consider source charge Q.
- Place test charge($q_t$) in the region surrounding it
- The force experienced by test charge is –

$$\mathbf{F}_t = \frac{Qq_t\hat{\mathbf{a}}_R}{4\pi\epsilon R^2}$$

- Express it as force per unit charge
- $\frac{\mathbf{F}_t}{q_t}$ = Electric Field Intensity(**E**)

$$\boxed{\mathbf{E} = \frac{Q\,\hat{\mathbf{a}}_R}{4\pi\epsilon R^2}}$$

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- Unit=Volts/meter

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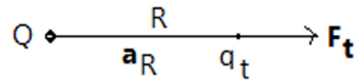


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- Unit=Volts/meter

* Principle of superposition

The electric field intensity ($\bar{E}$) due to 'n' pt charges at a pt in free space = vector sum of the ($\bar{E}$) due to each charge acting alone.

The charges are located by $\bar{r}_1, \bar{r}_2, \ldots \bar{r}_n$ & the field pt is located by radius vector $\bar{r}$

$$\bar{E} = \frac{Q_1}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_1)}{|r-r_1|^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_2)}{|r-r_2|^3} + \cdots \frac{Q_n}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}_n)}{|r-r_n|^3}$$

$$\boxed{\bar{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n} \frac{Q_k\,(r-r_k)}{|r-r_k|^3}} \longrightarrow (A)$$

Q) Two pt charges of 20nC & −20nC are situated at (1,0,0) & (0,1,0) in free space. Determine the $\bar{E}$ at (0,0,1)

$$\rightarrow \quad Q_1 = 20\,nC \text{ at } (1,0,0) \rightarrow r_1 = a_x$$
$$Q_2 = -20\,nC \text{ at } (0,1,0) \rightarrow r_2 = a_y$$
$$\Longrightarrow \quad r - r_1 = (a_z - a_x), \quad r - r_2 = (a_z - a_y)$$

$$\text{cal } E \text{ at } (0,0,1) \rightarrow r = \underline{a_z}$$

w.k.t.
$$E = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k (r - r_k)}{|r - r_k|^3} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{2} \frac{Q_k (r - r_k)}{|r - r_k|^3}$$

$$= \frac{1}{4\pi\varepsilon_0} \left[ \frac{20\times10^{-9} (a_z - a_x)}{(\sqrt{2})^3} + \frac{(-20\times10^{-9})(a_z - a_y)}{(\sqrt{2})^3} \right]$$

$$\boxed{E = -63.63\, a_x + 63.63\, a_y \quad V/m \quad (N/C)}$$

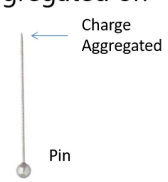
* Charge Configurations

1) Point charge.
2) line charge
3) Surface charge
4) Volume charge

(1) Point Charge

- Point charge: A point charge mean a charge whose dimensions are much smaller than other relevant dimensions

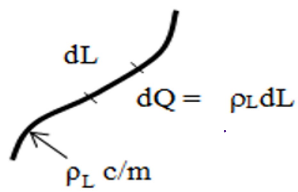
- Examples
- Charge aggregated on pin head




- Metallic spheres


2) Line charge

- It has filament like distribution of charge e.g. sharp beam in CRT, charged conductor of very small radius.



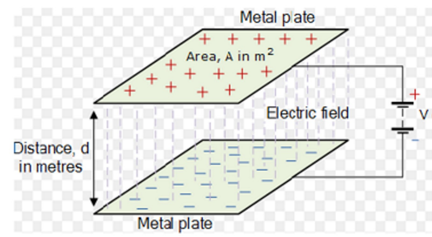

- Consider $dQ=\rho_L\, dL$
- We define the line charge density $\rho_L$ C/m as charge per unit length.
- Hence

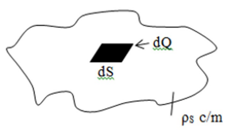
$$\therefore \rho_L = \lim_{dL \to 0} \frac{dQ}{dL} \text{ C/m}$$

3) Surface charge

- An electric charge accumulated on a surface is called surface charge.

- Example: Parallel plate capacitor

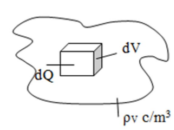



- Consider incremental charge $dQ=\rho_s\, ds$
- We define the surface charge density ($\rho_S$) as the charge per unit area.
- Hence $\therefore \rho_S = \lim_{dS \to 0} \frac{dQ}{dS}$ C/ m$^2$

4) Volume charge

Here charge is confined within a volume.
It is volume like distribution of charge
e.g. Region surrounding cathode in CRT




- Consider $dQ=\rho_v\, dV$
- We define the line charge density $\rho_v$ C/m as charge per unit volume.
- Hence

$$\rho_v = \lim_{dv \to 0} \frac{dQ}{dv} \text{ c/m}^3$$

Lec 04: Electric flux and Electric flux density

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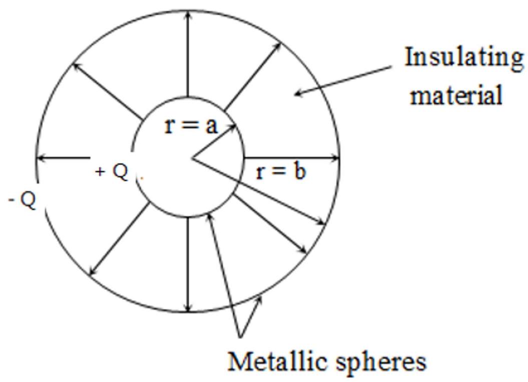
## Faraday's Experiment




- With the equipment dismantled, the inner sphere was given a known positive charge ( +Q)
- The hemisphere were then clamped together around the charged sphere with dielectric material between them.
- The outer sphere was discharged.
- The charge on inner surface of outer sphere is measured.

* Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere
* This was true regardless of the dielectric material separating the two spheres
* He concluded that, there was some sort of 'displacement' from the inner sphere to the

- From Faraday's experiment,
- Electric Flux ($\psi$ ) $\propto$ charge (Q)
- $\therefore\ \psi = KQ$
- In SI system, K = 1
- $\therefore \psi = Q$ coulomb

- Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere
- This was true regardless of the dielectric material separating the two spheres
- He concluded that, there was some sort of 'displacement' from the inner sphere to the outer which was independent of the medium
- We will refer this as displacement flux or simply electric flux($\psi$)

- From Faraday's experiment,
- Electric Flux ($\psi$ ) $\propto$ charge (Q)
- $\therefore \psi = KQ$
- In SI system, $K = 1$
- $\therefore \psi = Q$ coulomb

## Electric Flux Density( **D**)

- The electric flux density(**D**) at any point is defined as electric flux or electric displacement per unit area.
- It is a vector quantity.
- Its direction being taken as that of the normal to the surface element which makes the displacement through the element of area a maximum

- The flux density is-
- $\mathbf{D} = \frac{\text{Flux}}{\text{Area}} = \frac{Q}{4\pi R^2}\hat{\mathbf{a}}_r$

## Relation between **E** and **D**

E

The electric flux density ($\bar{D}$) at a point P which is at distance r meters from point charge Q is given as–

$$\bar{D} = \frac{Q}{4\pi r^2}\hat{a}_r \quad (1)$$

Similarly $\bar{E}$ at a point P in free space is-

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \quad (2)$$

From above equations

$\therefore\ \bar{D} = \varepsilon_0\,\bar{E}$ (for free space)

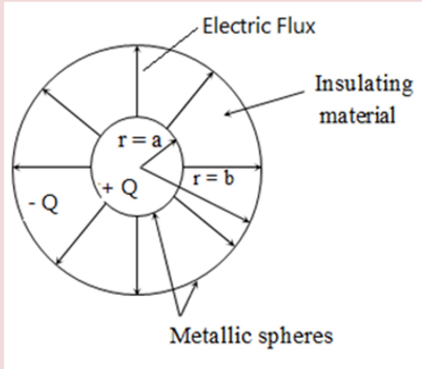
- The Electric field intensity(**E**) is dependent on the medium in which the charge is placed.
- Electric flux density(**D**) is independent of medium
- **D** provides the same information as **E** does

## Maxwell's Equation for Static Field

### Gauss's Law

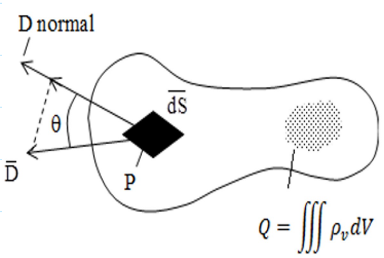
- From Faraday's experiment,
    - Electric Flux ($\psi$ ) $\propto$ charge (Q)
    - $\therefore \psi = KQ$
    - In SI system, $K = 1$
    - $\therefore \psi = Q$ coulomb
- Statement: The electric flux passing through any closed surface is equal to the total charge enclosed by that surface

- Faraday's Experiment




## Mathematical Form

- Consider volume like distribution of charge, surrounded by a closed surface of any shape




- Let D be the flux density on surface which will vary in magnitude and direction from point to point on the surface.
- The incremental flux($d\Psi$) through incremental surface (**dS**) are given as-
- $d\Psi = D_{normal} dS = D \cos\theta dS = \mathbf{D} \cdot \mathbf{dS}$

$\therefore$ total flux through the closed surface

$$\psi = \int d\psi = \oint_{\text{closed surface}} \bar{D} \cdot \bar{dS} \quad \text{——— } (3)$$

$= Q =$ charge enclosed.

# Lec 05: GAUSS'S LAW

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$$d\psi = D_{normal}\, ds$$

$$d\psi = D \cos\theta \cdot ds \quad \text{— (1)}$$

$$d\psi = \bar{D} \cdot \bar{ds} \quad \text{— (2)}$$

$$\psi = \int d\psi = \oint \bar{D} \cdot \bar{ds} \quad \text{— (3)}$$

closed surface

$= \underline{\underline{Q}} =$ charge enclosed

For volume charge density ($\rho_v$), we can write

$$\boxed{\oint_S \bar{D} \cdot \bar{ds} = \int_{vol} \rho_v \cdot dv \quad \text{— (4)}}$$

↳ Eqⁿ (4) represents

MAXWELL's EQ$^{N}$ → Integral form

* Differential or Point form of GAUSS's LAW

To obtain point form → use divergence theorem

⇓

relates surface integral with volume integral.

$$\oint_S \bar{A} \cdot \bar{ds} = \int_{vol} (\nabla \cdot \bar{A})\, dv$$

Applying Divergence theorem to eqⁿ (4)

$$\therefore \oint_S \bar{D} \cdot \bar{ds} = \int_{vol} (\nabla \cdot \bar{D}) \cdot dv = \int_{vol} \rho_v dv \quad \text{— (5)}$$

Comparing two volume integrals

$$\boxed{\nabla \cdot \bar{D} = \rho_v \quad \text{— (6)} \quad \text{— (Maxwell's eq}^n \text{ point-form)}}$$

# Divergence defined in all coordinate system

$$\text{div } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad \text{(cartesian)}$$

$$\text{div } \mathbf{D} = \frac{1}{r}\frac{\partial}{\partial r}(r D_r) + \frac{1}{r}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div } \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}$$

* Applications of Gauss's Law

used to determine $\bar{D}$ & $\bar{E}$ → symmetric charge distributions

derivations $\bar{D}$ & $\bar{E}$
- → Point charge
- → Uniform line charge
- → Infinite surface charge

(i) Point charge

$$\bar{D} = \frac{Q}{4\pi r^2}\,\bar{a_r}$$

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\bar{a_r} \qquad \{\vec{D} = \varepsilon_0 \vec{E}\}$$

(ii) Uniform line charge

$$\vec{D} = \frac{\rho_L}{2\pi\rho}\,\bar{a_\rho}$$

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\,\vec{a_\rho}$$

(iii) Infinite surface charge

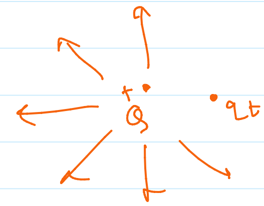
$$\vec{D} = \frac{\rho_s}{2}\,\vec{a_z}$$

$$\bar{E} = \frac{\rho_s}{\varepsilon_0 (2)}\,\bar{a_z}$$

$$\bar{E} = \frac{\rho_s^2}{\varepsilon_0 (2)} \bar{a}_2$$

→ (i) Work done

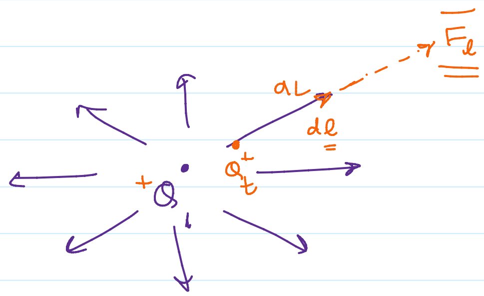
(ii) Potential difference




* WORK DONE

Consider a charge +Q → electric field present –

— " — a unit test charge $q_t$




⇒ +ve charge → $Q_1$ ↳ $\bar{E}$

test charge $Q_t$ placed in this field → it will move due to the repulsive force.

Let the movement of charge $Q_t$ → $dl$
& the direc. of movement → $\bar{a}_L$ (unit vector)
↳ in the direc. of $dl$

∴ According to Coulomb's law the force exerted by $E$ is given by.

$$\underline{\bar{F}} = Q_t \cdot \bar{E} \quad - (N) \quad \text{——— (1)}$$

Thus, the comp. of $\bar{F}$ in the direction of unit vector $\bar{a}_L$ is given by.

$$\boxed{\bar{F}_l = F \cdot \cos\theta = \bar{E} \cdot \bar{a}_L = Q_t \cdot \bar{E}\, \bar{a}_L} \quad \text{——— (2)}$$

This is the force responsible to move the charge $Q_t$ through the distance $dl$ in the direction of the field.

To keep the charge in equilibrium
↳ reqd to apply the force which is equal & opposite to the force exerted by the field in the direc. of $dl$.

field in the direc. of dL.

$$\boxed{\overline{F}_{applied} = -\overline{F}_{L} = -Q_t\,\overline{E}\cdot a_L \quad (N)} \qquad -(3)$$

In this case, the work is said to be done.

↳ is expenditure of energy which is given by the prod. of force & the distance

Hence, mathematically the differential work done by an external source in moving the charge $Q_t$ through a distance $dl$, against the direction of field $\overline{E}$ is given by

$$dW = \overline{F}_{applied} \times dl$$

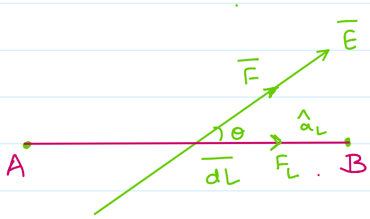
$$= -Q_t\,\overline{E}\; a_L\cdot dl \qquad (4)$$

But distance vector $dL$

$$\rightarrow dl\cdot a_L = dL \qquad (5)$$

$$\boxed{dW = -Q_t\,\overline{E}\cdot\overline{dL} \quad \text{Joules}} \qquad (6)$$

* Electric Potential(V)




- Consider a region of Electrostatic field (**E**)
- We have to move the charge from B to A
- The force we must apply is equal and opposite to force the experienced by charge Q, i.e., we have to do work.
- Suppose we have to move charge Q a distance **dL** in an electric field **E.** The force on Q is

**F** = Q**E** (1)

The component of force in the direction **dL,** which must be overcome is

$\mathbf{F}_L = \mathbf{F}.\hat{a}_L = Q\mathbf{E}.\hat{a}_L$ eq$^n$ (2)

We have to apply equal and opposite force to move the charge

**Fapply** = **-** Q**E**.$\hat{a}_L$ (3)

If it moves through distance dL, work done will be

$$dW = -Q\,\mathbf{E}.\mathbf{dL} \qquad (4)$$

Hence total work done or the potential energy required to move from B to A is -

$$W = -Q\int_{B\,(initial)}^{A(final)} \mathbf{E}\cdot\mathbf{dL} \qquad (5)$$

* Potential Difference (V) ⇒ [units Volts]

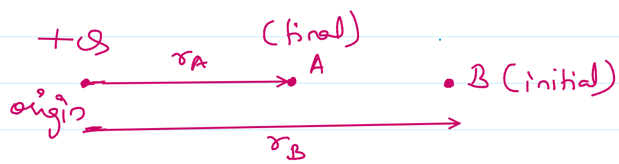
Potential difference is defined as work done in moving a unit +ve charge from one point to another in an electric field

$$\therefore \text{Potential difference } (V_{AB}) = \frac{W}{Q} = -\int_B^A \bar{E}\cdot\bar{dL}$$

**P1**:Find potential difference between points A and B at radial distance $r_A$ and $r_B$ from a point charge Q.

- Consider Charge Q at the origin of the spherical coordinate system
- The **E** is given as

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$




- The differential length is

$$\overline{dL} = dr\,\hat{a}_r$$

- By definition the potential difference is $\rightarrow V_{AB} = -\int_B^A \bar{E}\cdot\bar{dL} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\cdot dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$

$$\therefore V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B$$

where $V_B$ & $V_A$ are potentials at B & A

* **Absolute Potential( V) also called as Scalar potential**

- If we select infinity as reference with zero potential i.e.
- $r_B$ tends to infinity and $V_B$=0
- Hence the absolute potential of point A is –

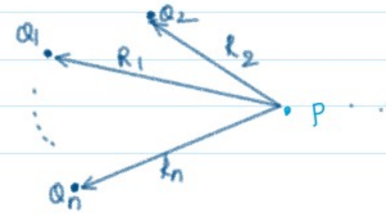
$$V = \frac{Q}{4\pi\varepsilon r}$$

* Potential field of a system of charges

Consider a system with n. number of pt charges $Q_1, Q_2, ----Q_n$

$$V_p = \frac{Q_1}{\;} + \frac{Q_2}{\;} + \cdots + \frac{Q_n}{\;}$$

$$V_p = \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 R_n}$$

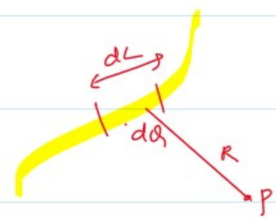



$$\boxed{V_p = \frac{1}{4\pi\varepsilon_0} \sum_{K=1}^{n} \frac{Q_K}{R_K}}$$

(i) 'V' due to line charge

diff charge on the line is given by

$$dQ = \rho_L \cdot dL$$




$$\therefore\ dV = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_L \cdot dL}{4\pi\varepsilon_0 R}$$

$\therefore$ The total potential at pt P due to line charge is

$$\boxed{V = \int dV = \int_L \frac{\rho_L\, dL}{4\pi\varepsilon_0 R}}$$

(ii) V due to surface charge

$$\boxed{V = \int_S \frac{\rho_s\, ds}{4\pi\varepsilon_0 R}}$$

(iii) V due to volume charge

$$\boxed{V = \int_V \frac{\rho_v\, dv}{4\pi\varepsilon_0 R}}$$

Lec 06: Work done and electric potential

20 January 2022 13:29

$$\boxed{dW = -Q_t \bar{E}\cdot \bar{dL} \quad \text{Joules}} \qquad (6)$$

The total work done → by integrating the diff work done over the distance from initial position to the final position

∴ From eqn (6) we have

$$W = \int_{\text{Initial}}^{\text{final}} dW = \int_{\text{Initial}}^{\text{final}} -Q\,\bar{E}\cdot\bar{dL}$$

$$\boxed{\therefore W = -Q\int_{\text{initial}}^{\text{final}} \bar{E}\cdot\bar{dL}} \qquad (7)$$

* Potential diff (V)

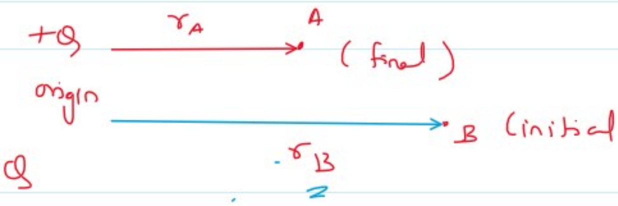
w.k.t. work done in moving the pt charge Q from pt B to pt A in the electric field $\bar{E}$ is given by

$$W = -Q\int_B^A \bar{E}\cdot\bar{dL} \qquad (1)$$

Potential diff (V) is defn as the work done per unit charge in moving the unit charge from B to A in the field $\bar{E}$

$$\boxed{V_{AB} = V = \frac{W}{Q} = -\int_B^A \bar{E}\cdot\bar{dL}} \qquad (2)$$

Eg:- Find the potential diff. betn pts A & B at radial distance $r_A$ & $r_B$ from the pt charge Q




spherical → $(r, \theta, \phi)$

Consider charge Q at origin

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

$$dL = \underline{dr\,\hat{a}_r} + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$

$$dL = dr\,\hat{a}_r$$

$$V_{AB} = -\int_B^A \bar{E}\cdot\bar{dL} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r\cdot[dr\cdot\hat{a}_r] = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}dr$$

$$\boxed{V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)}$$

Absolute or Scalar potential

It is often convenient to speak → absolute (scalar) potential

⇓ is possible

If we have selected specific ref-pt with zero potential
(It is customary to choose infinity as ref-pt.

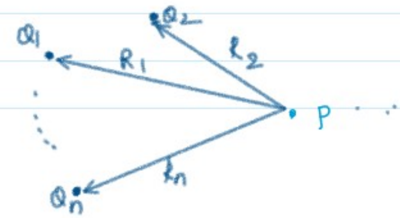
∴ The potential at any pt ($r_B \to \infty$) due to a pt charge Q located at origin is given by

$$\boxed{V = \frac{Q}{4\pi\varepsilon_0 r}}$$

* Potential field of a system of charges

Consider a system with n. number of pt charges $Q_1, Q_2, ---- Q_n$

$$V_p = \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 R_n}$$




$$\boxed{V_p = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{Q_k}{R_k}}$$

(i) 'V' due to line charge

diff charge on the line is given by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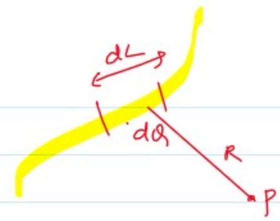



$$dQ = \rho_L \cdot dL$$

$$\therefore \quad dV = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_L \cdot dL}{4\pi\varepsilon_0 R}$$

$\therefore$ The total potential at pt P due to line charge is

$$\boxed{V = \int dV = \int_L \frac{\rho_L \, dL}{4\pi\varepsilon_0 R}}$$

(ii) V due to surface charge

$$\boxed{V = \int_S \frac{\rho_s \, dS}{4\pi\varepsilon_0 R}}$$

(iii) V due to volume charge

$$\boxed{V = \int_V \frac{\rho_v \, dv}{4\pi\varepsilon_0 R}}$$

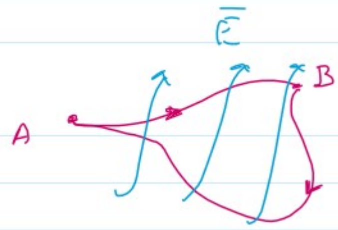
Lec 07: Relation between E and V, Current densities

24 January 2022 08:52

## Relationship between $\bar{E}$ & $\bar{V}$ — Maxwells equation

The poten. diff between pt A & B is independent of the path. Hence,

$$V_{AB} = -V_{BA} \quad \text{—} (1)$$




$$\because \; V_{AB} + V_{BA} = \oint_L \bar{E} \cdot \bar{dL} \quad \text{—} (2)$$

OR

$$\boxed{\oint_L \bar{E} \cdot \bar{dL} = 0} \quad \text{—} (3)$$

This shows that the line integral of $\bar{E}$ along a closed path (ref. fig). must be zero.

$\Rightarrow$ Physically this implies that no net work is done in moving a charge along the closed path. in an electrostatic field

Apply Stoke's theorem to eq$^n$ (3)

Stoke's theorem relates line integral with surface integral

$$\left[ \oint_L \bar{A} \cdot dL = \int_S (\nabla \times \bar{A}) \cdot \bar{ds} \right.$$

$$\because \oint_L E \cdot \bar{dL} = \int_S (\nabla \times E) \cdot ds = 0$$

$$\boxed{\therefore \; \nabla \times E = 0} \quad \text{———} (4)$$

Any vector field that satisfy eq$^n$ (3) & eq$^n$ (4) is said to be conservative or irrotational field

* Potential gradient

w.k.t. $\quad V = -\int \bar{E} \cdot \bar{dL}$

$$dV = -\bar{E} \cdot \bar{dL} = -E_x\,dx - E_y\,dy - E_z\,dz \quad (a)$$

$dv \rightarrow$ change in 'v' $\rightarrow$ from vector calculus we have

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} \cdot dy + \frac{\partial V}{\partial z} \cdot dz \quad \text{— (b)}$$

Comparing (a) & (b)

$$E_x = -\frac{\partial V}{\partial x} \; ; \; E_y = -\frac{\partial V}{\partial y} \; ; \; E_z = -\frac{\partial V}{\partial z}$$

$$\boxed{\therefore \text{ Thus, } \quad E = -\nabla V} \quad \text{—— (c)}$$

Hence $\bar{E}$ is the gradient of $V$. The $-ve$ sign shows that the direction of $\bar{E}$ is opposite to the direction in which $V$ increases.

**NOTE**

1) Cartesian coordinate system

$$\nabla V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z$$

2) cylindrical coordinate system

$$\nabla V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial}{\partial z} \hat{a}_z$$

3) spherical coordinate system

$$\nabla V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \cdot \hat{a}_\theta + \frac{1}{r \sin\theta} \cdot \frac{\partial V}{\partial \phi} \hat{a}_\phi$$

* Energy Density in Electrostatic field ($W_E$)

It is def. as energy / cubic meter

$$\therefore W_E = \frac{1}{2} \bar{D} \cdot \bar{E}$$

$$= \frac{1}{2} \varepsilon E^2$$

$$= \frac{1}{2} \frac{D^2}{\varepsilon} \quad \text{Joules.}$$

The energy stored in terms of energy density is

$$W = \int_{vol} w_E \, dV = \frac{1}{2} \int \bar{D} \cdot \bar{E} \, dv$$

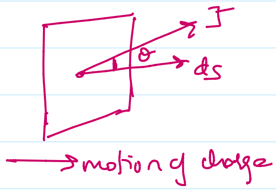
* Current, Current density and continuity equation

(i) Current → Electric charge in motion constitutes c/n

$$I = \frac{dQ}{dt} \quad \text{--- (1)}$$

(ii) Current density ($\bar{J}$)

$\bar{J}$ at a given pt is defined as the c/n through a unit normal area at that pt. The incremental amount of c/n $dI$ crossing an incremental surface $ds$ is




$$dI = J \, ds \cos\theta$$
$$= \bar{J} \cdot \overline{ds}$$

∴ The total c/n is given by

$$\boxed{I = \int_s \bar{J} \cdot \overline{ds}} \quad \text{--- (2)}$$

# Lec 08:Continuity equation

27 January 2022 10:29

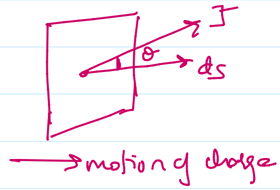
## * Current, Current density and Continuity equation

(i) Current → Electric charge in motion constitutes c/n

$$I = \frac{dQ}{dt} \quad \text{—— (1)}$$

(ii) Current density ($\bar{J}$)

$\bar{J}$ at a given pt is defined as the c/n through a unit normal area at that pt. The incremental amount of c/n $dI$ crossing an incremental surface $ds$ is

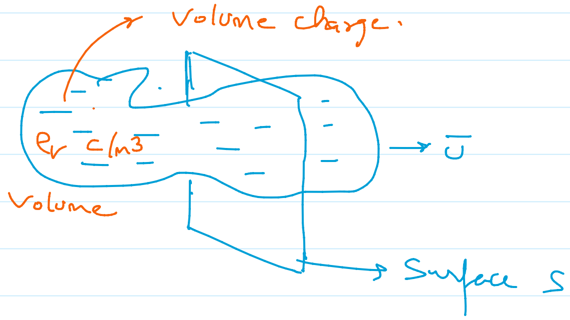



$$dI = J\,ds\cos\theta$$
$$= \bar{J}\cdot\bar{ds}$$

∴ The total c/n is given by

$$\boxed{I = \int_S \bar{J}\cdot\bar{ds}} \quad \text{—— (2)}$$

## * Convection c/n density




when charge passes through surface 's' it constitutes a convection c/n with density($\bar{J}$)

$$\bar{J} = \rho_v \bar{U} \quad \text{—— (3)}$$

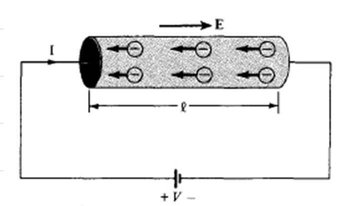
where $\rho_v$ → vol. charge density

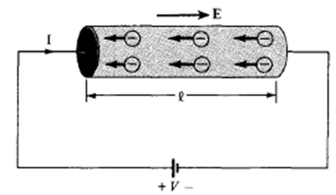
$\bar{U}$ → velocity to the right

* Do not satisfy Ohm's law

- It occurs when current flows through insulating medium such as liquid, vacuum etc. Example: Electron beam in CRT

## * Conduction c/n density






When an elec-field is applied to a conductor, conduction c/n occurs due to the drift motion of electrons.

∴ c/n density is given by

$$\bar{J} = \sigma \bar{E} \quad \text{—— (4)}$$

where

$\sigma \rightarrow$ conductivity of the material.

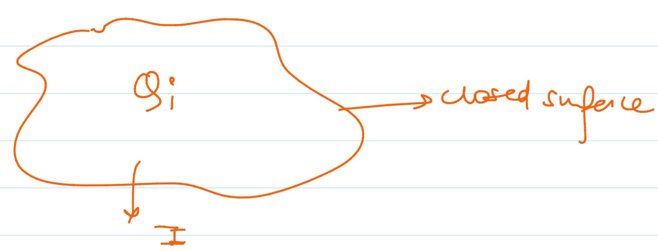
## Continuity Equation

**Conservation of Charge:-**

"The Principle of conservation of charge states that Charge can be neither created nor destroyed, Although equal amounts of positive and negative charge may be simultaneously created, obtained by seperation, destroyed or lost by recombination".

The c/n through the closed surface is given by

$$I = \oint_S \bar{J} \cdot \bar{ds} \quad \text{—— (1)}$$




This outward flow must be balanced by a decrease in +ve charge within the closed surface

$\boxed{Q \ \& \ \rho_v}$

$$I = \oint_S \bar{J} \cdot \bar{ds} = -\frac{d}{dt} Q_i \quad \text{—— (2)}$$

$$\boxed{\therefore I = \oint_S \bar{J} \cdot \bar{ds} = -\frac{d}{dt} \int_{vol} \rho_v \, dv \quad \text{—— (3)}}$$

→ Integral form of CONTINUITY EQUATION.

Apply divergence theorem

$$\therefore \oint_S \bar{J} \cdot d\bar{s} = \int_{Vol} (\nabla \cdot \bar{J})\, dv = -\frac{d}{dt} \int_{Vol} \rho_v\, dv$$

If the surface is kept constant, the derivative becomes partial derivative & may appear within the integral sign

$$\therefore \int_{vol} (\nabla \cdot \bar{J})\, ds = -\int \frac{\partial \rho_v}{\partial t}\, dv$$

Since this expression is true for any volume, it is true for increamental volume

$$\boxed{\therefore \nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{--- (4)}}$$

$\Rightarrow$ Point form of CONTINUITY EQ$^{N}$

## Poisson's & Laplace's Equation

Poisson's & Laplace's eq$^{n}$ are easily derived from Gauss's Law.

$$\nabla \cdot \bar{D} = \rho_v \quad \text{--- (1)}$$

w.k.t $\quad \bar{D} = \varepsilon \bar{E} \quad$ --- (2)

& also the gradient relationship for $\bar{E}$

$$\bar{E} = -\nabla V \quad \text{--- (3)}$$

Substituting (2) & (3) in (1)

$$\nabla \cdot \bar{D} = \nabla \cdot (\varepsilon \bar{E}) = \underbrace{\nabla \cdot (-\varepsilon \nabla V) = \rho_V}$$

For a homogeneous medium in which 'ε' is constant

$$\nabla \cdot \nabla V = -\frac{\rho_V}{\varepsilon}$$

$\Downarrow$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \longrightarrow (4)$$

This is known as POISSON'S EQ$^{N}$

A special case of the above eq$^n$ occurs when $\rho_v = 0$ (i.e. for a charge free region).

$\therefore$ Eq$^n$ (4) is written as,

$$\boxed{\nabla^2 V = 0} \longrightarrow (5)$$

LAPLACE'S EQ$^{N}$

i) RCS $\Rightarrow$ $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

ii) CCS $\Rightarrow$ $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

iii) SCS $\Rightarrow$ $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$

If a potential $V = x^2yz + Ay^3z$

(i) Find $A$ so that Laplace's eq$^n$ is satisfied.

(ii) With the value of 'A' determine elec. field at $(2, 1, -1)$

## Application of Laplace's and Poisson's Equation

- Using Laplace or Poisson's equation we can obtain:

1. Potential at any point in between two surface when potential at two surface are given.
2. We can also obtain capacitance between these two surface.

POYNTING THEOREM

27 February 2023 20:03

**Poynting Theorem :**

When electromagnetic waves propagate through space from their source to distant receiving points, there is a transfer of energy from the source to the receivers. In order to find the power in an electromagnetic waves, it is necessary to develop a power theorem, which is known as Poynting theorem. It can be obtained from Maxwell's equation as follows

Consider Maxwell's equation from modifies Ampere's circuit law,

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \dots (1)$$

Dot each side of equation (1) with $\overline{E}$

$$\overline{E}.(\nabla \times \overline{H}) = \overline{E}.\overline{J} + E.\frac{\partial \overline{D}}{\partial t} \qquad \dots (2)$$

Now make use of vector identity,

$$\nabla.(\overline{E} \times \overline{H}) = \overline{H}.\nabla \times \overline{E} - \overline{E}.\nabla \times \overline{H} \qquad \dots (3)$$

from (2) and (3)

$$\therefore \ \overline{H}.\nabla \times \overline{E} - \nabla.(\overline{E} \times \overline{H}) = \overline{E}.\overline{J} + \overline{E}.\frac{\partial \overline{D}}{\partial t} \qquad \dots (4)$$

but $\quad \nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$

and therefore,

$$-H.\frac{\partial \overline{B}}{\partial t} - \nabla.(\overline{E} \times \overline{H}) = \overline{E}.\overline{J} + \overline{E}.\frac{\partial \overline{D}}{\partial t} \qquad \dots (5)$$

or

$$-\nabla.(\overline{E} \times \overline{H}) = \overline{E}.\overline{J} + \varepsilon \overline{E}.\frac{\partial \overline{E}}{\partial t} + \mu H.\frac{\partial \overline{H}}{\partial t} \qquad \dots (6)$$

However

$$\varepsilon \bullet \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} = \frac{\varepsilon}{2}\frac{\partial E^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2}\right)$$

$$\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} = \frac{\mu}{2}\frac{\partial H^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2}\right)$$

Thus equation (6) becomes,

$$-\nabla \cdot (\bar{E}\times\bar{H}) = \bar{E}.\bar{J} + \frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) \qquad \dots (7)$$

Integrate equation (7) throughout a volume

$$-\int_{vol} \nabla.(\bar{E}\times\bar{H})dv = \int_{vol} (\bar{E}.\bar{J})dv + \frac{\partial}{\partial t}\int_{vol}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv \qquad \dots (8)$$

Apply divergence theorem to left hand side of equation (8)

$$-\oint_S (\bar{E}\times\bar{H}).\overline{ds} = \int_{vol} (\bar{E}.\bar{J})dv + \frac{\partial}{\partial t}\int_{vol}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv \qquad \dots (9)$$

The equation (9) can be written as

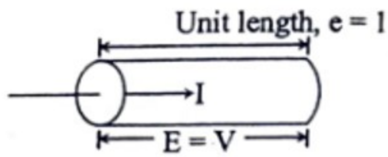
$$\underbrace{\oint_S (\bar{E}\times\bar{H}).\overline{ds}}_{\text{I term}} = \underbrace{-\int_{vol} (\bar{E}.\bar{J})dv}_{\text{II term}} \underbrace{- \frac{\partial}{\partial t}\int_{vol}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv}_{\text{III term}} \qquad \dots (10)$$

Equation (10) is referred to as Poynting theorem

**Physical Interpretation :**

II term : $\int_{vol} (\bar{E}.\bar{J})dv$

This term represents power dissipated in the volume V. Consider a conductor of cross sectional area A, carrying current I and voltage drop E per unit length. Therefore power loss per unit length is EI.




∴ Power dissipated per unit volume would be

$\frac{EI}{A}$ = E J watts per unit volume.

∴ In general, when $\bar{E}$ and $\bar{J}$ are not in same direction,

Power dissipated per unit volume $= \bar{E}.\bar{J}$

∴ The total power dissipated in a volume V would be

$$\int_v (\bar{E}.\bar{J})dv$$

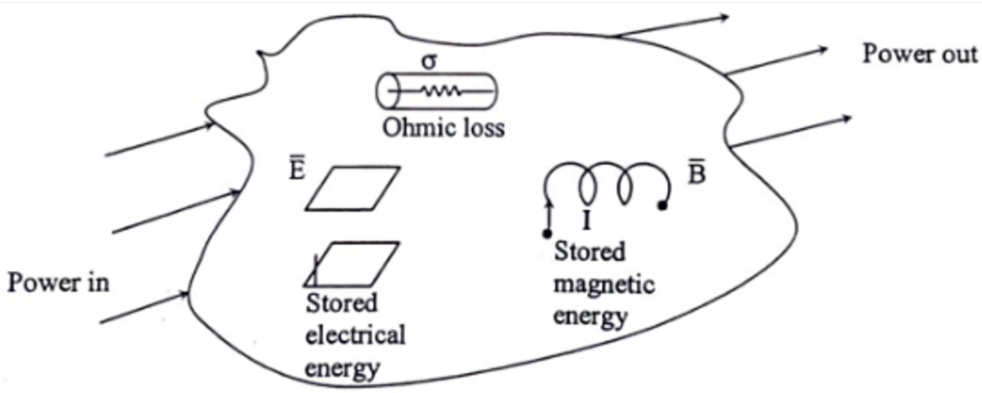
III term : $-\frac{\partial}{\partial t}\int_{vol}\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right)dv$

In electrostatic, the quantity ½ $\varepsilon E^2$ is energy density or stored energy per unit volume of the electric field. Also for the steady magnetic field the quantity ½ $\mu H^2$ represents the stored energy density of the magnetic field. The negative time derivative of this quantity represents the rate at which the stored energy in the volume is decreasing.

I[st] term : $\oint_s (\overline{E}\times\overline{H}).\overline{ds}$

The interpretation of I term follows from the application of the law of conservation of energy. The rate of energy dissipation in the volume v must equal the rate at which the stored energy in v is decreasing, plus the rate at which energy is entering the volume v from outside.

$\therefore \oint_s (\overline{E}\times\overline{H}).\overline{ds}$ represents outward flow of energy through the surface enclosing the volume and $-\oint_s (\overline{E}\times\overline{H}).\overline{ds}$ represents inward flow of energy through the surface enclosing the volume.




**Statement :** The net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the conduction loss :

**The Poynting Vector :**

The interpretation of poynting theorem, leads to the conclusion that the integral of $\overline{E}\times\overline{H}$ over any, closed surface gives the rate of energy flow through that surface. This cross product $\overline{E}\times\overline{H}$ is known as poynting vector $\overline{p}$,

$$\overline{P} = \overline{E}\times\overline{H} \qquad \dots (11)$$

The poynting vector has dimensions of watts per square meter. It is a measure of rate of energy flow per unit area at that point. The direction of flow is perpendicular to $\overline{E}$ and $\overline{H}$ in the direction of the vector $\overline{E}\times\overline{H}$.

In perfect dielectrics, $\overline{E}$ and $\overline{H}$ fields are given as

$$E_x = E_0 \cos(\omega t - \beta z)$$

$$H_y = \frac{E_0}{\eta}\cos(\omega t - \beta z)$$

The Poynting vector is

$$\overline{P} = \overline{E} \times \overline{H}$$

$$\therefore\ P_z \hat{a}_z = E_x \hat{a}_x \times H_y \hat{a}_y$$

$$\therefore\ P_z = \frac{E_0^2}{\eta}\cos^2(\omega t - \beta z)$$

To find the time average power density, integrate over one cycle and divide by the period $T = \frac{1}{f}$.

$$\therefore P_z, av = f\int_0^{1/f} \frac{E_0^2}{\eta}\cos^2(\omega t - \beta z)\,dt$$

$$= \frac{f}{2}\frac{E_0^2}{\eta}\int_0^{1/f}[1 + \cos(2\omega t - 2\beta z)dt$$

$$= \frac{f}{2}\frac{E_0^2}{\eta}\left[t + \frac{1}{2\omega}\sin(2\omega t - 2\beta z)\right]_0^{1/f}$$

$$= P_z, av = \frac{1}{2}\frac{E_0^2}{\eta}\ \omega/m^2 \qquad \ldots (12)$$

If medium is lossy dielectric,

$$E_x = E_0 e^{-\alpha z}\cos(\omega t - \beta z)$$

$$H_y = \frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta n)$$

$$\therefore\ P_z = \frac{E_0^2}{|\eta|}e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta n)$$

$$\therefore\ P_z, av = \frac{1}{T}\int_0^T P_z dt$$

$$\because\ \cos A \cos B = ½\,[\cos(A-B) + \cos(A+B)]$$

$$P_z, av = \frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos\theta_n$$

If we use, rms values instead of pear amplitudes, the factor of ½ would not be present. It can be shown that, the time average poynting vector can be also written as

$$P_z, av = \tfrac{1}{2} \operatorname{Re}(\overline{E} \times \overline{H})$$

Finally, the average power flowing through any are 'S' normal to the z–axis is

$$P_{ave} = \frac{1}{2}\frac{E_0^2}{\eta} S \text{ watt.} \quad \ldots (13)$$

For any given surface 'S', it is given by

$$P_{ave} = \int_S \overline{P_{av}} \cdot \overline{dS} \quad \ldots (14)$$

# Lec 09: Biot Savart's Law

27 January 2022 10:29

Q) If a potential $V = x^2yz + Ay^3z$
(i) Find $A$ so that Laplace's eqn is satisfied.
(ii) With the value of 'A' determine elec. field at $(2,1,-1)$

$\Rightarrow$

$$\boxed{V = x^2yz + Ay^3z}$$

(i) Find A

Laplace's eqn in CCS (RCS)

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{--- (1)}$$

$$\frac{\partial V}{\partial x} = 2xyz \; ; \quad \frac{\partial^2 V}{\partial x^2} = 2yz$$

$$\frac{\partial V}{\partial y} = x^2z + 3Ay^2z \; ; \quad \frac{\partial^2 V}{\partial y^2} = 0 + 6Ayz$$

$$\frac{\partial V}{\partial z} = x^2y + Ay^3 \; ; \quad \frac{\partial^2 V}{\partial z^2} = 0$$

Substituting above value's in eqn (1)

$$\nabla^2 V = 2yz + 6Ayz = 0$$

$$\boxed{\therefore A = -1/3}$$

(ii) $\bar{E}$ at $(2,1,-1)$

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \cdot \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z\right]$$

w.r.t.

$$V = x^2yz - \frac{1}{3}y^3z$$

$$\bar{E} = -(2xyz)\hat{a}_x + \left(x^2z - \frac{3y^2}{3}z\right)\hat{a}_y + \left(x^2y - \frac{y^3}{3}\right)\hat{a}_z$$

$$\boxed{\therefore \bar{E} \text{ at } (2,1,-1) = 4\hat{a}_x + 3\hat{a}_y - \frac{11}{3}\hat{a}_z}$$

* Steady Magnetic field

$\rightarrow$ Biot Savart's Law (Coulombs Law)

→ Ampere's circuit Law ( Gauss's Law)

Static Elec. field → charac. by $\bar{E}$ or $\bar{D}$

Likewise

Static mag. field → charac by $\bar{H}$ or $\bar{B}$

Electrostatic → produced by static or stationary charges

If charges are moving with const. velocity → $\underbrace{\text{static magnetic field}}^{\text{Magnetostatic field}}$ is produced

Magnetic field in free space due to direct c/n is considered.

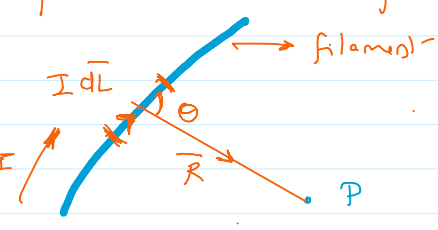
Two major Law's governing magnetostatic fields

→ 1) Biot–Savart Law ( Ampere Law)

→ 2) Ampere Circuital Law.

* Biot – Savart's Law.

It gives differential magnetic field intensity ($d\bar{H}$) due to differential c/n element ($I d\bar{L}$)




Mag. of $d\bar{H}$ at pt P is proportional to the product of c/n, diff length $dL$ & the sine of ∠le betn the filament & the line connecting differential length to the pt of interest P.
It is inversely prop. to the square of the distance from filament to point P. The constant of proportionality is given by ($1/4\pi$)

$$\boxed{d\bar{H} = \frac{I\, dL \sin\theta}{4\pi R^2}} \quad \text{——— (1)}$$

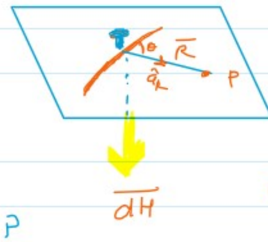
Direction of $dH$ is normal to the plane containing the diff element & the line drawn from the filament to the pt P.

In vector notation

In vector notation

cross product

$$\overline{dH} = \frac{I\,dL \times \hat{a}_R}{4\pi R^2} \qquad (2)$$




Right-hand screw

where

$\bar{a}_R \rightarrow$ unit vector from diff c/n element to pt P

$I\,\overline{dL} \Rightarrow$ differential c/n element

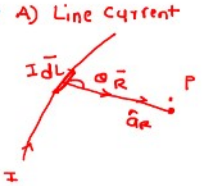
$R \Rightarrow$ distance of diff c/n element from pt P

∴ Biot-Savart Law in integral form

$$\bar{H} = \int \frac{I\,\overline{dL} \times \hat{a}_R}{4\pi R^2} \qquad (3)$$

## Biot-Savart for different current distributions

A) Line current

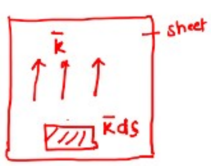



$$d\bar{H} = \frac{I\,d\bar{L} \times \hat{a}_R}{4\pi R^2}$$

Total magnetic field intensity ($\bar{H}$)

$$\bar{H} = \int_L \frac{I\,d\bar{L} \times \hat{a}_R}{4\pi R^2} \qquad (5)$$

B) surface current




$\bar{K}$ = surface current density (A/m)

(current per unit width)

$$\bar{H} = \int_S \frac{\bar{K}\,ds \times \hat{a}_R}{4\pi R^2} \qquad (6)$$

C) Volume current



$\bar{J}$ = volume current density $A/m^2$

$$\bar{H} = \int_{vol} \frac{\bar{J}\,dv \times \hat{a}_R}{4\pi R^2} \qquad (7)$$

$$I\,dL = \bar{K}\,ds = \bar{J}\,dv \qquad (4)$$

# Lec 10: Ampere's circuit law and Maxwell's equation

31 January 2022 21:59

Biot Savart's is also called as Ampere's law for the c/n element

Applications

(i) $\bar{H}$ due to infinitely long straight conductor

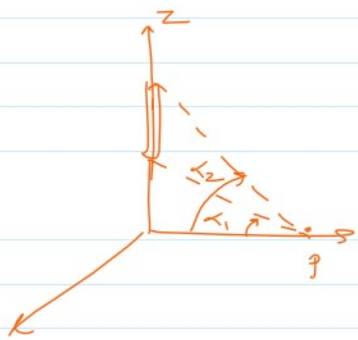
$\rightarrow \bar{H} = \frac{I}{2\pi r} \hat{a}_\phi$ A/m

$\rightarrow \bar{B} = \mu \bar{H} = \frac{\mu I}{2\pi r} \hat{a}_\phi$ wb/m$^2$

(ii) $\bar{H}$ due to straight conductor of finite length

$\bar{H} = \frac{I}{4\pi r} [\sin\alpha_2 - \sin\alpha_1] \hat{a}_\phi$ A/m

$\bar{B} = \mu \bar{H} = \frac{\mu I}{4\pi r} [\sin\alpha_2 - \sin\alpha_1] \hat{a}_\phi$ wb/m$^2$




(iii) $\bar{H}$ at the centre of a circular conductor

$\bar{H} = \frac{I}{2R} \hat{a}_N$ A/m

$\bar{B} = \mu \bar{H} = \frac{\mu I}{2R} \hat{a}_N$ wb/m$^2$

* Ampere's Circuital Law

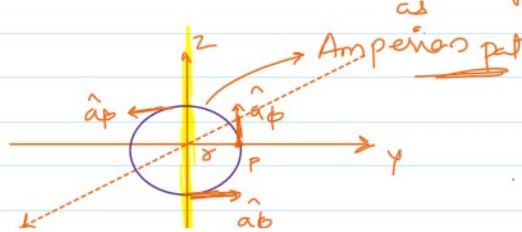
Statement:-

Ampere's circuital law states that the line integral of the tangential component of the magnetic field around a closed path is equal to the c/n enclosed by the path.

$$\oint \underbrace{\bar{H} \cdot \bar{dL}} = I_{enc.} \quad \text{—— (A)}$$

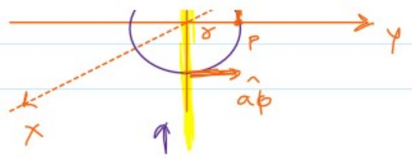
Proof:- Here $\bar{H} = H_\phi \hat{a}_\phi$

$\therefore \underbrace{\bar{dL} = r\, d\phi\, \hat{a}_\phi}$ —— (1)




While $\bar{H}$ obtained at pt P from Biot-Savart's

While $\bar{H}$ obtained at pt P from Biot-Savart's law due to infinitely long conductor is given by




I ( Iencl = I )

$$\bar{H} = \frac{I}{2\pi r}\hat{a}_\phi \quad \text{—(2)}$$

$$\therefore \bar{H}\cdot\bar{dl} = \frac{I}{2\pi r}\hat{a}_\phi \cdot (r\,d\phi\,\hat{a}_\phi)$$

$$= \frac{I}{2\pi r} r\,d\phi = \frac{I}{2\pi}\,d\phi$$

Integrating $\bar{H}\cdot\bar{dl}$ over the entire closed path

$$\oint \bar{H}\cdot\bar{dl} = \int_{\phi=0}^{2\pi} \frac{I}{2\pi}\,d\phi = \frac{I}{2\pi}\left[2\pi\right] = I$$

= c/n carried by the conductor

This proves that the integral $\bar{H}\cdot\bar{dl}$ along the closed path gives the direct c/n enclosed by that closed path

Appl$^n$

(i) $\bar{H}$ due to infinitely long straight conductor

(ii) $\bar{H}$ due to Infinite sheet of c/n

* Curl

Curl $\bar{H} = \nabla\times\bar{H} = \bar{J}$ ⇒ pt-form of ampere's circuital law.

w.k.t.

$$\oint \bar{H}\cdot\bar{dl} = I_{encl} \quad \text{—(1)}$$

The above eq$^n$ can also be expressed as

$$\oint \bar{H}\cdot\bar{dl} = I_{enclo} = \int_S \bar{J}\cdot\bar{ds} \quad \text{—(2)}$$

Applying stokes theorem to eq$^n$ (2)

$$\oint_L \bar{H}\cdot dL = \int_S (\nabla\times\bar{H})\cdot\bar{ds} = \int_S \bar{J}\cdot\bar{ds}$$

Comparing the surface integrals

Comparing the surface integrals

$$\nabla \times \bar{H} = \bar{J} \quad (3)$$

→ Maxwell's eqn in pt form

* Magnetic flux density ($\bar{B}$)

w.k.t.

$$\bar{B} = \mu_0 \bar{H} \quad (1)$$

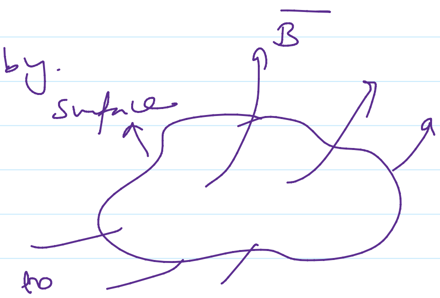
$\bar{D} = \epsilon_0 \bar{E}$

$\bar{B} = \mu_0 H$

where $\mu_0$ = permeability of free space
$= 4\pi \times 10^{-7}$ H/m

Mag. flux through an open surface 'S' is given by.

$$\phi = \int_S \bar{B} \cdot \bar{ds} \quad (3)$$




The flux penetrating a closed surface is equal to the flux leaving the closed surface. Therefore, for a closed surface,

$$\oint_S \bar{B} \cdot \bar{ds} = 0 \quad (4)$$

Applying divergence theorem

$$\oint_S \bar{B} \cdot \bar{ds} = \int_V (\nabla \cdot \bar{B}) \cdot dv = 0$$

or

$$\boxed{\nabla \cdot \bar{B} = 0} \quad (5)$$

→ This is called as pt-form or diff form of GAUSS'S LAW for MAGNETIC FIELD

4th Maxwell eqn

Maxwell's eqn for static field

Maxwells eqn for static field

magnetic & electric

| Differential (pt form) | Integral form | Remark |
|---|---|---|
| 1) $\nabla \cdot \bar{D} = \rho_v$ | $\oint_S \bar{D} \cdot \bar{ds} = \int_V \rho_v \, dv$ | → Gauss's Law |
| 2) $\nabla \cdot \bar{B} = 0$ | $\oint_S \bar{B} \cdot \bar{ds} = 0$ | → Non existence of mag. monopoles |
| 3) $\nabla \times \bar{E} = 0$ | $\oint_L \bar{E} \cdot \bar{dl} = 0$ | → Conservative nature of elec-static field |
| 4) $\nabla \times \bar{H} = \bar{J}$ | $\oint_L \bar{H} \cdot \bar{dl} = \int_S \bar{J} \cdot \bar{ds}$ | |

(1), (2): diverg

(3), (4): curl

Lec 11: Chapter 02

01 February 2022 09:30

* Scalar Magnetic Potential ($V_m$), vector potential ($\bar{A}$)

Mag. potential — Scalar ($V_m$)
— vector ($\bar{A}$)

① Scalar Mag. Potential ($V_M$)

The mag. field intensity ($\bar{H}$)

$$\bar{H} = -\nabla V_m \quad \text{—— (1)}$$

Minus sign → analogy to the electric potential

This definition must satisfy results of magnetic fields.
i.e., example.

$$\nabla \times \bar{H} = \bar{J} \quad \text{—— (2)}$$

from (1) & (2)

$$\nabla \times (-\nabla V_M) = \bar{J} = 0$$

w.r.t. curl of gradient of any scalar $= 0$

Thus to define $\bar{H}$ as gradient of a scalar mag. potential, the c/n density must be zero through the region in which scalar mag. potential is defined.

$$\therefore \bar{H} = (-\nabla V_M) \text{ if } \bar{J} = 0 \quad \text{—— (3)}$$

* vector mag potential ($\bar{A}$)

From magnetostatic field w.r.t,

$$\nabla \cdot \bar{B} = 0 \quad \text{—— (1)}$$

Also from vector calculus, divergence of curl of any vector field is zero

ie. $$\nabla \cdot (\nabla \times \bar{A}) = 0 \quad \text{—— (2)}$$

In order to satisfy eqⁿˢ (1) & (2) simultaneously

$$\bar{B} = \nabla \times \bar{A} \qquad —③$$

where $\bar{A}$ signifies a vector magnetic potential.

| | | |
|---|---|---|
| **1.0** | | **Introduction to Static fields** |
| | **1.1** | Charge, Coulomb's law, Charge configurations, Electric field intensity, Electric flux density, Gauss's law and applications, Current density, and Continuity equation |
| | **1.2** | Scalar Electric Potential, Potential gradient, Laplace's and Poison's equations |
| | **1.3** | Biot Savart Law, Ampere Circuit law, Gauss's law for magnetic field, Vector magnetic potential |
| **2.0** | | **Electromagnetic Field and Maxwell's Equations** |
| | **2.1** | Faraday's Law, Displacement current density, Maxwell's equation for time varying filed, Boundary conditions. |
| | **2.2** | EM wave propagation through lossy, perfect dielectric and conducting medium. |
| | **2.3** | Power in EM Wave: Poynting theorem and Poynting vector |

## Chapter 2:- Electromagnetic Field & Maxwell's Equations

electric field — magnetic fields

① static elec. field → produced by stationary electric charges

② static mag. field → motion of the electric charges with uniform velocity.

③ Time varying field → produced due to time varying c/n's

* Faradays Law ⇒ Maxwell eqn

He observed that

(i) when a closed path moves in a mag-field → emf gets induced & this may establish c/n in a suitable closed ckt

(ii) when closed path is kept fixed & the mag. field is varied

→ effect → commonly called → electromagnetic induction.

Faraday's law can be stated as

$$e.m.f = e = -N\frac{d\phi}{dt} \quad \text{volts} \qquad —①$$

$N$ → no. of turns (Assume $N=1$ → single turn)

$N \rightarrow$ no. of turns (Assume $N=1 \rightarrow$ single turn)

$e \rightarrow$ induced e.m.f.

$$\therefore e = -\frac{d\phi}{dt} \quad \text{—— (2)}$$

Lenz's Law $\rightarrow$ The direction of induced emf is such that it opposes the cause producing it. i.e. changes in mag. flux.

We already know that there must exist an electric field inside a conductor to sustain a c/n within the conductor. So we define the induced e.m.f. in a conductor in terms of induced elec. field intensity $(\bar{E})$ inside the conductor as

$$emf = e = \oint_L \bar{E} \cdot \bar{dL} \quad \text{—— (3)}$$

where path of integration is in the direction of the induced c/n

Also, the total flux enclosed by a closed path is expressed as

$$\phi = \int_S \bar{B} \cdot \bar{dS} \quad \text{—— (4)}$$

from eqn's (2), (3) & (4)

$$\boxed{e = -\frac{d\phi}{dt} = \oint_L \bar{E} \cdot \bar{dL} = -\frac{d}{dt}\int_S \bar{B} \cdot \bar{dS} \quad \text{—— (5)}}$$

# Lec 12: Maxwell's equations for time varying fields

04 February 2022 08:55

$$e = \oint \bar{E}\cdot\bar{dl} = -\frac{d}{dt}\int_s \bar{B}\cdot\bar{ds} \quad \text{——} (5)$$

There are 2 conditions for the induced e.m.f as explained below

Case(i) :- A closed ckt stationary, while a mag. flux density varying with time. (Time changing flux linking a stationary closed path.

$\therefore$ from eqn (5) we have

$$\oint \bar{E}\cdot\bar{dl} = -\int \frac{\partial \bar{B}}{\partial t}\cdot\bar{ds} \quad \text{——} (6)$$

This is similar to transformer action & emf $\Rightarrow$ transformer e.m.f.

Using Stokes theorem, a line integral can be converted to surface integral

$$\oint_s (\nabla\times\bar{E})\cdot\bar{ds} = -\int \frac{\partial \bar{B}}{\partial t}\cdot\bar{ds} \quad \text{——} (7)$$

Assuming that both the surface $\int^s$ taken over identical surfaces.

$$(\nabla\times\bar{E})\cdot\bar{ds} = -\frac{\partial \bar{B}}{\partial t}\cdot\bar{ds}$$

. Finally we have,

$$\boxed{\nabla\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}} \quad \text{——} (8)$$

Eqn (8) $\rightarrow$ represents one of the Maxwell's eqn.

If $\bar{B}$ is not varying with time, then eqn (6) & (8) $\rightarrow$ give the results obtained previously (electrostatic)

$$\therefore \oint \bar{E}\cdot\bar{dl} = 0 \quad \&$$

$$\nabla\times\bar{E} = 0$$

Case (ii) :-

A magnetic flux density stationary, while a closed path moving.

Here the emf induced is called as generator emf (motional emf)

Consider that a charge Q is moved in a mag. field $\bar{B}$ with a velocity $\bar{v}$. Then the force on a charge is given by

Consider that a charge $Q$ is moved in a mag. field $\bar{B}$ with a velocity $\bar{V}$. Then the force on a charge is given by

$$\bar{F} = Q\bar{V} \times \bar{B} \quad \text{—— (9)}$$

But the motional electric field intensity is defined as the force per unit charge.

$$\therefore \quad \bar{E}_m = \frac{\bar{F}}{Q} = \bar{V} \times \bar{B} \quad \text{—— (10)}$$

Thus the induced emf is given by.

$$\text{e.m.f} = \oint \bar{E}_m \cdot d\bar{L} = \oint (\bar{V} \times \bar{B}) \cdot d\bar{L} \quad \text{—— (11)}$$

Eqn (11) represents total emf induced when a conductor is moved in a uniform constant mag. field.

* Maxwell's eq from Ampere's Law

w.k.t. Ampere's Law in point form as

$$\nabla \times \bar{H} = \bar{J} \quad \text{—— (1)}$$

We will show, its inadequacy for time varying condn by taking divergence of each side

$$\nabla \cdot (\nabla \times \bar{H}) = \bar{\nabla} \cdot \bar{J} \quad \text{—— (2)}$$

But the divergence of the curl of any vector field is zero.

Hence $\quad \nabla \cdot \bar{J} = 0$

However from continuity eqn

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{—— (3)}$$

This is an unrealistic limitation for time varying field.
So eqn (1) must be modified before we accept it for time varying field

To do this, we add a term to eqn (1) so that it becomes

$$\nabla \times \bar{H} = \bar{J} + \bar{G} \quad \text{—— (4)}$$

Again taking the divergence

$$\nabla\cdot(\nabla\times\bar{H}) = \nabla\cdot\bar{J} + \nabla\cdot\bar{G} = 0$$

$$\nabla\cdot\bar{G} = -\nabla\cdot\bar{J} = \frac{\partial \rho_v}{\partial t}$$

But $\rho_v = \nabla\cdot\bar{D}$

$$\therefore \nabla\cdot\bar{G} = \frac{\partial}{\partial t}(\nabla\cdot\bar{D}) = \nabla\cdot\frac{\partial\bar{D}}{\partial t}$$

from which,

$$\bar{G} = \frac{\partial\bar{D}}{\partial t} \quad — (5)$$

Hence eqn (4) becomes

$$\boxed{\nabla\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t}} \quad — (6)$$

This is Maxwell's eqn (based on Amp-circuital Law) for time varying field

The term $\frac{\partial\bar{D}}{\partial t} = J_d \rightarrow$ known as displacement c/n density &

$\bar{J} \rightarrow$ conduction c/n density $(\bar{J} = \sigma\bar{E})$

For any arbitrary open surface s bounded by closed loop.

we can write eqn (6) as

$$\boxed{\oint_L \bar{H}\cdot d\bar{l} = \int_s \bar{J}\cdot d\bar{s} + \int_s \frac{\partial\bar{D}}{\partial t}\cdot d\bar{s}} \quad — (7)$$

Generalized form of Maxwell's Eqns (Time varying fields)

| Differential from | Integral from | Remark |
|---|---|---|
| (1) $\nabla\cdot\bar{D} = \rho_v$ | $\oint_s \bar{D}\cdot d\bar{s} = \int_V \rho_v\, dv$ | Gauss's Law |
| (2) $\nabla\cdot\bar{B} = 0$ | $\oint \bar{B}\cdot d\bar{s} = 0$ | Gauss's Law for mag. field |
| (3) $\nabla\times\bar{E} = -\frac{\partial B}{\partial t}$ | $\oint \bar{E}\cdot d\bar{l} = -\frac{\partial}{\partial t}\int \bar{B}\cdot d\bar{s}$ | Faraday's law |

(4) $\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$  $\oint \bar{H} \cdot \bar{dl} = \int_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right) \cdot \bar{dS}$  Modified form of A.C. Law

Boundary condition
- Electrostatic bound cond$^n$
  - $\bar{E} \rightarrow \bar{E}_t$ & $\bar{E}_n$
  - $\bar{D} \rightarrow \bar{D}_t$ & $\bar{D}_n$
- Mag. bound co
  - $\bar{H} \rightarrow \bar{H}_t + H_n$
  - $\bar{B} \rightarrow \vec{B}_{n}$ & $\vec{B}_t$

# Lec 13: Boundary conditions

08 February 2022 10:04

eg: $\bar{E} = |\bar{E}|\cos(\omega t + \phi)$ — mag; $2\pi f_0$; phase $\angle$

* Maxwell's eq$^n$ — Phasor representation (Time Varying field)

The partial derivative of a phasor w.r.t. time is equivalent to multiplying the corresponding phasor by $(j\omega)$

similarly → the second derivative is equivalent to $(j\omega)^2$

* Maxwell's eq$^n$ in Phasor form (Time Varying field)

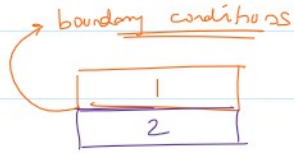
| Differential form | Integral form | PHASOR REPRESENTATION: Point form | Integral form |
|---|---|---|---|
| ① $\nabla \cdot \bar{D} = \rho_v$ | $\oint_S \bar{D}\cdot\bar{ds} = \int_V \rho_v\, dv$ | $\nabla \cdot D = \rho_v$ | $\oint_S \bar{D}\cdot\bar{ds} = \int_{vol} \rho_v\, dv$ |
| ② $\nabla \cdot \bar{B} = 0$ | $\oint \bar{B}\cdot\bar{ds} = 0$ | $\nabla\cdot\bar{B} = 0$ | $\oint_S \bar{B}\cdot\bar{ds} = 0$ |
| ③ $\nabla\times\bar{E} = -\frac{\partial B}{\partial t}$ | $\oint \bar{E}\cdot\bar{dl} = -\frac{\partial}{\partial t}\int \bar{B}\cdot\bar{ds}$ | $\nabla\times\bar{E} = -j\omega\bar{B}$ | $\oint_L \bar{E}\cdot\bar{dl} = -j\omega\int \bar{B}\cdot\bar{ds}$ |
| ④ $\nabla\times\bar{H} = \bar{J} + \frac{\partial D}{\partial t}$ | $\oint \bar{H}\cdot\bar{dl} = \int_S\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\cdot\bar{ds}$ | $\nabla\times\bar{H} = \bar{J} + j\omega\bar{D}$ | $\oint_L \bar{H}\cdot\bar{dl} = \int_S \bar{J}\cdot\bar{ds} + j\omega\int_S \bar{D}\cdot\bar{ds}$ |

PHASOR REPRESENT. →

* Boundary conditions

→ Integral form of Maxwell's eq$^n$

(i) Electrostatic Boundary conditions




$$\oint_L \bar{E}\cdot\bar{dl} = 0 \quad — ① \quad \&$$

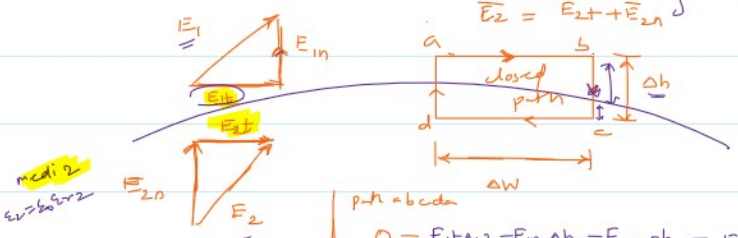
$$\oint \bar{D}\cdot\bar{ds} = Q_{enc} \quad — ②$$

$$\bar{E} = \bar{E}_t + \bar{E}_n \quad \& \quad \bar{D} = \bar{D}_t + \bar{D}_n \quad — ③$$

* Dielectric – Dielectric boundary cond$^n$

$$\left.\begin{aligned}\bar{E}_1 &= \bar{E}_{1t} + \bar{E}_{1n}\\ \bar{E}_2 &= \bar{E}_{2t} + \bar{E}_{2n}\end{aligned}\right\} — ④$$

Medium 1: $\varepsilon_1 = \varepsilon_0\varepsilon_{r_1}$




path abcda

$$0 = E_{1t}\Delta w - E_{1n}\Delta h - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + \frac{1}{2}E_{1n}\Delta h$$

As $\Delta h \to 0$

* Tangential component

$$⑤ \quad — \quad \boxed{\bar{E}_{1t} = \bar{E}_{2t}} \qquad \boxed{E_{1t} = E_{2t}}$$

Tangential comp. of $\bar{E}$ are same on the 2 sides of boundary

It is continuous across boundary [ $E_t$ undergoes no change on the boundary ]

Since $\bar{D} = \varepsilon \bar{E} = D_t + D_n$

$$\frac{D_{1t}}{\varepsilon_1} = \bar{E}_{1t} = \bar{E}_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

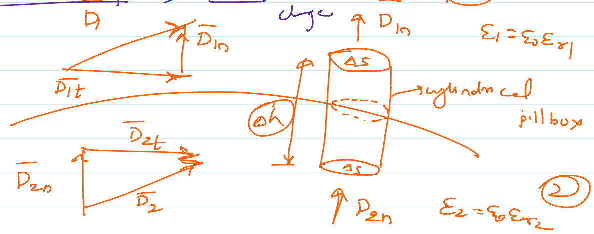
$$\boxed{\therefore \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}} \longrightarrow (6)$$

$D_t$ undergoes some change across the interface. Hence $D_t$ is said to be discontinuous across the interface

* Normal component $\longrightarrow$ $D_{1n}\Delta s - D_{2n}\Delta s = \Delta Q = \rho_s \Delta s$

$$\boxed{D_{1n} - D_{2n} = \rho_s} \longrightarrow \boxed{\rho_s = 0} \text{ if no free charge}$$

(i) $\boxed{D_{1n} = D_{2n}}$




$D_n$ undergoes no change at the boundary.

Since $\bar{D} = \varepsilon \bar{E}$

$$\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}}$$

Normal component of $\bar{E}$ is discontinuous at the boundary.

# bLec 14: Magnetic boundary conditions

10 February 2022 10:57

→ We make use of Gauss's Law for magnetic fields

$$\oint \bar{B} \cdot \bar{ds} = 0 \quad \text{—— (1)}$$

$\bar{B} \to B_t + B_n$

$\bar{H} \to H_t + H_n$

& Ampere ckt law

$$\oint \bar{H} \cdot \bar{dl} = I \quad \text{—— (2)}$$

$$\boxed{B_{1n} = B_{2n} \quad \text{or} \quad \mu_1 H_{1n} = \mu_2 H_{2n}}$$

normal component of $\bar{B}$ is continuous at the boundary whereas normal component of $\bar{H}$ is discontinuous at the boundary

$$\boxed{H_{1t} = H_{2t} \quad \text{or} \quad \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}}$$

The tangential component of $\bar{H}$ is continuous while that of $\bar{B}$ is discontinuous at the boundary.

(i) Derive wave equation (Helmholtz wave eqn)

(ii) Determine solution of vector wave equation.

* WAVE PROPAGATION

In the solution of any EM problem the fundamental relations that must be satisfied are the four

field equations

In the solution of any EM problem the fundamental relations that must be satisfied are the four field equations.

$$\nabla \times \overline{H} = \overline{J} + \frac{\delta \overline{D}}{\delta t}$$

$$\nabla \times \overline{E} = \frac{-\delta \overline{B}}{\delta t} = -\mu \frac{\delta \overline{H}}{\delta t}$$

$$\nabla . \overline{D} = \rho v$$

$$\nabla . \overline{B} = 0$$

and the characteristics of the medium in which the field exists.

$$\overline{D} = \varepsilon \overline{E}, \quad \overline{B} = \mu \overline{H} \quad \text{and} \quad \overline{J} = \sigma \overline{E}$$

The medium is homogeneous, isotropic and source free. A homogenous medium is one for which the quantities ε, μ and σ are constant throughout the medium. The medium is isotropic, if ε is scalar constant.

**Wave propagation in a conducting medium (lossy dielectrics)**

A lossy dielectric is a medium in which an EM wave loses power as it propagates due to poor conduction, ie. $\sigma \neq 0$. It is imperfect dielectric.

Consider charge free ($\rho v = 0$), a linear, isotropic homogenous and lossy dielectric medium. Maxwell's equation is phasor notation are

$$\nabla \times \overline{H} = \overline{J} + j\omega\varepsilon\overline{E} = \sigma\overline{E} + j\omega\varepsilon\overline{E} = (\sigma + j\omega\varepsilon)\overline{E} \quad \ldots (1)$$

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \quad \ldots (2)$$

$$\nabla . \overline{E} = 0 \quad \ldots (3)$$

$$\nabla . \overline{H} = 0 \quad \ldots (4)$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} = \overline{J} + (j\omega)\varepsilon E = \overline{J} + j\omega\varepsilon E = \sigma\overline{E} + j\omega\varepsilon\overline{E} = \boxed{\overline{E}(\sigma + j\omega\varepsilon)}$$

Taking curl of eqn (2)

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu\nabla \times \overline{H} \quad \ldots (5)$$

Apply vector identity,

$$\nabla \times \nabla \times \overline{A} = \nabla(\nabla . \overline{A}) - \nabla^2 \overline{A} \quad \ldots (6)$$

i.e. $\nabla \times \nabla \times \overline{E} = \nabla(\nabla . \overline{E}) - \nabla^2 \overline{E}$

since $\nabla . \overline{E} = 0$

$$\therefore \nabla \times \nabla \times \overline{E} = -\nabla^2 \overline{E} = -j\omega\mu\nabla \times \overline{H}$$

$$\nabla^2 \overline{E} = j\omega\mu(\sigma + j\omega\varepsilon)\overline{E} \quad \ldots (7)$$

Put $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \quad \ldots (8)$

$$\therefore \quad \nabla^2 \overline{E} - \gamma^2 \overline{E} = 0 \quad \ldots (9)$$

where γ is propagation constant. By similar procedure it can be shown that

$$\nabla^2 \overline{H} - \gamma^2 \overline{H} = 0 \quad \ldots (10)$$

Equation (9) and (10) are known as homogeneous vector Helmholtz equation or simply vector wave equation.

The propagation constant ($\gamma$) : Attenuation constant ($\alpha$) and phase constant or wave number ($\beta$).

The propagation constant $\gamma$ is a complex quantity.

$$\gamma = \alpha + j\beta \qquad \dots (11)$$

Consider equation (8)

$$\gamma^2 = j\,\omega\,\mu\,(\sigma + j\omega\varepsilon)$$

=

From equation (8) and equation (11) $\alpha$ and $\beta$ can be obtained.

Squaring equation (11)

$$\gamma^2 = \alpha^2 + 2j\alpha\beta - \beta^2 = (\alpha^2 - \beta^2) + 2j\alpha\beta \qquad \dots (12)$$

Equating real part of equation (8) and (12)

$$\alpha^2 - \beta^2 = -\omega^2\,\mu\,\varepsilon$$

$$\therefore \beta^2 - \alpha^2 = \omega^2\,\mu\,\varepsilon \qquad \dots (13)$$

Also $|\gamma^2| = \beta^2 + \alpha^2 \qquad \dots (14)$

$$|\gamma^2| = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2} \qquad \dots (15)$$

$$\therefore \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2} \qquad \dots (16)$$

Solving (13) and (16)

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right)} \qquad \dots (17)$$

and $$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right)} \qquad \dots (18)$$

$\gamma = \alpha + j\beta$

Also

$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

propagation constant

$$\gamma = \alpha + j\beta$$

attenuation constant

phase constant (wave number)

Solving (13) and (16)

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right)} \qquad \dots (17)$$

and

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right)} \qquad \dots (18)$$

* Solution of wave eqn

from eqn (9) we have.

$$\nabla^2 E = \gamma^2 E$$

$$\nabla^2 E - \gamma^2 E = 0 \qquad \{\text{from eqn (5)}\}$$

$(x, y, z)$

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

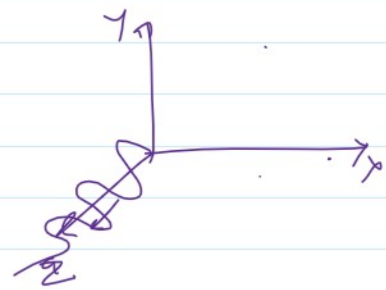
Expanding $\nabla^2$ and $\overline{E}$ in Cartesian co–ordinate system.

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \quad \text{and } \overline{E} = E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z\right) = \gamma^2\left(E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z\right)$$

Thus equation (9) is equivalent to three scalar equations

One for each component of $\overline{E}$ along $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$




The wave equation reduces to a very simple form if uniform plane wave is considered.

Consider uniform plane wave traveling in $+\hat{a}_z$ direction, $\overline{E}$ is independent of x and y and there is no component in the direction of propagation.

Therefore equation (9) reduces to,

$$\nabla^2\,\overline{E} = \frac{\partial^2\overline{E}}{\partial z^2} = \gamma^2\overline{E} \qquad \dots (19)$$

which is equivalent to

$$\frac{\partial^2}{\partial z^2}E_x = \gamma^2 E_x \qquad \dots 20(a)$$

$$\frac{\partial^2}{\partial z^2}E_y = \gamma^2 E_y \qquad \dots 20(b)$$

$$\frac{\partial^2}{\partial z^2}E_z = \gamma^2 E_z = 0 \qquad \dots 20(c)$$

$$\boxed{\nabla^2 E = \gamma^2 E}$$

$(E_x, E_y, E_z)$

In general for uniform plane wave traveling in z direction, $\overline{E}$ may have x and y component but not $E_z$.

Proof : In a region in which there is no charge density,

$$\nabla.\,\overline{E} = \frac{1}{\varepsilon}\,\nabla.\,\overline{D} = 0$$

i.e. $$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z} = 0$$

D & E

$\nabla \cdot E = 0$ $\quad D = \varepsilon E$

$$\boxed{E = \frac{D}{\varepsilon}}$$

For uniform plane in which $\overline{E}$ is independent of x and y, the first two terms of this relation are equal to zero so that it reduces to $\frac{\partial E_z}{\partial z} = 0$

There is no variation of $E_z$ in z direction. So second derivative then must be zero. It requires that

There is no variation of $E_z$ in z direction. So second derivative then must be zero. It requires that $E_z$ must be zero or constant. If it is constant, it would not be a part of wave motion. So $E_z$ must be zero.

Without loss of generality, we assume that E has only one component say $E_x$ knowing that result will be same for $E_y$.

$$\therefore \quad \frac{\partial^2 E_x}{\partial z^2} = \gamma^2 E_x \qquad \dots (21)$$

This is linear homogenous differential equation, by inspection its solution may be written as

$$E_x = E_0 e^{-\gamma z} + E_0' e^{+\gamma z} \qquad \dots (22)$$

where $E_0$ and $E_0'$ are constants. The fact that the field must be finite at infinity requires that $E_0' = 0$. The factor $e^{\gamma z}$ denotes wave traveling in $-\hat{a}_z$ where as $e^{-\gamma z}$ represents wave traveling in $+\hat{a}_z$ direction.

$$\therefore \quad E_x = E_0 e^{-\gamma z} \qquad \dots (23)$$

reinsert $e^{j\omega t}$ factor and reduce to trigonometric form by taking real part of it.

$$\therefore \quad E_x = \text{Re} E_0 e^{-\alpha z} e^{-j\beta z} e^{j\omega t}$$

$$E_x = E_0 e^{-\alpha z} \cos(\omega t - \beta z) \qquad \dots (24)$$

$$\therefore \overline{E}(z, t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{a}_x \qquad \dots (25)$$

→ prop. const

$\gamma = \alpha + j\beta$

attenuation const

phase const (wave number)

$\alpha \rightarrow$

$\beta \rightarrow$

**Calculation of $\overline{H}$ :**

Given $\overline{E}$, $\overline{H}$ can be calculated using Maxwell's equation,

$$\nabla \times \overline{E} = -j\omega\mu \overline{H}$$

Since $\overline{E}$ has only x component and is independent of x and y since wave is traveling in z–direction.

$$\therefore \nabla \times \overline{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -j\omega\mu \overline{H}$$

Here $E_x = E_0 e^{-\gamma z}$

$$\nabla \times \overline{E} = +\hat{a}_y \frac{\partial}{\partial z} E_x = -j\omega\mu H_y \hat{a}_y$$

$$= -\gamma e^{-\gamma z} E_0 \hat{a}_y = -j\omega\mu H_y \hat{a}_y$$

$$\therefore \quad H_y = \frac{\gamma}{j\omega\mu} E_0 e^{-\gamma z} \qquad \dots (26)$$

Reinsert $e^{j\omega t}$ and taking real part of it.

$$H_y = \frac{\text{Re}\,\gamma}{j\omega\mu} E_0 e^{j\omega t} e^{-\gamma z}$$

$$\therefore \quad H_y = \frac{\gamma E_0}{j\omega\mu} e^{-\alpha z} \cos(\omega z - \beta z) \qquad \dots (27)$$

# Lec 16 & 17: Solution to wave equation

15 February 2022 09:04

Intrinsic Impedance ($\eta$)

$\eta$ = Elec field / Mag field

Now taking ratio

$$\frac{E_x}{H_y} = \frac{E_0 e^{-\gamma z}}{\frac{\gamma E_0}{j\omega\mu} e^{-\gamma z}} = \frac{j\omega\mu}{\gamma}$$

$$= \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{(\sigma + j\omega\varepsilon)}} = \text{complex quantity}$$

$$= \eta \text{ (intrinsic impedance)}$$

where $\eta$ is complex quantity, which may be expressed as

$$\eta = \sqrt{\frac{j\omega\mu}{(\sigma + j\omega\varepsilon)}} = |\eta| < \theta_n = |\eta| e^{j\theta n}$$

$\therefore$ [redacted]

$$\therefore \quad \theta_n = \frac{1}{2}\tan^{-1}\frac{\sigma\omega\mu}{\omega^2\mu\varepsilon} = \frac{1}{2}\tan^{-1}\frac{\sigma}{\omega\varepsilon}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}$$

Hence the equation (27) may be written as –

$$H_y = \frac{E_0}{|\eta| < \theta_n} e^{\alpha z} \cos(\omega t - \beta z)$$

$$\overline{H}(z_n t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta n)\hat{a}_y \quad \dots (28)$$

$$\therefore \overline{E}(z, t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{a}_x \quad \dots (25)$$

From equation (25) and (28), as the wave propagates along $+\hat{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, hence $\alpha$ is known as the attenuation constant or factor of the medium. It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) or in decibels per meter (dB/m). An attenuation of 1 neper denotes a reduction to $e^{-1}$ of the original value whereas an increase of 1 neper indicates in increase by a factor of e.

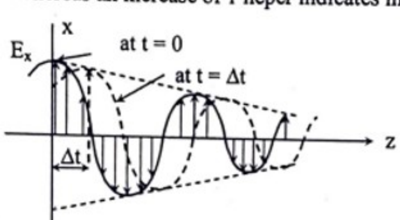



Fig : $\overline{E}$ field with y–comp. traveling in + $\hat{a}_x$ direction

The quantity $\beta$ is measure of the phase shift per length and is called the phase constant or wave number.

In terms of $\beta$, the velocity of propagation and wavelength are given by,

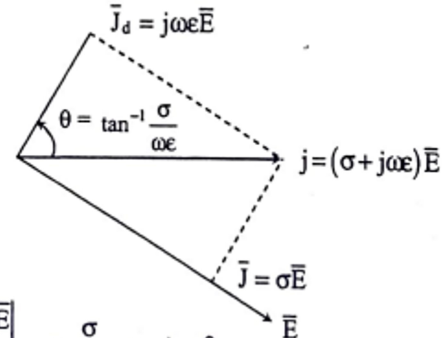
$$v = \frac{\omega}{\beta}, \quad \lambda = \frac{2\pi}{\beta} \quad \dots (29)$$

$$E_x = E_0 e^{-\alpha z} \cos(\omega t - \beta z) \quad \dots (24)$$

$$\therefore \quad H_y = \frac{\gamma E_0}{j\omega\mu} e^{-\alpha z} \cos(\omega t - \beta z) \quad \dots (27)$$

$$H_y = \frac{E_0}{|\eta| < \theta_n} e^{\alpha z} \cos(\omega t - \beta z)$$

From equation (24) and (27) it is clear that $\overline{E}$ and $\overline{H}$ are out of phase by $\theta_n$ at any instant of time due to the complex intrinsic impedance of the medium. The ratio of the magnitude of the conduction current density $\overline{J}$ to the displacement current density $\overline{J}_d$ in lossy medium is




$$\frac{|\overline{J}|}{|\overline{J}_d|} = \frac{|\sigma\overline{E}|}{|j\omega\varepsilon\overline{E}|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta \qquad \text{......(30)}$$

where $\tan\theta$ is known as the loss tangent and $\theta$ is the loss angle of the medium.

If $\tan\theta$ is very small ($\sigma << \omega\,\varepsilon$) the medium is said to be good dielectric.

If $\tan\theta$ is very large ($\sigma >> \omega\,\varepsilon$), the medium is good conductor

Also from the view point of wave propagation the characteristic behavior of medium depends not only on $\sigma$, $\varepsilon$, and $\mu$ but also on the frequency of operation.

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right)} \qquad \text{... (17)}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right)} \qquad \text{... (18)}$$

$$\eta = \sqrt{\frac{j\omega\mu}{(\sigma + j\omega\varepsilon)}} = |\eta| < \theta_n = |\eta|\, e^{j\theta_n}$$

**Wave Motion in Perfect (Lossless) Dielectrics :**

It is a special case of lossy dielectrics.

Here $\quad \sigma << \omega\,\varepsilon$,

$$\boxed{\therefore\ \sigma \cong 0,\ \ \varepsilon = \varepsilon_0\varepsilon_r,\ \ \mu = \mu_0\mu_r} \qquad \text{......(31)}$$

$$\theta_n = \frac{1}{2}\tan^{-1}\frac{\sigma\omega\mu}{\omega^2\mu\varepsilon} = \frac{1}{2}\tan^{-1}\frac{\sigma}{\omega\varepsilon}$$

Substituting above relations into equation (17) , (18)

$$\alpha = 0 \qquad \text{.......(32)}$$

$$\beta = \omega\sqrt{\mu\varepsilon} \qquad \text{.......(33)}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}$$

## WAVE PROPAGATION IN A GOOD CONDUCTOR

$\beta = \omega\sqrt{\mu\varepsilon}$ .......(35)

## WAVE PROPAGATION IN A GOOD CONDUCTOR

In perfect conductor,

$\sigma >> \omega\varepsilon$ so that $\sigma / \omega\varepsilon \to \infty$, that is

$[\sigma = \infty, \varepsilon = \varepsilon_0, \mu = \mu_0 \mu_r]$ .......(37)

Also,

Consider expression for $\alpha$ and $\beta$

Thus

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right)} \quad \text{and} \quad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right)}$$

Substitute $\frac{\sigma}{\omega\varepsilon} >> 1$,

$$\therefore \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \qquad \text{........(38)}$$

$$\therefore v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}} \qquad \text{........(39)}$$

$$\lambda = \frac{2\pi}{\beta} \qquad \text{........(40)}$$

The expression for $\gamma$ may be written as

$$\because \gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = j\omega\mu\sigma\left[1+\frac{j\omega\varepsilon}{\sigma}\right]$$

$$\because \frac{\sigma}{\omega\varepsilon} >> 1$$

$$\therefore \gamma^2 = j\omega\mu\sigma = \sqrt{\omega\mu\sigma\angle 90^\circ} \qquad \text{.......(41)}$$

$$\therefore \gamma = \sqrt{\omega\mu\sigma} \angle 45^\circ \qquad \text{.......(42)}$$

The intrinsic impedance of the conductor is

$$\because \frac{\sigma}{\omega\varepsilon} >> 1$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

Dividing the numerator and denominator by $j\omega\varepsilon$ we get,

$$\eta = \sqrt{\frac{j\omega\mu / j\omega\varepsilon}{(\sigma + j\omega\varepsilon)/j\omega\varepsilon}}$$

$$= \sqrt{\frac{\mu/\varepsilon}{1+\frac{\sigma}{j\omega\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{1}{1+\frac{\sigma}{j\omega\varepsilon}}}$$

$$\because \quad \frac{\sigma}{j\omega\varepsilon} >> 1$$

$$\therefore \quad 1+\frac{\sigma}{j\omega\varepsilon} = \frac{\sigma}{j\omega\sigma}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{1}{\frac{\sigma}{j\omega\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{j\omega\varepsilon}{\sigma}}$$

$$= \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$\eta = \sqrt{\frac{\mu\omega}{\sigma}} \angle 45° \qquad \text{.......(43)}$$

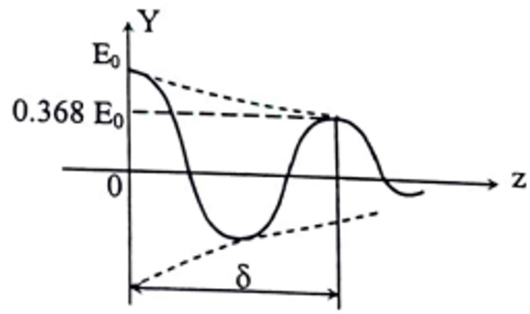
Thus in good conductor $\overline{E}$ leads $\overline{H}$ by 45°. If

$$\overline{E} = E_0 e^{-\alpha x} \cos(\omega t - \beta x)\hat{a}_y \qquad \text{.......(44)}$$

Then $$\overline{H} = \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha x} \cos(\omega t - \beta z - 45°)\hat{a}_z \qquad \text{.......(45)}$$

**Depth of Penetration:** In a medium which has a conductivity the wave is attenuated as it propagates owing to the losses which occur. In a good conductor at radio frequencies the rate of attenuation is very grate and the wave may penetrate only a very short distance before being reduced to a negligibly small percentage of its original strength.

As $\overline{E}$ and $\overline{H}$ wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$ The distance '$\delta$' shown in figure, through which the wave amplitude decreases by a factor $e^{-1}$ (about 37%) is called skin depth or penetration depth of the medium.




$$\therefore \quad E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

$$\text{or} \quad \delta = \frac{1}{\alpha}$$

For good conductors

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

To get some idea of the magnitude of the skin depth, consider copper ($\sigma = 5.8 \times 10^7$ ℧/m, $\mu = \mu_0$). The skin depth in copper at various frequencies is shown in table.

| Frequency (Hz) | 10 | 60 | 100 | 500 | $10^4$ | $10^8$ | $10^{10}$ |
|---|---|---|---|---|---|---|---|
| Skin depth (mm) | 20.8 | 8.6 | 6.6 | 2.99 | 0.66 | $6.6\times10^{-3}$ | $6.6\times10^{-4}$ |

The phenomenon whereby field intensity in a conductor rapidly decreases is known as skin effect. The field and associated currents are confined to a very thin layer of the conductor surface.

# Lec 18: Wave motion in different mediums

17 February 2022 09:19

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right)} \quad \text{— (1)} \qquad \ldots (17)$$

$\rightarrow 0$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right)} \quad \text{— (2)} \qquad \ldots (18)$$

$\rightarrow 0$

$$\theta_n = \frac{1}{2}\tan^{-1}\frac{\sigma\omega\mu}{\omega^2\mu\varepsilon} = \frac{1}{2}\tan^{-1}\frac{\sigma}{\omega\varepsilon} \quad \text{— (3)}$$

$\eta = |\eta|\angle\theta_n$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}} \quad \text{— (4)}$$

→ **Wave Motion in Perfect (Lossless) Dielectrics :**

It is a special case of lossy dielectrics.

Here $\sigma << \omega\varepsilon$,

$$\boxed{\therefore\ \sigma \cong 0,\ \ \varepsilon = \varepsilon_0\varepsilon_r,\ \ \mu = \mu_0\mu_r} \qquad \ldots\ldots(31)$$

Substituting above relations into equation (17) , (18)

$$\alpha = 0 \qquad \ldots\ldots(32)$$

$$\beta = \omega\sqrt{\mu\varepsilon} \qquad \ldots\ldots(33)$$

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \ldots\ldots(34)$$

$$\lambda = \frac{2\pi}{\beta} = \qquad \ldots\ldots(35)$$

Also, $$\eta = \sqrt{\frac{\mu}{\varepsilon}}\angle 0° \qquad \ldots\ldots(36)$$

Thus $\overline{E}$ and $\overline{H}$ are in time phase with each other.

## WAVE PROPAGATION IN A GOOD CONDUCTOR

In perfect conductor,

$\sigma >> \omega\varepsilon$ so that $\sigma / \omega\varepsilon \to \infty$, that is

$[\sigma = \infty, \varepsilon = \varepsilon_0, \mu = \mu_0 \mu_r]$ .......(37)

Consider expression for $\alpha$ and $\beta$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right)} \quad \text{and} \quad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right)}$$

Substitute $\frac{\sigma}{\omega\varepsilon} >> 1$,

$$\therefore \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \quad \text{.......(38)}$$

$$\therefore v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{.......(39)}$$

$$\lambda = \frac{2\pi}{\beta} \quad \text{.......(40)}$$

The expression for $\gamma$ may be written as

$$\because \gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = j\omega\mu\sigma\left[1+\frac{j\omega\varepsilon}{\sigma}\right]$$

$$\because \frac{\sigma}{\omega\varepsilon} >> 1$$

$$\therefore \gamma^2 = j\omega\mu\sigma = \sqrt{\omega\mu\sigma\angle 90^\circ} \quad \text{.......(41)}$$

$$\therefore \gamma = \sqrt{\omega\mu\sigma}\ \angle\ 45^\circ \quad \text{.......(42)}$$

The intrinsic impedance of the conductor is

$$\because \frac{\sigma}{\omega\varepsilon} >> 1$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

Dividing the numerator and denominator by $j\omega\varepsilon$ we get,

$$\eta = \sqrt{\frac{j\omega\mu / j\omega\varepsilon}{(\sigma + j\omega\varepsilon)/j\omega\varepsilon}}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}-1\right)}$$

$$\frac{\sigma}{\omega\varepsilon} >> 1$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\frac{\sigma}{\omega\varepsilon}-1\right)}$$

$$\frac{\sigma}{\omega\varepsilon}$$

$$\frac{\sigma}{\varepsilon}$$

$$= \sqrt{\frac{2\pi f\mu\sigma}{2}}$$

$$\eta = \sqrt{\frac{\mu/\varepsilon}{1+\frac{\sigma}{j\omega\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{1}{1+\frac{\sigma}{j\omega\varepsilon}}}$$

$$\because \frac{\sigma}{j\omega\varepsilon} >> 1$$

$$\therefore 1+\frac{\sigma}{j\omega\varepsilon} = \frac{\sigma}{j\omega\cancel{\sigma}\varepsilon}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{1}{\frac{\sigma}{j\omega\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{j\omega\varepsilon}{\sigma}}$$

$$= \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$\eta = \sqrt{\frac{\mu\omega}{\sigma}} \angle 45° \qquad \text{.......(43)}$$

Thus in good conductor $\overline{E}$ leads $\overline{H}$ by 45°. If

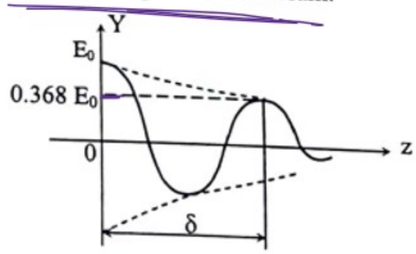
$$\overline{E} = E_0 e^{-\alpha x} \cos(\omega t - \beta x)\hat{a}_y \qquad \text{.......(44)}$$

Then $$\overline{H} = \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha x} \cos(\omega t - \beta z - 45°)\hat{a}_z \qquad \text{.......(45)}$$

great
high

**Depth of Penetration:** In a medium which has a conductivity the wave is attenuated as it propagates owing to the losses which occur. In a good conductor at radio frequencies the rate of attenuation is very grate and the wave may penetrate only a very short distance before being reduced to a negligibly small percentage of its original strength.

As $\overline{E}$ and $\overline{H}$ wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance 'δ' shown in figure, through which the wave amplitude decreases by a factor $e^{-1}$ (about 37%) is called skin depth or penetration depth of the medium.




$$\therefore \quad E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

or $$\delta = \frac{1}{\alpha}$$

For good conductors

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

To get some idea of the magnitude of the skin depth, consider copper ($\sigma = 5.8 \times 10^7$ ℧/m, $\mu = \mu_0$). The skin depth in copper at various frequencies is shown in table.

| Frequency (Hz) | 10 | 60 | 100 | 500 | $10^4$ | $10^8$ | $10^{10}$ |
|---|---|---|---|---|---|---|---|
| Skin depth (mm) | 20.8 | 8.6 | 6.6 | 2.99 | 0.66 | $6.6\times10^{-3}$ | $6.6\times10^{-4}$ |

The phenomenon whereby field intensity in a conductor rapidly decreases is known as skin effect. The field and associated currents are confined to a very thin layer of the conductor surface.

## Poynting Theorem :

When electromagnetic waves propagate through space from their source to distant receiving points, there is a transfer of energy from the source to the receivers. In order to find the power in an electromagnetic waves, it is necessary to develop a power theorem, which is known as Poynting theorem. It can be obtained from Maxwell's equation as follows

Consider Maxwell's equation from modifies Ampere's circuit law,

$$\nabla\times\overline{H} = \overline{J}+\frac{\partial\overline{D}}{\partial t} \qquad \ldots (1)$$

Dot each side of equation (1) with $\overline{E}$

$$\overline{E}.(\nabla\times\overline{H}) = \overline{E}.\overline{J}+E.\frac{\partial\overline{D}}{\partial t} \qquad \ldots (2)$$

Now make use of vector identity,

$$\nabla.(\overline{E}\times\overline{H}) = \overline{H}.\nabla\times\overline{E} - \overline{E}.\nabla\times\overline{H} \qquad \ldots (3)$$

from (2) and (3)

$$\therefore\ \overline{H}.\nabla\times\overline{E} - \nabla.(\overline{E}\times\overline{H}) = \overline{E}.\overline{J}+\overline{E}.\frac{\partial\overline{D}}{\partial t} \qquad \ldots (4)$$

but $\quad \nabla\times\overline{E} = -\frac{\partial\overline{B}}{\partial t}$

and therefore,

$$-H.\frac{\partial\overline{B}}{\partial t}-\nabla.(\overline{E}\times\overline{H}) = \overline{E}.\overline{J}+\overline{E}.\frac{\partial\overline{D}}{\partial t} \qquad \ldots (5)$$

or

$$-\nabla.(\overline{E}\times\overline{H}) = \overline{E}.\overline{J}+\varepsilon\overline{E}.\frac{\partial\overline{E}}{\partial t}+\mu H.\frac{\partial\overline{H}}{\partial t} \qquad \ldots (6)$$

However $\varepsilon\overline{E}$

$$\varepsilon\overline{E}.\frac{\partial\overline{E}}{\partial t} = \frac{\varepsilon}{2}\frac{\partial E^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2}\right)$$

$$\boxed{\overline{A}\cdot\frac{\partial\overline{A}}{\partial t} = \frac{1}{2}\frac{\partial A^2}{\partial t}}$$

$$\mu\overline{H}.\frac{\partial\overline{H}}{\partial t} = \frac{\mu}{2}\frac{\partial H^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2}\right)$$

Thus equation (6) becomes,

$$-\nabla.(\overline{E}\times\overline{H}) = \overline{E}.\overline{J}+\frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right) \qquad \ldots (7)$$

Integrate equation (7) throughout a volume

$$-\int_{vol}\nabla.(\overline{E}\times\overline{H})dv = \int_{vol}(\overline{E}.\overline{J})dv + \overset{\frac{d}{dt}}{\frac{\partial}{\partial t}}\int_{vol}\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right)dv \qquad \ldots (8)$$

Apply divergence theorem to left hand side of equation (8)

$$-\oint_s(\overline{E}\times\overline{H}).\overline{ds} = \int_{vol}(\overline{E}.\overline{J})dv+\frac{\partial}{\partial t}\int_{vol}\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right)dv \qquad \ldots (9)$$

The equation (9) can be written as

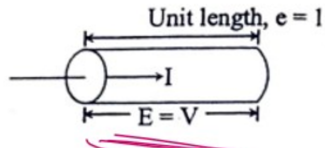
$$\underbrace{\oint_s(\overline{E}\times\overline{H}).\overline{ds}}_{\text{I term}} = \underbrace{-\int_{vol}(\overline{E}.\overline{J})dv}_{\text{II term}}\underbrace{-\frac{\partial}{\partial t}\int_{vol}\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right)dv}_{\text{III term}} \qquad \ldots (10)$$

Equation (10) is referred to as Poynting theorem

**Physical Interpretation :**

II term : $\int_{vol} (\overline{E}.\overline{J})dv$

This term represents power dissipated in the volume V. Consider a conductor of cross sectional area A, carrying current I and voltage drop E per unit length. Therefore power loss per unit length is EI.




$\therefore$ Power dissipated per unit volume would be

$\frac{EI}{A} = E\,J$ watts per unit volume.

$\therefore$ In general, when $\overline{E}$ and $\overline{J}$ are not in same direction,

Power dissipated per unit volume $= \overline{E}.\overline{J}$

$\therefore$ The total power dissipated in a volume V would be

$$\int_v (\overline{E}.\overline{J})dv$$

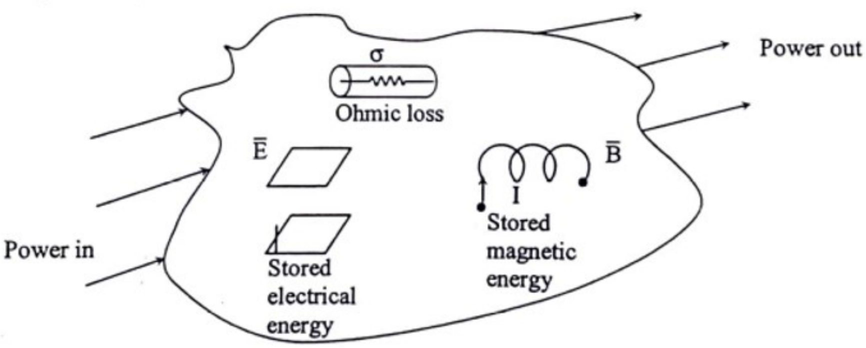
III term : $-\frac{\partial}{\partial t}\int_{vol}\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right)dv$

In electrostatic, the quantity $\boxed{\frac{1}{2}\varepsilon E^2}$ is energy density or stored energy per unit volume of the electric field. Also for the steady magnetic field the quantity $\boxed{\frac{1}{2}\mu H^2}$ represents the stored energy density of the magnetic field. The negative time derivative of this quantity represents the rate at which the stored energy in the volume is decreasing.

I[st] term : $\oint_s (\overline{E}\times\overline{H}).\overline{ds}$

The interpretation of I term follows from the application of the law of conservation of energy. The rate of energy dissipation in the volume v must equal the rate at which the stored energy in v is decreasing, plus the rate at which energy is entering the volume v from outside.

$\therefore \oint_s (\overline{E}\times\overline{H}).\overline{ds}$ represents outward flow of energy through the surface enclosing the volume and $-\oint_s (\overline{E}\times\overline{H}).\overline{ds}$ represents inward flow of energy through the surface enclosing the volume.




**Statement :** The net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the conduction loss :

**The Poynting Vector :**

The interpretation of poynting theorem, leads to the conclusion that the integral of $\overline{E}\times\overline{H}$ over any, closed surface gives the rate of energy flow through that surface. This cross product $\overline{E}\times\overline{H}$ is known as poynting vector $\overline{p}$,

$$\overline{P} = \overline{E}\times\overline{H} \qquad \ldots (11)$$

$a_x \times a_y = a_z$

$a_y \times a_y = a_x \times a_x = 0$

The poynting vector has dimensions of watts per square meter. It is a measure of rate of energy flow per unit area at that point. The direction of flow is perpendicular to $\overline{E}$ and $\overline{H}$ in the direction of the vector $\overline{E}\times\overline{H}$.

a. $a_x$

In perfect dielectrics, $\bar{E}$ and $\bar{H}$ fields are given as

$$E_x = E_0 \cos(\omega t - \beta z)$$

$$\boxed{H_y = \frac{E_0}{\eta}\cos(\omega t - \beta z)}$$

$$\eta = \frac{\bar{E}}{\bar{H}} \qquad \bar{H} = \frac{\bar{E}}{\eta}$$

$a_z$ $a_y$

The Poynting vector is

$$\bar{P} = \bar{E} \times \bar{H}$$

$$\therefore \; P_z \hat{a}_z = E_x \hat{a}_x \times H_y \hat{a}_y$$

$$\therefore \; P_z = \frac{E_0^2}{\eta}\cos^2(\omega t - \beta z)$$

To find the time average power density, integrate over one cycle and divide by the period $T = \frac{1}{f}$.

$$\therefore P_2, av = f\int_0^{1/f} \frac{E_0^2}{\eta}\cos^2(\omega t - \beta z)\,dt$$

$$= \frac{f}{2}\frac{E_0^2}{\eta}\int_0^{1/f}[1 + \cos(2\omega t - 2\beta z)dt$$

$$= \frac{f}{2}\frac{E_0^2}{\eta}\left[t + \frac{1}{2\omega}\sin(2\omega t - 2\beta z)\right]_0^{1/f}$$

$$= P_z, av = \frac{1}{2}\frac{E_0^2}{\eta}\ \omega/m^2 \qquad \dots (12)$$

If medium is lossy dielectric,

$$E_x = E_0 e^{-\alpha z}\cos(\omega t - \beta z)$$

$$H_y = \frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta n)$$

$$\therefore \; P_z = \frac{E_0^2}{|\eta|}e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta n)$$

$$\therefore \; P_z, av = \frac{1}{T}\int_0^T P_z dt$$

$$\because \; \cos A \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$P_z, av = \frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos\theta_n$$

If we use, rms values instead of pear amplitudes, the factor of ½ would not be present. It can be shown that, the time average poynting vector can be also written as

$$P_z, av = \tfrac{1}{2}\,\mathrm{Re}\,(\bar{E} \times \bar{H})$$

Finally, the average power flowing through any are 'S' normal to the z–axis is

$$P_{ave} = \frac{1}{2}\frac{E_0^2}{\eta}\ S \text{ watt.} \qquad \dots (13)$$

For any given surface 'S', it is given by

$$P_{ave} = \int_S \bar{P}_{av} . \bar{dS} \qquad \dots (14)$$

**Instantaneous, Average and Complex Poynting vector :**

In a. c. circuit, the instantaneous power $\tilde{W}$ is always given by the product of the instantaneous voltage $\tilde{V}$ and the instantaneous current $\tilde{I}$

$$\therefore \; \tilde{W} = \tilde{V}\,\tilde{I} \qquad \dots (15)$$

complex power

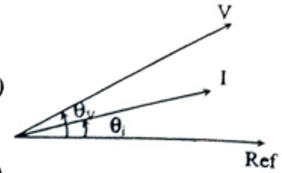
Let $\tilde{V} = |V|\cos(\omega t + \theta v)$ ... (16)

$\tilde{I} = |I|\cos(\omega t + \theta_i)$ ... (17)

$\therefore$ The instantaneous power,

$$\tilde{W} = |V||I|\cos(\omega t + \theta v)\cos(\omega t + \theta i)$$

$$= \frac{|V||I|}{2}\,[\cos(\theta_v - \theta_i) + \cos(2\omega t+\theta_v + \theta_i)] \quad \ldots (18)$$




Taking average of equation (18) over one cycle,

$$W_{av} = \frac{|V||I|}{2}\cos(\theta_v - \theta_i) = \frac{|V||I|}{2}\cos\theta \quad \ldots (19)$$

where $\theta = \theta_v - \theta_i$

and $$W_{react} = \frac{|V||I|}{2}\sin\theta \quad \ldots (20)$$

If V is voltage and I is current expressed in phasor, $W = \frac{1}{2}VI^*$

where $I^*$ is the complex conjugate of I.

$$W = W_{av} + jW_{react} \quad \ldots (21)$$

Thus $$W_{av} = \frac{1}{2}\mathrm{Re}[VI^*]$$

$$W_{react} = \frac{1}{2}I_m[VI^*]$$

In electromagnetic field theory, the instantaneous power flow per square meter is

$$\tilde{P} = \tilde{E} \times \tilde{H}$$

∴ the complex poynting vector $\overline{P}$ may be defined as –

$$\overline{P} = \frac{1}{2}\overline{E} \times \overline{H}^* \quad \ldots (22)$$

∴ the average and reactive parts of the power flow per square meter.

$$\overline{P}_{av} = \frac{1}{2}\mathrm{Re}[\overline{E} \times \overline{H}^*]$$ active

$$\overline{P}_{react} = \frac{1}{2}\mathrm{Im}[\overline{E} \times \overline{H}^*]$$

The product of $\overline{E}$ and $\overline{H}$ is a vector product only mutually perpendicular components of $\overline{E}$ and $\overline{H}$ contribute to the power flow, and the direction of the flow is normal to the plane containing $\overline{E}$ and $\overline{H}$.

**The Poynting Theorem in complex form :**

Maxwell's curl equations may be expressed in phasor form as

$$\nabla \times \overline{H} = (\sigma + j\omega\varepsilon)\overline{E} + \overline{J} \quad \text{and} \quad \ldots (24)$$

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \quad \ldots (25)$$

in which $\overline{J}$ represents non ohmic currents such as convection current or specified current source.

$$\therefore \ \nabla \cdot (\overline{E} \times \overline{H}^*) = \overline{H}^* . \nabla \times \overline{E} - \overline{E}.(\nabla \times \overline{H}^*)$$

$$= -j\omega\mu\overline{H}.\overline{H}^* - (\sigma - j\omega\varepsilon)\overline{E}.\overline{E}^* - \overline{E}.\overline{J}^* \quad \ldots (26)$$

Integrating over the volume V surrounding by surface 's', we have

$$\oint_s (\overline{E} \times \overline{H}^*).\overline{ds} = -j\omega\int_v (\mu\overline{H}.\overline{H}^* - \varepsilon\overline{E}.\overline{E}^*)dv - \int_v \sigma\overline{E}.\overline{E}^* dv - \int_v \overline{E}.\overline{J}^* dv$$

The time average stored energy densities (electric and magnetic) are given by

$$U_e = \frac{1}{4}\varepsilon\ \overline{E}.\overline{E}^*$$

$$U_m = \frac{1}{4}\mu\,\overline{H}.\overline{H}^* \quad \ldots (27)$$

The Poynting theorem now may be separated is to real and imaginary part and written as –

$$\mathrm{Re}\oint_s \overline{P}.\overline{ds} + \frac{1}{2}\int_v \sigma\overline{E}.\overline{E}^* dv = -\frac{1}{2}\mathrm{Re}\int_v \overline{E}.\overline{J}^* dv \quad \ldots (28)$$

$$I_m \oint \overline{P}.\overline{ds} + 2\omega\int_v (U_m - U_e)dv = -\frac{1}{2}I_m\int_v \overline{E}.\overline{J}^* dv \quad \ldots (29)$$

# Intrinsic Impedance

Now taking ratio

$$\frac{E_x}{H_y} = \frac{E_0 e^{-\gamma z}}{\frac{\gamma E_0}{j\omega\mu} e^{-\gamma z}} = \frac{j\omega\mu}{\gamma}$$

where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$$= \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{(\sigma + j\omega\varepsilon)}} = \text{complex quantity}$$

$$= \eta \text{ (intrinsic impedance)}$$

$e^{j\theta_n}$

$\eta = |\eta| \angle \theta_n$

where $\eta$ is complex quantity, which may be expressed as

$$\eta = \sqrt{\frac{j\omega\mu}{(\sigma + j\omega\varepsilon)}} = |\eta| < \theta_n = |\eta| e^{j\theta_n}$$

$\eta = \frac{E}{H}$

$\boxed{H = \frac{E}{\eta}}$

$|\eta| \angle \theta_n$

$$\therefore \quad \eta^2 = \frac{j\omega\mu(\sigma - j\omega\varepsilon)}{\sqrt{(\sigma + j\omega\varepsilon)(\sigma - j\omega\varepsilon)}} = \sqrt{\frac{\omega^2\mu\varepsilon + j\sigma\omega\mu}{\sigma^2 + \omega^2\varepsilon^2}}$$

$$\therefore \quad \theta_n = \frac{1}{2}\tan^{-1}\frac{\sigma\omega\mu}{\omega^2\mu\varepsilon} = \frac{1}{2}\tan^{-1}\frac{\sigma}{\omega\varepsilon}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}$$

$E =$

Hence the equation (27) may be written as –

$$H_y = \frac{E_0}{|\eta| < \theta_n} e^{\alpha z} \cos(\omega t - \beta z)$$

$$\overline{H}(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta n)\hat{a}_y \qquad \ldots (28)$$

$$\therefore \overline{E}(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{a}_x \qquad \ldots (25)$$

From equation (25) and (28), as the wave propagates along $+\hat{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, hence $\alpha$ is known as the attenuation constant or factor of the medium. It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) or in decibels per meter (dB/m). An attenuation of 1 neper denotes a reduction to $e^{-1}$ of the original value whereas an increase of 1 neper indicates in increase by a factor of e.

$\gamma = \alpha + j\beta$

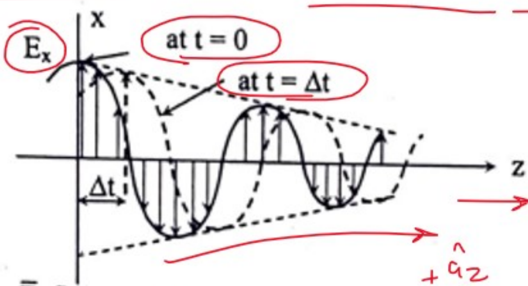



Fig : $\bar{E}$ field with ~~x~~-comp. traveling in + $\hat{a}_x$ direction $+\hat{a}_z$

The quantity $\beta$ is measure of the phase shift per length and is called the phase constant or wave number.

In terms of $\beta$, the velocity of propagation and wavelength are given by,

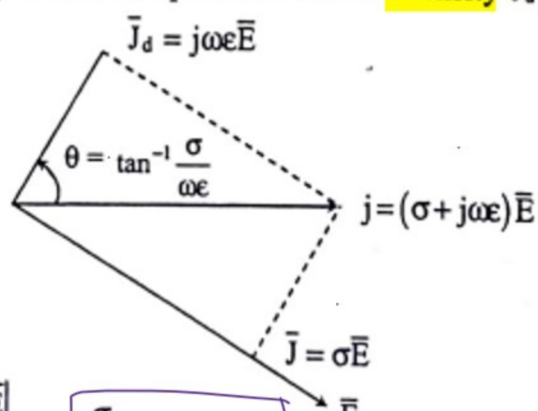
$$\boxed{v = \frac{\omega}{\beta}, \quad \lambda = \frac{2\pi}{\beta}} \qquad \dots (29)$$

$$E_x = E_0\, e^{-\alpha z} \cos(\omega t - \beta z) \qquad \dots (24)$$

$$\therefore \quad H_y = \frac{\gamma E_0}{j\omega\mu} e^{-\alpha z} \cos(\omega z - \beta z) \qquad \dots (27)$$

$$H_y = \frac{E_0}{|\eta| \angle \theta_n} e^{\alpha z} \cos(\omega t - \beta z)$$

From equation (24) and (27) it is clear that $\bar{E}$ and $\bar{H}$ are out of phase by $\theta_n$ at any instant of time due to the complex intrinsic impedance of the medium. The ratio of the magnitude of the conduction current density $\bar{J}$ to the displacement current density $\bar{J}_d$ in lossy medium is




$$\frac{|\bar{J}|}{|\bar{J}_d|} = \frac{|\sigma\bar{E}|}{|j\omega\varepsilon\bar{E}|} = \boxed{\frac{\sigma}{\omega\varepsilon} = \tan\theta} \qquad \dots\dots (30)$$

$$\frac{|\bar{J}|}{|\bar{J}_d|} = \frac{|\sigma\bar{E}|}{|j\omega\varepsilon\bar{E}|} = \boxed{\frac{\sigma}{\omega\varepsilon} = \tan\theta} \quad \bar{E} \qquad \ldots\ldots(30)$$

where $\tan\theta$ is known as the loss tangent and $\theta$ is the loss angle of the medium.

If $\tan\theta$ is very small ($\sigma << \omega\,\varepsilon$) the medium is said to be good dielectric.
If $\tan\theta$ is very large ($\sigma >> \omega\,\varepsilon$), the medium is good conductor

Also from the view point of wave propagation the characteristic behavior of medium depends not only on $\sigma$, $\varepsilon$, and $\mu$ but also on the frequency of operation. (← NOTES)

# Condition at a Boundary Surface

01 February 2023 12:08

**Generalized Form of Maxwell's Equation:**

| Differential form | Integral form | Remarks |
|---|---|---|
| $\nabla.\bar{D} = \rho_v$ | $\oint_S \bar{D}.\overline{dS} = \int_{vol} \rho_v dv$ | Gauss's Law |
| $\nabla.\bar{B} = 0$ | $\oint \bar{B}.\overline{dS} = 0$ | Gauss's Law for magnetic field |
| $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | $\oint \bar{E}.\overline{dL} = \frac{\partial}{\partial t} \int_S \bar{B}.\overline{dS}$ | Faraday's law |
| $\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$ | $\oint \bar{H}.\overline{dL} = \int_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right).\overline{dS}$ | Modified form of Ampere's circuital law |

**Maxwell's Equation in Phasor Form:**

| Point Form | Integral Form |
|---|---|
| $\nabla \times \bar{H} = \bar{J} + j\omega\bar{D}$ | $\oint_L \bar{H}.\overline{dL} = \int_S \bar{J}.\overline{dS} + j\omega \int_S \bar{D}.\overline{dS}$ |
| $\nabla \times \bar{E} = -j\omega\bar{B}$ | $\oint_L \bar{E}.\overline{dL} = -j\omega \int \bar{B}.\overline{dS}$ |
| $\nabla.\bar{D} = \rho_v$ | $\oint_S \bar{D}.\overline{dS} = \int_{vol} \rho_v dV$ |
| $\nabla.\bar{B} = 0$ | $\oint_S \bar{B}.\overline{dS} = 0$ |
| $\nabla.\bar{J} = -j\omega\rho_v$ | $\oint_S \bar{J}.\overline{dS} = -j\omega \int_{vol} \rho_v dV$ |

**Condition at a Boundary Surface:**
- Dielectric ($\varepsilon_{r1}$) – Dielectric ($\varepsilon_{r2}$)
- Conductor – Dielectric
- Conductor – free Space.

If the field exist in a region consisting of two different media, the conditions that must satisfy at the interface separating the media are called boundary conditions.

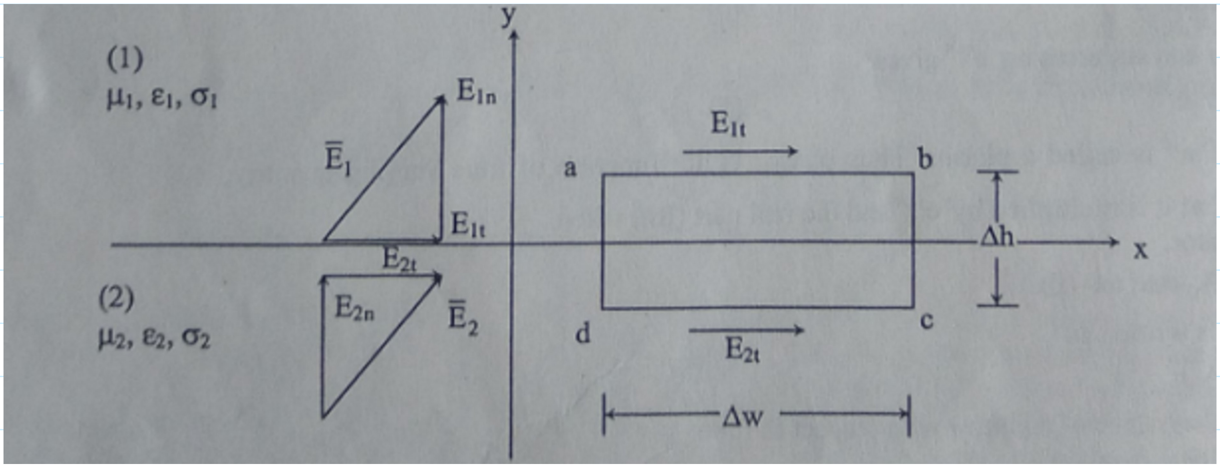
The integral form of Maxwell's equation can be used to determine boundary conditions.

**Tangential component of $\overline{E}$ and $\overline{H}$ :**

Let two dielectric media of permittivities of $\varepsilon_1$ and $\varepsilon_2$ and permeability $\mu_1$ and $\mu_2$ be separated by plane boundary as shown in figure. Let $\sigma_1$ and $\sigma_2$ be the conductivities of two media.

Consider Maxwell's equation from Faraday's law,

$$\oint \overline{E}.\overline{dL} = -\int_S \frac{\partial \overline{B}}{\partial t}.\overline{dS} \qquad \ldots(1)$$




Apply equation (1) to the closed path abcda assuming that the path is very small with respect to variation of $\overline{E}$.

$$E_{1t}\Delta w - \frac{1}{2}E_{1n}\Delta h - \frac{1}{2}E_{2n}\Delta h - E_{2t}\Delta w + \frac{1}{2}E_{2n}\Delta h + \frac{1}{2}E_{1n}\Delta h = -\frac{\partial B_Z}{\partial t}\Delta w \Delta h \qquad \ldots(2)$$

Where $B_Z$ is average magnetic flux density through rectangle $\Delta w \Delta h$. Now consider condition as the area of the rectangle is made to approach zero by reducing the height $\Delta h$.

$\therefore$ As $\Delta h \to 0$

$\therefore$ $E_{1t}\,\Delta w - E_{2t}\,\Delta w = 0$

$\therefore E_{1t} = E_{2t}$     the tangential component of E is continuous.

**Tangential component of $\overline{D}$**

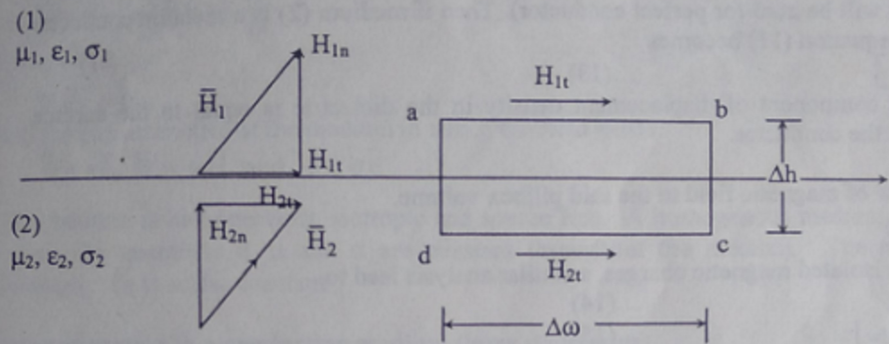
$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \leftarrow \text{(discontinuous)}$$

* Tangential Component of $\overline{H}$

Similarly, for the tangential component of $\bar{H}$ consider Maxwell's equation from Ampere's circuital law,

$$\oint_L \bar{H}.\overline{dL} = \int_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right).\overline{dS} \quad \ldots(4)$$

Apply to closed path abcda,




$$H_{1t}\Delta w - \frac{1}{2}H_{1n}\Delta h - \frac{1}{2}H_{2n}\Delta h - H_{2t}\Delta w + \frac{1}{2}H_{2n}\Delta h + \frac{1}{2}E_{1n}\Delta h = \left(\frac{\partial D_Z}{\partial t} + J_Z\right)\Delta h \Delta w \quad \ldots(5)$$

As $\Delta h \rightarrow 0$

$$H_{1t} - H_{2t} = 0$$

$$\boxed{H_{1t} = H_{2t}}$$

# Electric and Magnetic field boundary conditions

02 February 2023 08:48

**Generalized Form of Maxwell's Equation:**

| Differential form | Integral form | Remarks |
|---|---|---|
| $\nabla . \bar{D} = \rho_v$ | $\oint_S \bar{D}.\overline{dS} = \int_{vol} \rho_v dv$ | Gauss's Law |
| $\nabla . \bar{B} = 0$ | $\oint \bar{B}.\overline{dS} = 0$ | Gauss's Law for magnetic field |
| $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ | $\oint \bar{E}.\overline{dL} = \frac{\partial}{\partial t} \int_S \bar{B}.\overline{dS}$ | Faraday's law |
| $\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$ | $\oint \bar{H}.\overline{dL} = \int_S \left( \bar{J} + \frac{\partial \bar{D}}{\partial t} \right) \overline{dS}$ | Modified form of Ampere's circuital law |

**Maxwell's Equation in Phasor Form:**

| Point Form | Integral Form |
|---|---|
| $\nabla \times \bar{H} = \bar{J} + j\omega \bar{D}$ | $\oint_L \bar{H}.\overline{dL} = \int_S \bar{J}.\overline{dS} + j\omega \int_S \bar{D}.\overline{dS}$ |
| $\nabla \times \bar{E} = -j\omega \bar{B}$ | $\oint_L \bar{E}.\overline{dL} = -j\omega \int \bar{B}.\overline{dS}$ |
| $\nabla . \bar{D} = \rho_v$ | $\oint_S \bar{D}.\overline{dS} = \int_{vol} \rho_v dV$ |
| $\nabla . \bar{B} = 0$ | $\oint_S \bar{B}.\overline{dS} = 0$ |
| $\nabla . \bar{J} = -j\omega \rho_v$ | $\oint_S \bar{J}.\overline{dS} = -j\omega \int_{vol} \rho_v dV$ |

## Boundary conditions

If the field exist in a region consisting of two different media, the conditions that must satisfy at the interface separating the media are called boundary conditions.

## Electrostatic Boundary conditions

* Electrostatic Boundary conditions

If the field exists in a region consisting of two different media, the conditions that must satisfy at the Interface seperating the media are called boundary conditions. These conditions are helpful in determining the field on one side of the boundary if field on the other side is known.

To determine the boundary conditions, we need to use maxwell's eq^n for electrostatic fields.

$$\oint \bar{E} \cdot \bar{dL} = 0 \quad -(1) \quad \&$$

$$\oint \bar{D} \cdot \bar{ds} = Q_{encl} \quad -(2)$$

The electric field intensity & flux density is decomposed into two orthogonal components such as:

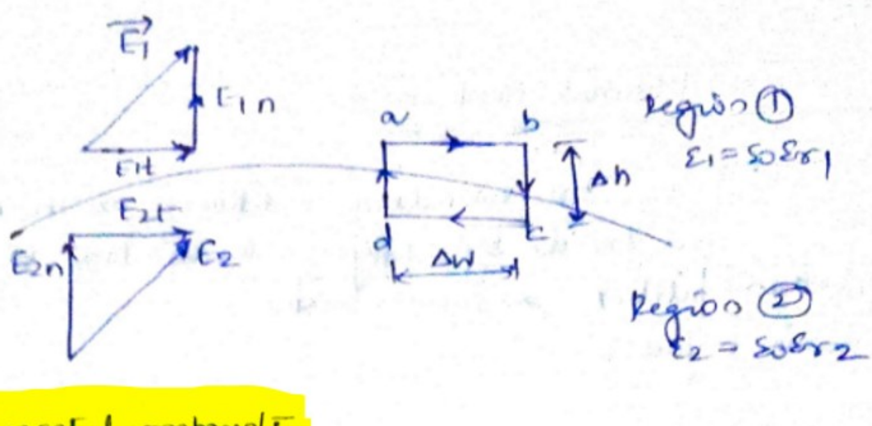
$$\bar{E} = \bar{E}_t + \bar{E}_n \quad \& \quad \bar{D} = \bar{D}_t + \bar{D}_n \quad -(3)$$

## (*) [A] Dielectric - dielectric boundary conditions:-

Consider the $\bar{E}$ field existing in a region consisting of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ & $\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$ as shown in fig. $\bar{E}_1$ & $\bar{E}_2$ in media 1 & 2 respectively can be decomposed as

$$\bar{E}_1 = \bar{E}_{1t} + \bar{E}_{1n}$$

$$\bar{E}_2 = \bar{E}_{2t} + \bar{E}_{2n} \quad -(4)$$




Tangential components

## Tangential components

Apply eqn ① to the closed path abcda assuming that the path is very small w.r.t. the variation of $\bar{E}$. we have

$$0 = E_{1t}\Delta w - \frac{1}{2}E_{1n}\frac{\Delta h}{2} - \frac{1}{2}E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + \frac{1}{2}E_{2n}\Delta h + \frac{1}{2}E_{1n}\Delta h$$

As $\Delta h \to 0$

$$\boxed{E_{1t} = E_{2t}} \quad \text{—} ⑤$$

Thus the tangential components of $\bar{E}$ are same on the two sides of boundary. In other words, it is continuous across boundary [ $E_t$ undergoes no change on the boundary ]

Since $\vec{D} = \varepsilon\vec{E} = \bar{D}_t + \bar{D}_n$

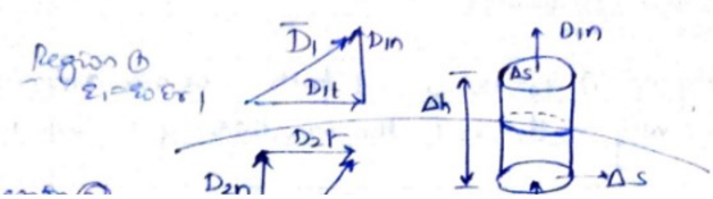
$$\therefore \frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

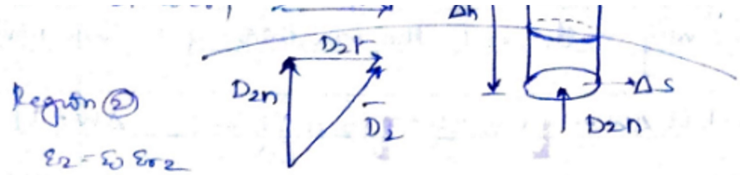
$$\therefore \boxed{\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}} \quad \text{———} ⑥$$

$D_t$ undergoes some changes across the interface. Hence $D_t$ is said to be discontinuous across the interface.

## Normal components

"The boundary conditions on the normal components are found by applying Gauss's law to the small (cylindrical pillbox as shown below. Gaussian surface)






The sides are again very short, & the flux leaving the top & bottom surfaces is the difference

$$D_{1n} \Delta S - D_{2n} \Delta S = \Delta Q = \rho_s \Delta S$$

from which

$$\boxed{D_{1n} - D_{2n} = \rho_s} \quad \text{--- (7)}$$

where $\rho_s$ is the free charge density placed delib-erately at the boundary. If no free charge exists at the interface $\rho_s = 0$, & eq^n (7) becomes

$$\boxed{D_{1n} = D_{2n}} \quad \text{--- (8)}$$

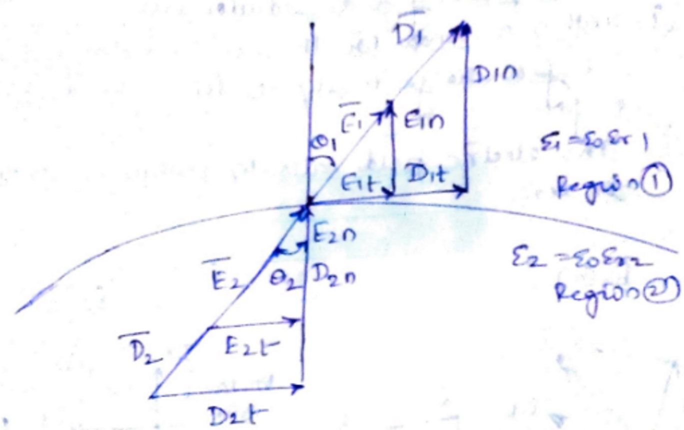
Thus the normal component of $\bar{D}$ is continuous across the interface, that is $D_n$ undergoes no change at the boundary. Since $\bar{D} = \varepsilon \bar{E}$

$$\therefore \boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}} \quad \text{--- (9)}$$

This shows that normal component of $\bar{E}$ is discontinuous at the boundary.

The boundary conditions are usually applied to
→ find the electric field on one side of the boundary given the field on the other side
→ to determine the 'refraction' of $\bar{E}$ across the interface

Consider $\bar{D}_1$ or $\bar{E}_1$ & $\bar{D}_2$ or $\bar{E}_2$ making <le $\theta_1$ & $\theta_2$ with the normal to the interface as illustrated in fig.




From fig

$$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$$

or

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \quad \text{— (10)}$$

Similarly

$$\varepsilon_1 E_1 \cos\theta_1 = D_{1n} = D_{2n} = \varepsilon_2 E_2 \cos\theta_2$$

$$\therefore \varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2 \quad \text{— (11)}$$

Dividing eqn (10) by eqn (11)

$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2} \quad \text{— (12)}$$

$$\therefore \frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_0 \varepsilon_{r1}}{\varepsilon_0 \varepsilon_{r2}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \quad \text{— (13)}$$

This is the law of refraction of the electric field at a boundary of free charge (since $\rho_s = 0$ is assumed at the interface).

## Magnetostatic Boundary conditions

* Magnetic boundary conditions

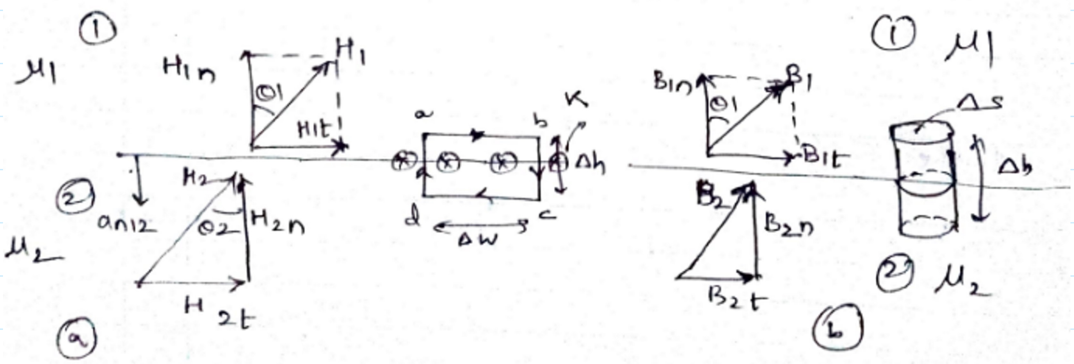
We define magnetic boundary conditions as the conditions that H (or B) field must satisfy at the boundary between two different media. Our ~~derivations~~ derivations here are similar to those for electrostatic fields. We make use of Gauss's law for magnetic fields.

$$\oint \bar{B}\cdot\bar{ds} = 0 \quad \text{--- (1)}$$

& Ampere's ckt law

$$\oint \bar{H}\cdot d\bar{L} = I \quad \text{--- (2)}$$

Consider the boundary between two magnetic media (1) & (2), characterized, respectively, by $\mu_1$ & $\mu_2$ as shown in fig below.




Apply eqⁿ (1) to the pillbox (Gaussian surface) of fig (b) & allowing $\Delta h \to 0$, we obtain

$$B_{1n}\Delta S - B_{2n}\Delta S = 0 \quad \text{--- (3)}$$

Thus

$$\boxed{\bar{B}_{1n} = \bar{B}_{2n}} \quad \text{or} \quad \mu_1 H_{1n} = \mu_2 H_{2n} \quad \text{--- (4)}$$

Since $B = \mu H$. eq (4) shows that the normal comp. of B is continuous at the boundary. It also shows that the normal comp. of $\bar{H}$ is discontinuous at the boundary; $\bar{H}$ undergoes some change at the interface.

Similarly, we apply eqⁿ (2) to the closed path abcda of fig (a), where surface c/n K on the boundary is assumed normal to the path. We obtain

$$K \cdot \Delta w = H_{1t} \cdot \Delta w + H_{1n} \frac{\Delta h}{2} + H_{2n} \cdot \frac{\Delta h}{2} - H_{2t} \cdot \Delta w - H_{2n} \cdot \frac{\Delta h}{2} - H_{1n} \cdot \frac{\Delta h}{2} \quad \text{—— (5)}$$

As $\Delta h \to 0$, eqⁿ (5) leads to,

$$H_{1t} - H_{2t} = K \quad \text{—— (6)}$$

This shows that the tangential component of H is also discontinuous. Eqⁿ (6) may be written in terms of B as

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K \quad \text{—— (7)}$$

In the general case eqⁿ (6) becomes

$$(\bar{H}_1 - \bar{H}_2) \times a_{n12} = K \quad \text{—— (8)}$$

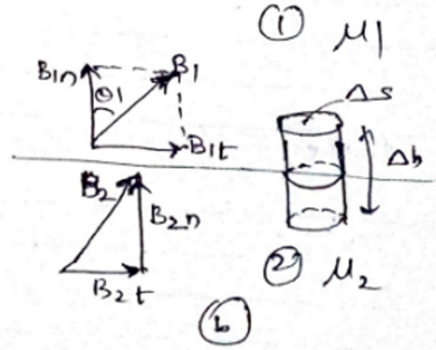
where $a_{n12}$ is a unit vector normal to the Interface & is directed from medium (1) to medium (2). If the boundary is free of c/n or the media are not conductors (for K is free c/n density), K=0 & eqⁿ (6) becomes

$$\boxed{H_{1t} = H_{2t}} \quad \text{or} \quad \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \quad \text{—— (9)}$$

Thus the tangential component of $\bar{H}$ is continuous while that of B is discontinuous at the boundary.

If the field make an angle $\theta$ with the normal to the interface, eq$^n$ (4) results in

$$B_1 \cos\theta_1 = B_{1n} = B_{2n} = B_2 \cos\theta_2 \quad \text{——— (10)}$$




while eq$^n$ (5) produces.

$$\frac{B_1}{\mu_1} \sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2} \sin\theta_2 \quad \text{——— (11)}$$

Dividing eq$^n$ (11) by eq$^n$ (10)

$$\frac{\tan\theta_1}{\mu_1} = \frac{\tan\theta_2}{\mu_2}$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}} \quad \text{——— (12)}$$

which is the law of refraction for magnetic flux lines at a boundary with no surface c/n.

[B] Conductor – Dielectric boundary conditions

consider perfect conductor ($\sigma \to \infty, \rho_c \to 0$)

The properties of good conductor are

(i) within a conductor there is zero charge density & a surface charge density resides on the exterior surface.

(ii) The electric field intensity within a conductor is zero fig (a)

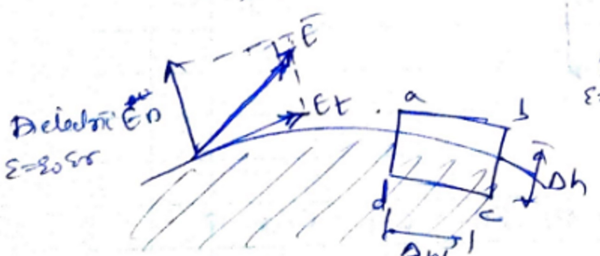
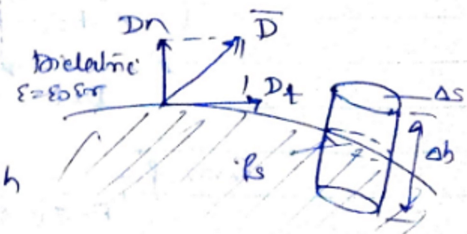



fig (b)




Consider conductor in an external field $\bar{E}$. Decompose $\bar{E}$ into two components such as:-

$$\bar{E} = \bar{E}_t + \bar{E}_n$$

where $E_t$ — tangential & $E_n$ is normal component

Apply $\oint \bar{E} \cdot \bar{dl} = 0$ to closed path abcda,

$$E_t \Delta w - \frac{1}{2} E_n \Delta h + \frac{1}{2} E_n \Delta h = 0$$

$$\therefore \; E_t = 0 \quad \text{—— (14)}$$

Since $\bar{D} = \varepsilon \bar{E}$

$$D_t = 0 \quad \text{—— (15)}$$

As far as normal component is concern apply Gauss's law to the pillbox shown in fig (b)

$$\oint \bar{D} \cdot \bar{ds} = Q.$$

$$\Delta Q = D_n \Delta S - 0 \, \Delta S = \rho_s \Delta S$$

$$\therefore \; D_n = \rho_s \quad \text{—— (16)}$$

or

$$\varepsilon_0 E_n = \rho_s \quad \text{—— (17)}$$

# Introduction to EM waves

11 February 2022 09:15

Waves are a means for transferring energy or information from one place to another

Electromagnetic waves as the name suggests, are a means for transferring electromagnetic energy

Plane wave :- If each field has same direction at every pt in any plane ⊥ar to the direction of wave travel, it is called as plane wave

Uniform Plane Wave :- If the field has the same direction and magnitude at every point in any plane ⊥ar to the direction of wave travel, it is called as Uniform plane wave

Electromagnetic (EM) waves.

All forms of EM energy share 3 fundamental characteristics

(i) They all travel at high velocity

(ii) In travelling, they assume the properties of waves.

(iii) They radiate outward from a source.

* EM wave propagation through various media

(i) Free Space ($\sigma = 0$ ; $\varepsilon = \varepsilon_0$, $\mu = \mu_0$)

(ii) Lossless dielectric ($\sigma << \omega\varepsilon$, $\varepsilon = \varepsilon_0\varepsilon_r$, $\mu = \mu_0\mu_r$)

(iii) Lossy dielectric ( $\sigma \neq 0$ , $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$)

(iv) Good conductors ( $\sigma = \infty$ , $\varepsilon = \varepsilon_0$, $\mu = \mu_0 \mu_r$ ; $\sigma >> \varepsilon\omega$ )

Lossy dielectric is the most general case & will be discussed first.

Extra Numericals

27 January 2022 10:29

Q1) Consider 2 pt charges $Q_1 = 2\,mC$ at $(-3,7,4)$ & $Q_2 = 5\,mC$ at $(2,4,-1)$.
Find force $F_2$ on $Q_2$ ($F_2$ & $F_1$)   ?   Q

⇒

(i) Force $F_2$ on $Q_2$ is given by

$$F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\hat{a}_{12}$$

$$R_{12} = r_2 - r_1 = (2\hat{a}_x + 4\hat{a}_y - \hat{a}_z) - (-3\hat{a}_x + 7\hat{a}_y + 4\hat{a}_z)$$
$$= 5\hat{a}_x - 3\hat{a}_y - 5\hat{a}_z$$

$$\hat{a}_{12} = \frac{r_2 - r_1}{|R_{12}|} = \frac{5\hat{a}_x - 3\hat{a}_y - 5\hat{a}_z}{\sqrt{25+9+25}} = \frac{5\hat{a}_x - 3\hat{a}_y - 5\hat{a}_z}{7.6811}$$

$$\therefore \hat{a}_{12} = 0.651\,\hat{a}_x - 0.391\,\hat{a}_y - 0.651\,\hat{a}_z$$

$$\therefore \bar{F}_2 = \frac{2\times10^{-3}\times5\times10^{-3}}{4\pi\times8.85\times10^{-12}\times(7.6811)^2}(0.651\hat{a}_x - 0.391\hat{a}_y - 0.651\hat{a}_z)$$

$$\bar{F}_2 = \frac{10\times10^{-6}}{6561.45\times10^{-12}}(0.651\hat{a}_x - 0.391\hat{a}_y - 0.651\hat{a}_z)$$

$$= 1524.05\,(0.651\hat{a}_x - 0.391\hat{a}_y - 0.65\hat{a}_z)$$

$$\boxed{\bar{F}_2 = 992.158\,\hat{a}_x - 595.91\,\hat{a}_y - 990.63\,\hat{a}_z \quad N}$$

Note:- As $\hat{a}_{21} = -\hat{a}_{12}$
$\therefore \bar{F}_1 = -\bar{F}_2$

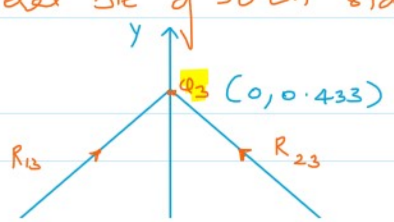
(ii) Force $F_1$ on $Q_1$

$$\boxed{\therefore \bar{F}_1 = -992.158\,\hat{a}_x + 595.91\,\hat{a}_y + 990.63\,\hat{a}_z \quad N}$$

Q2) Three pt charges $Q_1 = 10^{-6}\,C$; $Q_2 = 10^{-6}\,C$ & $Q_3 = 0.5\times10^{-6}\,C$ are located at the corners of an equilateral Δle of 50 cm side. Determine force on charge $Q_3$

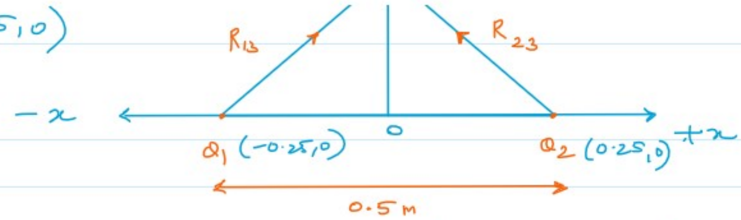
⇒

$Q_1 = 10^{-6}\,C$ at $(-0.25,)$
$Q_2 = 10^{-6}\,C$ at $(0.25, 0)$
$Q_3 = 0.5\times10^{-6}\,C$ at



$Q_2 = 10^{-6}\,C$ at $(0.25, 0)$

$Q_3 = 0.5\times10^{-6}\,C$ at $(0, 0.433)$




Force on charge $Q_3$ is given by

$$\overline{F_3} = \underline{\underline{\overline{F_{13}}}} + \underline{\underline{\overline{F_{23}}}}$$

Calculating unit vectors

$$\hat{a}_{13} = \frac{r_3 - r_1}{|R_{13}|} = \frac{\overline{R_{13}}}{|R_{13}|}$$

$$\overline{R_{13}} = r_3 - r_1 = 0.433\hat{a}_y - (-0.25\hat{a}_x) = 0.25\hat{a}_x + 0.433\hat{a}_y$$

$$\therefore \hat{a}_{13} = \frac{0.25\hat{a}_x + 0.433\hat{a}_y}{\sqrt{(0.25^2)+(0.433^2)}} = \frac{0.25\hat{a}_x + 0.433\hat{a}_y}{0.499}$$

$$\therefore \hat{a}_{13} = 0.5\hat{a}_x + 0.8677\hat{a}_y$$

Similarly $\overline{R_{23}} = r_3 - r_2 = 0.433a_y - 0.25a_x$

$$\therefore \hat{a}_{23} = \frac{\overline{R_{23}}}{|R_{23}|} = \frac{-0.25a_x + 0.433a_y}{\sqrt{(-0.25)^2 + (0.433)^2}}$$

$$\hat{a}_{23} = -0.5\hat{a}_x + 0.8677\hat{a}_y$$

$$F_{13} = \frac{Q_1 Q_3}{4\pi\varepsilon_0 R_{13}^2}\hat{a}_{13} = \frac{10^{-6}\times 0.5\times10^{-6}}{4\pi\times 8.85\times10^{-12}\times(0.499)^2}\cdot(0.5\hat{a}_x + 0.8677\hat{a}_y)$$

$$= 0.0181(0.5\hat{a}_x + 0.8677\hat{a}_y) = (9.05\hat{a}_x + 15.7\hat{a}_y)\text{ mN}$$

$$\boxed{\therefore F_{13} = 9.05\hat{a}_x + 15.7a_y \text{ mN}}$$

$$F_{23} = \frac{Q_2 Q_3}{4\pi\varepsilon_0\cdot R_{23}^2}\hat{a}_{23} = \frac{10^{-6}\times0.5\times10^{-6}}{4\pi\times8.85\times10^{-12}\times(0.499)^2}\times(-0.5\hat{a}_x + 0.8677\hat{a}_y)$$

$$= 0.0181(-0.5\hat{a}_x + 0.8677\hat{a}_y)$$

$$= -9.05\hat{a}_x + 15.7\hat{a}_y$$

$\therefore \; F_{23} = -9.05\,\hat{a}_x + 15.7\,\hat{a}_y$ mN

$\therefore$ Using the principle of superposition

$$\bar{F}_3 = \bar{F}_{13} + \bar{F}_{23}$$

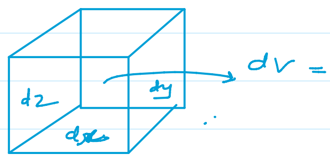
$$= \cancel{9.05\,\hat{a}_x} + 15.7\,\hat{a}_y - \cancel{9.05\,\hat{a}_x} + 15.7\,\hat{a}_y$$

$$\boxed{F_3 = 31.4\,\hat{a}_y \text{ mN}}$$

Q3) For $x, y$ & $z$ +ve ; Let $\rho_v = 40xyz$ $C/m^3$. Find the total charge within the region bounded by $0 \le x, y, z \le 2$

$\Rightarrow$

$$Q = \iiint_{z\,y\,x} \rho_v \, dv$$




$$= \int_0^2 \int_0^2 \int_0^2 40\,xyz \cdot dx\,dy\,dz$$

$$= 40 \left[\frac{x^2}{2}\right]_0^2 \left[\frac{y^2}{2}\right]_0^2 \left[\frac{z^2}{2}\right]_0^2$$

$$= \frac{40}{8}\left(\overset{8}{\cancel{64}}\right) = 320 \text{ C}$$

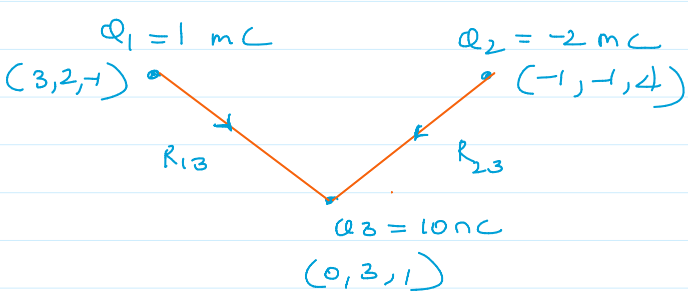
$$\boxed{\therefore Q = 320 \text{ C}}$$

dx

Q4) Two point charges 1mC & −2mC are located at (3,2,−1) & (−1,−1,4) respectively. Calculate the electric force on a 10 nC charge located at (0,3,1) & the electric field at that point.

$\Rightarrow$

Let $Q_3 = 10$ nC




(i) Find force $F_3$

(ii) Find $\bar{E}$

(i)

(ii) Find E $\quad\quad\quad Q_3 = 10\,nC$

$(0,3,1)$

(i)

$$\bar{F}_3 = \bar{F}_{13} + \bar{F}_{23}$$

$$F_{13} = \frac{Q_1 Q_3}{4\pi\varepsilon_0 R_{13}^2} \cdot \hat{a}_{13} = \frac{Q_1 Q_3}{4\pi\varepsilon_0 |R_{13}|^3} (r_3 - r_1)$$

$$r_3 - r_1 = 3\hat{a}_y + \hat{a}_z - (3a_x + 2\hat{a}_y - \hat{a}_z)$$

$$= -3a_x + \hat{a}_y + 2\hat{a}_z \quad ; \quad R_{13} = \sqrt{9+1+4} = (\sqrt{14})$$

$$F_{13} = \frac{1\times10^{-3} \times 10 \times 10^{-9}}{4\pi \times 8.85 \times 10^{-12} \times (\sqrt{14})^3} (-3\hat{a}_x + \hat{a}_y + 2\hat{a}_z)$$

$$F_{13} = 1.7165 \times 10^{-3} (-3\hat{a}_x + \hat{a}_y + 2\hat{a}_z)$$

$$F_{13} = 5.15\hat{a}_x + 1.7165\hat{a}_y + 3.433\hat{a}_z \ mN$$

Similarly

$$F_{23} = \frac{Q_2 Q_3}{4\pi\varepsilon_0 |R_{23}|^3} \times (r_3 - r_2)$$

$$r_3 - r_2 = 3\hat{a}_y + \hat{a}_z - (-\hat{a}_x - \hat{a}_y + 4\hat{a}_z)$$

$$r_3 - r_2 = \hat{a}_x + 4\hat{a}_y - 3\hat{a}_z \quad \text{Also} \quad |R_{32}| = \sqrt{1+16+9} = \sqrt{26}$$

$$F_{23} = \frac{-2 \times 10^{-3} \times 10 \times 10^{-9}}{4\pi \times 8.85 \times 10^{-12} \times (\sqrt{26})^3} (\hat{a}_x + 4\hat{a}_y - 3\hat{a}_z)$$

$$F_{23} = -1.356 \times 10^{-3} (\hat{a}_x + 4\hat{a}_y - 3\hat{a}_z)$$

$$F_{23} = -1.356\hat{a}_x - 5.424\hat{a}_y + 4.068\hat{a}_z \ mN$$

$$\therefore F_3 = F_{13} + F_{23}$$

$$= (5.15\hat{a}_x + 1.7165\hat{a}_y + 3.433\hat{a}_z) + (-1.356\hat{a}_x - 5.424\hat{a}_y + 4.068\hat{a}_z)$$

$$\boxed{\bar{F}_3 = 3.794\hat{a}_x - 3.707\hat{a}_y + 7.501\hat{a}_z \ mN}$$

(iii) Electric field is given by

$\cdots = \cdots 4 a_x - 3.707 a_y + 7.501 a_z \cdots$

(ii) Electric field is given by

$$\bar{E} = \frac{\bar{F}_3}{Q_3} = \frac{(3.794\,\hat{a}_x - 3.707\,\hat{a}_y + 7.501\,\hat{a}_z)\ \text{mN}}{10 \times 10^{-9}}$$

$$\bar{E} = (0.3794\,\hat{a}_x - 0.3707\,\hat{a}_y + 0.7501\,\hat{a}_z) \times 10^{6}$$

$$\boxed{\bar{E} = 379.4\,\hat{a}_x - 370.7\,\hat{a}_y + 750.1\,\hat{a}_z \quad \frac{\text{kV}}{\text{N}}}$$

Q5) Find $\bar{E}$ at the origin if the following charge distributions are present in free space

(i) point charge of 12 nC at (2, 0, 6)

(ii) Uniform line charge density of 3 nC/m at $x = -2$, $y = 3$

(iii) Uniform surface charge density of 0.2 nC/m$^2$ at $x = 2$

→

(i) $$\bar{E}_Q = \frac{Q}{4\pi\varepsilon_0 |R_1|^3}(r_0 - r_1) = \frac{12 \times 10^{-9}}{4\pi \times 8.85 \times 10^{-12} \times (\sqrt{40})^3} \times (-2\hat{a}_x - 6\hat{a}_z)$$

$$\bar{E}_Q = 0.4265\,(-2\hat{a}_x - 6\hat{a}_z) = -0.853\,\hat{a}_x - 2.559\,\hat{a}_z \quad \text{V/m}$$

$$\boxed{\therefore\ \bar{E}_Q = -0.853\,\hat{a}_x - 2.559\,\hat{a}_z \quad \text{V/m}}$$

(ii) $$\bar{E}_L = \frac{\rho_L}{2\pi\varepsilon_0 |R_1|^3}(r_0 - r_2) = \frac{3 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}\,(\sqrt{13})^3}(2\hat{a}_x - 3\hat{a}_y)$$

$$\boxed{\bar{E}_L = 2.302\,\hat{a}_x - 3.451\,\hat{a}_y \quad \text{V/m}}$$

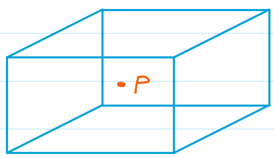
(iii) $$\bar{E}_S = \frac{\rho_S}{2\varepsilon_0}\,\hat{a}_n = \frac{0.2 \times 10^{-9}}{2 \times 8.85 \times 10^{-12}}(-2\hat{a}_x)$$

$$\boxed{\overline{E_S} = -22.598\,\hat{a}_x \ \text{V/m}}$$

$$\boxed{\begin{aligned} \therefore\ \overline{E} &= \overline{E}_Q + \overline{E}_L + \overline{E}_S \\ \overline{E} &= -21.598\,\hat{a}_x - 3.451\,\hat{a}_y - 2.559\,\hat{a}_z \ \text{V/m} \end{aligned}}$$

Q6) Find the total charge enclosed in a cube defined by a volume of $10^{-10}\,m^3$ located at $(2,3,4)$. How much flux leaves this volume.

$\Rightarrow$




$$Q = \int_V \rho_v \, dv = \int_V (\nabla \cdot \overline{D})\, dv$$

$$\nabla \cdot \overline{D} = \frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} D_z \Rightarrow \left\{ \left( \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \right) \cdot \left( D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z \right) \right\}$$

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- Experimental law deals with force between two point charges
- The law states that the force F between two point charges $Q_1$ and $Q_2$ is-
  - directly proportional to the products ($Q_1 \times Q_2$) of the charges
  - Inversely proportional to square of distance (R) between them
  - along the line joining them.
  - depends upon medium
  - depends upon the nature of charges (+ or – ).


- The force is expressed as –
- $\mathrm{F} = \frac{Q_1 Q_2}{4\pi\epsilon R^2}$
- where ε is permittivity of the medium. For free space $\varepsilon = \varepsilon_0\varepsilon_r = \varepsilon_0 = 8.854 \times 10^{-12}$ F/m i.e. $\varepsilon_r = 1$

# Vector Form:

- The force **$F_2$** on $Q_2$ due to $Q_1$ is given by-

$$\therefore \bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

Where $\bar{a}_{12}$ is unit vector in the direction of $\bar{R}_{12}$

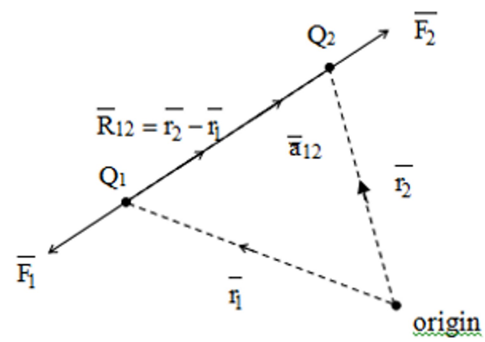
$$\bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}$$

Similarly, force $\bar{F}_1$ on $Q_1$ due $Q_2$ is given by

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}\bar{a}_{21}$$

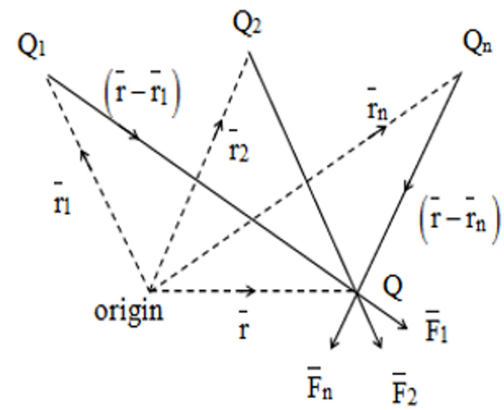
$$= \frac{-Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

$$\bar{F}_1 = -\bar{F}_2$$



# Principle of Superposition:

- Consider system of n point charges. The total force is given as-




$$\bar{F} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \ldots + \bar{F}_n$$

$$= \frac{QQ_1\left(\bar{r}-\bar{r}_1\right)}{4\pi\varepsilon_0\left(\left|\bar{r}-\bar{r}_1\right|\right)^3} + \frac{QQ_2\left(\bar{r}-\bar{r}_2\right)}{4\pi\varepsilon_0\left(\left|\bar{r}-\bar{r}_2\right|\right)^3} + \ldots + \frac{QQ_n\left(\bar{r}-\bar{r}_n\right)}{4\pi\varepsilon_0\left(\left|\bar{r}-\bar{r}_n\right|\right)^3}$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{Q_k\left(\bar{r}-\bar{r}_k\right)}{\left|\bar{r}-\bar{r}_k\right|^3}$$