Module 2

CHAPTER 7

Maxwell's Equations

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7.1 INTRODUCTION

- In previous chapters, we have studied static electric field and static magnetic fields. The source of static electric field is charge while the source of static magnetic field is current through filament. These fields are not changing with respect to time, hence these are called as static fields.
- In this chapter we will study time varying fields and observe that :
  - i) time varying electric field can be produced by time varying magnetic field and
  - ii) time varying magnetic field can be produced by time varying electric field.
- The equations describing the relations between changing electric and magnetic fields are known as Maxwell's equations.
- Maxwell's equations are extensions of the known work of Gauss, Faraday and Ampere. There are two forms of each Maxwell equation namely integral form and differential or point form.
- Maxwell's equations in the integral form govern the interdependence of certain field (E,D,B,H) and source quantities (charge and current) associated with regions in space, that is contours, surfaces and volumes.
- The differential forms of Maxwell's equations, however relate characteristics of the field vectors at a given point to one another and to the source densities at that point.
- The Maxwell's equations provides the mathematical background for the study of electromagnetic waves, transmission lines and antenna. Thus, the applications of the electromagnetic fields can be seen by studying Maxwell's equations.
- In the previous chapters we have studied Divergence theorem and Stoke's theorem.

Divergence theorem is used to convert closed surface integral into volume integral and using stoke's theorem, closed line integral is converted into surface integral.

Divergence Theorem $$\oint \bar{A}\cdot d\bar{s} = \int_V (\nabla\cdot\bar{A})\, dv$$

Stoke's Theorem $$\oint \bar{A}\cdot d\bar{l} = \int_s (\nabla\times\bar{A})\cdot d\bar{s}$$

- These relations are frequently required in this chapter. Some of the relations we studied in previous chapters are called as Maxwell's equations for static field. These are summarized below.

7.2 MAXWELL'S EQUATIONS FOR STATIC FIELDS

- In order to get clear understanding between time varying and static fields, before we start discussing time varying fields, the equations governing static fields are as follows :

7.2.1 Faraday's Law

Statement

The emf is induced in a loop when the magnetic flux is changing in the vicinity of it.

- Since for static field, flux is not changing with respect to time then the emf induced in a loop is zero. The emf induced in a loop is expressed is expressed in terms of electric field as,

$$\text{emf} = \oint \bar{E}\cdot d\bar{l}$$

- Thus for static field, Faraday's law is expressed mathematically as

$$\oint \bar{E}\cdot d\bar{l} = 0 \qquad ...(7.2.1)$$

- This relation is called as *integral form of the Maxwell's equation derived from Faraday's law for the static field.* This relation stands because the work done in a closed path is equal to zero. Using Stoke's theorem,

$$\oint \bar{E}\cdot d\bar{l} = \int_s (\nabla\times\bar{E})\cdot d\bar{s}$$

$$\therefore \int_s (\nabla\times\bar{E})\cdot d\bar{s} = 0$$

but $d\bar{s} \neq 0$, therefore $\nabla\times\bar{E} = 0$ ...(7.2.2)

- This relation is called as *point form of the Maxwell's equation derived from Faraday's law for the static field.*

7.2.2 Ampere's Law

Statement

The line integral of H around a single closed path is equal to the current enclosed by that path.

- The line integral of H around a single closed path is given as,

$$\oint \bar{H}\cdot d\bar{l}$$

- The current I can be expressed in terms of J as,

$$I = \int_S \bar{J}\cdot d\bar{s}$$

- Then, mathematically Ampere's law is given as

$$\oint_S \bar{H} \cdot d\bar{l} = I = \int \bar{J} \cdot d\bar{s} \quad ...(7.2.3)$$

- This is called as *integral form of Maxwell's equation derived from Ampere's law forstatic field.*
- To relate $\bar{H}$ with $\bar{J}$ convert line integral into surface integral using Stoke's theorem as

$$\oint \bar{H} \cdot d\bar{l} = \int_S (\nabla \times \bar{H}) \cdot d\bar{s}$$

$$\therefore \int_S (\nabla \times \bar{H}) \cdot d\bar{s} = \int_S \bar{J} \cdot d\bar{s}$$

or $$\nabla \times \bar{H} = \bar{J} \quad ...(7.2.4)$$

- This is called as *point form of Maxwell's equation derived from Ampere's law for static field.*

7.2.3 Gauss's Law (For Electric)

Statement

The total electric flux crossing the closed surface is equal to the total charge enclosed by that surface.

- The electric flux through the closed surface is,

$$\Psi = \oint_S \bar{D} \cdot d\bar{s}$$

- The charge can be expressed in terms of ρ_v as,

$$Q_{encl.} = \int_V \rho_v \, dv$$

- Then the Gauss law for electric is expressed mathematically as,

$$\boxed{\oint \bar{D} \cdot d\bar{s} = Q_{encl} = \int_V \rho_v \, dv} \quad ...(7.2.5)$$

- This relation is called as *integral form of the Maxwell's equation derived from Gauss's law for the static field.*
- To relate $\bar{D}$ with ρ_v convert surface integral into volume integral using Divergence theorem as,

$$\oint_S \bar{D} \cdot d\bar{s} = \int_V (\nabla \cdot \bar{D}) \, dv$$

$$\therefore \int (\nabla \cdot \bar{D}) \, dv = \int_V \rho_v \, dv$$

or $$\boxed{\nabla \cdot \bar{D} = \rho_v} \quad ...(7.2.6)$$

- This is called as point form of Maxwell's equation derived from Gauss's law for the static field.

7.2.4 Gauss's Law (For Magnetic)

Statement

The total magnetic flux crossing the closed surface is equal to zero.

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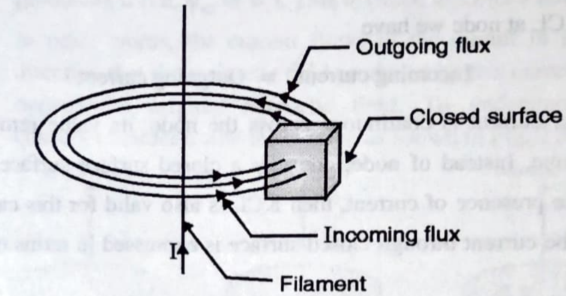
- The reason for this is, magnetic flux lines are always closed in nature. Due to which a closed surface in the presence of these lines will have same number of incoming and outgoing flux lines. The incoming flux, if considered positive then the outgoing flux is negative, resulting in total flux crossing closed surface equal to zero.



- In case of electric field, the positive and negative charge separately exist and electric flux originates from positive charge and terminates on negative charge. Thus electric flux line have start and end points. This is not the case for magnetic lines. The magnetic field lines are always closed in nature. There is no starting and end point, this is called as non- existence of monopole in magnetic field.
- Mathematically it is given as,

$$\oint \bar{B} \cdot d\bar{s} = 0 \quad ...(7.2.7)$$

- This is called as integral form of Maxwell's equation derived from Gauss's law for magnetic field for static field.
- Using Divergence theorem $\oint_S \bar{B} \cdot d\bar{s} = \int_V (\nabla \cdot \bar{B}) \, dv$

$$\therefore \int_V (\nabla \cdot \bar{B}) \, dv = 0$$

i.e. $$\nabla \cdot \bar{B} = 0 \quad ...(7.2.8)$$

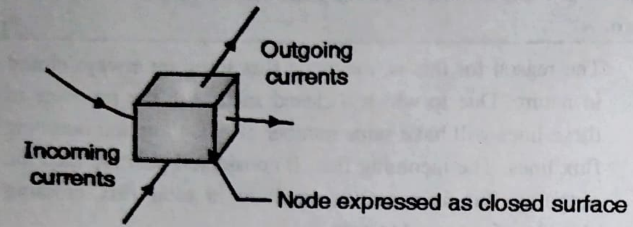
- This is called as *point form of Maxwell's equation derived from Gauss's law for Magnetic field for static field.*

7.2.5 The Continuity Equation for Current

UQ. Write short note on : Maxwell's equation for steady fields. **MU - May 10, 5 Marks**

Statement

The total current crossing the closed surface is equal to zero.



From KCL at node we have

$$\text{Incoming current} = \text{Outgoing current}$$

- i.e. current is continuous across the node, its value remain same. Instead of node, consider a closed surface surface in the presence of current, then KCL is also valid for this case. The current through closed surface is expressed in terms of J as

$$I_{total} = \oint \bar{J} \cdot \overline{ds}$$

- But incoming current if considered positive then outgoing current is negative, resulting in total current crossing the closed surface equal to zero.

Mathematically, $$\boxed{\oint \bar{J} \cdot \overline{ds} = 0} \quad ...(7.2.9)$$

- This is called as integral form of Maxwell's equation derived from continuity equation for static field.

Using Divergence theorem, $$\oint \bar{J} \cdot \overline{ds} = \int_v (\nabla \cdot \bar{J})\, dv$$

$$\therefore \quad \int_v (\nabla \cdot \bar{J})\, dv = 0$$

i.e. $$\nabla \cdot \bar{J} = 0 \quad ...(7.2.10)$$

- This is called as *point form of Maxwell's equation derived from continuity equation for static field.*

Maxwell's Equations for Static Field

Maxwell's equations for static field are summarized in Table 7.2.1.

Table 7.2.1 : Maxwell's equations for static fields

Law	Integral form	Point form
Ampere's	$\oint \bar{H} \cdot d\bar{l} = \int_s \bar{J} \cdot \overline{ds}$	$\nabla \times \bar{H} = \bar{J}$
Faraday's	$\oint \bar{E} \cdot d\bar{l} = 0$	$\nabla \times \bar{E} = 0$
Gauss law (electric)	$\oint \bar{D} \cdot \overline{ds} = \int \rho_v\, dv$	$\nabla \cdot \bar{D} = \rho_v$
Gauss law (magnetic)	$\oint \bar{B} \cdot \overline{ds} = 0$	$\nabla \cdot \bar{B} = 0$
Continuity equation	$\oint \bar{J} \cdot \overline{ds} = 0$	$\nabla \cdot \bar{J} = 0$

Note : In this article we have studied Maxwell's equations for static field. In practice field always vary with time. Maxwell's equations for this field has to be written separately, these are not same for static fields. Only Gauss law, for electric and magnetic is so powerful that it does not require any modification. All other laws require modification.

7.3 FARADAY'S LAW

UQ. State and explain Faraday-Lenz's law. Obtain point form and integral form of Maxwell's equation for induced motional and transformer emf. **MU - May 10, 10 Marks**

UQ. Derive Faraday's Law with suitable applications. **(MU - Q. 1(c), May 19, 5 Marks)**

- In 1820, Oersted demonstrated that when a current is passed through filament it deflects the compass needle.
- It proved that a current is producing the magnetic field. Then Faraday thought that if current could produce a magnetic field, then a magnetic field should be able to produce a current.
- He worked on this problem over a period of 10 years. In 1831 he proved it experimentally. Refer Fig. 7.3.1.
  - o He wound two separate windings on the iron toroid.
  - o A galvanometer was placed in one circuit and a battery in the other.
  - o Upon closing the switch in the battery circuit he noted a momentary deflection of the galvanometer.
  - o When the switch was open, the galvanometer was again deflected momentarily in the opposite direction.
  - o The deflection of the galvanometer proved that a current is produced in the galvanometer circuit.

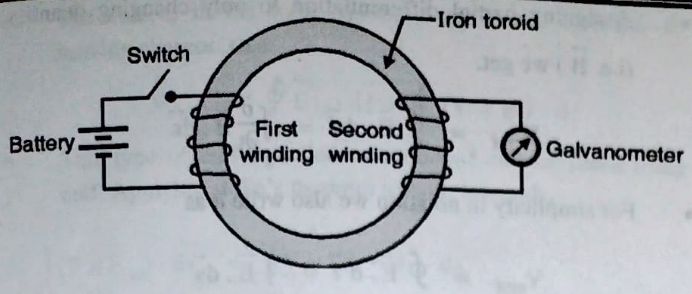


Fig. 7.3.1 : Faraday's experiment

- The current is produced in the second winding whenever the switch is closed or opened. What happens is when the switch is closed the current flows through the first coil producing changing magnetic field, which induces an emf (V_{emf} measured in volts) which is responsible for current in the galvanometer circuit. Similar action takes place at the time of opening the switch.
- To flow a current in the galvanometer circuit, an emf (called as electromotive force) is required to be generated in the second circuit. Thus Faraday's discovery was :

The electromotive force, V_{emf} around a closed circuit is equal to the negative of time rate of change of magnetic flux enclosed by that path.

Mathematically,

$$V_{emf} = \oint \overline{E_f} \cdot d\,\overline{l} = -\frac{d\psi_m}{dt} \quad ...(7.3.1(a))$$

If there are N number of turns in that circuit, then

$$V_{emf} = \oint \overline{E_f} \cdot d\,\overline{l} = -N\frac{d\psi_m}{dt} \quad ...(7.3.1(b))$$

where ψ_m = the magnetic flux through each turn.

N = number of turns.

E_f = induced electric field in the closed circuit which is the cause of emf in that circuit.

- Note the difference between the electric field (E_f) and the field due to charge (also called electrostatic field) E_e.

1. The electrostatic field (E_e) is conservative giving,

$$\oint E_e \cdot d\,\overline{l} = 0 \quad ...(7.3.2(a))$$

2. But the emf produced field (E_f) is non conservative

$$\oint \overline{E_f} \cdot d\,\overline{l} \neq 0 \quad ...(7.3.2(b))$$

- The flux can be obtained from flux density using

$$\psi_m = \int_s \overline{B} \cdot d\overline{s}$$

- Introducing this in Equation (7.3.1(a)) and simply replacing E_f by E we get,

$$V_{emf} = \oint \overline{E} \cdot d\,\overline{l} = -\frac{d}{dt}\int_s \overline{B} \cdot d\,\overline{s} \quad ...(7.3.3)$$

- One should be very careful about the direction of $d\overline{l}$ and the direction of $d\overline{s}$. These are related using right hand, where the fingers of our right hand indicate the direction of the closed path, and our thumb indicates the direction of ds. A flux density $\overline{B}$ in the direction of $d\overline{s}$ and increasing with time (d/dt of it is positive) thus produces an average value of $\overline{E}$ which is opposite (due to negative sign on right hand side) to the positive direction about the closed path.

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7.3.1 Lenz's Law

- The negative sign on the right hand side shows that the induced voltage acts in such a way as to oppose the cause producing it (i.e. ψ_m or B). This is called as Lenz's law.
- In other words, the current flows in the circuit in such a direction that the magnetic field produced by this current will oppose the original magnetic field. To understand the concept consider a simple situation as shown in Fig. 7.3.2.

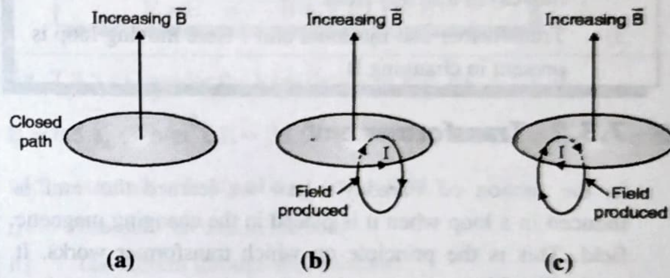

Fig. 7.3.2

- In Fig. 7.3.2(a), the flux density (or flux) is assumed increasing vertically, which induces emf (V_{emf}) in the closed path resulting in current I in that path.
- This current in turn produces a magnetic field. The relation between the direction of current and the magnetic field is obtained using right hand thumb rule. Which states that if we grip the current filament in our right hand such that the direction of thumb is in the direction of current then the direction of fingers indicate the direction of field.
- In Fig. 7.3.2(b), if we assume the current to be anticlockwise then the magnetic field produced at the center of the loop will be upward which will assist the original field.
- In Fig. 7.3.2(c), when the current is clockwise, the magnetic field at the center will be downward, opposing the original vertically up increasing field. According to Lenz's law this is the current direction of current.
- The emf (V_{emf}) on the left side of Equation (7.3.3) is non zero when the right hand side is not zero. The non zero value of $\frac{d}{dt}\int \overline{B} \cdot d\overline{s}$ may result from any of the following situations :

1. A time changing flux and a path is stationary.

2. A time changing path and the magnetic field ($\bar{B}$ or ψ_m) is steady.
3. A combination of the two.

- The first situation is observed in case of transformer so the situation is called as **transformer action.**
- The second situation is observed in case of electric generator so it is called as **electric generator action.**
- Thus the Faraday's law stated earlier is also expressed as :

Faraday's law states that the voltage around the closed path can be generated by a time changing magnetic flux through a fixed path (transformer action) or by a time changing path in a steady magnetic field (electric generator action).

- As stated above emf induced is due to transformer action or electric generator action or combined effect of two. This will give rise to three types of emfs.

1) Transformer emf : B changes with time and loop is stationary.
2) Motional emf : Here area change (otherwords loop moves) in a steady field.
3) Transformer and motional emf : Here moving loop is present in changing B.

7.3.2 Transformer emf

- In the section on Faraday's Law we learned that emf is induced in a loop when it is placed in the changing magnetic field. This is the principle on which transformer works. It works like this.
- In the transformer alternating current flowing in primary produces alternating flux which is confined to the core. The stationary turns cuts this flux resulting in emf across each turn of the secondary. And we get voltage (V_2) across secondary. This emf is named as transformer emf.

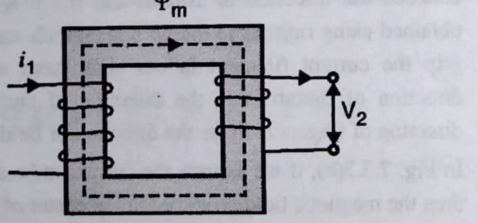

Fig. 7.3.3 : Transformer principle

- Notice that transformer emf is due to stationary loop and changing magnetic field.
- Thus on the right side of Equation (7.3.3), can be altered by changing normal differentiation to partial differentiation.

$$V_{emf} = \oint \bar{E} \cdot d\bar{l} = -\frac{\partial}{\partial t} \int_s \bar{B} \cdot d\bar{s}$$

- Assigning partial differentiation to only changing quantity (i.e. $\bar{B}$) we get,

$$V_{emf} = \oint \bar{E} \cdot d\bar{l} = -\int_s \frac{\partial}{\partial t} \bar{B} \cdot d\bar{s}$$

- For simplicity in notation we also write it as

$$V_{emf} = \oint \bar{E} \cdot d\bar{l} = -\int_s \dot{\bar{B}} \cdot d\bar{s}$$

Where $\dot{\bar{B}}$ is the partial differential of $\bar{B}$ with respect to t.

- Applying stoke's theorem to line integral

$$\oint \bar{E} \cdot d\bar{l} = \int_s (\nabla \times \bar{E}) \cdot d\bar{s} = -\int_s \dot{\bar{B}} \cdot d\bar{s}$$

or

$$\boxed{\nabla \times \bar{E} = -\dot{\bar{B}}}$$

7.3.3 Motional emf

- We know the generator principle. In the generator there are permenant magnets producing steady magnetic field. In the presence of this field a rator attached to the loop is kept. When the rotor rotates, the loop rotates cutting the magnetic field. This results in emf induced in a loop. This emf is called motional emf. Notice the word motion in motional reminding us that the loop is in motion.

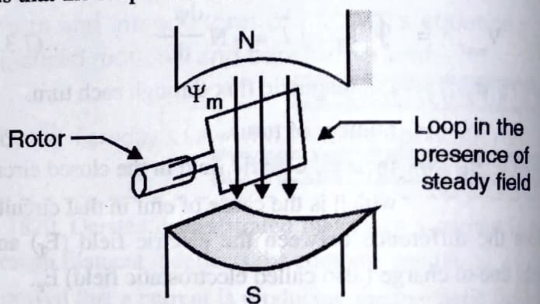

Fig. 7.3.4 : Generator principle

- The emf in the loop will in turn gives rise to current in the loop, and then the loop will experience a force. The force on a charge moving with uniform velocity $\bar{V}$ in a magnetic field $\bar{B}$ is

$$\bar{F}_m = Q\bar{V} \times \bar{B}$$

- The field obtained from it is called motional electric field (E_m).

$$\bar{E}_m = \frac{\bar{F}_m}{Q} = \bar{V} \times \bar{B}$$

- The current in the loop can be considered to be due to moving charges, so

$$V_{emf} = \oint \bar{E}_m \cdot d\bar{l} = \oint (\bar{V} \times \bar{B}) \cdot d\bar{l}$$

- This type of emf is called as motional emf or flux-cutting emf. Applying stoke's theorem to both integrals,

$$\int_S (\nabla \times \bar{E}_m) \cdot d\bar{s} = \int_S \nabla \times (\bar{V} \times \bar{B}) \cdot d\bar{s}$$

or $$\nabla \times \bar{E}_m = \nabla \times (\bar{V} \times \bar{B})$$

- Another example of the motional emf is shown in Fig. 7.3.5. Here the conducting rod is moving between a pair of rails. The current is induced in the rod which in presence of field will exert force on the rod.

- The magnetic field (B) in the Fig. 7.3.5 is perpendicular to the page (into) and the rod is moving in the plane of the paper. The two directions are perpendicular to each other. So the equation.

$$\bar{F}_m = Q\bar{V} \times \bar{B}$$

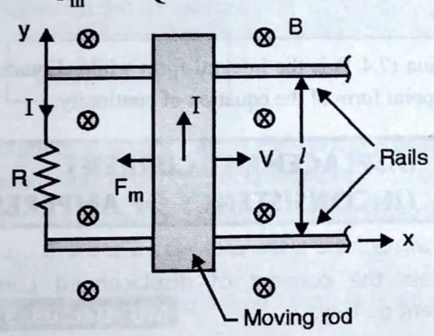

Fig. 7.3.5

reduces to $$F_m = I\,l \times B \Rightarrow I\,l\,B. \quad \text{and}$$

$$V_{emf} = VBl$$

- To apply Equation (7.3.3) is not always easy, some care must be exercised. Remember the following points :

1. The integral in Equation (7.3.3) is zero along the portion of the loop where $\bar{V} = 0$. Thus $d\bar{l}$ is taken along the portion of the loop that is cutting the field (along the rod in Fig. 7.3.5), where $\bar{V}$ has non zero value.

2. The direction of the induced current is the same as that of $\bar{E}_m$ or $\bar{V} \times \bar{B}$. The limits of the integral in Equation (7.3.3) are selected in the opposite direction to the induced current thereby satisfying Lenz's law. In Equation (7.3.3), for example, the integration over l is along $-\bar{a}_y$ where as induced current flows in the rod along $\bar{a}_y$.

3. **Moving Loop in time varying field**

In this case the magnetic field is changing as well as the loop is moving. Then applying superposition,

$$V_{emf} = \oint \bar{E} \cdot d\bar{l} = -\int_S \dot{\bar{B}} \cdot d\bar{s} + \oint_l (\bar{V} \times \bar{B}) \cdot d\bar{l}$$

Applying Stoke's theorem to every line integral, gives

$$\int_S (\nabla \times \bar{E}) \cdot d\bar{s} = -\int_S \dot{\bar{B}} \cdot d\bar{s} + \int_S \nabla \times (\bar{V} \times \bar{B}) \cdot d\bar{s}$$

$$\boxed{\nabla \times \bar{E} = -\dot{\bar{B}} + \nabla \times (\bar{V} \times \bar{B})}$$

7.3.4 Solved Example on Faraday's Law

Important Formulae

$$\nabla \times \bar{E} = -\dot{\bar{B}}$$

$$\nabla \times \bar{E} = -\dot{\bar{B}} + \nabla \times (\bar{V} \times \bar{B})$$

Ex. 7.3.1 : Consider the loop shown in Fig. Ex. 7.3.1. If $\bar{B} = 0.5\,\bar{a}_z$ (Wb/m^2), R = 20 Ω, l = 10 cm, and the rod is moving with a constant velocity of 8 $\bar{a}_x$ (m/s), find.

i) The induced emf in the rod.

ii) The current through the resistance.

iii) The motional force on the rod.

iv) The power dissipated by the resistor.

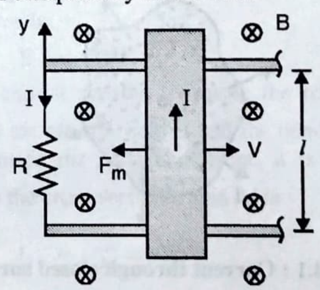

Fig. Ex. 7.3.1

☑ **Soln. :**

i) **To find the induced emf : Given, $\bar{B} = 0.5\,\bar{a}_z$, $\bar{V} = 8\,\bar{a}_x$ and the conductor is along y-axis,**

$$V_{emf} = \int (\bar{V} \times \bar{B}) \cdot d\bar{l} = VBl = 8\,(0.5)\,(0.1) = 0.4\text{ V} \quad \text{...Ans.}$$

ii) **To find the current**

$$I = \frac{V_{emf}}{R} = \frac{0.4}{20} = 20\text{ (mA)} \quad \text{...Ans.}$$

iii) To find the motional force

$$\bar{F}_m = I\bar{l} \times \bar{B} = 20 \times 10^{-3} (0.1\, \bar{a}_y \times 0.5\, \bar{a}_z) = -\bar{a}_x \text{ (mN)}$$

...Ans.

iv) The power dissipated by the resistor

$$P = I^2 R = (20 \times 10^{-3})^2 \times 20 = 8 \text{ (mW)}$$

...Ans.

7.4 MODIFICATION OF CONTINUITY EQUATION OF CURRENT

- **The fundamental property of electrical charge is that it can neither be created nor destroyed. If a charge disappears from one point it must reappear at another point. This postulate is called as "Conservation of charge".**
- **Consider a volume ∇v located inside a conducting media.**

The current density $\bar{J}$ is a vector having the direction of current flow. ***The current coming out from any volume can be thought of as a rate of decrease of charge in that volume.*** **Mathematically, current is expressed as**

$$i = -\frac{dq}{dt} \quad \text{...(i)}$$

- **Also charge is expressed in terms of ρ_v,**

$$q = \int_V \rho_v \, dv$$

$$\therefore \quad i = -\frac{dq}{dt} = -\frac{d}{dt} \int_V \rho_v \, dv$$

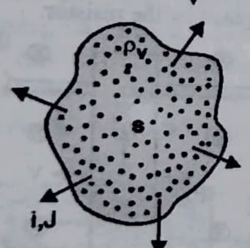

Fig. 7.4.1 : Current through closed surface

- **But current i through closed surface can also be obtained as,**

$$i = \oint_S \bar{J} \cdot \bar{ds} \quad \text{...(ii)}$$

- **If the region of integration is stationary i.e. volume is not changing with time (because we assume only ρ_v is changing with time). Then d/dt in above equation can be written as partial differentiation ∂/∂t. Thus, comparing Equations (i) and (ii),**

$$\oint \bar{J} \cdot \bar{ds} = -\frac{\partial}{\partial t} \int \rho_v \, dv = -\int \frac{\partial \rho_v}{\partial t} dv = -\int \dot{\rho}_v \, dv$$

$$\boxed{\oint_S \bar{J} \cdot \bar{ds} = -\int_V \dot{\rho}_v \, dv} \quad \text{...(7.4.1)}$$

- The dot (·) over ρ_v indicates that it is a partial differentiation with respect to t.
- To relate $\bar{J}$ with ρ_v we have to convert surface integral into volume integral using Divergence theorem as

$$\oint \bar{J} \cdot \bar{ds} = \int_V (\nabla \cdot \bar{J}) \, dv$$

$$\therefore \quad \int_V (\nabla \cdot \bar{J}) \, dv = -\int_V \dot{\rho}_v \, dv$$

- Since both sides are volume integrals, we can compare to write,

$$\boxed{\nabla \cdot \bar{J} = -\dot{\rho}_v} \quad \text{...(7.4.2)}$$

"Equation (7.4.2) is the time varying form for the equation of continuity".

- Equation (7.4.1) is the integral form while Equation (7.4.2) is the point form of the equation of continuity.

7.5 DISPLACEMENT CURRENT (INCONSISTENCY OF AMPERES LAW)

UQ. Explain the concept of displacement current or current density. **MU - Dec. 09, 5 Marks**

UQ. What is the discrepancy in Ampere's law ? How was it removed by Maxwell ? **MU - May 10, 5 Marks**

UQ. Modified form of Ampere's Circuital law. **(MU - Dec. 11, 5 Marks)**

- During our study of magnetostatics we formulated the point form of Ampere's law as,

$$\nabla \times \bar{H} = \bar{J} \quad \text{...(i)}$$

- Taking the divergence of both sides of this equation we get,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}$$

but according to vector identity, divergence of a curl is zero, therefore

$$\nabla \cdot (\nabla \times \bar{H}) = 0$$

i.e. $$\nabla \cdot \bar{J} = 0$$

- This result is not consistent with the continuity equation $(\nabla \cdot \bar{J} = \dot{\rho}_v)$ i.e. statement of **Ampere's law is inconsistent for time varying fields and some modification is required in** it. Suppose we add an unknown term $\bar{G}$ to (i), then

$$\nabla \times \bar{H} = \bar{J} + \bar{G} \quad ...(ii)$$

- Taking divergence of both sides we have

$$\nabla \cdot (\nabla \times H) = \nabla \cdot (\bar{J} + \bar{G}) = 0$$

i.e. $\nabla \cdot \bar{J} + \nabla \cdot \bar{G} = 0$

or $\nabla \cdot \bar{G} = -\nabla \cdot \bar{J}$

- but from the continuity equation

$$\nabla \cdot \bar{J} = -\dot{\rho}_v,$$

we get, $\nabla \cdot \bar{G} = -\nabla \cdot \bar{J} = -(-\dot{\rho}_v) = +\dot{\rho}_v \quad ...(iii)$

- We know the point form of Gauss's law as, $\nabla \cdot \bar{D} = \rho_v$
- Taking differentiation of both sides $\nabla \cdot \dot{D} = \dot{\rho}_v$

Putting this value in Equation (iii), we get

$$\nabla . \bar{G} = \nabla . \dot{\bar{D}}$$

or $\bar{G} = \dot{\bar{D}} \quad ...(iv)$

Using Equations (ii) and (iv) we get,

$$\boxed{\nabla \times \bar{H} = \bar{J} + \dot{\bar{D}}} \quad ...(A)$$

- In the above equation $\bar{J}$ and $\bar{D}$ are added means units of both are same, this can be checked as follows :
- As we have expressed current as surface integral of current density (J), it implies that the current density is having unit of A/m^2. Consider now the unit of $\dot{D}$. As D is nothing but the flux density its unit is C/m^2 and the unit $\dot{D}$ of will be $C/s/m^2$.
- As the basic unit of ampere (A) is C/s, therefore D is having unit of A/m^2.
- Thus $\dot{D}$ is also a current density and is called as **displacement current density,** while J is called as **conduction current density.** The displacement current density is denoted by J_d or J_{disp}, while conduction current density is denoted by J_c or J_{cond}.

Thus, the definition of total current density can be formed as :

$$\bar{J}_{total} = \bar{J}_{cond.} + \bar{J}_{disp}$$

7.6 PHYSICAL SIGNIFICANCE OF DISPLACEMENT CURRENT

UQ. Explain the concept of displacement current or current density. **MU - Dec. 09, 5 Marks**

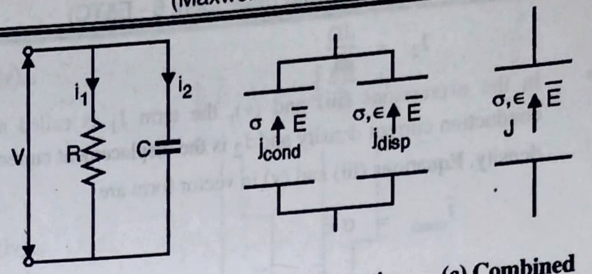

Fig. 7.6.1 : Physical significance of J_d

- Consider a resistor and capacitor in parallel as shown in Fig. 7.6.1(a). The voltage V is applied across the parallel combination.
- The nature of the current flow through the resistor is different from that through the capacitor.
- The constant voltage across a resistor produces a continuous flow of current of constant value given by,

$$i_1 = V/R \quad ...(i)$$

- This current is called as **conduction current.**
- The current flows through the capacitor only when voltage across it is changing i.e.

$$i_2 = \frac{dQ}{dt} = C\frac{dV}{dt} \quad ...(ii)$$

- This current is the **displacement current.** Physically the displacement current is not a "current" in the sense that there is no flow of a physical quantity like charges.
- Now consider the resistor and capacitor elements each occupies a volume as shown in Fig. 7.6.1 (b). Fringing of the field is neglected. Inside each element the electric field E equals, given by

$$E = V/d$$

- Then the current density J_1, inside the resistor element is product of the electric field E and the conductivity σ of the medium inside the resistor element. It is also equal to i_1 divided by the cross-sectional area A, or

$$J_1 = \frac{i_1}{A} = \sigma E \quad ...(iii)$$

- The capacitance of a parallel plate capacitor is C = ∈A/d, where A is the area of the plates and d is the spacing between them. Substituting this value for C, and V = Ed, into Equation (ii) yields,

$$i_2 = \frac{\in Ad}{d}\frac{dE}{dt} = \in A\frac{dE}{dt} \quad ...(iv)$$

- Dividing Equation (iv) by the area A gives a current density inside a capacitor. Thus,

$$J_2 = \frac{i_2}{A} = \in \frac{dE}{dt}$$

We know $D = \in E$

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$$\therefore \quad J_2 = \frac{dD}{dt} \quad \text{...(v)}$$

- In the expressions (iii) and (v), the term J_1 is called as conduction current density and J_2 is the displacement current density. Equations (iii) and (v) in vector form are

$$\bar{J}_{cond} = \sigma \bar{E} \quad \text{...(vi)}$$

$$\bar{J}_{disp} = \epsilon \frac{d\bar{E}}{dt} = \frac{d\bar{D}}{dt} \quad \text{...(vii)}$$

- Now, instead of having two separate elements in parallel, one of which acts like a pure resistance and the other like a pure capacitance, there is only one element which has both capacitance and resistance as shown in Fig. 7.6.1(c), where there is a capacitor filled with a conducting dielectric so that both conduction and displacement currents are present.
- Then the total current density is $\bar{J}_{total} = \bar{J}_{cond} + \bar{J}_{disp}$
- Then the total current I is

$$I = \int_s \bar{J}_{total} \cdot d\bar{s} = \int_s (\bar{J}_{cond} + \bar{J}_{disp}) \cdot d\bar{s}$$

7.7 MAXWELL'S EQUATIONS IN INTEGRAL AND POINT FROM (MAXWELL'S EQUATIONS FOR TIME VARYING FIELDS)

UQ. Derive Maxwell's equation in point form and integral form.
(MU - Dec. 10, May 11, Dec. 14, Dec. 15, 10 Marks)

UQ. Explain Maxwell's equations in differential and integral form for time-varying field.
(MU - Dec. 12, 10 Marks)

UQ. Write maxwell's equation in a) point form b) integral-form. Explain the significance of each equation. **(MU - May 15, 10 Marks)**

- Maxwell's equations for time varying fields in integral form and in point form consists of a set of four laws resulting from several experimental findings and a purely mathematical contribution.
- Each equation, we shall learn to understand their physical significance as well as their mathematical formulation.
- In this chapter two new concepts will be introduced.
  1. The electric field is produced by a *changing magnetic field* (Time varying magnetic field).
  2. The magnetic field is produced by a *changing electric field* (Time varying electric field).
- The first concept was introduced by Michael Faraday and the second by James Maxwell, they form the basic equations of electromagnetic theory.
- In the set of five Maxwell's equations, one equation is derived from Ampere's law, one from Faraday's law and two derived from Gauss's law and one is derived from continuity equation are grouped together, are called Maxwell's equations. Each equation we will study in detail separately in the following section.

7.8 FARADAY'S LAW IN INTEGRAL AND POINT FORM

UQ. Derive Maxwell's equation in point form and integral form.
(MU - Dec. 10, May 11, Dec. 14, Dec. 15, 10 Marks)

UQ. Explain Maxwell's equations in differential and integral form for time-varying field.
(MU - Dec. 12, 10 Marks)

UQ. Write Maxwell's equation in a) point form b) integral-form. Explain the significance of each equation. **(MU - May 15, 10 Marks)**

UQ. State and explain Faraday's law in both the integral and differential form. Explain the shortcomings of each of the form. **(MU - May 14, 5 Marks)**

- Michael Faraday in 1831, discovered experimentally that a current was induced in a conducting loop when a magnetic flux linking the loop is changed. The current which is induced indicates the existence of a voltage or an electromotive force (emf).

The quantitative relationship between the induced emf and the rate of change of flux linkage is known as **Faraday's law.**

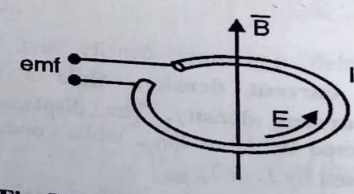

Fig. 7.8.1 : Changing $\bar{B}$ produces emf

Statement of Faraday's law

In a closed path (loop) the electric potential (emf) is developed due to time - varying magnetic field in the vicinity of that closed path. Mathematically,

$$\text{emf} = \oint_c \bar{E} \cdot d\bar{l} = \frac{-d\psi_m}{dt} \; (V) \quad \text{...(i)}$$

- The negative sign in the expression indicates a Lenz's law, which is : The current induced in a loop flows in such a direction as to oppose the cause producting it. The letter, ψ_m is the magnetic flux and can be related with the flux density $\bar{B}$ as :

$$\psi_m = \int_S \bar{B} \cdot d\bar{s} \qquad \text{...(ii)}$$

- Then from Equations (i) and (ii), $\oint_C \bar{E} \cdot d\bar{l} = -\frac{d}{dt} \int_S \bar{B} \cdot d\bar{s}$
- Actually emf is generated by a time changing magnetic flux through a fixed path (transformer action) or by a time varying path in a steady magnetic field. (i.e. generator action).
- If we consider a stationary path, the magnetic flux is only time varying quantity on right hand side and a partial derivative may be taken under the integral sign.

i.e. $$\oint \bar{E} \cdot d\bar{l} = -\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

or $$\boxed{\oint_C \bar{E} \cdot d\bar{l} = -\int_S \dot{\bar{B}} \cdot d\bar{s}} \qquad \text{...(7.8.1)}$$

- This is called as integral form of Maxwell's equation derived from Faraday's law. The dot over B expresses partial differentiation with respect to t. Applying Stokes theorem to convert line integral on left side of above equation to surface integral :

$$\oint_C \bar{E} \cdot d\bar{l} = \oint_S (\nabla \times \bar{E}) \cdot d\bar{s}$$

$$\therefore \quad \oint_C (\nabla \times \bar{E}) \cdot d\bar{s} = -\int_S \dot{\bar{B}} \cdot d\bar{s}$$

Comparing we write, $$\boxed{\nabla \times \bar{E} = -\dot{\bar{B}}} \qquad \text{...(7.8.2)}$$

- This is a point form or differential form of Maxwell's equation derived from Faraday's law.
- In converting to the differential form from integral form, the line integral of $\bar{E}$ around the closed path C is replaced by the curl of $\bar{E}$, the surface integral of $\partial\bar{B}/\partial t$ over the surface s bounded by C is replaced by $\partial\bar{B}/\partial t$ itself as shown :

$$\oint_C \bar{E} \cdot d\bar{l} = -\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

7.9 AMPERE'S LAW IN INTEGRAL AND POINT FORM

UQ. Derive Maxwell's equation in point form and integral form. **(MU - Dec. 10, May 11, Dec. 14, Dec. 15, 10 Marks)**

UQ. Explain Maxwell's equations in differential and integral form for time-varying field. **(MU - Dec. 12, 10 Marks)**

UQ. Write Maxwell's equation in a) point form b) integral-form. Explain the significance of each equation. **(MU - May 15, 10 Marks)**

UQ. Write integral form of Ampere's law and interpret the same. **(MU - Dec. 15, 5 Marks)**

Statement of Ampere's law

The line integral of $\bar{H}$ around a closed path is equal to the current enclosed by the path.

Mathematically $$\oint \bar{H} \cdot d\bar{l} = I \qquad \text{...(i)}$$

Where, both conduction and displacement currents are present, the current enclosed is the total current.

$$I = \int \bar{J}_{total} \cdot d\bar{s} = \int (\bar{J}_{cond} + \bar{J}_{disp}) \cdot d\bar{s} \qquad \text{...(ii)}$$

- The first term on RHS is the conduction current and the second term is the displacement current. Thus, the conduction current is

$$i_c = \int_S \bar{J}_{cond} \cdot d\bar{s}$$

- Usually $\bar{J}$ is used for $\bar{J}_{cond}$, then the conduction current is

$$i_c = \int_S \bar{J} \cdot d\bar{s}$$

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- The displacement current is the second term i.e.

$$i_d = \int_S \bar{J}_{disp} \cdot d\bar{s}$$

- The displacement current density is $\dot{D}$, therefore the displacement current is also written as,

$$i_d = \int_S \dot{\bar{D}} \cdot d\bar{s}$$

- Equation (ii) can now be written as

$$I = i_c + i_d = \int_S (\bar{J} + \dot{\bar{D}}) \cdot d\bar{s}$$

Putting the value in Equation (i)

$$\boxed{\oint \bar{H} \cdot d\bar{l} = \int_S (\bar{J} + \dot{\bar{D}}) \cdot d\bar{s}} \quad ...(7.9.1)$$

- This is the integral form of Maxwell's equation derived from Ampere's law.

Differential form

Using Stoke's theorem we get

$$\oint_C \bar{H} \cdot d\bar{l} = \int_S (\nabla \times \bar{H}) \cdot d\bar{s}$$

or

$$\int_S (\nabla \times \bar{H}) \cdot d\bar{s} = \int_S (\bar{J} + \dot{\bar{D}}) \cdot d\bar{s}$$

- Both sides of above expression are surface integrals, comparing we get,

$$\boxed{\nabla \times \bar{H} = \bar{J} + \dot{\bar{D}}} \quad ...(7.9.2)$$

- This is the point form or differential form of Maxwell's equation derived from Ampere's law.
- In converting to the differential form from integral form, the line integral of $\bar{H}$ around the closed path C is replaced by curl of $\bar{H}$.
- And the surface integral of $(\bar{J} + \dot{\bar{D}})$ over the surface s bounded by C is replaced by $(\bar{J} + \dot{\bar{D}})$ itself as shown :

$$\oint \bar{H} \cdot d\bar{l} = \int_S (\bar{J} + \dot{\bar{D}}) \cdot d\bar{s}$$

$$\nabla \times \bar{H} = \bar{J} + \dot{\bar{D}}$$

7.10 GAUSS'S LAW IN INTEGRAL AND POINT FORM

> **UQ.** Derive Maxwell's equation in point form and integral form.
> **(MU - Dec. 10, May 11, Dec. 14, Dec. 15, 10 Marks)**
>
> **UQ.** Explain Maxwell's equations in differential and integral form for time-varying field.
> **(MU - Dec. 12, 10 Marks)**
>
> **UQ.** Write Maxwell's equation in a) point form b) integral-form. Explain the significance of each equation.
> **(MU - May 15, 10 Marks)**

- Thus far we have derived Maxwell's equations in integral form involving the line integrals of $\bar{E}$ and $\bar{H}$ around the closed path, that is, Faraday's law and Ampere's circuital law, respectively.
- The remaining two Maxwell's equations in integral form, namely, Gauss's law for the electric field and Gauss's law for the magnetic filed, are concerned with the closed surface integrals of $\bar{D}$ and $\bar{B}$, respectively.

7.10.1 Gauss's Law for the Electric Field

Statement

> **The total flux crossing the closed surface is equal to the total charge enclosed by the closed surface.**

Mathematically,

$$\oint_S \bar{D} \cdot d\bar{s} = Q \quad ...(i)$$

- The charge 'Q' can be written in terms of ρ_v as :

$$Q = \int_V \rho_v \, dv \quad ...(ii)$$

From Equations (i) and (ii) we have,

$$\boxed{\oint_S \bar{D} \cdot d\bar{s} = \int_V \rho_v \, dv} \quad ...(7.10.1)$$

- This is called as integral form of Maxwell's equation derived from Gauss's law.
- Applying divergence theorem to convert surface integral on the left hand side to volume integral

$$\oint_S \bar{D}\cdot \bar{ds} = \int_V (\nabla\cdot\bar{D})\, dv$$

Then from Table (7.2.1), $\int_V (\nabla\cdot\bar{D})\, dv = \int_V \rho_v\, dv$

- Both sides of the above equation are volume integrals, comparing

$$\boxed{\nabla\cdot\bar{D} = \rho_v} \quad \text{...(7.10.2)}$$

- This is called as **point form of Maxwell's equation derived from Gauss's law.**

7.10.2 Gauss's Law for the Magnetic Field

Statement

In case of magnetic field the total outgoing magnetic flux through a closed surface is equal to zero.

Mathematically,

$$\boxed{\oint_S \bar{B}\cdot\bar{ds} = 0} \quad \text{...(7.10.3)}$$

- This is called as integral form of Maxwell's equation from Gauss's law for magnetic field.
- This equation results from the fact that the magnetic flux lines are continuous. Using divergence theorem to convert surface integral to volume integral Equation (7.10.3) becomes

$$\oint_S \bar{B}\cdot\bar{ds} = \int_V (\nabla\cdot\bar{B})\, dv$$

$$\therefore \quad \boxed{\int(\nabla\cdot\bar{B})\, dv = 0}$$

but dv can not be zero.

$$\therefore \quad \boxed{\nabla\cdot\bar{B} = 0} \quad \text{...(7.10.4)}$$

- Equation (7.10.4) is called as point form or differential form of Maxwell's equation derived from Gauss's law for magnetic field.

7.11 GENERAL SET OF MAXWELL'S EQUATIONS

UQ. Explain significance of Maxwell's equation. **(MU - Dec. 09, 5 Marks)**

UQ. Write short note on : Maxwell's equation for time varying fields **(MU - May 10, 5 Marks)**

UQ. Derive Maxwell's equation in point form and integral form. **(MU - Dec. 10, May 11, Dec. 14, Dec. 15, 10 Marks)**

UQ. Explain Maxwell's equations in differential and integral form for time-varying field. **(MU - Dec. 12, 10 Marks)**

UQ. Write Maxwell's equation in a) point form b) integral-form. Explain the significance of each equation. **(MU - May 15, 10 Marks)**

UQ. State the Maxwell's equations for good dielectric in integral and point form. Also state their significance. **(MU - Dec. 16, 5 Marks)**

UQ. Explain in brief Maxwell's Equation for time varying field in Integral and Point format, also give their significance. **(MU - Q. 3(a), Dec. 18, 10 Marks)**

UQ. Explain Point and Integral format of Time Varying field Maxwell's Equation with appropriate examples. **(MU - Q. 3(b), May 19, 10 Marks)**

UQ. Derive Maxwell's equation for time varying fields in point and integral form and explain its significance. **(MU - Q. 3(a), Dec. 19, 10 Marks)**

- The general set of Maxwell's equations are given in Table 7.11.1.

Table 7.11.1 : Maxwell's equations, general set for time varying fields

Law	Integral form	Point from
Ampere's law	$\oint \bar{H}\cdot d\bar{l} = \int_S (\bar{J}+\dot{\bar{D}})\cdot\bar{ds}$	$\nabla\times\bar{H} = \bar{J}+\dot{\bar{D}}$
Faraday's law	$\oint \bar{E}\cdot d\bar{l} = -\int_S \dot{\bar{B}}\cdot\bar{ds}$	$\nabla\times\bar{E} = -\dot{\bar{B}}$
Gauss's law : electric field	$\oint_S \bar{D}\cdot\bar{ds} = \int_V \rho_v\cdot dv$	$\nabla\cdot\bar{D} = \rho_v$

Law	Integral form	Point from
Gauss's law : magnetic field	$\oint_S \bar{B} \cdot d\bar{s} = 0$	$\nabla \cdot \bar{B} = 0$
Continuity equation	$\oint_S \bar{J} \cdot d\bar{s} = -\int_V \dot{\rho}_v \, dv$	$\nabla \cdot \bar{J} = -\dot{\rho}_v$

- The above set of equations is called as general set of Maxwell's equation. Sometimes media in the problem may be free space or good dielectric or good conductor.

Physical Significance of Maxwell's Equations

i) First two Maxwell's equations says :
The time varying electric (magnetic) field produces a time varying magnetic (electric) field.

ii) These electric field and magnetic fields are perpendicular to each other.

7.12 MAXWELL'S EQUATIONS FOR DIFFERENT TYPES OF MEDIA

UQ. Derive Maxwell's equation in point form and integral form for free space. **(MU - May 12, 10 Marks)**

UQ. State and explain Maxwell's equation in free space in integral and differential form. Hence explain the difference between conduction and displacement current. **(MU - May 14, 10 Marks)**

UQ. State the Maxwell's equations for good dielectric in integral and point form. Also state their significance. **(MU - Dec. 16, 5 Marks)**

- In the previous section we studied general set of Maxwell's equations. Depending upon the media involved in the problem we should change these equations. The media may be

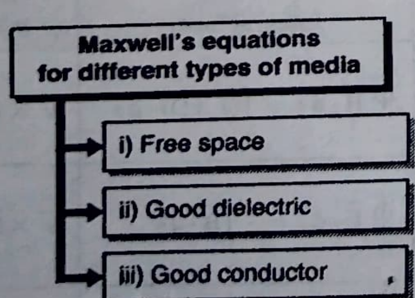

Fig. 7.12.1 : Maxwell equation for different types of media

- Let us discuss properties of these media so that using these we can modify Maxwell's equations.

1) Free space

- Free space is a space without charges, so it will not have any charge density.

$$\rho_v = 0$$

- Due to absence of charges there is no conduction through space giving zero conduction density.

$$\bar{J} = 0$$

- So for free space,

$$\bar{J} = 0 \text{ and } \rho_v = 0$$

2) Good dielectric

- Dielectric also is a medium without charges and thus there is no charge density

$$\rho_v = 0$$

- For a good dielectric, the displacement current is greater than conduction current and thus conduction current ($\bar{J}$) can be neglected

$$\bar{J} = 0$$

- So for good dielectric

$$\bar{J} = 0 \text{ and } \rho_v = 0$$

3) Good conductor

- In normal state good conductor will not have any charge (temperature effect is neglected) giving

$$\rho_v = 0$$

- In good conductor the conduction current (J) is greater than displacement current (J_d D) and D can be neglected.

$$J_d = \dot{D} = 0$$

So, for good conductor

$$\dot{D} = 0 \text{ and } \rho_v = 0$$

$$\boxed{\begin{aligned} &\textbf{Free space : } \bar{J} = 0, \rho_v = 0 \\ &\textbf{Good dielectric : } \bar{J} = 0, \rho_v = 0 \\ &\textbf{Good conductor : } \dot{D} = 0, \rho_v = 0 \end{aligned}}$$

Depending upon media the Maxwell's equations change. It is as given below :

For a free space

$$\bar{J} = 0 \text{ and } \rho_v = 0$$

- Then, the set of Maxwell's Equation for free space is as shown in Table 7.12.1.

Table 7.12.1 : Maxwell's equations for free space in time varying field

Point from	Integral form
$\nabla \times \bar{H} = \dot{\bar{D}}$	$\oint \bar{H} \cdot d\bar{l} = \int_S \dot{\bar{D}} \cdot d\bar{s}$
$\nabla \times \bar{E} = -\dot{\bar{B}}$	$\oint_C \bar{E} \cdot d\bar{l} = -\int_S \dot{\bar{B}} \cdot d\bar{s}$
$\nabla \cdot \bar{D} = 0$	$\oint_S \bar{D} \cdot d\bar{s} = 0$
$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot d\bar{s} = 0$

For good dielectrics

$\bar{J} = 0$ and $\rho_v = 0$

Applying these conditions the Maxwell's equations for good dielectrics are given in Table 7.12.2.

Table 7.12.2 : Maxwell's equations for good dielectrics

Point from	Integral form
$\nabla \times \bar{H} = \dot{\bar{D}}$	$\oint \bar{H} \cdot d\bar{l} = \int_S \dot{\bar{D}} \cdot d\bar{s}$
$\nabla \times \bar{E} = -\dot{\bar{B}}$	$\oint_C \bar{E} \cdot d\bar{l} = -\int \dot{\bar{B}} \cdot d\bar{s}$
$\nabla \cdot \bar{D} = 0$	$\oint_S \bar{D} \cdot d\bar{s} = 0$
$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot d\bar{s} = 0$

For good conductors

$J >> \dot{D}$ and $\rho_v = 0$

- Applying these conditions, Maxwell's equations for good conductors are as given in Table 7.12.3.

Table 7.12.3 : Maxwell's equations for good conductors

Point from	Integral form
$\nabla \times \bar{H} = \bar{J}$	$\oint \bar{H} \cdot d\bar{l} = \int_S \bar{J} \cdot d\bar{s}$
$\nabla \times \bar{E} = -\dot{\bar{B}}$	$\oint_C \bar{E} \cdot d\bar{l} = -\int_S \dot{\bar{B}} \cdot ds$
$\nabla \cdot \bar{D} = 0$	$\oint_S \bar{D} \cdot d\bar{s} = 0$
$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot d\bar{s} = 0$
$\nabla \cdot \bar{J} = 0$	$\oint_S \bar{J} \cdot d\bar{s} = 0$

Comparison between conduction and displacement current

Sr. No.	Conduction current	Displacement current
1.	It is the current flowing through a conductor.	It is the current flowing through a dielectric.
2.	It is denoted by i_C.	It is denoted by i_d.
3.	In a good conductor $i_C >> i_d$.	In a good dielectric $i_d >> i_C$.

7.13 MAXWELL'S EQUATION FOR HARMONICALLY VARYING FIELDS (PHASOR FORM)

- Field vectors that vary with space coordinates and are sinusoidal functions of time are called as harmonically varying fields. Harmonically varying fields are important from practical point of view since all voltage and current sources in practice are sinusoidal in nature. Therefore fields generated by these sources will also be sinusoidal.
- Let us see how we can write Maxwell's equations for the special case of sinusoidal variation with time. Consider the electric flux density D vary with time and is given by,

$$D = D_0 e^{j\omega t}$$

- Remember, $e^{j\omega t} = \cos \omega t + j \sin \omega t$, Thus multiplying the field by $e^{j\omega t}$ is making the field vary with time harmonically.

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In the above expression D_0 is the value of D at t = 0 and it is the function of x, y, z while D is a function of x, y, z and t, then

$$\dot{D} = \frac{\partial D}{\partial t} = D_0\, j\omega\, e^{j\omega t}$$

i.e. $$\dot{D} = j\omega D$$

- **Here we can say that for harmonically varying field dot over the field (·) [i.e. partial differentiation w.r.t. t] can be replaced by multiplying the field by jω. Now, Maxwell's equations for harmonically varying field can be written as :**

(i) Ampere's law

The point form $\nabla \times \bar{H} = \bar{J} + j\omega\bar{D} = \sigma\bar{E} + j\omega\epsilon\bar{E}$

i.e. $\nabla \times \bar{H} = (\sigma + j\omega\epsilon)\,\bar{E}$

The integral form $\oint \bar{H}\cdot d\bar{l} = (\sigma + j\omega\epsilon) \int_S \bar{E}\,.\,d\bar{s}$

(ii) Faraday's law

The point form $\nabla \times \bar{E} = -\dot{\bar{B}} = -j\omega\bar{B} = -j\omega\mu\bar{H}$

i.e. $\nabla \times \bar{E} = -j\omega\mu\bar{H}$

The integral form $\oint \bar{E}\cdot d\bar{l} = -j\omega\mu \int_S \bar{H}\,.\,d\bar{s}$

(iii) Gauss's law

The point form $\nabla \cdot \bar{D} = \rho_v$

The integral form $\oint \bar{D}\cdot d\bar{s} = \int_V \rho_v\, dv$

(iv) Gauss's law for magnetic field

The point form $\nabla \cdot \bar{B} = 0$

The integral form $\oint \bar{B}\cdot d\bar{s} = 0$

(v) Continuity equation

The point form $\nabla \cdot \bar{J} = -j\omega\rho_v$

The integral form $\oint \bar{J}\cdot d\bar{s} = -j\omega \int_V \rho_v\, dv$

The above relations are summarized in Table 7.13.1.

Table 7.13.1 : Maxwell's equations for harmonically varying fields (phasor form)

Point from	Integral form
$\nabla \times \bar{H} = (\sigma + j\omega\epsilon)\,\bar{E}$	$\oint \bar{H}\,.\,d\bar{l} = (\sigma + j\omega\epsilon)\int \bar{E}\cdot d\bar{s}$
$\nabla \times \bar{E} = -j\omega\mu\,\bar{H}$	$\oint \bar{E}\,.\,d\bar{l} = -j\omega\mu \int \bar{H}\cdot d\bar{s}$
$\nabla \cdot \bar{D} = \rho_v$	$\oint \bar{D}\cdot d\bar{s} = \int_V \rho_v\, dv$
$\nabla \cdot \bar{B} = 0$	$\oint \bar{B}\cdot d\bar{s} = 0$
$\nabla \cdot \bar{J} = -j\omega\rho_v$	$\oint \bar{J}\cdot d\bar{s} = -j\omega \int \rho_v\, dv$

7.14 TYPE 1 PROBLEMS : BASED ON FIELD CONVERSION

From given electric (magnetic) obtain the magnetic (electric) field. This can be done by using first two equations in Table 7.11.1.

$$\nabla \times \bar{H} = \bar{J} + \dot{\bar{D}}$$

and for free space $\nabla \times \bar{H} = \dot{\bar{D}}$

$$\nabla \times \bar{E} = -\dot{\bar{B}}$$

Ex. 7.14.1 : Given $\bar{E} = E_m \sin(\omega t - \beta z)\bar{a}_y$ in free space find $\bar{D}$, $\bar{B}$ and $\bar{H}$ displacement current density. Sketch $\bar{E}$ and $\bar{H}$ at t = 0.

☑ **Soln. :**

Given : $\bar{E} = E_m \sin(\omega t - \beta z)\bar{a}_y = E_y\,\bar{a}_y$

We have, $$\bar{D} = \epsilon_0 \bar{E} = \epsilon_0 E_m \sin(\omega t - \beta z)\bar{a}_y \quad (C/m^2)$$

We know the Maxwell's equation $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

$$\text{LHS} = \nabla \times \bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix}$$

$$= -\bar{a}_x \frac{\partial E_y}{\partial z} + \bar{a}_z \frac{\partial E_y}{\partial x}$$

$$= -\bar{a}_x \frac{\partial E_y}{\partial z} \qquad (\because E_y \text{ is not a function of } z)$$

$$= \beta E_m \cos(\omega t - \beta z)\,\bar{a}_x = \text{RHS} = -\dot{B}$$

$$\therefore \quad \dot{\bar{B}} = -\beta E_m \cos(\omega t - \beta z)\,\bar{a}_x \quad \text{...Ans.}$$

$\bar{B}$ is obtained by integrating $\dot{B}$ with respect to t as

$$\bar{B} = \int \dot{\bar{B}}\,dt$$

$$= -\int \beta E_m \cos(\omega t - \beta z)\,\bar{a}_x\,dt$$

$$= -\frac{\beta E_m}{\omega} \sin(\omega t - \beta z)\,\bar{a}_x \;(T)$$

$$\text{Now} \quad \bar{H} = \frac{\bar{B}}{\mu_0} = -\left(\frac{\beta E_m}{\mu_0 \omega}\right) \sin(\omega t - \beta z)\,\bar{a}_x$$

$$= -H_m \sin(\omega t - \beta z)\,\bar{a}_x \;(A/m)$$

$$\text{Where,} \quad H_m = \frac{\beta E_m}{\mu\omega}$$

At t = 0, we get, $\bar{E} = E_m \sin(-\beta z)\,\bar{a}_y = -E_m \sin(\beta z)\,\bar{a}_y$

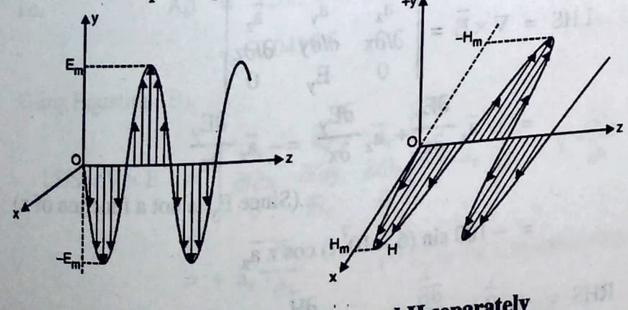
$$\bar{H} = -H_m \sin(-\beta z)\bar{a}_x = H_m \sin(\beta z)\,\bar{a}_x$$

...Ans.

Displacement current density is

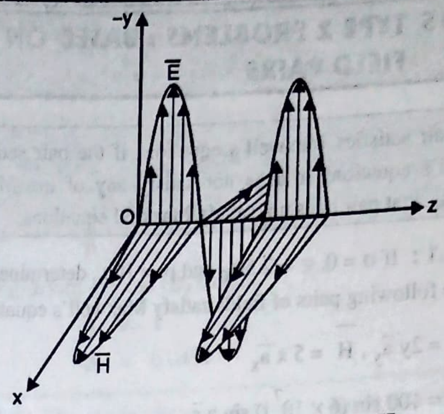
$$\bar{J}_d = \dot{\bar{D}} = \epsilon_0\,\dot{\bar{E}} = \epsilon_0\,\omega\,E_m \cos(\omega t - \beta z)\,\bar{a}_y \quad \text{...Ans.}$$

$\bar{E}$ and $\bar{H}$ at t = 0 are plotted separately as shown in Fig. Ex. 7.14.1(a). While drawing these figures remember that both $\bar{E}$ and $\bar{H}$ are sinusoidal functions of z and unit vectors $\bar{a}_y$ and $\bar{a}_x$ gives directions of E and H vectors respectively. These directions are shown by arrows in the diagram. The combined plot is shown in Fig. Ex. 7.14.1(b) in which the y-axis is shown above for the convenience in plotting.



(a) Showing variation of E and H separately

Fig. Ex. 7.14.1 Contd...



(b) Shows combined plot of E and H

Fig. : Illustrating Ex. 7.14.1

The combined plot clearly indicates that E and H are perpendicular to each other and also perpendicular to direction of propagation. This is a TEM wave.

Ex. 7.14.2 : Given that, $\bar{H} = H_m\,e^{j(\omega t + \beta z)}\,\bar{a}_x$ (A/m) in free space, find $\bar{E}$.

☑ **Soln. : Given**

$$\bar{H} = H_m\,e^{j(\omega t + \beta z)}\,\bar{a}_x = H_x\,\bar{a}_x$$

$$\text{Where,} \quad H_x = H_m\,e^{j(\omega t + \beta z)}$$

We have for free space, $\nabla \times \bar{H} = \dot{\bar{D}} = \dfrac{d\bar{D}}{dt}$

$$\text{LHS} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & 0 & 0 \end{vmatrix} = +$$

$$\bar{a}_y \frac{\partial H_x}{\partial z} - \bar{a}_z \frac{\partial H_x}{\partial y}$$

$$= \bar{a}_y \frac{\partial H_x}{\partial z} \qquad (\because H_x \text{ is not a function of y})$$

$$= \frac{\partial}{\partial z} H_m e^{j(\omega t + \beta z)}\,\bar{a}_y = j\beta H_m e^{j(\omega t + \beta z)}\,\bar{a}_y$$

$$\text{LHS} = j\beta H_m e^{j(\omega t + \beta z)}\,\bar{a}_y = \text{RHS} = \dot{\bar{D}}$$

$$\therefore \quad \dot{\bar{D}} = j\beta H_m e^{j(\omega t + \beta z)}\,\bar{a}_y$$

Flux density $\bar{D}$ is obtained by integrating $\dot{\bar{D}}$ w. r. t. time as,

$$\bar{D} = \int \dot{\bar{D}}\,dt = \int j\beta H_m e^{j(\omega t + \beta z)}\,\bar{a}_y\,dt$$

$$= j\beta H_m \frac{e^{j(\omega t + \beta z)}}{j\omega}\,\bar{a}_y = \frac{\beta}{\omega} H_m e^{j(\omega t + \beta z)}\,\bar{a}_y \;(C/m^2)$$

$$\text{Now} \quad \bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{\beta H_m\,e^{j(\omega t + \beta z)}}{\epsilon_0 \omega} \;(V/m) \quad \text{...Ans.}$$

7.15 TYPE 2 PROBLEMS : BASED ON FIELD PAIRS

Given pair satisfies Maxwell's equations if the pair satisfies all Maxwell's equations it does not satisfy any of the Maxwell's equations. that pair does not satisfy Maxwell equations.

Ex. 7.15.1 : If $\sigma = 0, \epsilon = 2.5\epsilon_0$ and $\mu = 10\mu_0$ determine whether or not the following pairs of fields satisfy Maxwell's equation.

(a) $\bar{E} = 2y\,\bar{a}_y$, $\bar{H} = 5x\,\bar{a}_x$

(b) $\bar{E} = 100 \sin(6\times 10^7 t) \sin z\,\bar{a}_y$,

$\bar{H} = -0.1328 \cos(6\times 10^7 t)\ \cos z\,\bar{a}_x$

(c) $\bar{D} = (z + 6\times 10^7 t)\,\bar{a}_x$,

$\bar{B} = (-754 z - 4.52\times 10^7 t)\,\bar{a}_y$.

☑ **Soln. :**

For convenience the order of Maxwell's equations is changed. Maxwell's equations in point form are

$$\nabla\cdot\bar{B} = 0 \quad \text{... (i)}$$

$$\nabla\cdot\bar{D} = \rho_v \quad \text{... (ii)}$$

$$\nabla\times\bar{H} = \dot{\bar{D}} \quad (\text{as } \bar{J} = \sigma\bar{E} = 0, \because \sigma = 0) \quad \text{... (iii)}$$

$$\nabla\times\bar{E} = -\dot{\bar{B}} \quad \text{... (iv)}$$

(a) For the first pair of fields

Given : $\bar{E} = 2y\,\bar{a}_y = E_y\,\bar{a}_y$ and $\bar{H} = 5x\,\bar{a}_x = H_x\,\bar{a}_x$

From first Maxwell's Equation,

$$\nabla\cdot\bar{B} = 0$$

i.e. $$\mu(\nabla\cdot\bar{H}) = 0 \quad (\because \bar{B} = \mu\bar{H})$$

$$\text{LHS} = \mu(\nabla\cdot\bar{H}) = \mu\left[\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right]$$

$$= \mu\frac{\partial H_x}{\partial x} \quad (\because H_y = H_z = 0)$$

$$= 10\,\mu_0(5) = 50\mu_0 \neq \text{RHS} \quad \text{...Ans.}$$

Thus, this pair of fields does not satisfy Equation (i), so it does not satisfy Maxwell's equations.

(b) For the second pair of fields

Given : $\bar{E} = 100 \sin(6\times 10^7 t) \sin z\,\bar{a}_y = E_y\,\bar{a}_y$ and

$\bar{H} = -0.1328 \cos(6\times 10^7 t) \cos z\,\bar{a}_x = H_x\,\bar{a}_x$

(i) First Maxwell's equation is,

$$\nabla\cdot\bar{B} = 0$$

$$\text{LHS} = \nabla\cdot\bar{B} = \mu(\nabla\cdot\bar{H})$$

$$= 10\,\mu_0\left[\frac{\partial H_x}{\partial x}\right] \quad \text{...}(\because H_y = H_z = 0)$$

$$= 0 = \text{RHS}.$$

$\therefore$ The second pair satisfy first Maxwell's equation.

(ii) The second equation is $\nabla\cdot\bar{D} = \rho_v$

$$\text{LHS} = \nabla\cdot\bar{D} = \epsilon(\nabla\cdot\bar{E}) = 2.5\,\epsilon_0(\nabla\cdot\bar{E})$$

$$= 2.5\,\epsilon_0\frac{\partial E_y}{\partial y} = 0$$

But ρ_v can be equal to zero, then LHS = RHS.

$\therefore$ The pair (b) satisfies second Maxwell's equation.

(iii) Now check the third Maxwell's equation

$$\nabla\times\bar{H} = \dot{\bar{D}}$$

$$\text{LHS} = \nabla\times\bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & 0 & 0 \end{vmatrix} = +\bar{a}_y\frac{\partial H_x}{\partial z} - \bar{a}_z\frac{\partial H_x}{\partial y}$$

$$= \bar{a}_y\frac{\partial H_x}{\partial z} \quad \text{... (Since } H_x \text{ is not a function of y)}$$

$$= +0.1328\cos(6\times 10^7 t)\sin z\,\bar{a}_y$$

Now,

$$\text{RHS} = \dot{\bar{D}} = \frac{\partial\bar{D}}{\partial t} = \epsilon\frac{\partial\bar{E}}{\partial t} = 2.5\epsilon_0\frac{\partial\bar{E}}{\partial t}$$

$$= 2.5\times 0.854\times 10^{-12}\times 100\times 6\times 10^7\times\cos(6\times 10^7 t)\sin z\,\bar{a}_y$$

$$= 0.1328\cos(6\times 10^7 t)\sin z\,\bar{a}_y = \text{LHS}$$

Thus, second pair satisfies third Maxwell's equation.

(iv) Now check whether the fourth Maxwell's equation is satisfied. The fourth Maxwell's equation is

$$\nabla\times\bar{E} = -\dot{\bar{B}}$$

$$\text{LHS} = \nabla\times\bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix}$$

$$= -\bar{a}_x\frac{\partial E_y}{\partial z} + \bar{a}_z\frac{\partial E_y}{\partial x} = -\bar{a}_x\frac{\partial E_y}{\partial z}$$

...(Since E_y is not a function of x)

$$= -100\sin(6\times 10^7 t)\cos z\,\bar{a}_x$$

$$\text{RHS} = -\dot{\bar{B}} = -\frac{\partial\bar{B}}{\partial t} = -10\mu_0\frac{\partial\bar{H}}{\partial t}$$

$$= (-10)\times 4\pi\times 10^{-7}\times(+0.1328)\times 6\times 10^7\sin(6\times 10^7 t)\cos z\,\bar{a}_x$$

$$= -100\sin(6\times 10^7 t)\cos z\,\bar{a}_x = \text{LHS}$$

i.e. fourth Maxwell's equation is also satisfied. Thus the second pair satisfies all Maxwell's equations.

(c) Solve it yourself. The answer is - third pair satisfies Maxwell's equations.

Ex. 7.15.2 : What values of A and β are required of two fields

$\bar{E} = 120\pi \cos(10^6 \pi t - \beta x)\, \bar{a}_y$ (V/m) and

$\bar{H} = A \cos(10^6 \pi t - \beta x)\, \bar{a}_z$ (A/m)

Satisfy Maxwell's in a medium where $\epsilon_r = 4$ and $\sigma = 0$

☑ Soln. :

Given : $\bar{E} = 120\pi \cos(10^6 \pi t - \beta x)\, \bar{a}_y = E_y\, \bar{a}_y$

$$\bar{H} = A \cos(10^6 \pi t - \beta x)\, \bar{a}_z = H_z\, \bar{a}_z$$

We know $\nabla \times \bar{H} = \dot{\bar{D}} = \epsilon \dot{\bar{E}}$ $(J = \sigma E = 0, \because \sigma = 0)$ ...(A)

and $\nabla \times \bar{E} = -\dot{\bar{B}} = -\mu \dot{\bar{H}}$ ...(B)

From Equation (A),

$$\text{LHS} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & H_z \end{vmatrix} = \bar{a}_x \frac{\partial H_z}{\partial y} - \bar{a}_y \frac{\partial H_z}{\partial x}$$

$$= -\bar{a}_y \frac{\partial H_z}{\partial x} \quad ...(\because H_z \text{ is not a function of } y)$$

$$= -\bar{a}_y \{A\beta \sin(10^6 \pi t - \beta x)\} = \text{RHS} = \epsilon \dot{\bar{E}}$$

$$\therefore \quad \dot{\bar{E}} = -\frac{A\beta}{\epsilon} \sin(10^6 \pi t - \beta x)\, \bar{a}_y \quad ...(i)$$

Differentiating given $\bar{E}$,

$$\dot{\bar{E}} = -120\pi \times 10^6 \pi \sin(10^6 \pi t - \beta x)\, \bar{a}_y \quad ...(ii)$$

Comparing Equations (i) and (ii) we get,

$$\frac{A\beta}{\epsilon} = 120\pi \times 10^6 \pi$$

i.e. $$A\beta = 120\pi \times 10^6 \pi \times \epsilon_0 \epsilon_r$$

$$= 0.0419 \quad ...(C)$$

Using Equation (B),

$$\text{LHS} = \nabla \times \bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -\bar{a}_x \frac{\partial E_y}{\partial z} + \bar{a}_z \frac{\partial E_y}{\partial x}$$

$$= +\bar{a}_z \frac{\partial E_y}{\partial x} \quad ...(\because E_y \text{ is not a function of } z)$$

$$= \bar{a}_z \{120\,\pi\beta \sin(10^6 \pi t - \beta x)\} = \text{RHS}$$

$$= -\mu \dot{\bar{H}}$$

$$\therefore \quad \dot{\bar{H}} = -\frac{1}{\mu} \times 120\,\pi\beta \sin(10^6 \pi t - \beta x)\, \bar{a}_z \quad ...(iii)$$

Differentiating given $\bar{H}$,

$$\dot{\bar{H}} = -A \times 10^6 \pi \times \sin(10^6 \pi t - \beta x)\, \bar{a}_z \quad ...(iv)$$

Comparing Equations (iii) and (iv) we get,

$$\frac{1}{\mu} 120\,\pi\beta = A \times 10^6 \pi$$

i.e. $$\frac{A}{\beta} = 23.87$$

Solving Equations (C) and (D) we get

$$A = 1 \quad \text{and}$$

$$\beta = 0.0419 \quad ...\text{Ans.}$$

Ex. 7.15.3 : The electric field vector in free space is given by

$\bar{E} = E_m \sin\alpha \cos(\omega t - \beta z)\, \bar{a}_y$ (V/m)

(i) Find corresponding $\bar{H}$;

(ii) Under what condition do these field satisfy all of Maxwell's equations ?

☑ Soln. :

(i) To find $\bar{H}$

Given : $\bar{E} = E_m \sin\alpha \cos(\omega t - \beta z)\, \bar{a}_y = E_y\, \bar{a}_y$

We have $\nabla \times \bar{E} = -\dot{\bar{B}}$

$$\text{LHS} = \nabla \times \bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -\bar{a}_x \frac{\partial E_y}{\partial z}$$

... (Since E_y is not a function of x)

$$= -\bar{a}_x E_m \sin\alpha\, (\beta) \sin(\omega t - \beta z)$$

i.e. $$\dot{\bar{B}} = \bar{a}_x (\beta E_m) \sin\alpha \sin(\omega t - \beta z) \quad ...\text{Ans.}$$

Integrating $\dot{\bar{B}}$ with respect to t,

$$\bar{B} = \int \dot{\bar{B}}\, dt = \bar{a}_x \int (\beta E_m) \sin\alpha \sin(\omega t - \beta z) dt$$

$$= -\bar{a}_x \frac{\beta E_m}{\omega} \sin\alpha \cos(\omega t - \beta z)$$

and since $$\bar{B} = \mu_0 \bar{H}$$

$$\bar{H} = \frac{\bar{B}}{\mu_0} = -\bar{a}_x \left(\frac{\beta E_m}{\omega \mu_0}\right) \sin\alpha \cos(\omega t - \beta z) \text{ (A/m)}$$

$$= -H_x\, \bar{a}_x \quad ...\text{Ans.}$$

(ii) To find condition : For free space we have $\nabla \times \bar{H} = \dot{\bar{D}}$

$$\text{LHS} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -H_x & 0 & 0 \end{vmatrix}$$

$$= -\bar{a}_y \frac{\partial}{\partial z} (+H_x)$$

$$\nabla \times \bar{H} = -\frac{\beta E_m}{\omega \mu_0} \sin \alpha(+\beta) \sin(\omega t - \beta z)\, \bar{a}_y$$

$$= -\frac{\beta^2 E_m}{\omega \mu_0} \sin \alpha \sin(\omega t - \beta z)\, \bar{a}_y$$

$$= \text{RHS} = \dot{\bar{D}} \qquad ...(i)$$

From given $\bar{E}$,

$$\bar{D} = \epsilon_0 \bar{E}$$

or $\dot{\bar{D}} = \epsilon_0 \dot{\bar{E}} = -\epsilon_0\, \omega . E_m \sin \alpha \sin(\omega t - \beta z)\, \bar{a}_y$ ...(ii)

Comparing Equations (i) and (ii) $-\beta^2/(\omega\mu_0) = -\omega\epsilon_0$

or $\beta = \omega\sqrt{\mu_0 \epsilon_0}$ ...Ans.

This is one condition and secondly it is obvious that $\sin \alpha$ should not be equal to zero since it will make the fields zero.

Ex. 7.15.4 : A certain material has $\sigma = 0$ and $\epsilon_r = 1$.

If $H = 4 \sin(10^6 t - 0.01z)\, \bar{a}_y$ (A/m). Make use of Maxwell's equation to find μ_r.

☑ **Soln. :**

Given : $H = 4 \sin(10^6 t - 0.01z)\bar{a}_y = H_y\, \bar{a}_y$

We have $\nabla \times \bar{H} = \dot{\bar{D}}$ ... (Since $\bar{J} = \sigma \bar{E} = 0$)

$$\text{LHS} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & H_y & 0 \end{vmatrix}$$

$$= -\bar{a}_x\left(\frac{\partial H_y}{\partial z}\right) + \bar{a}_z\left(\frac{\partial H_y}{\partial x}\right)$$

$$= -\frac{\partial H_y}{\partial z}\, \bar{a}_x \qquad \text{... (Since } H_y \text{ is not a function of y)}$$

$$= -4\cos(10^6 t - 0.01z)(-0.01)\, \bar{a}_x = \text{RHS} = \dot{\bar{D}}$$

$$\therefore \quad \dot{\bar{D}} = 0.04 \cos(10^6 t - 0.01z)\, \bar{a}_x$$

$$\therefore \quad \bar{D} = \int \dot{\bar{D}}\, dt = \frac{0.04 \sin(10^6 t - 0.01z)}{10^6}\, \bar{a}_x$$

$$= 0.04 \times 10^{-6} \sin(10^6 t - 0.01z)\, \bar{a}_x$$

and $\bar{E} = \dfrac{0.04 \times 10^{-6}}{\epsilon_0} \sin(10^6 t - 0.01z)\, \bar{a}_x = E_x\, \bar{a}_x$

The other Maxwell equation is, $\nabla \times \bar{E} = -\dot{\bar{B}}$

$$\text{LHS} = \nabla \times \bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \bar{a}_y \frac{\partial E_x}{\partial z} - \bar{a}_z \frac{\partial E_x}{\partial x}$$

$$= +\bar{a}_y \frac{\partial E_x}{\partial z} \qquad \text{... (Since } E_x \text{ is not a function of y)}$$

$$\text{LHS} = \bar{a}_y\left[\frac{0.04 \times 10^{-6} \times 0.01}{\epsilon_0} \cos(10^6 t - 0.01z)\right]$$

$$\text{LHS} = -45.24 \cos(10^6 t - 0.01z)\, \bar{a}_y = \text{RHS} = -\dot{\bar{B}}$$

$$\therefore \quad \dot{\bar{B}} = 45.24 \cos(10^6 t - 0.01z)\, \bar{a}_y$$

$$\therefore \quad \bar{B} = \int \dot{\bar{B}}\, dt = \frac{45.24 \sin(10^6 t - 0.01z)}{10^6}\, \bar{a}_y \qquad ...(A)$$

But $\bar{B} = \mu \bar{H}$

$$\therefore \quad \bar{H} = \frac{\bar{B}}{\mu_0 \mu_r} = \frac{45.24 \sin(10^6 t - 0.01z)}{10^6 \times 4\pi \times 10^{-7} \times \mu_r}\, \bar{a}_y$$

$$= \frac{36}{\mu_r} \sin(10^6 t - 0.01z)\, \bar{a}_y$$

Comparing calculated H with given H we get

$$\therefore \quad \mu_r = \frac{36}{4} = 9 \qquad ...\text{Ans.}$$

Ex. 7.15.5 : Let $\mu = 10^{-5}$ H/m, $\epsilon = 4 \times 10^{-9}$ F / m, $\sigma = 0$, $\rho_V = 0$. Find K (including units) so that each of the following pairs of field satisfies Maxwell's equations :

(i) $\bar{D} = 6\bar{a}_x - 2y\, \bar{a}_y + 2z\, \bar{a}_z$ nC/m^2

$\bar{H} = Kx\, \bar{a}_x - 10\, y\, \bar{a}_y - 25z\, \bar{a}_z$ A/m

(ii) $\bar{E} = (20y - Kt)\, \bar{a}_x$ V/m

$\bar{H} = (y + 2 \times 10^6\, t)\, \bar{a}_z$ V/m.

☑ **Soln. :**

(i) We have,

$$\nabla \cdot \bar{B} = 0 \qquad \text{i.e.} \quad \nabla \cdot \bar{H} = 0$$

Now $\nabla \cdot \bar{H} = \dfrac{\partial H_x}{\partial x} + \dfrac{\partial H_y}{\partial y} + \dfrac{\partial H_z}{\partial z}$

$$= \frac{\partial}{\partial x}(Kx) + \frac{\partial}{\partial y}(-10\, y) + \frac{\partial}{\partial z}(-25\, z)$$

$$= K - 10 - 25 = 0$$

$$\therefore \quad K = 35\ (\text{A/m}^2)$$

Since K is present in the expression of H, Kx should have a unit of A/m or K should be expressed as (A/m^2).

(ii) Given E is E_x component which is a function of y while H is H_z component which is a function of y.

Using Maxwell's equation,

$$\nabla \times \bar{H} = \bar{J} + \dot{\bar{D}}$$

But given $\sigma = 0$ the $\bar{J} = \sigma \bar{E} = 0$

$$\therefore \quad \nabla \times \bar{H} = \dot{\bar{D}}$$

i.e. $$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & H_z \end{vmatrix} = \epsilon \frac{\partial}{\partial t} \bar{E}$$

$$\bar{a}_x \frac{\partial H_z}{\partial y} = \epsilon \frac{\partial}{\partial t} (20y - Kt)\, \bar{a}_x$$

$$\frac{\partial}{\partial y} (y + 2 \times 10^6 t) = \epsilon \frac{\partial}{\partial t} (20y - Kt)$$

$$1 = \epsilon (-K)$$

or $$K = -\frac{1}{\epsilon} = -\frac{1}{4 \times 10^{-9}}$$

$$= -0.25 \times 10^9 \text{ (V/m·s)}$$

Since Kt is present in the expression of E, K should be (V/m·s).

7.16 TYPE 3 PROBLEMS : BASED ON FARADAY'S LAW

Following examples are based on faraday's law. If the magnetic flux density B is given, we can find current in the loop with resistance R by following procedure.

$$\bar{B} \xrightarrow{\Psi_m = \int_s \bar{B} \cdot d\bar{s}} \phi \xrightarrow{\text{Emf} = -\frac{d\Psi_m}{dt}} \text{emf} \xrightarrow{i = \frac{\text{emf}}{R}} \text{current}$$

UEx. 7.16.1

(MU - May 11, 10 Marks, Q. 1(b), Dec. 19, 5 Marks

The circular loop conductor having a radius of 0.15 m is placed in the x-y plane. This loop consists of a resistance of 20 Ω, if the magnetic flux density is $\bar{B} = 0.5 \sin (10^3 t)\, \bar{a}_z$ (Tesla), Find the current flowing through this loop.

☑ **Soln. :**

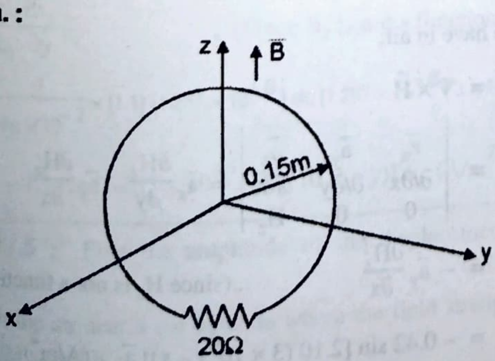


Fig. : Illustrating Ex. 7.16.1

For the given surface in x - y plane

$$d\bar{s} = r\, dr\, d\phi\, \bar{a}_z$$

The magnetic flux is,

$$\Psi_m = \int_s \bar{B} \cdot d\bar{s} = \int_0^{2\pi} \int_0^{0.15} 0.5 \sin (10^3 t)\, \bar{a}_z \cdot r\, dr\, d\phi\, \bar{a}_z$$

$$\Psi_m = 0.5 \sin (10^3 t) \left(\frac{r^2}{2}\right)_0^{0.15} (\phi)_0^{2\pi}$$

$$= 35.34 \times 10^{-3} \sin (10^3 t) \text{ (Wb)}$$

The emf induced in the loop is,

$$\text{emf} = \frac{-d\Psi_m}{dt} = -35.34 \times 10^{-3} \times 10^3 \cos (10^3 t)$$

$$= -35.34 \cos (10^3 t) \quad \text{(Volts)}$$

The current flowing in the loop is,

$$i = \frac{\text{emf}}{R} = \frac{-35.34 \cos (10^3 t)}{20}$$

$$i = -1.767 \cos (10^3 t) \text{ (Amp)}$$ ...Ans.

Ex. 7.16.2 : An area of 0.65 m^2 in the z = 0 plane encloses a filamentary conductor. Find the induced voltage if

$$\bar{B} = 0.05 \cos 10^3 t \left(\frac{\bar{a}_y + \bar{a}_z}{\sqrt{2}}\right) \text{ (T)}$$

☑ **Soln. :** For the given z = 0 surface $d\bar{s} = ds\, \bar{a}_z$

The magnetic flux from given $\bar{B}$ is,

$$\Psi_m = \int_s \bar{B} \cdot d\bar{s} = \int_s 0.05 \cos 10^3 t \left(\frac{\bar{a}_y + \bar{a}_z}{\sqrt{2}}\right) \cdot ds\, \bar{a}_z$$

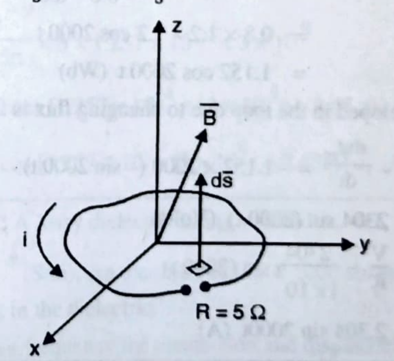


Fig. : Illustrating Ex. 7.16.2

$$\Psi_m = \int_s 0.05 \cos 10^3 t \left(\frac{ds}{\sqrt{2}}\right)$$

$$\Psi_m = \left(\frac{0.05}{\sqrt{2}}\right) \cos 10^3 t \int_s ds = \frac{0.05}{\sqrt{2}} \cos 10^3 t\, [\, s\,]$$

Since area given = 0.65 m^2 = s

$$\therefore \quad \Psi_m = -\frac{0.05 \times 0.65}{\sqrt{2}} \cos 10^3 t = 0.0229 \cos 10^3 t \text{ (Wb)}$$

The voltage induced in the loop is obtained by,

$$V = -\frac{d\Psi_m}{dt} = +0.0229 \times 10^3 \sin 10^3 t$$

$$V = 23 \sin 10^3 t \text{ (Volts)}$$ ...Ans.

Here B is decreasing in the First half cycle of the cosine function so that I must flow in the direction shown.

Ex. 7.16.3 : The sides of a square loop in the z = 0 plane are located at x = ± 0.6 m and y = ± 0.6 m. There exists a uniform time varying magnetic field given by

$$\bar{B} = (0.2\,\bar{a}_x - 0.4\,\bar{a}_y + 0.8\,\bar{a}_z) \cos(2000\,t)\ (\text{Wb/m}^2)$$

If the total resistance of the square loop is 1 kΩ. Find the current though the loop.

☑ **Soln. :**

From the given $\bar{B}$ we can find flux as

$$\psi_m = \int \bar{B} \cdot d\bar{s}$$

for the surface in z= 0, $d\bar{s} = dxdy\,\bar{a}_z$

$$\bar{B} \cdot d\bar{s} = 0.8\,dx\,dy\,\cos(2000\,t)$$

$$\therefore \quad \psi_m = \int_{-0.6}^{0.6}\int_{-0.6}^{0.6} 0.8\,dxdy\cos 2000\,t$$

$$= 0.8\cos 2000\,t \int_{-0.6}^{0.6}\int_{-0.6}^{0.6} dxdy$$

$$= 0.8 \times 1.2 \times 1.2 \cos 2000\,t$$

$$= 1.152 \cos 2000\,t\ (\text{Wb})$$

The emf developed in the loop due to changing flux is :

$$V = -\frac{d\psi_m}{dt} = -1.152 \times 2000\,(-\sin 2000\,t)$$

$$= 2304 \sin(2000\,t)\ (\text{Volts})$$

$$\therefore \quad i = \frac{V}{R} = \frac{2304}{1 \times 10^3} \sin(2000\,t)$$

$$= 2.304 \sin 2000t\ (\text{A}) \quad \text{...Ans.}$$

Ex. 7.16.4 : Determine the e.m.f. induced about the path r = 0.5, z = 0 and t = 0, if B = 0.01 sin 377t T.

☑ **Soln. :**

Given : $r = 0.5, z = 0,\ t = 0$

$B = 0.01 \sin 377\,t\ T$

$$\psi_m = \text{Magnetic flux}$$

$$\psi_m = B \times \text{Area}$$

$$= 0.01 \sin 377\,t \times \pi\,(0.5)^2$$

$$= 7.853 \times 10^{-3} \sin 377\,t\ (\text{Wb})$$

Induced e.m.f. according Lenz's law

$$V_{emf} = -\frac{d\psi_m}{dt}$$

$$= -\frac{d}{dt}\,[7.853 \times 10^{-3} \sin 377\,t]$$

$$V_{emf} = -7.853 \times 10^{-3}\,[377 \cos 377\,t]$$

At t = 0

$$V_{emf\,(t=0)} = -7.853 \times 10^{-3} \times 377 \times (1) \quad \text{...cos 0 = 1}$$

$$V_{emf} = -2.9609\ \text{V}$$

7.17 TYPE 4 PROBLEMS : BASED ON DISPLACEMENT CURRENT DENSITY

Following examples are based on calculation of displacement current density J_d. These calculations require following relations :

$$\bar{J}_d = \dot{\bar{D}} = \epsilon\,\dot{\bar{E}} = \epsilon\left(\frac{\dot{J}}{\sigma}\right)$$

For free space or dielectric,

$$\bar{J}_d = \nabla \times \bar{H} = \frac{1}{\mu}(\nabla \times \bar{B})$$

Ex. 7.17.1 : Find displacement current density for the field $\bar{E} = 300 \sin 10^9\,t$ V/m.

☑ **Soln. :**

The displacement current density,

$$\bar{J}_d = \dot{\bar{D}} = \epsilon\dot{\bar{E}} = \epsilon_0\,(10^9)\,(300)\cos 10^9\,t$$

$$= (8.854 \times 10^{-12})\,(10^9)\,(300)\,(\cos 10^9\,t)$$

$$\bar{J}_d = 2.6562 \cos 10^9\,t\ \text{A/m}^2$$

Ex. 7.17.2 : Find the displacement current density next to your radio, in air, where the local FM station provides a carrier having

$$\bar{H} = 0.2 \cos[2.10\,(3 \times 10^8\,t - x)]\,\bar{a}_z \quad (\text{A/m})$$

☑ **Soln. :**

We have in air,

$$\bar{J}_d = \nabla \times \bar{H}$$

$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & H_z \end{vmatrix} = \bar{a}_x \frac{\partial H_z}{\partial y} - \bar{a}_y \frac{\partial H_z}{\partial x}$$

$$= -\bar{a}_y \frac{\partial H_z}{\partial x} \quad \text{...(since } H_z \text{ is not a function of y)}$$

$$\bar{J}_d = -0.42 \sin[2.10\,(3 \times 10^8\,t - x)]\,\bar{a}_y \quad (\text{A/m}^2) \quad \text{...Ans.}$$

Ex. 7.17.3 : The magnetic field intensity in free space is given as $\bar{H} = H_0 \sin\theta\ \bar{a}_y$ A/m, where $\theta = (\omega t - \beta z)$ and β is constant. Determine the current density vector $\bar{J}$.

☑ **Soln. :**

From given $\quad \bar{H} = H_0 \sin(\omega t - \beta z)\,\bar{a}_y$

Considering the given $\bar{H}$ as,

$$\bar{H} = H_0 \sin(\omega t - \beta z)\,\bar{a}_y = H_y\,\bar{a}_y$$

The given H_y is the function of only z and t, so all differentiations.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

Also, $H_x = H_z = 0$

We have,

$$\bar{J}_d = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix}$$

$$= -\bar{a}_x \frac{\partial H_y}{\partial z}$$

$$\bar{J}_d = \beta H_0 \cos(\omega t - \beta z)\,\bar{a}_x \ \text{A/m}^2$$

Ex. 7.17.4 : Find $\bar{J}_d$ in air space within a large power distribution transformer where

$$\bar{B} = 1.1 \cos[1.257 \times 10^{-6} (3 \times 10^8 t - y)]\ \bar{a}_x \quad (\text{Wb/m}^2)$$

☑ **Soln. :**

Given : $\bar{B} = 1.1 \cos[1.257 \times 10^{-6} (3 \times 10^8 t - y)]\ \bar{a}_x = B_x\ \bar{a}_x$

We have,

$$\bar{J}_d = \nabla \times \bar{H} = \frac{1}{\mu_0} \nabla \times \bar{B} = \frac{1}{\mu_0} \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{\mu_0}\left(\bar{a}_y \frac{\partial B_x}{\partial z} - \bar{a}_z \frac{\partial B_x}{\partial y}\right)$$

$$= -\frac{1}{\mu_0} \frac{\partial B_x}{\partial y} \bar{a}_z \quad \text{...(Since } B_x \text{ is not a function z.)}$$

$$= -\frac{1}{4\pi \times 10^{-7}} \times (1.1)(1.257 \times 10^{-6}) \sin[1.257 \times 10^{-6} (3 \times 10^8 t - y)]\ \bar{a}_z$$

$$\bar{J}_d = -1.1 \sin[1.257 \times 10^{-6} (3 \times 10^8 t - y)]\ \bar{a}_z \ (\text{A/m}^2) \ \text{...Ans.}$$

Ex. 7.17.5 : Find the amplitude of the displacement current density :

(i) in the air near a car antenna where the field strength of the FM signal is

$$\bar{E} = 80 \cos(6.277 \times 10^8 t - 2.092\ y)\ \bar{a}_z \quad (\text{V/m})$$

(ii) inside a capacitor where $\epsilon_r = 600$ and

$$\bar{D} = 3 \times 10^{-6} \sin(6 \times 10^6 t - 0.3468\ x)\ \bar{a}_z \quad (\text{C/m}^2)$$

☑ **Soln. :**

(a) The field strength is given as

$$\bar{E} = 80 \cos(6.277 \times 10^8 t - 2.092\ y)\ \bar{a}_z \quad (\text{V/m}) \ \text{...Ans.}$$

The displacement current density,

$$\bar{J}_d = \dot{\bar{D}} = \epsilon \dot{\bar{E}}$$

$$= \epsilon_0 \times 80 \times (-6.277 \times 10^8) \sin(6.277 \times 10^8 t - 2.092\ y)\ \bar{a}_z$$

$$= -0.44 \sin(6.277 \times 10^8 t - 2.092\ y)\ \bar{a}_z \ (\text{A/m}^2) \ \text{...Ans.}$$

(b) Inside the capacitor the flux density is given as

$$\bar{D} = 3 \times 10^{-6} \sin(6 \times 10^6 t - 0.3468\ x)\ \bar{a}_z \quad (\text{C/m}^2) \ \text{...Ans.}$$

The displacement current density is,

$$\bar{J}_d = \dot{\bar{D}} = 3 \times 10^{-6} \times 6 \times 10^6 \cos(6 \times 10^6 t - 0.3468\ x)\ \bar{a}_z$$

$$\bar{J}_d = 18 \cos(6 \times 10^6 t - 0.3468\ x)\ \bar{a}_z \ (\text{A/m}^2) \ \text{...Ans.}$$

Ex. 7.17.6 : Find $\bar{J}_d$ inside a large oil filled power capacitor where $\epsilon_r = 6$ and

$$\bar{E} = 100 \sin[1.257 \times 10^{-6} (3 \times 10^8 t - 2.45z)]\ \bar{a}_x \ \text{kV/m.}$$

☑ **Soln. :**

Given : $\bar{H} = \cos(2x)\cos(\omega t - \beta y)\ \bar{a}_x = H_x\ \bar{a}_x$

$$\bar{J}_d = \dot{\bar{D}} = \epsilon \dot{\bar{E}} = \epsilon_0 \epsilon_r \frac{\partial \bar{E}}{\partial t}$$

$$\bar{J}_d = \frac{10^{-9}}{36\pi} \times 6 \times 1.257 \times 10^{-6} \times 3 \times 10^8 \times 100 \cos[1.257 \times 10^{-6} \times (3 \times 10^8 t - 2.45\ z)]\ \bar{a}_x$$

$$\bar{J}_d = 2.0 \cos[1.257 \times 10^{-6} (3 \times 10^8 t - 2.45z)]\ \bar{a}_x \ (\text{A/m}^2) \text{...Ans.}$$

Ex. 7.17.7 : A lossy dielectric has $\mu_r = 1$ and $\epsilon_r = 1$, $\sigma = 2 \times 10^{-8}$ (S/m). An electric field $\bar{E} = 2000 \sin \omega t\ \bar{a}_z$ V/m at a certain point in the dielectric

(i) at what frequency the conduction and displacement current densities be equal ?

(ii) at this frequency calculate the instantaneous displacement current density.

☑ **Soln. :**

(i) To find frequency

We have the displacement current density,

$$\bar{J}_d = \dot{\bar{D}} = \epsilon \dot{\bar{E}} = j\omega \epsilon \bar{E}$$

and the conduction current density, $J_c = \sigma \bar{E}$

Two current densities are equal when $\sigma = \omega\epsilon$

$$\therefore \quad \omega = \frac{\sigma}{\epsilon} = \frac{2 \times 10^{-8}}{10^{-9} / 36\pi} = 2.26 \times 10^3 \ (\text{rad/sec.})$$

$$\therefore \quad f = \frac{\omega}{2\pi} = 360 \text{ Hz} \quad \text{...Ans.}$$

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(ii) The displacement current density

$$\bar{J}_d = \dot{\bar{D}} = \epsilon \frac{\partial \bar{E}}{\partial t} = (10^{-9}/36\pi) \times 2000 \cos (2.26 \times 10^3 t)$$

$$= 40 \cos (2.26 \times 10^3 t) \quad (\mu A/m^2) \qquad ...Ans.$$

Ex. 7.17.8 : Moist soil has a conductivity of 10^{-3} S/m and $\epsilon_r = 2.5$ find J_c and J_d where $E = 6.0 \times 10^{-6} \sin (9.0 \times 10^9 t)$ (V/m).

☑ **Soln. :**

We have the conduction current density

$$J_c = \sigma E = 10^{-3} \times 6.0 \times 10^{-6} \sin (9.0 \times 10^9 t)$$

$$= 6.0 \times 10^{-9} \sin (9.0 \times 10^9 t) \ (A/m^2) \qquad ...Ans.$$

The displacement current density is

$$\bar{J}_d = \dot{\bar{D}} = \epsilon \frac{\partial \bar{E}}{\partial t}$$

$$= \epsilon_0 \epsilon_r \times (9 \times 10^9) \times (6.0 \times 10^{-6} t) \cos (9.0 \times 10^9 t)$$

$$= 1.19 \times 10^{-6} \cos (9.0 \times 10^9 t) \quad (A/m^2) \qquad ...Ans.$$

Ex. 7.17.9 : If $V = V_m \sin \omega t$ is the voltage applied to a capacitor and i is the current flowing through it, then show that the displacement current through the capacitor is equal to the conduction current.

OR

Show that the displacement current in the dielectric of a parallel plate capacitor is equal to the conduction current in the leads.

☑ **Soln. :**

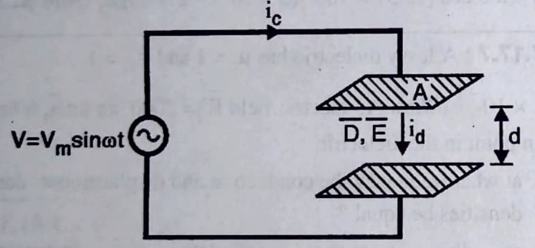

Fig. : Illustrating Ex. 7.17.9

Conduction and displacement currents are defined as :

Conduction current : This is the current flowing through external circuit i.e. wires attached to the plates of capacitor.

Displacement current : It is the current passing through dielectric between the plates.

We know that current flows through leads of the capacitor (i.e. i_c) only when voltage across it changes.

i.e. $$i_c = C \frac{dV}{dt} = C\omega V_m \cos \omega t$$

where, $$C = \frac{\epsilon A}{d}$$

We have to prove that this current is equal to current through dielectric given by,

$$i_d = \int_S \bar{J}_d \cdot d\bar{s} = \int_S \dot{\bar{D}} \cdot d\bar{s}$$

The flux density D between the plates is normal to the plates i.e. $\dot{\bar{D}}$ and $d\bar{s}$ are in same direction. Therefore, the dot product changes to multiplication giving,

$$i_d = \int_S \dot{D}\, ds = \int_S \frac{\partial D}{\partial t} ds = \int_S \frac{\partial (\epsilon E)}{\partial t} ds \qquad (\because D = \epsilon E)$$

$$= \epsilon \int_S \frac{\partial E}{\partial t} ds$$

The electric field is related with potential through E = V/d

$$\therefore \quad i_d = \epsilon \int_S \frac{\partial}{\partial t} \left(\frac{V}{d}\right) ds = \frac{\epsilon}{d} \int_S \frac{\partial V}{\partial t} ds$$

Since $\frac{\partial V}{\partial t}$ is independent of ds, it can be treated constant and thus

$$i_d = \frac{\epsilon}{d} \frac{\partial V}{\partial t} \int_S ds$$

The integral in the expression is nothing but area of the plates i.e. A. Then,

$$i_d = \frac{\epsilon}{d} A \frac{\partial V}{\partial t}$$

Since only V is chancing w. r. t. time

$$i_d = \frac{\epsilon A}{d} \frac{dV}{dt} = C \frac{dV}{dt} = C\omega V_m \cos \omega t$$

Which is same as conduction current i_c.

$$\therefore \quad i_c = i_d$$

UEx. 7.17.10 **MU - Q. 1(d), Dec. 18, 5 Marks**

For parallel plates capacitor with plate area 10 cm^2 and plates separation 3 mm has voltage of $100 \sin 10^3 t$ V applied to its plates. Calculate displacement current density ($\epsilon = 2\epsilon_0$)

☑ **Soln. :**

Given :

A = Plate area = $10 \text{ cm}^2 = (10 \times 10^{-2})^2 \ (m^2)$

d = separation = $3 \text{ mm} = 3 \times 10^{-3}$ (m)

V = $100 \sin 10^3 t$ (volts)

and $\epsilon = 2\epsilon_0$

The displacement current density is

$$J_d = \frac{i_d}{A}$$

But $$i_d = i_c = C \frac{dV}{dt}$$

$$\therefore \; J_d = \frac{1}{A}\left(C\frac{dV}{dt}\right) = \frac{1}{A}\left(\frac{\in A}{d}\frac{dV}{dt}\right) = \frac{\in}{d}\frac{dV}{dt} = \frac{2\in_0}{d}\frac{dV}{dt}$$

$$= \frac{2\times(10^{-9}/36\pi)}{3\times10^{-3}} \times 100\times10^3 \cos(10^3 t)$$

$$\therefore \; J_d = 5.89\times10^{-4} \cos(10^3 t)\ (A/m^2)$$

Ex. 7.17.11 : Find the displacement current within a parallel plate capacitor where $\in = 100\in_0$, $A = 0.01\ m^2$, $d = 0.05$ mm and the capacitor voltage is 100 sin (2000 πt) volts.

☑ **Soln. :** The displacement current through capacitor is

$$I_d = \int \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s} = \frac{\partial D}{\partial t}\int ds = \frac{\partial D}{\partial t} s \quad ...(A)$$

If the applied voltage is V, the electric field intensity E is given by

$$E = V/d$$

but $$D = \in E = \in\left(\frac{V}{d}\right)$$

$$\therefore \; \dot{D} = \partial D/\partial t = \frac{\in}{d}\frac{\partial V}{\partial t} = \frac{\in}{d} \times 100 \times 2000\,\pi \times \cos(2000\,\pi t)$$

$$= \frac{100\times(10^{-9}/36\,\pi)}{0.05\times10^{-3}} \times 100\times2000\,\pi\times\cos(2000\,\pi t)$$

$$= 11.11 \cos(2000\,\pi t)$$

Putting this value in (A),

$$I_d = \left(\frac{\partial D}{\partial t}\right) s = 11.11\times0.01\times\cos(2000\,\pi t)$$

$$= 0.11 \cos 2000\,\pi t \text{ (Amp)} \quad ...\text{Ans.}$$

Ex. 7.17.12 : (i) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\frac{\sigma}{\omega\in}$ for the applied field $E = E_m \cos \omega t$.

Assume $\mu = \mu_0$.

(ii) What is the amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$, where τ is real.

☑ **Soln. :**

(i) The conduction current density is given by

$$J_c = \sigma E = \sigma E_m \cos \omega t$$

The displacement current density is,

$$J_d = \dot{D} = \in \frac{\partial}{\partial t} E = -\omega\in E_m \sin \omega t$$

$$\therefore \; \frac{|J_c|}{|J_d|} = \frac{\sigma E_m}{\omega\in E_m} = \frac{\sigma}{\omega\in} \quad ...\text{Ans.}$$

(ii) For the given electric field

$$J_c = \sigma E = \sigma E_m e^{-t/\tau}$$

and $$J_d = \dot{D} = \in\dot{E} = \frac{\in E_m}{-\tau} e^{-t/\tau}$$

$$\therefore \; \frac{|J_c|}{|J_d|} = \frac{\sigma E_m}{\in E_m/\tau} = \frac{\sigma\tau}{\in} \quad ...\text{Ans.}$$

Ex. 7.17.13 : In a lossy dielectric medium, conduction current density $J_c = 0.02 \sin 10^9 t\ (A/m^2)$. Find the displacement current density if $\sigma = 10^3$ S/m and $\in_r = 6.5$.

☑ **Soln. :**

For lossy dielectric medium

$$\frac{|\bar{J}_c|}{|\bar{J}_d|} = \frac{\sigma}{\omega\in}$$

$$\therefore \; \bar{J}_d = \frac{\omega\in J_c}{\sigma} = \frac{10^9\times6.5\times8.854\times10^{-12}\times0.02}{10^3}$$

$$\therefore \; \bar{J}_d = 1.151\times10^{-6}\ A/m^2 = 1.151\ \mu A/m^2$$

as $\bar{J}_d$ and $\bar{J}_c$ are perpendicular to each other in time domain

$$\therefore \; \bar{J}_d = 1.151 \cos 10^9 t\ \mu A/m^2$$

Ex. 7.17.14 : A circular cross-section conductor of radius 2 mm carries a current $i_c = 2.5 \sin(5\times10^8 t)\ \mu A$. What is the amplitude of the displacement current density if $\sigma = 35$ Ms/m and $\in_r = 1$.

☑ **Soln. :**

Comparing given i_c with sinusoidal current expression

$$i_c = 2.5 \sin(5\times10^8 t)\ \mu A = I_0 \sin \omega t$$

We get $$\omega = 5\times10^8$$

We have $$\frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\in} = \frac{35\times10^6}{5\times10^8\times\left(\frac{10^{-9}}{36\,\pi}\right)}$$

$$= 791.28\times10^7$$

Now the conduction current density is obtained as,

$$|J_c| = \frac{i_c}{\pi r^2} = \frac{2.5\times10^{-6}}{\pi(2\times10^{-3})^2}$$

$$= 0.199 \text{ Amp. and}$$

$$|J_d| = \frac{|J_c|}{791.28\times10^7}$$

$$= \frac{0.199}{791.28\times10^7}$$

$$|J_d| = 25.20\ pA/m^2$$

Ex. 7.17.15 : $\bar{H} = H_x(\omega t - \beta z)\,\hat{a}_x$ exists within a dielectric of permittivity $\in$. Estimate the corresponding displacement current density and then find the charge density, electric field corresponding to $\bar{H}$ field.

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☑ **Soln. :**

Considering the given $\bar{H}$ as,

$$\bar{H} = H_{x0} \cos(\omega t - \beta z) \qquad \bar{a}_x = H_x \bar{a}_x$$

The given H_x is a function of only z and t, so all differentiations

$$\partial/\partial x = \partial/\partial y = 0 .$$

Also $H_y = H_z = 0$

To find displacement current density,

$$\bar{J}_d = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & \partial/\partial z \\ H_x & 0 & 0 \end{vmatrix} = \bar{a}_y \frac{\partial H_x}{\partial z}$$

$$= +\beta H_{x0} \sin(\omega t - \beta z)\, \bar{a}_y$$

But the displacement current density is also given by,

$$\bar{J}_d = \dot{\bar{D}}$$

$$\therefore \quad \dot{\bar{D}} = \beta H_{x0} \sin(\omega t - \beta z)\, \bar{a}_y$$

$$\therefore \quad \bar{D} = \int \dot{\bar{D}}\, dt = \frac{-\beta H_{x0} \cos(\omega t - \beta z)}{\omega}\, \bar{a}_y \ (C/m^2)$$

The electric field is given by

$$\bar{D} = \in \bar{E}$$

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\in} = \frac{-\beta H_{x0}}{\omega \in} \cos(\omega t - \beta z)\, \bar{a}_y \qquad (V/m)$$

To find charge density,

$$\rho_v = \nabla \cdot \bar{D} = \frac{\partial D_y}{\partial y} \qquad (\because\ D_x = D_z = 0)$$

$$\therefore \quad \rho_v = 0 \qquad (\because\ D_y \text{ is not a function of y})$$

7.18 BOUNDARY CONDITIONS USING MAXWELL'S EQUATIONS

- In a single continuous media the electric or magnetic fields do not change the magnitude or direction of it. For example in a single continuous media some electric field is given as,

$$\bar{E} = \bar{a}_x + 2\bar{a}_y + 3\bar{a}_z \ (V/m)$$

Then magnitude of it is, $|\bar{E}| = \sqrt{1^2 + 2^2 + 3^2}$ and

Direction is given by unit vector

$$\bar{a}_E = \frac{\bar{a}_x + 2\bar{a}_y + 3\bar{a}_z}{\sqrt{1^2 + 2^2 + 3^2}}$$

- In a given media, the magnitude and direction remain same.
- But when this field will go from one media to second media, the direction and magnitude of it will change. This phenomena is similar to light when goes from one media to second media it changes its path.

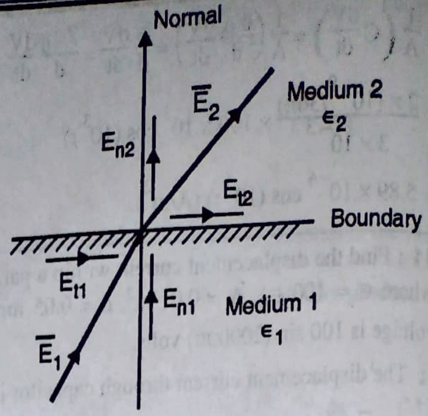

Fig. 7.18.1 : Boundary relations

- Let electric field in medium 1 is $\bar{E}_1$ while in medium 2 it is $\bar{E}_2$. Consider $\bar{E}_1$ is known and we have to determine $\bar{E}_2$. Any electric field can be resolved into two parts,
  (i) Tangential to the boundary i.e. $\bar{E}_t$ and
  (ii) Normal to the boundary i.e. $\bar{E}_n$.
- From the given $\bar{E}$ we can find $\bar{E}_t$ and $\bar{E}_n$ or reverse i.e. from $\bar{E}_t$ and $\bar{E}_n$ we can find $\bar{E}$. In the figure given $\bar{E}_1$ is resolved into $\bar{E}_{t1}$ and $\bar{E}_{n1}$.
- If we have some relation by which knowing $\bar{E}_{t1}$ we can determine $\bar{E}_{t2}$ and knowing $\bar{E}_{n1}$ we can determine $\bar{E}_{n2}$, then $\bar{E}_2$ can be obtained from E_{t2} and E_{n2}. The relation between tangential components i.e. $\bar{E}_{t1}$ and $\bar{E}_{t2}$ or relation between normal components $\bar{E}_{n1}$ and $\bar{E}_{n2}$ are called as **boundary relations** or boundary conditions. Boundary conditions are obtained separately for electric and magnetic fields.
- Maxwell's equations in the point form cannot be used but integral forms can always be used to determine what happens at the boundary surface between different media.
- First we will derive boundary conditions for tangential electric components and normal components field and then for magnetic field.

7.18.1 Boundary Conditions for Electric Field

- Let us consider two semi infinite dielectric media as shown in Fig. 7.18.2.
- Suppose y = 0 forms the boundary of medium 1 and 2 and these media have constants $\in_1$ and $\in_2$ respectively (σ_1 and σ_2 equal to zero).

- First we shall determine the boundary conditions for tangential components.

☞ Tangential components

- Consider a rectangular path of length Δx and width Δy and placed half in each medium. Let the tangential components to the boundary be E_{t1} and E_{t2}, in medium 1 and 2 respectively.

We know that

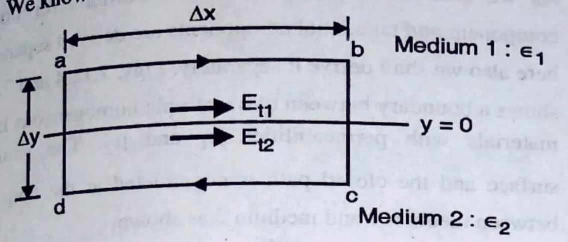

Fig. 7.18.2 : Tangential component

$$\oint \bar{E} \cdot d\bar{l} = -\int_s \dot{\bar{B}} \cdot d\bar{s}$$

- Applying this equation to rectangular path shown in figure,

$$\text{RHS} = -\int_s \dot{\bar{B}} \cdot d\bar{s} = -\int_s \frac{\partial}{\partial t}(B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z) \cdot dxdy\,\bar{a}_z$$

$$= -\int_0^{\Delta y}\int_0^{\Delta x} \frac{\partial B_z}{\partial t}\,dxdy = -\dot{B}_z\,\Delta x\,\Delta y$$

- Here B_z is the average magnetic flux density through the rectangle $\Delta x\,\Delta y$. The LHS can be split into four integrals as shown.

$$\text{LHS} = \oint \bar{E} \cdot d\bar{l}$$

$$= \int_a^b \bar{E} \cdot d\bar{l} + \int_b^c \bar{E} \cdot d\bar{l} + \int_c^d \bar{E} \cdot d\bar{l} + \int_d^a \bar{E} \cdot d\bar{l}$$

- In order to find relation between fields, we must go close to the boundary i.e. let $\Delta y \to 0$. Its effect is, the RHS reduces to zero and two integrals on LHS will reduce to zero. Thus we have

$$\text{RHS} = 0$$

$$\text{and LHS} = \int_a^b \bar{E} \cdot d\bar{l} + \int_c^d \bar{E} \cdot d\bar{l}$$

- We will solve each integral separately.

$$\int_a^b \bar{E} \cdot d\bar{l} = \int_0^{\Delta x} E_{t1}\,\bar{a}_x \cdot dx\,\bar{a}_x = E_{t1}\,\Delta x$$

$$\int_c^d \bar{E} \cdot d\bar{l} = \int_{\Delta x}^{0} E_{t2}\,\bar{a}_x \cdot dx\,\bar{a}_x = -E_{t2}\,\Delta x$$

$$\therefore \text{LHS} = (E_{t1} - E_{t2})\,\Delta x = \text{RHS} = 0$$

$$\therefore E_{t1} = E_{t2}$$

- Since directions of both are same we write

$$\boxed{\bar{E}_{t1} = \bar{E}_{t2}\,.} \quad \text{...(7.18.1)}$$

i.e. tangential components of electric field adjacent to the boundary on both sides are equal and in same direction or in simple words tangential components are ***continuous*** across the boundary of two dielectrics. This is the boundary condition for electric field.

We have,

$$E_{t1} = E_{t2}$$

i.e. $$\epsilon_1 D_{t1} = \epsilon_2 D_{t2}$$

or $$\frac{D_{t1}}{\epsilon_2} = \frac{D_{t2}}{\epsilon_1} \text{ i.e. } \frac{D_{t1}}{D_{t2}} = \frac{\epsilon_2}{\epsilon_1} \quad \text{... (7.18.2)}$$

- Since media 1 and 2 are having different permittivity i.e. $\epsilon_1 \neq \epsilon_2$, then $D_{t1} \neq D_{t2}$ i.e. tangential components of $\bar{D}$ are not continuous across an interface.

☞ Normal components

- In order to find relation between the normal components of the fields at a boundary, consider a small box half in each medium with its top face in medium 1 and bottom face in medium 2, as shown in Fig. 7.18.3. The face have an area $\Delta x\,\Delta y$ and height Δz. Applying Gauss's law to the box shown in figure.

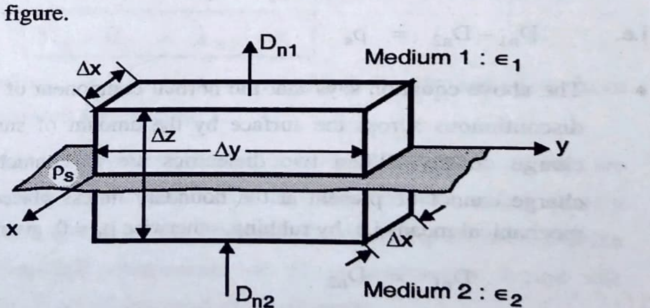

Fig. 7.18.3 : Normal components

$$\oint \bar{D} \cdot d\bar{s} = Q_{enclosed} \quad \text{...(a)}$$

- Consider the boundary plane has surface charge density ρ_s. The area of this plane enclosed within the box is $\Delta x\,\Delta y$. Then charge on this plane enclosed within the box is

$$Q_{enclosed} = \rho_s\,\Delta x\,\Delta y = \text{RHS}$$

- The closed surface in figure is consists of six surfaces so the LHS of Equation (a) is calculated as,

$$\text{LHS} = \oint \bar{D}\cdot d\bar{s} = \int_{Top} \bar{D}\cdot d\bar{s} + \int_{Bottom} + \int_{Left} + \int_{Right} + \int_{Front} + \int_{Back}.$$

- The flux densities D_{n1} and D_{n2} will approach the boundary, if $\Delta z \to 0$
- Then areas of left, right, front and back surfaces are reduced to zero and LHS will reduce to,

$$\text{LHS} = \int_{Top} \bar{D}\cdot d\bar{s} + \int_{Bottom} \bar{D}\cdot d\bar{s}$$

- Now we will solve each integral separately.

$$\int_{Top} \bar{D}\cdot d\bar{s} = \int_0^{\Delta y}\int_0^{\Delta x} D_{n1}\,\bar{a}_z \cdot dxdy\,\bar{a}_z$$

$$= \int_0^{\Delta y}\int_0^{\Delta x} D_{n1}\,dxdy = D_{n1}\,\Delta x\,\Delta y.$$

Similarly,

$$\int_{Bottom} \bar{D}\cdot d\bar{s} = \int_0^{\Delta y}\int_0^{\Delta x} D_{n2}\,\bar{a}_z \cdot dxdy\,(-\bar{a}_z)$$

$$= -D_{n2}\,\Delta x\,\Delta y.$$

$$\therefore \quad \text{LHS} = (D_{n1} - D_{n2})\,\Delta x\,\Delta y$$

$$= \text{RHS} = \rho_s\,\Delta x\,\Delta y$$

i.e. $$D_{n1} - D_{n2} = \rho_s$$

- The above equation says that the normal component of $\bar{D}$ is discontinuous across the surface by the amount of surface charge density. When two dielectrics are in contact the charge cannot be present at the boundary unless placed by mechanical means i.e. by rubbing, otherwise $\rho_s = 0$, giving

$$D_{n1} = D_{n2}$$

Since directions of D_{n1} and D_{n2} is same then

$$\boxed{\bar{D}_{n1} = \bar{D}_{n2}}$$

i.e. the normal component of the flux density is continuous across the charge-free boundary between two dielectrics. It follows that

$$D_{n1} = D_{n2}$$

i.e. $$\epsilon_1 E_{N1} = \epsilon_2 E_{N2}$$

or $$\frac{E_{n1}}{E_{n2}} = \frac{\epsilon_2}{\epsilon_1}$$

i.e. normal $\bar{E}$ is discontinuous.

7.18.2 Magnetic Boundary Conditions

- As we seen in electric, boundary conditions for normal component and tangential components are derived separately, here also we shall derive it separately. Figs. 7.18.4 and 7.18.5 shows a boundary between two isotropic homogeneous linear materials with permeabilities μ_1 and μ_2. The gaussian surface and the closed path is constructed at the boundary between medium 1 and medium 2 as shown.

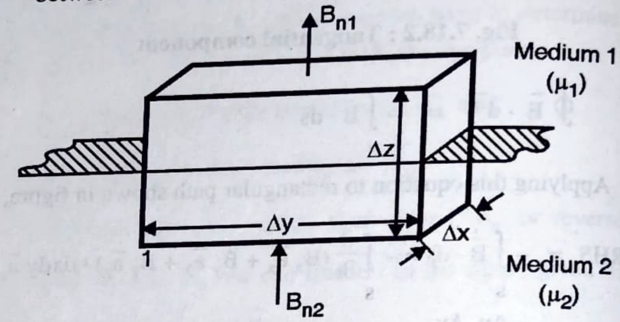

Fig. 7.18.4 : Normal magnetic components

Boundary condition for normal components of $\bar{B}$

- From the Gauss's law for magnetic we have,

$$\oint \bar{B}\cdot d\bar{s} = 0$$

- Since the closed surface consists of six surfaces we write,

$$\text{LHS} = \oint \bar{B}\cdot d\bar{s}$$

$$= \int_{Top} \bar{B}\cdot d\bar{s} + \int_{Bottom} + \int_{Left} + \int_{Right} + \int_{Front} + \int_{Back}$$

- In order to approach B_{n1} and B_{n2} to the boundary, let $\Delta z \to 0$
- Then areas of left, right, front and back surfaces are reduced to zero and LHS will reduce to,

$$\text{LHS} = \int_{Top} \bar{B}\cdot d\bar{s} + \int_{Bottom} \bar{B}\cdot d\bar{s}$$

Let us solve each integral separately

$$\int_{Top} \bar{B}\cdot d\bar{s} = \int_0^{\Delta y}\int_0^{\Delta x} B_{n1}\,\bar{a}_z\cdot dxdy\,\bar{a}_z$$

$$= \int_0^{\Delta y}\int_0^{\Delta x} B_{n1}\,dxdy = B_{n1}\,\Delta x\,\Delta y$$

$$\int_{Bottom} \bar{B}\cdot d\bar{s} = \int_0^{\Delta y}\int_0^{\Delta x} B_{n2}\,\bar{a}_z\cdot dxdy\,(-\bar{a}_z)$$

$$= -\int_0^{\Delta y}\int_0^{\Delta x} B_{n2}\,dxdy = -B_{n2}\,\Delta x\,\Delta y$$

$\therefore$ $\quad LHS = (B_{n1}-B_{n2})\,\Delta x\,\Delta y = RHS = 0$

i.e. $\quad B_{n1}-B_{n2} = 0$

or $\quad B_{n1} = B_{n2}$.

- Since the directions of both are also same we write

$$\boxed{\bar{B}_{n1} = \bar{B}_{n2}}$$

- In words, the normal components of B is continuous across the boundary.

☞ Tangential components

- The boundary condition for the tangential components of magnetic field is obtained by the application of Ampere's law around a closed rectangular path as shown in Fig. 7.18.5. In the figure x-axis is shown perpendicular to the page.

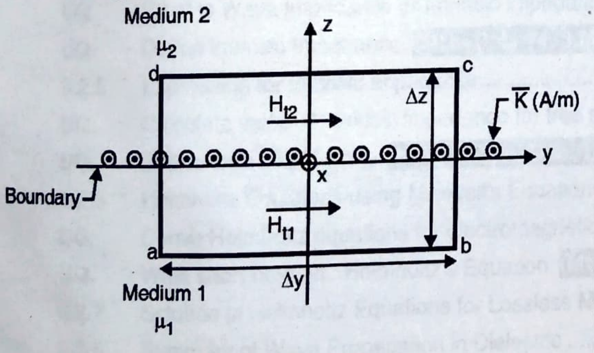

Fig. 7.18.5 : Tangential magnetic components

- Consider a linear current density K is along the boundary.

From Ampere's law

$$\oint \bar{H}\cdot d\bar{l} = I_{enclosed}$$

The RHS is obtained as,

$$RHS = I_{enclosed} = K\,\Delta y$$

The LHS is obtained by

$$LHS = \oint \bar{H}\cdot d\bar{l} = \int_a^b \bar{H}\cdot d\bar{l} + \int_b^c + \int_c^d + \int_d^a$$

- The magnetic field intensities H_{t1} and H_{t2} will approach the boundary if, $\Delta z \to 0$ then sides bc and da will approach zero. This reduces LHS as,

$$LHS = \oint \bar{H}\cdot d\bar{l} = \int_a^b \bar{H}\cdot d\bar{l} + \int_c^d \bar{H}\cdot d\bar{l}$$

We solve each integral separately

$$\int_a^b \bar{H}\cdot d\bar{l} = \int_0^{\Delta y} H_{t1}\,\bar{a}_y\cdot dy\,\bar{a}_y = H_{t1}\int_0^{\Delta y} dy = H_{t1}\,\Delta y$$

Similarly,

$$\int_c^d \bar{H}\cdot d\bar{l} = \int_{\Delta y}^{0} H_{t2}\,\bar{a}_y\cdot dy\,\bar{a}_y = H_{t2}\int_{\Delta y}^{0} dy = -H_{t2}\,\Delta y$$

$\therefore$ $\quad LHS = (H_{t1}-H_{t2})\,\Delta y = RHS = K\,\Delta y$

$\therefore$ $\quad H_{t1}-H_{t2} = K$

- Thus the tangential component of H is discontinuous at the interface, where a free surface current exists. We cannot convert this equation directly in vector form as

$$\bar{H}_{t1}-\bar{H}_{t2} = \bar{K}$$

- This is because direction of $\bar{H}$ and direction of K cannot be same, it must be perpendicular. In order to bring it into expression, we introduce a unit vector on the right side as

$$\boxed{\bar{H}_{t1}-\bar{H}_{t2} = \bar{a}_{N12}\times\bar{K}}$$

where, $\bar{a}_{N12}$ = unit vector normal to the boundary from medium 1 to medium 2.

- When the conductivities of both media are finite, currents are defined by volume current densities and free surface current do not exist on the interface. Hence K equals zero and the tangential components of H is continuous across the boundary of almost all physical media.

i.e. $\boxed{\bar{H}_{t1} = \bar{H}_{t2}}$

- Boundary relations for electric and magnetic are now summarized as shown below :

	Electric	Magnetic
Tangential	$\bar{E}_{t1} = \bar{E}_{t2}$	$\bar{H}_{t1}-\bar{H}_{t2} = \bar{a}_{N12}\times\bar{K}$
Normal	$\bar{D}_{n1} = \bar{D}_{n2}$	$\bar{B}_{n1} = \bar{B}_{n2}$

UEx. 7.18.1 **MU - Q. 4(a), Dec. 18, 10 Marks**

List boundary conditions for time varying field if given that :

$\bar{D} = 50\,\bar{a}_x + 80\,\bar{a}_y - 30\,\bar{a}_z\ \text{nC/m}^2$

In region $x \geq 0$ where $\epsilon = 2.1\,\epsilon_0$. Find electric charge density for region $x \leq 0$ where $\epsilon = 7.6\,\epsilon_0$.

☑ **Soln. :**

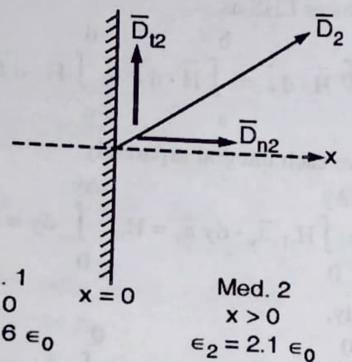

Fig. Ex. 7.18.1

Since x axis is normal to the boundary

$$\bar{D}_{n2} = 50\,\bar{a}_x$$

$$\bar{D}_{t2} = 80\,\bar{a}_y - 30\,\bar{a}_z$$

$$\therefore\ \bar{E}_{t2} = \frac{\bar{D}_{t2}}{\epsilon_2} = \frac{80\,\bar{a}_y - 30\,\bar{a}_z}{2.1\,\epsilon_0}$$

From boundary condition

$$\bar{E}_{t1} = \bar{E}_{t2}$$

$$\therefore\ \bar{D}_{t1} = \epsilon_1\,\bar{E}_{t1} = 7.6\,\epsilon_0\left(\frac{80\,\bar{a}_y - 30\,\bar{a}_z}{2.1\,\epsilon_0}\right)$$

$$= 3.62\,(80\,\bar{a}_y - 30\,\bar{a}_z) = 289.6\,\bar{a}_y - 108.69\bar{a}_z$$

Also

$$\bar{D}_{n1} = \bar{D}_{n2} = 50\,\bar{a}_x$$

$$\therefore\ \bar{D}_1 = \bar{D}_{n1} + \bar{D}_{t1}$$

$$\therefore\ \bar{D}_1 = 50\,\bar{a}_x + 289.6\,\bar{a}_y - 108.6\,\bar{a}_z\ (\text{nC/m}^2)$$

Chapter Ends...

□□□

Module 2

CHAPTER 8

Electromagnetic Wave Propagation

8.1 INTRODUCTION

- In the previous chapter we learned Maxwell's equations for time varying fields. These equations form the basis to study the basic principles of electromagnetic wave propagation.
- In this chapter we are not going to deal with how waves are generated but we shall see how the wave travels from the source into surrounding media. This media can be roughly divided into two types :

1. Lossless media.   2. Lossy media.

These media are further classified as shown in Fig. 8.1.1.

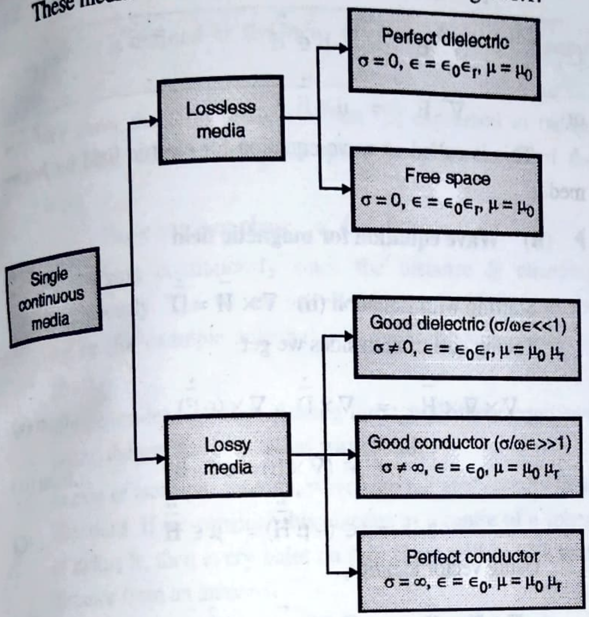

Fig. 8.1.1 : Classification of media

Here each notation has the usual meaning, where

σ = Conductivity

ϵ = Permittivity of media = $\epsilon_0\epsilon_r$,

ϵ_0 = Permittivity of free space

$= 8.854 \times 10^{-12}$ (F/m) $\simeq \dfrac{10^{-9}}{36\pi}$ (F/m)

ϵ_r = Relative permittivity,

μ = Permeability of media = $\mu_0\mu_r$,

μ_0 = Permeability of free space

$= 4\pi \times 10^{-7}$ (H/m)

μ_r = Relative permeability.

In this chapter we shall study how the wave propagates in a single continuous media, which can be one of the media explained above.

8.2 WAVE PROPAGATION IN LOSSLESS MEDIA

Lossless media is the media with no loss. It has conductivity $\sigma = 0$.

Lossless media is either perfect dielectric with the property,

$$\sigma = 0, \qquad \epsilon = \epsilon_0\epsilon_r, \qquad \mu = \mu_0\mu_r$$

or free space with the property

$$\sigma = 0, \qquad \epsilon = \epsilon_0, \qquad \mu = \mu_0$$

- **We will obtain the results for perfect dielectric and then putting $\mu_r = 1$ and $\epsilon_r = 1$ in the results, we get results for free space.**
- The term σ represents loss. For $\sigma = 0$, means no loss, so energy of the wave when propagates remain constant. We will prove it mathematically. This requires wave equations to be derived.

8.2.1 Wave Equations for Lossless Media

UQ. Derive equations for uniform plane waves in time domain in free space in terms of $\bar{E}, \bar{D}, \bar{H}$ and $\bar{B}$.
MU - May 11, 10 Marks

UQ. Derive the wave equation from Maxwell's equations for free space, charge free region.
MU - Dec. 11, 10 Marks

UQ. Derive wave equation in free space.
MU - May 14, Dec. 15, 5 Marks

UQ. Derive wave equation for good dielectric medium.
MU - Dec. 14, Dec. 15, 5 Marks

UQ. Explain Wave equation for free space.
(MU - Q. 1(a), May 19, 5 Marks)

- Though propagation of a wave is a physical phenomena it is explained after deriving certain mathematical expressions called **wave equations.** These are derived in this section.
- Before we derive complex wave equation for general case, let us derive simple wave equation for dielectric media containing no charges and no conduction currents.
- To derive wave equations in dielectric we start with Maxwell's equations for dielectric, which are :

$$\nabla \times \bar{E} = -\dot{\bar{B}} \quad \text{...(i)}$$

$$\nabla \times \bar{H} = \dot{\bar{D}} \quad \text{...(ii)}$$

$$\nabla \cdot \bar{D} = 0 \text{ or } \nabla \cdot \bar{E} = 0 \quad (\because \bar{D} = \epsilon\bar{E}) \quad \text{...(iii)}$$

$$\nabla \cdot \bar{B} = 0 \text{ or } \nabla \cdot \bar{H} = 0 \quad (\because \bar{B} = \mu\bar{H}) \quad \text{...(iv)}$$

Also we know that, $\bar{D} = \epsilon\bar{E}$

Module 2

for free space, $\epsilon = \epsilon_0$

$$\bar{B} = \mu \bar{H}$$

for free space, $\mu = \mu_0$

$$\bar{J} = \sigma \bar{E}$$

for free space, $\sigma = 0$

The constants ϵ, μ and σ are independent of time. Single differentiation and double differentiation of both sides of above equations gives

$$\dot{\bar{D}} = \epsilon \dot{\bar{E}}$$

$$\text{and} \quad \ddot{\bar{D}} = \epsilon \ddot{\bar{E}} \qquad \text{...(v)}$$

$$\dot{\bar{B}} = \mu \dot{\bar{H}}$$

$$\text{and} \quad \ddot{\bar{B}} = \mu \ddot{\bar{H}} \qquad \text{...(vi)}$$

$$\dot{\bar{J}} = \sigma \dot{\bar{E}}$$

$$\text{and} \quad \ddot{\bar{J}} = \sigma \ddot{\bar{E}} \qquad \text{...(vii)}$$

We have, $\nabla \times \bar{H} = \dot{\bar{D}}$

differentiating both sides partially w. r. t time we get

$$\frac{\partial}{\partial t}(\nabla \times \bar{H}) = \frac{\partial}{\partial t}\dot{\bar{D}} = \ddot{\bar{D}}$$

(double dot is double differentiation partially w. r. t time)

i.e. $$\nabla \times \frac{\partial}{\partial t}\bar{H} = \ddot{\bar{D}}$$

or $$\nabla \times \dot{\bar{H}} = \ddot{\bar{D}} \qquad \text{...(viii)}$$

Similarly from,

$$\nabla \times \bar{E} = -\dot{\bar{B}}$$

We get, $$\nabla \times \dot{\bar{E}} = -\ddot{\bar{B}} \qquad \text{...(ix)}$$

There are two types of wave equations,

(A) Wave equation for electric field
(B) Wave equation for magnetic field

(i) Wave equations for electric field : To derive wave equation for electric field we start with Equation (i).

(ii) Wave equations for magnetic field : To derive wave equation for magnetic field we start with Equation (ii)

Both electric and magnetic wave equations are in similar forms.

(A) Wave equation for electric field

Starting with Equation (i) : $\nabla \times \bar{E} = -\dot{\bar{B}}$

Taking curl of both sides,

$$\nabla \times \nabla \times \bar{E} = -\nabla \times \dot{\bar{B}} = -\nabla \times (\mu \dot{\bar{H}}) \qquad \text{...from (vi)}$$

$$= -\mu (\nabla \times \dot{\bar{H}}) = -\mu (\ddot{\bar{D}}) \qquad \text{...from (viii)}$$

$$= -\mu (\epsilon \ddot{\bar{E}}) = -\mu \epsilon \ddot{\bar{E}} \qquad \text{...(A)}$$

Using vector identity :

$$\nabla \times \nabla \times \bar{E} = \nabla (\nabla \cdot \bar{E}) - \nabla^2 \bar{E} \qquad \text{...(x)}$$

but, $$\nabla \cdot \bar{E} = 0 \qquad \text{...from (iii)}$$

then Equation (x) reduces to,

$$\nabla \times \nabla \times \bar{E} = -\nabla^2 \bar{E} \qquad \text{...(B)}$$

Comparing Equation (A) and (B),

i.e. $$-\nabla^2 \bar{E} = -\mu \epsilon \ddot{\bar{E}}$$

or $$\nabla^2 \bar{E} = \mu \epsilon \ddot{\bar{E}} \qquad \text{...(8.2.1)}$$

This is called as wave equation for electric field for lossless media.

(B) Wave equation for magnetic field

Starting with Equation (ii) : $\nabla \times \bar{H} = \dot{\bar{D}}$

Taking curl of both sides we get,

$$\nabla \times \nabla \times \bar{H} = \nabla \times \dot{\bar{D}} = \nabla \times (\epsilon \dot{\bar{E}}) \qquad \text{...from (v)}$$

$$= \epsilon (\nabla \times \dot{\bar{E}}) = \epsilon (-\ddot{\bar{B}}) \qquad \text{...from (ix)}$$

$$= \epsilon (-\mu \ddot{\bar{H}}) = -\mu \epsilon \ddot{\bar{H}} \qquad \text{...(C)}$$

Using vector identity,

$$\nabla \times \nabla \times \bar{H} = \nabla (\nabla \cdot \bar{H}) - \nabla^2 \bar{H} \qquad \text{...(xi)}$$

But $$\nabla \cdot \bar{H} = 0 \qquad \text{...from (iv)}$$

Then Equation (xi) reduces to,

$$\nabla \times \nabla \times \bar{H} = -\nabla^2 \bar{H} \qquad \text{...(D)}$$

Comparing Equations (C) and (D) we get

$$-\nabla^2 \bar{H} = -\mu \epsilon \ddot{\bar{H}}$$

$$\nabla^2 \bar{H} = \mu \epsilon \ddot{\bar{H}} \qquad \text{...(8.2.2)}$$

"Equation (8.2.2) is expression for wave equation for magnetic field, for lossless media."

(C) Wave Equations for Free Space

- Noticing the difference in properties of dielectric and free space, just by replacing μ and ϵ by μ_0 and ϵ_0, wave equations for free space can be obtained.

Wave equations for lossless media	
Dielectric	**Free space**
$\nabla^2 \bar{E} = \mu\epsilon \ddot{\bar{E}}$	$\nabla^2 \bar{E} = \mu_0\epsilon_0 \ddot{\bar{E}}$
$\nabla^2 \bar{H} = \mu\epsilon \ddot{\bar{H}}$	$\nabla^2 \bar{H} = \mu_0\epsilon_0 \ddot{\bar{H}}$

Using these wave equations we can study propagation field, for lossless media.

8.2.2 Properties of Plane Waves

UQ. Define uniform plane wave.
MU - May 13, 3 Marks, Q. 1(e), Dec. 18, 3 Marks)

- The waves which have studied in previous articles all are uniform plane waves. In order to understand what are these waves and the importance of it, let us see its properties first.

Wavefront

A wavefront is defined as the locus of points having the same phase.

- We know the term phase constant (β) expressed in rad/m. When the wave travels in a space the phase angle of the wave.

  Phase angle or phase = β × distance

  is charging continuously since the distance is changing continuously. The waves travelled same distance from the source (for example antenna) will undergo same change in phase.
- If we consider a surface passing through all these equiphase points, the surface is called as **wavefront.**
- In case of isotropic antenna, waves are radiated equally in all directions. If we consider this antenna as a centre of a sphere of radius R, then every point on the sphere will be at a same distance from an antenna.
- Since distance travelled by all waves to reach the sphere is same, all waves will undergo same change in phase angle (equal to βR). These waves are said to have same phase and the sphere is called as **spherical wavefront.** Or the waves are called as spherical waves.

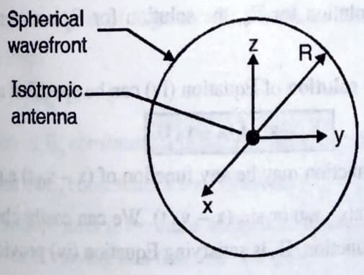

Fig. 8.2.1 : Showing spherical wavefront

In order to understand the uniform plane waves now you should know plane waves. It is defined as :

Plane Waves

- A plane wave is a constant frequency waves whose waveforms are infinite parallel planes of constant amplitude normal to the phase velocity vector (direction of propagation).
- By extension, the term is also used to describe waves that are approximately plane waves in a localized region of space. Plane waves does not exist in practice because a source of infinite in extent would be required to create it and practical wave sources are always finite in extent. But if we are far enough from a source, the wavefront becomes almost spherical and a very small portion of the surface of a giant sphere is very nearly a plane. Plane wave is also defined as :
- **If electric and magnetic field of a wave lies in a plane perpendicular to the direction of wave travel, the wave is called as plane wave.**

  Now let us see what is uniform plane wave.

Uniform plane waves

If the field has the same direction and magnitude at every point in any plane perpendicular to the direction of wave travel, the wave is called as uniform plane wave. This is as shown in Fig. 8.2.2.

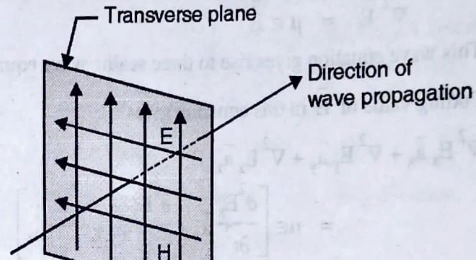

Fig. 8.2.2

Why uniform plane waves ?

- A simple and very useful type of wave that serves a building block in the study of electromagnetic waves consists of electric and magnetic fields that are perpendicular to each other and to the direction of propagation of wave.
- Also they are uniform in amplitudes perpendicular to the direction of propagation. These waves are nothing but uniform plane waves.
- Further more the principle of guiding of electromagnetic waves along transmission lines and waveguides and principles of many other wave phenomena can be studied basically in terms of uniform plane waves. Hence it is very important that we understand the principles of uniform plane wave propagation.

8.2.3 Uniform Plane Wave Propagation (Solution of Wave Equation)

UQ. Give the solution of wave equation for free space.
MU - May 15, 5 Marks

Module 2

- In this section we will see how the wave propagates in perfect dielectric and what is the velocity of it ? We start the analysis by wave equations derived in the last section.

The wave equations for electric and magnetic fields are :

$$\nabla^2 \bar{E} = \mu \in \ddot{\bar{E}} \qquad ...(i)$$

and

$$\nabla^2 \bar{H} = \mu \in \ddot{\bar{H}} \qquad ...(ii)$$

- As the form of Equations (i) and (ii) is same, once we make the analysis of one equation, the result can be applied to other equation directly.

Let the electric and magnetic field intensities are defined in cartesian system as

$$\bar{E} = E_x \bar{a}_x + E_y \bar{a}_y + E_z \bar{a}_z$$

$$\bar{H} = H_x \bar{a}_x + H_y \bar{a}_y + H_z \bar{a}_z$$

Starting with Equation (i)

$$\nabla^2 \bar{E} = \mu \in \ddot{\bar{E}}$$

This wave equation gives rise to three scalar wave equation.

Putting value of $\bar{E}$ in this equation gives :

$$\nabla^2 E_x \bar{a}_x + \nabla^2 E_y \bar{a}_y + \nabla^2 E_z \bar{a}_z = \mu \in \left[\frac{\partial^2 E_x}{\partial t^2} \bar{a}_x + \frac{\partial^2 E_y}{\partial t^2} \bar{a}_y + \frac{\partial^2 E_z}{\partial t^2} \bar{a}_z \right]$$

Two vectors are equal when x, y and z components of vectors are identical, comparing two sides, we get scalar wave equations as :

$$\nabla^2 E_x = \mu \in \frac{\partial^2 E_x}{\partial t^2},$$

$$\nabla^2 E_y = \mu \in \frac{\partial^2 E_y}{\partial t^2},$$

and

$$\nabla^2 E_z = \mu \in \frac{\partial^2 E_z}{\partial t^2}$$

One Dimensional Wave Equations

- The solution of these three differential equations can be obtain on similar lines. Let's consider for the sake of simplicity that the time varying electric field intensity is a function of x only, so that all $\partial/\partial y$ and $\partial/\partial z$ concern with E are zero. The value of ∇^2 is also simplified as shown below :

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \Rightarrow \frac{\partial^2}{\partial x^2}$$

The above three equations containing $\nabla^2 E_x$, $\nabla^2 E_y$ and $\nabla^2 E_z$ can be written using reduced expression for ∇^2 as

$$\frac{\partial^2 E_x}{\partial x^2} = \mu \in \frac{\partial^2 E_x}{\partial t^2} \qquad ...(iii)$$

$$\frac{\partial^2 E_y}{\partial x^2} = \mu \in \frac{\partial^2 E_y}{\partial t^2} \qquad ...(iv)$$

$$\frac{\partial^2 E_z}{\partial x^2} = \mu \in \frac{\partial^2 E_z}{\partial t^2} \qquad ...(v)$$

Notice that Equations (iii), (iv) and (v) are scalar wave equations. In these equations E_x, E_y and E_z are changing w.r.t. only one dimension (x), and hence are called as **one dimensional wave equations.** According to Maxwell's equation for lossless media, we have

$$\nabla \cdot \bar{D} = 0 \quad \text{then} \quad \nabla \cdot \bar{E} = 0 \quad (\therefore \bar{D} = \in \bar{E})$$

Expanding div.E using expression for E in cartesian system, we get

$$\nabla \cdot \bar{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

$$\therefore \quad \nabla \cdot \bar{E} = \frac{\partial E_x}{\partial x} \Rightarrow 0 \qquad (\because \partial/\partial y = \partial/\partial z = 0)$$

In the above equation differentiation of E_x is zero means E_x is either constant or zero. But if we accept that E_x is constant then we are not sticking with the condition that E is a variable of x. Thus, we are forced to conclude that

$$E_x = 0 \qquad ...(8.2.3)$$

The above equation shows that the **electric field intensity varying w.r.t x does not have the x component.** So that the electric field vector which we initially assumed in terms of E_x, E_y and E_z becomes,

$$\bar{E} = E_y \bar{a}_y + E_z \bar{a}_z \qquad ...(vi)$$

Also Equation (iii) is totally vanished. Now consider Equations (iv) and (v), which are the differential Equations for E_y and E_z having same form. By solving Equation (iv) we get electric field E_y. The form of Equations (iv) and (v) is same therefore once we obtain solution for E_y, the solution for E_z can very easily be obtained.

The **general solution** of Equation (iv) can be specified as

$$\boxed{E_y = f(x - v_0 t)} \qquad ...(vii)$$

This function may be any function of $(x - v_0 t)$ e.g. it may be $\sqrt{x - v_0 t}$ or $e^{(x - v_0 t)}$ or $\sin(x - v_0 t)$. We can easily check that for any of this function, E_y is satisfying Equation (iv) provided that

$$v_0 = \frac{1}{\sqrt{\mu \in}} \qquad ...(8.2.4)$$

For the sake of simplicity assume

$$\boxed{E_y = \sin(x - v_0 t)} \qquad ...(8.2.5)$$

If we plot this function w. r. t. x at different timings, we get

at t = 0, $E_y = \sin x$

at $t = t_1$, $t_1 > 0$, $E_y = \sin(x - x_1)$

where, $x_1 = v_0 t_1$

at $t = t_2$, $t_2 > t_1 > 0$,

$E_y = \sin(x - x_2)$ where $x_2 = v_0 t_2$ etc.

All these sinusoidal functions are plotted in Fig. 8.2.3.

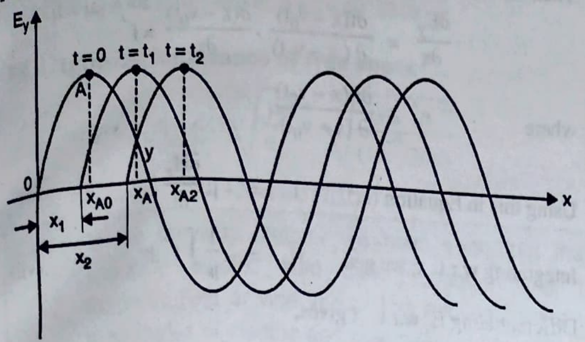

Fig. 8.2.3 : Showing propagation of wave

- By observing variation of E_y w.r.t x at different timings we can conclude that at time t = 0, the sinusoidal positive starts at x = 0. At $t = t_1$ the sinusoidal positive going starts at $x = x_1 = v_0\ t_1$. Similarly at time t_2, sinusoid starts at $x = x_2 = v_0\ t_2$.
- If we consider any point on the waveform at t = 0, e.g. A, it is at x_{A0} position. Then at $t = t_1$, it is at x_{A1} ; at $t = t_2$, its position changes to x_{A2}.
- Here we observe that as time progresses, point A moves forward in x direction. This is true for all points on the first waveform or in general the phenomenon of variation of E_y is propagating in x direction.
- We have $x = v_0 t$, where x is measured in meters while t is measured in seconds, then the unit of v_0 is m/s. Thus v_0 represents a velocity of wave.
- The similar type of analysis can be made for Equation (v) to show that the variation of E_z w.r.t. x also constitute a wave.

The above analysis of Equation (i) concludes that for E as a function of x only,

$$E_x = 0$$

- Variation of E_y constitutes a wave travelling in x-direction.
- Variation of E_z constitutes a wave travelling in x-direction.

On the similar lines if we make analysis of Equation (ii), then we can write for H varying as a function of x only,

- $H_x = 0$
- Variation of H_y constitutes a wave travelling in x-direction.
- Variation of H_z constitutes a wave travelling in x-direction.

Phase Velocity

Definition :

- It is defined as the velocity of some point in the sinusoidal waveform.
- For example it is the velocity of point A in the Fig. 8.2.3. This point covers a distance from x_{A0} to x_{A1} in time t = 0 to t_1. It means it has some velocity, we call it as phase velocity.

Now we have

$x_1 = v_0 t_1$, $x_2 = v_0 t_2$, and so on. In general $x = v_0 t$

or $v_0 = \frac{x}{t} \Rightarrow \frac{\text{meters}}{\text{second}}$ i.e. it is a velocity.

From Equation (8.2.4)

$$\boxed{v_0 = \frac{1}{\sqrt{\mu\varepsilon}} = \text{phase velocity}}$$

Module 2

Note :

- A wave consists of electric and magnetic fields.
- When the wave travels in a particular direction, the fields associated with it also travel in same directions.
- By assuming that E and H is a function of only x, we have made the wave travel in x-direction.
- If we want the wave travel in y-or z-direction; the assumption should be, E and H vary only w.r.t, y or z respectively.
- For a wave in y-direction, we can prove that

$$E_y = H_y = 0$$

and E_x, E_z, H_x, H_z constitutes a wave in y-direction.

- Similarly, for a wave in z-direction.

$$E_z = H_z = 0$$

and, E_x, E_y, H_x, H_y constitutes a wave in z-direction.

- All field components travel with a velocity,

$$\upsilon_0 = \frac{1}{\sqrt{\mu\varepsilon}}$$

8.2.4 Intrinsic Impendence or Characteristic Impedance of Lossless Medium

UQ. What is Wave Impedance or Intrinsic Impedance ? **MU - May 09, 5 Marks**

UQ. Define intrinsic Impedance. **MU - Dec. 14, 3 Marks**

Significance of Intrinsic Impedance

- We have the unit of electric and magnetic field as (V/m) and (A/m) respectively. If we take the ratio.

$\frac{E}{H} \Rightarrow \frac{V/m}{A/m} = V/A$, which is a unit of impedance.

- At some point on the circuit board, having the knowledge of voltage and current at that point, the impedance at that point can be determined simply by taking the ratio of them, this idea can be extended further.
- The wave in a media is associated with the electric and magnetic field with it. At some point in the media, by taking

the ratio of E and H, the impedance of media at that point can be obtained. This impedance is called as **intrinsic impedance** denoted by (η). It is defined as,

Intrinsic Impedance (or Characteristic Impedance) of the Lossless Media

- It is the ratio of the magnitudes of E and H for a plane (TEM) wave in an unbounded medium. Mathematically,

$$\boxed{\eta = \text{Intrinsic impedance} = \frac{\text{Magnitude of E}}{\text{Magnitude of H}}}$$

8.2.5 Expression for Intrinsic Impedance

UQ. Calculate value of intrinsic impedance for free space. **(MU - May 09, Dec. 14, 5 Marks)**

UQ. Define term : TEM wave **(MU - Q. 1(e), Dec. 18, 2 Marks)**

For a wave propagating in x- direction we have

$$E_x = H_x = 0 \quad \text{and}$$

$$\partial/\partial y = \partial/\partial z = 0$$

Then electric and magnetic fields are given by,

$$\bar{E} = E_y \bar{a}_y + E_z \bar{a}_z \qquad ...(i)$$

$$\bar{H} = H_y \bar{a}_y + H_z \bar{a}_z \qquad ...(ii)$$

Using Maxwell's equation,

$$\nabla \times \bar{E} = -\dot{\bar{B}} = -\mu \dot{\bar{H}} \text{ , we get}$$

$$\nabla \times E = \begin{vmatrix} a_x & a_y & a_z \\ \partial/\partial x & 0 & 0 \\ 0 & E_y & E_z \end{vmatrix} = -\mu \frac{\partial}{\partial t} [H_y \bar{a}_y + H_z \bar{a}_z]$$

Solving determinant in above equation we get,

$$-\frac{\partial E_z}{\partial x} \bar{a}_y + \frac{\partial E_y}{\partial x} \bar{a}_z = -\mu \left[\frac{\partial H_y}{\partial t} \bar{a}_y + \frac{\partial H_z}{\partial t} \bar{a}_z \right] \qquad ...(a)$$

Similarly, using Maxwell's equation for $\nabla \times \bar{H} = \dot{\bar{D}} = \epsilon \dot{\bar{E}}$ we get,

$$-\frac{\partial H_z}{\partial x} \bar{a}_y + \frac{\partial H_y}{\partial x} \bar{a}_z = \epsilon \left[\frac{\partial E_y}{\partial t} \bar{a}_y + \frac{\partial E_z}{\partial t} \bar{a}_z \right] \qquad ...(b)$$

From Equations (a) and (b) we have following relation between components of E and components of H

$$\frac{\partial E_z}{\partial x} = \mu \frac{\partial H_y}{\partial t} \qquad ...(iii)$$

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t} \qquad ...(iv)$$

$$-\frac{\partial H_z}{\partial x} = \epsilon \frac{\partial E_y}{\partial t} \qquad ...(v)$$

$$\frac{\partial H_y}{\partial x} = \epsilon \frac{\partial E_z}{\partial t} \qquad ...(vi)$$

Now if we assume solution for E_y as,

$$E_y = f(x - v_0 t).$$

Then differentiation w.r.t. x gives

$$\frac{\partial E_y}{\partial x} = \frac{\partial f(x - v_0 t)}{\partial (x - v_0 t)} \cdot \frac{\partial (x - v_0 t)}{\partial x} = f'$$

where $$f' = \frac{\partial f (x - v_0 t)}{\partial (x - v_0 t)}$$

Using this in Equation (iv), $$f' = -\mu \frac{\partial H_z}{\partial t}$$

Integrating w.r.t t, we get, $$H_z = -\frac{1}{\mu} \int f' dt \qquad ...(vii)$$

Differentiating E_y w.r.t. t gives,

$$\frac{\partial E_y}{\partial t} = \frac{\partial f(x - v_0 t)}{\partial (x - v_0 t)} \cdot \frac{\partial (x - v_0 t)}{\partial t} = (-\upsilon_0) f'$$

or $$E_y = \int -v_0 f' dt = -\upsilon_0 \int f' dt \qquad ...(viii)$$

Dividing Equation (viii) by Equation (vii) we get,

$$\frac{E_y}{H_z} = \frac{v_0}{1/\mu} = \mu v_0$$

But, $$v_0 = \frac{1}{\sqrt{\mu \epsilon}}$$

$$\therefore \quad \frac{E_y}{H_z} = \mu \cdot \frac{1}{\sqrt{\mu \epsilon}} = \sqrt{\frac{\mu}{\epsilon}} = \eta \qquad ...(ix)$$

where $$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

By assuming expression for E_z as $E_z = f(x - v_0 t)$ and giving the similar treatment as for E_y we get,

$$\frac{E_z}{H_y} = -\sqrt{\frac{\mu}{\epsilon}} = -\eta \qquad ...(x)$$

Combining Equations (ix) and (x) we get,

$$\frac{E_y}{H_z} = -\frac{E_z}{H_y} = \eta$$

or $$E_y = \eta H_z$$

and $$E_z = -\eta H_y$$

Using Equation (i) $$E = E_y a_y + E_z a_z$$

$$= \eta H_z \bar{a}_y - \eta H_y \bar{a}_z$$

Taking mod of both sides :

i.e. $$|E| = \eta \sqrt{H_y^2 + H_z^2}$$

i.e. $$|\bar{E}| = \eta |\bar{H}|$$

or $$\eta = \frac{|\bar{E}|}{|\bar{H}|} = \sqrt{\frac{\mu}{\epsilon}} \qquad ...(8.2.6)$$

- As E is measured in Volts/m and H is measured in Amp/m, then the unit of η will be volts / Amp i.e. Ω and it is called as **intrinsic impedance of the media.** By putting $\mu = \mu_0 = 4\pi \times 10^{-7}$ and $\epsilon = \epsilon_0 = 10^{-9}/36\pi$.

Intrinsic impedance of free space

we get, $$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{(10^{-9}/36\pi)}}$$

$$\simeq 120\,\pi = 377(\Omega)$$

Note : In the previous chapter we have seen that the changing electric field can produce changing magnetic field or vice versa. The relation between magnitudes of electric and magnetic field quantities in a particular media depends upon the intrinsic impedance of that media.

1. $\eta > 1$ indicates that the electric field is always greater than magnetic field. In free space it is 377 times greater than H. That is E always dominates H. Because of this most of the relations in electromagnetics are expressed in terms of E.
2. η is real indicates that E and H are in phase.

Ex. 8.2.1 : For electromagnetic wave prove that $\bar{E} \cdot \bar{H} = 0$ and $\bar{E} \times \bar{H}$ is having the direction of propagation of wave.

Soln. :

Assuming electromagnetic wave propagating in x direction, we can write electric and magnetic field vector as,

$$\bar{E} = E_y \bar{a}_y + E_z \bar{a}_y \quad ...(a)$$

and $$\bar{H} = H_y \bar{a}_y + H_z \bar{a}_z \quad ...(b)$$

The relation between the field components is,

$$\frac{E_y}{H_z} = \eta$$

i.e. $$E_y = \eta H_z \quad ...(c)$$

and $$\frac{E_z}{H_y} = -\eta \quad \text{i.e. } E_z = -\eta H_y \quad ...(d)$$

The dot product of $\bar{E}$ and $\bar{H}$ results in

$$\bar{E} \cdot \bar{H} = E_y H_y + E_z H_z$$

Using Equations (c) and (d) we get,

$$\bar{E} \cdot \bar{H} = \eta H_z H_y - \eta H_y H_z = 0$$

The zero dot product signifies that E and H fields are perpendicular to each other.

Now $$\bar{E} \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & E_y & E_z \\ 0 & H_y & H_z \end{vmatrix} = (E_y H_z - E_z H_y)\,\bar{a}_x$$

Again using Equations (c) and (d), we get

$$\bar{E} \times \bar{H} = \left(\eta H_z^2 + \eta H_y^2\right) \bar{a}_x$$

$$= \eta |H|^2 \,\bar{a}_x$$

i.e. the vector $\bar{E} \times \bar{H}$ is having direction $\bar{a}_x$ with its magnitude equal to η times square of magnetic field intensity. Since the direction of propagation is assumed to be x, hence $\bar{E} \times \bar{H}$ is having the same direction.

TEM Wave

In this problem notice that the electromagnetic wave is a TEM wave (transverse electromagnetic wave). It is defined as:

TEM wave is a wave for which electric field (E), magnetic field (H) and direction of propagation are perpendicular to each other. Mathematically

$$\bar{E} \cdot \bar{H} = 0$$

$$\bar{E} \times \bar{H} \Rightarrow \text{in the direction of propagation}$$

Module 2

8.2.6 Helmholtz Equations using Maxwell's Equations (Wave Equations in Frequency Domain)

UQ. Derive Helmholtz equations for electromagnetic fields in free space. **MU - Dec. 16, 5 Marks**

UQ. Write short note on : Helmholtz's Equation **(MU - Q. 6(b), May 19, 5 Marks)**

A) Helmholtz equation for electric field
B) Helmholtz equation for magnetic field

- We have Maxwell's equations for harmonically varying fields in lossless media ($\bar{J} = 0$ and $\rho_v = 0$) as follows :

$$\nabla \times \bar{E} = -j\omega\bar{B} = -j\omega\mu\bar{H} \quad (\because \bar{B} = \mu\bar{H}) \quad ...(i)$$

$$\nabla \times \bar{H} = j\omega\bar{D} = j\omega\epsilon\bar{E} \quad (\because \bar{D} = \epsilon\bar{E}) \quad ...(ii)$$

$$\nabla \cdot \bar{D} = 0 \quad \text{i.e. } \nabla \cdot \bar{E} = 0 \quad ...(iii)$$

$$\nabla \cdot \bar{B} = 0 \quad \text{i.e. } \nabla \cdot \bar{H} = 0 \quad ...(iv)$$

A) Helmholtz equation for electric field

Taking curl of Equation (i),

$$\nabla \times \nabla \times \bar{E} = \nabla \times (-j\omega\mu\bar{H}) = -j\omega\mu\,(\nabla \times \bar{H})$$

Putting value of $\nabla \times \bar{H}$ from Equation (ii),

$$\nabla \times \nabla \times \bar{E} = -j\omega\mu\,(j\omega\epsilon\bar{E}) = -j^2\omega^2\mu\epsilon\bar{E}$$

i.e. $$\nabla \times \nabla \times \bar{E} = \omega^2 \mu \epsilon \bar{E} \quad (\because j^2 = -1) \quad ...(v)$$

From vector identity we have,

$$\nabla \times \nabla \times \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E}$$

But from Equation (iii), $\nabla \cdot \bar{E} = 0$ then above equation reduces to,

$$\nabla \times \nabla \times \bar{E} = -\nabla^2 \bar{E} \quad ...(vi)$$

Comparing Equations (v) and (vi),

$$\nabla \times \nabla \times \bar{E} = \omega^2 \mu \epsilon \bar{E} = -\nabla^2 \bar{E}$$

i.e. $$\boxed{\nabla^2 \bar{E} = -\omega^2 \mu \epsilon \bar{E}} \quad ...(8.2.7)$$

This is the Helmholtz equation for electric field.

B) Helmholtz equation for magnetic field

Taking curl of Equation (ii), $\nabla \times \nabla \times \bar{H} = \nabla \times (j\omega\epsilon \bar{E})$

$$= j\omega\epsilon\,(\nabla \times \bar{E})$$

Putting value of $\nabla \times \bar{E}$ from Equation (i),

$$\nabla \times \nabla \times \bar{H} = j\omega\epsilon\,(-j\omega\mu\bar{H}) = -j^2 \omega^2 \mu\epsilon \bar{H}$$

i.e. $$\nabla \times \nabla \times \bar{H} = \omega^2 \mu\epsilon \bar{H} \quad ...(vii)$$

From vector identity we have

$$\nabla \times \nabla \times \bar{H} = \nabla(\nabla \cdot \bar{H}) - \nabla^2 \bar{H}$$

But from Equation (iv), $\nabla \cdot \bar{H} = 0$, then the above equation reduces to

$$\nabla \times \nabla \times \bar{H} = -\nabla^2 \bar{H} \quad ...(viii)$$

Comparing Equation (vii) with Equation (viii),

$$\nabla \times \nabla \times \bar{H} = \omega^2 \mu\epsilon \bar{H} = -\nabla^2 \bar{H}$$

i.e. $$\boxed{\nabla^2 \bar{H} = -\omega^2 \mu\epsilon \bar{H}} \quad ...(8.2.8)$$

This is the Helmholtz equation for magnetic field.

Equations (8.2.7) and (8.2.8) are also called as wave equations in frequency domain for lossless media.

Significance of Helmholtz equations is discussed in the next section.

8.2.7 Solution of Helmholtz Equations for Lossless Media

- In the section 8.2.3 we learned uniform plane wave propagation in losseless media, for which free space is a special case. In this medium the conduction current is almost absent in comparison with the displacement current. Such a medium may be treated as a perfect dielectric or lossless medium ($\sigma = 0$).

- In all previous sections we made analysis in space domain and in time domain. Instead of working in time domain if we work in frequency domain the analysis is more simplified. For this purpose we start the analysis by using Helmholtz equations.

- For the uniform plane wave travelling in x-direction, the electric and magnetic field intensities are varying w.r.t x only. We have Helmholtz equations as,

$$\nabla^2 \bar{E} = -\omega^2 \mu\epsilon \bar{E} = -\beta^2 \bar{E} \quad ...(i)$$

$$\nabla^2 \bar{H} = -\omega^2 \mu\epsilon \bar{H} = -\beta^2 \bar{H} \quad ...(ii)$$

where, $\beta = \omega\sqrt{\mu\epsilon}$

For variation in only x direction the Helmholtz equations can be changed by using

$$E_x = H_x = 0 \text{ and all } \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$$

Thus, for the electric field intensity we get,

$$\frac{\partial^2 E_y}{\partial x^2} = -\beta^2 E_y \quad ...(iii)$$

and $$\frac{\partial^2 E_z}{\partial x^2} = -\beta^2 E_z \quad ...(iv)$$

Considering variation of y component only

$$\frac{\partial^2 E_y}{\partial x^2} = -\beta^2 E_y \quad ...(v)$$

- Equation (v) is a second order differential equation whose solution can be assumed having the form

$$E_y = E_{y0}\, e^{-j\beta x} \quad ...(vi)$$

- This equation gives us how E_y is varying w.r.t. x. The corresponding sinusoidal time varying field is obtained by multiplying $E_y(x)$ by $e^{j\omega t}$ and then taking its real part or imaginary part. The time variation is obtained as,

$$E_y(x,t) = E_{y0}\, e^{-j\beta x} \times e^{j\omega t} = E_{y0} \times e^{j(\omega t - \beta x)} \quad ...(vii)$$

The sinusoidal variation if obtained by taking the real part of Equation (vii)

$$\tilde{E}_y(x, t) = \text{R.P.}\{E_y(x, t)\} = E_{y0} \cos(\omega t - \beta x) \quad ...(viii)$$

The plot of $\tilde{E}_y(x,t)$ for different timings is as shown in Fig. 8.2.4.

A) Velocity of a wave
B) Phase constant and wavelength
C) Intrinsic impedance or wave impedance

A) Velocity of a wave

- Fig. 8.2.4 shows that the wave is travelling in + x direction. If we fix our attention on a particular point (a point of a particular phase) on the wave, we set

$\cos(\omega t - \beta x) = \text{constant}$ or $\omega t - \beta x = \text{constant}$

Then the differentiation of it w.r.t time gives,

$$\omega - \beta \frac{dx}{dt} = 0 \quad \text{or} \quad \frac{dx}{dt} = \frac{\omega}{\beta} \qquad ...(ix)$$

Since x represents a distance, then its differentiation with time represents a velocity. Thus,

$$v = \frac{\omega}{\beta} \qquad ...(8.2.9)$$

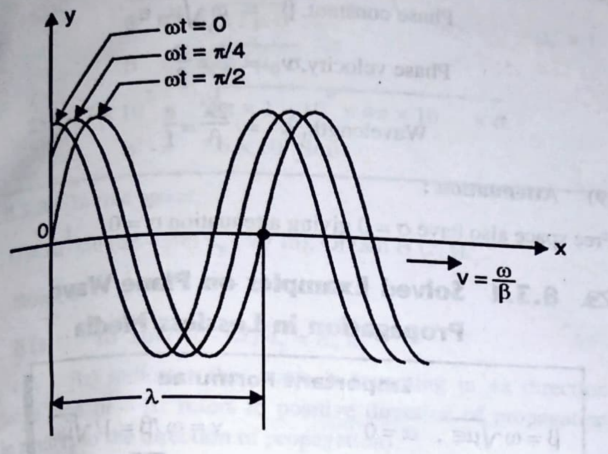

Fig. 8.2.4 : **Showing propagation of plane wave in x-direction**

This velocity of some point in the sinusoidal waveform is called the **phase velocity**. Putting value of β as $\omega\sqrt{\mu\epsilon}$ we get,

$$v = \frac{1}{\sqrt{\mu\epsilon}} \qquad ...(8.2.10)$$

B) Phase constant and wavelength :

- The constant β is called the phase shift constant and is a measure of the phase shift in radians per unit length. Another important quantity observed with wave is wavelength λ, defined as the distance over which the sinusoidal waveform passes through a full cycle of 2π radians. Thus, we can write

$$\beta\lambda = 2\pi$$

or

$$\lambda = 2\pi / \beta \qquad ...(8.2.11)$$

Using the value of β from Equation (8.2.9) and $\omega = 2\pi f$, we get

$$\lambda = \frac{2\pi}{\omega / v} = \frac{2\pi v}{2\pi f}$$

or

$$\lambda = v/f \qquad ...(8.2.12)$$

In general we can conclude that for the sinusoidal time variation, variation w.r.t. space is also sinusoidal.

C) Intrinsic impedance or wave impedance

The expression for intrinsic impedance remain same as ;

$$\eta = \sqrt{\mu/\epsilon}$$

- Before we start solving some more examples on wave propagation in free space, remember the following forms of a uniform plane wave :

$$E_0 \cos(\omega t - \beta z) \text{ or } E_0 \sin(\omega t - \beta z)$$

$$E_0\, e^{j(\omega t - \beta z)} \qquad ...(8.2.13)$$

$$E_0 \sin(z - v_0 t)$$

- All these forms represent wave traveling in + z direction. When negative sign is changed by positive sign, the wave travels in – z direction. When z is replaced by x, wave travels in x direction and so on.
- By replacing $\mu \to \mu_0$ and $\epsilon \to \epsilon_0$ in the above analysis we get results for free space.
- From Fig. 8.2.4 it is clear that, as the wave travels amplitude of wave remain constant. It is due to the fact that for lossless media conductivity (σ) is zero, giving no loss of energy.

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8.2.8 Summary of Wave Propagation in Dielectric

I.	Wave equations	(i) For electric : $\nabla^2 \bar{E} = \mu\epsilon \ddot{\bar{E}}$ (ii) For magnetic : $\nabla^2 \bar{H} = \mu\epsilon \ddot{\bar{H}}$
II.	TEM wave in general	(i) $\bar{E} \cdot \bar{H} = 0$ (ii) $\bar{E} \times \bar{H} \Rightarrow$ in the direction of propagation.
III.	Field components for a wave in x-direction	(i) $E_x = H_x = 0$ (ii) E_y, E_z, H_y, H_z represents a wave in x-direction.
IV.	Relation between field components.	$\frac{E_y}{H_z} = -\frac{E_z}{H_y} = \eta$
V.	Velocity of a wave	$v = \frac{1}{\sqrt{\mu\epsilon}}$
VI.	Intrinsic impendence	$\eta = \sqrt{\mu/\epsilon}$
VII.	Helmholtz equations	(i) For electric : $\nabla^2 \bar{E} = -\omega^2 \mu\epsilon \bar{E}$ (ii) For magnetic : $\nabla^2 \bar{H} = -\omega^2 \mu\epsilon \bar{H}$
VIII.	Properties of wave	(i) Attenuation, $\alpha = 0$ (ii) Phase constant, $\beta = \omega\sqrt{\mu\epsilon}$ (iii) Phase velocity, $v = \omega/\beta = \frac{1}{\sqrt{\mu\epsilon}}$ (iv) Wave length, $\lambda = 2\pi/\beta = v/f$

8.3 WAVE PROPAGATION IN FREE SPACE

UQ. Derive wave equation in free space. **MU - May 14, 5 Marks**

UQ. Derive Helmholtz equations for electromagnetic fields in free space. **MU - Dec. 16, 5 Marks**

UQ. Write short note on : Wave propagation in free space. **(MU - Q. 6(d), Dec. 19, 5 Marks)**

Look at the properties of dielectric media and free space.

For dielectric : $\sigma = 0,\quad \epsilon = \epsilon_0 \epsilon_r,\quad \mu = \mu_0 \mu_r$

For free space : $\sigma = 0,\quad \epsilon = \epsilon_0,\quad \mu = \mu_0$

By replacing ϵ by ϵ_0 and μ by μ_0 we obtain the results for free space from wave propagation in dielectric media (Refer section (8.2.8)).

(1) Wave equations are

$$\nabla^2 \bar{E} = \mu_0 \epsilon_0 \ddot{\bar{E}} \text{ and } \nabla^2 \bar{H} = \mu_0 \epsilon_0 \ddot{\bar{H}}$$

(2) For a wave in x-direction

$$E_x = H_x = 0$$

and, E_y, E_z, H_y, H_z constitutes a traveling wave in + x direction.

(3) Velocity of a wave

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Substituting values of μ_0 and ϵ_0, v $= \dfrac{1}{\sqrt{4\pi \times 10^{-7} \times 10^{-9}/36\pi}}$

$$= 3 \times 10^8 \text{ (m/s)}$$

This is nothing but velocity of light. Thus waves travel in free space with a velocity of light, denoted by c.

$$\therefore \quad v = c = 3 \times 10^8 \text{ (m/s)}.$$

Let us compare the two velocities, velocity in free space and in dielectric. The velocity in dielectric is,

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0 \mu_r \epsilon_0 \epsilon_r}} = \frac{1/\sqrt{\mu_0 \epsilon_0}}{\sqrt{\mu_r \epsilon_r}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

Almost for all dielectrics, $\mu_r = 1$, giving

$$v = \frac{c}{\sqrt{\epsilon_r}}$$

The number ϵ_r for any dielectric is greater than one, means v is always less than c.

(4) Intrinsic impedance

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{10^{-9}/36\pi}} = 120\pi \simeq 377\ (\Omega)$$

(5) Similar to dielectric, waves in free space are TEM waves :

$\bar{E} \cdot \bar{H} = 0$ and $\bar{E} \times \bar{H} \Rightarrow$ in the direction of propagation.

(6) Relation between field components travelling in + x direction :

$$\frac{E_y}{H_z} = -\frac{E_z}{H_y} = 120\pi$$

(7) Helmholtz equations :

$$\nabla^2 \bar{E} = -\omega^2 \mu_0 \epsilon_0 \bar{E} \text{ and } \nabla^2 \bar{H} = -\omega^2 \mu_0 \epsilon_0 \bar{H}$$

(8) The properties of wave :

Phase constant, $\beta = \omega\sqrt{\mu_0 \epsilon_0}$

Phase velocity, $v = c = \dfrac{\omega}{\beta}$

Wavelength, $\lambda = \dfrac{2\lambda}{\beta} = \dfrac{c}{f}$

(9) Attenuation :

Free space also have $\sigma = 0$ giving attenuation $\alpha = 0$.

8.3.1 Solved Examples on Plane Wave Propagation in Lossless Media

Important Formulae

$\beta = \omega\sqrt{\mu\epsilon}$, $\alpha = 0$	$v = \omega/\beta = 1/\sqrt{\mu\epsilon}$
$\lambda = 2\pi/\beta = v/f$	$\eta = \sqrt{\mu\epsilon}$

Ex. 8.3.1 : Find the velocity of a plane wave in a lossless medium having a relative permittivity of 5 and relative permeability of 2.

☑ **Soln. :**

Given : $\epsilon_r = 5$; $\mu_r = 2$

For lossless medium, the velocity of plane wave is given by,

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{(\mu_0 \mu_r)(\epsilon_0 \epsilon_r)}}$$

$$v = \frac{1}{\sqrt{(4\pi \times 10^{-7} \times 2 \times \frac{10^{-9}}{36\pi} \times 5)}}$$

$$= 0.9487 \times 10^8 \text{ m/s}$$

Ex. 8.3.2 : An uniform plane wave is travelling at a velocity of 2.5×10^5 m/s having wavelength $\lambda = 0.25$ mm in a non-magnetic good conductor. Calculate the frequency of wave and the conductivity of a medium.

☑ **Soln. :**

i) To find frequency of wave :

The velocity of propagation is given by,

$$v = f\lambda$$

$$\therefore \quad f = \frac{v}{\lambda} = \frac{2.5 \times 10^5}{0.25 \times 10^{-3}} = 1 \times 10^9 \text{ Hz} = 1 \text{ GHz}$$

ii) To find conductivity of a medium :

Velocity of propagation is also given by,

$$v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta}$$

$$\therefore \quad \beta = \frac{2\pi f}{v} = \frac{2\times\pi\times 10^9}{2.5\times 10^5} = 25.1327\times 10^3$$

As given material is good conductor, the phase velocity β is given by,

$$\beta = \sqrt{\pi f \mu \sigma}$$

$$\beta = \sqrt{\pi f \mu_0 \sigma} \qquad \mu_r = 1$$

$$\therefore \quad 25.1327\times 10^3 = \sqrt{\pi\times 1\times 10^9\times 4\pi\times 10^{-7}\times\sigma}$$

$$\therefore \quad \sigma = 1.6\times 10^5 \text{ S/m}$$

Ex. 8.3.3 : In free space,

$E(z, t) = 10^3 \sin(\omega t - \beta z)\,\bar{a}_y$ (V / m). Obtain H (z, t).

☑ **Soln. :**

$$E(z, t) = 10^3 \sin(\omega t - \beta z)\,\bar{a}_y = E_y\,\bar{a}_y$$

$(\omega t - \beta z)$ indicates that wave is travelling in +z direction (negative sign in $-\beta z$ refers to positive direction of propagation while z refers to the direction of propagation).

For +z direction we have, $\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \eta$

i.e. $E_x = \eta H_y$

and $E_y = -\eta H_x$

Since given electric field has E_y component, it will result in H_x.

$$\therefore \quad \frac{E_y}{H_x} = -\eta$$

or

$$H_x = -\frac{E_y}{\eta} = -\frac{E_y}{120\pi} \qquad \ldots(\eta = 120\pi, \text{ for free space})$$

$$= \frac{-10^{-3}}{120\pi}\sin(\omega t - \beta z)$$

$$= -2.65\sin(\omega t - \beta z) \text{ (A/m)} \qquad \text{...Ans.}$$

Ex. 8.3.4 : An electric field E in free space is given as,

$\bar{E} = 1000\cos(10^8 t - \beta y)\,\bar{a}_z$ (V/m)

Find β, λ, $\bar{H}$ at P (0.2, 2.5, 0.4) at 10 nsec.

☑ **Soln. :**

Given : $\bar{E} = 1000\cos(10^8 t - \beta y)\,\bar{a}_z = E_z(y, t)\,\bar{a}_z$

Where $E_z = 1000\,(10^8 t - \beta y)$

From which we get,

$$\omega = 10^8 \text{ (rad/s)}$$

Also in free space,

$$v = c = 3\times 10^8 \text{ (m/s)}$$

and

$$\eta = 120\pi\ (\Omega)$$

(i) To find β :

$$\beta = \frac{\omega}{v} = \frac{10^8}{3\times 10^8} = 1/3 \text{ (rad/m)} \qquad \text{...Ans.}$$

(ii) To find λ :

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{(1/3)} = 6\pi = 18.85 \text{ (m)} \qquad \text{...Ans.}$$

So we write

$$\bar{E} = 100\cos\left(10^8 t - \frac{1}{3}y\right)\bar{a}_z = E_z\,\bar{a}_z$$

(iii) To find $\bar{H}$:

Given expression for E has only E_z component, travelling in + y direction. The electric and magnetic field components for wave in y direction are related by,

$$\frac{E_z}{H_x} = -\frac{E_x}{H_z} = \eta$$

Since E has only E_z component, it gives only H_x.

$$\therefore \quad H_x = \frac{E_z}{\eta} = \frac{1000\cos(10^8 t - \beta y)}{120\pi}$$

$$= 2.65\cos(10^8 t - \beta y)$$

$$\therefore \quad \bar{H} = H_x\,\bar{a}_x = 2.65\cos(10^8 t - \beta y)\,\bar{a}_x$$

$\bar{H}$ at P(0.2, 2.5, 0.4) and t = 10 ns is

$$\bar{H} = 2.65\cos\left(10^8\times 10\times 10^{-9} - \frac{1}{3}\times(2.5)\right)\bar{a}_x$$

$$= 2.65\cos(1 - 0.833)\,\bar{a}_x$$

$$= 2.65\cos(0.167)\,\bar{a}_x$$

The number 0.167 in the bracket is in radians. Converting in degrees

$$0.167 \text{ rad} = 0.167\times\frac{180}{\pi} = 9.568^\circ$$

$$\therefore \quad \bar{H} = 2.65\cos(9.568^\circ)\,\bar{a}_x$$

$$= 2.61\,\bar{a}_x \text{ (A / m)} \qquad \text{...Ans.}$$

UEx. 8.3.5 MU - Dec. 11, 12 Marks

A lossless dielectric medium has $\sigma = 0$, $\mu_r = 1$ and $\epsilon_r = 4$. An electromagnetic wave has magnetic field expressed as $\bar{H} = -0.1\cos(\omega t - z)\,\bar{a}_x + 0.5\sin(\omega t - z)\,\bar{a}_y$ (A/m) Find : (i) phase constant (ii) angular velocity (iii) wave impedance (iv) the components of electric field intensity of the wave.

☑ **Soln. :**

Given : $\bar{H} = -0.1\cos(\omega t - z)\,\bar{a}_x + 0.5\sin(\omega t - z)\,\bar{a}_y$

$$= \bar{H}_1 + \bar{H}_2$$

where, $\bar{H}_1 = -0.1\cos(\omega t - z)\,\bar{a}_x = H_1\cos(\omega t - z)\,\bar{a}_{H1}$

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$$\bar{H}_2 = 0.5 \sin(\omega t - z)\,\bar{a}_y = H_2 \sin(\omega t - z)\,\bar{a}_{H2}$$

From the given expression it is clear that the wave is propagating in + z direction, and phase constant $\beta = 1$. For the direction of propagation, $\bar{a}_\beta = \bar{a}_z$.

i) To find phase constant

As explained above, Phase constant,

$$\beta = 1 \quad \text{...Ans.}$$

ii) To find angular velocity

The phase velocity is given by,

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0(1)\,\epsilon_0(4)}} = \frac{c}{\sqrt{4}} = 1.5\times10^8 \text{ (m/s)}$$

we have $v = \frac{\omega}{\beta}$

$$\omega = v\beta = 1.5\times10^8\times1 = 1.5\times10^8 \text{ (rad/s)} \quad \text{...Ans.}$$

iii) To find wave impedance

The wave impedance or intrinsic impedance is,

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0(1)}{\epsilon_0(4)}} = \frac{120\pi}{\sqrt{4}} = 60\pi\ (\Omega)$$

iv) To find wave components of $\bar{E}$

Let the electric field be

$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

where,

$$\bar{E}_1 = E_1 \cos(\omega t - z)\,\bar{a}_{E1}$$

$$\bar{E}_2 = E_2 \sin(\omega t - z)\,\bar{a}_{E2}$$

E_1 and E_2 are obtained using,

$$\eta = \frac{E_1}{H_1} \rightarrow E_1 = \eta H_1 = 60\pi(0.1)$$

$$= 6\pi \text{ (V/m)}$$

and

$$\eta = \frac{E_2}{H_2} \rightarrow E_2 = \eta H_2$$

$$= 60\pi(0.5) = 30\pi \text{ (V/m)}$$

The directions $\bar{a}_{E1}$ and $\bar{a}_{E2}$ are obtained using

$\bar{a}_{E1}\times\bar{a}_{H1} = \bar{a}_\beta$	$\bar{a}_{E2}\times\bar{a}_{H2} = \bar{a}_\beta$
$\bar{a}_{E1}\times(-\bar{a}_x) = \bar{a}_z$	$\bar{a}_{E2}\times\bar{a}_y = \bar{a}_z$
$\rightarrow \bar{a}_{E1} = \bar{a}_y$	$\rightarrow \bar{a}_{E2} = \bar{a}_x$

So the electric field intensity is,

$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

$$= 6\pi\cos(\omega t - z)\,\bar{a}_y + 30\pi\sin(\omega t - z)\,\bar{a}_x$$

$$\bar{E} = 6\pi\cos(1.5\times10^8 t - z)\,\bar{a}_y + 30\pi\sin(1.5\times10^8 t - z)\,\bar{a}_x \text{ (V/m)} \quad \text{...Ans.}$$

Ex. 8.3.6 : A 150 MHz uniform plane wave in free space is travelling in $\bar{a}_x$ direction. The electric field intensity has a maximum amplitude of $200\,\bar{a}_y + 400\,\bar{a}_z$ (V/m) at P (10, 30, – 40) at t = 0. Find

i) Phase constant, ii) Wavelength,
iii) Phase velocity, iv) Characteristic impedance and
v) E (x, y, z)

Given : $\mu_0 = 4\pi\times10^{-7}$ and $\epsilon_0 = 10^{-9}/36\pi$

☑ **Soln. :**

Given : $f = 150\times10^6$ Hz

The wave is travelling in free space so velocity is

$$v = c = 3\times10^8 \text{ (m/s)}$$

The wave is travelling in x-direction, so

$$\bar{a}_\beta = \bar{a}_x$$

i) To find phase constant

We have,

$$v = \frac{\omega}{\beta} \rightarrow$$

$$\beta = \frac{\omega}{v} = \frac{2\pi\times150\times10^6}{3\times10^8} = \pi \text{ (rad/m)} \quad \text{...Ans.}$$

ii) To find wavelength

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\pi} = 2 \text{ (m)} \quad \text{...Ans.}$$

iii) To find phase velocity

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$

$$= 3\times10^8 \text{ (m/s)} \quad \text{...Ans.}$$

This value is very obvious since the wave is travelling in free space.

iv) To find characteristic impedance

$$\eta = \frac{\sqrt{\mu}}{\epsilon} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\ (\Omega) \quad \text{...Ans.}$$

This is nothing but intrinsic impedance of free space.

v) To find E

Since at t = 0, the amplitude is maximum it means the variation of E is in cosine manner and the wave is in x-direction, so

$$\bar{E} = (200\,\bar{a}_y + 400\,\bar{a}_z)\cos(\omega t - \beta x)$$

Putting the values of ω and β,

$$\bar{E} = (200\,\bar{a}_y + 400\,\bar{a}_z)\cos(300\pi\times10^6 t - \pi x) \text{ (V/m)} \quad \text{...Ans.}$$

UEx. 8.3.7 MU - Q. 3(b). Dec. 18, 10 Marks

Magnetic field component of an EM wave propagating through a non-magnetic medium ($\mu = \mu_0$) is :

$\bar{H} = 25 \sin(2 \times 10^8 t + 6x)\, \bar{a}_y$ mA/m

Determine

- The direction of wave propagation
- The permittivity
- Electric field

Soln. :

Expression for a wave in – x direction is $\bar{H} = H_0 \sin(\omega t + \beta x)\, \bar{a}_y$

Comparing with given expression,

(i) Direction of wave is – x

(ii) $\omega = 2 \times 10^8$ rad/s, $\beta = 6$

We have,

$$v = \frac{\omega}{\beta} = \frac{2 \times 10^8}{6} = \frac{c}{\sqrt{\epsilon_r}}$$

$$\therefore \sqrt{\epsilon_r} = \frac{3 \times 10^8 \times 6}{2 \times 10^8} = 9 \Rightarrow \epsilon_r = 81$$

The intrinsic impedance

$$\eta = \frac{120\,\pi}{\sqrt{\epsilon_r}} = \frac{40}{3}\pi = 41.89\ (\Omega)$$

But

$$\eta = \frac{|E|}{|H|} \Rightarrow |E| = \eta \times |H| = 41.89 \times 25 \times 10^{-3}$$

$$= 1.05\ (\text{V/m})$$

Also

$$\bar{a}_E \times \bar{a}_H = \bar{a}_\beta$$

$$\therefore \bar{a}_E \times \bar{a}_y = -\bar{a}_x \Rightarrow \bar{a}_E = \bar{a}_z$$

Then the electric field is,

$$\bar{E} = |E| \sin(2 \times 10^8 t + 6x)\, \bar{a}_z$$

$$\bar{E} = 1.05 \sin(2 \times 10^8\, t + 6x)\, \bar{a}_z\ (\text{V/m})$$

Ex. 8.3.8 : Define uniform plane wave. The electric field of a uniform plane wave in free space is given by,

$E(z, t) = \bar{a}_x\, 100 \sin(\omega t - \beta z)$ V/m.

(i) Find magnetic field.

(ii) Time average power flows through a rectangular area 3 cm × 15 cm is the z-plane.

Soln. :

i) **To find magnetic field H**

For this we required to calculate η, But as the medium is free space, the η is

$$\eta_0 = 377$$

and

$$\bar{E} = 100 \sin(\omega t - \beta z)\, \bar{a}_x \text{ V/m}$$

Then

The amplitude of $\bar{H}$ is given by,

$$H = \frac{E_0}{\eta_0} = \frac{100}{377}$$

The direction of the $\bar{H}$ is given as

$$\bar{a}_E \times \bar{a}_H = \bar{a}_\beta$$

$$\bar{a}_x \times \bar{a}_H = \bar{a}_z$$

This gives $\quad \bar{a}_H = \bar{a}_y$

Hence

$$\bar{H} = \frac{E}{\eta_0} \bar{a}_y = \frac{100}{377} \sin(\omega t - \beta z)\, \bar{a}_y \text{ V/m}$$

$$= 0.265 \sin(\omega t - \beta z)\, \bar{a}_y \text{ V/m}$$

ii) The average power flowing through any area is given by :

$$P_{avg} = \frac{E^2}{2\eta} S \text{ [watts]}$$

$$P_{avg} = \frac{(100)^2}{2 \times 377} [3 \times 10^{-2} \times 15 \times 10^{-2}]$$

$$= 59.68 \text{ mW}$$

UEx. 8.3.9 MU - May 10, May 12. 5 Marks

Give solution to the wave equation in perfect dielectric for a wave travelling in z-direction, which has only x-component of E-field.

Soln. :

Given :

Wave is travelling in z-direction and has only E_x component of field.

$$\bar{E} = E_x \bar{a}_x$$

The wave equation

$$\nabla^2 \bar{E} = \mu \epsilon \ddot{\bar{E}}$$

reduces to

$$\nabla^2 E_x = \mu \epsilon \frac{\partial^2}{\partial t^2} E_x \quad ...(A)$$

Also for a wave in z-direction, $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0.$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \rightarrow \frac{\partial^2}{\partial z^2}$$

Now Equation (A) becomes

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \epsilon \frac{\partial^2 E_x}{\partial t^2}$$

The solution of this equation will be

$$E_x = \sin(z - v_0 t) \quad ...(B)$$

where, $\quad v_0 = \frac{1}{\sqrt{\mu \epsilon}}$ = velocity of a wave

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Equation (B) is the required solution.

UEx. 8.3.10 MU - May 10, 10 Marks

A uniform plane waves propagates in a medium with $\epsilon_r = 4, \sigma = 0, \mu_r = 1$. The E-field has only x-component which is sinusoidal with a frequency of 100 MHz and has maximum value of 0.1 mV/m at $t = 0, z = 1/8$. Find instantaneous expression for E(z, t) and H(z, t). Also find the location where E_x is positive maximum at t = 10 ns.

☑ **Soln. :**

Given : $\epsilon_r = 4, \sigma = 0, \mu_r = 1$

$f = 100$ MHz $\rightarrow \omega = 2\pi f = 200\pi \times 10^6$ rad / m

$$\beta = \omega\sqrt{\mu\epsilon} = 200\pi \times 10^6 \sqrt{\mu_0 \epsilon_0 \epsilon_r} = \frac{200\pi \times 10^6}{3\times 10^8}\sqrt{4}$$

$$\rightarrow \beta = \frac{4\pi}{3} \text{ (rad/m)}$$

If we consider expression for E as

$$E_x(t) = E_{max}\cos(\omega t - \beta z)$$
$$= 0.1\times 10^{-3}\cos(\omega t - \beta z) \quad \text{...(i)}$$

$$\text{At } t = 0 \rightarrow E_x(0) = 0.1\times 10^{-3}\cos(-\beta z)$$
$$= 0.1\times 10^{-3}\cos(\beta z)$$
$$= 0.1\times 10^{-3}\cos\left(\frac{4\pi}{3}z\right)$$

It is given that E has maximum value of 0.1 mV / m at z = 1/8. That is $E_x(0)$ at z = 1/8 must be 0.1×10^{-3}. Putting the value of z,

$$E_x(0) = 0.1\times 10^{-3}\cos\left(\frac{4\pi}{3}\times\frac{1}{8}\right)$$
$$= 0.1\times 10^{-3}\cos\left(\frac{\pi}{6}\right) \neq 0.1\times 10^{-3}$$

Thus Equation (i) requires modification. Let

$$E_x(t) = 0.1\times 10^{-3}\cos(\omega t - \beta z + \theta)$$

We find θ by using the fact that for $E_x(t)$ to be maximum cosine function should have 1 value. For this the bracketed term must be zero or 2π etc.

$$\cos(\omega t - \beta z + \theta) = 1 \rightarrow \omega t - \beta z + \theta = 0$$

Putting the values of t = 0, β and z,

$$0 - \frac{4\pi}{3}\times\frac{1}{8} + \theta = 0$$
$$\rightarrow \theta = \frac{\pi}{6}$$

So the expression for E is

$$E(z,t) = 0.1\times 10^{-3}\cos\left(200\pi\times 10^6 t - \frac{4\pi}{3}z + \frac{\pi}{6}\right)$$
$$= E_x(z,t)$$

For a wave in z-direction E and H are related using

$$\frac{E_x}{H_y} = \eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0\epsilon_r}} = \frac{120\pi}{\sqrt{\epsilon_r}}$$
$$= 60\pi\ (\Omega)$$

or

$$H_y(z,t) = \frac{E_x}{60\pi} = \frac{0.1\times 10^{-3}}{60\pi}\cos\left(200\pi\times 10^6 t - \frac{4\pi}{3}z + \theta\right)$$

To find the position where E_x is positive maximum at t = 10 ns, set the cosine term equal to 1.

$$\cos\left(200\pi\times 10^6\times 10\times 10^{-9} - \frac{4\pi}{3}z + \frac{\pi}{6}\right) = 1$$

$$\rightarrow 200\pi\times 10^6\times 10\times 10^{-9} - \frac{4\pi}{3}z + \frac{\pi}{6} = 0$$

$$\rightarrow 2\pi - \frac{4\pi}{3}z + \frac{\pi}{6} = 0$$

$$\rightarrow \frac{4\pi}{3}z = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$$

$$\rightarrow z = \frac{13}{8}\text{ (m)}$$

UEx. 8.3.11 MU - Dec. 09, 10 Marks

Given that the electric field intensity of an electro-magnetic wave in non conducting dielectric medium with permittivity $\epsilon = 9\epsilon_0$ and permeability μ_0 is $\bar{E}(z,t) = \bar{a}_y 5\cos(10^9 t - \beta z)$ V/m Find the magnetic flied intensity $\bar{H}$ and the value of β.

☑ **Soln. :**

The $-\beta z$ term in the given expression for $\bar{E}$ shows that the wave is travelling in + z-direction.

The ratio of E and H component is

$$\frac{E_x}{H_y} = \eta = -\frac{E_y}{H_x}$$

The given electric field has only E_y term, therefore, the magnetic field consists of only H_x component, given by $H_x = -(E_y/\eta)$

Given : $\mu = \mu_0, \epsilon = 9\epsilon_0$. The intrinsic impedance is calculated as,

$$\therefore \quad \eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{9\epsilon_0}} = \frac{1}{3}\sqrt{\frac{\mu_0}{\epsilon_0}}$$
$$= \frac{1}{3}\times 120\pi = 40\pi\ (\Omega)$$

$$\therefore \quad H_x = -\frac{E_y}{\eta} = -\frac{5}{40\pi}\cos(10^9 t - \beta z)$$

$$\therefore \quad \bar{H} = -\frac{1}{8\pi}\cos(10^9 t - \beta z)\,\bar{a}_x \quad \text{(A/m)}$$

From the given expression, $\omega = 10^9$ (rad/sec) or

$$f = \frac{\omega}{2\pi} = \frac{10^9}{2\pi}\text{ (Hz)}$$

The velocity of propagation is, $v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{3\sqrt{\mu_0\epsilon_0}} = \frac{v_0}{3}$

$$= 10^8 \text{ (m/s)} \quad \text{...Ans.}$$

The phase constant is, $\beta = \frac{\omega}{v} = \frac{10^9}{10^8} = 10$ (rad / sec)

...Ans.

UEx. 8.3.12 MU - May 09, 10 Marks

A 10 GHz plane wave travelling in free space has an amplitude of $E_x = 10$ V/m. Find v, λ, β, η and the amplitude and direction of H.

☑ **Soln. :** We know that velocity is obtained by,

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ (m/s)}$$

We have $\lambda = \frac{v}{f} = \frac{3 \times 10^8}{10 \times 10^9} = 30 \times 10^{-3}$ (m)

We know that $\beta = \frac{\omega}{v} = \frac{2\pi \times 10 \times 10^9}{3 \times 10^8}$

$$= 0.209 \times 10^3 \text{ (rad / m)}$$

The intrinsic impedance is obtained by,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi\ (\Omega)$$

Let us assume that the wave is travelling in +z direction, so that

$$\frac{E_x}{H_y} = \eta$$

The component of H in $\bar{a}_y$ direction is,

$$H_y = \frac{E_x}{\eta} = \frac{10}{120\pi} = 26.53 \text{ (mA/m)}$$

UEx. 8.3.13 MU - Q. 4(b), May 19, 10 Marks

Aircraft antenna radiates Electric field in air ($\sigma = 0, \mu = \mu_0, \epsilon = \epsilon_0$) which is $\bar{E} = 25 \cos(10^9 t + 0.33 x)\, \bar{a}_y$ KV/m find out following terms related with this EM System :

- Propagation constant (k)
- Phase velocity
- Intrinsic Impedance (η)
- Average Poyting Power
- Magnetic Field ($\bar{H}$)

☑ **Soln. :**

Given : $\bar{E} = 25 \cos(10^9 t + 0.33 x)\, \bar{a}_y$ (kV/m)

$\sigma = 0, \mu = \mu_0, \epsilon = \epsilon_0$

The velocity is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c = 3 \times 10^8 \text{ (m/s)}$$

(1) $\beta = \frac{\omega}{v} = \frac{10^9}{3 \times 10^8} = 3.33$ (rad/m)

(2) $v = 3 \times 10^8$ (m/s)

(3) $\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\ (\Omega)$

(4) $P_{avg} = \frac{|E|^2}{2\eta} = \frac{25^2}{2 \times 120\pi} = 0.829$ (W)

(5) $\eta = \frac{|\bar{E}|}{|\bar{H}|} \Rightarrow |\bar{H}| = \frac{|\bar{E}|}{120\pi} = \frac{25}{120\pi} = 0.066$ (A/m)

Also, $\bar{a}_E \times \bar{a}_H = \bar{a}_\beta \Rightarrow \bar{a}_y \times \bar{a}_H = -\bar{a}_x$

$\therefore \bar{a}_H = -\bar{a}_z$

Then the magnetic field is

$$\bar{H} = -0.066 \cos(10^9 t + 0.33 x)\, \bar{a}_z \text{ (A/m)}$$

Module 2

8.4 WAVE PROPAGATION IN LOSSY MEDIA (CONDUCTING MEDIA)

- This media is characterized by ($\sigma \neq 0, \epsilon = \epsilon_0 \epsilon_r, \mu = \mu_0 \mu_r$).
In sections 8.2 and 8.3 we have studied wave propagation in lossless media. Due to zero loss ($\sigma = 0$), the amplitude of these waves remain constant (does not decrease).
- Then it is very obvious that when wave travels in lossy media ($\sigma \neq 0$), the energy of the wave should continuously decrease, causing amplitude of the wave decrease continuously.
- Except the amplitude, all other properties of the wave, namely v, λ. β are same as that for lossless media. To obtain the results mathematically first obtain wave equations for this media.

8.4.1 Wave Equations for Lossy Media

UQ. Derive general wave equations for $\bar{E}$ and $\bar{H}$-fields.

MU - May 10, May 12, Dec. 12, 5/10 Marks

Starting with Maxwell's equations,

$$\nabla \times \bar{H} = \bar{J} + \dot{\bar{D}} \quad \text{... (i)}$$

$$\nabla \times \bar{E} = -\dot{\bar{B}} \quad \text{... (ii)}$$

$$\nabla \cdot \bar{D} = 0 \quad \text{i.e. } \nabla \cdot \bar{E} = 0 \quad \text{... (iii)}$$

$$\nabla \cdot \bar{B} = 0 \quad \text{i.e. } \nabla \cdot \bar{H} = 0 \quad \text{... (iv)}$$

also

$$\bar{D} = \epsilon \bar{E} \quad \therefore \dot{\bar{D}} = \epsilon \dot{\bar{E}} \quad \text{... (v)}$$

$$\bar{B} = \mu \bar{H} \quad \therefore \dot{\bar{B}} = \mu \dot{\bar{H}} \quad \text{... (vi)}$$

$$\bar{J} = \sigma \bar{E} \quad \therefore \dot{\bar{J}} = \sigma \dot{\bar{E}} \quad \text{... (vii)}$$

(A) Wave equation for electric field
(B) Wave equation for magnetic field

(A) Wave equation for electric field

We have, $\nabla \times \bar{E} = -\dot{\bar{B}}$

Taking curl of both sides we get,

$$\nabla \times \nabla \times \bar{E} = -\nabla \times \dot{\bar{B}} = -\nabla \times (\mu \dot{\bar{H}})$$

$$= -\mu (\nabla \times \dot{\bar{H}}) = -\mu (\dot{\bar{J}} + \ddot{\bar{D}})$$

$$= -\mu (\sigma \dot{\bar{E}} + \epsilon \ddot{\bar{E}}) \quad \text{...(a)}$$

Using vector identity :

$$\nabla \times \nabla \times \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E}$$

but $\nabla \cdot \bar{E} = 0$

Then vector identity reduces to

$$\nabla \times \nabla \times \bar{E} = -\nabla^2 \bar{E} \quad \text{... (b)}$$

Comparing Equation (a) with Equation (b) we get,

$$\nabla^2 \bar{E} = \mu (\sigma \dot{\bar{E}} + \epsilon \ddot{\bar{E}}) \quad \text{... (8.4.1)}$$

This is called as general wave equation for electric field in lossy media.

(B) Wave equation for magnetic field

We have $\nabla \times \bar{H} = \bar{J} + \dot{\bar{D}}$

Taking curl of both sides

$$\nabla \times \nabla \times \bar{H} = \nabla \times (\bar{J} + \dot{\bar{D}}) = (\nabla \times \bar{J}) + (\nabla \times \dot{\bar{D}})$$

$$= [\nabla \times (\sigma \bar{E})] + [\nabla \times (\epsilon \dot{\bar{E}})] \quad \text{...from (v) and (vii)}$$

$$= \sigma (\nabla \times \bar{E}) + \epsilon (\nabla \times \dot{\bar{E}})$$

$$= \sigma (-\dot{\bar{B}}) + \epsilon (-\ddot{\bar{B}})$$

$$= -\mu \sigma \dot{\bar{H}} - \epsilon \mu \ddot{\bar{H}} \quad \text{... from (vi)}$$

$$= -\mu (\sigma \dot{\bar{H}} + \epsilon \ddot{\bar{H}}) \quad \text{... (c)}$$

Using vector identity : $\nabla \times \nabla \times \bar{H} = \nabla (\nabla \cdot \bar{H}) - \nabla^2 \bar{H}$

but $\nabla \cdot \bar{B} = 0 \quad \therefore \nabla \cdot \bar{H} = 0$

Then vector identity reduces to :

$$\nabla \times \nabla \times \bar{H} = -\nabla^2 \bar{H} \quad \text{... (d)}$$

Comparing Equation (c) with Equation (d) we get,

$$\nabla^2 \bar{H} = \mu (\sigma \dot{\bar{H}} + \epsilon \ddot{\bar{H}}) \quad \text{... (8.4.2)}$$

This is called as **general wave equation for magnetic field in lossy media.**

8.4.2 Helmholtz Equations for Lossy Media (Wave Equations in Phasor form or in Frequency Domain)

For the sinusoidal time varying fields the wave equation for the conducting media can be written by replacing dot (·) by jω as

$$\nabla^2 \bar{E} = \mu (j\omega\sigma + j\omega \cdot j\omega\epsilon) \bar{E}$$

i.e. $$\nabla^2 \bar{E} = j\omega\mu (\sigma + j\omega\epsilon) \bar{E} \quad \text{... (8.4.3)}$$

Similarly, we get,

$$\nabla^2 \bar{H} = j\omega\mu (\sigma + j\omega\epsilon) \bar{H} \quad \text{... (8.4.4)}$$

Let $$\boxed{\gamma^2 = j\omega \mu (\sigma + j\omega\epsilon)}$$

This term is a complex number and under root of it is also complex.

i.e. $\gamma \Rightarrow$ complex number $= \alpha + j\beta$

The significance of α and β, will be seen in the next section. Then Equations (8.4.3) and (8.4.4) are rewritten as,

$$\boxed{\begin{aligned} \nabla^2 \bar{E} &= j\omega\mu (\sigma + j\omega\epsilon) \bar{E} = \gamma^2 \bar{E} \\ \nabla^2 \bar{H} &= j\omega\mu (\sigma + j\omega\epsilon) \bar{H} = \gamma^2 \bar{H} \end{aligned}} \quad \text{...(8.4.5)}$$

*"Equation (8.4.5) is expression for **Helmholtz equations** for lossy dielectric."*

These equations are referred as Helmholtz equations or wave equations in phasor form or wave equations in frequency domain for lossy media.

Letting the term $\sigma = 0$, the above equations reduce

to $$\nabla^2 \bar{E} = -\omega^2 \mu\epsilon \bar{E} = -\beta^2 \bar{E}$$

and $$\nabla^2 \bar{H} = -\omega^2 \mu \epsilon \bar{E} = -\beta^2 \bar{H}$$

Which are Helmholtz equations for lossless media, already derived in section 8.2.6.

8.4.3 Expression for α and β for Lossy Media

In the previous section we have come across the term γ_2 which is

$$\gamma^2 = j\omega \mu (\sigma + j\omega\epsilon)$$

but $$\gamma = \alpha + j\beta$$

$$(\alpha + j\beta)^2 = j\omega \mu (\sigma + j\omega\epsilon) = j\omega \mu \sigma - \omega^2 \mu\epsilon$$

expanding square bracket $\alpha^2 - \beta^2 + 2j\alpha\beta = j\omega\mu\sigma - \omega^2\mu\epsilon \quad \text{...(i)}$

Equating real and imaginary parts we get,

$$\beta^2 - \alpha^2 = \omega^2 \mu\epsilon \quad \text{...(ii)}$$

and $$2\alpha\beta = \omega\mu\sigma \quad \text{...(iii)}$$

Now, $$\alpha = \omega\mu\sigma/(2\beta) \quad \text{...(iv)}$$

Putting Equation (iv) in Equation (ii),

$$\text{we get } \beta^2 - \frac{\omega^2\mu^2\sigma^2}{4\beta^2} = \omega^2\mu\epsilon$$

$$4\beta^4 - 4\omega^2\mu\epsilon\,\beta^2 - \omega^2\mu^2\sigma^2 = 0$$

Solving the quadratic equation for β^2 we get,

$$\beta^2 = \frac{4\,\omega^2\mu\epsilon \pm \sqrt{16\,\omega^4\mu^2\epsilon^2 + 16\,\omega^2\mu^2\sigma^2}}{8}$$

Taking only positive root,

$$\beta^2 = \frac{\omega^2\mu\epsilon}{2} + \frac{\omega^2\mu\epsilon}{2}\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}$$

i.e. $$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}+1\right)} \quad ...(8.4.6)$$

Similarly, we can obtain expression for α as

i.e. $$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)} \quad ...(8.4.7)$$

8.4.4 Intrinsic Impedance for Lossy Media

- Consider the electromagnetic wave is propagating in x direction, then

$$\bar{E} = E_y\,\bar{a}_y + E_z\,\bar{a}_z \quad ...(i)$$

and $$\bar{H} = H_y\,\bar{a}_y + H_z\,\bar{a}_z \quad ...(ii)$$

For sinusoidal time varying field, Maxwell's equation are :

$$\nabla\times\bar{H} = \bar{J} + j\omega\bar{D} = (\sigma + j\omega\epsilon)\bar{E} \quad ...(iii)$$

and $$\nabla\times\bar{E} = -j\omega\bar{B} = -j\omega\mu\bar{H} \quad ...(iv)$$

Expanding Equation (iv) and remembering that $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$ and $E_x = 0$

$$\nabla\times\bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & 0 & 0 \\ 0 & E_y & E_z \end{vmatrix} = -j\omega\mu(H_y\bar{a}_y + H_z\,\bar{a}_z)$$

i.e. $$-\frac{\partial}{\partial x}E_z\,\bar{a}_y + \frac{\partial}{\partial x}E_y\,\bar{a}_z = -j\omega\mu\,(H_y\,\bar{a}_y + H_z\,\bar{a}_z)$$

comparing coefficients :

$$\frac{\partial E_z}{\partial x} = j\omega\mu H_y \quad ...(v)$$

and $$\frac{\partial E_y}{\partial x} = -j\omega\mu H_z \quad ...(vi)$$

- Equations (v) and (vi) are having same form, once we obtain solution for one, we can easily write solution for other. Equation (vi) is a differential equation whose solution can be assumed as

$$E_y = E_{yo}\,e^{-\gamma x} \quad \text{where } \gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\epsilon)} \quad ...(8.4.8)$$

Then $$\frac{\partial E_y}{\partial x} = -\gamma\,E_{yo}\,e^{-\gamma x} = -\gamma E_y$$

But form Equation (vi) $$\frac{\partial E_y}{\partial x} = -\gamma E_y = -j\omega\mu H_z$$

i.e. $$\frac{E_y}{H_z} = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

Similarly, we can prove that

$$\frac{E_z}{H_y} = -\sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

The quantity in the radical is denoted by η.

Thus $$E_z = -\eta H_y \quad\text{and}\quad E_y = \eta H_z$$

Squaring and adding, we get,

$$E_y^2 + E_z^2 = \eta^2(H_y^2 + H_z^2)$$

i.e. $$|E|^2 = \eta^2|H|^2$$

$\therefore$ $$\eta = \frac{|E|}{|H|}$$

- This is called as **intrinsic impedance** of the medium. Thus, for the sinusoidal time varying field the intrinsic impedance of media is given by,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}\;(\Omega) \quad ...(8.4.9)$$

This η can be expressed in magnitude an angle form

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

$$= \sqrt{\frac{j\omega\mu}{j\omega\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)}} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1 - j\frac{\sigma}{\omega\epsilon}}}$$

$$= \frac{\sqrt{\mu/\epsilon}}{\sqrt{\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}\angle\tan^{-1}\left(-\frac{\sigma}{\omega\epsilon}\right)}}$$

$$= \frac{\sqrt{\mu/\epsilon}}{\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/4}}\angle\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right) = |\eta|\angle\theta_n$$

where, $$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/4}}$$ and

$$\theta_n = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right) \text{ or } \tan(2\theta_n) = \frac{\sigma}{\omega\epsilon}$$

- In electrical circuit the voltage (V) and current (I) are in phase only when impedance is resistive. If it is reactive then some phase difference exist between V and I. Since E and H are related through η, when it is complex then phase difference θ_n is present between two fields. So when

$$E = E_0 e^{-\alpha x} \cos(\omega t - \beta x)$$

then the magnetic field is

$$H = H_0 e^{-\alpha x} \cos(\omega t - \beta x - \theta_n)$$

Where $H_0 = \dfrac{E_0}{|\eta|}$ and $\theta_n = \tan^{-1}\left(\dfrac{\sigma}{\omega\epsilon}\right)$

8.4.5 Behaviour of a Wave in Lossy Media

The Helmholtz equations for this media are

$$\nabla^2 \bar{E} = j\omega\mu(\sigma + j\omega\epsilon)\bar{E} = \gamma^2 \bar{E} \quad ...(i)$$

and $$\nabla^2 \bar{H} = j\omega\mu(\sigma + j\omega\epsilon)\bar{H} = \gamma^2 \bar{H} \quad ...(ii)$$

where $$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) \quad ...(8.4.10)$$

- In Equation (i) assume that E is varying w.r.t. x only then we get, y component of E as

$$\frac{\partial^2 E_y}{\partial x^2} = \gamma^2 E_y \quad ...(iii)$$

The above differential equation has one possible solution as

$$E_y = E_{y0} e^{-\gamma x} \quad ...(8.4.11)$$

In time varying form

$$\tilde{E}_y(x,t) = \text{R.P. } [E_y e^{j\omega t}] = \text{R.P. of } [E_{y0} e^{-\gamma x} \cdot e^{j\omega t}] \quad ...(iv)$$

In Equation (8.4.4), γ^2 has real as well as imaginary term, therefore, assuming $\gamma = \alpha + j\beta$, we get,

$$\tilde{E}_y(x,t) = \text{R.P. of } [E_{y0} e^{-\alpha x} e^{-j\beta x} e^{j\omega t}]$$

$$= \text{R.P. of } [E_{y0} e^{-\alpha x} e^{j(\omega t - \beta x)}]$$

$$= e^{-\alpha x} \text{ R.P. of } [E_{y0} e^{j(\omega t - \beta x)}]$$

$$= e^{-\alpha x} E_{y0} \cos(\omega t - \beta x) \quad ...(v)$$

- Let us rewrite the result for lossless dielectric

$$\tilde{E}_y = E_{y0} \cos(\omega_t - \beta_x)$$

- Comparing it with Equation (v) in this section, we find an additional term $e^{-\alpha x}$ in the expression for lossy dielectric. In the above equation the amplitude of cosine term is E_{y0} which is a constant, but for lossy dielectric it is $E_{y0}\, e^{-\alpha x}$ Equation (v).
- The term α is a positive constant (will be proved in next section) and when wave propagates in + x direction, x increases continuously.
- The result is $e^{-\alpha x}$ term decreases continuously. It means when wave travels in lossy media, the amplitude the wave ($E_{y0} e^{-\alpha x}$) decreases continuously or in other words the wave gets attenuated in the direction of propagation. It is shown in Fig. 8.4.1.

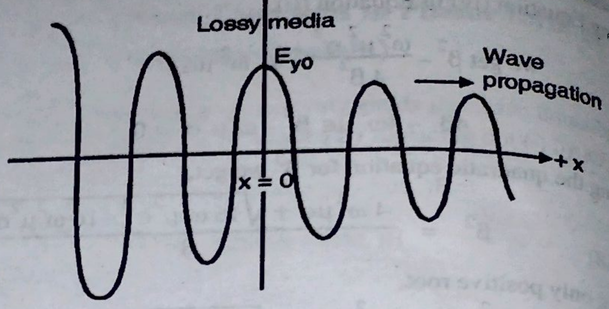

Fig. 8.4.1 : Wave propagation in lossy dielectric

- The rate of attenuation is decided by a constant α called as **attenuation constant** and β is called as **phase shift constant.** Because attenuation and phase shift is associated with propagation, the term γ, (which consists of α and β) is called **propagation constant.**
- Except the attenuation the other wave properties remain the same as for the pure dielectric media.

i.e. $\beta = 2\pi/\lambda$ and $v = f\lambda = \omega/\beta$

8.4.6 The Terms γ, α and β

UQ. Define the following terms :
1) Propagation constant
2) Attenuation constant
3) Phase constant

MU - May 16, 12 Marks

(i) Propagation constant (γ)

When the wave is propagating in a lossy media, it results in a wave equations. (Equations (8.4.3), (8.4.4))

$$\nabla^2 \bar{E} = j\omega\mu(\sigma + j\omega\epsilon)\bar{E}$$

$$\nabla^2 \bar{H} = j\omega\mu(\sigma + j\omega\epsilon)\bar{H}$$

Letting the term

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$$

We get

$$\nabla^2 \bar{E} = \gamma^2 \bar{E}$$

and $$\nabla^2 \bar{H} = \gamma^2 \bar{H}$$

where γ is called as propagation constant.

The solution of these differential equations will be in the form (Equation (8.4.11))

$$E = E_0 e^{-\gamma x} \quad ...(i)$$

Since γ_2 is a complex number, then γ is also complex.

$$\gamma = \alpha + j\beta \quad ...(ii)$$

Where α = attenuation constant (Np/m)

β = phase constant (rad/m).

Putting Equation (ii) in Equation (i) we get,

$$E = E_0 e^{-(\alpha + j\beta)x}$$

$$= E_0 e^{-\alpha x} \cdot e^{-j\beta x} \quad ...(iii)$$

This expression says that when the wave propagates in + x direction in a lossy media.

(i) It gets attenuated due to α

(ii) Its phase angle changes due to β.

(ii) Attenuation constant (α)

- As seen in Equation (iii) above, when the wave is travelling in a lossy media it gets attenuated. The rate of attenuation is decided by α. Hence α is called an attenuation constant.
- More the value of α, more the attenuation is. It is measured in (Np/m).
- For lossless media obviously α = 0. The value of α is given by Equation (8.4.7).

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)}$$

(iii) Phase constant (β)

From Equation (iii) it is clear that when the wave propagates in + x direction, its phase angle continuously changes. This is due to the term β called as phase constant. It is measured in (rad/m). The value of β is given by Equation (8.4.6).

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}+1\right)}$$

☞ Significance of propagation constant

In general γ (= α + jβ) signifies that when the wave propagates it gets attenuated in amplitude and the phase angle of the wave changes.

☞ Significance of attenuation constant

The non zero value of α in a lossy media signifies that the amplitude of wave decreases continuously as the wave propagates.

☞ Significance of phase constant

When the wave is propagating its phase is continuously changing indicated by β.

8.5 MATHEMATICAL DEFINITION OF THE DIELECTRIC MEDIA AND CONDUCTING MEDIA

UQ. What is "loss Tangent". Explain how it classifies lossless dietlectrics, lossy Dielectric and good conductor. **MU - Dec. 14, 5 Marks**

In electromagnetics, materials are divided roughly into two classes

(1) Conductors and (2) Dielectrics or insulators.

- The dividing line between the two classes is not sharp and some media (the earth 'for example) are considered as conductors in one part of the radio frequency range, but as dielectric (with loss) in another part of the range.

In Maxwell's first equation : $\nabla \times \bar{H} = \sigma\bar{E} + j\omega\epsilon\bar{E}$

- The first term on the RHS is the conduction current density and the second term is the displacement current density. If we take ratio of conduction current density to displacement current density we get, the ratio as,

$$\frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\epsilon}$$

Naturally

For good conductor : $\frac{\sigma}{\omega\epsilon} \gg 1$ ...(8.5.1)

For good dielectric : $\frac{\sigma}{\omega\epsilon} \ll 1$

and $\frac{\sigma}{\omega\epsilon} = 1$ is considered as dividing line between conductor and dielectric.

Computing values of α and β from the given data using Equations (8.4.11) and (8.4.6) definitely is time consuming and labourious. Instead of that I am sure you will be very happy if you have simplified expressions for α and β. The simplification can definitely be done provided that the medium is a good conductor or good dielectrics. Now we will concentrate on modifying Equations (8.4.6) and (8.4.7) as follows.

The ratio $J_c / J_d = \frac{\sigma}{\omega_\epsilon}$ is called as loss tangent.

8.6 WAVE PROPAGATION IN GOOD DIELECTRIC

UQ. Write short note on : Wave propagation through lossy dielectrics. **(MU - Q. 6(d), Dec. 18, 5 Marks)**

Using Equation (8.5.1) we can obtain modified expressions for α, β velocity (v), and intrinsic impedance (η) as follows :

8.6.1 Modification of α and β for Good Dielectric

Using the definition given in Equation (8.5.1), α and β can be modified as given below. As for good dielectric (σ/ωϵ) << 1, we can use the approximation as

$$\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} = \left(1+\frac{\sigma^2}{2\omega^2\epsilon^2}\right)$$

Since, for a << 1, $\sqrt{1+a^2} = \left(1+\frac{a^2}{2}\right)$

Using this in expression of α, we get

$$\alpha \cong \omega\sqrt{\frac{\mu\epsilon}{2}\left(1+\frac{\sigma^2}{2\omega^2\epsilon^2}-1\right)}$$

i.e. $$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \quad ...(8.6.1)$$

Now for $$\beta \cong \omega\sqrt{\frac{\mu\epsilon}{2}\left(1+\frac{\sigma^2}{2\omega^2\epsilon^2}+1\right)}$$

$$= \omega\sqrt{\mu\epsilon\left(1+\frac{\sigma^2}{4\omega^2\epsilon^2}\right)}$$

Now using approximation

$$\sqrt{1+a^2} = \left(1+\frac{a^2}{2}\right)$$

$$\beta = \omega\sqrt{\mu\epsilon}\left[1+\frac{\sigma^2}{8\,\omega^2\epsilon^2}\right] \quad ...(8.6.2)$$

For free space ($\sigma = 0, \mu = \mu_0, \epsilon = \epsilon_0$)

For free space we have conductivity σ = 0. Thus Equations (8.6.1) and (8.6.2) reduces to

$$\alpha = 0$$

$$\beta = \omega\sqrt{\mu_0\epsilon_0}$$

and $$\gamma = \alpha + j\beta = j\beta$$

8.6.2 Velocity in Good Dielectric

Now the velocity of wave,

$$v = \omega/\beta = \frac{1}{\sqrt{\mu\epsilon}\left[1+\frac{\sigma^2}{8\,\omega^2\epsilon^2}\right]}$$

Again using approximation for a << 1 as $\frac{1}{1+a} = (1-a)$

$$v = \frac{1}{\sqrt{\mu\epsilon}}\left(1-\frac{\sigma^2}{8\,\omega^2\epsilon^2}\right)$$

i.e. $$v = v_0\left[1-\frac{\sigma^2}{8\,\omega^2\epsilon^2}\right] \quad ...(8.6.3)$$

8.6.3 η in Good Dielectric

Taking jωε common from denominator

$$\eta = \sqrt{\frac{\mu/\epsilon}{[1+(\sigma/j\omega\epsilon)]}} \quad ...(A)$$

For a good dielectric, (σ / ωε) << 1, Then

$$\eta = \sqrt{\frac{\mu}{\epsilon}}\left[1-\frac{\sigma}{j\omega\epsilon}\right] \quad ...\text{using}$$

approximation, $\frac{1}{1+a} = 1-a$

or $$\eta = \sqrt{\frac{\mu}{\epsilon}}\left[1+j\frac{\sigma}{2\,\omega\epsilon}\right] \quad ...(8.6.4)$$

whenever a small amount of loss is present in the intrinsic impedance a small reactive component gets added.

For free space ($\sigma = 0, \mu = \mu_0, \epsilon = \epsilon_0$)

For free space Equation (8.6.4) reduces to

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \Omega$$

8.7 WAVE PROPAGATION IN GOOD CONDUCTOR

In the previous sections we studied the propagation is good dielectric, free space and lossy dielectric. Its time now study propagation in good conductor.

For good dielectrics we have, (σ / ωε) >> 1

8.7.1 To find α and β for Good Conductor

Using expression for γ as

$$\gamma^2 = j\omega\mu\,[\sigma + j\omega\epsilon] = j\omega\mu\sigma\left[1+\frac{j\omega\epsilon}{\sigma}\right]$$

$$= j\omega\mu\sigma \quad \text{as ... } (\sigma/\omega\epsilon) >> 1 \quad ...(a)$$

i.e. $$\gamma = \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma\angle 90}$$

or $$\gamma = \sqrt{\omega\mu\sigma}\ \angle 45° \quad ...(8.7.1)$$

Now $$\gamma = \alpha + j\beta \quad ...(b)$$

Squaring Equation (b) and equating with Equation (a)

$$\gamma^2 = \alpha^2 - \beta^2 + 2j\alpha\beta = j\omega\mu\sigma$$

comparing real parts $\alpha^2 - \beta^2 = 0$ i.e. $\alpha = \beta$

and comparing imaginary terms

$$2\alpha\beta = \omega\mu\sigma$$

i.e. $$2\alpha^2 = \omega\mu\sigma$$

∴ $$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} \quad ...(8.7.2)$$

and $$\beta = \sqrt{\frac{\omega\mu\sigma}{2}} \quad ...(8.7.3)$$

8.7.2 Velocity of Propagation for Good Conductor

$$v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega\mu\sigma/2}}$$

or

$$v = \sqrt{\frac{2\omega}{\mu\sigma}} \qquad ...(8.7.4)$$

8.7.3 Intrinsic Impedance for Good Conductor

Now for a good conductor, $(\sigma/\omega\epsilon) >> 1$, then Equation (A) becomes

$$\eta \approx \sqrt{\frac{\mu}{\epsilon} \times \frac{j\omega\epsilon}{\sigma}}$$

$$= \sqrt{\frac{j\omega\mu}{\sigma}} \quad \text{...because 1 can be neglected.}$$

$$\therefore \quad \eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

$$(\text{As } \sqrt{jy} = \sqrt{y\angle\pi/2} = \sqrt{y}\angle\pi/4) \quad ...(8.7.5)$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}}(1 + j1)$$

Then $\quad \text{Re}(\eta) = \sqrt{\frac{\omega\mu}{2\sigma}}$

and $\quad \text{Im}(\eta) = \sqrt{\frac{\omega\mu}{2\sigma}}$

- Note that η has phase angle 45°, which reflects on to the phase difference between E and H

$$E = E_0 e^{-\alpha x}\cos(\omega t - \beta x)$$

$$H = \frac{E_0}{|\eta|} e^{-\alpha x}\cos(\omega t - \beta x - \pi/4)$$

- While solving the problems on finding α, β, γ, η we can use simplified expressions which saves the time as well as labour. But be careful because these expressions are used only if conditions for good conductor or good dielectric are satisfied, which are

for good conductor $\frac{\sigma}{\omega\epsilon} >> 1$ and for good dielectrics $\frac{\sigma}{\omega\epsilon} << 1$

- The general expressions which we derived earlier as well as simplified expressions for α, β, γ, v and η are given below for fast reference while solving the problems.

<table>
<tr><th colspan="2">General Expressions</th></tr>
<tr><td colspan="2">$E_y = E_{y0} e^{-\gamma x}$; $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$
$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)}$; $\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}+1\right)}$
$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$</td></tr>
<tr><th colspan="2">Simplified Expressions</th></tr>
<tr><th>For good dielectrics $\left(\frac{\sigma}{\omega\epsilon} >> 1\right)$</th><th>For good conductor $\left(\frac{\sigma}{\omega\epsilon} << 1\right)$</th></tr>
<tr><td>$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\omega}}$
$\beta = \omega\sqrt{\mu\epsilon}\left[1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right]$
$v = v_0\left[1-\frac{\sigma^2}{8\omega^2\epsilon^2}\right]$
$\eta = \sqrt{\frac{\mu}{\epsilon}}\left[1+j\frac{\sigma}{2\omega\epsilon}\right]$</td><td>$\gamma = \sqrt{\omega\mu\sigma}\angle 45^\circ$
$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$
$v = \sqrt{\frac{2\omega}{\mu\sigma}}$
$\eta = \sqrt{\frac{\mu\omega}{\sigma}}\angle 45^\circ$</td></tr>
</table>

UEx. 8.7.1 **MU - Dec. 02, 10 Marks**

A medium has the following parameters :

$\mu_r = 10$, $\epsilon_r = 2.5$, $\sigma = 10^{-4}$ mho / m. Determine α, β, λ, v, η for 1 GHz.

☑ **Soln. :** We have propagation constant

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$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \sqrt{j \times 2\pi \times 1 \times 10^9 \times 10 \times 4\pi \times 10^{-7} \times \left(10^{-4} + j \times 2\pi \times 1 \times 10^9 \times 2.5 \times \frac{10^{-9}}{36\pi}\right)}$$

$$= \sqrt{j\,78956.8\,(10^{-4} + j\,0.1391)}$$

$$= \sqrt{78956.8 \angle 90° \,(0.1391 + \angle 89.96°)} = 104.80 \angle 89.98° = 0.0366 + j\,104.80$$

But $\gamma = \alpha + j\beta$ $\therefore$ $\alpha = 0.0366$ (Np/m) and $\beta = 104.80$ (rad/m)

Now, $\lambda = \frac{2\pi}{\beta} = 0.05995$ (m) $\quad v = \frac{\omega}{\beta} = \frac{2\pi \times 1 \times 10^9}{104.80} = 56.95 \times 10^6$ (m/s)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\frac{j \times 2\pi \times 1 \times 10^9 \times 4\pi \times 10^{-7} \times 10}{10^{-4} + j \times 2\pi \times 1 \times 10^9 \times 8.854 + 10^{-12} \times 2.5}}$$

$$\eta = \sqrt{\frac{78956.8 \angle 90°}{0.1391 \angle 89.96°}} = 753.41 \angle 0.02 = 753.41 + j\,0.263\ (\Omega)$$ ...Ans.

UEx. 8.7.2 MU - May 14, Dec. 15, 10 Marks

A media has the following properties $\mu_r = 8$, $\epsilon_r = 2$, $\sigma = 10^{-4}$ mho/m at 2 GHz. Determine :

(i) Attenuation constant (ii) Attenuation constant in dB (iii) Phase constant (iv) Propagation constant
(v) Wavelength (vi) Phase velocity (vii) Intrinsic impedance (viii) Refractive index
(ix) Loss tangent (x) Is the media behaving like a conductor or dielectric?

☑ **Soln. :** f = 2 GHz $\rightarrow \omega = 2\pi f = 1.2566 \times 10^{10}$ (rad/s)

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \sqrt{j \times 1.257 \times 10^{10} \times 8 \times 4\pi \times 10^{-7}\left(10^{-4} + j1.257 \times 10^{10} \times 2 \times \frac{10^{-9}}{36\pi}\right)}$$

$\gamma = 0.0377 + j\,167.67 = \alpha + j\beta$

(i) $\therefore \alpha = 0.0377$ (Np/m)

(ii) $(\alpha)_{dB} = 8.686 \times 0.0377 = 0.3275$ (dB)

(iii) $\beta = 167.67$ (rad/m)

(iv) $\gamma = \alpha + j\beta = 0.0377 + j\,167.67\ (m^{-1})$

(v) $\lambda = \frac{2\pi}{\beta} = 0.0375$ (m)

(vi) $v_p = \frac{\omega}{\beta} = \frac{1.2566 \times 10^{10}}{167.67} = 74.9 \times 10^6$ (m/s)

(vii) $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$

(viii) Refractive index,

$= 753.47 + j0.1693\ (\Omega)$

$$n = \frac{c}{v} = \frac{3 \times 10^8}{74.9 \times 10^6} = 4$$

(ix) loss tangent, $\tan \Delta = \frac{\sigma}{\omega\epsilon} = \frac{10^{-4}}{1.257 \times 10^{10} \times 2 \times \frac{10^{-9}}{36\pi}} = 0.000449$

(x) Since $\sigma/\omega\epsilon \ll 1$, the medium is a lossy dielectric.

8.8 SKIN DEPTH (DEPTH OF PENETRATION)

UQ. What do you mean by depth of penetration ?
MU - May 14, Dec. 14, Dec. 15, 5 Marks

UQ. Write a short note on : Skin effect.
(MU - Q. 6(f), Dec. 19, 5 Marks)

- In a good conductor where σ is very large, both α and β are also large. Thus, as the wave progresses through a medium which has conductivity, the wave is attenuated because of losses.
- In a good conductor at radio frequencies the rate of attenuation is very great and the wave may penetrate only a very short distance before being reduced to a negligibly small percentage of its original strength.

8.8.1 Definition of Depth of Penetration

The term depth of penetration is defined as that depth in which the wave has been attenuated to 1/e i.e. approximately 37% of its original value. The depth of penetration is also called as skin depth.

- Consider a wave that is transmitted into the conducting medium. Let x = 0 is the boundary of conducting medium with x increasing positively into the conducting medium as shown in Fig. 8.8.1.

We have,
$$E_y = E_{y0}\, e^{-\gamma x}$$
i.e.
$$E_y = \underbrace{E_0\, e^{-\alpha x}}_{\text{Amplitude}} \times \underbrace{e^{-j\beta x}}_{\text{Phase}}$$

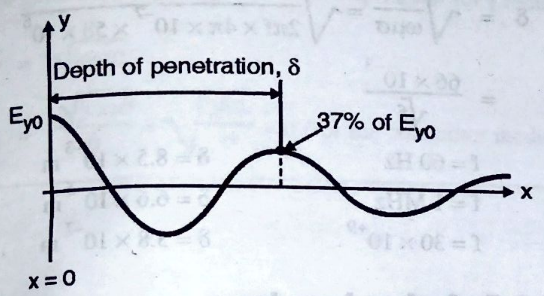


Fig. 8.8.1 : Showing attenuation of a wave in lossy medium

- It gives variation of E_y in both magnitude and phase as a function of x. The $e^{-\alpha x}$ term indicates attenuation. Thus, when

$$x = 0,$$
$$E_y = E_{y0}$$

and at
$$x = x_1 = 1/\alpha,$$
$$E_y = (E_{y0}/e)$$

i.e. we can say that at $\alpha x_1 = 1$, the wave amplitude is 1/e times its value at x = 0. Thus, by definition, x_1 is the depth of penetration, denoted by δ.

i.e
$$\alpha\delta = 1$$
or
$$\delta = 1/\alpha$$

Thus, the general expression for depth of penetration using Equation (8.4.7) is

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)}} \quad ...(8.8.1)$$

For good conductor, using the modified expression of α

$$\delta = \frac{1}{\alpha} \simeq \sqrt{\frac{2}{\omega\mu\sigma}} \quad ...(8.8.2)$$

Significance of Skin Depth (Effect of Skin Depth)

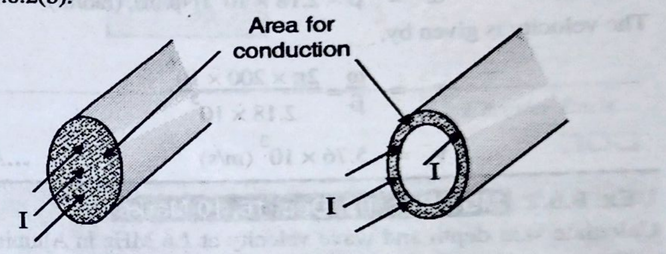
Skin effect

At low frequencies the current is uniformly distributed through the cross section of the conductor. But as the frequency is increased it will have tendency to flow only near to the skin of the conductor. This effect is called skin effect. From the expression of δ in Equation (8.8.2) it is clear that :

At low frequencies - δ is large

At high frequencies - δ is small

When low frequency current is passed through conductor, almost total area of the conductor carries the current as shown in Fig. 8.8.2(a). But if the frequency is increased due to δ is small, the current will flow through the conductor in a region near to the conductor surface (which is the skin of the conductor) as shown in Fig. 8.8.2(b).



(a) Low frequency (b) High frequency current through conductor

Fig. 8.8.2

8.8.2 Solved Examples on Depth of Penetration

Important Formulae

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)}}$$

$$\delta = \frac{1}{\alpha} \simeq \sqrt{\frac{2}{\omega\mu\sigma}}$$

Ex. 8.8.1 Calculate intrinsic impedance η, the propagation constant γ, the wave velocity v for a conducting medium in which $\sigma = 60$ Ms/m, $\mu_r = 1$, at a frequency f = 200 MHz.

Soln. : It is given that the medium is a conductor. Let us check whether it is a good conductor or not.

$$\frac{\sigma}{\omega\epsilon} = \frac{60\times 10^6}{2\pi\times 200\times 10^6\times(10^{-9}/36\pi)}$$
$$= 5.4\times 10^9 >> 1$$

So the given media is a good conductor. Using approximate relations,

$$\gamma = \sqrt{\omega\mu\sigma}\,\angle 45^\circ$$

$$= \sqrt{2\pi \times 200 \times 10^6 (4\pi \times 10^{-7}) \times 60 \times 10^6}\,\angle 45^\circ$$

$$= 307.81 \times 10^3 \angle 45^\circ\ (m^{-1})$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ = \sqrt{\frac{2\pi \times 200 \times 10^6 \times 4\pi \times 10^{-7}}{60 \times 10^6}}$$

$$\angle 45^\circ = 5.130 \times 10^{-3} \angle 45^\circ\ (\Omega)$$

The calculated γ can be written as,

$$\gamma = 307.81 \times 10^3 \angle 45^\circ$$

$$= 2.18 \times 10^5 + j\,2.18 \times 10^5$$

Equating with $\alpha + j\beta$ we get,

$$\alpha = \beta = 2.18 \times 10^5\ \text{(Np/m), (rad/m)}$$

The velocity is given by,

$$v = \frac{\omega}{\beta} = \frac{2\pi \times 200 \times 10^6}{2.18 \times 10^5}$$

$$v = 5.76 \times 10^3\ \text{(m/s)} \qquad \text{...Ans.}$$

UEx. 8.8.2 MU - Dec. 09, Dec. 16, 10 Marks

Calculate skin depth and wave velocity at 1.6 MHz in Aluminium with conductivity 38.2 mS/m and $\mu_r = 1$.

☑ **Soln. :**

Given : $\sigma = 40$ mS/m, $\quad \mu_r = 1, f = 2$ MHz

Initially check that Aluminium is a good conductor

$$\frac{\sigma}{\omega\epsilon} = \frac{38.2 \times 10^6}{2\pi \times 1.6 \times 10^6 \times \left(\frac{10^{-9}}{36\pi}\right)} = 4.3 \times 10^{11} >> 1$$

Thus aluminium is a good conductor, for which

$$\gamma = \sqrt{\omega\mu\sigma}\,\angle 45^\circ$$

$$= \sqrt{(2\pi \times 1.6 \times 10^6 \times 4 \times \pi \times 10^{-7} \times 1 \times 38.2 \times 10^6)}\,\angle 45^\circ$$

$$= \sqrt{4.83 \times 10^8}\,\angle 45^\circ = 21.967 \times 10^3 \angle 45^\circ$$

$$\gamma = 15.45 \times 10^3 + j15.45 \times 10^3$$

We know

$$\gamma = \alpha + j\beta$$

By comparing real and imaginary terms we get,

$$\alpha = 15.45 \times 10^3\ N_p/m$$

$$\beta = 15.45 \times 10^3\ \text{rad/m}$$

Skin depth for aluminium is given by,

$$\delta = \frac{1}{\alpha} = \frac{1}{15.45 \times 10^3} = 64.73\ \mu m$$

Velocity of wave propagation is given by,

$$v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\pi \times 1.6 \times 10^6}{15.45 \times 10^3} = 651.53\ m/s$$

Ex. 8.8.3 : Find the depth of penetration if for copper $\mu_r = 1$, $\sigma = 58$ Mmho/m at different frequencies f = 60 Hz, 1 MHz and 30 GHz.

☑ **Soln. :**

First of all check whether Cu is a good conductor at the highest frequency,

i.e.

$$\frac{\sigma}{\omega\epsilon} = \frac{58 \times 10^6}{2\pi \times 30 \times 10^9 \times (10^{-9}/36\pi)}$$

$$= 34.8 \times 10^5 \geq 1$$

Thus, Cu is a good conductor. Using the expression for δ for good conductor.

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi f \times 4\pi \times 10^{-7} \times 58 \times 10^6}}$$

$$= \frac{66 \times 10^{-3}}{\sqrt{f}}$$

at

$f = 60$ Hz $\qquad \delta = 8.5 \times 10^{-3}$ m

$f = 1$ MHz $\qquad \delta = 6.6 \times 10^{-5}$ m

$f = 30 \times 10^{+9}$ $\qquad \delta = 3.8 \times 10^{-7}$ m $\qquad$ ...Ans.

8.8.3 Surface Impedance

The surface impedance is defined as the ratio of the tangential component of the electric field E_{tan} at the surface of a conductor and the linear current density J_s which flows as a result of this electric field. It is denoted by Z_s.

$$Z_s = \frac{E_{tan}}{J_s}$$

The current density J_s is the current per meter width flowing in thin sheet.

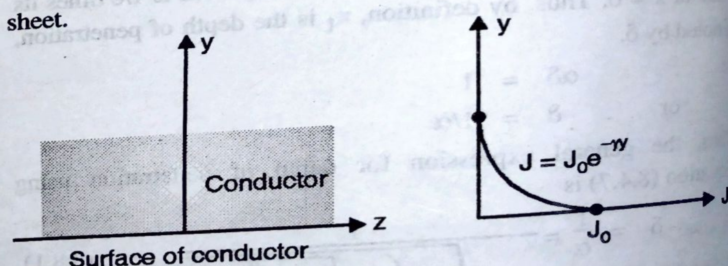


Fig. 8.8.3

In the Fig. 8.8.3, conductor is a flat plate with its surface at y = 0 plane. The current distribution in the y direction is

$$J = J_0\, e^{-\gamma y}$$

where, J_0 is the current density at the surface. Also it is assumed that the thickness of the conductor is very much greater than depth of penetration, so that there is no reflection from the back surface of the conductor.

The total current density,

$$J_s = \int_0^\infty J\,dy = J_0 \int_0^\infty e^{-\gamma y}\,dy = \frac{-J_0}{\gamma}\left(e^{-\gamma y}\right)_0^\infty$$

$$= \frac{J_0}{\gamma}$$

But

$$J_0 = \sigma E_{tan}$$

$$\therefore\ Z_s = \frac{E_{tan}}{J_s} = \frac{E_{tan}}{J_0/\gamma} = \frac{E_{tan}}{\sigma E_{tan}/\gamma} = \frac{\gamma}{\sigma}$$

Now

$$\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\epsilon)}$$

Since, for good conductor, $\frac{\omega\epsilon}{\sigma} << 1$ the expression for γ reduces to

$$\gamma = \sqrt{j\omega\mu\sigma}$$

$$\therefore\ Z_s = \frac{\sqrt{j\omega\mu\sigma}}{\sigma} = \sqrt{\frac{j\omega\mu}{\sigma}} = \eta \text{ (for the conductor medium)}$$

Thus, for a good conductors, the surface impedance of a plane conductor that is very much thicker than the skin depth is just equal to the characteristic impedance of the conductor.

$$\therefore\ Z_s = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\,\angle 45^\circ = \sqrt{\frac{\omega\mu}{2\sigma}}\,(1 + j1)$$

Separating real and imaginary parts

Surface resistance $= R_s = \sqrt{\omega\mu/2\sigma}$

and Surface reactance $= X_s = \sqrt{\omega\mu/2\sigma}$

For good conductor the depth of penetration is $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$

Using the expression for δ, the surface resistance becomes

$$\therefore\ \boxed{R_s = \frac{1}{\sigma\delta}} \quad ...(8.8.3)$$

Module 2

Chapter Ends...

□□□

Module 2

CHAPTER 9

Poynting Vector

9.1 INTRODUCTION

In the transmitter the wave is generated by a source, then it travels to a distant receiving point. Whenever a wave travels from one point to other, transfer of energy takes place. There is a simple and direct relation between the rate of the energy transfer and the amplitudes of electric and magnetic intensities associated with a travelling electromagnetic wave.

- This relation is known as Poynting vector after I.H. Poynting.
- Many of us think of the Poynting vector as a "Pointing" vector. This homonym, while accidental, is correct.

9.2 POYNTING THEOREM

UQ. Define Poynting vector. Obtain the integral form of Poynting theorem and explain each term.

MU - May 09, Dec. 09, May 12, 10 Marks

UQ. State the Poynting theorem. Write its final expression hence explain the meaning of each term.

MU - May 14, Dec. 16, 5 Marks, Q. 4(a), Dec. 19, Q. 5(b), Dec. 18, 10 Marks

UQ. State Poynting theorem. What is Poynting vector?

MU - May 15, 5 Marks

UQ. Describe Poynting Theorem and explain various terms associated with the same.

MU - May 16, 5 Marks

UQ. Derive Poynting Vector and explain effects of medium parameters on EM power with suitable diagram. **(MU - Q. 5(b), May 19, 10 Marks)**

UQ. Write short note on : Poynting theorem.

MU - May 11, Dec. 12, 10 Marks

UQ. State the Poynting theorem and explain meaning of each term.

MU - Dec. 15, 5, Marks, Q. 4(a), Dec. 19, 10 Marks

9.2.1 Statement of Poynting Theorem

Vector product of electric intensity $\bar{E}$ and magnetic intensity $\bar{H}$ at any point is a measure of the rate of energy flow per unit area at that point.

$$\bar{P} = \bar{E} \times \bar{H} \text{ (Watts/m}^2)$$

9.2.2 Mathematical Expression of Poynting Theorem

Mathematically the theorem can be expressed as,

$$\bar{P} = \bar{E} \times \bar{H}$$

The direction of flow is perpendicular to $\bar{E}$ and $\bar{H}$ in the direction of the vector $\bar{E} \times \bar{H}$. The relation can be obtained from Maxwell's equations as follows. The magnetomotive force equation is,

$$\nabla \times \bar{H} = \bar{J} + \dot{\bar{D}} \qquad \text{...(9.2.1)}$$

or

$$\bar{J} = \nabla \times \bar{H} - \epsilon \dot{\bar{E}}$$

In order to convert $\bar{J}$ from the dimension of current density to power per unit volume, multiply both sides of the above expression by $\bar{E}$ we have,

$$\bar{E} \cdot \bar{J} = \bar{E} \cdot (\nabla \times \bar{H}) - \epsilon \bar{E} \cdot \dot{\bar{E}} \qquad \text{... (9.2.2)}$$

We have vector identity,

$$\nabla \cdot (\bar{P} \times \bar{Q}) = \bar{Q} \cdot (\nabla \times \bar{P}) - \bar{P} \cdot (\nabla \times \bar{Q})$$

where, $\bar{P}$ and $\bar{Q}$ are any vector functions. Replacing $\bar{P}$ by $\bar{E}$ and $\bar{Q}$ by $\bar{H}$,

$$\nabla \cdot (\bar{E} \times \bar{H}) = \bar{H} \cdot (\nabla \times \bar{E}) - \bar{E} \cdot (\nabla \times \bar{H})$$

Putting the value of $\bar{E} \cdot (\nabla \times \bar{H})$ from above equation in Equation (9.2.2),

$$\bar{E} \cdot \bar{J} = \bar{H} \cdot (\nabla \times \bar{E}) - \nabla \cdot (\bar{E} \times \bar{H}) - \epsilon \bar{E} \cdot \dot{\bar{E}}$$

Using the second Maxwell's equation $\nabla \times \bar{E} = -\mu \dot{\bar{H}}$

in the expression for $\bar{E} \cdot \bar{J}$ we get,

$$\bar{E} \cdot \bar{J} = -\mu \bar{H} \cdot \dot{\bar{H}} - \epsilon \bar{E} \cdot \dot{\bar{E}} - \nabla \cdot (\bar{E} \times \bar{H})$$

Now, $$\bar{H} \cdot \dot{\bar{H}} = \frac{1}{2} \frac{\partial}{\partial t} H^2 \quad \text{and} \quad \bar{E} \cdot \dot{\bar{E}} = \frac{1}{2} \frac{\partial}{\partial t} E^2$$

So that, $$\bar{E} \cdot \bar{J} = -\frac{\mu}{2} \frac{\partial}{\partial t} H^2 - \frac{\epsilon}{2} \frac{\partial}{\partial t} E^2 - \nabla \cdot (\bar{E} \times \bar{H})$$

Integrating over a volume v,

$$\int_v (\bar{E} \cdot \bar{J})\, dv = -\frac{\partial}{\partial t} \int_v \left(\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2} \right) dv - \int_v \nabla \cdot (\bar{E} \times \bar{H})\, dv \qquad \text{...(9.2.3)}$$

Using the divergence theorem the last term can be changed from volume integral to surface integral,

$$\int_v \nabla \cdot (\bar{E} \times \bar{H})\, dv = \oint_s (\bar{E} \times \bar{H}) \cdot d\bar{s}$$

Then Equation (9.2.3) becomes,

$$\int_V (\bar{E} \cdot \bar{J})\, dv = -\frac{\partial}{\partial t} \int_V \left(\frac{\mu}{2} H^2 + \frac{\epsilon}{2} E^2\right) dv - \oint_S (\bar{E} \times \bar{H}) \cdot \overline{ds} \quad ...(9.2.4)$$

Equation (9.2.4) shows that the instantaneous power flow contains stored energy and the total power flowing in the volume.

9.2.3 A Physical Interpretation of Poynting Theorem

A physical interpretation of the above equation leads to some interesting conclusions :

(i) The term $\int_V (\bar{E} \cdot \bar{J})\, dv$ on the left hand side represents the (instantaneous) **power dissipated** in the **volume v**. This result is obtained as a generalization of Joule's law.

- A conductor of cross-sectional area A, carrying a current I and having a voltage drop E per unit length (V/m) will have a power loss of EI watt per unit length.
- The power dissipated per unit volume will be EI/A = EJ watt per unit volume. In this case V and $\bar{J}$ are in the same direction.
- In general where this may not be true, the power dissipated per unit volume would still be given by the product of $\bar{J}$ and the component of $\bar{E}$ having the same direction as $\bar{J}$ or $\bar{E} \cdot \bar{J}$.
- The total power dissipated in the volume v would be

$$\int_V (\bar{E} \cdot \bar{J})\, dv \quad ...(9.2.5)$$

When E in this expression represents the electric intensity required to produce the current density $\bar{J}$ in the conducting medium, the Equation (9.2.5) represents power dissipated as ohmic (I^2R) loss.

- However, if $\bar{E}$ is an electric intensity due to a source of power, e.g., due to a battery, then the power represented by Equation (9.2.5) would be used up in driving the current against the battery voltage and hence charging the battery.
- If the direction of $\bar{E}$ were opposite to that $\bar{J}$ the dissipated power represented by Equation (9.2.5) would be negative. In this case, the battery would be generating electric power.

(ii) The first term $-\frac{\partial}{\partial t} \int_V \left(\frac{\mu}{2} H^2 + \frac{\epsilon}{2} E^2\right) dv$ on the right hand side of Equation (9.2.4) without the negative time derivative **represents the stored (electric and magnetic) energy in the volume.**

- **The negative time derivative of this quantity then represents the rate at which the stored energy in the volume is decreasing.**

(iii) The interpretation of the remaining term $-\oint_S (\bar{E} \times \bar{H}) \cdot \overline{ds}$ follows from the application of the law of conservation of energy.

- **The rate of energy dissipation in the volume v must equal the rate at which the stored energy in v is decreasing plus the rate at which energy is entering the volume v from outside.**
- The term $-\oint_S (\bar{E} \times \bar{H}) \cdot \overline{ds}$
- Therefore represents the rate of flow of energy inward through the surface of the volume. Then this expression without the negative sign, i.e.

$$\int_S (\bar{E} \times \bar{H}) \cdot \overline{ds}$$

- Represents the **rate of flow of energy outward through the surface enclosing the volume.** Thus, the integral of $\bar{E} \times \bar{H}$ over any surface gives the rate of energy flow through that surface. It is seen that the vector

$$\bar{P} = \bar{E} \times \bar{H} \quad ...(9.2.6)$$

has the dimensions of **watt / sq.m.**

Equation (9.2.6) is the expression Poynting vector of $\bar{P}$.

9.2.4 Significance of Poynting Vector

GQ. Explain significance of Poynting vector. **(5 Marks)**

- The Poynting vector is expressed as,

$$\bar{P} = \bar{E} \times \bar{H}$$

- Its unit is watt/sq.m.
- In Poynting theorem the vector product $\bar{P} = \bar{E} \times \bar{H}$ at any point is a measure of the rate of energy flow per unit area at that point.
- The direction of flow is perpendicular to $\bar{E}$ and $\bar{H}$ in the direction of the vector $\bar{E} \times \bar{H}$. The direction of the vector $\bar{P}$ indicates the direction of the instantaneous power flow at the point, and many of us think of the Poynting vector as a "Pointing" vector. This homonym, while accidental, is correct.

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> **Note :** Poynting vector physically denotes the power density leaving the given volume in time varying field.

- The energy stored in the system is represented by energy storing elements like capacitor and inductor. The power dissipated is accounted by resistor. The diagrammatic representation is shown in Fig. 9.2.1.
- From the physical interpretation now it is clear that the Poynting theorem relates, the incoming energy, outgoing energy, energy stored and dissipated in the system. In other words the **"Poynting theorem gives the energy balance in the system or it proves law of conservation of energy".**

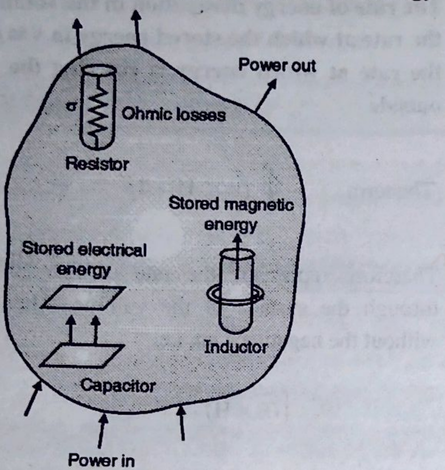

Fig. 9.2.1 : Power balance for EM fields

9.3 APPLICATIONS OF POYNTING VECTOR

The Poynting vector is used in EM wave analysis as discussed below.

9.3.1 Power Flow for a Plane Wave

> **GQ.** Derive expression for power flow for a plane wave. **(5 Marks)**

We know that energy density due to electric and magnetic fields is given by

$$\frac{1}{2}(\in E^2 + \mu H^2)$$

When a wave is travelling it travels with a velocity $v_0 = \frac{1}{\sqrt{\mu \in}}$

For a wave moving with a velocity v_0 the rate of flow of energy per unit area would be

$$P = \frac{1}{2}(\in E^2 + \mu H^2)\, v_0$$

The electric field (E) and magnetic field (H) for a plane wave are related by

$$\frac{E}{H} = \sqrt{\frac{\mu}{\in}}$$

Using this equation, the equation previous to it can be written as

$$P = \frac{1}{2}\left(\in \sqrt{\frac{\mu}{\in}} EH + \mu \sqrt{\frac{\in}{\mu}} EH\right) v_0 = \left(\frac{EH}{v_0}\right) v_0$$

$$= \bar{E} \times \bar{H}$$

This gives us power flow for a plane wave.

9.3.2 Power Flow in a Concentric Cable

- Consider a concentric cable with conductors having negligible resistance. The radius of the inner conductor is 'a' and the inside radius of the out conductor is 'b'.
- This cable is used to transfer power to a load resistance R along a cable. The voltage between conductors is V and a steady current I is flowing in the inner and outer conductor.

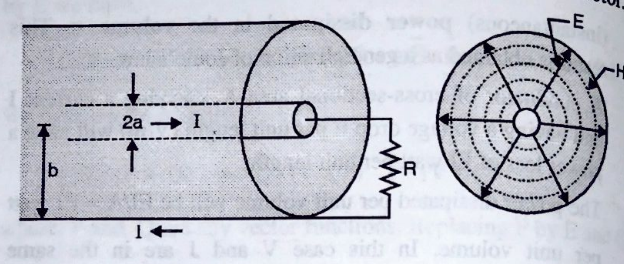

Fig. 9.3.1 : Power flow in a cable

- The magnetic field strength H will be directed in circles about its axis. By Ampere's law

$$\oint \bar{H} \cdot d\bar{l} = I$$

- Since, H is constant along any of the circular paths so

$$\oint \bar{H} \cdot d\bar{l} = 2\pi\rho H$$

where r is the radius of the circle path considered.

$$\therefore \quad H = \frac{I}{2\pi\rho}$$

- The voltage between two conductors can be shown to be

$$V = \frac{q}{2\pi\in} \log\frac{b}{a}$$

where q is the charge per unit length. The electric field is radial and is given by

$$E = \frac{q}{2\pi\in\rho}$$

$\therefore$ E and V are related by $\quad E = \frac{V}{\rho \log (b/a)}$

The Poynting vector is $\quad \bar{P} = \bar{E} \times \bar{H}$

Since, E and H are everywhere perpendicular the cross product simplifies to P = E H.

- The power flow along the cable is obtained by integrating Poynting vector over any cross-sectional surface.

$$W = \int_s (\bar{E}\times\bar{H})\cdot d\bar{s} = \int_a^b \frac{V}{a \log (b/a)}\left(\frac{I}{2\pi\rho}\right) 2\pi\rho\, d\rho$$

$$= \frac{VI}{\log (b/a)}\int_a^b \frac{d\rho}{\rho} = VI$$

- This is the well-known result that the power flow along the cable is the product of the voltage and current.

9.4 INSTANTANEOUS, AVERAGE AND COMPLEX POYNTING VECTOR

We know that in an AC circuit, the instantaneous power $\tilde{W}$ is always given by the product of the instantaneous voltage $\tilde{V}$ and the instantaneous current I.

$$\tilde{W} = \tilde{V}\tilde{I}$$

The quantities $\tilde{V}$ and $\tilde{I}$ are obtained by multiplying V and I by $e^{j\omega t}$ and taking real part i.e.

$$\tilde{V} = \text{Re}\{V e^{j\omega t}\} = \text{Re}\{|V| e^{j\theta_v} e^{j\omega t}\}$$

$$= |V| \cos(\omega t + \theta_v) \quad ...(i)$$

$$\tilde{I} = \text{Re}\{I e^{j\omega t}\} = \text{Re}\{|I| e^{j\theta_i} e^{j\omega t}\} = |I| \cos(\omega t + \theta_i) \quad ...(ii)$$

Now, the instantaneous power is expressed as

$$\tilde{W} = |V||I| \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

$$= \frac{|V||I|}{2}[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)] \quad ...(iii)$$

The right hand side of above equation consists of an average part and an oscillating part. If we define $\theta_v - \theta_i = \theta$ (see Fig. 9.4.1) we may write the average power as,

$$W_{av} = \frac{|V||I|}{2}\cos\theta \quad ...(iv)$$

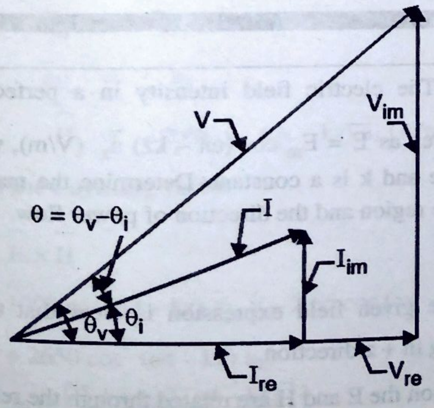

Fig. 9.4.1 : To find average and complex vector

Another useful quantity is the "reactive power" or more accurately, the reactive volt-amperes (VAR).

$$W_{react} = \frac{|V||I|}{2}\sin\theta$$

Since, θ is the phase angle between the voltage and the current, W_{av} and W_{react} are the *in-phase* and *out-of-phase* components of the volt-amp product.

Consider now the complex power W, defined as one-half the product of V and the complex conjugate of I.

$$W = \frac{1}{2}VI^* = \frac{1}{2}|V| e^{j\theta_v}\, |I| e^{-j\theta_i} = \frac{|V||I|}{2}e^{j\theta}$$

$$= W_{av} + jW_{react}$$

The average and reactive power may be recovered from the complex power by taking the real and imaginary parts.

$$W_{av} = \frac{1}{2}\text{Re}\{VI^*\}$$

$$W_{react} = \frac{1}{2}\text{Im}\{VI^*\}$$

9.4.1 Complex Poynting Vector

In electromagnetic field theory there are relations similar to those between the Poynting vector, the electric field strength and the magnetic field strength. The instantaneous power flow per square

Instantaneous : $\tilde{P} = \tilde{E}\times\tilde{H}$ ...(9.4.1)

Complex Poynting vector $\bar{P}$ can be expressed as

Complex : $\bar{P} = \frac{1}{2}\bar{E}\times\bar{H}^*$ ...(9.4.2)

from which we may obtain the average and reactive parts of the power flow per square meter.

Average or active : $P_{av} = \frac{1}{2}\text{Re}\{\bar{E}\times\bar{H}^*\}$ ...(9.4.3)

Reactive : $P_{react} = \frac{1}{2}\text{Im}\{\bar{E}\times\bar{H}^*\}$ ...(9.4.4)

The product of **E** and **H** in equation is a vector product. We know, mutually perpendicular components of **E** and **H** contribute to power flow and the direction of the flow is normal to the plane containing **E** and **H**. In rectangular co-ordinates, the complex flow of power per unit area normal to the y-z plane is

$$P_x = \frac{1}{2}(E_y H_z^* - E_z H_y^*)$$

with corresponding expressions for the other directions. In spherical co-ordinates, the outward (radial) flow of complex power per unit area is

$$P_r = \frac{1}{2}(E_\theta H_\phi^* - E_\phi H_\theta^*)$$

9.5 AVERAGE POYNTING VECTOR EXPRESSION

- As per the poynting theorem, the term $-\oint(\bar{E} \times \bar{H}) \cdot \bar{ds}$ represents the rate of energy flow inward through the surface of volume. Here negative sign indicates inward flow.
- So the term $+\oint(\bar{E} \times \bar{H}) \cdot \bar{ds}$ represents the energy flow in the outward direction through the surface enclosing the volume.
- The rate of energy flow at any point is given by the vector product of $\bar{E}$ and $\bar{H}$. It is denoted by $\bar{P}$ and $\bar{P}$ is called as the Poynting vector.

$$\therefore \quad \bar{P}_z = \bar{E} \times \bar{H} \quad \text{...(i)}$$

- The direction of this energy flow is perpendicular to both $\bar{E}$ and $\bar{H}$ which is in the direction of vector $\bar{E} \times \bar{H}$.
- Consider uniform plane wave travelling in the positive z direction.

$$\therefore \quad \bar{P}_z = P_z \bar{a}_z \quad \text{...(ii)}$$

This wave is having $\bar{E}$ component in the x direction,

$$\bar{E} = E_x \bar{a}_x \quad \text{...(iii)}$$

and $\bar{H}$ component in the y direction,

$$\bar{H} = H_y \bar{a}_y \quad \text{...(iv)}$$

Putting Equations (ii), (iii) and (iv) in Equation (i) to get

$$\bar{P} = P_z \bar{a}_z = (E_x \bar{a}_x) \times (H_y \bar{a}_y)$$

In case of perfect dielectric, $\bar{E}$ and $\bar{H}$ are given as

$$E_x = E_{x0} \cos(\omega t - \beta z) \quad \text{...(v)}$$

and

$$H_y = H_{y0} \cos(\omega t - \beta z) \quad \text{...(vi)}$$

But $\quad H_{y0} = \dfrac{E_{x0}}{\eta}$

Thus, we have Equation (vi) as, $H_y = \dfrac{E_{x0}}{\eta} \cos(\omega t - \beta z) \quad \text{...(vii)}$

$$\therefore \quad \bar{P}_z = \bar{E} \times \bar{H} = \frac{E_{x0}^2}{\eta} \cos^2(\omega t - \beta z)\, \bar{a}_z \quad \text{...(viii)}$$

By taking integration of Equation (viii) over one cycle and dividing by the period (T), we can have time average power density which is denoted by $P_{z,\,avg}$, we get,

$$\bar{P}_{z,\,avg} = \frac{E_{x0}^2}{2\eta} \bar{a}_z \ (\text{W/m}^2) \quad \text{...(9.5.1)}$$

The magnitude of it is

$$P_{z,\,avg} = \frac{E_{x0}^2}{2\eta} \quad \text{...(9.5.2)}$$

$$P = \int_s \bar{P}_{z,\,avg} \cdot \bar{ds}$$

In general average power in watts crossing the given area is,

$$\boxed{P = \int_s \bar{P}_{avg} \cdot \bar{ds} \quad \text{(W)}} \quad \text{...(9.5.3)}$$

For the special case when power flow is normal to the area,

$$P = \int_s P_{av}\, ds = P_{av} \int ds = P_{avg}\, S \quad \text{...(9.5.4)}$$

Thus, the average power flowing through any area 'S' is obtained by,

or

$$\boxed{P_{z,\,avg} = \frac{E_{x0}^2}{2\eta} S \quad \text{(Watts)}} \quad \text{...(9.5.5)}$$

9.6 SOLVED EXAMPLES ON POYNTING THEOREM

Important Formulae

$$\bar{P} = \bar{E} \times \bar{H} \qquad P = \frac{1}{2} E \times H^*$$

$$P_{avg} = \frac{1}{2} \text{Re}\{E \times H^*\}$$

$$P_{react} = \frac{1}{2} \text{Im}\{E \times H^*\}$$

$$P = \int_s \bar{P}_{avg} \cdot \bar{ds} \quad \text{(W)}$$

$$P_{z,\,avg} = \frac{E_{x0}^2}{2\eta} s \ \text{(Watts)}$$

Ex. 9.6.1 : The electric field intensity in a perfect dielectric medium is given as $\bar{E} = E_m \cos(\omega t - kz)\, \bar{a}_x$ (V/m), where E_m is the peak value and k is a constant. Determine the magnetic filed intensity in the region and the direction of power flow.

☑ **Soln. :**

(i) From the given field expression is clear that the wave is travelling in + z direction.

For this direction the E and H are related through the relation

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \eta$$

Since the given component of E is E_x, it will result in H_y.

$$H_y = \frac{E_x}{\eta} = \frac{E_m}{\eta} \cos(\omega t - kz)$$

or $$\bar{H} = H_y \bar{a}_y = \frac{E_m}{\eta} \cos(\omega t - kz)\, \bar{a}_y \quad \text{(A/m)}$$

Note that for perfect dielectric ($\sigma = 0$), the intrinsic impedance is real indicating E and H are in phase.

(ii) The direction of power flow is obtained using Poynting vector

$$\bar{P} = \bar{E} \times \bar{H} = E_x \bar{a}_x \times H_y \bar{a}_y = E_x H_y \bar{a}_z$$

Thus the power flow is in z-direction. It should be, because the wave is in + z direction.

Ex. 9.6.2 : A plane wave has electric field intensity E with $E_0 \cos(\omega t - kz)\, \bar{a}_y$ with $E_0 = 1000$ V/m and f = 300 MHz. For its propagation in free space (lossless)

(i) Calculate the Poynting vector.

(ii) Instantaneous and time average power density in the wave.

☑ **Soln. :**

From the given field expression the wave is travelling in + z direction and electric field has E_y component. From free space properties

$$\upsilon = c = 3 \times 10^8 \text{ m/s, and } \eta = 120\pi$$

We calculate phase constant k

$$k = \frac{\omega}{c} = \frac{2\pi \times 300 \times 10^6}{3 \times 10^8} = 2\pi \text{ (rad/m)}$$

To find the Poynting vector we require $\bar{H}$. In general form

$$\bar{H} = H_0 \cos(\omega t - kz)\, \bar{a}_H$$

where $$H_0 = \frac{E_0}{\eta} = \frac{1000}{120\pi} = 2.65 \text{ (A/m)}.$$

and $$\bar{a}_E \times \bar{a}_H = \bar{a}_z$$

i.e. $$\bar{a}_y \times \bar{a}_H = \bar{a}_z$$

or $$\bar{a}_H = -\bar{a}_x$$

then $$\bar{H} = -2.65 \cos(\omega t - kz)\, \bar{a}_x \text{ (A/m)} \quad \text{...Ans.}$$

(i) To find the Poynting vector

$$\bar{P} = \bar{E} \times \bar{H}$$

$$= 1000 \cos(\omega t - kz)\, \bar{a}_y \times -2.65 \cos(\omega t - kz)\, \bar{a}_x$$

$$= +2650 \cos^2(\omega t - kz)\, \bar{a}_z$$

$$= 2650 \left[\frac{1 + \cos 2(\omega t - kz)}{2}\right] \bar{a}_z$$

$$= 1325\,[1 + \cos(2\omega t - 2kz)]\, \bar{a}_z \quad \text{(W/m}^2\text{)} \quad \text{...Ans.}$$

(ii) Instantaneous and time average power density

The Poynting vector itself gives the instantaneous power density. So it is

$$\bar{P} = 1325\,[1 + \cos(2\omega t - 2kz)]\, \bar{a}_z \quad \text{(W/m}^2\text{)}$$

To obtain average power density

$$\bar{P}_{av} = \frac{1}{2\eta} E_0^2\, \bar{a}_z = \frac{1}{2 \times 120\pi} \times 1000^2\, \bar{a}_z$$

$$= 1325\, \bar{a}_z \text{ (W/m}^2\text{)} \quad \text{...Ans.}$$

You can see this value in the $\bar{P}$ as a dc term.

Ex. 9.6.3 : In free space $E(z, t) = 100 \sin(\omega t - \beta z)\, a_x$ V/m. Find the total power passing through a square area of side 25 mm in the z = 0 plane.

☑ **Soln. :**

From the given field expression the wave is travelling in + z direction. The average power in this wave is

$$P_{z\,avg} = \frac{1}{2} \frac{E_{x0}^2}{\eta}$$

For free space $\eta = 120\pi$ and given $E_{x0} = 100$ V/m

$$P_{z\,avg} = \frac{1}{2} \times \frac{100^2}{120\pi} = 13.26 \text{ W/m}^2 = P_{avg}$$

The power passing through a square is given by,

$$P_z = P_{avg} \times S = 13.26 \times (25 \times 10^{-3})^2 = 8.289 \text{ mW}$$

Ex. 9.6.4 : In a non magnetic material ($\epsilon_r \neq 0, \mu = \mu_0, \sigma = 0$)

(i) Find E using Maxwell equations.

(ii) Find poynting vector.

(iii) Find the time average power crossing the surface $x = 1, 0 < y < 2, 0 < z < 3$ m.

Given, $\bar{H} = 30 \cos(2\pi \times 10^8 t - 6x)\, \bar{a}_y$ (mA/m)

☑ **Soln. :**

Given : $$\bar{H} = 30 \cos(2\pi \times 10^8 t - 6x)\, \bar{a}_y$$

$$= H_0 \cos(2\pi \times 10^8 t - 6x)\, \bar{a}_H \left(\frac{\text{mA}}{\text{m}}\right)$$

From which we get,

$$\omega = 2\pi \times 10^8 \text{ (rad/s)},$$

$$\beta = 6$$

and the wave is travelling in + x direction, i.e. $\bar{a}_\beta = \bar{a}_x$. To obtain electric field we require intrinsic impedance it is obtained as follows :

$$\beta = \omega\sqrt{\mu\epsilon}$$

dividing by μ

$$\frac{\beta}{\mu} = \omega\sqrt{\frac{\epsilon}{\mu}}$$

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but for a lossless media ($\because$ $\sigma = 0$), the intrinsic impedance

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

$\therefore$ Equation previous to above equation becomes

$$\frac{\beta}{\mu} = \frac{\omega}{\eta}$$

or $\eta = \frac{\omega\mu}{\beta} = \frac{2\pi \times 10^8 \times 4\pi \times 10^{-7}}{6} = 131.59\ (\Omega)$ ...Ans.

(i) To find $\bar{E}$

The electric field can be written as,

$$\bar{E} = E_0 \cos(2\pi \times 10^8 t - 6x)\, \bar{a}_E$$

The amplitude E_0 is obtained using,

$$\eta = \frac{E_0}{H_0}$$

$\therefore$ $E_0 = \eta H_0 = 131.59 \times 30 \times 10^{-3} = 3.948$

To obtain $\bar{a}_E$,

$\bar{a}_E \times \bar{a}_H = \bar{a}_\beta$ i.e. $\bar{a}_E \times \bar{a}_y = \bar{a}_x$

$\therefore$ $\bar{a}_E = -\bar{a}_z$

$\therefore$ $\bar{E} = -3.948 \cos(2\pi \times 10^8 t - 6x)\, \bar{a}_z$ (V/m) ...Ans.

(ii) To find Poynting vector

$$\bar{P} = \bar{E} \times \bar{H}$$

$$= -3.948 \cos(2\pi \times 10^8 t - 6x)\, \bar{a}_z \times 30 \cos(2\pi \times 10^8 t - 6x)\, \bar{a}_y \times (10^{-3})$$

$\bar{P} = 0.1184 \cos^2(2\pi \times 10^8 t - 6x)\, \bar{a}_x$ (W/m^2) ...Ans.

(iii) To find average power

Average power density is

$$\bar{P}_{avg} = \frac{1}{2}\eta H_0^2 \bar{a}_x = \frac{1}{2} \times 131.59 \times (30 \times 10^{-3})^2$$

$$= 0.0592\, \bar{a}_x\ (\text{W/m}^2)$$

Average power crossing the given area in x = 1 plane is,

$$P = \int_s \bar{P}_{avg} \cdot \bar{ds} = \int_s 0.0592\, \bar{a}_x \cdot dy\, dz\, \bar{a}_x$$

$$= \int_0^3 \int_0^2 0.0592\, dy\, dz$$

$P = 0.0592 \times 2 \times 3 = 0.3535$ (W) ...Ans.

UEx. 9.6.5 MU - May 15. 10 Marks

In a non magnetic medium

$\bar{E} = 4 \sin(2\pi \times 10^7 t - 0.8x)\, \bar{a}_z$ (V/m). Find,

(i) ϵ_r, η.

(ii) The time average power carried by the wave.

(iii) The total power crossing 100 cm^2 of plane $2x + y = 5$.

Soln. : From the given field expression, the wave is in +x direction for which,

$$E_0 = 4,\ \omega = 2\pi \times 10^7, \qquad \beta = 0.8$$

The velocity of the wave in this media is

$$v = \frac{\omega}{\beta} = \frac{2\pi \times 10^7}{0.8} = 0.78 \times 10^8\ (\text{m/s}) \neq c$$

It implies that the medium is not a free space.

(i) To find ϵ_r, η

$$\beta = \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_0 \mu_r \epsilon_0 \epsilon_r} = \frac{\omega}{c}\sqrt{\epsilon_r}$$

Notice that for a non-magnetic medium $\mu_r = 1$

$\therefore$ $\sqrt{\epsilon_r} = \frac{\beta c}{\omega}$

or $\epsilon_r = \frac{\beta^2 c^2}{\omega^2} = \frac{0.8^2 \times (3 \times 10^8)^2}{(2\pi \times 10^7)^2} = 14.59$

we have, $\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_r}} = \frac{120\pi}{\sqrt{\epsilon_r}} = 98.7\ (\Omega)$ ...Ans.

(ii) To find the time average power

For the wave in x-direction, the poynting vector will be in the same direction, giving

$$\bar{P}_{avg} = \frac{E_0^2}{2\eta}\bar{a}_x = \frac{4^2}{2 \times 98.7}\bar{a}_x = 81.1\, \bar{a}_x\ (\text{mW/m}^2)$$

(iii) To find power crossing given area

The unit vector normal to the area given is,

$$\bar{a}_n = \frac{2\bar{a}_x + \bar{a}_y}{\sqrt{5}}$$

This direction is not the same as P_{avg}. So to find the power

$$P = \int \bar{P}_{avg} \cdot \bar{ds} = \int 81.1\, \bar{a}_x \cdot ds \frac{2\bar{a}_x + \bar{a}_y}{\sqrt{5}}$$

$$= \int \frac{2 \times 81.1}{\sqrt{5}} ds = 72.53 \int ds = 72.53 \times (100 \times 10^{-4})$$

$P = 725.4$ (μW) ...Ans.

Chapter Ends...

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