University of Mumbai

Electromagnetics and Antenna

(Course Code : ECC601)

Semester VI - Electronics and Telecommunication Engineering

Strictly as per the New Syllabus (REV-2019 'C' Scheme) of Mumbai University w.e.f. academic year 2021-2022

Sandeep Wali

Assistant Professor
Electronics Department,
Vidyalankar Institute of Technology (VIT),
Mumbai.

TECH-NEO PUBLICATIONS

Where Authors Inspire Innovation

A Sachin Shah Venture

M6-77

Syllabus...

University of Mumbai

Course Code	Course Name	Teaching Scheme (Hrs.)			Credits Assigned			
		Theory	Practical	Tutorial	Theory	Practical	Tutorial	Total
ECC601	Electromagnetics and Antenna	03	-	-	03	-	-	03

Course Code	Course Name	Examination Scheme							
		Theory Marks				Exam. Duration (in Hrs)	Term Work	Practical and Oral	Total
		Internal assessment			End Sem. Exam				
		Test 1	Test 2	Avg. of Test 1 and Test 2					
ECC601	Electromagnetics and Antenna	20	20	20	80	03	-	-	100

Pre-requisites :

1. Vector Calculus
2. Fundamental concepts of electricity and magnetism

Course Objective : The objective of the course is to make student familiar with Maxwell's equation and its usefulness to describe different electromagnetic phenomena such as wave propagation, radiations from antenna etc.

Course Outcome : Student will be able to :

1. Students will be able to describe electromagnetics field including static and dynamic in terms of Maxwell's equations.
2. Students will be able to apply Maxwell's equation to solve various electromagnetic phenomenon such as electromagnetic wave propagation in different medium, power in EM wave.
3. Students will derive the field equations for the basic radiating elements and describe basic antenna parameters like radiation pattern, directivity, gain etc.
4. Students will be able to implement different types of the antenna structures such as Antenna arrays, Microstrip antenna and reflector antenna etc.

Module No.	Unit No.	Topics	Hrs.
1.0		**Introduction to Static fields**	06
	1.1	Charge, Coulomb's law, Charge configurations, Electric field intensity, Electric flux density, Gauss's law and applications, Current density, and Continuity equation. **(Refer Chapters 1, 2 and 3)**	
	1.2	Scalar Electric Potential, Potential gradient, Laplace's and Poison's equations. **(Refer Chapters 4 and 5)**	
	1.3	Biot Savart Law, Ampere Circuit law, Gauss's law for magnetic field, Vector magnetic Potential. **(Refer Chapter 6)**	
2.0		**Electromagnetic Field and Maxwell's Equations**	09
	2.1	Faraday's Law, Displacement current density, Maxwell's equation for time varying field, Boundary conditions. **(Refer Chapter 7)**	
	2.2	EM wave propagation through lossy, perfect dielectric and conducting medium. **(Refer Chapter 8)**	
	2.3	Power in EM Wave : Poynting theorem and Poynting vector. **(Refer Chapter 9)**	

Module No.	Unit No.	Topics	Hrs.
3.0		**Basic of Antennas**	08
	3.1	Basic concepts : Radiation mechanism, Near field and far field radiation, retarded potential. **(Refer Chapter 10)**	
	3.2	Antenna Parameters : Isotropic antenna, Radiation pattern, radiation intensity, Beam width, directivity, Gain, beam efficiency, bandwidth, polarization, Input impedance, Antenna efficiency, Radiation resistance, Loss resistance, aperture concept, FRIIS transmission formula. **(Refer Chapter 11)**	
	3.3	Wire Elements : Infinitesimal dipole, Wire dipole, Monopole antennas : radiation field derivations and related parameters, Introduction to loop antenna. **(Refer Chapter 12)**	
4.0		**Antenna Arrays**	06
	4.1	Linear arrays of two isotropic point sources, linear arrays of N elements, Principle of pattern multiplication.	
	4.2	Introduction to Planner and circular arrays, Introduction to array synthesis using Binomial array. **(Refer Chapter 13)**	
5.0		**Types of Antennas**	06
	5.1	Yagi antenna, Broadband antenna like Helical and Log Periodic antenna. Horn Antennas : E-Plane Sectoral Horn, H-Plane Sectoral Horn, Pyramidal Horn and Conical Horn. **(Refer Chapters 14 and 15)**	
	5.2	Reflector Antennas : Plane Reflectors, Corner Reflectors and Parabolic Reflector. **(Refer Chapter 16)**	
	5.3	Patch Antenna : Microstrip antenna, Feeding Techniques, Introduction to design of Microstrip antenna (Rectangular and circular patch). **(Refer Chapter 17)**	
6.0		**Electromagnetic Wave Propagation**	04
	6.1	Ground Wave Propagation, Sky Wave Propagation and Space Wave Propagation. **(Refer Chapters 18 and 19)**	
		Total	39

Internal Assessment (20-Marks)

Internal Assessment (IA) consists of two class tests of 20 marks each. IA-1 is to be conducted on approximately 40% of the syllabus and IA-2 will be based on remaining contents (approximately 40% syllabus but excluding contents covered in IA-I). Duration of each test shall be one hour. Average of the two tests will be considered as IA marks.

End Semester Examination (80-Marks)

Weightage to each of the modules in end-semester examination will be proportional to number of respective lecture hours mentioned in the curriculum.

1. Question paper will comprise of **total 06** questions, each carrying **20 marks.**
2. **Question No : 01 will be compulsory** and based on entire syllabus wherein **4 to 5 sub-questions will be asked.**
3. Remaining questions will be mixed in nature and randomly selected from all the modules.
4. **Total 04 questions** need to be attempted.

□□□

Index

MODULE 1

MODULE 2

MODULE 3

MODULE 4

MODULE 5

MODULE 6

◈ Multiple Choice Questions (MCQ's)

Module 1

CHAPTER 1

Vectors

1.1 INTRODUCTION

If you refer syllabus of different universities for the subject of electromagnetics, it may not be including this chapter of vectors. Does it mean that electromagnetic subject can be studied without vectors ? Answer is yes as well as no. Naturally the question may arise in your mind that **What is the role of vectors in electromagnetics** ? Note that electromagnetics is the study of electric and magnetic fields and their applications. Many of the field quantities are vectors. To study these fields we should have knowledge of vectors.

It is possible to study electric and magnetic fields without the use of vector, as most of you may have done in basic physics course. But the use of vector in the study of electromagnetic field theory saves time and provides economy of thought. In addition, the vector form gives a clear understanding of physical laws which is described by mathematics. For this reason, the first chapter of this book is devoted to vector analysis.

At the observation point the field which we observe may not be because of single source. The total field at the point is a obtained by adding fields due to different sources. This addition is not normal addition, it is a vector addition. If the directions of fields are opposite it is vector subtraction. Thus one should know how to add or subtract vectors. For this reason vector addition is added in this chapter. Other operations like product is also included.

In electromagnetics del operator plays very important role. This operator can operate on scalar or vector. We get different operations like gradient, divergence and curl. It is included in this chapter.

To specify the direction of a vector or for specifying position of source or observation point in a three dimensional space, three numbers are needed. These numbers depend on the choice of coordinate system. The most commonly used orthogonal coordinate systems are cartesian, cylindrical and spherical coordinate system. It is added as one of the part of this chapter.

1.2 SCALARS AND VECTORS

Scalar

A quantity that has only magnitude and algebraic sign is called **scalar** quantity. For example, mass, density, pressure, volume etc. are scalar quantities.

Vector

Vector is directed line i.e. a line having both magnitude and direction. The vector has a starting point and end point. The direction is shown by arrow. In Fig. 1.2.1, vector starts at A and ends at B. This vector is written as $\overrightarrow{AB}$ or $\overline{AB}$.

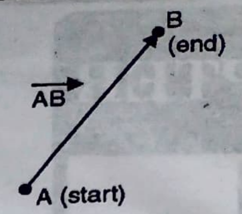


Fig. 1.2.1 : Graphical representation of vector ($\overline{AB}$)

The first letter in $\overline{AB}$ indicates the start (or tail) and the second letter is the end (or head) of the vector. The arrow or bar over the head of the letter is used to indicate vector. We write the vector as

$$\text{Vector} = (\text{Magnitude or length}) \times \text{direction}$$

i.e. $$\overline{AB} = (|\overline{AB}| \text{ or } AB) \times \bar{a}_{AB} \quad ...(1.2.1)$$

where, $|\overline{AB}|$ or AB = magnitude or length of vector.

$\bar{a}_{AB}$ = unit vector in the direction AB.

> **Note :** $\bar{a}$ is used to indicate unit vector i.e. a vector whose magnitude is one, it gives only direction. And subscript AB indicates the direction from A to B.

From Equation (1.2.1), unit vector is obtained as :

$$\bar{a}_{AB} = \frac{\overline{AB}}{|\overline{AB}|} \quad ...(1.2.2)$$

i.e. vector divided by its magnitude gives unit vector.

In most of the problems, we need vector or unit vector when coordinates of two points are given to us. To obtain vector from the given points the method is :

Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be the two points given (Fig. 1.2.1), then vector AB is given by

$$\overline{AB} = (x_2 - x_1)\,\bar{a}_x + (y_2 - y_1)\,\bar{a}_y + (z_2 - z_1)\,\bar{a}_z \quad ...(1.2.3)$$

In Equation (1.2.3), notice that always the bracket values are **end coordinates minus starting coordinates.** In vector AB, A is the starting point and B is the end point of the vector. Also $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ are the unit vectors in cartesian coordinate system.

The magnitude or length of $\overline{AB}$ is nothing but distance between A and B, using distance formula

$$\therefore \quad |\overline{AB}| \text{ or } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad ...(1.2.4)$$

Unit vector, vector and its magnitude

$$\bar{a}_{AB} = \frac{\overline{AB}}{|\overline{AB}|}$$

$$\overline{AB} = (x_2 - x_1)\,\bar{a}_x + (y_2 - y_1)\,\bar{a}_y + (z_2 - z_1)\,\bar{a}_z$$

$$\therefore \quad |\overline{AB}| \text{ or } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Ex. 1.2.1 : Find the $\overline{AB}$ if A (2, 1, 3) and B (–2, – 4, 5), also find unit vector AB.

Soln. :

$$\overline{AB} = (-2-2)\,\bar{a}_x + (-4-1)\,\bar{a}_y + (5-3)\,\bar{a}_z$$

$$= -4\,\bar{a}_x - 5\,\bar{a}_y + 2\,\bar{a}_z$$

The unit vector is,

$$\bar{a}_{AB} = \frac{\overline{AB}}{|\overline{AB}|} = \frac{-4\,\bar{a}_x - 5\,\bar{a}_y + 2\,\bar{a}_z}{\sqrt{(-4)^2 + (-5)^2 + (2)^2}}$$

$$= \frac{-4\,\bar{a}_x - 5\,\bar{a}_y + 2\,\bar{a}_z}{\sqrt{45}}$$

1.3 ADDITION AND SUBTRACTION OF VECTORS

In the last section we have seen how to find vector. Now if two or more vectors are given, operations like addition or subtraction can be performed on them. Two vectors $\bar{A}$ and $\bar{B}$, in Fig. 1.3.1(a) which are not in the same direction nor in opposite direction determine a plane. Their sum is another vector $\bar{C}$ in the same plane, $\bar{C} = \bar{A} + \bar{B}$ can be obtained graphically by two methods :

1. By the Parallelogram Rule
2. By the Head to Tail Rule

1.3.1 Parallelogram Rule

Both vectors $\bar{A}$ and $\bar{B}$ are drawn from a common origin O (origin refers to start of the vector) and a parallelogram is completed. The diagonal vector of the parallelogram is the required vector $\bar{C}$, as shown in Fig. 1.3.1(b).

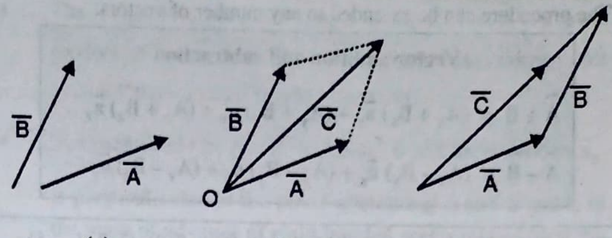


Fig. 1.3.1 : Addition of vectors

1.3.2 Head to Tail Rule

By beginning the second vector B from the head of the first vector $\bar{A}$ and completing the triangle, as shown in Fig. 1.3.1(c), we obtain $\bar{C}$.

The vector addition obeys the commutative and associative laws.

Commutative law : $\bar{A} + \bar{B} = \bar{B} + \bar{A}$

Associative law : $\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$

Vector subtraction can be defined in terms of vector addition as : $\bar{A} - \bar{B} = \bar{A} + (-\bar{B})$,

where, $-\bar{B}$ is the negative of vector $\bar{B}$.

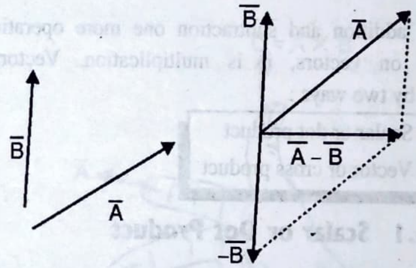


Fig. 1.3.2 : Subtraction of vectors

Thus, to subtract a vector $\bar{B}$ from a vector $\bar{A}$, reversed $\bar{B}$ is added with $\bar{A}$. This is shown in Fig. 1.3.2.

If $\quad \bar{A} = A_x\,\bar{a}_x + A_y\,\bar{a}_y + A_z\,\bar{a}_z$

and $\quad \bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$

Then, while adding the vectors, add the components in the same direction.

$$\bar{A} + \bar{B} = (A_x + B_x)\,\bar{a}_x + (A_y + B_y)\,\bar{a}_y + (A_z + B_z)\,\bar{a}_z \quad \text{...(1.3.1)}$$

Similarly, subtracting the components in the same direction,

$$\bar{A} - \bar{B} = (A_x - B_x)\,\bar{a}_x + (A_y - B_y)\,\bar{a}_y + (A_z - B_z)\,\bar{a}_z \quad \text{...(1.3.2)}$$

The procedure can be extended to any number of vectors.

Vector addition and subtraction

$$\bar{A}+\bar{B} = (A_x+B_x)\,\bar{a}_x + (A_y+B_y)\,\bar{a}_y + (A_z+B_z)\,\bar{a}_z$$

$$\bar{A}-\bar{B} = (A_x-B_x)\,\bar{a}_x + (A_y-B_y)\,\bar{a}_y + (A_z-B_z)\,\bar{a}_z$$

Ex. 1.3.1 : Four points are A (2, 3, –1), B (1, 5, 2), C (3, 1, – 5), D (1, 2, 3), find $\overline{AB}+\overline{CD}$; $\overline{AB}-\overline{CD}$.

☑ **Soln. :**

The vectors $\overline{AB}$ and $\overline{CD}$ are :

$$\overline{AB} = (1-2)\,\bar{a}_x + (5-3)\,\bar{a}_y + [2-(-1)]\,\bar{a}_z$$

$$= -\bar{a}_x + 2\,\bar{a}_y + 3\,\bar{a}_z$$

$$\overline{CD} = (1-3)\,\bar{a}_x + (2-1)\,\bar{a}_y + [3-(-5)]\,\bar{a}_z$$

$$= -2\,\bar{a}_x + \bar{a}_y + 8\,\bar{a}_z$$

Now $\overline{AB}+\overline{CD} = -3\,\bar{a}_x + 3\,\bar{a}_y + 11\,\bar{a}_z$

and $\overline{AB}-\overline{CD} = \bar{a}_x + \bar{a}_y - 5\,\bar{a}_z$

1.4 PRODUCT OF VECTORS

Like addition and subtraction one more operation can be performed on vectors, it is multiplication. Vectors can be multiplied by two ways :

(i) Scalar or dot product
(ii) Vector or cross product

1.4.1 Scalar or Dot Product

Let $\bar{A} = A\,\bar{a}_A$ and $\bar{B} = B\,\bar{a}_B$, be two vectors as shown in Fig. 1.4.1.

Then, $\bar{A}\cdot\bar{B} = AB\cos\theta$ ...(1.4.1)

It is called the dot product of vectors $\bar{A}$ and $\bar{B}$, where θ is the angle between them. In Equation (1.4.1),

A = magnitude of $\bar{A}$,

B = magnitude of $\bar{B}$

θ = angle between $\bar{A}$ and $\bar{B}$

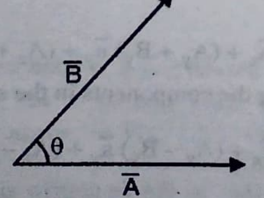


Fig. 1.4.1 : Vectors for scalar product

There are different cases :

(i) $\bar{A}$ and $\bar{B}$ are parallel,

i.e. $\theta = 0$, $\cos\theta = 1$ $\therefore\ \bar{A}\cdot\bar{B} = AB$

(ii) $\bar{A}$ and $\bar{B}$ are perpendicular

i.e. $\theta = \pi/2$ $\cos\theta = 0\ \therefore$ $\bar{A}\cdot\bar{B} = 0$

(iii) $\bar{A}$ and $\bar{B}$ are opposite

i.e. $\theta = \pi$ $\cos\theta = -1$ $\therefore$ $\bar{A}\cdot\bar{B} = -AB$

(iv) Dot product of unit vectors

$$\bar{a}_x\cdot\bar{a}_x = \bar{a}_y\cdot\bar{a}_y = \bar{a}_z\cdot\bar{a}_z = 1;$$

$$\bar{a}_x\cdot\bar{a}_y = \bar{a}_x\cdot\bar{a}_z = \bar{a}_y\cdot\bar{a}_z = 0$$

When vectors are given as

$$\bar{A} = A_x\,\bar{a}_x + A_y\,\bar{a}_y + A_z\,\bar{a}_z\,;$$

$$\bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$$

Then, $\bar{A}\cdot\bar{B} = (A_x\,\bar{a}_x + A_y\,\bar{a}_y + A_z\,\bar{a}_z)\cdot(B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z)$

Using dot products of unit vectors we get,

$$\bar{A}\cdot\bar{B} = A_xB_x + A_yB_y + A_zB_z \quad ...(1.4.2)$$

☞ Length of projection

The definition of dot product can be used to find length of projection of one vector on other. Let $\bar{F}$ and $\bar{G}$ are the vectors making angle θ as shown in Fig. 1.4.2.

We know that projection of G on F is G cos θ.

The same result can also be obtained by using vectors as,

$$PQ = \bar{G}\cdot\bar{a}_F$$

$$= G\times 1\cos\theta \quad \text{...using definition of dot product}$$

$$= G\cos\theta$$

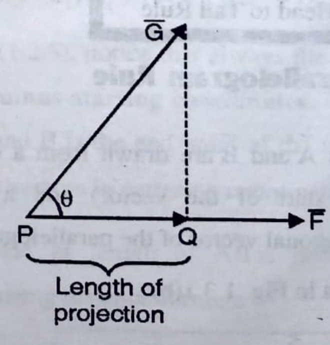


Fig. 1.4.2 : Projection of vector

Vector projection

Vector projection of $\overline{G}$ on $\overline{F}$ is,

$$\overline{PQ} = \text{length of projection} \times \text{unit vector in the direction } \overline{F}$$

i.e. $$\overline{PQ} = (\overline{G} \cdot \bar{a}_F)\,\bar{a}_F$$

Thus, length of projection of $\overline{G}$ in the direction $\overline{F}$ is

$$\textbf{Length of projection} = \overline{G} \cdot \bar{a}_F \qquad \text{...(1.4.3)}$$

and vector projection in the direction $\overline{F}$ is

$$\textbf{Vector projection} = (\overline{G} \cdot \bar{a}_F)\,\bar{a}_F \qquad \text{...(1.4.4)}$$

These result are important and required while transforming a vector from one coordinate system to another, which we will deal with in the article on coordinate system.

Dot product

$$\overline{A} \cdot \overline{B} = A\,B\cos\theta = \overline{B} \cdot \overline{A}$$

$$\overline{A} \cdot \overline{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\textbf{Length of projection} = \overline{G} \cdot \bar{a}_F$$

$$\textbf{Vector projection} = (\overline{G} \cdot \bar{a}_F)\,\bar{a}_F$$

$$\bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_z = 1$$

$$\bar{a}_x \cdot \bar{a}_y = \bar{a}_x \cdot \bar{a}_z = \bar{a}_y \cdot \bar{a}_z = 0$$

Ex. 1.4.1 : Vectors are given as $\overline{A} = 5\,\bar{a}_x + 4\,\bar{a}_y + 3\,\bar{a}_z$ and $\overline{B} = 2\,\bar{a}_x + 3\,\bar{a}_y + 4\bar{a}_z$.

Find (a) $\overline{A} \cdot \overline{B}$ (b) angle between $\overline{A}$ and $\overline{B}$

Soln. :

(a) $$\overline{A} \cdot \overline{B} = (5 \times 2) + (4 \times 3) + (3 \times 4) = 34$$

[Refer Equation (1.4.2)]

(b) Angle between $\overline{A}$ and $\overline{B}$ is obtained by using Equation (1.4.1)

$$\overline{A} \cdot \overline{B} = AB\cos\theta$$

where, $$A = \sqrt{5^2 + 4^2 + 3^2} = \sqrt{50},$$

$$B = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \text{ and } \overline{A} \cdot \overline{B} = 34$$

Putting in Equation (1.4.1), $34 = \sqrt{50} \times \sqrt{29} \times \cos\theta$

$\therefore$ $\cos\theta = 0.892$ or $\theta = 26.76°$

1.4.2 Vector Product or Cross Product

Vector Product or Cross Product is defined as,

$$\overline{A} \text{ cross } \overline{B} = \overline{A} \times \overline{B} = AB\sin\theta\,\bar{a}_n \qquad \text{...(1.4.5)}$$

- The × sign does not mean simple multiplication, it is a cross product of two vectors. The term $\bar{a}_n$ indicates a normal unit vector, thus A cross B results in a vector.
- The **magnitude** of $\overline{A} \times \overline{B}$ is **AB sin θ** and the **direction** $\bar{a}_n$ **is perpendicular to the plane containing** $\overline{A}$ **and** $\overline{B}$ and is in the sense of **advance of right handed screw** rotated form the **first vector** (i.e. $\overline{A}$) **to the second vector** (i.e. $\overline{B}$) through the **smaller angle** between their positive directions.
- In the Fig. 1.4.3(a), the screw will move upward. Thus, the direction of $\overline{A} \times \overline{B}$ is upward.

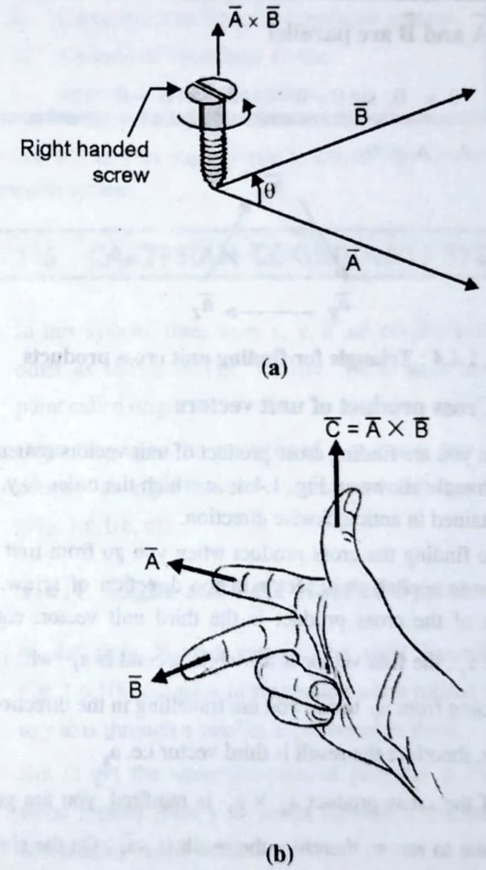


Fig. 1.4.3 : Figure illustrating direction of $\overline{A} \times \overline{B}$

Simplest way

- The simplest way to obtain the direction of $\overline{A} \times \overline{B}$ is using right hand as shown in Fig. 1.4.3(b).
- In Fig. 1.4.3 when the first finger is in the direction of first vector ($\overline{A}$), the second finger is in the direction of second vector ($\overline{B}$) then the thumb indicates the direction of cross product ($\overline{A} \times \overline{B}$).

- When the order of cross product is changed i.e. instead of $\bar{A} \times \bar{B}$ if it is $\bar{B} \times \bar{A}$, then right handed screw in the Fig. 1.4.3(a) advances in downward direction that is $-\bar{a}_n$. Because of this, commutative law does not apply to the cross product.

Therefore, $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$ ...(1.4.6)

1.4.3 Different Cases in Vector Product

(i) $\bar{A}$ and $\bar{B}$ are parallel
(ii) Cross product of unit vectors

(i) $\bar{A}$ and $\bar{B}$ are parallel

i.e. $\theta = 0$, $\sin 0 = 0$ $\therefore \bar{A} \times \bar{B} = 0$

Similarly, $\bar{A} \times \bar{A} = 0$

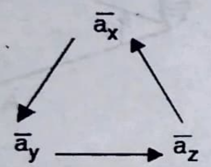


Fig. 1.4.4 : Triangle for finding unit cross products

(ii) Cross product of unit vectors

- When you are finding cross product of unit vectors remember the triangle shown in Fig. 1.4.4, in which the order x, y, z is maintained in anticlockwise direction.
- While finding the cross product when you go from first unit vector to second unit vector in the direction of arrow, the result of the cross product is the third unit vector. e.g. in $\bar{a}_y \times \bar{a}_z$, the first vector is $\bar{a}_y$ while second is $\bar{a}_z$, when you are going from $\bar{a}_y$ to $\bar{a}_z$, you are travelling in the direction of arrow, therefore the result is third vector i.e. $\bar{a}_x$.
- But if the cross product $\bar{a}_z \times \bar{a}_y$ is required, you are going opposite to arrow, therefore the result is $-\bar{a}_x$. On the similar lines we can find other cross products. The different possible cross products are listed below :

When vectors are given as, $\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$;

$\bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$ then,

$$\bar{A} \times \bar{B} = (A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z) \times (B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z)$$

$$= A_x B_x (\bar{a}_x \times \bar{a}_x) + A_x B_y (\bar{a}_x \times \bar{a}_y) + A_x B_z (\bar{a}_x \times \bar{a}_z)$$

$$+ A_y B_x (\bar{a}_y \times \bar{a}_x) + A_y B_y (\bar{a}_y \times \bar{a}_y) + A_y B_z (\bar{a}_y \times \bar{a}_z)$$

$$+ A_z B_x (\bar{a}_z \times \bar{a}_x) + A_z B_y (\bar{a}_z \times \bar{a}_y) + A_z B_z (\bar{a}_z \times \bar{a}_z)$$

By using cross product of unit vectors,

$$\bar{A} \times \bar{B} = A_x B_y \bar{a}_z - A_x B_z \bar{a}_y - A_y B_x \bar{a}_z$$

$$+ A_y B_z \bar{a}_x - A_z B_x \bar{a}_y - A_z B_y \bar{a}_x$$

$$= (A_y B_z - A_z B_y) \bar{a}_x + (A_z B_x - A_x B_z) \bar{a}_y$$

$$+ (A_x B_y - A_y B_x) \bar{a}_z$$

This can be conveniently expressed in determinant form as :

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad ...(1.4.7)$$

1.4.4 Application of Cross Product

One of the application of cross product is finding area of the parallelogram. When two vector, $\overline{AB}$ and $\overline{AC}$ are given, then area of parallelogram formed by two vectors is

Area of parallelogram $= |\overline{AB} \times \overline{AC}|$

Area of the triangle ABC $= \frac{1}{2} |\overline{AB} \times \overline{AC}|$ ...(1.4.8)

Cross product of unit vectors

$$\bar{A} \times \bar{B} = A B \sin \theta \, \bar{a}_n$$

$$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\bar{a}_x \times \bar{a}_x = \bar{a}_y \times \bar{a}_y = \bar{a}_z \times \bar{a}_z = 0$$

$\bar{a}_x \times \bar{a}_y = \bar{a}_z$, $\quad \bar{a}_y \times \bar{a}_x = -\bar{a}_z$

$\bar{a}_y \times \bar{a}_z = \bar{a}_x$, $\quad \bar{a}_z \times \bar{a}_y = -\bar{a}_x$

$\bar{a}_z \times \bar{a}_x = \bar{a}_y$, $\quad \bar{a}_x \times \bar{a}_z = -\bar{a}_y$

Ex. 1.4.2 : Two vectors are represented by

$\bar{A} = 2\bar{a}_x + 2\bar{a}_y + 0\bar{a}_z$;

$\bar{B} = 3\bar{a}_x + 4\bar{a}_y - 2\bar{a}_z$

Find $\bar{A} \times \bar{B}$ and show that $\bar{A} \times \bar{B}$ is at right angle to $\bar{A}$.

Soln. :

From Equation (1.4.7),

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 2 & 2 & 0 \\ 3 & 4 & -2 \end{vmatrix} = \bar{a}_x (-4-0) - \bar{a}_y (-4-0) + \bar{a}_z (8-6)$$

$$\bar{A}\times\bar{B} = -4\bar{a}_x + 4\bar{a}_y + 2\bar{a}_z$$

The vector $\bar{A}\times\bar{B}$ is at right angle to $\bar{A}$ then

$$(\bar{A}\times\bar{B})\cdot\bar{A} = 0$$

$$\text{L.H.S} = (-4\bar{a}_x + 4\bar{a}_y + 2\bar{a}_z)\cdot(2\bar{a}_x + 2\bar{a}_y)$$
$$= -8+8=0=\text{R.H.S.}$$

$\therefore$ $\bar{A}\times\bar{B}$ is at right angle to $\bar{A}$.

Ex. 1.4.3 : Given vectors

$$\bar{A} = 3\bar{a}_x + 4\bar{a}_y + \bar{a}_z, \quad \bar{B} = 2\bar{a}_y - 5\bar{a}_z$$

Find the angle between $\bar{A}$ and $\bar{B}$ using (i) dot product (ii) cross product.

☑ **Soln. :**

The magnitudes of vectors are,

$$A = |\bar{A}| = \sqrt{3^2+4^2+1^2} = \sqrt{26}$$
$$B = |\bar{B}| = \sqrt{2^2+(-5)^2} = \sqrt{29}$$

i) Using dot product

$$\bar{A}\cdot\bar{B} = AB\cos\theta$$

Here the dot product is obtained using

$$\bar{A}\cdot\bar{B} = (3\times0)+(4\times2)+(1\times-5)=3$$

$$\therefore\quad 3 = \sqrt{26}\sqrt{29}\cos\theta \rightarrow \cos\theta = \frac{3}{\sqrt{26}\sqrt{29}}$$
$$= 0.1093$$
$$\theta = \cos^{-1}(0.1093) = 83.73°$$

ii) Using cross product

$$\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{vmatrix}$$
$$= \bar{a}_x(-20-2) - \bar{a}_y(-15-0) + \bar{a}_z(6-0)$$
$$= -22\bar{a}_x + 15\bar{a}_y + 6\bar{a}_z$$

we have, $\bar{A}\times\bar{B} = AB\sin\theta\,\bar{a}_z$

Taking mod of both sides,

$$|\bar{A}\times\bar{B}| = AB\sin\theta$$

$$\text{or}\quad \sin\theta = \frac{|\bar{A}\times\bar{B}|}{AB} = \frac{\sqrt{(-22)^2+(15)^2+6^2}}{\sqrt{26}\sqrt{29}}$$

$$\sin\theta = \frac{\sqrt{745}}{\sqrt{26}\sqrt{29}} = 0.994$$

$$\text{or}\quad \theta = \sin^{-1}(0.994) = 83.73°$$

1.5 COORDINATE SYSTEMS

- The laws of electromagnetics are invariant with coordinate system. When we solve problems in electromagnetics, for example, finding electric field at a point, we desire the fields.
- The position is expressed in terms of coordinates. The coordinates are specified by the coordinate system.
- The number of coordinate systems like rectangular, cylindrical, spherical, elliptical, spheroidal, paraboloidal etc. are present but following three are most common and fulfil our requirements :

1. Cartesian or rectangular coordinate system
2. Cylindrical coordinate system
3. Spherical coordinate system

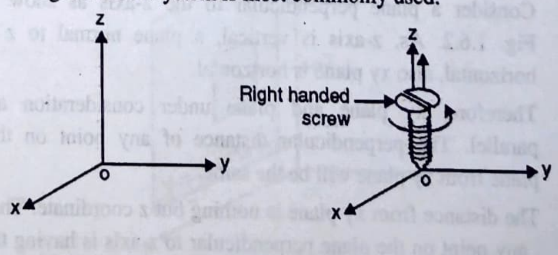
We will discuss each system in detail. The simplest of these is cartesian system.

1.6 CARTESIAN COORDINATE SYSTEM

- In this system, three axes x, y, z are perpendicular to each other as shown in Fig. 1.6.1(a). These axes intersect at a point called origin.
- Depending upon the positions of x, y, z axes two systems are defined as right handed system and left handed system (Fig. 1.6.1(b, c)).

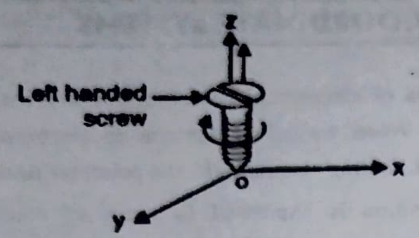
1.6.1 Right and Left Handed Systems

- In the right handed system, the right handed screw in Fig. 1.6.1(b) progress in z direction when rotated from x-axis to y axis through a smaller angle between them.
- But to get the same direction of progress in Fig. 1.6.1(c), when rotated from x to y-axis through a smaller angle, the screw must be left handed screw.
- Right handed system is most commonly used.



(a) Axis position (b) Right handed system

Fig. 1.6.1 (Contd...)



(c) Left handed system

Fig. 1.6.1 : Cartesian system

- Remember x and y axis are horizontal while z-axis is vertical also they are mutually perpendicular to each other. These three axes meet at a point called origin.
- Now see how we can define a point in cartesian system.

1.6.2 Point in Cartesian System

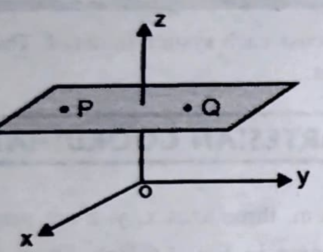


Fig. 1.6.2 : z = Constant plane

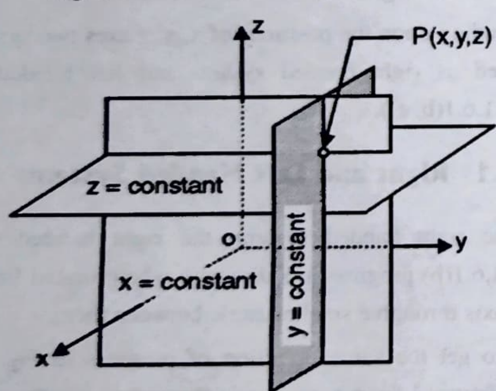


Fig. 1.6.3 : Point is defined by three planes

- Consider a plane perpendicular to the z-axis as show in Fig. 1.6.2. As, z-axis is vertical, a plane normal to z is horizontal, also xy plane is horizontal.
- Therefore, xy plane and plane under consideration are parallel. The perpendicular distance of any point on this plane from xy plane will be the same.
- The distance from xy plane is nothing but z coordinate. Thus , any point on the plane perpendicular to z-axis is having the same z coordinate.
- In other words we can say that for a plane perpendicular to z-axis, z coordinate of any point on the plane is constant or simply the plane is defined as **z = constant plane.**
- Similarly, a plane perpendicular to x-axis can be defined as **x = constant** plane and a plane perpendicular to y-axis is **y = constant** plane.
- Remember, x = constant and y = constant planes are vertical planes, while z = constant plane is horizontal plane.
- Consider, x = constant and y = constant planes. These two vertical planes intersect and the intersection is a vertical line.
- To obtain a point take the third plane, i.e. z = constant plane i.e. horizontal plane. The vertical line and horizontal plane intersects and the intersection is a point [point P in Fig. 1.6.3].
- Thus, point P is a intersection of x = constant, y = constant and z = constant planes and coordinates of it can be written as (x, y, z).

> **Any point in cartesian system is the intersection of**
>
> **x = constant, y = constant and z = constant planes.**
>
> **Point in cartesian system : (x, y, z)**

- These planes are mutually perpendicular to each other. This is a very important concept.
- So, if the coordinates of a point are (2, –1, 3) it means, it is the intersection of x = 2, y = –1 and z = 3 planes.

1.6.3 Unit Vectors in Cartesian System

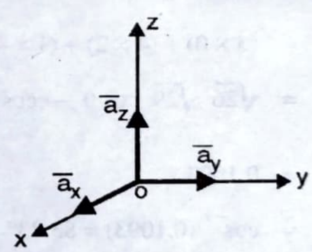


Fig. 1.6.4 : Unit vectors

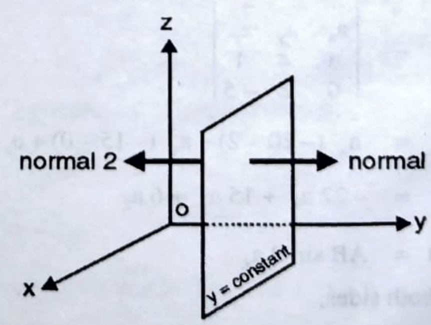


Fig. 1.6.5 : Unit vector $\bar{a}_y$

- Unit vectors in cartesian system are $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$. These vectors are in the positive x, y and z directions as shown in Fig. 1.6.4. Instead of $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ many times different notations are used like or $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$.

- We are using first notations for unit vectors in cartesian system throughout this book. There is one more way to describe directions of unit vectors.

Unit vectors in cartesian system : $\bar{a}_x$, $\bar{a}_y$, $\bar{a}_z$

- Although this discussion is not required in this article but, if you understand, it will be easier to imagine directions of unit vectors in cylindrical and spherical coordinate systems. It is like this.
- We know that a plane normal to y axis is y = constant plane, then $\bar{a}_y$ is a unit vector normal to it. Two normals are possible as shown in Fig. 1.6.5.
- But out of these two normals you take that normal which is in the increasing direction of y i.e. first normal in the figure.
- Similarly, $\bar{a}_x$ and $\bar{a}_z$ are unit vectors perpendicular to x = constant and z = constant plane respectively, also don't forget that they are in increasing directions of x and z respectively. Thus,

Unit vectors are always normal to the respective surfaces and they are in the increasing directions of coordinates.

1.6.4 Differential Lengths, Areas, and Volume

(A) Differential lengths

- Now, consider point P (x, y, z) in Fig. 1.6.6. If we increase the x-coordinate by differential amount dx (i.e. imagine that point P is pulled in x direction by a length dx), it results in length dx.
- Similarly by increasing y and z coordinates by differential amounts will result in lengths dy and dz. This is shown in Fig. 1.6.6.
- Thus differential lengths in cartesian system are :

Differential lengths in cartesian system : dx, dy, dz.

(B) Differential areas

- Now by increasing each coordinate of point P by differential amounts results in point P′ whose coordinates are (x + dx, y + dy, z + dz).
- The differential lengths dy and dz forms area 1 which is dydz. Similarly, lengths dx and dz forms area 2 which is dxdz and area 3 which is the result of dx and dy is dxdy. (Refer Fig. 1.6.6).

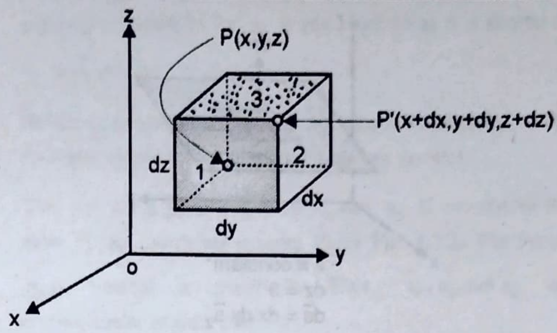


Fig. 1.6.6 : Differential increments in coordinates

Thus differential areas in cartesian system are :

Surface 1 : $ds_1 = dy\,dz$ ...(x = constant)

Surface 2 : $ds_2 = dx\,dz$ ...(y = constant)

Surface 3 : $ds_3 = dx\,dy$ ...(z = constant)

Many times in analysis we require vector areas, defined as,

Vector area = (Area) × unit vector normal to it.

For example, if the given area is in x = constant surface, unit vector normal to it is $\bar{a}_x$

$$\bar{ds} = dy\,dz\ \bar{a}_x$$

Noticing that unit vectors $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ are unit vectors normal to x = constant, y = constant and z = constant surfaces respectively, we write vector areas for surfaces in Fig. 1.6.7.

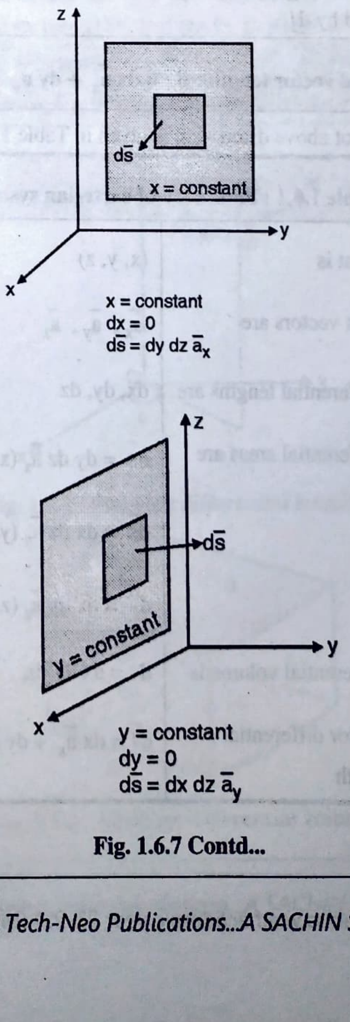


Fig. 1.6.7 Contd...

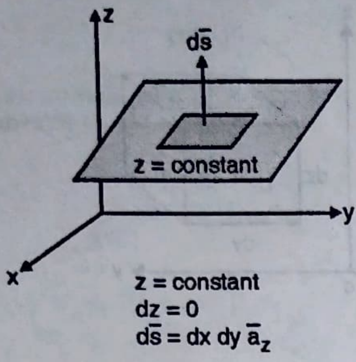


Fig. 1.6.7 : Differential vector areas

(C) Differential volume

When we multiply three differential lengths dx, dy and dz , it forms a differential volume.

Differential volume in cartesian : dv = dx dy dz

(D) Differential vector length

When we find vector $\overrightarrow{PP}$ which is the result of three differential increments in the coordinates, we get,

$$\overrightarrow{PP} = (x + dx - x)\ \bar{a}_x + (y + dy - y)\ \bar{a}_y + (z + dz - z)\ \bar{a}_z$$

$$= dx\ \bar{a}_x + dy\ \bar{a}_y + dz\ \bar{a}_z$$

This is denoted by $\bar{dl}$,

Differential vector length : $\bar{dl} = dx\ \bar{a}_x + dy\ \bar{a}_y + dz\ \bar{a}_z$

The summary of above discussion is given in Table 1.6.1.

Table 1.6.1 : Summary of cartesian system

1.	Point is	(x, y, z)
2.	Unit vectors are	$\bar{a}_x$, $\bar{a}_y$, $\bar{a}_z$
3.	Differential lengths are	dx, dy, dz
4.	Differential areas are	$\bar{ds} = dy\, dz\, \bar{a}_x$ (x = constant) $\bar{ds} = dx\, dz\, \bar{a}_y$ (y = constant) $\bar{ds} = dx\, dy\, \bar{a}_z$ (z = constant)
5.	Differential volume is	dv = dx dy dz.
6.	Vector differential length	$\bar{dl} = dx\ \bar{a}_x + dy\ \bar{a}_y + dz\ \bar{a}_z$

1.7 CYLINDRICAL COORDINATE SYSTEM

1.7.1 Point in Cylindrical System

- Just like in cartesian system, as we obtain a point by intersection of three mutually perpendicular plane surfaces, in this system also a point is obtained by intersection of three surfaces mutually perpendicular to each other.
- But the difference is that all three surfaces are not plane surfaces, one is cylindrical and other two are plane surfaces.
- In the cylindrical system again we are taking x, y, z-axis for reference. Now imagine a hollow vertical cylinder of radius r placed such that axis of the cylinder coincides with z-axis.
- If you take any point on the cylindrical surface it is at a same distance r from the axis therefore we define the cylindrical surface as **r = constant surface.**
- Consider now a plane vertical surface of which one edge coincides with z-axis. We can rotate this plane about z-axis. The angle of rotation ϕ is measured from xz plane as shown in Fig. 1.7.1.
- When we take any point on this plane, every time the angle of the point with xz plane is ϕ. Therefore we define this plane as **ϕ = constant plane.**
- Thus, we have two new surfaces, one is r = constant and the other is ϕ = constant surface. The intersection of these two surfaces is a vertical line.
- Remember we want a point. The situation is similar to in Cartesian system. Hence we need one more surface, take **z = constant plane.**
- The intersection of vertical line with z = constant plane is a point. This point is on r = constant, ϕ = constant and z = constant surfaces and this point can be written as (r, ϕ, z).
- The three surfaces in cylindrical system are again mutually perpendicular to each other.
- One important thing about angle ϕ is, ϕ = constant plane can make full rotation about z-axis i.e. ϕ can vary from 0 to 360°.

Point in cylindrical system : (r, ϕ, z)

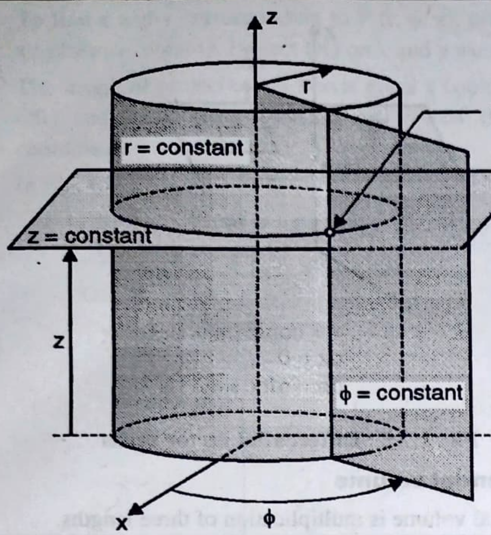


Fig. 1.7.1 : Cylindrical system

1.7.2 Unit Vectors in Cylindrical System

- The unit vectors in this system are, $\bar{a}_r$, $\bar{a}_\phi$, and $\bar{a}_z$. These unit vectors are not along any axis because there is no r or ϕ axis in cylindrical system.
- **The unit vectors are normal to the respective surfaces and in the increasing directions of the coordinates.**
- Thus, the unit vector $\bar{a}_r$ is normal to r = constant surface i.e. normal to the cylinder at given point and in the increasing direction of r.
- As r increases away from axis of the cylinder (i.e. z-axis), the unit vector is away from z-axis.
- The unit vector $\bar{a}_\phi$ is normal to ϕ = constant surface and in the increasing direction of ϕ. Angle ϕ is increasing away from x-axis in anticlockwise direction.

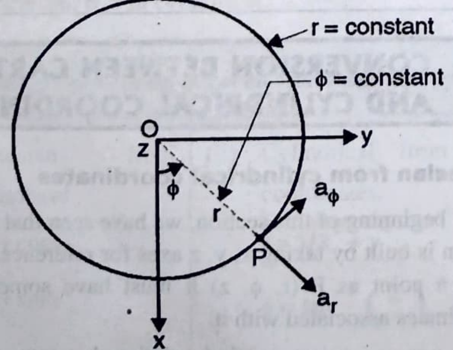


Fig. 1.7.2 : Showing $\bar{a}_r$, $\bar{a}_\phi$ vectors

- Separate explanation for $\bar{a}_z$ is not required as it is similar to $\bar{a}_z$ in Cartesian.
- Notice that the vectors $\bar{a}_r$ and $\bar{a}_\phi$ both are horizontal as the r = constant and ϕ = constant surfaces are vertical.
- You can get a perfect idea of $\bar{a}_r$ and $\bar{a}_\phi$ if we obtain top view of the coordinate system. Refer Fig. 1.7.2. The vector $\bar{a}_z$ is vertical at point P. Thus, $\bar{a}_r$, $\bar{a}_\phi$ and $\bar{a}_z$ are perpendicular to each other.

Unit vectors in cylindrical system : $\bar{a}_r$, $\bar{a}_\phi$, $\bar{a}_z$

1.7.3 Differential Lengths, Areas and Volume

Now let us find expressions for differential lengths, areas and volume.

(A) Differential lengths

- Consider point P(r, ϕ, z) in Fig. 1.7.3. Increasing the first coordinate by differential amount dr results in differential length dr.
- When ϕ is increased by differential amount dϕ, this will result in an arc with radius r and angle substended dϕ. Then the length of the arc is r dϕ. And finally when the third coordinate increased by dz it results in length dz.
- Hence, the differential lengths in cylindrical system are :

Differential lengths in cylindrical system : dr, r dϕ, dz

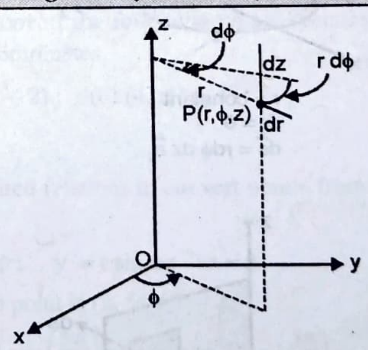


Fig. 1.7.3 : Showing differential lengths

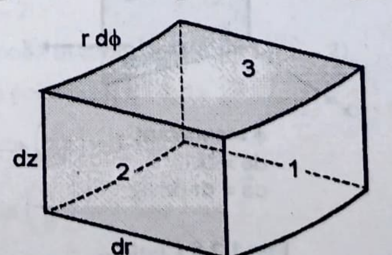


Fig. 1.7.4 : Showing differential volume

(B) Differential areas

- Let us find the differential areas and volume. Take a look at Fig. 1.7.4 in which the differential lengths are enlarged for better understanding.
- For surface (1), it is a curved surface, a part of cylindrical surface, i.e. r is constant, then dr = 0 and ds = r$d\phi$ dz.
- For surface (2), which is a vertical plane i.e. ϕ is constant, then $d\phi$ = 0 and ds = dr dz. For surface (3), which is horizontal, (z is constant) therefore dz = 0, the remaining differential lengths are dr and r$d\phi$. Thus differential area is dr r$d\phi$.
- When we write it in sequence r, ϕ, z it is ds = r dr $d\phi$.

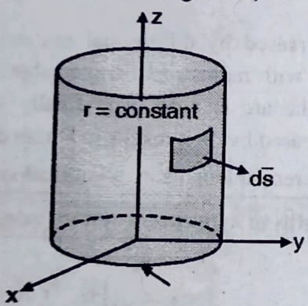
Differential areas

$ds_1 = r d\phi\, dz$ ...(r = constant)

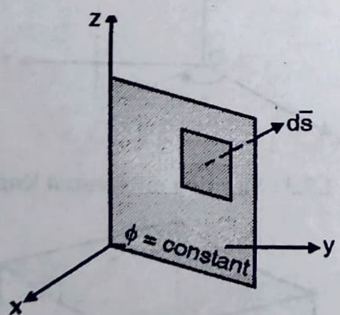
$ds_2 = dr\, dz$ ...(ϕ = constant)

$ds_3 = r dr\, d\phi$ ...(z = constant)

- Vector areas are obtained using similar procedure as in cartesian system. (Refer Fig. 1.7.5)

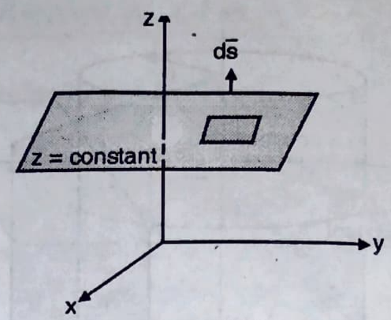


r = constant
dr = 0
$d\bar{s} = r d\phi\, dz\, \bar{a}_r$



ϕ = constant
$d\phi$ = 0
$d\bar{s} = dr\, dz\, \bar{a}_\phi$

Fig. 1.7.5 Contd...



z = constant
dz = 0
$d\bar{s} = r dr\, d\phi\, \bar{a}_z$

Fig. 1.7.5 : Differential vector areas

(C) Differential volume

The differential volume is multiplication of three lengths.

Differential volume in cylindrical system : dv = r dr $d\phi$ dz

The summary of the above discussion is given in Table 1.7.1.

Table 1.7.1 : Summary of cylindrical system

1.	Point is	(r, ϕ, z)
2.	Unit vector are	$\bar{a}_r$, $\bar{a}_\phi$, $\bar{a}_z$
3.	Differential lengths are	dr, r$d\phi$, dz
4.	Differential areas are	ds = r$d\phi$dz $\bar{a}_r$ ...(r = constant) ds = drdz $\bar{a}_\phi$ .... (ϕ = constant) ds = rdr$d\phi$ $\bar{a}_z$ ...(z = constant)
5.	Differential volume is	dv = r dr $d\phi$ dz
6.	Vector differential length	$d\bar{l} = dr\, \bar{a}_r + r\, d\phi\, \bar{a}_\phi + dz\, \bar{a}_z$

1.8 CONVERSION BETWEEN CARTESIAN AND CYLINDRICAL COORDINATES

(i) Cartesian from cylindrical coordinates

- In the beginning of this section, we have seen that cylindrical system is built by taking x, y, z axes for reference. When we write a point as P (r, ϕ, z) it must have some Cartesian coordinates associated with it.
- Here, we are going to find relation between r, ϕ, z in cylindrical and x, y, z in cartesian.

- To find x and y corresponding to P (r, ϕ, z), project pt. P on xy plane i.e. point Q. Project OQ on x and y axes.
- The length of projection on x-axis gives x coordinate (length OR) and length of projection on y axis (OS) gives y coordinate.
- In Fig. 1.8.1,

$$OR = OQ \cos\phi$$

i.e. $$x = r\cos\phi \quad \text{...(i)}$$

and $$OS = r\cos(90-\phi)$$

i.e. $$y = r\sin\phi \quad \text{...(ii)}$$

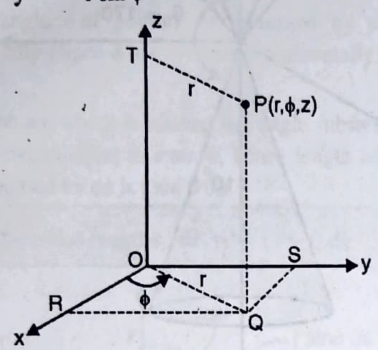


Fig. 1.8.1 : To show co-ordinate transformation

And z in cartesian is same as z in cylindrical

$$\therefore \quad z = z \quad \text{...(iii)}$$

(ii) Cylindrical from cartesian coordinates

- You can also make a reverse transformation i.e. point in cartesian can be transformed in cylindrical as follows :
- Squaring (i) and (ii) and adding ,

$$x^2 + y^2 = r^2(\cos^2\phi + \sin^2\phi) = r^2$$

or $$r = \sqrt{x^2+y^2} \quad \text{...(iv)}$$

- Dividing (ii) by (i)

$$\frac{y}{x} = \frac{r\sin\phi}{r\cos\phi} = \tan\phi$$

$$\therefore \quad \phi = \tan^{-1}\frac{y}{x} \quad \text{...(v)}$$

and z in cartesian = z in cylindrical

$$\therefore \quad z = z \quad \text{...(vi)}$$

Table 1.8.1 : Conversion between Cartesian and cylindrical

(i) Cartesian from cylindrical	(ii) Cylindrical from cartesian coordinates.
$x = r\cos\phi$	$r = \sqrt{x^2+y^2}$
$y = r\sin\phi$	$\phi = \tan^{-1}\left(\frac{y}{x}\right)$
$z = z$	$z = z$

Ex. 1.8.1 : Convert the following points specified in cartesian into cylindrical coordinates

(a) (0, – 2, 2), (b) ($\sqrt{3}$, 1, – 1) and (c) ($-\sqrt{2}, \sqrt{2}$, 3).

☑ **Soln. :**

To convert the points from cartesian to cylindrical, use relations

$$r = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}\frac{y}{x}, \quad z = z$$

(a) The given point is (0, – 2, 2)

then, $$r = \sqrt{0^2 + (-2)^2} = \sqrt{4} = 2; \quad \phi = \tan^{-1}\left(\frac{-2}{0}\right) = -\frac{\pi}{2}$$

but as y = – 2, the angle is $3\pi/2$ and z = 2.

Therefore, the point in cylindrical is (2, $3\pi/2$, 2)

(b) The given point is ($\sqrt{3}$, 1, – 1)

then, $$r = \sqrt{(\sqrt{3})^2 + (\sqrt{1})^2} = 2,$$

$$\phi = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \text{ and } z = -1$$

Therefore the point in cylindrical is (2, $\pi/6$, – 1).

(c) The given point is ($-\sqrt{2}, \sqrt{2}$, 3)

then, $$r = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = 2,$$

$$\phi = \tan^{-1}\frac{\sqrt{2}}{-\sqrt{2}} = -\frac{\pi}{4} \text{ and } z = 3$$

But as x coordinate is – ve, the angle is $3\pi/4$.

Therefore, the point in cylindrical is (2, $3\pi/4$, 3).

Ex. 1.8.2 : Convert the following points specified in cylindrical into cartesian coordinates

(a) (2, $5\pi/3$, – 2) ; (b) (4, $\pi/6$, 1)

☑ **Soln. :**

The required relations to convert points from cylindrical into cartesian are :

$$x = r\cos\phi; \quad y = r\sin\phi; \quad z = z$$

(a) The given point is (2, $5\pi/3$, – 2)

then, $$x = 2\cos\left(\frac{5\pi}{3}\right) = 1, \quad y = 2\sin\left(\frac{5\pi}{3}\right) = -1.73$$

and $z = -2$

Therefore, the point in cartesian is (1, – 1.73, – 2)

(b) The given point is (4, $\pi/6$, 1)

then, $$x = 4\times\cos\left(\frac{\pi}{6}\right) = 3.46,$$

$$y = 4\times\sin\left(\frac{\pi}{6}\right) = 2 \text{ and } z = -2$$

Therefore, the point in cartesian is (3.46, 2, 1).

1.9 SPHERICAL COORDINATE SYSTEM

1.9.1 Point in Spherical System

- For spherical coordinate system also x, y, z axes are used for reference. Imagine a sphere of radius r with center at origin.
- Any point on the sphere is at the same distance r from origin, therefore the shperical surface is defined as r = constant surface.
- Now consider a line from origin making angle θ with z-axis. Rotate this line about z-axis fixing the end at the origin.
- This forms a cone with angle θ, this conical surface is defined as a θ = constant surface.
- When a sphere with center at origin intersects with a vertical cone with vertex at origin, the intersection is a horizontal circle with radius equal to r sin θ [see Fig. 1.9.1].

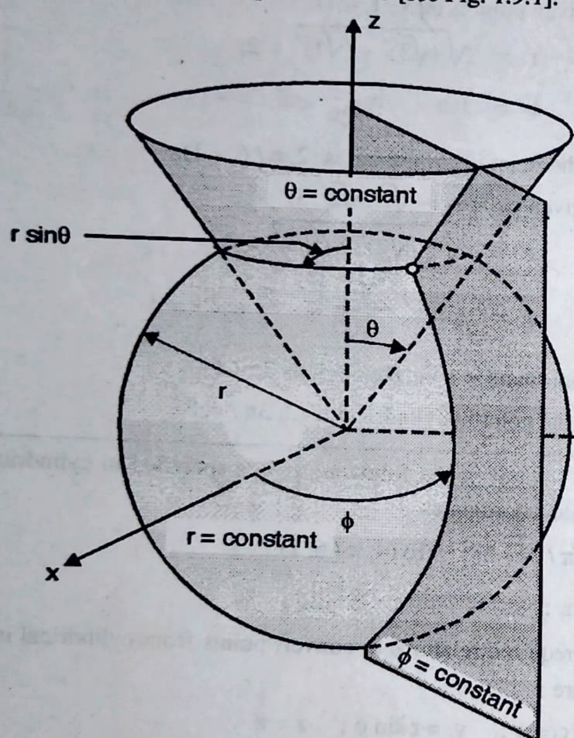


Fig. 1.9.1 : Spherical Coordinates

- We want to locate a point in spherical coordinate system. Imagine a ϕ = constant plane similar to in cylindrical system.
- A horizontal circle with center on z-axis, intersects ϕ = constant plane. The intersection is a point.

Point in spherical system : (r, θ, ϕ)

Because r = constant, θ = constant and ϕ = constant surface intersects at a point, the point is defined as (r, θ, ϕ). In spherical system variation of angle θ is from 0 to 180° and variation of ϕ is from 0 to 360°.

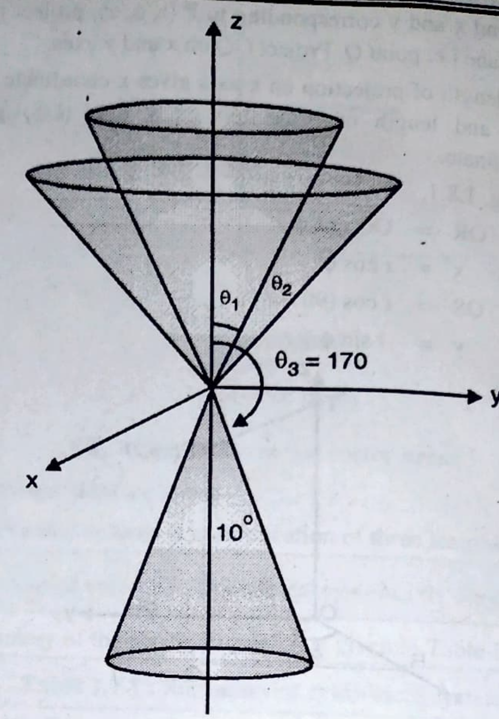


Fig. 1.9.2 : Variation of θ from 0 to 180° only

About the variation of angle θ

- Above we mentioned that the variation of angle θ is from 0 to 180° only, why ? The range of limits is from 0 to 180°, this fact can be best understood from the Fig. 1.9.2.
- By increasing angle θ slowly from zero, separate cones are formed for different angles. This is shown in Fig. 1.9.2. Let $\theta = \theta_3 = 170°$, the cone formed is shown in the figure.
- Now when θ is made 190°, the cone formed is similar to cone formed with θ = 170°.
- Thus, the cone repeats, for any angle greater than 180°. Hence, there is no point in increasing θ above 180°.

1.9.2 Unit Vectors in Spherical System

Unit vectors are $\bar{a}_r$, $\bar{a}_\theta$ and $\bar{a}_\phi$ perpendicular to r = constant, θ = constant and ϕ = constant surfaces respectively.

Unit vectors in spherical system : $\bar{a}_r$, $\bar{a}_\theta$, $\bar{a}_\phi$

1.9.3 Differential Lengths, Areas and Volume

(A) Differential lengths

- Now let us find the expressions for the differential lengths, differential areas and differential volume in spherical coordinate system.

- When the first coordinate i.e. r is increased by differential amount, the differential length is dr.
- When the second coordinate θ, is increased by differential amount dθ, it forms an arc with radius r and angle substended dθ. Therefore the length of the arc is rdθ.
- Now the third coordinate is increased by differential amount dϕ. When angle ϕ is increased by dϕ it results in an arc. To find length of the arc, refer Fig. 1.9.4. As we know the intersection of r = constant and θ = constant surfaces is a **horizontal circle of radius rsin θ.**
- When angle ϕ of point P is increased by dϕ, P rotates horizontally (since ϕ always varies horizontally) by an angle dϕ.
- Thus, the arc which is formed has angle substended equal to dϕ and radius equal to r sin θ, hence length of arc is when ϕ is increased by dϕ is r sin θ dϕ.

Differential lengths : dr, rdθ, r sin θ dϕ

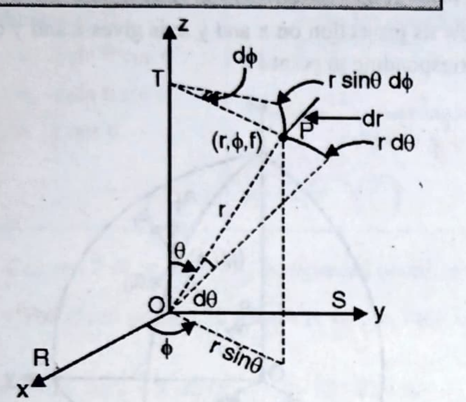


Fig. 1.9.3 : Differential lengths in spherical system

(B) Differential areas

- To find differential areas and volume see Fig. 1.9.5, in which differential lengths are shown enlarged for the sake of understanding.

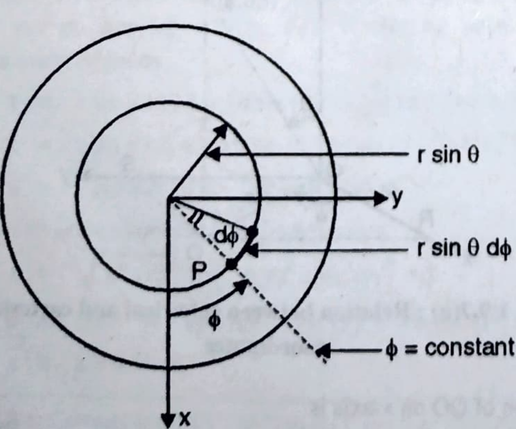


Fig. 1.9.4 Contd...

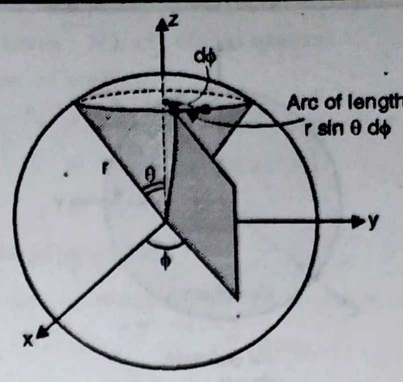


Fig. 1.9.4 : Illustration of arc length r sin θ dϕ

For surface (1); r is constant

$\therefore \quad dr = 0 \quad$ and $\quad ds = r^2 \sin\theta \, d\theta \, d\phi$

For surface (2); θ is constant

$\therefore \quad d\theta = 0 \quad$ and $\quad ds = r \sin\theta \, dr \, d\phi$

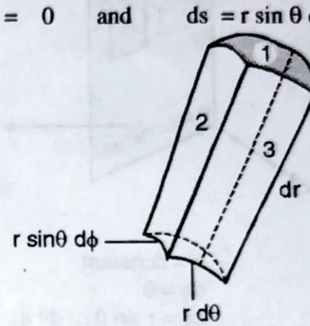


Fig. 1.9.5 : 3-d picture using differential lengths

For surface (3); ϕ is constant

$\therefore \quad d\phi = 0 \quad$ and $\quad ds = r \, dr \, d\theta$

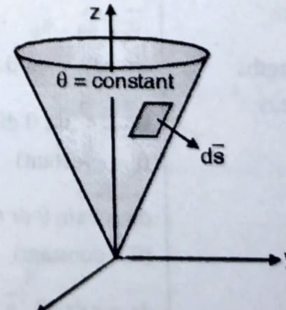
Thus, differential areas in spherical system are :

$$ds = r^2 \sin\theta \, d\theta \, d\phi \qquad \text{...(r = constant)}$$

$$ds = r \sin\theta \, dr \, d\phi \qquad \text{...(}\theta\text{ = constant)}$$

$$ds = r \, dr \, d\theta \qquad \text{...(}\phi\text{ = constant)}$$

Vector areas are again obtained by multiplying area and unit vector normal to it. (Refer Fig. 1.9.6)

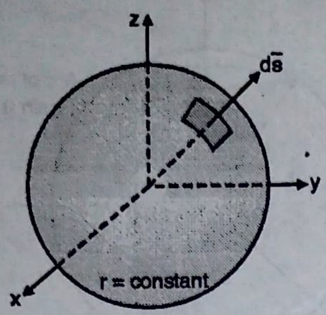


θ = constant

$d\theta = 0$

$\bar{ds} = r \sin\theta \, dr \, d\phi \, \bar{a}_\theta$

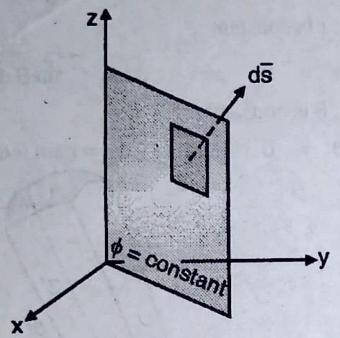
Fig. 1.9.6 Contd...



r = constant

dr = 0

$d\bar{s} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$



ϕ = constant

$d\phi = 0$

$d\bar{s} = r \sin\theta \, dr \, d\theta \, \bar{a}_\phi$

Fig. 1.9.6 : Differential vector areas

(C) Differential volume

The differential volume is multiplication of there lengths,

The differential volume : $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$

Summary of spherical system is given in Table 1.9.1.

Table 1.9.1 : Summary of spherical system

1.	Point is	(r, θ, ϕ)
2.	Unit vectors are	$\bar{a}_r, \bar{a}_\theta, \bar{a}_\phi$
3.	Differential lengths	$dr, r d\theta, r \sin\theta \, d\phi$.
4.	Differential areas	$d\bar{s} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$ (r = constant) $d\bar{s} = r \sin\theta \, dr \, d\phi \, \bar{a}_\theta$ (θ = constant) $d\bar{s} = r \, dr \, d\theta \, \bar{a}_\phi$ (ϕ = constant)
5.	Differential volume	$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$
6.	Vector differential length	$d\bar{l} = dr \, \bar{a}_r + r d\theta \, \bar{a}_\theta + r \sin\theta \, d\phi \, \bar{a}_\phi$

1.9.4 Conversion between Cartesian and Spherical Coordinates

(i) Cartesian from spherical coordinates

- The unit vectors in spherical system are $\bar{a}_r, \bar{a}_\theta$ and $\bar{a}_\phi$. These unit vectors are perpendicular to r = constant, θ = constant, and ϕ = constant surface respectively and in the increasing directions of the r, θ and ϕ respectively. In Fig. below only the first octant is shown for simplicity, and the unit vectors are also shown in Fig. 1.9.7.
- When point P (r, θ, ϕ) is present in Cartesian system, it must have corresponding cartesian coordinates.
- Here we are going to find relation between spherical coordinates and Cartesian coordinates.
- To find x and y corresponding to point P, project point P in xy plane (Fig. 1.9.7(a)). The projection is $OQ = r \sin\theta$.
- Now its projection on x and y axis gives x and y coordinates corresponding to point P.

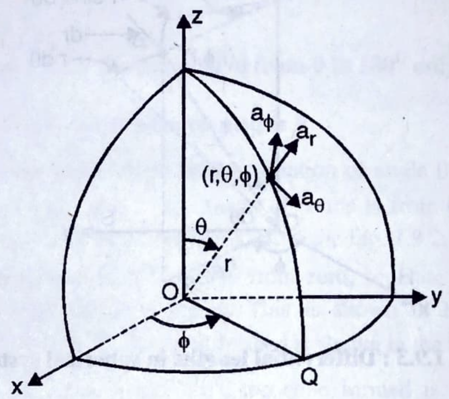


Fig. 1.9.7 : Unit vectors in spherical system

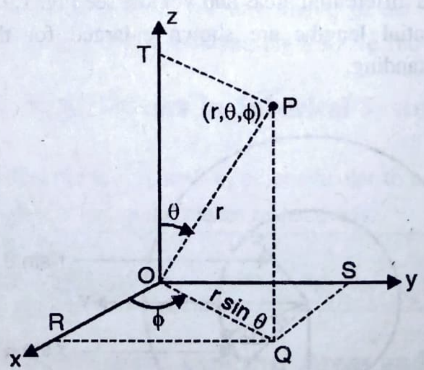


Fig. 1.9.7(a) : Relation between spherical and cartesian coordinates

Projection of OQ on x-axis is

$$OR = OQ \cos\phi = r \sin\theta \cos\phi$$

$$\therefore \quad x = r \sin\theta \cos\phi \quad \text{...(i)}$$

Projection of OQ on y-axis is

$$OS = OQ \sin\phi = r\sin\theta\sin\phi$$

$$\therefore \quad y = r\sin\theta\sin\phi \quad ...(ii)$$

Projection of OP on z-axis gives z coordinates of P.

Projection of OP on z-axis = OT = OP cos θ

$$\therefore \quad z = r\cos\theta \quad ...(iii)$$

(ii) Spherical from Cartesian coordinates

The reverse transformation is obtained from (i), (ii) and (iii), it is

$$r = \sqrt{x^2+y^2+z^2}\,;$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right);$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

Table 1.9.2 : Conversion between Cartesian and spherical

i) Cartesian from spherical coordinates.	ii) Spherical from cartesian coordinates.
$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$r = \sqrt{x^2+y^2+z^2}$ $\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

Ex. 1.9.1 : Convert P (8, π / 3, π / 6) in cartesian coordinates.

☑ **Soln. :** The given point is in spherical system then using the relations,

$$x = r\sin\theta\cos\phi = 8\sin(\pi/3)\cos(\pi/6) = 6,$$

$$y = r\sin\theta\sin\phi = 8\sin(\pi/3)\sin(\pi/6) = 3.46,$$

$$z = r\cos\theta = 8\cos(\pi/3) = 4$$

Thus the point in cartesian is (6, 3.64, 4).

Ex. 1.9.2 : Convert P (10, π / 6, π / 3) in cylindrical.

☑ **Soln. :**

Since direct conversion from spherical to cylindrical is not derived, we go step by step by first converting point P into cartesian using relations.

$$x = r\sin\theta\cos\phi = 10\sin(\pi/6)\cos(\pi/3) = 2.5\text{ m},$$

$$y = r\sin\theta\sin\phi = 10\sin(\pi/6)\sin(\pi/3) = 4.33\text{ m},$$

$$z = r\cos\theta = 10\cos(\pi/6) = 8.66\text{ m}.$$

Now convert into cylindrical coordinates using

$$r = \sqrt{x^2+y^2} = \sqrt{(2.5)^2+(4.33)^2} = 5$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(4.33/2.5) = 59.99°,$$

$$z = z = 8.66\text{ m}$$

Ex. 1.9.3 : Convert P (3, 45°, 60°) in (i) Cartesian (ii) Cylindrical co-ordinate system.

☑ **Soln. : Given :** P(3, 45°, 60°) in spherical

(i) Conversion in Cartesian

$$x = r\sin\theta\cos\phi = 1.06\text{ (m)}$$

$$y = r\sin\theta\sin\phi = 1.84\text{ (m)}$$

$$z = r\cos\theta = 2.12\text{ (m)}$$

(ii) Conversion in cylindrical

It can be done by using Cartesian coordinates in part (i)

$$r = \sqrt{x^2+y^2} = 2.12\text{ (m)}$$

$$\phi = \tan^{-1}(y/x) = 60.0°$$

$$z = z = 2.12\text{ (m)}$$

1.10 SUMMARY OF THREE COORDINATE SYSTEMS

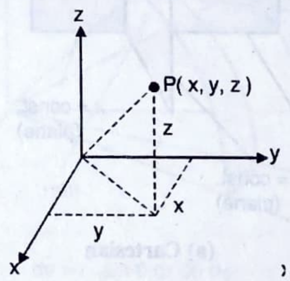
$$x = r\cos\phi \qquad r = \sqrt{x^2+y^2} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$= r\sin\theta\cos\phi \qquad = r\sin\theta \qquad = \sqrt{r^2+z^2}$$

$$y = r\sin\phi \qquad \phi = \tan^{-1}(y/x) \qquad \theta = \tan^{-1}(\sqrt{x^2+y^2}/z$$

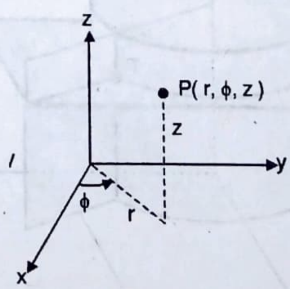
$$= r\sin\theta\sin\phi \qquad\qquad = \tan^{-1}(r/z))$$

$$z = r\cos\theta$$



$-\infty < x < \infty$
$-\infty < y < \infty$
$-\infty < z < \infty$

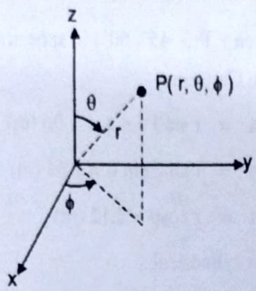
(a)



$0 \le r < \infty$
$0 \le \phi \le 2\pi$

(b)

Fig. 1.10.1 Contd...

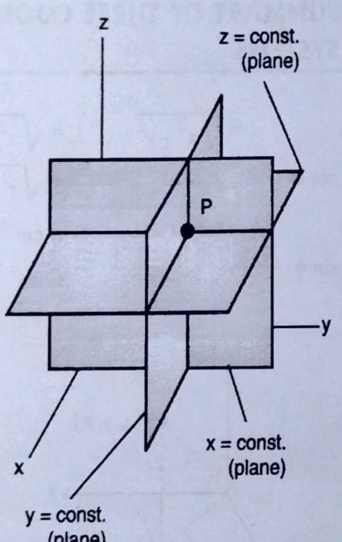


$0 \le r < \infty$

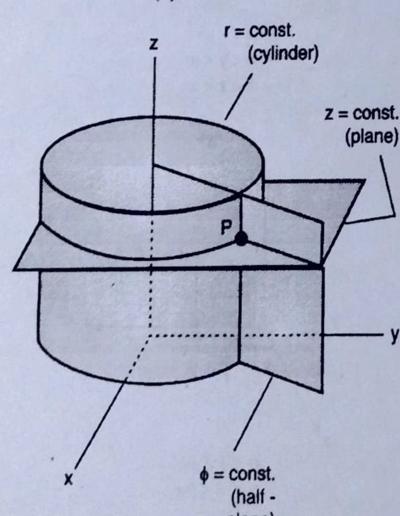
$0 \le \theta \le \pi$

(c)

Fig. 1.10.1 : Defining point P (a) P (x, y, z) in Cartesian coordinates; (b) P (r ϕ, z) in circular coordinates; and (c) P (r, θ, ϕ) in spherical coordinates

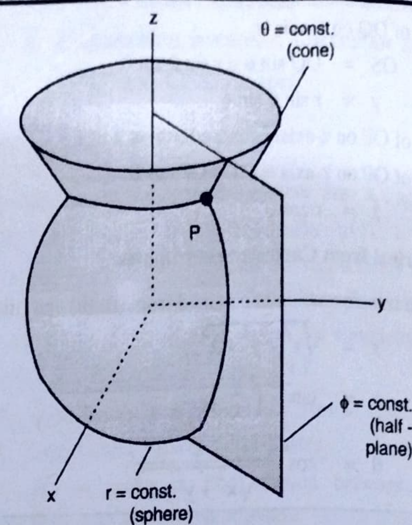


(a) Cartesian



(b) Cylindrical

Fig. 1.10.2 Contd...



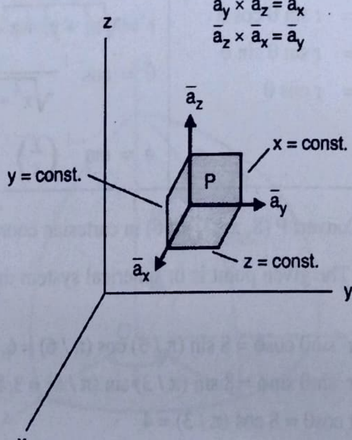
(c) Spherical coordinates

Fig. 1.10.2

$\bar{a}_x \times \bar{a}_y = \bar{a}_z$

$\bar{a}_y \times \bar{a}_z = \bar{a}_x$

$\bar{a}_z \times \bar{a}_x = \bar{a}_y$

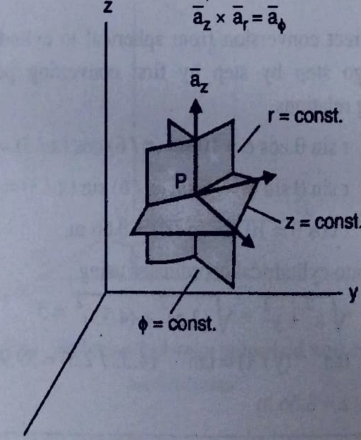


(a) Cartesian

$\bar{a}_r \times \bar{a}_\phi = \bar{a}_z$

$\bar{a}_\phi \times \bar{a}_z = \bar{a}_r$

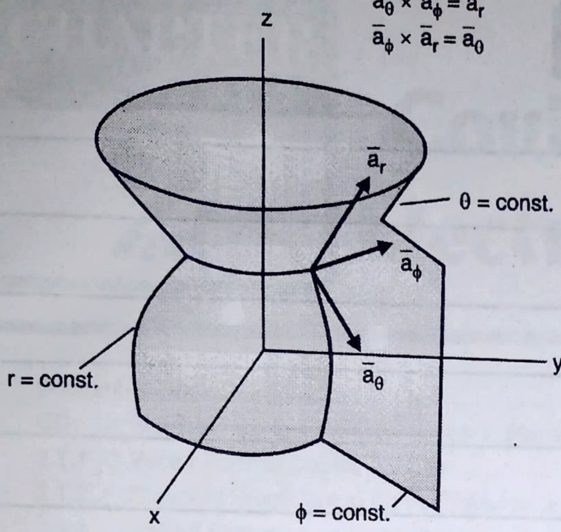
$\bar{a}_z \times \bar{a}_r = \bar{a}_\phi$



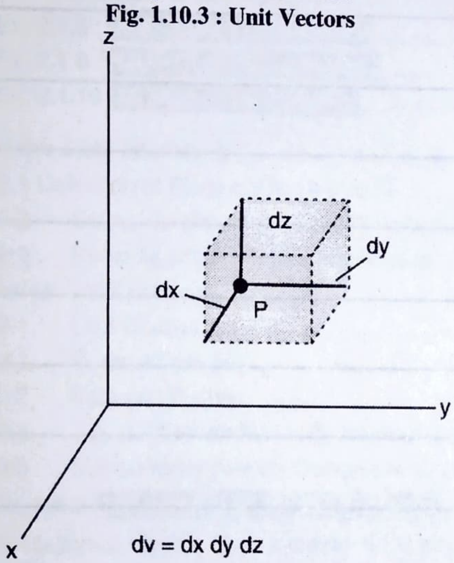
(b) Cylindrical

Fig. 1.10.3 Contd...

$$\bar{a}_r \times \bar{a}_\theta = \bar{a}_\phi$$
$$\bar{a}_\theta \times \bar{a}_\phi = \bar{a}_r$$
$$\bar{a}_\phi \times \bar{a}_r = \bar{a}_\theta$$

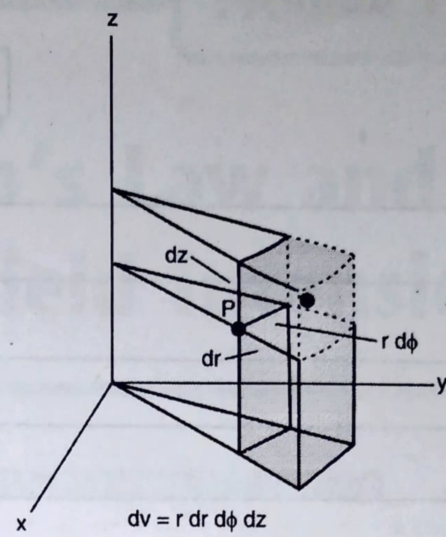


(c) Spherical coordinates
Fig. 1.10.3 : Unit Vectors



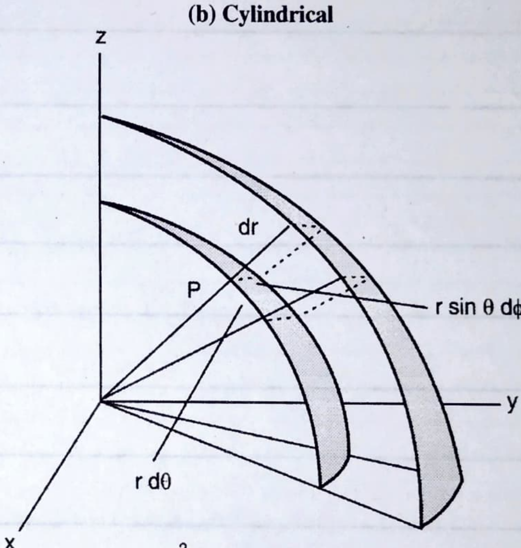
dv = dx dy dz

(a) Cartesian
Fig. 1.10.4 Contd...



dv = r dr dφ dz

(b) Cylindrical



$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$

(c) Spherical coordinates
Fig. 1.10.4 : Differential elements of vector length, vector area and scalar volume

Chapter Ends...

❑❑❑

Module 1

CHAPTER 2 Coulomb's Law and Electric Field Intensity

Introduction

- In basic course of physics you have already studied that between two point charges either a force of attraction or repulsion exists depending upon the nature of charges.
- In the course we calculated only magnitude of the force, it was not expressed as vector. But force is a vector quantity and acts along the line joining two point charges. The law used to calculate the force is called Coulomb's law.
- If number of charges are present in media then force on any charge due to other charges can be obtained separately using Coulomb's law and then the total force is obtained by adding all forces. This is called principle of superposition.
- In general the charges may be in motion and we require to calculate the force between them. This is a complicated situation and the solution can be obtained only after you know how to find force between static charges. So in the initial part of this chapter stress is given to solve number of problems on Coulomb's law.
- After becoming master of solving problems of force, the concept is extended to define electric field. In electrostatics we consider electric field generated by charges at rest and the field do not change with time.
- In electrostatics we do not consider magnetic field and then the electrostatic relations are very simple. Note that the electric field can also be produced by other types of charges like line charge, surface charge or volume charge.
- Once we get formula for electric field due to point charge, we can extend this idea to obtain electric field due to other types of charges.
- In this chapter after solving problems on Coulomb's law, number of problems are solved on electric field.

2.1 COULOMB'S LAW

UQ. State and explain Coulomb's Law in Electrostatics.
MU - May 09, 2 Marks, Dec. 12, 5 Marks

Statement

Coulomb's law states - between two point charges there is a force of attraction or repulsion depending upon the nature of charges.

If the charges are like charges there is a force of repulsion and in case of unlike charges it is force of attraction.

The force is a vector quantity which has both magnitude and direction. The magnitude of force between two electrical charges is given by Coulomb law.

Experiments conducted by Coulomb showed that the following hold for two charged bodies that are very small in size compared to their separation so that they can be considered as point charge :

1. The magnitude of the force is proportional to the product of the magnitudes of the charges.
2. The magnitude of the force is inversely proportional to the square of the distance between the charges.
3. The magnitude of the force depends on the medium.

Thus, if we consider two point charges Q_1 (C) and Q_2 (C) separated by a distance R (m) in free space, the force is given mathematically as,

$$F = K\frac{Q_1 Q_2}{R^2} \text{ (N)}$$

For free space the constant of proportionality (K) is given by

$$K = \frac{1}{4\pi\epsilon_0}$$

Where, ϵ_0 is known as the permittivity of free space. It's value is

$$\epsilon_0 = 8.854\times10^{-12} \approx 10^{-9}/36\pi \text{ (F/m)}$$

We shall use throughout this book a value of $10^{-9}/36\pi$ for ϵ_0, because by using this figure two π's in the expression F gets cancel which simplifies the calculations.

$$\therefore \quad F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \text{ (N)}$$

2.1.1 Vector Form of Coulomb's Law

Force is a vector quantity and acts along the line joining the two charges therefore the above expression is to be multiplied by a unit vector along the line. Thus, vector form is

$$\text{Force on } Q_2 \text{ is, } \bar{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\bar{a}_{12} \quad ...(2.1.1)$$

$$\text{and on } Q_1 \text{ is} \quad \bar{F}_1 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2}\bar{a}_{21} \quad ...(2.1.2)$$

Since, $\bar{a}_{12} = -\bar{a}_{21}$, We can write

$$\bar{F}_1 = -\bar{F}_2 \quad ...(2.1.3)$$

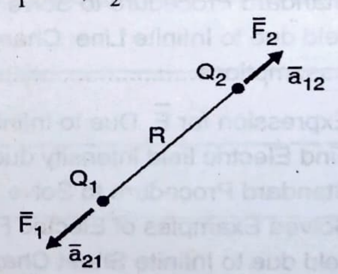


Fig. 2.1.1 : Force experienced by Q_1 and Q_2

The unit of force is Newton (N). The force given by Equations (2.1.1) and (2.1.2) is applicable only for point charges. Usually in the problems on Coulomb's law, position of the point charges is given, charges are given and may be asked to find force on a particular charge.

2.1.2 Standard Procedure to Solve Problems on Coulomb's Law

- **Step 1 :** Draw a neat sketch.
- **Step 2 :** Find vector joining these charges and pointing towards the charge on which force is to be determined. For example, while finding force on Q_1 i.e. $\bar{F}_1$ arrow of the vector points towards Q_1, while finding force on Q_2 i.e. $\bar{F}_2$, the arrow of the vector points towards Q_2.
- **Step 3 :** Find unit vector in the direction of vector and length of the vector (i.e. distance between charges).
- **Step 4 :** Use Coulomb's law to find force between point charges.

2.1.3 Solved Examples on Point Charge

Important Formulae

$$\bar{F}_2 = \frac{Q_1 Q_2}{4 \pi \epsilon_0 R_{12}^2} \bar{a}_{12}$$

$$\bar{F}_1 = -\bar{F}_2$$

Ex. 2.1.1 : A point charge $Q_1 = 2$ mC is located in free space at $P_1(-3, 7, -4)$ while $Q_2 = 5$ mC is at $P_2(2, 4, -1)$.

Find $\bar{F}_2$ and $\bar{F}_1$.

☑ **Soln. :**

Force acting on Q_2 due to Q_1 is $\bar{F}_2$. It is directed from Q_1 to Q_2. The vector joining charges 1 and 2 and pointing towards P_2 is

$$\bar{R}_{12} = 5\bar{a}_x - 3\bar{a}_y + 3\bar{a}_z$$

and
$$\bar{a}_{12} = \frac{5\bar{a}_x - 3\bar{a}_y + 3\bar{a}_z}{\sqrt{43}}$$

Force on Q_2 is,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4 \pi \epsilon_0 R_{12}^2} \bar{a}_{12}$$

$$= \frac{2 \times 10^{-3} \times 5 \times 10^{-3}}{4\pi \times 10^{-9} / 36\pi \times (\sqrt{43})^2} \times \left(\frac{5\bar{a}_x - 3\bar{a}_y + 3\bar{a}_z}{\sqrt{43}}\right)$$

$$\bar{F}_2 = 319.183\,(5\bar{a}_x - 3\bar{a}_y + 3\bar{a}_z)$$

$$\bar{F}_2 = 1.596\,\bar{a}_x - 0.958\,\bar{a}_y + 0.958\,\bar{a}_z \text{ (kN)}$$

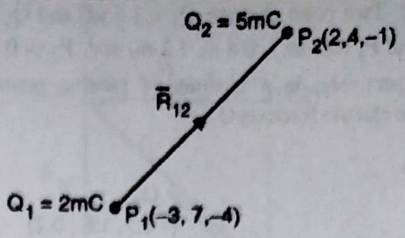


Fig. : Illustrating Ex. 2.1.1

The force on Q_1 is $\bar{F}_1$, obtained from $\bar{F}_2$ as $\bar{F}_1 = -\bar{F}_2$

So
$$\bar{F}_1 = -(1.595\,\bar{a}_x - 0.958\,\bar{a}_y + 0.958\,\bar{a}_z)$$

$$\bar{F}_1 = -1.595\,\bar{a}_x + 0.958\,\bar{a}_y - 0.958\,\bar{a}_z \text{ (kN)}$$

Ex. 2.1.2 : Point charges $Q_1 = 300$ μC located at (1, –1, –3) m experiences a force $\bar{F}_1 = 8\bar{a}_x - 8\bar{a}_y + 4\bar{a}_z$ N due to point charge Q_2 at (3, – 3, – 2) m. Determine Q_2.

☑ **Soln. :** Vector joining point 2 to 1 is $\bar{R}_{21}$, given by,

$$\bar{R}_{21} = -2\bar{a}_x + 2\bar{a}_y - \bar{a}_z$$

$$\therefore \quad \bar{a}_{21} = \frac{-2\bar{a}_x + 2\bar{a}_y - \bar{a}_z}{3}$$

The force is calculated as :

$$\bar{F}_1 = \bar{F}_{21} = \frac{Q_1 Q_2}{4 \pi \epsilon_0 R_{21}^2} \bar{a}_{21}$$

It is given that,

$$\bar{F}_1 = 8\bar{a}_x - 8\bar{a}_y + 4\bar{a}_z \text{ (N)}$$

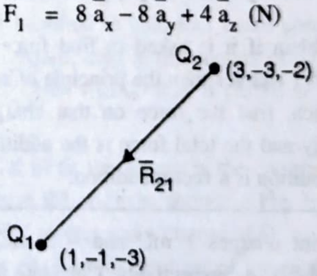


Fig. : Illustrating Ex. 2.1.2

Then,

$$8\bar{a}_x - 8\bar{a}_y + 4\bar{a}_z = \frac{300 \times 10^{-6} \times Q_2}{4\pi \times 10^{-9} / 36\pi \times (3)^2} \times \frac{-2\bar{a}_x + 2\bar{a}_y - \bar{a}_z}{3}$$

i.e. $8\bar{a}_x - 8\bar{a}_y + 4\bar{a}_z = 10^5 \times Q_2(-2\bar{a}_x + 2\bar{a}_y - \bar{a}_z)$

i.e. $-4(-2\bar{a}_x + 2\bar{a}_y - \bar{a}_z) = 10^5 \times Q_2(-2\bar{a}_x + 2\bar{a}_y - \bar{a}_z)$

Comparing, $Q_2 = -40$ μC ...Ans.

Ex. 2.1.3 : Two point charges, $Q_1 = 1.6\ \mu C$ and $Q_2 = -2.2\ \mu C$ are located at P_1 (0.5 m, – 0.8 m, 1.2 m) and P_2 (– 0.7 m, 1.5 m, 0.7 m) respectively, in a medium of relative permittivity 2.6, calculate the electric force on Q_1.

☑ **Soln. :**

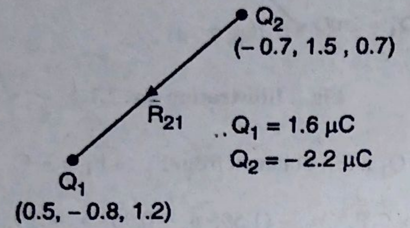


Fig. Ex. 2.1.3

$$\bar{R}_{21} = 1.2\ \bar{a}_x - 2.3\ \bar{a}_y + 0.5\ \bar{a}_z$$

$$\bar{a}_{R21} = \frac{1.2\ \bar{a}_x - 2.3\ \bar{a}_y + 0.5\ \bar{a}_z}{\sqrt{1.2^2 + 2.3^2 + 0.5^2}} = \frac{1.2\ \bar{a}_x - 2.3\ \bar{a}_y + 0.5\ \bar{a}_z}{2.642}$$

Force on Q_1 is

$$\bar{F}_{21} = \frac{Q_1 Q_2}{4\pi \epsilon_0 R_{21}^2}\ \bar{a}_{R21}$$

$$= \frac{1.6 \times 10^{-6} \times (-2.2 \times 10^{-6})}{4\pi \left(\frac{10^{-9}}{36\pi}\right) \times 2.642^2} \left(\frac{1.2\ \bar{a}_x - 2.3\ \bar{a}_y + 0.5\ \bar{a}_z}{2.642}\right)$$

$$= -0.00172\ (1.2\ \bar{a}_x - 2.3\ \bar{a}_y + 0.5\ \bar{a}_z)$$

$$\bar{F}_{21} = -\ 0.002064\ \bar{a}_x + 0.003956\ \bar{a}_y - 0.00086\ \bar{a}_z\ \text{(N)}$$

2.1.4 Principle of Superposition

In the problem if it is asked to find force on a particular charge due to other charges then the principle of superposition can be used. In which, find the force on that charge due to other charges separately and the total force is the addition of all forces. Obviously this addition is a vector addition.

Ex. 2.1.4 : Point charges 1 mC and – 2 mC are located at (3, 2, –1) and (–1, –1, 4) respectively. Calculate the electric force on a 10 nC charge located at (0, 3, 1).

☑ **Soln. :**

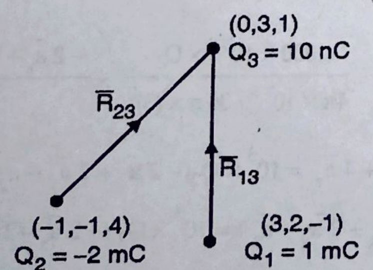


Fig. : Illustrating Ex. 2.1.4

Required vectors and unit vectors are :

$$\bar{R}_{13} = -3\ \bar{a}_x + \bar{a}_y - 2\ \bar{a}_z$$

$$\therefore \quad \bar{a}_{R13} = \frac{-3\ \bar{a}_x + \bar{a}_y - 2\ \bar{a}_z}{\sqrt{14}}$$

$$\bar{R}_{23} = \bar{a}_x + 4\ \bar{a}_y - 3\ \bar{a}_z$$

$$\therefore \quad \bar{a}_{R23} = \frac{\bar{a}_x + 4\bar{a}_y - 3\ \bar{a}_z}{\sqrt{26}}$$

Force on Q_3 due to Q_1 is,

$$\bar{F}_{13} = \frac{Q_1 Q_3}{4\pi \epsilon_0 R_{13}^2} \times \bar{a}_{R13}$$

$$= \frac{(1 \times 10^{-3})\ (10 \times 10^{-9})}{4\pi \times \frac{10^{-9}}{36\pi} \times (\sqrt{14})^2} \times \left(\frac{-3\ \bar{a}_x + \bar{a}_y + 2\ \bar{a}_z}{\sqrt{14}}\right)$$

$$= -5.16\ \bar{a}_x + 1.72\ \bar{a}_y + 3.44\ \bar{a}_z\ \text{(mN)}$$

Force on Q_3 due to Q_2 is,

$$\bar{F}_{23} = \frac{Q_2 Q_3}{4\pi \epsilon_0 R_{23}^2}\ \bar{a}_{R23}$$

$$= \frac{(-2 \times 10^{-3})\ (10 \times 10^{-9})}{4\pi \times \frac{10^{-9}}{36\pi} \times (\sqrt{26})^2} \times \left(\frac{\bar{a}_x + 4\bar{a}_y - 3\ \bar{a}_z}{\sqrt{26}}\right)$$

$$= -1.36\ \bar{a}_x - 5.44\ \bar{a}_y + 4.08\ \bar{a}_z\ \text{(mN)}$$

Using superposition total force on Q_3 is,

$$\bar{F}_3 = \bar{F}_{13} + \bar{F}_{23} = -6.52\ \bar{a}_x - 3.72\ \bar{a}_y + 7.52\ \bar{a}_z\ \text{(mN)} \quad \text{...Ans.}$$

UEx. 2.1.5 **(MU - May 11, 10 Marks)**

Four like charges of 30 μC each are located at four corners of a square, the diagonal of which measures 8 m. Find the force on a 150 μC charge located 3 m above the centre of the square.

☑ **Soln. :** Given square can be placed in coordinate system as shown in figure. I have selected this orientation, as this position resemble with the orientation in previous problem. Refer Fig. Ex. 2.1.5.

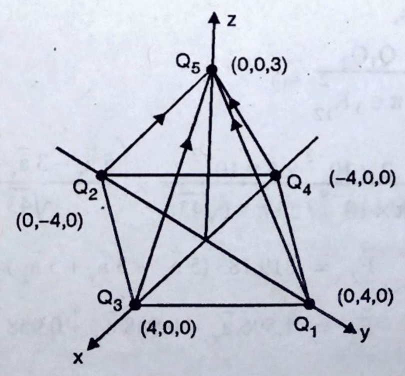


Fig. : Illustrating Ex. 2.1.5

Required vectors and unit vectors are :

$$\bar{R}_{15} = -4\bar{a}_y + 3\bar{a}_z \; ; \qquad \bar{a}_{15} = \frac{-4\bar{a}_y + 3\bar{a}_z}{5};$$

$$\bar{R}_{25} = 4\bar{a}_y + 3\bar{a}_z \; ; \qquad \bar{a}_{25} = \frac{4\bar{a}_y + 3\bar{a}_z}{5};$$

$$\bar{R}_{35} = -4\bar{a}_x + 3\bar{a}_z \; ; \qquad \bar{a}_{35} = \frac{-4\bar{a}_x + 3\bar{a}_z}{5}$$

$$\text{and}\bar{R}_{45} = 4\bar{a}_x + 3\bar{a}_z \; ; \qquad \bar{a}_{45} = \frac{4\bar{a}_x + 3\bar{a}_z}{5}.$$

Force on Q_5 can be calculated on the similar lines as in the previous problem. The answer is,

$$\bar{F}_5 = 0.324 \times 3\,\bar{a}_z \times 4$$

$$\bar{F}_5 = 3.888\,\bar{a}_z \text{ (N).} \qquad \text{...Ans.}$$

Ex. 2.1.6 : Four points charges of 3 nC each are placed at four corners of a square 2 meters in side. Find the force acting on each charge.

☑ **Soln. :**

We can place the square in cartesian coordinate system as shown in Fig. Ex. 2.1.6. Different vectors and unit vectors required are :

$$\bar{R}_{14} = 2\,\bar{a}_y$$

$$\therefore \quad \bar{a}_{14} = \bar{a}_y \; ;$$

$$\bar{R}_{24} = -2\,\bar{a}_x + 2\,\bar{a}_y$$

$$\therefore \quad \bar{a}_{24} = \frac{-\bar{a}_x + \bar{a}_y}{\sqrt{2}}$$

$$\bar{R}_{34} = -2\,\bar{a}_x$$

$$\therefore \quad \bar{a}_{34} = -\bar{a}_x$$

The force on Q_4 due to Q_1 is :

$$\bar{F}_{14} = \frac{Q_1 Q_4}{4\pi\epsilon_0 R_{14}^2}\,\bar{a}_{14}$$

$$\bar{F}_{14} = \frac{3\times10^{-9} \times 3\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} \times (2)^2}\,(\bar{a}_y) = 20.25\,\bar{a}_y \text{ (nN)}$$

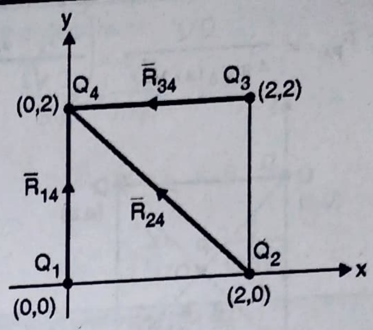


Fig. : Illustrating Ex. 2.1.6

The force on Q_4 due to Q_2 is :

$$\bar{F}_{24} = \frac{Q_2 Q_4}{4\pi\epsilon_0 R_{24}^2}\,\bar{a}_{24} = \frac{3\times10^{-9} \times 3\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} \times (2\sqrt{2})^2} \times \left(\frac{-\bar{a}_x + \bar{a}_y}{\sqrt{2}}\right)$$

$$= 7.16\,(-\bar{a}_x + \bar{a}_y) \text{ (nN)}$$

Force on Q_4 due to Q_3 is :

$$\bar{F}_{34} = \frac{Q_3 Q_4}{4\pi\epsilon_0 R_{34}^2}\,\bar{a}_{34} = \frac{3\times10^{-9} \times 3\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} \times (2)^2}\,(-\bar{a}_x)$$

$$= -20.25\,\bar{a}_x \text{ (nN)}$$

The total force using principle of superposition is

$$\bar{F}_4 = \bar{F}_{14} + \bar{F}_{24} + \bar{F}_{34}$$

$$\bar{F}_4 = 27.41\,(-\bar{a}_x + \bar{a}_y) \text{ (nN)} \qquad \text{...Ans.}$$

Ex. 2.1.7 : It is required to hold four equal point charges 40 μC Coulomb each in equilibrium at the corners of a square. Calculate the point charge, which will do this, if placed at the centre of the square.

☑ **Soln. :** Let us fit the square in the coordinate system such that one corner is at the origin as shown in Fig. Ex. 2.1.7. Let side of the square be 'a' meters and a charge of Q' (= 40 μC) is placed at the centre. For equilibrium resultant of forces acting on any one charge, say A, due to all other charges must be zero. Force acting on charge at A due to remaining charges is obtained as :

$$\bar{F}_{DA} = \frac{QQ}{4\pi\epsilon_0 a^2}(-\bar{a}_y),$$

$$\bar{F}_{BA} = \frac{QQ}{4\pi\epsilon_0 a^2}(-\bar{a}_x),$$

$$\bar{F}_{CA} = \frac{QQ}{4\pi\epsilon_0 (\sqrt{2}a)^2} \times \left(\frac{-\bar{a}_x - \bar{a}_y}{\sqrt{2}}\right)$$

and $\bar{F}_{PA} = \dfrac{QQ'}{4\pi\epsilon_0 (a/\sqrt{2})^2} \times \left(\dfrac{-\bar{a}_x - \bar{a}_y}{\sqrt{2}}\right)$

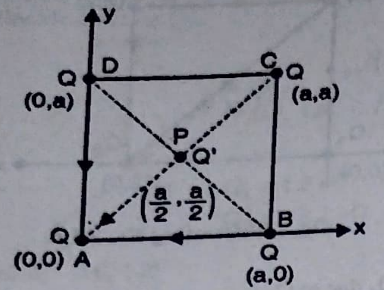


Fig. : Illustrating Ex. 2.1.7

For equilibrium, the resultant of these forces at A must be zero.

i.e. $\bar{F}_A = \bar{F}_{DA} + \bar{F}_{BA} + \bar{F}_{CA} + \bar{F}_{PA} = 0$

i.e. $\dfrac{Q}{4\pi\epsilon_0 a^2}\left\{-Q\bar{a}_y - Q\bar{a}_x - Q\left(\dfrac{\bar{a}_x + \bar{a}_y}{2\sqrt{2}}\right) - Q'\sqrt{2}(\bar{a}_x + \bar{a}_y)\right\} = 0$

i.e. $-\bar{a}_x\left[Q + \dfrac{Q}{2\sqrt{2}} + Q'\sqrt{2}\right] - \bar{a}_y\left[Q + \dfrac{Q}{2\sqrt{2}} + Q'\sqrt{2}\right] = 0$

i.e. $Q + \dfrac{Q}{2\sqrt{2}} + Q'\sqrt{2} = 0$

i.e. $Q' = -\left(Q + \dfrac{Q}{2\sqrt{2}}\right) \times \dfrac{1}{\sqrt{2}} = (-0.9571)\,Q\ (C)$

Putting $Q = 40\ \mu C$,

$Q' = -0.9571 \times 40 \times 10^{-6}$

$= 38.28\ (\mu C)$ ...Ans.

UEx. 2.1.8 (MU - Dec. 09, 10 Marks)

It is required to hold three equal point charges of + Q each in equilibrium at the corners of an equilateral triangle. Calculate the point charge which will do this if placed at the centre of a triangle.

☑ **Soln. :** Let us place the equilateral triangle as shown in Fig. Ex. 2.1.8. Assume side of the triangle be 'a', and the charge required for equilibrium be Q'.

We know that $AP = BP = CP = a/\sqrt{3}$

$\therefore$ y coordinate of P is $= a/\sqrt{3} \times \cos 60° = a/2\sqrt{3}$

Thus, coordinates of different points are :

$A\,(0, 0)$; $B\,(a, 0)$; $C\left(\dfrac{a}{2}, \dfrac{\sqrt{3}}{2}a\right)$; $P\left(\dfrac{a}{2}, \dfrac{a}{2\sqrt{3}}\right)$

For equilibrium total force acting on any charge must be zero. Force on Q at A due to charge at B is,

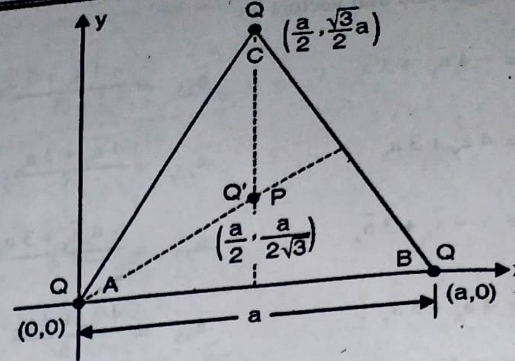


Fig. : Illustrating Ex. 2.1.8

$\bar{F}_{BA} = \dfrac{QQ}{4\pi\epsilon_0 a^2}(-\bar{a}_x)$

Force on Q at A due to charge at C is,

$\bar{F}_{CA} = \dfrac{QQ}{4\pi\epsilon_0 a^2} \times \left(\dfrac{-\frac{a}{2}\bar{a}_x - \frac{\sqrt{3}a}{2}\bar{a}_y}{a}\right)$

$= \dfrac{QQ}{4\pi\epsilon_0 a^2} \times \left(-\dfrac{\bar{a}_x}{2} - \dfrac{\sqrt{3}}{2}\bar{a}_y\right)$

and force on Q at A due to charge at P is,

$\bar{F}_{PA} = \dfrac{QQ'}{4\pi\epsilon_0 (a/\sqrt{3})^2} \times \left(\dfrac{-\frac{a}{2}\bar{a}_x - \frac{a}{2\sqrt{3}}\bar{a}_y}{\frac{a}{\sqrt{3}}}\right)$

Total force on Q at A must be zero for equilibrium. That is,

$\therefore\ \bar{F}_A = \bar{F}_{BA} + \bar{F}_{CA} + \bar{F}_{PA} = 0$

$= \dfrac{-Q}{4\pi\epsilon_0 a^2} \times [Q\bar{a}_x + Q/2\,\bar{a}_x + Q\cdot\sqrt{3}/2\,\bar{a}_y + 3Q'(\sqrt{3}/2\,\bar{a}_x + 1/2\,\bar{a}_y)] = 0$

i.e. $\bar{a}_x\left(Q + \dfrac{Q}{2} + 3Q'\dfrac{\sqrt{3}}{2}\right) + \bar{a}_y\left(\dfrac{Q\sqrt{3}}{2} + \dfrac{3Q'}{2}\right) = 0$

i.e. $Q + \dfrac{Q}{2} + 3Q'\dfrac{\sqrt{3}}{2} = 0$ and $\dfrac{Q\sqrt{3}}{2} + \dfrac{3Q'}{2} = 0$

Solving any one we get, $Q' = -\dfrac{Q}{\sqrt{3}}$ (C). ...Ans.

Ex. 2.1.9 : Two point charges Q_1 and Q_2 are located at (1, 2, 0) and (2, 0, 0) respectively. Find the relation between Q_1 and Q_2 such that the total force on a test charge at the point P (– 1, 1, 0) will have (i) no x-component; (ii) no y-component.

☑ **Soln. :**

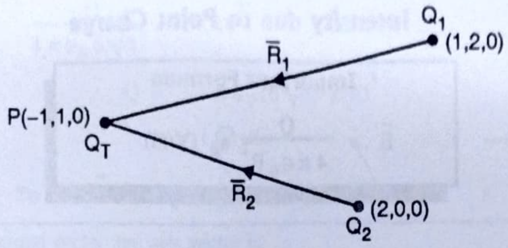


Fig. : Illustrating Ex. 2.1.9

Let Q_T is the test charge. Required vectors and unit vectors are,

$$\bar{R}_1 = -2\,\bar{a}_x - \bar{a}_y \; ; \qquad \therefore \; \bar{a}_{R1} = \frac{-2\,\bar{a}_x - \bar{a}_y}{\sqrt{5}}$$

$$\bar{R}_2 = -3\,\bar{a}_x + \bar{a}_y \; ; \qquad \therefore \; \bar{a}_{R2} = \frac{-3\,\bar{a}_x + \bar{a}_y}{\sqrt{10}}$$

Force acting on Q_T due to Q_1 is :

$$\bar{F}_1 = \frac{Q_1 Q_T}{4\pi\epsilon_0 R_1^2}\,\bar{a}_{R1} = \frac{Q_1 Q_T}{4\pi\epsilon_0 (\sqrt{5})^2} \times \left(\frac{-2\,\bar{a}_x - \bar{a}_y}{\sqrt{5}}\right)$$

$$= \frac{Q_1 Q_T}{4\pi\epsilon_0 (5\sqrt{5})}\,(-2\,\bar{a}_x - \bar{a}_y)$$

Force acting on Q_T due to Q_2 is :

$$\bar{F}_2 = \frac{Q_2 Q_T}{4\pi\epsilon_0 R_2^2}\,\bar{a}_{R2}$$

$$= \frac{Q_2 Q_T}{4\pi\epsilon_0 (\sqrt{10})^2} \times \left(\frac{-3\,\bar{a}_x + \bar{a}_y}{\sqrt{10}}\right)$$

$$\bar{F}_2 = \frac{Q_2 Q_T}{4\pi\epsilon_0 (10\sqrt{10})}\,(-3\,\bar{a}_x + \bar{a}_y)$$

Total force on Q_T is :

$$\bar{F}_T = \bar{F}_1 + \bar{F}_2$$

$$= \frac{Q_1 Q_T}{4\pi\epsilon_0 (5\sqrt{5})}\,(-2\,\bar{a}_x - \bar{a}_y) + \frac{Q_2 Q_T}{4\pi\epsilon_0 (10\sqrt{10})}\,(-3\,\bar{a}_x + \bar{a}_y)$$

$$\bar{F}_T = \frac{Q_T}{4\pi\epsilon_0}\left\{\left[\frac{-2Q_1}{5\sqrt{5}} - \frac{3Q_2}{10\sqrt{10}}\right]\bar{a}_x + \left[\frac{-Q_1}{5\sqrt{5}} + \frac{Q_2}{10\sqrt{10}}\right]\bar{a}_y\right\}$$

(i) For no x-component

$$\frac{-2Q_1}{5\sqrt{5}} - \frac{3Q_2}{10\sqrt{10}} = 0$$

$$4\sqrt{2}\,Q_1 + 3Q_2 = 0$$

$$\frac{Q_1}{Q_2} = \frac{-3}{(4\sqrt{2})} \qquad \text{...Ans.}$$

(ii) For no y-component

$$\frac{-Q_1}{5\sqrt{5}} + \frac{Q_2}{10\sqrt{10}} = 0$$

$$-2\sqrt{2}\,Q_1 + Q_2 = 0$$

$$\frac{Q_1}{Q_2} = \frac{1}{(2\sqrt{2})} \qquad \text{...Ans.}$$

UEx. 2.1.10 MU - May 14, 5 Marks.

Four 40 nC charges are located at A (1,0,0), B (– 1,0,0), C (0,1,0) and D (0,– 1,0). Determine the total force on the charge at A.

☑ **Soln. :**

$$\bar{R}_1 = 2\,\bar{a}_x \; ; \; \bar{a}_{R1} = \bar{a}_x$$

$$\bar{R}_2 = \bar{a}_x - \bar{a}_y \; ; \; \bar{a}_{R2} = \frac{\bar{a}_x - \bar{a}_y}{\sqrt{2}}$$

$$\bar{R}_3 = \bar{a}_x + \bar{a}_y \; ; \; \bar{a}_{R3} = \frac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}$$

$$\bar{F}_{total} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3$$

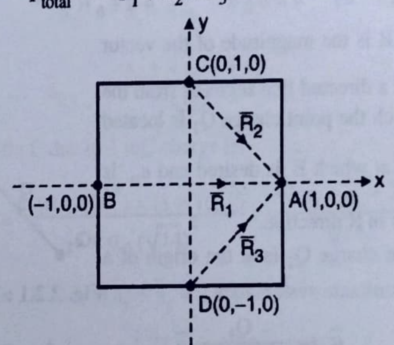


Fig. Illustrating Ex. 2.1.10

$$= \frac{40 \times 40 \times 10^{-18}}{4\pi \times \frac{10^{-9}}{36\pi} \times (2)^2}\,\bar{a}_x + \frac{40 \times 40 \times 10^{-18}}{4\pi \times \frac{10^{-9}}{36\pi} \times (\sqrt{2})^2}\left(\frac{\bar{a}_x - \bar{a}_y}{\sqrt{2}}\right)$$

$$+ \frac{40 \times 40 \times 10^{-18}}{4\pi \times \frac{10^{-9}}{36\pi} \times (\sqrt{2})^2}\left(\frac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}\right)$$

$$= \frac{40 \times 40 \times 10^{-18} \times 9}{10^{-9}}\left\{\frac{\bar{a}_x}{4} + \frac{\bar{a}_x - \bar{a}_y}{2\sqrt{2}} + \frac{\bar{a}_x + \bar{a}_y}{2\sqrt{2}}\right\}$$

$$= 16 \times 9 \times 10^{-7}\left\{\frac{1}{4} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right\}\bar{a}_x$$

$$\bar{F}_{total} = 13.78 \times 10^{-6}\,\bar{a}_x \text{ (N)}$$

2.2 ELECTRIC FIELD INTENSITY ($\bar{E}$)

Consider a positive electric charge Q_1 is fixed in position. If another positive point charge is brought into the vicinity of Q_1, it is acted upon by a force. If this second charge is moved slowly around a charge Q_1, we note that there exists everywhere a force on this second charge.

In other words the second charge is displaying the existence of a force in a region surrounding the charge Q_1. The region where forces act is called as field.

2.2.1 Definition of Electric Field Intensity

GQ. Define and derive electric field intensity due to point charge. **(5 Marks)**

If second charge is a positive test charge Q_T which is sufficiently small so as not to disturb significantly the field of Q_1, then the electric field intensity $\bar{E}$, due to Q_1 is defined as the force per unit charge on Q_T. That is

$$\bar{E} = \frac{\bar{F}_T}{Q_T} = \frac{1}{Q_T} \times \frac{Q_1 Q_T}{4\pi\epsilon_0 R^2}\bar{a}_R = \frac{Q_1}{4\pi\epsilon_0 R^2}\bar{a}_R \text{ (N/C)}$$

Here, R is the magnitude of the vector $\bar{R}$, which is a directed line segment from the point at which the point charge Q_1 is located to the point at which $\bar{E}$ is desired and $\bar{a}_R$ is a unit vector in $\bar{R}$ direction.

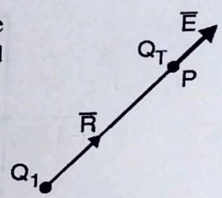


Fig. 2.2.1 : To find E

Let the charge Q_1 is at the origin of a spherical coordinate system then,

$$\bar{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\bar{a}_r$$

In general, electric field intensity due to point charge Q is

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_R \text{ (V/m)} \qquad ...(2.2.1)$$

According to definition of electric field the unit of it is N/C but throughout this book we will use the unit of Electric field, E as Volts/m, (V / m). Why ? This will be clear in the article on scalar potential.

Note :

(i) The force on the test charge is dependent upon the strength of the test charge, but the electric field intensity is not.

(ii) If the charge on the test charge is allowed to approach zero, then the force acting on it becomes zero, but the force per unit charge remains constant, that is, the electric field due to the charge Q is considered to exist, whether or not there is a test charge to detect its presence.

2.2.2 Solved Examples on Electric Field Intensity due to Point Charge

Important Formula

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_R \text{ (V/m)}$$

Ex. 2.2.1 : Find the electric field $\bar{E}$ at (0, 3, 4) m due to a point charge Q = 0.5 μC placed at the origin.

☑ **Soln. :**

In the Fig. Ex. 2.2.1, the vector joining charge Q and the point where we desire field is,

$$\bar{R} = (3-0)\bar{a}_y + (4-0)\bar{a}_z = 3\bar{a}_y + 4\bar{a}_z$$

$$\therefore R = \sqrt{3^2+4^2} = 5 \text{ and } \bar{a}_R = \frac{3\bar{a}_y + 4\bar{a}_z}{5}$$

Putting the values in the expression for $\bar{E}$

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_R$$

$$= \frac{0.5 \times 10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi} \times 25} \times \left(\frac{3\bar{a}_y + 4\bar{a}_z}{5}\right)$$

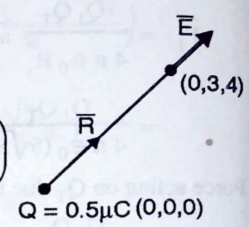


Fig. : Illustrating Ex. 2.2.1

$$\bar{E} = 108\bar{a}_y + 144\bar{a}_z \text{ (V/m)} \qquad \text{...Ans.}$$

Ex. 2.2.2 : A charge Q located at the origin in free space produces a field for which E_z = 1 kV/m at point P (– 2, 1, – 1).

(a) Find Q (b) Find $\bar{E}$ at M (1, 6, 5) in cartesian coordinates

☑ **Soln. :**

(a) To find Q

To find $\bar{E}$ at P the required vector and unit vector is :

$$\bar{R} = -2\bar{a}_x + \bar{a}_y - \bar{a}_z \ ;$$

$$\bar{a}_R = \frac{-2\bar{a}_x + \bar{a}_y - \bar{a}_z}{\sqrt{6}}$$

The electric field at P is :

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_R$$

$$= \frac{Q}{4\pi\epsilon_0 (\sqrt{6})^2} \times \left(\frac{-2\bar{a}_x + \bar{a}_y - \bar{a}_z}{\sqrt{6}}\right)$$

Since the z component E_z is of value 1 kV/m, we find

$$\frac{-Q}{4\pi\epsilon_0 6\sqrt{6}} = 1\times10^3$$

$$\therefore\ Q = -4\pi\epsilon_0(6\sqrt{6})\times10^3$$

$$= -1.63\ \mu C \qquad \text{...Ans.}$$

(b) To find $\bar{E}$ at M (1, 6, 5)

Required vector and unit vector is,

$$\bar{R}_1 = \bar{a}_x + 6\bar{a}_y + 5\bar{a}_z\ ;$$

$$\bar{a}_{R1} = \frac{\bar{a}_x + 6\bar{a}_y + 5\bar{a}_z}{\sqrt{62}}$$

$\bar{E}$ at M in cartesian coordinates is :

$$\bar{E}_1 = \frac{Q}{4\pi\epsilon_0 R_1^2}\bar{a}_{R1}$$

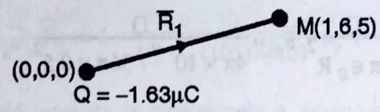


Fig. : Illustrating Ex. 2.2.2

$$\bar{E}_1 = \frac{-1.63\times10^{-6}}{4\pi\times\left(\frac{10^{-9}}{36\pi}\right)\times(\sqrt{62})^2}\times\left(\frac{\bar{a}_x + 6\bar{a}_y + 5\bar{a}_z}{\sqrt{62}}\right)$$

$$= -30\bar{a}_x - 180\bar{a}_y - 150\bar{a}_z\ (V/m). \qquad \text{...Ans.}$$

2.2.3 Principle of Superposition of Fields

GQ. Explain principle of superposition of fields. **(5 Marks)**

In the region when more than one point charge is present then to find the total or resultant field at a point principle of superposition can be applied.

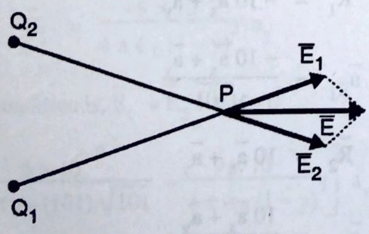


Fig. 2.2.2 : Illustration of Superposition principle

The total or resultant field at a point is the vector sum of the individual component fields at the point. In the Fig. 2.2.2, the field intensity of the charge Q_1 at the point P is $\bar{E}_1$ and field due to charge Q_2 is $\bar{E}_2$. The total field at P due to both charges is the vector sum of $\bar{E}_1$ and $\bar{E}_2$. The procedure can be applied to any number of charges.

Ex. 2.2.3 : Point charges 1 mC and – 2 mC are located at (3, 2, –1) and (–1, –1, 4) respectively. Calculate the electric force on a 10 nC charge located at (0, 3, 1) and the electric field intensity at that point.

☑ **Soln. : To find the force**

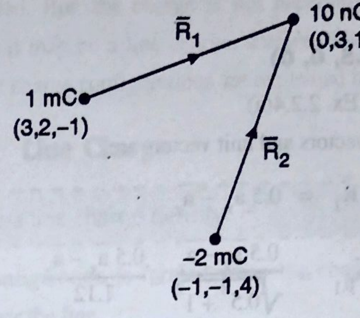


Fig. : Illustrating Ex. 2.2.3

Required vectors and unit vectors are

$$\bar{R}_1 = -3\bar{a}_x + \bar{a}_y + 2\bar{a}_z$$

$$\therefore\ \bar{a}_{R1} = \frac{-3\bar{a}_x + \bar{a}_y + 2\bar{a}_z}{\sqrt{14}}$$

$$\bar{R}_2 = \bar{a}_x + 4\bar{a}_y - 3\bar{a}_z$$

$$\therefore\ \bar{a}_{R2} = \frac{\bar{a}_x + 4\bar{a}_y - 3\bar{a}_z}{\sqrt{26}}$$

Force on 10 nC due to 1 mC charge is,

$$\bar{F}_1 = \frac{(10\times10^{-9})\times(1\times10^{-3})}{4\pi\epsilon_0(\sqrt{14})^2}\times\frac{-3\bar{a}_x + \bar{a}_y + 2\bar{a}_z}{\sqrt{14}}$$

$$= 1.72(-3\bar{a}_x + \bar{a}_y + 2\bar{a}_z)\times10^{-3}$$

$$= -5.16\bar{a}_x + 1.72\bar{a}_y + 3.44\bar{a}_z\ (mN)$$

Force on 10 nC due to – 2 mC charge is,

$$\bar{F}_2 = \frac{(10\times10^{-9})\times(-2\times10^{-3})}{4\pi\epsilon_0(\sqrt{26})^2}\times\frac{\bar{a}_x + 4\bar{a}_y - 3\bar{a}_z}{\sqrt{26}}$$

$$= -1.36(\bar{a}_x + 4\bar{a}_y - 3\bar{a}_z)\times10^{-3}$$

$$= -1.36\bar{a}_x - 5.44\bar{a}_y + 4.08\bar{a}_z\ (mN)$$

The force is obtained by superposition,

$$\bar{F} = \bar{F}_1 + \bar{F}_2$$

$$= -6.52\bar{a}_x - 3.72\bar{a}_y + 7.52\bar{a}_z\ (mN) \qquad \text{...Ans.}$$

The electric field is,

$$\bar{E} = \frac{\bar{F}}{Q} = \frac{(-6.52\bar{a}_x - 3.72\bar{a}_y + 7.52\bar{a}_z)\times10^{-3}}{10\times10^{-9}}$$

$$= -652\bar{a}_x - 372\bar{a}_y + 752\bar{a}_z\ (kV/m) \qquad \text{...Ans.}$$

Ex. 2.2.4 : Point charges of 120 nC are located at A (0, 0, 1) and B (0, 0, – 1) in free space.

(i) Find E at P (0.5, 0, 0)

(ii) What single charge at origin would provide the identical field strength.

Soln. :

(I) E at P (0.5, 0, 0)

Refer Fig. Ex. 2.2.4(a)

Required vectors and unit vectors are :

$$\bar{R}_1 = 0.5\,\bar{a}_x - \bar{a}_z$$

$$\bar{a}_{R1} = \frac{0.5\,\bar{a}_x - \bar{a}_z}{\sqrt{0.5^2 + 1^2}} = \frac{0.5\,\bar{a}_x - \bar{a}_z}{1.12}$$

$$\bar{R}_2 = 0.5\,\bar{a}_x + \bar{a}_z$$

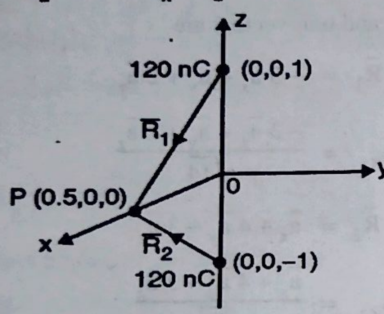


Fig. : Illustrating Ex. 2.2.4(a)

$$\therefore \quad \bar{a}_{R2} = \frac{0.5\,\bar{a}_x + \bar{a}_z}{1.12}$$

$\bar{E}$ due to change at (0, 0, 1) is :

$$\bar{E}_1 = \frac{Q_1}{4\pi\epsilon_0 R_1^2}\,\bar{a}_{R1}$$

$$= \frac{120\times 10^{-9}}{4\pi\times(10^{-9}/36\pi)\times(1.12)^2}\times\left(\frac{0.5\,\bar{a}_x - \bar{a}_z}{1.12}\right)$$

$$= 768.72\,(0.5\,\bar{a}_x - \bar{a}_z)\ (\text{V/m})$$

$\bar{E}$ due to charge at (0, 0, – 1) is :

$$\bar{E}_2 = \frac{Q_2}{4\pi\epsilon_0 R_2^2}\,\bar{a}_{R2}$$

$$= \frac{120\times 10^{-9}}{4\pi\times(10^{-9}/36\pi)\times(1.12)^2}\times\left(\frac{0.5\,\bar{a}_x + \bar{a}_z}{1.12}\right)$$

$$= 768.72\,(0.5\,\bar{a}_x + \bar{a}_z)\ (\text{V/m})$$

Total $\bar{E}$ is, $\quad \bar{E}_T = \bar{E}_1 + \bar{E}_2 = 2\times 768.72\times 0.5\,\bar{a}_x$

$$= 768.72\,\bar{a}_x\ (\text{V/m}) \qquad \text{...Ans.}$$

(II) To find charge Q at origin

Let charge Q is placed at origin which produces same electric field. Refer Fig. Ex. 2.2.4(b). Required vector and unit vector is :

$$\bar{R} = 0.5\,\bar{a}_x \ ; \quad \therefore\ \bar{a}_R = \bar{a}_x$$

Field due to Q is,

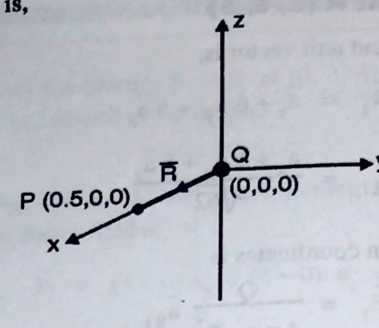


Fig. : Illustrating Ex. 2.2.4(b)

$$\bar{E}_Q = \frac{Q}{4\pi\epsilon_0 R^2}\,\bar{a}_R = \frac{Q}{4\pi\times 10^{-9}/36\pi\times 0.5^2}\,\bar{a}_x\ (\text{V/m})$$

We have to find Q such that $\bar{E}_Q = \bar{E}_T$.

$$\therefore \quad \frac{9\times Q}{10^{-9}\times 0.5^2}\,\bar{a}_x = 768.72\,\bar{a}_x$$

$$\therefore \quad Q = 21.35\ \text{nC} \qquad \text{...Ans.}$$

Ex. 2.2.5 : Two point charges of 6 nC are located at points P_1 (10, 0) and P_2 (– 10, 0). Find the coordinates of point P_3 (x, y) where the third point charge of – 9 nC is to be located, so that the total electric field at point P_4 (0, 1) is zero.

Soln. :

From Fig. Illustrating Ex. 2.2.5,

$$\bar{R}_1 = -10\,\bar{a}_x + \bar{a}_y$$

$$\therefore \quad \bar{a}_{R1} = \frac{-10\,\bar{a}_x + \bar{a}_y}{\sqrt{101}}$$

$$\bar{R}_2 = 10\,\bar{a}_x + \bar{a}_y$$

$$\therefore \quad \bar{a}_{R2} = \frac{10\,\bar{a}_x + \bar{a}_y}{\sqrt{101}}$$

The electric field intensity due to charge at P_1 is,

$$\bar{E}_1 = \frac{Q_1}{4\pi\epsilon_0 R_1^2}\,\bar{a}_{R1} = \frac{6\times 10^{-9}}{4\pi\epsilon_0(101)}\left(\frac{-10\,\bar{a}_x + \bar{a}_y}{\sqrt{101}}\right)$$

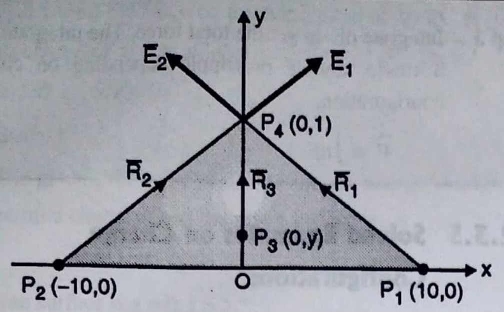


Fig. : Illustrating Ex. 2.2.5

The electric field intensity due to charge at P_2 is,

$$\bar{E}_2 = \frac{6 \times 10^{-9}}{4 \pi \epsilon_0 (101)} \left(\frac{10\, \bar{a}_x + \bar{a}_y}{\sqrt{101}} \right)$$

The total electric field intensity because of charges at P_1 and P_2 is obtained by Superposition as,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = 2 \times \frac{6 \times 10^{-9}}{4 \pi \epsilon_0 (101) \sqrt{101}}\, \bar{a}_y \ (\text{V/m})$$

This field intensity is having only $\bar{a}_y$ component. To make electric field intensity zero at (0, 1), the third point charge of – 9 nC must be placed such that it will result in $\bar{E}_3$ having $-\bar{a}_y$ component. For this, – 9 nC charge must be placed on y axis and below point P_4 as shown in Fig. Ex. 2.2.5. Let P_3 is having coordinates (0, y).

The electric field due to – 9 nC charge at P_3 is,

$$\bar{E}_3 = \frac{Q_3}{4 \pi \epsilon_0 R_3^2}\, \bar{a}_{R3}$$

Here $\bar{R}_3 = (1 - y)\, \bar{a}_y$;

$\bar{a}_{R3} = \bar{a}_y$.

Thus, $$\bar{E}_3 = \frac{-9 \times 10^{-9}}{4 \pi \epsilon_0 (1 - y)^2}\, \bar{a}_y$$

As the given condition is, $\bar{E}_1 + \bar{E}_2 + \bar{E}_3 = 0$

$$\left[2 \times \frac{6 \times 10^{-9}}{4 \pi \epsilon_0 (101) \sqrt{101}} - \frac{9 \times 10^{-9}}{4 \pi \epsilon_0 (1 - y)^2} \right] \bar{a}_y = 0$$

$$\frac{12}{101 \sqrt{101}} - \frac{9}{(1 - y)^2} = 0$$

Solving we get,

$$1 - y = \pm 27.59$$

i.e. $y = 26.59$ or 28.59

As explained above P_3 must be below point P_4, then possible value is y = – 26.59. Hence point P_3 is (0, – 26.59). ...Ans.

Module 1

2.3 CHARGE CONFIGURATIONS

In the above problems we learned computing force due to point charges. For the point charge, Equation (2.1.1) can be directly applied. But, the charge is not necessarily be always a point charge, it may be a line charge, surface charge or a volume charge. These charge configurations are explained in brief below.

2.3.1 Line Charge

GQ. Express line charge density. **(2 Marks)**

This configuration is formed when the charge is uniformly distributed over the line.

Since charge is uniformly distributed, charge density which is charge per unit length is constant. This charge density is denoted by ρ_l, where ρ stands for charge density and subscript l is used for line charge.

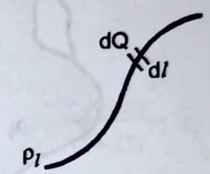


Fig. 2.3.1(a) : Line charge

The line charge density is expressed as :

$$\rho_l = \frac{\text{Total charge spread}}{\text{Length of line}} \ (\text{C/m})$$

As ρ_l is a constant, when we multiply ρ_l by a differential length dl, we get a differential charge dQ. In the problems of line charge we consider it as point charge.

Thus,

$$dQ = \rho_l \, dl \qquad ...(2.3.1)$$

2.3.2 Surface Charge

GQ. Express Surface charge density. **(2 Marks)**

When the charge is uniformly distributed over the surface, surface charge configuration is formed.

The surface charge density is expressed as,

$$\rho_s = \frac{\text{Total charge spread}}{\text{Area of surface}} \ (\text{C/m}^2)$$

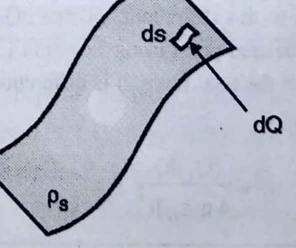


Fig. 2.3.1(b) : Surface charge

The differential charge is obtained by multiplying ρ_s by differential area ds :

$$dQ = \rho_s \, ds \qquad ...(2.3.2)$$

2.3.3 Volume Charge

GQ. Express volume charge density. **(2 Marks)**

Here the charge is uniformly filled in the volume. The charge density associated is ρ_v, given by,

$$\rho_v = \frac{\text{Charge filled in the volume}}{\text{Volume}} \quad (C/m^3)$$

The differential charge dQ is,

$$dQ = \rho_v \, dv \qquad ...(2.3.3)$$

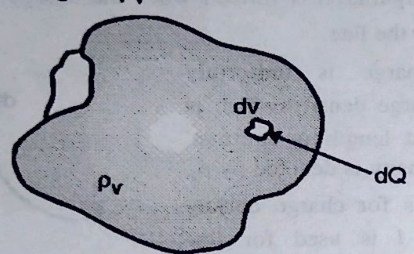


Fig. 2.3.1(c) : Volume charge

In the Equations (2.3.1), (2.3.2) and (2.3.3), the terms dl, ds or dv already we have seen in coordinate systems.

Integrating dQ we obtain total charge.

i.e. $$Q = \int dQ \qquad ...(2.3.4)$$

The integral is single, double or triple depending upon whether the charge is line charge, surface or volume charge.

2.3.4 Standard Procedure to Solve Problems on Coulomb's Law using Charge Configurations

The procedure for finding force due to above charge configurations is explained below :

- **Step 1 :** Create a small charge dQ using the Equations (2.3.1), (2.3.2) or (2.3.3) depending upon the charge configuration given. Consider this differential charge dQ as a point charge.
- **Step 2 :** Due to this differential charge dQ find the force at the desired point using Equation (2.3.5). This force is not the total force, it is differential force, denoted by $d\bar{F}$.

Thus, $$d\bar{F} = \frac{Q_1 \, dQ}{4\pi\epsilon_0 R^2} \bar{a}_R \qquad ...(2.3.5)$$

- **Step 3 :** Integrate $d\bar{F}$ to get the total force. The integration is a single, double or tripple depending on charge configuration.

$$\bar{F} = \int d\bar{F}$$

2.3.5 Solved Examples on Charge Configurations

Important Formulae

Line Charge $dQ = \rho_l \, dl$ Surface Charge $dQ = \rho_s \, ds$

$$Q = \int dQ$$

Volume Charge $dQ = \rho v \, dv$

Differential force $d\bar{F} = \dfrac{Q_1 \, dQ}{4\pi\epsilon_0 R^2} \bar{a}_R$ $\quad \bar{F} = \int d\bar{F}$

Ex. 2.3.1 : The volume charge density $\rho_v = \rho_o e^{-|x|-|y|-|z|}$ exists over all free space. Calculate the total charge present.

☑ **Soln. :** The charge can be expressed in terms of ρ_v :

$$Q = \int_V \rho_v \, dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_0 e^{-|x|-|y|-|z|} dx\,dy\,dz$$

$$Q = \rho_0 \, 2\times\int_0^{\infty} e^{-x} dx \times 2\times\int_0^{\infty} e^{-y} dy \times 2\times \int_0^{\infty} e^{-z} dz$$

$$= 8\rho_0 \left(\frac{e^{-x}}{-1}\right)_0^{\infty}\left(\frac{e^{-y}}{-1}\right)_0^{\infty}\left(\frac{e^{-z}}{-1}\right)_0^{\infty}$$

$$Q = 8\rho_0 (1\times1\times1) = 8\rho_0 \qquad ...\text{Ans.}$$

UEx. 2.3.2 **MU - Dec. 10, Dec. 12, 5 Marks**

Calculate the total charge within the indicated volume $0 \le r \le 0.1$, $0 \le \phi \le \pi$, $2 \le z \le 4$; given $\rho_v = r^2 z^2 \sin 0.6\,\phi$.

☑ **Soln. :**

Given : $\rho_v = r^2 z^2 \sin(0.6\,\phi)$

The charge within the volume is,

$$Q = \int \rho_v \, dv = \int_2^4 \int_0^{\pi} \int_0^{0.1} r^2 z^2 \sin(0.6\,\phi)\, r\,dr\,d\phi\,dz$$

$$= \left(\frac{r^4}{4}\right)_0^{0.1} \left(\frac{z^3}{3}\right)_2^4 \left(\frac{-\cos 0.6\phi}{0.6}\right)_0^{\pi}$$

$$= \frac{1}{4}\times\frac{1}{3}\times\frac{1}{0.6}(0.1)^4 (4^3 - 2^3)(1-\cos 0.6\,\pi)$$

$$Q = 1.018\times10^{-3} = 1.018 \text{ (mC)}$$

Ex. 2.3.3 : Find the force on a point charge of 50 μC at (0, 0, 5) due to a charge of 500 π μC that is uniformly distributed over the circular disk $r \le 5$, $z = 0$.

☑ **Soln. :**

The given problem consists of surface charge configuration. A differential charge dQ on the given surface is,

$$dQ = \rho_s \, ds$$

The given surface is $z = 0, r \le 5$,

$$\therefore \quad ds = r \, dr \, d\phi$$

Hence, $$dQ = \rho_s \, r \, dr \, d\phi$$

The charge density can be found from the given data as,

$$\rho_s = \frac{Q_T}{A} = \frac{500 \times \pi \times 10^{-6}}{\pi (5)^2}$$

$$= 0.2 \times 10^{-4} \, (C/m^2)$$

The force on 50 μC charge at (0, 0, 5) acts along the line joining dQ and 50 μC charge. To find unit vector in this direction, consider charge dQ is placed at (r, φ, 0). The cylindrical coordinates corresponding to (0, 0, 5) are (0, φ, 5).

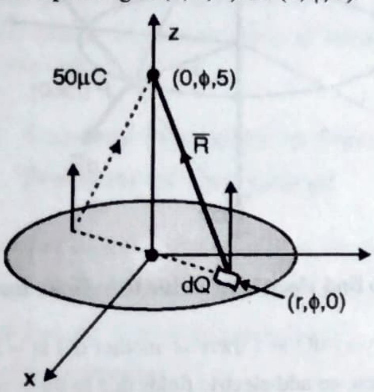


Fig. : Illustrating Ex. 2.3.3

Required vector and unit vector are :

$$\bar{R} = -r\,\bar{a}_r + 5\,\bar{a}_z \;;$$

$$\bar{a}_R = \frac{-r\,\bar{a}_r + 5\,\bar{a}_z}{\sqrt{r^2 + 25}}$$

The differential force acting on 50 μC charge due to dQ is (using Equation 2.3.1)

$$d\bar{F} = \frac{dQ \times Q}{4 \pi \epsilon_0 R^2} \bar{a}_R$$

$$d\bar{F} = \frac{50 \times 10^{-6} \times dQ}{4\pi \times \frac{10^{-9}}{36\pi} \times (r^2 + 25)} \times \left(\frac{-r\,\bar{a}_r + 5\,\bar{a}_z}{\sqrt{r^2 + 25}} \right)$$

$$d\bar{F} = \frac{450 \times 10^3 \times \rho_s \, r \, dr \, d\phi}{(r^2 + 25)^{3/2}} (-r\,\bar{a}_r + 5\,\bar{a}_z)$$

When a differential charge dQ is present at (r, φ, 0), another dQ is also present at diametrically opposite side (because the charge is uniformly distributed on the circular disk). This second dQ is also acting on 50 μC charge.

When we add forces due to these two dQ's, the horizontal components ($\bar{a}_r$) of the forces get cancelled as they are in opposite direction. Then the summation of two forces results in only vertical component i.e. $\bar{a}_z$ component.

This is true for every charge dQ present on surface of the disk. Thus, when we add forces due to all dQ's it results in only $\bar{a}_z$ component. The total force obtained by integrating only vertical component of dF is,

$$\bar{F} = \iint d\bar{F}$$

$$= \int_0^{2\pi} \int_0^{5} 450 \times 10^3 \times 0.2 \times 10^{-4} \times 5 \times \frac{r \, dr \, d\phi}{(r^2 + 25)^{3/2}} \bar{a}_z$$

$$\bar{F} = 90 \pi \bar{a}_z \int_0^5 \frac{r \, dr}{(r^2 + 25)^{3/2}}$$

Putting $r^2 + 25 = t$, $2r \, dr = dt$ | limits : $r \to 0$, $t \to 25$; $r \to 5$, $t \to 50$

$$\therefore \bar{F} = 90 \pi \bar{a}_z \int_{25}^{50} \frac{dt/2}{t^{3/2}}$$

$$= 45 \pi \bar{a}_z \int_{25}^{50} t^{-3/2}$$

$$dt = 45 \pi \bar{a}_z \left[\frac{t^{-1/2}}{-1/2} \right]_{25}^{50}$$

$$\bar{F} = -90 \pi \bar{a}_z \left[\frac{1}{\sqrt{50}} - \frac{1}{\sqrt{25}} \right]$$

$$\bar{F} = 16.56 \, \bar{a}_z \text{ (N)}$$ ...Ans.

2.4 CALCULATION OF ELECTRIC FIELD INTENSITY FOR VARIOUS CHARGE DISTRIBUTIONS

In the above problems we have seen the electric field due to point charges Equation (2.2.1) is applicable only for point charges. When charge configurations like line charge, surface charge or volume charge is given, we cannot use Equation (2.2.1) directly for finding $\bar{E}$.

2.4.1 Standard Procedure to Solve Problems on Continuous Charge Distribution

The procedure is outlined below :

- **Step 1 :** Create a small charge dQ using the expression $\rho_l\, dl$ or ρ_s ds or ρ_v dv depending upon the charge configuration given. Consider this dQ as a point charge.
- **Step 2 :** Due to this small charge dQ find the electric field at the desired point using $d\bar{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\bar{a}_R$

  This electric field is not the total electric field, but a small or differential field $d\bar{E}$.
- **Step 3 :** Integrate $d\bar{E}$ to get the total electric field $\bar{E}$.

Some configurations like infinite line charge, infinite sheet charge are the standard configurations. The electric field intensities due to these configurations are obtained in the next section.

2.5 ELECTRIC FIELD DUE TO INFINITE LINE CHARGE

2.5.1 Assumptions

(i) Consider the infinite line charge is placed along z-axis from $-\infty$ to $+\infty$.

(ii) Consider this line charge as axis of the cylinder in cylindrical coordinates, so use cylindrical coordinates.

(iii) The point where we desire the field is in xy plane at (r, ϕ, 0).

2.5.2 Expression for $\bar{E}$ Due to Infinite Line Charge

UQ. Find Electric field intensity due to infinite line charge.
MU - May 09, May 12, 10 Marks

The given charge configuration is a line charge, not a point charge. Therefore we cannot use the formula.

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_R$$

Since this is applicable only to point charge. In order to use this formula, create a small charge dQ on the given charge configuration at a distance z from origin at (0, 0, z).

To create a small charge requires a small section of this line charge. For the line charge along z-axis the small section is of length dl = dz. Then the differential charge dQ is,

$$dQ = \rho_l\, dl = \rho_l\, dz$$

We can treat it as a point charge. The coordinates of dQ are (0, 0, z) in cartesian, but when we use cylindrical coordinate system it becomes (0, ϕ, z).

The electric field due to charge dQ acts along the line joining dQ and point P i.e. in the direction of $\bar{R}$, given by,

$$\bar{R} = r\bar{a}_r - z\bar{a}_z \text{ and } \bar{a}_R = \frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}$$

Thus, the differential electric field due to dQ is

$$d\bar{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\left(\frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}\right)$$

$$d\bar{E} = \frac{\rho_l\, dz}{4\pi\epsilon_0 (r^2+z^2)}\left(\frac{r\bar{a}_r - z\bar{a}_z}{\sqrt{r^2+z^2}}\right)$$

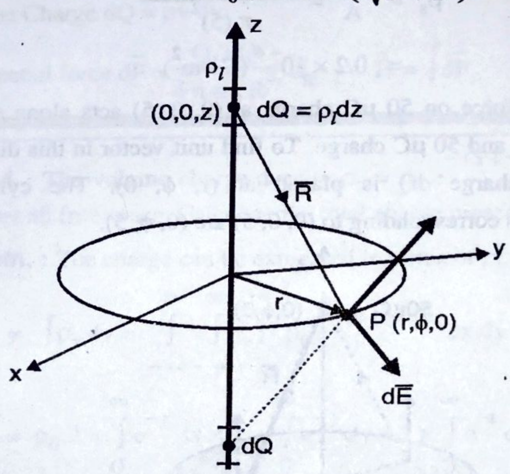


Fig. 2.5.1 : To find electric field due to in finite line charge

Since for every dQ at z there is another dQ at $-$ z, as shown in Fig. 2.5.1. When we add electric fields due to these two charges, we can observe that vertical components cancel each other i.e. $\bar{a}_z$ components gets cancel while horizontal components i.e. $\bar{a}_r$ components are added.

Thus when we integrate the expression $d\bar{E}$, it results in only $\bar{a}_r$ component. Then,

$$\bar{E} = \int d\bar{E}$$

i.e. $$\bar{E} = \int_{-\infty}^{\infty} \frac{\rho_l\, r\, dz}{4\pi\epsilon_0 (r^2+z^2)^{3/2}}\bar{a}_r$$

To solve integral :

Put, $z = r\tan t$, $dz = r\sec^2 t\, dt$ | limits : $z \to -\infty$; $t \to -\pi/2$; $z \to \infty$; $t \to \pi/2$

The term $(r^2+z^2)^{3/2} = [r^2 + (r\tan t)^2]^{3/2} = r^3 \sec^3 t$

Then the electric field $\bar{E}$ becomes,

$$\bar{E} = \int_{-\pi/2}^{\pi/2} \frac{\rho_l r^2 \sec^2 t \, dt}{4 \pi \epsilon_0 (r^3 \sec^3 t)} \bar{a}_r$$

$$= \frac{\rho_l}{4 \pi \epsilon_0 r} \bar{a}_r \int_{-\pi/2}^{\pi/2} \cos t \, dt$$

$$= \frac{\rho_l}{4 \pi \epsilon_0 r} \bar{a}_r [\sin t]_{-\pi/2}^{\pi/2} = \frac{\rho_l}{2 \pi \epsilon_0 r} \bar{a}_r$$

$$\bar{E} = \frac{\rho_l}{2 \pi \epsilon_0 r} \bar{a}_r \quad \text{...due to line charge along z-axis}$$

Thus, the electric field due to a infinite line charge not along z-axis, the general expression is,

$$\bar{E} = \frac{\rho_l}{2 \pi \epsilon_0 R} \bar{a}_R \text{ (V / m)} \quad \text{...(2.5.1)}$$

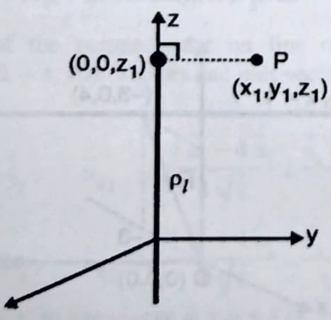
In the above result, R is the length of perpendicular from the desired point to the line charge and $\bar{a}_R$ is the unit vector in the direction of perpendicular towards the desired point.

"Equation (2.5.1) is the expression for $\bar{E}$ at general point P due to uniform charge distribution along an infinite straight line with uniform line charge density ρ_l."

2.5.3 Standard Procedure to Solve Problems of Line Charge

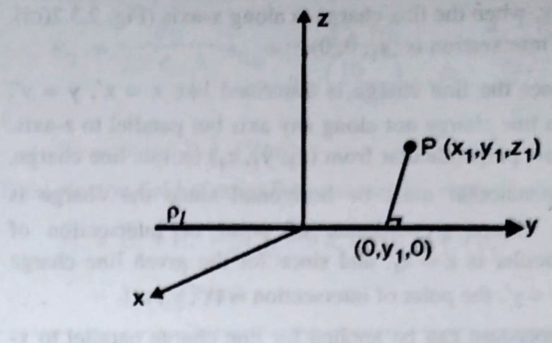
The procedure to solve problems of line charge is as follows :

- **Step 1 :** Draw a perpendicular from the desired point to the line charge and find the foot of perpendicular. To find this point will need more explanation :

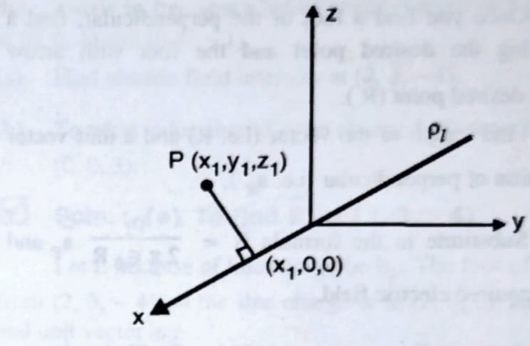


(a)

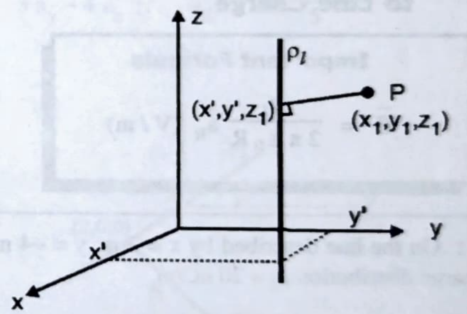
Fig. 2.5.2 (Contd...)



(b)



(c)



(d)

Fig. 2.5.2 : Finding point of intersection of perpendicular on line charge

Let the point where we desire the electric field be (x_1, y_1, z_1) and the line charge is along z-axis (Fig. 2.5.2(a)). When we draw perpendicular on line charge remember you are drawing perpendicular on z-axis (which is vertical), thus perpendicular on it is horizontal.

Hence z-coordinate of point P and the point of intersection is same (they are at same height). Also line charge is along z-axis hence x and y coordinate of any point on z-axis is zero. This means point of intersection is $(0, 0, z_1)$.

Similarly when the line charge is along y-axis (Fig. 2.5.2(b)), y coordinate of point P and point of intersection of perpendicular on y-axis same and thus coordinates of intersection is $(0, y_1, 0)$.

Similarly, when the line charge is along x-axis (Fig. 2.5.2(c)), point of intersection is $(x_1, 0, 0)$.

Sometimes the line charge is described like $x = x'$, $y = y'$. This is a line charge not along any axis but parallel to z-axis. Now draw perpendicular from (x_1, y_1, z_1) on this line charge, the perpendicular must be horizontal since the charge is vertical. Hence z-coordinate of point of intersection of perpendicular is $z = z_1$, and since for the given line charge $x = x'$, $y = y'$, the point of intersection is (x', y', z_1).

Same procedure can be applied for line charge parallel to x-axis or y-axis.

- **Step 2 :** Once you find a foot of the perpendicular, find a vector joining the desired point and the foot with arrow towards the desired point ($\bar{R}$).
- **Step 3 :** Find length of the vector (i.e. R) and a unit vector in the direction of perpendicular (i.e. $\bar{a}_R$).
- **Step 4 :** Substitute in the formula $\bar{E} = \dfrac{\rho_l}{2\pi\epsilon_0 R}\bar{a}_R$ and obtain the required electric field.

2.5.4 Solved Examples of Electric Field Due to Line Charge

Important Formula

$$\bar{E} = \frac{\rho_l}{2\pi\epsilon_0 R}\bar{a}_R \ (\text{V/m})$$

Ex. 2.5.1 : On the line described by x = 2 m, y = –4 m there is a uniform charge distribution $\rho_l = 20$ nC/m.

Determine $\bar{E}$ at (–2, –1, 4) m.

Soln. : Since for the line charge x and y are constants, it is parallel to z-axis. The foot of perpendicular from (–2, –1, 4) on the line charge is at (2, – 4, 4). Refer Fig. Ex. 2.5.1.

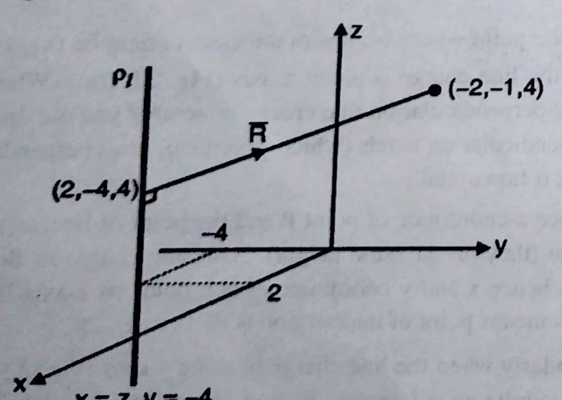


Fig. : Illustrating Ex. 2.5.1

Required vector and unit vector is :

$$\bar{R} = -4\bar{a}_x + 3\bar{a}_y \ ; \ \bar{a}_R = \frac{-4\bar{a}_x + 3\bar{a}_y}{5}$$

The electric field intensity $\bar{E}$ due to infinite line charge is,

$$\bar{E} = \frac{\rho_l}{2\pi\epsilon_0 R}\bar{a}_R$$

$$\bar{E} = \frac{20\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times5}\left(\frac{-4\bar{a}_x + 3\bar{a}_y}{5}\right)$$

$$= -57.6\,\bar{a}_x + 43.2\,\bar{a}_y \ (\text{V/m}) \quad \text{...Ans.}$$

Ex. 2.5.2 : A uniform line charge, $\rho_l = 25$ nC / m lies on the line x = – 3, z = 4 in free space. Find $\bar{E}$ in cartesian components at (i) the origin ; (ii) P_1 (2, 15, 3).

Soln. :

(i) To find $\bar{E}$ at origin

Since for the line charge x and z are constants, it is parallel to y-axis. The foot of perpendicular from origin on line charge is at (– 3, 0, 4). Required vector and unit vector is :

$$\bar{R} = 3\bar{a}_x - 4\bar{a}_z$$

$$\therefore \quad \bar{a}_R = \frac{3\bar{a}_x - 4\bar{a}_z}{5}$$

The electric field at origin due to line charge is :

$$\bar{E} = \frac{\rho_l}{2\pi\epsilon_0 R}\bar{a}_R = \frac{25\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times5}\left(\frac{3\bar{a}_x - 4\bar{a}_z}{5}\right)$$

$$\bar{E} = 54\,\bar{a}_x - 72\,\bar{a}_z \ (\text{V/m}) \quad \text{...Ans.}$$

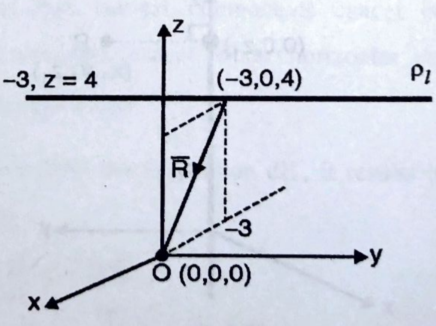


Fig. : Illustrating Ex. 2.5.2(a)

(ii) To find the electric field at P_1 (2, 15, 3)

The foot of perpendicular from point P on the line charge is at (– 3, 15, 4). Required vector and unit vector is :

$$\bar{R}_1 = 5\bar{a}_x - \bar{a}_z \qquad \therefore \ \bar{a}_{R1} = \frac{5\bar{a}_x - \bar{a}_z}{\sqrt{26}}$$

The electric field at P_1 is obtained as,

$$\bar{E} = \frac{\rho_l}{2\pi\epsilon_0 R}\bar{a}_{R1} = \frac{25\times 10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times\sqrt{26}}\left(\frac{5\bar{a}_x - \bar{a}_z}{\sqrt{26}}\right)$$

$$\bar{E} = 86.55\,\bar{a}_x - 17.31\,\bar{a}_z \text{ (V/m)} \qquad \text{...Ans.}$$

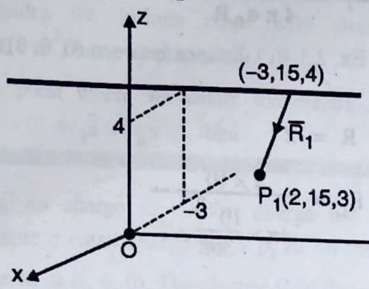


Fig. : Illustrating Ex. 2.5.2(b)

Ex. 2.5.3 : Two uniform line charges of density $\rho_l = 4$ nC/m in the x = 0 plane at y = ± 4 m. Find $\bar{E}$ at (4, 0, 10) m.

☑ **Soln. :** Two line charges are in x = 0 plane i.e. in y-z plane as shown in Fig. Ex. 2.5.3. These line charges are parallel to z-axis.

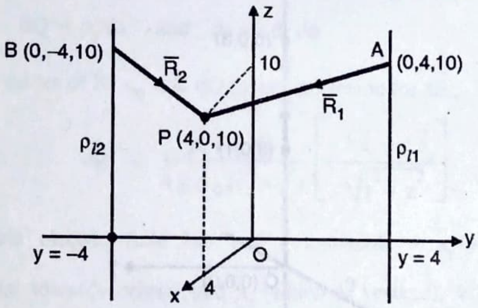


Fig. : Illustrating Ex. 2.5.3

The foot of the perpendicular on line charges is at A (0, 4, 10) and B (0, – 4, 10). Vectors and unit vectors towards point P are,

$$\bar{R}_1 = 4\bar{a}_x - 4\bar{a}_y, \qquad \bar{a}_{R1} = \frac{4\bar{a}_x - 4\bar{a}_y}{4\sqrt{2}} = \frac{\bar{a}_x - \bar{a}_y}{\sqrt{2}}$$

$$\bar{R}_2 = 4\bar{a}_x + 4\bar{a}_y, \qquad \bar{a}_{R2} = \frac{4\bar{a}_x + 4\bar{a}_y}{4\sqrt{2}} = \frac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}$$

The electric field due to line charge at y = + 4 is,

$$\bar{E}_1 = \frac{\rho_{l1}}{2\pi\epsilon_0 R_1}\bar{a}_{R1} = \frac{4\times 10^{-9}}{2\pi\left(\frac{10^{-9}}{36\pi}\right)(4\sqrt{2})}\times\frac{\bar{a}_x - \bar{a}_y}{\sqrt{2}}$$

$$= 9\,\bar{a}_x - 9\,\bar{a}_y \text{ (V/m)}$$

Similarly electric field due to line charge at y = – 4 is,

$$\bar{E}_2 = \frac{\rho_{l2}}{2\pi\epsilon_0 R_2}\bar{a}_{R2} = \frac{4\times 10^{-9}}{2\pi\left(\frac{10^{-9}}{36\pi}\right)4\sqrt{2}}\times\frac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}$$

$$= 9\,\bar{a}_x + 9\,\bar{a}_y \text{ (V/m)}$$

Total electric field at point P is obtained by superposition,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = (9\,\bar{a}_x - 9\,\bar{a}_y) + (9\,\bar{a}_x + 9\,\bar{a}_y)$$

$$= 18\,\bar{a}_x \text{ (V/m)} \qquad \text{...Ans.}$$

Ex. 2.5.4 : An infinite uniform line charge $\rho_l = 2$ nC / m lies along the x-axis in free space, while point charges of 8 nC are located at (0, 0, 1) and (0, 0, – 1).

(a) Find electric field intensity at (2, 3, – 4).

(b) To what value should ρ_l be changed to cause $\bar{E}$ to be zero at (0, 0, 3).

☑ **Soln. : (a) To find $\bar{E}$ at (2, 3, – 4)**

Let E because of line charge be $\bar{E}_l$. The foot of perpendicular from (2, 3, – 4) on the line charge is at (2, 0, 0). Required vector and unit vector is :

$$\bar{R} = 3\,\bar{a}_y - 4\,\bar{a}_z \;;\therefore\; \bar{a}_R = \frac{3\,\bar{a}_y - 4\,\bar{a}_z}{5}$$

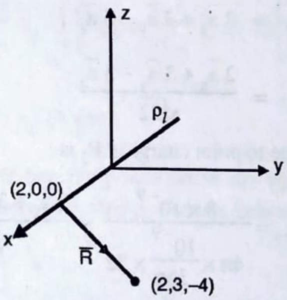


Fig. (a) : $\bar{E}$ due to line charge

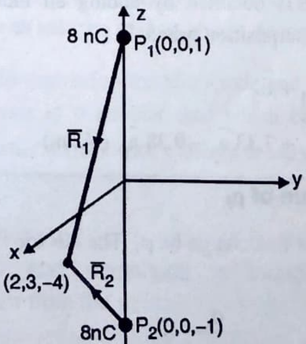


Fig. (b) : $\bar{E}$ due to point charge

Fig. Ex. 2.5.4

Then the electric field due to line charge is :

$$\bar{E}_l = \frac{\rho_l}{2\pi\epsilon_0 R}\bar{a}_R = \frac{2\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times5}\times\frac{3\bar{a}_y-4\bar{a}_z}{5}$$

$$= 4.32\,\bar{a}_y - 5.76\,\bar{a}_z \text{ (V/m)}$$

Let $\bar{E}$ due to point charge at P_1 be $\bar{E}_1$. In the Fig. Ex. 2.5.4(b),

$$\bar{R}_1 = 2\bar{a}_x + 3\bar{a}_y - 5\bar{a}_z$$

$$\therefore \bar{a}_{R1} = \frac{2\bar{a}_x + 3\bar{a}_y - 5\bar{a}_z}{\sqrt{38}}$$

The electric field due to point charge at P_1 is :

$$\bar{E}_1 = \frac{Q_1}{4\pi\epsilon_0 R_1^2}\bar{a}_{R1}$$

$$= \frac{8\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\times38}\times\frac{2\bar{a}_x+3\bar{a}_y-5\bar{a}_z}{\sqrt{38}}$$

i.e. $\bar{E}_1 = 0.614\,\bar{a}_x + 0.921\,\bar{a}_y - 1.535\,\bar{a}_z$ (V/m)

Let $\bar{E}$ due to point charge at P_2 to $\bar{E}_2$. In the Fig. Ex. 2.5.4(b),

$$\bar{R}_2 = 2\bar{a}_x + 3\bar{a}_y - 3\bar{a}_z\ ;$$

$$\bar{a}_{R2} = \frac{2\bar{a}_x + 3\bar{a}_y - 3\bar{a}_z}{\sqrt{22}}$$

The electric field due to point charge at P_2 is :

$$\bar{E}_2 = \frac{Q_2}{4\pi\epsilon_0 R_2^2}\bar{a}_{R2} = \frac{8\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\times22}\times\frac{2\bar{a}_x+3\bar{a}_y-3\bar{a}_z}{\sqrt{22}}$$

i.e. $\bar{E}_2 = 1.396\,\bar{a}_x + 2.094\,\bar{a}_y - 2.094\,\bar{a}_z$ (V/m)

Now total $\bar{E}$ is obtained by adding all individual electric fields i.e. using Superposition principle.

$$\bar{E} = \bar{E}_l + \bar{E}_1 + \bar{E}_2$$

$$= 2.01\,\bar{a}_x + 7.33\,\bar{a}_y - 9.38\,\bar{a}_z \text{ (V/m)} \quad \text{...Ans.}$$

(b) To find value of ρ_l

Let density of line charge be ρ_l. The electric field due to line charge is,

$$\bar{E}_l = \frac{\rho_l}{2\pi\epsilon_0 R}\bar{a}_R$$

In the Fig. Ex. 2.5.4(c), the length of perpendicular from (0, 0, 3) on line charge is R = 3 and $\bar{a}_R = \bar{a}_z$.

$$\therefore \bar{E}_l = \frac{\rho_l}{2\pi\times10^{-9}/36\pi\times3}\bar{a}_z$$

$$= 6\times10^{-9}\rho_l\,\bar{a}_z \text{ (V/m)}$$

The electric field due to point charge at (0, 0, 1) is,

$$\bar{E}_1 = \frac{Q}{4\pi\epsilon_0 R^2}\bar{a}_R$$

In the Fig. Ex. 2.5.4(c) distance between (0, 0, 3) and charge at (0, 0, 1) is,

R = 2 and $\bar{a}_R = \bar{a}_z$

$$\therefore \bar{E}_1 = \frac{8\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\times2^2}$$

$$\bar{a}_R = 18\,\bar{a}_z$$

Similarly, E due to point charge at (0, 0, – 1) is,

$$\bar{E}_2 = \frac{8\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\times4^2}$$

$$\bar{a}_z = 4.5\,\bar{a}_z$$

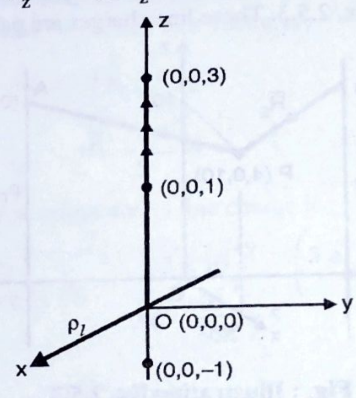


Fig. (c) : Illustrating Ex. 2.5.4

It is given that $\bar{E}_l + \bar{E}_1 + \bar{E}_2 = 0$

$$\therefore (6\times10^{-9}\rho_l + 18 + 4.5)\,\bar{a}_z = 0$$

i.e. $\rho_l = -3.75\times10^{-9}$

or $\rho_l = -3.75$ (nC/m) ...Ans.

⏭ 2.6 ELECTRIC FIELD DUE TO INFINITE SHEET CHARGE

UQ. Develop an expression for E due to a uniform charge density ρ_s on an infinite sheet.

MU - May 10, Dec. 10, Dec. 11, 8 Marks

UQ. Derive an expression for electric field intensity due to infinite surface charge at any point P on z-axis, if surface is placed at z = 0 plane of ρ_S C/m^2.

MU - May 11, 10 Marks

Assumptions

(i) Consider the infinite sheet with charge density $\rho_s\ (C/m^2)$ be in xy plane.

(ii) The point where we desire the electric field is on z axis (0, ϕ, z) refer Fig. 2.6.1.

The given charge is a sheet charge not a point charge therefore create a small charge $dQ = \rho_s\ ds$ on the sheet. Let this small charge be at (r, ϕ, 0). The electric field due to dQ acts along the vector.

$$\bar{R} = -r\,\bar{a}_r + z\,\bar{a}_z\ ;\ \bar{a}_R = \frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}}$$

The electric field due to charge dQ is :

$$\overline{dE} = \frac{dQ}{4\pi\epsilon_0 R^2}\,\bar{a}_R$$

Where, $dQ = \rho_s\,ds$ and $ds = r\,dr\,d\phi$

Putting values of R, a_R and dQ in the expression for dE ,

$$\overline{dE} = \frac{\rho_s\,r\,dr\,d\phi}{4\pi\epsilon_0(r^2+z^2)}\left[\frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}}\right]$$

This electric field has two components – $\bar{a}_r$ (which is horizontal towards origin) and $\bar{a}_z$ (which is vertical). When one charge dQ is at (r, ϕ, 0), because the charge is uniformly distributed, another charge dQ is also present at diametrically opposite to dQ at (r, ϕ, 0).

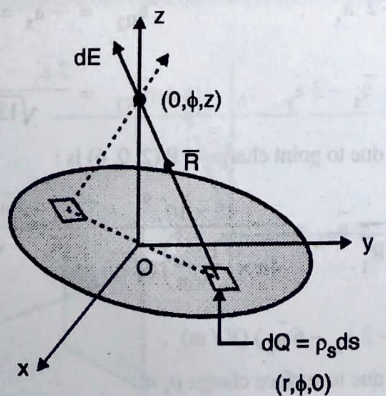


Fig. 2.6.1 : Finding $\bar{E}$ due to sheet charge

The electric field intensities due to these two charges when added, the horizontal components of $\overline{dE}$ gets cancelled and vertical components remain. To find the total electric field due to sheet charge, electric field intensities due to all small charges forming sheet charge are to be added.

In the addition process, the horizontal components gets cancelled and only vertical components remain. The total electric field is obtained by integrating dE with only vertical component. Thus,

$$\bar{E} = \int_0^{2\pi}\int_0^{\infty}\frac{\rho_s\,r\,dr\,d\phi}{4\pi\epsilon_0(r^2+z^2)^{3/2}}\,(z\,\bar{a}_z)$$

(The ∞ limit in the above integration is due to the fact that the infinite sheet charge is obtained when radius r extends to infinity.)

$$\bar{E} = \frac{\rho_s z}{4\pi\epsilon_0}\times(2\pi)\,\bar{a}_z\int_0^{\infty}\frac{r\,dr}{(r^2+z^2)^{3/2}}$$

To find integration :

Put	$r^2 + z^2 = t$	limits : $r \to 0;\ t \to z^2$
	$2r\,dr = dt$	$r \to \infty;\ t \to \infty$

Then the above expression becomes

$$\bar{E} = \frac{\rho_s z}{2\epsilon_0}\,\bar{a}_z\int_{z^2}^{\infty}\frac{dt/2}{t^{3/2}} = \frac{\rho_s z}{4\epsilon_0}\,\bar{a}_z\,(-2)\left[\frac{1}{\sqrt{t}}\right]_{z^2}^{\infty} \quad ...(2.6.1)$$

or $$\bar{E} = \frac{\rho_s}{2\epsilon_0}\,\bar{a}_z\ (V/m)$$

This result is true for points above the xy plane. When the point where we desire the electric field is below xy plane the unit vector changes to $-\bar{a}_z$.

In general, $$\bar{E} = \frac{\rho_s}{2\epsilon_0}\,\bar{a}_n \quad ...(2.6.1(a))$$

"This is expression for an electric field $\bar{E}$ due to infinite sheet of charge with uniform surface charge density."

"This is also expression for Magnitude and direction of $\bar{E}$ at a point on the axis of a circular disc which carriers a uniform charge over its entire surface with a charge density of ρ_s C/m^2."

Note :

(i) The electric field is normal to the plane of the charge and magnitude is independent of the distance from the plane.

(ii) In infinite surface charge problems there is no need to find vector. Only unit vector normal to the surface towards observation point is sufficient.

Ex. 2.6.1 : The surface charge density of 00.1 × 10^{-9} C/m^2 exist on the plane z = 3. Find $\bar{E}$ at (i) P_1 (1, 2, 5) (ii) P_2 (– 1, 3, 2).

☑ **Soln. :** The electric field due to infinite surface charge is,

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_n$$

where $\bar{a}_n$ is a unit vector normal to the infinite surface charge. For the given z = 3 surface unit vector normal is,

$$\bar{a}_n = \pm \bar{a}_z$$

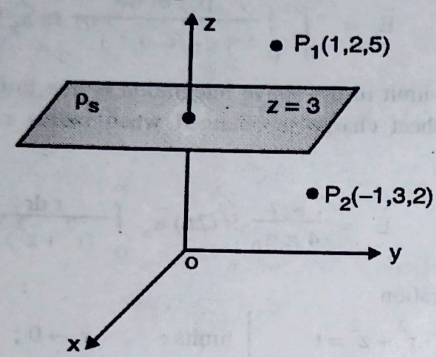


Fig. : Illustrating Ex. 2.6.1

If the point is above the plane, unit vector is pointing upward then positive sign is used. If the point is below then negative sign is used. Point P_1 has z-coordinate greater then 3, hence it is above the plane. While P_2 has z-coordinate less than 3, hence it is below the plane. Therefore,

At point P_1 (1, 2, 5) :

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_n = \frac{0.1\times10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\bar{a}_z = 1.8\pi\times\bar{a}_z \text{ (V / m)...Ans.}$$

At point P_2 (– 1, 3, 2) :

$$\bar{E} = \frac{0.1\times10^{-9}}{2\times\frac{10^{-9}}{36\pi}}(-\bar{a}_z)$$

$$\bar{E} = -1.8\,\pi\,\bar{a}_z \text{ (V / m)} \quad \text{...Ans.}$$

Ex. 2.6.2 : A sheet of charge lies in y-z plane at x = 0 and has uniform surface charge density of 5 pC/m^2. Find the electric field at a point P (– 5, 0, 0) on x axis.

☑ **Soln. :** The infinite surface charge is in x = 0 plane and the point of interest is at P (– 5, 0, 0). The point P lies on the back side of surface charge. Then the unit vector normal towards P is,

$$\bar{a}_n = -\bar{a}_x$$

The Field to due infinite surface charge

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_n = \frac{5\times10^{-12}}{2\times(10^{-9}/36\pi)}(-\bar{a}_x)$$

$$\bar{E} = -0.283\,\bar{a}_x \text{ (V/m)} \quad \text{...Ans.}$$

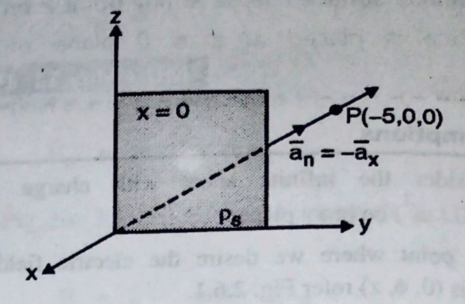


Fig. : Illustrating Ex. 2.6.2

Ex. 2.6.3 : Show that $\nabla\cdot\bar{E}$ is zero for the field of a uniform sheet charge.

☑ **Soln. :** For the uniform sheet charge in the xy plane, the electric field is,

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_z = E_z\,\bar{a}_z$$

Taking the divergence,
$$\nabla\cdot\bar{E} = \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}$$

$$= 0 \quad \text{...since } E_z \text{ is constant.}$$

Ex. 2.6.4 : Find $\bar{E}$ at the origin if the following charge distributions are present in free space,

(i) point charge 12 nC at P (2, 0, 6),

(ii) uniform line charge density 3 nC/m at x = – 2, y = 3,

(iii) uniform surface charge density 0.2 nC/m^2 at x = 2.

☑ **Soln. :**

Required vectors and unit vectors are :

$$\bar{R}_1 = 2\,\bar{a}_x - 6\,\bar{a}_z\;; \qquad \bar{a}_{R1} = \frac{-2\,\bar{a}_x - 6\,\bar{a}_z}{\sqrt{40}}$$

$$\bar{R}_2 = -2\,\bar{a}_x\;; \qquad \bar{a}_{R2} = -\bar{a}_x = \bar{a}_n$$

$$\bar{R}_3 = 2\,\bar{a}_x - 3\,\bar{a}_y\;; \qquad \bar{a}_{R3} = \frac{2\,\bar{a}_x - 3\,\bar{a}_y}{\sqrt{13}}$$

Electric field due to point charge at P (2, 0, 6) is :

$$\bar{E}_Q = \frac{Q}{4\pi\epsilon_0 R_1^2}\bar{a}_{R1} = \frac{12\times10^{-9}}{4\pi\times\left(\frac{10^{-9}}{36\pi}\right)(\sqrt{40})^2}\times\left(\frac{-2\,\bar{a}_x-6\,\bar{a}_z}{\sqrt{40}}\right)$$

$$= 0.427\,(-2\,\bar{a}_x - 6\,\bar{a}_z)\text{ (V / m)}$$

Electric field due to surface charge ρ_s is :

$$\bar{E}_s = \frac{\rho_s}{2\epsilon_0}\bar{a}_n = \frac{0.2\times10^{-9}}{2\times\left(\frac{10^{-9}}{36\pi}\right)}(-\bar{a}_x)$$

$$\bar{E}_s = -11.31\,\bar{a}_x \text{ (V/m)}$$

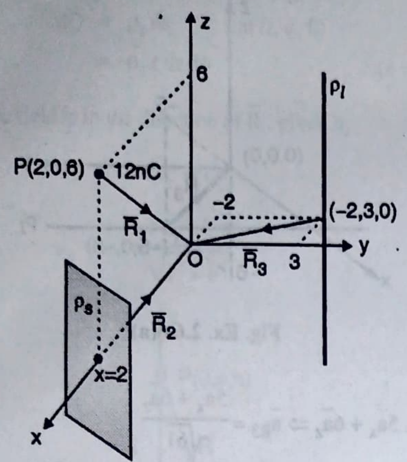


Fig. : Illustrating Ex. 2.6.4

Electric field due to line charge ρ_l is :

$$\bar{E}_l = \frac{\rho_l}{2\pi\epsilon_0 R_3}\bar{a}_{R3}$$

$$= \frac{3\times10^{-9}}{2\pi\times\left(\frac{10^{-9}}{36\pi}\right)(\sqrt{13})}\times\left(\frac{2\bar{a}_x - 3\bar{a}_y}{\sqrt{13}}\right)$$

$$= 4.152\,(2\bar{a}_x - 3\bar{a}_y) \text{ (V/m)}$$

Using principle of Superposition the total electric field is :

$$\bar{E} = \bar{E}_Q + \bar{E}_s + \bar{E}_l$$

$$= -3.86\,\bar{a}_x - 12.456\,\bar{a}_y - 2.562\,\bar{a}_z \text{ (V/m)}. \quad \text{...Ans.}$$

UEx. 2.6.5 (MU - Q. 2(a), Dec. 18, 10 Marks)

A sheet charge of $\rho_s = 2\ \text{nC/m}^2$ located at x = 2 in free space and line charge $\rho_l = 20$ nC/m is located at x = 1 and z = 4, find electric field at the origin and direction of electric field at (4, 5, 6).

☑ **Soln. :**

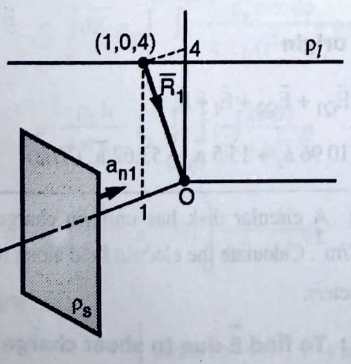


Fig. Ex. 2.6.5

To find E at origin

$$\bar{R}_1 = -\bar{a}_x - 4\bar{a}_z \Rightarrow \bar{a}_{R1} = \frac{-\bar{a}_x - 4\bar{a}_z}{\sqrt{17}}$$

$$\bar{a}_n = -\bar{a}_x$$

E due to surface charge at 'O',

$$\bar{E}_{s0} = \frac{\rho_s}{2\epsilon_0}\bar{a}_{n1} = \frac{2\times10^{-9}}{2\times\left(\frac{10^{-9}}{36\pi}\right)}(-\bar{a}_x)$$

$$= -113.1\,\bar{a}_x \text{ (V/m)}$$

E due to line charge at 'O'

$$\bar{E}_{l0} = \frac{\rho l}{2\pi\epsilon_0 R_1}\bar{a}_{R1}$$

$$= \frac{20\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times\sqrt{17}}\times\left(\frac{-\bar{a}_x - 4\bar{a}_z}{\sqrt{17}}\right)$$

$$= \frac{20\times18}{17}(-\bar{a}_x - 4\bar{a}_z)$$

$$= -21.18\,\bar{a}_x - 84.71\,\bar{a}_z \text{ (V/m)}$$

Total E at 'O'

$$\bar{E}_0 = \bar{E}_{s0} + \bar{E}_{l0} = -134.28\,\bar{a}_x - 84.71\,\bar{a}_z \text{ (V/m)}$$

To find E at (4, 5, 6)

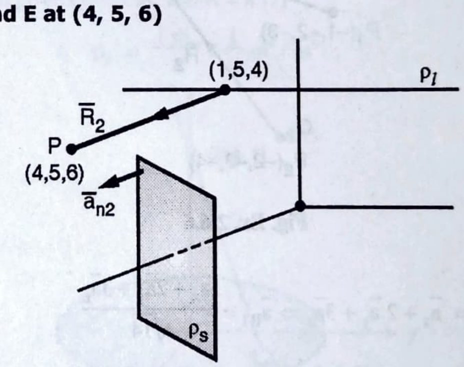


Fig. Ex. 2.6.5(a)

$$\bar{R}_2 = 3\bar{a}_x + 2\bar{a}_z \Rightarrow \bar{a}_{R2} = \frac{3\bar{a}_x + 2\bar{a}_z}{\sqrt{13}}$$

$$\bar{a}_{n2} = \bar{a}_x$$

E due to line charge at (4,5,6)

$$\bar{E}_{lp} = \frac{\rho_l}{2\pi\times\epsilon_0 R_2}\bar{a}_{R2} = \frac{20\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times\sqrt{13}}\left(\frac{3\bar{a}_x + 2\bar{a}_z}{\sqrt{13}}\right)$$

$$= \frac{18\times20}{13}(3\bar{a}_x + 2\bar{a}_z)$$

$$\therefore \bar{E}_{lp} = 83.07\,\bar{a}_x + 55.38\,\bar{a}_z$$

E due to surface charge at (4, 5, 6)

$$\bar{E}_{sp} = \frac{\rho_s}{2\epsilon_0}\bar{a}_{n2} = \frac{2\times 10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\times\bar{a}_x = 113.1\,\bar{a}_x$$

Total E at (4, 5, 6)

$$\bar{E}_P = \bar{E}_{lp} + \bar{E}_{sp} = 197.17\,\bar{a}_x + 55.38\,\bar{a}_z \text{ (V/m)}$$

Direction of E at P is

$$\bar{a}_{EP} = \frac{\bar{E}_P}{|\bar{E}_P|} = \frac{197.17\,\bar{a}_x + 55.38\,\bar{a}_z}{\sqrt{197.17^2 + 55.38^2}}$$

$$\bar{a}_{EP} = 0.966\,\bar{a}_x + 0.27\,\bar{a}_z$$

UEx. 2.6.6 (MU - Q. 3(a), May 19, 10 Marks)

Find out total Electric field at Origin because of following charge distributions :

Point charge of 20 nC placed at (– 1, – 2, – 3)

Point charge of 50 nC placed at (– 2, – 3, – 4)

Uniform infinite line charge of 2 nC/m placed at x = – 5, z = – 6

Uniform infinite surface charge of 0.5 nC/m^2 placed at z = – 5

☑ **Soln. : E due to point charges**

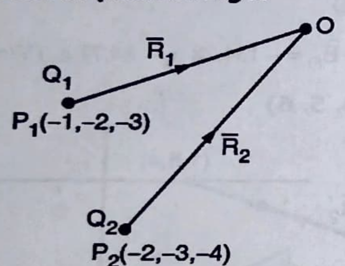


Fig. Ex. 2.6.6

$$\bar{R}_1 = \bar{a}_x + 2\,\bar{a}_y + 3\bar{a}_z \Rightarrow \bar{a}_{R1} = \frac{\bar{a}_x + 2\bar{a}_y + 3\bar{a}_z}{\sqrt{14}}$$

$$\bar{R}_2 = 2\bar{a}_x + 3\,\bar{a}_y + 4\bar{a}_z \Rightarrow \bar{a}_{R2} = \frac{2\bar{a}_x + 3\bar{a}_y + 4\bar{a}_z}{\sqrt{29}}$$

$$\bar{E}_{Q1} = \frac{Q_1}{4\pi\epsilon_0 R_1^2}\bar{a}_{R1} = \frac{20\times 10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\times(\sqrt{14})^2}\left(\frac{\bar{a}_x + 2\bar{a}_y + 3\bar{a}_z}{\sqrt{14}}\right)$$

$$= 3.43\,(\bar{a}_x + 2\bar{a}_y + 3\bar{a}_z)$$

$$\bar{E}_{Q2} = \frac{Q_2}{4\pi\epsilon_0 R_2^2}\,\bar{a}_{R2}$$

$$\bar{E}_{Q2} = \frac{50\times 10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\times(\sqrt{29})^2}\left(\frac{2\bar{a}_x + 3\bar{a}_y + 4\bar{a}_z}{\sqrt{29}}\right)$$

$$\bar{E}_{Q2} = 2.88\,(2\bar{a}_x + 3\bar{a}_y + 4\bar{a}_z) \text{ (V/m)}$$

E due to line charge

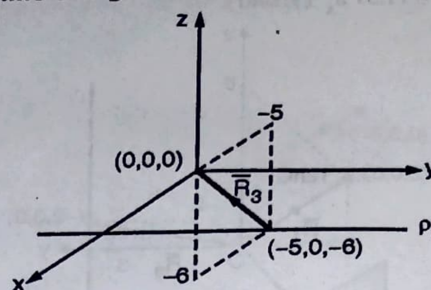


Fig. Ex. 2.6.6(a)

$$\bar{R}_3 = 5\bar{a}_x + 6\bar{a}_z \Rightarrow \bar{a}_{R3} = \frac{5\bar{a}_x + 6\bar{a}_z}{\sqrt{61}}$$

$$\bar{E}_l = \frac{\rho_l}{2\pi\epsilon_0 R_3}\bar{a}_{R3} = \frac{2\times 10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times\sqrt{61}}\left(\frac{5\bar{a}_x + 6\,\bar{a}_z}{\sqrt{61}}\right)$$

$$\bar{E}_l = 0.59\,(5\bar{a}_x + 6\bar{a}_z) \text{ (V/m)}$$

E due to surface charge

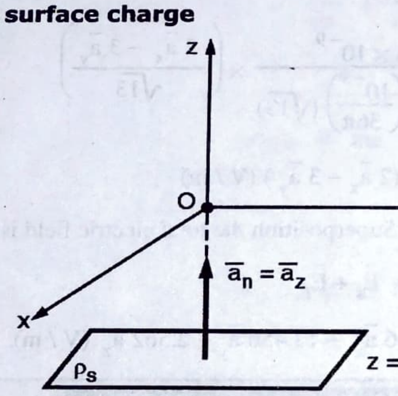


Fig. Ex. 2.6.6(b)

$$\bar{E}_s = \frac{\rho_s}{2\epsilon_0}\bar{a}_n = \frac{0.5\times 10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\times\bar{a}_z = 28.27\,\bar{a}_z$$

Total E at origin

$$\bar{E} = \bar{E}_{Q1} + \bar{E}_{Q2} + \bar{E}_l + \bar{E}_s$$

$$\bar{E} = 10.96\,\bar{a}_x + 15.5\,\bar{a}_y + 53.62\,\bar{a}_z \text{ (V/m)}$$

Ex. 2.6.7 : A circular disk has uniform charge distribution of density ρ_0 C/m^2. Calculate the electric field along its axis, radius of disk is 'a' meters.

☑ **Soln. : To find $\bar{E}$ due to sheet charge**

Consider a circular disk is placed in xy plane with center at origin and the point of interest is on Z axis at a height h from the disk.

The given surface has continuous charge distribution, the differential charge dQ is created on it. It is given by,

$$dQ = \rho_s \, ds \quad \text{at } (r, \phi, 0)$$

$$= \rho_s \, r \, dr \, d\phi \quad \quad ...(z = \text{constant})$$

The electric field is in the direction of $\bar{R}$, given by,

$$\bar{R} = -r\bar{a}_r + h\bar{a}_z$$

$$\therefore \quad \bar{a}_R = \frac{-r\bar{a}_r + h\bar{a}_z}{\sqrt{r^2 + h^2}}$$

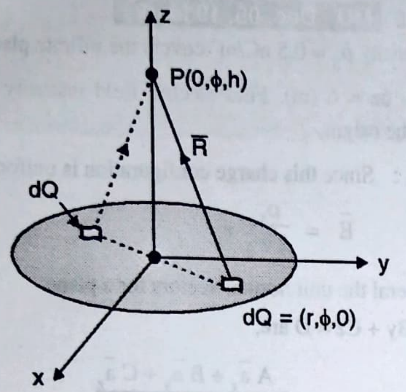


Fig. : Illustrating Ex. 2.6.7

The electric field at point P due to dQ is

$$d\bar{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\bar{a}_R = \frac{\rho_s r\, dr\, d\phi}{4\pi\epsilon_0 (r^2 + h^2)}\left(\frac{-r\bar{a}_r + h\bar{a}_z}{\sqrt{r^2 + z^2}}\right)$$

$$= \frac{\rho_s r\, dr\, d\phi}{4\pi\epsilon_0 (r^2 + h^2)^{3/2}}(-r\bar{a}_r + h\bar{a}_z)$$

When we consider one charge dQ at (r, ϕ, 0), there is one more dQ present at diametrically opposite side as shown in the figure. By adding fields due to these two dQ's horizontal components of the fields cancel each other and only vertical component remains. Thus, when we integrate $d\bar{E}$, it has only $\bar{a}_z$ component.

$$\therefore \quad \bar{E} = \int_s d\bar{E} = \int_0^{2\pi}\int_0^a \frac{\rho_s r\, dr\, d\phi}{4\pi\epsilon_0 (r^2 + h^2)^{3/2}}(h\bar{a}_z)$$

$$= \frac{\rho_s h}{4\pi\epsilon_0}\bar{a}_z \int_0^{2\pi}\int_0^a \frac{r\, dr\, d\phi}{(r^2 + h^2)^{3/2}}$$

$$\bar{E} = \frac{\rho_s h}{4\pi\epsilon_0}\bar{a}_z (2\pi)\int_0^a \frac{r\, dr}{(r^2 + h^2)^{3/2}}$$

To solve the integral.

Put,	$r^2 + h^2 = t$	limits : $r \to 0$; $t \to h^2$
	$\therefore$ $2r\,dr = dt$	$r \to a$; $t \to a^2 + h^2$
i.e.	$r\,dr = dt/2$	

The integral becomes :

$$\int_{h^2}^{a^2+h^2} \frac{(dt/2)}{t^{3/2}} = \frac{1}{2}\left[\frac{t^{-1/2}}{-1/2}\right]_{h^2}^{a^2+h^2}$$

$$= \left[\frac{1}{\sqrt{t}}\right]_{a^2+h^2}^{h^2} = \frac{1}{h} - \frac{1}{\sqrt{a^2 + h^2}}$$

Putting this value in expression of $\bar{E}$

$$\bar{E} = \frac{\rho_s h}{4\pi\epsilon_0}\bar{a}_z (2\pi)\left[\frac{1}{h} - \frac{1}{\sqrt{a^2 + h^2}}\right]$$

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{h}{\sqrt{a^2 + h^2}}\right]\bar{a}_z \text{ (V/m)}. \quad \text{...Ans.}$$

Ex. 2.6.8 : A flat disk of 1 m radius centered at the origin and situated in the xy plane has a charge of 1 nC uniformly distributed over its surface find $\bar{E}$ at (0, 0, 1).

☑ **Soln. :** This problem is similar to previous problem, only difference is ρ_s is not given directly, then how to proceed ? The Soln. is very easy. Remember how we define ρ_s, it is

$$\rho_s = \frac{Q_T}{A}$$

where, Q_T is the total charge spread on the area A. In the problem given $Q_T = 1$ nC and $A = \pi (1)^2$.

$$\therefore \quad \rho_s = \frac{1 \text{ nC}}{\pi} = \frac{1}{\pi} \text{ nC/m}^2$$

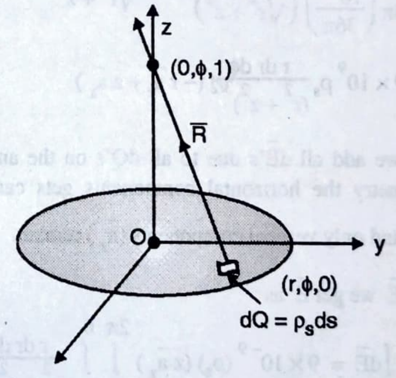


Fig. : Illustrating Ex. 2.6.8

Using the result of the previous problem we get E.

[Ans. : 5.27 $\bar{a}_z$ (V / m)]

Ex. 2.6.9 : A thin annular disc of inner radius 'a' and outer radius 'b' carries a uniform surface charge density ρ_s. Determine the electric field intensity at any point on the z-axis.

☑ **Soln. :** Let the annular ring is placed in xy plane with center at origin. The point on z-axis in cylindrical system is P (0, ϕ, z). Consider a small charge dQ is at (r, ϕ, 0).

Required vector and unit vector is :

$$\bar{R} = -r\,\bar{a}_r + z\,\bar{a}_z$$

$$\bar{a}_R = \frac{-r\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}}$$

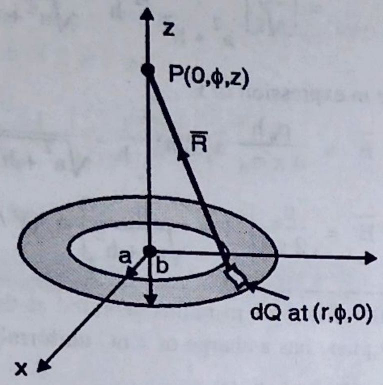


Fig. : Illustrating Ex. 2.6.9

The differential charge dQ on the annular disc in xy plane is :

$$dQ = \rho_s\,ds = \rho_s\,r\,dr\,d\phi$$

The differential field due to dQ is :

$$d\bar{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\,\bar{a}_R$$

$$= \frac{\rho_s\,r\,dr\,d\phi}{4\pi\left(\frac{10^{-9}}{36\pi}\right)\left(\sqrt{r^2+z^2}\right)^2} \times \frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$d\bar{E} = 9\times10^9\,\rho_s\frac{r\,dr\,d\phi}{(r^2+z^2)^{3/2}}\,(-r\,\bar{a}_r + z\,\bar{a}_z)$$

When we add all $d\bar{E}$'s due to all dQ's on the annular disc, due to symmetry the horizontal components gets cancel i.e. a_r components and only vertical components $(\bar{a}_z)$ remain.

Integrating $d\bar{E}$ we get $\bar{E}$ as :

$$\bar{E} = \int d\bar{E} = 9\times10^{-9}\,(\rho_s)\,(z\,\bar{a}_z)\int_0^{2\pi}\int_a^b \frac{r\,dr\,d\phi}{(r^2+z^2)^{3/2}}$$

$$\bar{E} = 9\times10^{-9}\,(\rho_s)\,(z\,\bar{a}_z)\,(\phi)_0^{2\pi}\int_a^b \frac{r\,dr}{(r^2+z^2)^{3/2}}$$

$$= 18\,\pi\times10^9\,(\rho_s)\,(z\,\bar{a}_z)\int_a^b \frac{r\,dr}{(r^2+z^2)^{3/2}}$$

o solve integral :

Put	$r^2+z^2=t$	limits :	$r\to a,\ t\to a^2+z^2$
i.e.	$2r\,dr = dt$		$r\to b,\ t\to b^2+z^2$

$$\therefore\ \bar{E} = 18\,\pi\times10^9\,(\rho_s)\,(z\,\bar{a}_z)\int_{(a^2+z^2)}^{(b^2+z^2)} \frac{\frac{dt}{2}}{t^{3/2}}$$

$$= 18\,\pi\times10^9\,(\rho_s)\,(z\,\bar{a}_z)\frac{1}{2}\left[\frac{t^{-1/2}}{-1/2}\right]_{(a^2+z^2)}^{(b^2+z^2)}$$

$$\bar{E} = 18\,\pi\times10^9\,(\rho_s)\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right] z\,\bar{a}_z\ \ (V/m)$$

...Ans.

UEx. 2.6.10 **MU - Dec. 09, 10 Marks**

A charge density $\rho_S = 0.5\ nC/m^2$ covers the infinite plane $-x + 3y - 6z = 6$ (m). Find electric field intensity on the side containing the origin.

☑ **Soln. :** Since this charge configuration is uniform sheet,

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\,\bar{a}_n$$

In general the unit normal vectors for a plane $Ax + By + Cz = D$ are,

$$a_n = \pm\frac{A\,\bar{a}_x + B\,\bar{a}_y + C\,\bar{a}_z}{\sqrt{A^2+B^2+C^2}}$$

Therefore, the unit normal vectors for the plane given are:

$$a_n = \pm\frac{(-\bar{a}_x + 3\,\bar{a}_y - 6\,\bar{a}_z)}{\sqrt{46}}$$

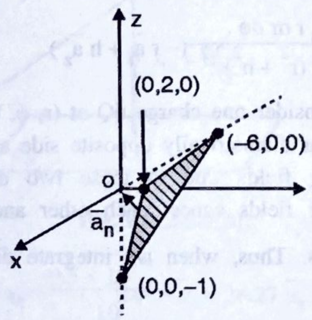


Fig. : Illustrating Ex. 2.6.10

Intersections of the plane with the coordinate axes is obtained from $-x + 3y - 6z = 6$ as,

(0, 2, 0)	...y-axis intersection
(– 6, 0, 0)	...x-axis intersection
(0, 0, – 1)	...z-axis intersection

It is clear from the orientation of the plane that the unit vector in the side containing origin is having negative $\bar{a}_y$ component. This negative sign is obtained by taking negative sign in the above equation. That is,

$$\bar{a}_n = -\frac{(-\bar{a}_x + 3\,\bar{a}_y - 6\,\bar{a}_z)}{\sqrt{46}}$$

i.e. $$\bar{a}_n = \frac{\bar{a}_x - 3\,\bar{a}_y + 6\,\bar{a}_z}{\sqrt{46}}$$

Putting this value in the expression for $\bar{E}$

$$\bar{E} = \frac{\rho_s}{2\,\epsilon_0}\,\bar{a}_n$$

$$= \frac{0.5 \times 10^{-9}}{2 \times \frac{10^{-9}}{36\pi}} \times \frac{\bar{a}_x - 3\,\bar{a}_y + 6\,\bar{a}_z}{\sqrt{46}}$$

$$\bar{E} = 28.75\left(\frac{\bar{a}_x - 3\,\bar{a}_y + 6\,\bar{a}_z}{\sqrt{46}}\right)$$

$$= 4.168\,(\bar{a}_x - 3\,\bar{a}_y + 6\,\bar{a}_z)\ (\text{V/m}).$$ **...Ans.**

Chapter Ends...

❑❑❑

Module 1

CHAPTER 3

Electric Flux Density and Gauss's Law

Module 1

3.1 INTRODUCTION

The electric field about a point charge is

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\bar{a}_r$$

- This equation shows that at any point the electric field strength E depends not only upon the magnitude and position of the charge Q, but also upon the dielectric constant of the medium (air, oil and others) in which the field is measured.
- It is desired to associate with the charge Q a second electrical quantity that will be independent of the medium involved.
- The second quantity is called as **electric displacement or electric flux** and is designated by the symbol ψ.

3.2 ELECTRIC FLUX

- To understand what is ψ ? Faraday's experiment (Fig. 3.2.1) with the concentric spheres will help us.
- The experiment is as follows :
- A sphere with charge Q was placed within, but not touching a larger hollow sphere. If there is any charge initially present on the outer sphere in order to remove it, the outer sphere was 'earthed' momentarily and then the inner sphere was removed.
- The charge remaining on the outer sphere was then measured. This charge was found to be equal (and of opposite sign) to the charge on the inner sphere.

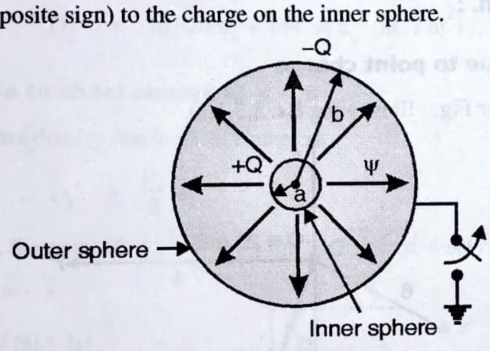


Fig. 3.2.1 : Faraday's experiment

- This is true for all sizes of the spheres and for all types of dielectric media between the spheres.
- Here the question arises, even though there is no electrical connection between two spheres, how the charge is developed on the outer sphere ? The answer is, there must be some electrical displacement taking place from inner sphere to outer sphere through the media between the two spheres. The amount of this displacement depends only upon the magnitude of the charge on the inner sphere.
- The electric displacement is denoted by ψ.

Thus, Electric displacement = $\psi = Q(C)$ ...(3.2.1)

- The electric displacement is also called as displacement flux or simply **electric flux.**
- It is measured in Coulombs.

3.2.1 Characteristic / Properties of Flux Lines

The electric flux or also called flux lines have some typical characteristics :

(i) Since the flux ψ is equal to Q, the flux is measured in Coulombs.

(ii) The electric flux is independent of medium.

(iii) The electric flux lines are directed lines which have starting point and end point.

(iv) These flux lines start from positive charge and terminate on the negative charge, Fig. 3.2.2(a).

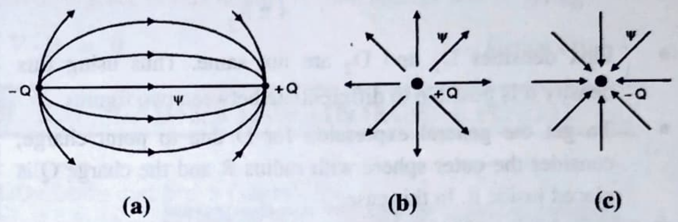


Fig. 3.2.2 : Direction of field lines

(v) If only positive charge is present in the system, then flux lines originate from the charge and terminate at infinity, Fig. 3.2.2(b).

(vi) If the system consists of only negative charge, then flux lines terminate on the charge originating from infinity, Fig. 3.2.2(c).

3.3 ELECTRIC FLUX DENSITY ($\bar{D}$)

GQ. Explain electric flux density. **(5 Marks)**

- In the above experiment, let the inner sphere is reduced its size so that it becomes a point charge 'Q'. Consider two different situations in which outer spheres are having radius R_1 and R_2.
- In both cases the total flux crossing the outer sphere is equal to charge on the inner sphere which is 'Q' Coulombs.

$$\Psi = Q$$

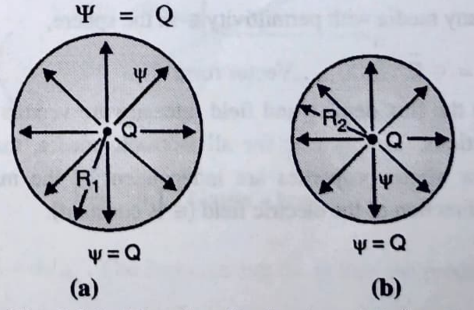


Fig. 3.3.1 : Flux density due to point charge

- Though situations in Figs. 3.3.1(a) and (b) are different, flux ψ remain same. Thus using flux concept we cannot differentiate between two figures. In other words sometimes flux concept is not sufficient to describe the situation.

Definition : Flux density (D), given by :

$$D = \frac{\text{Flux}}{\text{Area}} \ (C/m^2)$$

3.3.1 Vector Form of Electric Flux Density (Relation between $\bar{D}$ and $\bar{E}$)

- In Fig. 3.3.1(a) and (b) the flux is crossing different areas.

For Fig. 3.3.1(a) : $D_1 = \dfrac{\psi = Q}{4\pi R_1^2}$ and

For Fig. 3.3.1(b) : $D_2 = \dfrac{\psi = Q}{4\pi R_2^2}$

- Flux densities D_1 and D_2 are not same. Thus using flux density it is possible to differentiate between two figures.
- To get the general expression for D due to point charge, consider the outer sphere with radius R and the charge Q is placed inside it. In this case,

Flux through outer sphere = ψ = Q

$\therefore$ Flux density, $D = \dfrac{\psi = Q}{4\pi R^2}$

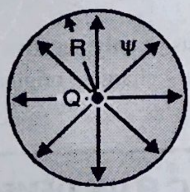


Fig. 3.3.1(c) : $\bar{D}$ due to point charge

- At any point on the outer sphere at a distance R from point charge Q, the electric field is,

$$E = \frac{Q}{4\pi\epsilon_0 R^2}$$

Comparing D and E, $D = \epsilon_0 E$

In vector form $\bar{D} = \epsilon_0 \bar{E}$ ...(3.3.1)

For any media with permittivity ϵ in the sphere,

$\bar{D} = \epsilon \bar{E}$ ...Vector form ...(3.3.2)

- Here the flux density and field intensity are vectors in same directions. This is true for all isotropic media, that is the media whose properties are independent of the magnitude and direction of the electric field (ϵ is constant).

- Using Equation (3.3.1), we can obtain $\bar{D}$ due to different types of charge configurations provided $\bar{E}$ for them is known. For example,

For point charge at origin :

$$\bar{D} = \frac{Q}{4\pi r^2}\bar{a}_r \ (C/m^2) \quad ...(3.3.3)$$

For infinite line charge :

$$\bar{D} = \frac{\rho_l}{2\pi r}\bar{a}_r \ (C/m^2) \quad ...(3.3.4)$$

For infinite surface charge :

$$\bar{D} = \frac{\rho_s}{2}\bar{a}_n \ (C/m^2) \quad ...(3.3.5)$$

For volume charge density, ρ_v :

$$\bar{D} = \int_v \frac{\rho_v\, dv}{4\pi R^2}\bar{a}_R \ (C/m^2) \quad ...(3.3.6)$$

3.3.2 Solved Examples on Flux Density

Ex. 3.3.1 : Find the flux density at a point A (6, 4, – 5) caused by :

(i) a point charge of 20 mC at the origin.

(ii) a uniform line charge $\rho_l = 20\ \mu C/m$ on the z-axis.

(iii) a uniform charge density $\rho_s = 60\ \mu C/m^2$ at a plane x = 8.

Soln. :

(i) $\bar{D}$ due to point charge

Refer Fig. : Illustrating Ex. 3.3.1(a).

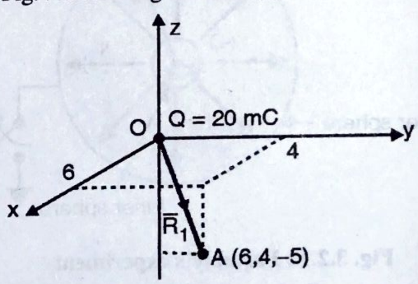


Fig. : Illustrating Ex. 3.3.1(a)

We know that $\bar{D}$ due to point charge is,

$$\bar{D}_1 = \frac{Q}{4\pi R_1^2}\bar{a}_{R1}$$

Required vector and unit vector is,

$$\bar{R}_1 = 6\bar{a}_x + 4\bar{a}_y - 5\bar{a}_z \ ; \quad \bar{a}_{R1} = \frac{6\bar{a}_x + 4\bar{a}_y - 5\bar{a}_z}{\sqrt{77}}$$

$$\therefore \quad \bar{D}_1 = \frac{20\times 10^{-3}}{4\pi\times 77}\left(\frac{6\bar{a}_x + 4\bar{a}_y - 5\bar{a}_z}{\sqrt{77}}\right)$$

$$\therefore \quad \bar{D}_1 = 14.16\,\bar{a}_x + 9.44\,\bar{a}_y - 11.8\,\bar{a}_z\ (\mu C/m^2) \qquad \text{...Ans.}$$

(ii) $\bar{D}$ due to line charge

Refer Fig. : Illustrating Ex. 3.3.1(b).

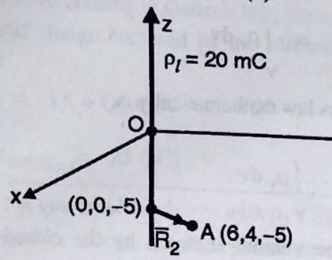


Fig. : Illustrating Ex. 3.3.1(b)

The perpendicular from (6, 4, – 5) on line charge along z-axis is at (0, 0, – 5). The vector joining (0, 0, – 5) to (6, 4, – 5) is,

$$\bar{R}_2 = 6\,\bar{a}_x + 4\,\bar{a}_y \ ; \ \bar{a}_{R2} = \frac{6\,\bar{a}_x + 4\,\bar{a}_y}{\sqrt{52}}$$

$\bar{D}$ at (6, 4, – 5) due to line charge is :

$$\bar{D}_2 = \frac{\rho_l}{2\pi R_2}\,\bar{a}_{R2}$$

$$= \frac{20 \times 10^{-6}}{2\pi \times \sqrt{52}} \left(\frac{6\,\bar{a}_x + 4\,\bar{a}_y}{\sqrt{52}} \right)$$

$$\bar{D}_2 = 367.26\,\bar{a}_x + 244.84\,\bar{a}_y\ (nC/m^2) \qquad \text{...Ans.}$$

(iii) $\bar{D}$ due to sheet charge at x = 8

The flux density due to sheet charge is,

$$\bar{D}_s = \frac{\rho_s}{2}\,\bar{a}_n$$

Since, the point (6, 4, – 5) is on the backside of sheet charge at x = 8, $\bar{a}_n = -\bar{a}_x$

$$\therefore \quad \bar{D}_s = \frac{60 \times 10^{-6}}{2}(-\bar{a}_x) = -30\,\bar{a}_x\ (\mu C/m^2) \qquad \text{...Ans.}$$

Total flux density is

$$\bar{D} = \bar{D}_1 + \bar{D}_2 + \bar{D}_3$$

$$\bar{D} = 351.42\,\bar{a}_x + 254.28\,\bar{a}_y - 11.8\,\bar{a}_z\ (\mu C/m^2) \qquad \text{...Ans.}$$

Ex.3.3.2 : Prove that the divergence of electric field and that of electric flux density in a charge free region is zero.

☑ **Soln. :**

We can consider flux density due to any type of charge distribution.

For point charge at origin, $\bar{D} = \frac{Q}{4\pi r^2}\,\bar{a}_r = D_r\,\bar{a}_r$

Divergence of it in spherical (Since r in above expression is in spherical)

$$\nabla \cdot \bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(D_\phi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) \quad (\because D_\theta = D_\phi = 0)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\frac{r^2 Q}{4\pi r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi}\right) = 0$$

Thus divergence of flux density is zero. Also $\bar{D} = \epsilon\,\bar{E}$ giving

$$\nabla \cdot E = 0 \qquad \text{... Hence Proved}$$

3.4 GAUSS'S LAW (INTEGRAL FORM)

UQ. State and prove Gauss's law.

MU - Dec. 09, Dec. 10, Dec. 12, 5 Marks

UQ. State Gauss's law for electrostatic fields.

MU - May 10, 3 Marks

UQ. State and explain Gauss's Law.

(MU - Q. 1(f), May 19, Q. 1(e), Dec. 19, 5 Marks)

Statement : The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

3.4.1 Mathematical Expression for Gauss's Law

- Let a positive charge Q is enclosed by a closed surface of any shape. At point P consider an differential element of surface ds and let $\bar{D}$ makes an angle θ with $\bar{ds}$ as shown in Fig. 3.4.1.

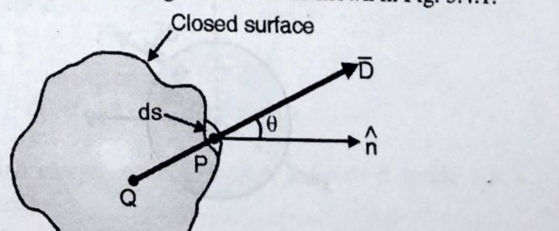


Fig. 3.4.1 : Gauss's law

- Here, $\bar{ds} = ds\,\bar{a}_n$. The flux crossing $\bar{ds}$ is then the product of the normal component of $\bar{D}$ and ds .

$\therefore \quad d\psi \;=\;$ flux crossing $ds = D_{normal}\, ds = D \cos\theta\, ds$

$= \bar{D} \cdot \bar{ds}$ ...using definition of dot product

The total flux crossing through the closed surface is,

$$\psi = \oint_s d\psi = \oint_s \bar{D} \cdot \bar{ds} = \epsilon \oint_s \bar{E} \cdot \bar{ds}$$

Here, $\oint_s$ means integral over a closed surface.

Mathematically Gauss's law can now be expressed as :

$$\boxed{\oint \bar{D} \cdot \bar{ds} = Q_{enclosed}}$$

☞ Proof

Consider now a positive charge Q situated at the center of an imaginary sphere of radius r as shown in Fig. 3.4.2.

- The total flux passing through the sphere is $\psi = \oint \bar{D} \cdot \bar{ds}$.
- Since a point charge is situated at the centre, the flux density is,

$$\bar{D} = \frac{Q}{4\pi r^2} \bar{a}_r$$

The differential area on the spherical surface is,

$$\bar{ds} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

Hence,
$$\psi = \int_0^{2\pi}\int_0^{\pi} \left(\frac{Q}{4\pi r^2} \bar{a}_r\right) \cdot (r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r)$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{Q}{4\pi} \sin\theta\, d\theta\, d\phi \quad \text{...(as } \bar{a}_r \cdot \bar{a}_r = 1)$$

$$= \frac{Q}{4\pi} [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} = \frac{Q}{4\pi} \times 2 \times 2\pi = Q\ (C)$$

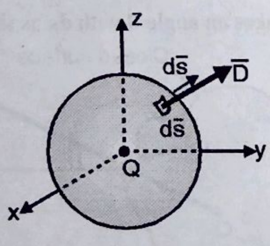


Fig. 3.4.2 : $\bar{D}$ due to Point charge

- This means that Q Coulombs of electric flux is crossing the surface if Q is the total or net charge enclosed. Thus Gauss law is proved.
- We could have obtained the result for this special case more simply by multiplying $Q / 4\pi r^2$ by the area of the sphere $(4\pi r^2)$. But if the charge is not at the center of the sphere, or if there is a distribution of charge enclosed by a surface of arbitrary shape, we should use integration.
- In general the charge can be expressed in terms of ρ_v as,

$$Q = \int_V \rho_v\, dv$$

Then the Gauss law mathematically is,

$$\oint \bar{D} \cdot \bar{ds} = \int_V \rho_v\, dv \qquad \text{...(3.4.1)}$$

- Where v is the volume enclosed by the closed surface s. Notice that in Equation (3.4.1),
- The LHS of this equation gives the total flux crossing the closed surface, while
- The RHS gives the total charge enclosed by this closed surface.

Equation (3.4.1) is called, the integral form of Maxwell's equation derived from the Gauss's law.

☞ Maxwell's Equations

- In electromagnetism, Maxwell's equations are a set of four partial differential equations that describe the properties of electric and magnetic fields and relate them to their sources, charge density and current density.
- These equations are used to show that the light is an electromagnetic wave. Individually, the equations are known as,

(i) Gauss's law for electric,
(ii) Gauss's law for magnetic,
(iii) Faraday's law of induction,
(iv) Ampere's law with Maxwell's correction.

- The Gauss law which we studied in this section was formulated by Carl Friedrich Gauss in 1835.
- The Gauss law for electric and magnetic was used by Maxwell's as it is in the set of Maxwell's equations.
- Using the set of four equations, it is possible to make analysis of a wave in any type of media, unbounded media, waveguide or optical fiber.
- Because of this work, though each equation is derived by different person, the set of equations is called as Maxwell's equations.
- Each equation has two forms, one is integral form and the other is point form.
- The point form of Gauss law is derived in the next part of this chapter. The equations related to other law shall be studied in other chapters.

3.4.2 Solved Examples on Gauss's Law

Ex. 3.4.1 : Three point charges, $Q_1 = 10$ nC, $Q_2 = 20$ nC, and $Q_3 = -5$ nC are enclosed by surface s. How much flux crosses s ?

☑ **Soln. :** According to Gauss's law, the total flux crossing s is equal to total charge enclosed by the surface s. The total charge enclosed is,

$$Q_T = Q_1 + Q_2 + Q_3 = 10 + 20 - 5 = 25 \text{ (nC)}$$

$$\therefore \quad \Psi_{\text{crossing s}} = 25 \text{ (nC)} \qquad \text{...Ans.}$$

Ex. 3.4.2 : A uniform line charge with $\rho_l = 3\ \mu C / m$ lies along the x-axis. What flux crosses a spherical surface centered at the origin with r = 3 m ?

☑ **Soln. :** The length of the line charge enclosed within the sphere is equal to $2 \times 3 = 6$m. Thus, the charge enclosed is,

$$Q = \rho_l \times l = 3 \times 10^{-6} \times 6 = 18\ \mu C$$

According to Gauss's law,

$$\Psi = Q = 18\ \mu C \qquad \text{...Ans.}$$

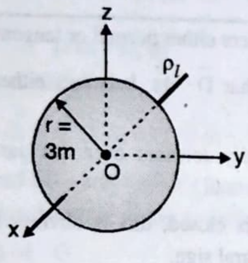


Fig. : Illustrating Ex. 3.4.2

Ex. 3.4.3 : Three point charges are located in air : + 0.008 μC at (0, 0) m, + 0.005 μC at (3, 0) m and at (0, 4) m there is a charge of – 0.009 μC. Compute the total flux over a sphere of 5 m radius with centre at (0, 0).

☑ **Soln. :** The sphere of 5 m radius will enclose all three charges in the system. Thus the total charge enclosed is,

$$Q_{\text{encl.}} = (+0.008 + 0.005 - 0.009) \times 10^{-6} = 4 \times 10^{-9} \text{ (C)}$$

According to Gauss law,

$$\text{Flux} = \text{charge enclosed} = 4 \times 10^{-9} \text{ (C)} \qquad \text{...Ans.}$$

Ex. 3.4.4 : A line charge on the X axis has its one end at the origin and the other end at 2.4 m. The charge density of the line is defined by $\rho_l = 2e^{-0.4x}$ μC/m. Find the flux out of a closed surfaced enclosing the line charge.

☑ **Soln. :**

$$\psi = Q_{\text{enclosed}} = \int \rho_l\, dl = \int_0^{2.4} 2e^{-0.4x}\, dx$$

$$\psi = 2\left(\frac{e^{-0.4x}}{-0.4}\right)_0^{2.4} = 3.086 (\mu C)$$

UEx. 3.4.5 (MU - May 10, 8 Marks)

A uniform line charge with $\rho_l = 50\ \mu C/m$ lies along the x-axis. What flux per unit length Ψ / L, crosses the portion of the z = – 3 m plane bounded by y = ± 2 m ?

☑ **Soln. :** The line charge and the given plane in front view looks as shown in Figure.

The angle subtended by the strip is θ where

$$\frac{\theta}{2} = \tan^{-1}\left(\frac{2}{3}\right) = 33.69°$$

$$\theta = 2 \times 33.69° = 67.38°$$

Consider a length L on the line charge. The change over this length is

$$Q = \rho_l L = 50 \times 10^{-6} \times L$$

The flux will be same as that of charge and is uniformly distributed around the line charge i.e. in 360°. The flux passing through the strip is proportional to the angle subtended by it.

$$50 \times 10^{-6} \times L \rightarrow 360°$$

$$\psi = ? \leftarrow 67.38°$$

$$\psi = \frac{50 \times 10^{-6} (L) \times 67.38°}{360°}$$

or $$\frac{\psi}{L} = \frac{50 \times 10^{-6} \times 67.38}{360} = 9.36\ \mu C/m$$

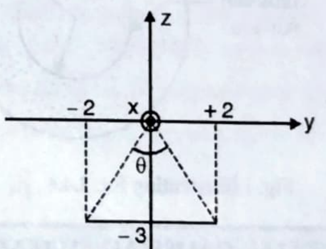


Fig. : Illustrating Ex. 3.4.5

UEx. 3.4.6 (MU - May 09, 10 Marks)

A spherical charge distribution is given by,

$\rho_v = \rho_0 \left[1 - \frac{r^2}{a^2}\right]$ For $0 \le r \le a$. Find $\bar{E}$ at $\frac{r}{a} = 0.745$.

☑ **Soln. :**

Given : $\rho_v = \rho_0 \left[1 - \frac{r^2}{a^2}\right]$ for $0 \le r \le a$

(i) To find charge enclosed for r ≤ a (since E inside r = a is asked)

$$Q_{\text{encl.}} = \int_V \rho_v\, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho_0 \left[1 - \frac{r^2}{a^2}\right] r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$Q_{\text{encl.}} = \rho_0 \left\{ (-\cos\theta)_0^{\pi} (\phi)_0^{2\pi} \left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]_0^r \right\}$$

$$Q_{encl} = 4\pi\rho_0\left[\frac{r^3}{3}-\frac{r^5}{5a^2}\right]$$

(ii) **To find D using Gauss law**

D at a distance r from center of the sphere such that $(r \le a)$ (since E at $r = 0.745$ a is asked which is inside $r = a$) is

$$Q_{encl.} = D_r(4\pi r^2)$$

$$\therefore \quad D_r = \frac{Q_{encl.}}{4\pi r^2} = \frac{1}{4\pi r^2}\left[4\pi\rho_0\left(\frac{r^3}{3}-\frac{r^5}{5a^2}\right)\right]$$

or $$E_r = D_r/\epsilon_0 = \frac{\rho_0}{\epsilon_0}\left(\frac{r}{3}-\frac{r^3}{5a^2}\right)$$

or $$\bar{E} = E_r\bar{a}_r = \frac{\rho_0}{\epsilon_0}\left(\frac{r}{3}-\frac{r^3}{5a^2}\right)\bar{a}_r \text{ (V/m)}$$

$\bar{E}$ at r = 0.745 a

$$= \frac{\rho_0}{\epsilon_0}\left(\frac{0.745a}{3}-\frac{(0.745\,a)^3}{5a^2}\right)\bar{a}_r$$

$$= \frac{\rho_0\,a}{\epsilon_0}(0.166)\,\bar{a}_r$$ ...Ans.

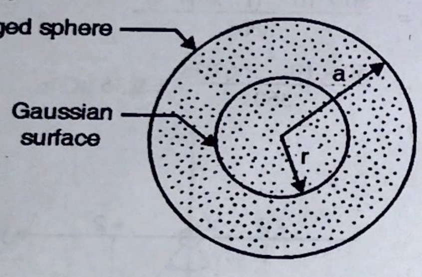


Fig. : Illustrating Ex. 3.4.6

3.5 SPECIAL GAUSSIAN SURFACES

- We know the Guass's law, $\psi = Q = \oint_s \bar{D}\cdot\bar{ds}$
- The left hand side of it is electric flux and can be determined by having the knowledge of $\bar{D}$ and $\bar{ds}$. Its not like Gauss law is used to find only flux.
- The same law can also be used to determine D of some known charge distributions.
- By using some conditions the problem of finding D is simplified. Before stating these conditions let us have look in the expression of Gauss's law.
- The integral in the expression has dot product term. There are two extreme cases of dot product :

(i) When $\bar{D}$ and $\bar{ds}$ are normal $\bar{D}\cdot\bar{ds} = 0$

...$(\bar{a}\cdot\bar{b} = 0$; when $\theta = 90°)$

(ii) When $\bar{D}$ and $\bar{ds}$ are parallel $\bar{D}\cdot\bar{ds} = D\,ds$

...$(\bar{a}\cdot\bar{b} = ab$; when $\theta = 0)$

- For the first case the integral goes to zero. For the second case if $D \neq 0$, and is constant then,

$$\oint_s \bar{D}\cdot\bar{ds} = \oint_s D\,ds = D\oint_s ds = DS$$

- Which is product of required D and the surface area over with D is existing.
- This product is equal to charge enclosed. Thus, when charge enclosed is known and area of surface (s) is known, D due to this charge configuration can be determined. Now, it's time to specify the conditions which simplifies the problem of determining D.

3.5.1 Condition for Special Gaussian Surfaces

1. D is everywhere either normal or tangential to the closed surface, so that $\bar{D}\cdot\bar{ds}$ becomes either D ds or zero, respectively.
2. D is essentially constant over that part of the surface where D is normal.
3. The surface is closed, this is obvious because see the circle on integral sign.

The type of surface or surface elements which satisfies these conditions are called as **Special Gaussian Surfaces.**

- While solving problems of finding $\bar{D}$ due to given charge configuration, you should have knowledge of the symmetry, with the help of which a proper closed surface can be selected. Remember,
- While dealing with the problems of positive point charge, electric field of it is directed radially outward from the point charge i.e. $\bar{a}_r$ in spherical system and
- While dealing with the problems of line charge electric field is directed radially outward from the line charge, i.e. $\bar{a}_r$ in cylindrical system. These things we have already derived in earlier sections.

3.6 APPLICATIONS OF GAUSS'S LAW

UQ. Explain applications of Gauss's Law.
(MU - Q. 1(f), May 19, 5 Marks)

Let us apply now Gauss's law to find D or E due to some symmetric charge distributions.

3.6.1 To Find $\bar{D}$ due to a Point Charge Q at the Origin

- The field due to point charge Q at the origin is in radial direction i.e. $\bar{a}_r$ (in spherical) direction. The only component of D is D_r. Also this field is the function of r only.
- Having knowledge of this symmetry, we can select a surface which satisfies the conditions of Gaussian surface. The surface in question is obviously a spherical surface centered at the origin and of any radius r.

Charge enclosed within Gaussian surface

- Since charge Q is at the center of the spherical surface, the only charge enclosed by the Gaussian surface is the charge Q (C).

i.e. $$Q_{encl.} = Q \quad \text{...(i)}$$

Flux through the closed surface

- The field D has only D_r Component in $\bar{a}_r$ direction and $\bar{ds}$ for sphere is also in same direction ($\bar{ds} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$)

$$\therefore \quad \bar{D}\cdot\bar{ds} = D\,ds = D_r\,ds$$

The flux through the closed surface,

$$\psi = \oint \bar{D}\cdot\bar{ds} = \oint D_r\,ds = D_r \oint ds = D_r\,(4\pi r^2) \quad \text{...(ii)}$$

From Equations (i) and (ii), using Gauss's law,

Total flux through closed surface = charge enclosed

i.e. $D_r\,(4\pi r^2) = Q$

or $$D_r = \frac{Q}{4\pi r^2}$$

or $$\bar{D} = D_r\,\bar{a}_r = \frac{Q}{4\pi r^2}\,\bar{a}_r \ (\text{C/m}^2)$$

The field intensity is,

$$\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}\,\bar{a}_r \quad \text{...(3.6.1)}$$

> *"Equation (3.6.1) is the expression for flux density due to point charge."*

This agrees with the result derived in chapter (2).

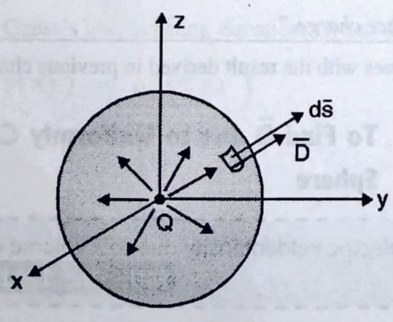


Fig. 3.6.1 : D due to point charge

3.6.2 To Find $\bar{D}$ due to a Uniform Line Charge

> **UQ.** Using Gauss's law, derive expression for intensity due to an infinite line charge. **MU - May 10, 3 Marks**

- Let us consider this uniform line charge with distribution ρ_l (C / m) lying along the z-axis.
- From the previous knowledge, the field due to this line charge is radially outward from the line charge ($\bar{a}_r$ in cylindrical). That is D has only D_r component.
- Also D is only a function of r.
- The special Gaussian surface for this line charge is a closed right circular cylinder whose axis is the z-axis. The closed surface is a right circular cylinder of length L, which consists of top and bottom plane surfaces and a cylindrical surface.

Charge enclosed within Gaussian surface

- The length of the line charge enclosed within the cylinder of length L is L (m). Thus the charge enclosed within the cylinder is,

$$Q_{encl.} = \text{Charge density} \times \text{length of line charge} = \rho_l\,L \quad \text{...(i)}$$

Flux through the closed surface

- Here the Gaussian surface is the cylinder with radius r (Refer Fig. 3.6.2). This closed surface consists of three surfaces. The $\bar{ds}$ for the top and bottom of this cylinder is in $\bar{a}_z$ direction, but D is in $\bar{a}_r$ direction. So for these two sides,

$$\bar{D}\cdot\bar{ds} = 0$$

- For cylindrical surface $\bar{D}$ and $\bar{ds}$ are in same $\bar{a}_r$ direction, causing

$$\bar{D}\cdot\bar{ds} = D\,ds = D_r\,ds$$

The flux through closed surface is,

$$\psi = \oint \bar{D}\cdot\bar{ds} = \int_{\text{Top}} \bar{D}\cdot\bar{ds} + \int_{\text{Bottom}} \bar{D}\cdot\bar{ds} + \int_{\text{Cylin.}} \bar{D}\cdot\bar{ds}$$

Since first two integral are zero,

$$\psi = \int_{\text{Cylin.}} \bar{D}\cdot\bar{ds} = \int_{\text{Cylin.}} D_r\,ds = D_r \int_{\text{Cylin.}} ds = D_r\,(2\pi r L) \quad \text{...(ii)}$$

From Gauss's law, using Equations (i) and (ii),

Total flux through closed surface = Charge enclosed

i.e. $D_r\,(2\pi r L) = \rho_l\,L$

or $$D_r = \frac{\rho_l}{2\pi r}$$

or $$\bar{D} = D_r\,\bar{a}_r = \frac{\rho_l}{2\pi r}\,\bar{a}_r \ (\text{C/m}^2)$$

Now, the electric field intensity due to infinite line charge is,

$$\bar{E} = \frac{\bar{D}}{\epsilon_0}$$

i.e.
$$\bar{E} = \frac{\rho_l}{2\pi\epsilon_0 r}\bar{a}_r \qquad ...(3.6.2)$$

"Equation (3.6.2) is the expression for field intensity due to uniform line charge."

This agrees with the result derived in previous chapter.

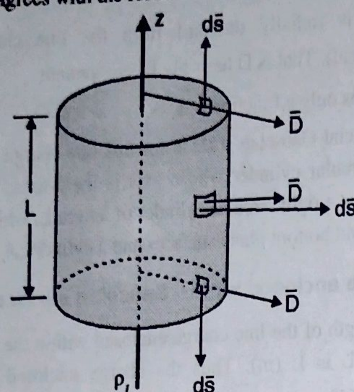


Fig. 3.6.2 : D due to line charge

3.6.3 To Find $\bar{D}$ due to Infinite Surface Charge

UQ. Using Gauss's law, find an expression for flux density due to an infinite uniform surface charge.
MU - Dec. 09, Dec. 10, Dec. 12, 5 Marks

- Let us consider this infinite surface charge with density ρ_s (C/m^2) is lying in z = 0 plane.
- From the previous knowledge, the field due to this surface charge is normal to the surface, towards the point of interest.
- The flux density is independent of distance of point.
- The special Gaussian surface for this charge configuration is a rectangular box, half above and half below the surface charge. Refer Fig. 3.6.3.

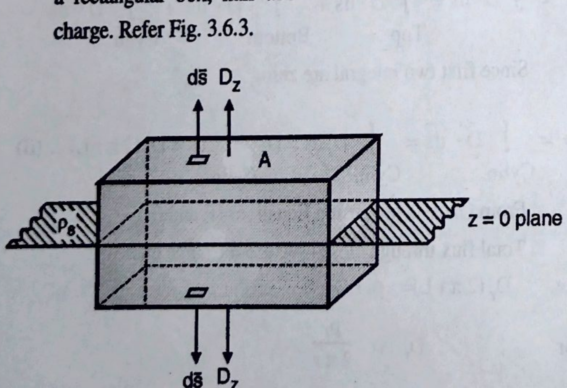


Fig. 3.6.3 : D due to surface charge

Charge enclosed within the Gaussian surface

The area of the surface charge enclosed within the box is A(m^2). Therefore, the charge enclosed is,

$$Q_{encl.} = \text{charge density} \times \text{area enclosed} = \rho_s A \qquad ...(i)$$

The flux crossing the closed surface

For the four vertical sides of the box the area $\bar{ds}$ is horizontal, but the flux density D is vertical, causing for each vertical side,

$$\bar{D}\cdot\bar{ds} = 0$$

For top and bottom side, the direction of $\bar{D}$ and $\bar{ds}$ is same, resulting in,

$$\bar{D}\cdot\bar{ds} = D\,ds = D_z\,ds$$

The flux through Gaussian surface is,

$$\psi = \oint \bar{D}\cdot\bar{ds} = \int_{Top} \bar{D}\cdot\bar{ds} + \int_{Bottom} + \int_{Left} + \int_{Right} + \int_{Front} + \int_{Back}$$

Since $\bar{D}\cdot\bar{ds}$ for last four integral is zero,

$$\psi = \int_{Top} \bar{D}\cdot\bar{ds} + \int_{Bottom} \bar{D}\cdot\bar{ds} = D_z \int_{Top} ds + D_z \int_{Bottom} ds$$

$$= D_z A + D_z A = 2 A D_z \qquad ...(ii)$$

Using Equations (i) and (ii), and Gauss's law,

Total flux crossing the closed surface = Charge enclosed

i.e. $2 A D_z = \rho_s A$

or $D_z = \dfrac{\rho_s}{2}$

or $\bar{D} = D_z \bar{a}_z = \dfrac{\rho_s}{2}\bar{a}_z$ (C/m^2)

The field intensity due to this infinite surface charge is,

$$\bar{E} = \frac{\bar{D}}{\epsilon_0} \quad \text{i.e.} \quad \bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_z \qquad ...(3.6.3)$$

"Equation (3.6.3) is the expression for field intensity due to infinite surface charge."

This agrees with the result derived in previous chapter.

3.6.4 To Find $\bar{D}$ due to Uniformly Charged Sphere

UQ. Find electric field intensity due to a volume charge.
MU - Dec. 12, 10 Marks

- Consider a sphere of radius 'a' with a uniform charge density, ρ_v (C/m^2) inside it.

- The field due to this charge will be in radial direction, that is $\bar{D}$ has only D_r component.
- Due to spherical symmetry, this field will be a function of only r.
- Having knowledge of this symmetry, the Gaussian surface for this problem is a sphere.

Expression for $\bar{D}$ and $\bar{E}$ in all regions of charged sphere

- There are two regions of interest, one is inside the sphere (r < a) and the other is outside (r > a).

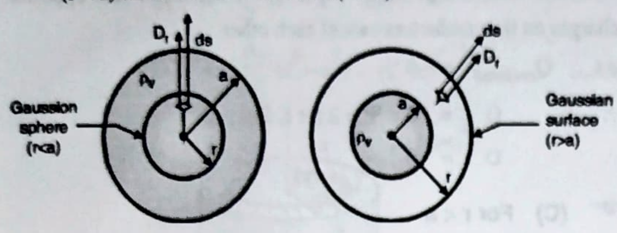


Fig. 3.6.4 : $\bar{D}$ for charged sphere

(A) For r < a

Charge enclosed within Gaussian surface

The charge enclosed by the Gaussian sphere is,

$$Q_{encl} = \int_v \rho_v\, dv = \rho_v \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho_v \left(\frac{4}{3}\pi r^3\right) \text{ (C)} \qquad \text{...(i)}$$

Flux through the closed surface

Since D and ds are both in radial direction,

$$\bar{D}\cdot \bar{ds} = D\,ds = D_r\, ds$$

The flux through the closed surface,

$$\Psi = \oint \bar{D}\cdot \bar{ds} = \oint D_r\, ds = D_r \oint ds = D_r\,(4\pi r^2) \qquad \text{...(ii)}$$

Using Gauss's law, equating Equations (i) and (ii),

$$D_r\,(4\pi r^2) = \rho_v\left(\frac{4}{3}\pi r^3\right)$$

This gives, $D_r = \frac{r}{3}\rho_v$

or $\bar{D} = D_r \bar{a}_r = \frac{r}{3}\rho_v\, \bar{a}_r$

The electric field intensity is,

$$\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{r}{3\epsilon_0}\rho_v\,\bar{a}_r \text{ (V/m)} \qquad \text{...For } r < a$$

(B) For r > a

Charge enclosed within Gaussian surface

The Gaussian sphere with r > a will totally enclose the charged sphere (r = a). Thus the total charge enclosed by the Gaussian sphere is the total charge present inside the charged sphere. Then,

$$Q_{encl.} = \int_v \rho_v\, dv = \rho_v \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho_v\left(\frac{4}{3}\pi a^3\right) \qquad \text{...(iii)}$$

Flux through the closed surface

The directions of $\bar{D}$ and $\bar{ds}$ are same for the Gaussian surface, then

$$\bar{D}\cdot \bar{ds} = D\,ds = D_r\, ds$$

The flux through the closed surface is,

$$\Psi = \oint \bar{D}\cdot \bar{ds} = \oint D_r\, ds$$

$$= D_r \oint ds = D_r\,(4\pi r^2) \qquad \text{...(iv)}$$

Equations (iii) and (iv), using Gauss's law,

$$D_r\,(4\pi r^2) = \rho_v\left(\frac{4}{3}\pi a^3\right)$$

or $D_r = \frac{a^3}{3r^2}\rho_v$

or $\bar{D} = D_r\bar{a}_r = \frac{a^3}{3r^2}\rho_v\,\bar{a}_r \text{ (C/m}^2)$

The electric field due to this charge configuration is,

$$\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{a^3}{3\epsilon_0 r^2}\rho_v\,\bar{a}_r \text{ (V/m)} \qquad \text{...For } r > a$$

The above results are summarized below.

$$\bar{D} = \frac{r}{3}\rho_v\,\bar{a}_r \quad \text{for } r < a \qquad \text{...(3.6.4(a))}$$

$$\bar{D} = \frac{a^3}{3r^2}\rho_v\,\bar{a}_r \quad \text{for } r > a \qquad \text{...(3.6.4(b))}$$

> *"Equation (3.6.4(a)) and (3.6.4(b)) are the expressions for $\bar{D}$ inside and outside the charged sphere."*

The plot of it can be made as shown in Fig. 3.6.5.

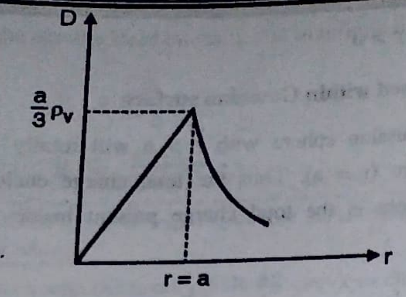


Fig. 3.6.5

3.6.5 To Find D in all Regions of Coaxial Cable

Consider two coaxial cylindrical conductors, the inner of radius 'a' and outer of radius 'b', each is infinite in extent.

Let the charge density on the outer surface of the inner conductor is ρ_s.

Expression for $\bar{D}$ in different regions of a co-axial cable

Symmetry shows that only radial component of D (a_r in cylindrical) and it is a function of only r.

(A) For a < r < b

Here the Gaussian surface is a cylinder with radius r.

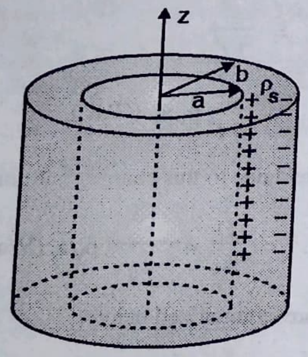


Fig. 3.6.6 : D due to coaxial cable

The charge on the inner conductor of length L is :

$$Q = \int_0^L \int_0^{2\pi} \rho_s \, a \, d\phi \, dz = 2\pi a L \rho_s$$

The flux crossing cylinder with radius r is :

$$\Psi = D_r (2\pi r L) = Q = 2\pi a L \rho_s$$

$$\therefore \quad \bar{D} = a \frac{\rho_s}{r} \bar{a}_r$$

This result can be expressed in terms of ρ_l where,

$$\rho_l = 2\pi a \rho_s$$

$$\therefore \quad \bar{D} = \frac{\rho_l}{2\pi r} \bar{a}_r$$

The result is identical with that of the infinite line charge.

(B) For r > b

We know the charge on the outer cylinder is equal and opposite to charge on the inner cylinder, that is,

$$2\pi b L \rho_{s,\,outer} = -2\pi a L \rho_{s,\,inner}$$

or

$$\rho_{s,\,outer} = -\frac{a}{b} \rho_{s,\,inner}$$

Consider the Gussian surface as a cylinder with radius r > b. Then the total charge enclosed is zero since equal and opposite charges on the conductors cancel each other.

$$\therefore \quad Q_{enclosed} = 0$$

$$\therefore \quad Q = 0 = \Psi = 2\pi r L D$$

$$\therefore \quad D = 0$$

(C) For r < a

As the charge is present on the outer surface of the inner conductor, then the Gaussian cylinder with r < a, will not include any charge, resulting in,

$$D = 0$$

The above results are summarized below :

$$\bar{D} = \frac{\rho_l}{2\pi r} \bar{a}_r \quad \text{for } a < r < b \quad ...(3.6.5(a))$$

$$\bar{D} = 0 \quad \text{for } r > b \quad ...(3.6.5(b))$$

$$\bar{D} = 0 \quad \text{for } r < a \quad ...(3.6.5(c))$$

"Equation (3.6.5 (a)),(3.6.5(b)), (3.6.5(c)) are the expressions for $\bar{D}$ in different regions of co-axial cable."

Ex. 3.6.1 : Obtain electric field in all regions due to following charge distributions in free space. $\rho(r, \phi, z) = 0; \quad 0 < r < a$

$= \rho_0; \quad a < r < b$

$= 0; \quad b < r < \infty$

Soln. :

There are three regions where we can find $\bar{D}$, which are,

$r < a$; $a < r < b$; $r > b$

(A) For r < a

Consider a right circular cylinder of length L with axis along z-axis and r < a, then this cylinder encloses a charge.

$$Q_{enclosed} = 0$$

The flux through Gaussion cylinder with radius r is,

$$\psi = \oint \bar{D} \cdot \bar{ds} = D_r \oint ds = D_r (2\pi r L)$$

Using Gauss law

$$D_r (2 \pi r L) = 0 \quad \text{i.e. } D_r = 0 \quad \text{...Ans.}$$

(B) For $a < r < b$

Consider a right circular cylinder of length L with axis along z-axis and $a < r < b$. This cylinder encloses a charge.

$$Q_{enclosed} = \pi (r^2 - a^2) \times L \times \rho_0$$

The Flux through Gaussion cylinder is,

$$\psi = (2 \pi r L) D_r$$

From Guass's law

$$\pi (r^2 - a^2) L \times \rho_0 = D_r (2 \pi r L)$$

$$\therefore \quad D_r = \frac{\rho_0}{2r} (r^2 - a^2) \text{ and}$$

$$\bar{D} = \frac{\rho_0}{2r} (r^2 - a^2) \bar{a}_r (C / m^2) \quad \text{...Ans.}$$

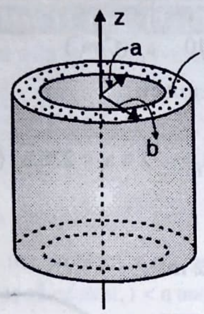


Fig. : Illustrating Ex. 3.6.1

(C) For $r > 4$

Consider a right circular cylinder of length L and $r > b$. This cylinder encloses a charge

$$Q_{enclosed} = \pi (b^2 - a^2) \times L \times \rho_0 = (b^2 - a^2) \pi \rho_0 L$$

The flux through Gaussian cylinder is

$$\psi = D_r (2 \pi r L)$$

From Gauss's law, $(b^2 - a^2) \pi \rho_0 L = D_r (2 \pi r L)$

$$\therefore \quad D_r = \frac{b^2 - a^2}{2r} \rho_0$$

and

$$\bar{D} = \frac{(b^2 - a^2)}{2r} \rho_0 \bar{a}_r (C / m^2) \quad \text{...Ans.}$$

Ex. 3.6.2 : The volume in cylindrical coordinates between $r = 2$ m and $r = 4$ m contains a uniform charge density ρ_v (C / m^3). Find $\bar{D}$ in all regions using Gauss's law.

☑ **Soln. :** There are three regions where we can find $\bar{D}$, which are,

$r < 2$; $2 < r < 4$; $r > 4$

For $r < 2$

Consider a right circular cylinder of length L with axis along z-axis and $r < 2$, then this cylinder encloses a charge.

$$Q_{enclosed} = 0 \quad \therefore 0 = D (2 \pi r L) \text{ i.e. } \bar{D} = 0 \quad \text{...Ans.}$$

For $2 < r < 4$

Consider a right circular cylinder of length L with axis along z-axis and $2 < r < 4$. This cylinder encloses a charge.

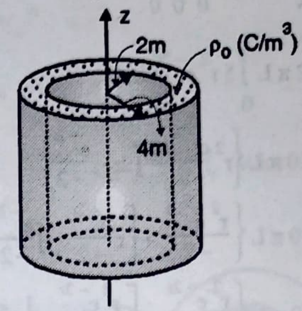


Fig. : Illustrating Ex. 3.6.2

$$Q_{enclosed} = \pi (r^2 - 2^2) \times L \times \rho_v$$

From Guass's law $\pi (r^2 - 2^2) L \times \rho_v = D (2 \pi r L)$

$$\therefore \quad D = \frac{\rho_v}{2r} (r^2 - 4) \text{ and } \bar{D} = \frac{\rho_v}{2r} (r^2 - 4) \bar{a}_r (C / m^2) \quad \text{...Ans.}$$

For $r > 4$

Consider a right circular cylinder of length L and $r > 4$. This cylinder encloses a charge

$$Q_{enclosed} = \pi (4^2 - 2^2) \times L \times \rho_v = 12 \pi \rho_v L$$

From Gauss's law,

$$12 \pi \rho_v L = D (2 \pi r L)$$

$$\therefore \quad D = \frac{6 \rho_v}{r} \text{ and}$$

$$\bar{D} = \frac{6 \rho_v}{r} \bar{a}_r (C / m^2) \quad \text{...Ans.}$$

UEx. 3.6.3 (MU - May 11, Dec. 11, 10 Marks)

A charge configuration is given by $\rho_v = 5r e^{-2r}$ (C/m^3). Find $\bar{D}$ using Gauss's law.

☑ **Soln. :** Assuming the variable r in the expression of ρ_v as radius r in cylindrical system, we see that ρ_v is a function of r only not ϕ and z.

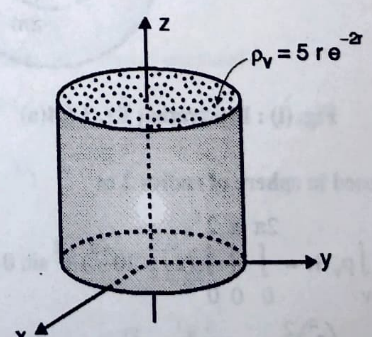


Fig. : Illustrating Ex. 3.6.3

Consider the right circular cylinder of radius r, height L and axis coincides with the z-axis as special gaussian surface. For this surface, charge enclosed is,

$$Q_{enclosed} = \int_v \rho_v \, dv = \int_0^L \int_0^{2\pi} \int_0^r \rho_v \cdot r \, dr \, d\phi \, dz$$

$$= 2\pi L \int_0^r 5 r^2 e^{-2r} dr$$

$$= 10\pi L \left\{ r^2 \frac{e^{-2r}}{-2} - \int 2r \cdot \frac{e^{-2r}}{-2} \right\}$$

$$= 10\pi L \left\{ \frac{r^2 e^{-2r}}{-2} + \left[r \frac{e^{-2r}}{-2} - \int \frac{e^{-2r}}{-2} \right] \right\}$$

$$Q_{enclosed} = 10\pi L \left\{ \frac{r^2 e^{-2r}}{-2} + \left[\frac{r^2 e^{-2r}}{-2} - \frac{1}{4} e^{-2r} \right] \right\}_0^r$$

$$Q_{enclosed} = 5\pi L \left\{ e^{-2r} \left(-r^2 - r - \frac{1}{2} \right) + \frac{1}{2} \right\}$$

Flux through Gaussian cylinder is,

$$\psi = D_r (2\pi r L)$$

From Gauss's law,

$$\therefore \quad D_r \, 2\pi r L = 5\pi L \left\{ e^{-2r} \left(-r^2 - r - \frac{1}{2} \right) + \frac{1}{2} \right\}$$

or $$D_r = \frac{2.5}{r} \left\{ e^{-2r} \left(-r^2 - r - \frac{1}{2} \right) + \frac{1}{2} \right\}$$

or $$\bar{D} = \frac{2.5}{r} \left\{ e^{-2r} \left(-r^2 - r - \frac{1}{2} \right) + \frac{1}{2} \right\} \bar{a}_r \, (C/m^2) \quad \text{...Ans.}$$

Ex. 3.6.4 : A charge distribution in free space has $\rho_v = 2 r \text{ nC/m}^3$ in spherical coordinates, for $0 \le r \le 10$ m and zero otherwise. Determine $\bar{E}$ at r = 2 m and r = 12 m.

☑ **Soln. :**

(a) To find $\bar{E}$ at r = 2 m

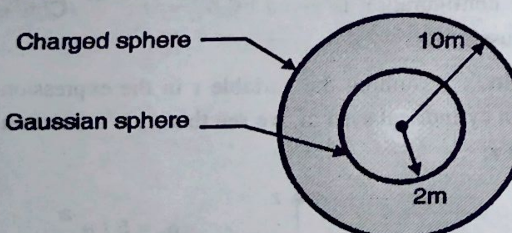


Fig. (i) : Illustrating Ex. 3.6.4(a)

Charge enclosed in sphere of radius 2 m

$$Q_{encl.} = \int_v \rho_v \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{2} (2 r \times 10^{-9}) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{encl.} = 2 \times \left(\frac{r^4}{4} \right)_0^2 (-\cos\theta)_0^{\pi} \, (\phi)_0^{2\pi} \times 10^{-9}$$

$$Q_{encl.} = 32\pi \times 10^{-9} \text{ (C)} \quad \text{...Ans.}$$

Flux through r = 2 sphere

Due to spherical symmetry D will have only D_r component, the sphere will also have $\bar{ds}$ in $\bar{a}_r$ direction, giving

$$\bar{D} \cdot \bar{ds} = D \, ds = D_r \, ds$$

The flux is obtained by,

$$\psi = \oint \bar{D} \cdot \bar{ds} = \oint D_r \, ds = D_r \oint ds = D_r \times 4\pi (2)^2 \text{ (C)}$$

Using Gauss's law,

Flux through closed surface = Charge enclosed

$$\therefore \quad D_r \times 4\pi (2)^2 = 32\pi \times 10^{-9}$$

or $$D_r = 2 \times 10^{-9}$$

or $$\bar{D} = D_r \bar{a}_r = 2 \times 10^{-9} \bar{a}_r \, (C/m^2)$$

The electric field intensity is,

$$\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{2 \times 10^{-9}}{\left(\frac{10^{-9}}{36\pi} \right)} \bar{a}_r = 72\pi \bar{a}_r = 226 \bar{a}_r \text{ (V/m)} \quad \text{...Ans.}$$

(b) To find $\bar{E}$ at r = 12 m

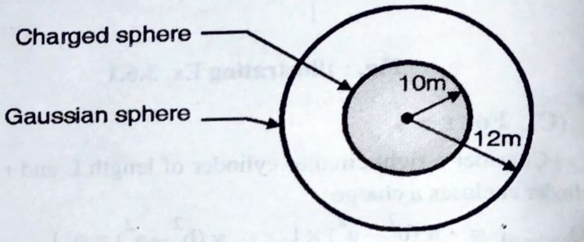


Fig. (ii) : Illustrating Ex. 3.6.4(b)

Charge enclosed in sphere of radius 12 m

This charge is equal to charge present inside sphere of radius 10 m.

$$Q_{encl.} = \int_v \rho_v \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{10} (2 r \times 10^{-9}) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 2 \times \left(\frac{r^4}{4} \right)_0^{10} (-\cos\theta)_0^{\pi} \, (\phi)_0^{2\pi} \times 10^{-9}$$

$$= 2\pi \times 10^4 \times 10^{-9} \text{ (C)} \quad \text{...Ans.}$$

Flux through r = 12 sphere

Again $\bar{D}$ will have only D_r component and the direction is same as for $\bar{ds}$, giving

$$\bar{D} \cdot \bar{ds} = D \, ds = D_r \, ds$$

The flux is obtained by,

$$\psi = \oint \bar{D} \cdot \bar{ds} = \oint D_r \, ds$$

$$= D_r \oint ds = D_r \, 4\pi (12)^2$$

Using Gauss's law,

$$\text{Flux} = \text{Charge enclosed}$$

i.e. $$D_r \, 4\pi (12)^2 = 2\pi \times 10^4 \times 10^{-9}$$

or $$D_r = \frac{1}{2} \frac{10^4}{12^2} \times 10^{-9}$$

or $$E_r = \frac{D_r}{\epsilon_0} = \frac{1}{\left(\frac{10^{-9}}{36\pi}\right)} \times \frac{1}{2} \frac{10^4}{12^2} \times 10^{-9} = 3927$$

or $$\bar{E} = E_r \bar{a}_r = 3927 \, \bar{a}_r \text{ (V/m)}$$ ...Ans.

UEx. 3.6.5 (MU - May 11, Dec. 11, 10 Marks)

A charge distribution with spherical symmetry has density,

$$\rho_v = \frac{\rho_0 r}{a} \qquad 0 \le r \le a$$

$$= 0 \qquad r > a$$

Determine $\bar{E}$ everywhere.

☑ **Soln. :** In the problem a charged sphere is of radius a. There are two regions of interest, r < a and r > a.

(i) $\bar{E}$ for r < a

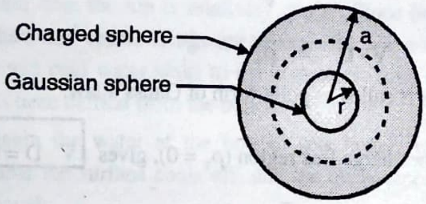


Fig. : Illustrating Ex. 3.6.5(i)

Charge enclosed in sphere of radius r (m)

$$Q_{encl.} = \int_V \rho_v \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \frac{\rho_0 r}{a} (r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

$$= \frac{\rho_0}{a} \left(\frac{r^4}{4}\right)_0^r (-\cos\theta)_0^{\pi} (\phi)_0^{2\pi} = \frac{\rho_0 \pi r^4}{a} \text{ (C)}$$ ...Ans.

Flux through Gaussian sphere of radius r

Due to spherical symmetry, $\bar{D}$ and $\bar{ds}$ are in same direction $\bar{a}_r$, causing

$$\bar{D} \cdot \bar{ds} = D \, ds = D_r \bar{a}_r$$

Flux is given by,

$$\psi = \oint \bar{D} \cdot \bar{ds} = \oint D_r \, ds = D_r \oint ds = D_r (4\pi r^2)$$

Using Gauss's law,

$$\text{Flux} = \text{Charge enclosed}$$

i.e. $$D_r (4\pi r^2) = \frac{\rho_0 \pi r^4}{a}$$

or $$D_r = \frac{\rho_0 r^2}{4a}$$

or $$E_r = \frac{D_r}{\epsilon_0} = \frac{\rho_0 r^2}{4\epsilon_0 a}$$

and, $$\bar{E} = E_r \bar{a}_r = \frac{\rho_0 r^2}{4\epsilon_0 a} \bar{a}_r \text{ (V/m)}$$ ...Ans.

(ii) $\bar{E}$ for r > a

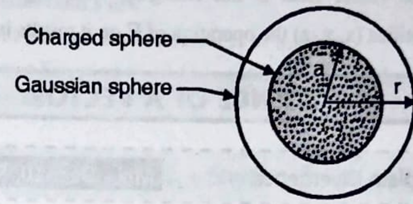


Fig. : Illustrating Ex. 3.6.5(ii)

Charge enclosed in sphere of radius r

This Gaussian sphere will enclose the total charge present in charged sphere of radius a, so

$$Q_{encl.} = \int \rho_v \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{\rho_0 r}{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{encl.} = \frac{\rho_0}{a} \left(\frac{r^4}{4}\right)_0^R (-\cos\theta)_0^{\pi} (\phi)_0^{2\pi} = \rho_0 \pi a^3 \text{ (C)}$$ ...Ans.

Flux through Gaussian surface

Due to spherical symmetry, D has only D_r component. Also $\bar{D}$ and $\bar{ds}$ are in same $\bar{a}_r$ direction, causing

$$\bar{D} \cdot \bar{ds} = D \, ds = D_r \, ds$$

The flux is given by,

$$\psi = \oint \bar{D} \cdot \bar{ds} = \oint D_r \, ds = D_r \oint ds = D_r (4\pi r^2)$$

Using Gauss's law,

$$\text{Flux} = \text{Charge enclosed}$$

i.e. $$D_r (4\pi r^2) = \rho_0 \pi a^3$$

or $$D_r = \frac{\rho_0 a^3}{4 r^2}$$

or $$E_r = \frac{D_r}{\epsilon_0} = \frac{\rho_0 a^3}{4\epsilon_0 r^2}$$

and, $$\bar{E} = E_r \bar{a}_r = \frac{\rho_0 a^3}{4\epsilon_0 r^2} \bar{a}_r \text{ (V/m)}$$ ...Ans.

3.7 THE DEL OPERATOR (∇)

- The del operator is defined as

$$\nabla = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z$$

- This is a vector and can operate on a scalar as well as vector.
- It is defined only in Cartesian system.
- It does not have a specific form in other coordinate systems.
- It can operate on any vector field for which appropriate partial derivatives exist.
- If the vector field is not changing with respect to space quantities (x, y, z) the operation of ∇ on it results in zero.

3.8 DIVERGENCE OF A VECTOR

UQ. Explain Divergence. **MU - Dec. 10, 5 Marks**

3.8.1 Definition

The divergence of a vector field $\bar{A}$ at a point is the net outward flux of $\bar{A}$ per unit volume as the volume about the point shrinks to zero.

It is abbreviated as div $\bar{A}$.

Mathematically, $\text{div}\,\bar{A} = \nabla\cdot\bar{A} = \lim_{\Delta v\to 0}\dfrac{\oint_s \bar{A}\cdot d\bar{s}}{\Delta v}$ ...(3.8.1)

The numerator in Equation (3.8.1) represents the net outward flux, is an integral over the entire surface s that bounds the volume.

3.8.2 Expression for Divergence in Cartesian

Let the vector in Cartesian system be

$$\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$$

Its divergence is

$$\text{div}\,\bar{A} = \nabla\cdot\bar{A}$$

$$= \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right)\cdot\left(A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z\right)$$

Using the definition of dot product

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

3.9 POINT FORM OF GAUSS'S LAW

UQ. Prove that the divergence of electric field and that of electric flux density in a charge free region is zero. **(MU - May 09, 10 Marks)**

The integral form of a Gauss law is

$$\oint \bar{D}\cdot d\bar{s} = \int_V \rho_v\, dv \quad ...(1)$$

We cannot compare D and ρ_v directly as D is associated with surface integral and ρ_v is with volume integral.

But the closed surface integral can be changed into the volume integral using Divergence theorem.

$$\oint \bar{A}\cdot d\bar{s} = \int_V (\nabla\cdot\bar{A})\, dv$$

Applying it to Equation (1)

$$\oint \bar{D}\cdot d\bar{s} = \int_V (\nabla\cdot\bar{D})\, dv = \int_V \rho_v\, dv$$

i.e. $$\int_V (\nabla\cdot\bar{D})\, dv = \int_V \rho_v\, dv$$

or $$\nabla\cdot\bar{D} = \rho_v \quad ...(3.9.1)$$

This is called as point form of Gauss's law.

In the charge free region ($\rho_v = 0$), gives $\boxed{\nabla\cdot\bar{D} = 0}$

The expressions for div $\bar{D}$ in three coordinate systems are :

$$\nabla\cdot\bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \quad \text{...(Cartesian)} \quad ...(3.9.2)$$

$$\nabla\cdot\bar{D} = \frac{1}{r}\frac{\partial}{\partial r}(r D_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(D_\phi) + \frac{\partial}{\partial z}(D_z) \quad ...(3.9.3)$$

...(Cylindrical)

$$\nabla\cdot\bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(D_\phi) \quad \text{...(Spherical)} \quad ...(3.9.4)$$

UEx. 3.9.1 MU - Q. 1(a), Dec. 18, 5 Marks

Calculate charge density due to electric flux density

$\bar{D} = 4r\sin\phi\,\bar{a}_r + 2r\cos\phi\,\bar{a}_\phi + 2z^2\,\bar{a}_z\ C/m^2$.

Soln. :

Given : $\bar{D} = 4r\sin\phi\,\bar{a}_r + 2r\cos\phi\,\bar{a}_\phi + 2z^2\,\bar{a}_z$

$= D_r\bar{a}_r + D_\phi\bar{a}_\phi + D_z\bar{a}_z$

The charge density is

$$\rho_v = \nabla \cdot \bar{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(D_\phi) + \frac{\partial}{\partial z}(D_z)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(4r^2 \sin \phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(2r \cos \phi) + \frac{\partial}{\partial z}(2z^2)$$

$$= \frac{1}{r}(8r \sin \phi) + \frac{1}{r}(-2r \sin \phi) + (4z)$$

$$= 8 \sin \phi - 2 \sin \phi + 4z$$

$$\rho_v = 6 \sin \phi + 4z \text{ C/m}^3$$

3.10 CURRENTS CONVECTION AND CONDUCTION CURRENTS

Before we see what is meant by these two currents, let us see what physics says about these two words convection and conduction.

Convection

- Convection is heat transferred by an air (or water) current. The air heats up, moves to another place, heats up that place.
- Here's an example to explain convection. Imagine a pan of water on a hotplate.
- The water is getting heated from below. The lower parts of the water get hot because they are touching the hot bottom of the pan (conduction coming next).
- The water near the top is relatively cooler. Since hot things are lighter and colder things are denser, the hot water rises the top and cold water sinks to the bottom. So, this way, the heat has been carried from the bottom to the top.
- Now again the water at the bottom sets hotter, while the water near the surface cools off, and the cycle repeats itself continuously.

Conduction

- Conduction is heat moving from one solid object to another.
- A hot piece of metal is place against a cold piece of metal and heat moves from one to the other.
- Conduction is transfer of heat by the atoms of a solid or liquid. When something is hot its atoms are vibrating furiously.
- In a solid, all the atoms are "connected" to each other by spring like forces. So, when one atom starts jiggling, so will its neighbours just like a wave travelling down a slinky toy.
- How fast the juggling (heat) get transferred depends on how strongly connected the atoms are. We know that atoms in a metal (like copper or silver) are more tightly bonded than atoms in plastic or rubber or sand or glass.
- That's why metals are better conductors of heat.
- Also bonding in liquids is extremely weak (there is almost no bonding at all) that's why liquids depend on convection, rather than conduction to transfer heat. Now let us talk about currents.

Convection current

- It does not involve conductors.
- So does not satisfy ohm's law.
- This current flows through an insulting medium such as liquid, rarefied gas or a vacuum.
- Well know example is a vacuum tube in which electrons emitted by cathode flows to anode in a vacuum.

Conduction current

- It requires conductor.
- It satisfies ohm's law.
- Current through any conductor is of this type.
- Note that Equation (3.10.1) is applicable for both these types of currents.

3.10.1 Relationship between $\bar{J}$, ρ_v and $\bar{v}$

- Fig. 3.10.1 shows a rectangular box with edges parallel to coordinate axes. Inside the box, volume charge density ρ_v is present and is moving in the x direction, i.e., only x component of velocity is present.
- In a time interval Δt, the element of charge has moved a distance Δx as shown in Fig. 3.10.1.
- The charge ΔQ which is moved through a reference plane perpendicular to the direction of motion is given by,

$$\Delta Q = \rho_v \times \text{volume} = \rho_v \times \Delta x \times s$$

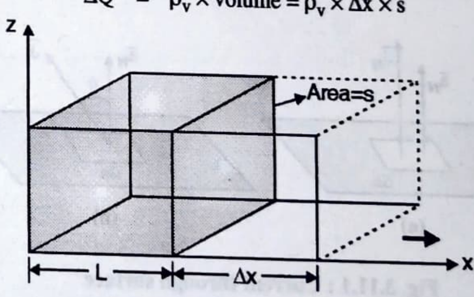


Fig. 3.10.1 : Relation between J and ρ_v

- Divide both sides of this equation by Δt as the movement of charge takes place within time Δt.

$$\frac{\Delta Q}{\Delta t} = \rho_v \times s \times \frac{\Delta x}{\Delta t}$$

Now, $$I = \frac{\Delta Q}{\Delta t} \text{ and } v_x = \frac{\Delta x}{\Delta t}$$

Hence, $$I = \rho_v \times s \times v_x$$

- Let us define now current density as current per unit area. Then the current density in x-direction is

$$J_x = \frac{I}{s} = \rho_v v_x$$

or in general $$\boxed{\bar{J} = \rho_v \bar{v}} \quad ...(3.10.1)$$

3.11 CURRENT DENSITY

- From the basics of physics we know current is the rate of charge with respect to time.

$$I = \frac{dQ}{dt}$$

- The unit of current is ampere (A). Current density is denoted by the letter $\bar{J}$ and is defined as the current (I) crossing an area (s) perpendicularly. It is a vector quantity with the units of A/m^2.

Hence, $\bar{J} = \frac{I}{s}\bar{a}$

where $\bar{a}$ is a unit vector perpendicular to the area s.

- The increment of current dI crossing an incremental surface ds normal to the current density is (Refer Fig. 3.11.1 (a))

$$dI = J_N\, ds$$

But when the current density is not perpendicular we should take normal component of current (Refer Fig. 3.11.1 (b)).

$$dI = J\cos\theta\, ds = \bar{J}\cdot d\bar{s}$$

Then the current through total surface, (s) is obtained by integration.

$$\boxed{I = \int_s \bar{J}\cdot d\bar{s}} \quad ...(3.11.1)$$

This equation is valid for convection as well as conduction currents.

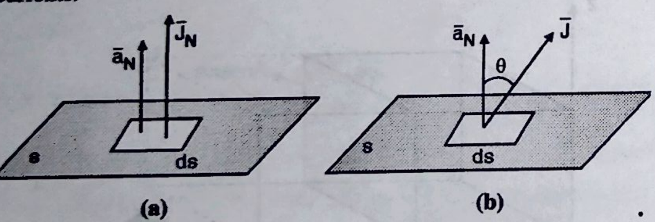


Fig. 3.11.1 : Current through surface

3.12 THE CONTINUITY EQUATION FOR CURRENT

UQ. Obtain point format of Continuity equation. (MU - Q. 1(b), Dec. 18, 5 Marks)

- The current through the closed surface is given by,

$$I = \oint_s \bar{J}\cdot d\bar{s}$$

According to law of conservation of charge, the outward flow of positive charge must be balanced by a decreases of positive charge within the closed surface.

- If Q_i is the charge inside the closed surface, then the rate of decrease is $-\frac{dQ_i}{dt}$. Hence,

$$\boxed{I = \oint_s \bar{J}\cdot d\bar{s} = -\frac{dQ_i}{dt}} \quad ...(3.12.1)$$

- Above equation is the integral from of continuity equation. According to divergence theorem,

$$\oint_s \bar{J}\cdot d\bar{s} = \int_V (\bar{\nabla}\cdot\bar{J})\, dv$$

- Now the charge enclosed Q_i can be expressed in terms of ρ_v, so

$$-\frac{d}{dt}(Q_i) = -\frac{d}{dt}\int dQ_i = -\frac{d}{dt}\int_V \rho_v\, dv$$

Hence, $\int_{vol} (\bar{\nabla}\cdot\bar{J})\, dv = -\frac{d}{dt}\int_{vol} \rho_v\, dv = -\int_{vol} \frac{\partial \rho_v}{\partial t}\, dv$

Here it is assumed that only ρ_v is changing w.r.t. time and dv is not changing.

- This effect is obtained by changing normal differentiation to partial differentiation.

Hence, we have

$$\boxed{\bar{\nabla}\cdot\bar{J} = -\frac{\partial \rho_v}{\partial t}} \quad ...(3.12.2)$$

Which is the differential or point form of continuity equation. This equation indicates that the current or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

3.12.1 Examples on Current Density

Important Formula

$$\boxed{I = \int_s \bar{J}\cdot d\bar{s}}$$

Ex. 3.12.1 : The vector current density is given as,

$\bar{J} = (4/r^2)\cos\theta\,\bar{a}_r + 20\,e^{-2r}\sin\theta\,\bar{a}_\theta - r\sin\theta\cos\phi\,\bar{a}_\phi\ (A/m^2)$

(a) Find $\bar{J}$ at r = 3, θ = 0, φ = π.

(b) Find the total current passing through the spherical cap r = 3, 0° < θ < 20°, 0 < φ < 2π in the direction.

☑ **Soln. :**

(a) Substituting r = 3, θ = 0, φ = π in the expression of $\bar{J}$, we have

$$\bar{J} = (4/9)\cos 0\,\bar{a}_r + 20\,e^{-6}\sin 0\,\bar{a}_\theta - 3\sin 0\cos\pi\,\bar{a}_\phi$$

$$= 0.444\,\bar{a}_r\ (A/m^2)$$ ...Ans.

(b) To the surface r = 3, $\bar{a}_r$ is perpendicular. Hence, $d\bar{s} = r^2\sin\theta\, d\theta\, d\phi\, \bar{a}_r$.

Now, $\bar{J}\cdot d\bar{s} = \left(\frac{4}{r^2}\right)\cos\theta \times r^2\sin\theta\, d\theta\, d\phi$

$= 4\sin\theta\cos\theta\, d\theta\, d\phi = 2\sin 2\theta\, d\theta\, d\phi$

$\therefore I = \int \bar{J}\cdot d\bar{s} = \int_0^{20^\circ}\int_0^{2\pi} 2\sin 2\theta\, d\theta\, d\phi$

$= 2\left[\frac{-\cos 2\theta}{2}\right]_0^{20^\circ}[\phi]_0^{2\pi} = 1.47\ A$

...Ans.

Chapter Ends...

❑❑❑

Module 1

CHAPTER 4

Electric Potential

4.1 INTRODUCTION

- The objective of this chapter is to study potential, denoted by V. The source of potential may be point charge or other types of charge configuration. Initially we shall obtain expression for the potential due to point charge and then this idea can be extended to obtain V due to other types of charge configurations.

  Since the basic definition of scalar potential is

$$\text{Potential} = \frac{\text{Work done}}{\text{Charge}}$$

- It is necessary to have knowledge of work done. In the next section we shall study work done and then go for the potential.

4.2 ENERGY EXPENDED (WORK DONE)

4.2.1 Expression for Work Done

- Energy expended is nothing but work done. In this article, we are going to calculate how much work is required to be done in moving a point charge in electric field.
- We have the basic formula for work done

$$\boxed{\text{Work done} = \text{Force applied} \times \text{displacement}}$$

- A charge Q experiences a force in an electric field $\bar{E}$. If we attempt to move the charge against the electric field, we have to apply a force equal and opposite to that exerted by the field. This requires us to do work.
- If we wish to move the charge in the direction of the field, work done is negative because we do not do the work, the field does.

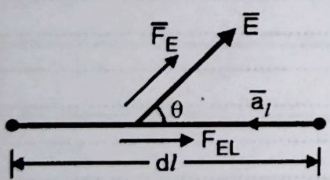


Fig. 4.2.1 : Finding work done

- Suppose we wish to move a charge Q, a distance dl in an electric field $\bar{E}$ as shown in Fig. 4.2.1. The force on Q due to electric field is,

$$\bar{F}_E = Q\bar{E}$$

- Subscript E indicates that this force is applied by electric field E, not due to charge.
- The component of this force in the direction $d\bar{l}$ is :

$$F_{El} = F_E \cos\theta = \bar{F}_E \cdot \bar{a}_l = Q\bar{E} \cdot \bar{a}_l$$

- When we are moving a charge Q a distance 'dl', then the force which we must apply is equal and opposite to the force due to field.

$$F_{appl} = -F_{El} = -Q\bar{E} \cdot \bar{a}_l$$

- The work done is force multiplied by distance. The work done in moving a charge Q by differential length dl is differential work, given by,

$$dW = F_{appl}\, dl = (-Q\bar{E} \cdot \bar{a}_l)\, dl$$

but $\quad dl\,\bar{a}_l = d\bar{l}$

so that $\quad dW = -Q\bar{E} \cdot d\bar{l}$

- This is the **incremental works** done in moving a charge Q in the presence of the electric field $\bar{E}$ by a differential length dl.
- The total work required to move the charge a finite distance between two points from initial position to final position is obtained by integrating dW, i.e.

$$W = \int dW$$

so that
$$W = -Q \int_{\text{initial}}^{\text{final}} \bar{E} \cdot d\bar{l} \text{ (Joules)} \qquad ...(4.2.1)$$

4.2.2 Expression for $d\bar{l}$ in Three Co-ordinate Systems

- Work done is measured in Joules.
- The expressions for $d\bar{l}$ in three coordinate systems are :

$$\boxed{\begin{aligned} d\bar{l} &= dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z && \text{...(Cartesian)} \\ d\bar{l} &= dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z && \text{...(Cylindrical)} \\ d\bar{l} &= dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi && \text{...(Spherical)} \end{aligned}} \quad ...(4.2.2)$$

4.2.3 Work Done Along Different Paths

- Suppose we want two move a point charge Q from $x = x_2$ to $x = x_1$ in an electric field $\bar{E}$. There are different paths possible as shown in Fig. 4.2.2(a).
- Because the initial and final positions for these paths are same, work done must be same for any path choosen. That is, **work done is independent of the path.**

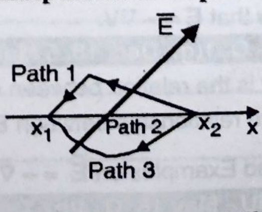


Fig. 4.2.2(a) : Work done along different paths

Module 1

4.2.4 Work Done in a Loop (Conservative Property of an Electric Field)

UQ. What do you understand by "conservative field" ? MU - May 10, 2 Marks

- Now suppose the charge is moved around any closed loop as shown in Fig. 4.2.2(b).
- Because initial position is x_3 which is equal to final position, the work done is zero.

$$W = -Q \int_{initial}^{final} \bar{E} \cdot d\bar{l} = 0$$

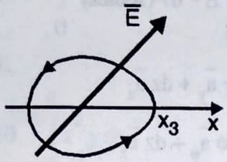


Fig. 4.2.2(b) : Conservative field

That is work done in moving the charge around any closed loop is zero. This can be written as,

$$\oint \bar{E} \cdot d\bar{l} = 0 \qquad ...(4.2.3)$$

- A small circle is placed on the integral sign indicates the closed nature of the path. The above equation is true for static fields. Such a field is called as **conservative** field or irrotational field. If an electric field varies with time, it need not be conservative.
- Equation (4.2.3) is called as integral form of **Maxwell's equation** derived from Faraday's law for static field. For time varying field this expression requires modification. Faraday's law and its modification is studied in chapter on Maxwell's equations.
- Applying Stoke's theorem to integral form in Equation (4.2.3).

$$\oint \bar{E} \cdot d\bar{l} = \int_s (\nabla \times \bar{E}) \cdot d\bar{s}$$

- For conservative field when LHS of above equation is zero, then the RHS is also zero, giving

$$\int_s (\nabla \times \bar{E})\, d\bar{s} = 0$$

or

$$\nabla \times \bar{E} = 0 \qquad ...(4.2.4)$$

This equations is referred to as point form of **Maxwell's equation** derived from Faraday's law for static field. For time varying fields this relation requires modification.

Note : There are two views that expression is Equation (4.2.3) can be seen :

(i) Since the work done in a loop is zero, the integral in Equation (4.2.3) is zero.

(ii) We know E has a unit of V/m and dl is expressed in meters. Combinely $\bar{E} \cdot d\bar{l}$ is having unit of (V/m × m) volts. The integration is nothing but summation. Thus the integral of $\bar{E} \cdot d\bar{l}$ gives summation of voltages. The circle on the integral sign indicates summation in a loop, means $\oint \bar{E} \cdot d\bar{l} = 0$. indicates summation of voltages in a loop is zero. What is this statement for ? This is nothing but Kirchoff's Voltage Law (KVL).

4.2.5 Important Properties of Work Done

Using the relations derived in the pervious section we can state important properties of work done. These relations are :

$$W = -Q \int_{initial}^{final} \bar{E} \cdot d\bar{l} \qquad ...(i)$$

$$W = -Q \oint \bar{E} \cdot d\bar{l} = 0 \qquad ...(ii)$$

Properties

i) When the work is required to calculate while moving the point charge from point B to A then initial and final positions of charge are B and A respectively and Equation (i) becomes

$$W \text{ (from B to A)} = -Q \int_B^A \bar{E} \cdot d\bar{l}$$

ii) When the direction of path ($d\bar{l}$) is against the electric field then $\bar{E} \cdot dl = E\, dl \cos(180) = -E\, dl$. This negative sign makes the work done to be positive. Then the external source is said to be doing the work.

iii) But when the path selected ($d\bar{l}$) is in the direction of electric field then $\bar{E} \cdot d\bar{l} = E\, dl \cos(0) = +E\, dl$. This results in work done to be negative. Then it is said that the field itself is doing the work, no external source is required.

iv) The selected path, if perpendicular to field then $\bar{E} \cdot d\bar{l} = E\, dl \cos(90) = 0$. This makes work done to be zero.

v) From Equation (ii), when the charge is moved in a loop then the total work done is zero.

vi) The work done is independent of the path, it depends only on the initial and final positions of the path.

Important Formulae

$$W = -Q \int_{initial}^{final} \bar{E} \cdot d\bar{l} = 0$$

$$\oint \bar{E} \cdot d\bar{l} = 0 \qquad \nabla \times \bar{E} = 0$$

Ex. 4.2.1 : Prove that in moving a point charge Q, if the path selected is always perpendicular to the electric field the work done is zero.

☑ **Soln. :**

The work done is given by,

$$W = -Q \int_{initial}^{final} \bar{E} \cdot d\bar{l}$$

Here $d\bar{l}$ is a differential vector in the direction of path selected. Thus if $\bar{E}$ and $d\bar{l}$ are perpendicular to each other according to property of dot product,

$$\bar{E} \cdot d\bar{l} = 0 \qquad \therefore \quad W = 0$$ ...Ans.

Ex. 4.2.2 : Let $\bar{E} = 400\ \bar{a}_x - 300\ \bar{a}_y + 500\ \bar{a}_z$ (V/m) in the neighbourhood of point P (6, 2, – 3). Find the incremental work done in moving a 4C charge at distance of 1 mm in the direction specified by,

(i) $\bar{a}_x + \bar{a}_y + \bar{a}_z$; (ii) $-2\,\bar{a}_x + 3\,\bar{a}_y - \bar{a}_z$

☑ **Soln. :**

The incremental work done is :

$$dW = -Q\,\bar{E} \cdot d\bar{l} \qquad \text{where} \qquad d\bar{l} = dl\ \bar{a}_l$$

(i) To find work done in the direction of $\bar{a}_x + \bar{a}_y + \bar{a}_z$

Given : $\bar{a}_l = (\bar{a}_x + \bar{a}_y + \bar{a}_z)/\sqrt{3}$

$$\therefore \quad d\bar{l} = 1 \times 10^{-3}\,(\bar{a}_x + \bar{a}_y + \bar{a}_z)/\sqrt{3}$$

The work done is :

$$dW = -4 \times (400\,\bar{a}_x - 300\,\bar{a}_y + 500\,\bar{a}_z) \cdot 1 \times 10^{-3}\,(\bar{a}_x + \bar{a}_y + \bar{a}_z)/\sqrt{3}$$

$$= 4 \times 10^{-3}\,(400 - 300 + 500)/\sqrt{3}$$

$$= -2.4/\sqrt{3} = -1.39\ \text{(J)}$$ ...Ans.

(ii) To find work done in the direction of $-2\,\bar{a}_x + 3\,\bar{a}_y - \bar{a}_z$

Given : $d\bar{l} = (-2\,\bar{a}_x + 3\,\bar{a}_y - \bar{a}_z)/\sqrt{14}$

$$\therefore \quad d\bar{l} = 1 \times 10^{-3}\,(-2\,\bar{a}_x + 3\,\bar{a}_y - \bar{a}_z)/\sqrt{14}$$

The work done is :

$$dW = -4 \times (400\,\bar{a}_x - 300\,\bar{a}_y + 500\,\bar{a}_z) \cdot \frac{1 \times 10^{-3}}{\sqrt{14}}\,(-2\,\bar{a}_x + 3\,\bar{a}_y - \bar{a}_z)$$

$$dW = -4 \times 10^{-3}\,(-800 - 900 - 500)/\sqrt{14}$$

$$= 8.8/\sqrt{14} = 2.35\ \text{(J)}$$ ...Ans.

4.2.6 Examples on Work done

Important Formulae

$$W = -Q \int_{initial}^{final} \bar{E} \cdot d\bar{l}\ \text{(Joules)}$$

$$d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z \quad \text{...(Cartesian)}$$

$$d\bar{l} = dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z \quad \text{...(Cylindrical)}$$

$$d\bar{l} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi \quad \text{...(Spherical)}$$

UEx. 4.2.3 **MU - May 11, 10 Marks**

Evaluate work done in bringing a charge of 5 μC from origin to P (2, – 1, 4) through field $\bar{E} = 2xyz\ \bar{a}_x + x^2 z\ \bar{a}_y + x^2 y\ \bar{a}_z$ (V / m) through the path

(i) Straight line segments (0, 0, 0) to (2, 0, 0) to (2, – 1, 0) to (2, – 1, 4)

(ii) Straight line x = – 2y, z = 2x.

☑ **Soln. :**

(i) Using straight line segments

Straight line segments given in the problem are shown in Fig. Ex. 4.2.3.

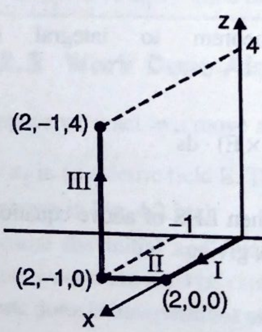


Fig. : Illustrating Ex. 4.2.3

For segment I : $d\bar{l} = dx\,\bar{a}_x$; $y = z = 0$

For segment II : $d\bar{l} = dy\,\bar{a}_y$; $x = 2, z = 0$

For segment III : $d\bar{l} = dz\,\bar{a}_z$; $x = 2, y = -1$

To find work done along different segments :

For segment I,

$$W_1 = -Q \int_{\text{initial}}^{\text{final}} \bar{E} \cdot d\bar{l}$$

$$W_1 = -5\times 10^{-6} \int_0^2 (2xyz\,\bar{a}_x + x^2 z\,\bar{a}_y + x^2 y\,\bar{a}_z) \cdot dx\,\bar{a}_x$$

$$W_1 = -5\times 10^{-6} \int_0^2 2xyz\,dx = 0 \qquad \text{...(since, } y = z = 0)$$

For segment II,

$$W_2 = -5\times 10^{-6} \int_0^{-1} x^2 z\,dy = 0 \qquad \text{...(since, } z = 0)$$

For segment III,

$$W_3 = -5\times 10^{-6} \int_0^4 x^2 y\,dz$$

$$= -5\times 10^{-6} \times (2)^2 \times (-1)\,[z]_0^4 \quad \text{...(since, } x = 2, y = -1)$$

$$W_3 = 80\ \mu\ \text{Joules}$$

Total work done is,

$$W = W_1 + W_2 + W_3 = 0 + 0 + 80\times 10^{-6}$$

$$= 80\ \mu\ \text{Joules.} \qquad \text{...Ans.}$$

(ii) Work done along straight line x = – 2y and z = 2x

Because it is a direct path to go from (0, 0, 0) to (2, – 1, 4) all coordinates are changing.

$$\therefore \quad d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

and work done is,

$$W = -5\times 10^{-6} \int_{(0,0,0)}^{(2,-1,4)} 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$$

To change expression in the integral sign in terms of any one coordinate, let it be x, use given relations.

$$x = -2y \quad \text{or} \quad y = -x/2 \quad \therefore\ dy = -dx/2$$

$$z = 2x \quad \text{or} \quad dz = 2\,dx$$

Writing the expression in the integral sign in terms of x and putting limits for x,

$$W = -5\times 10^{-6} \int_0^2 2x\left(-\frac{x}{2}\right) \cdot 2x\,dx$$

$$+ x^2 \cdot 2x\left(-\frac{dx}{2}\right) + x^2\left(-\frac{x}{2}\right) 2dx$$

$$= -5\times 10^{-6} \int_0^2 -2x^3\,dx - x^3\,dx - x^3\,dx$$

$$W = -5\times 10^{-6} \int_0^2 -4x^3\,dx$$

$$W = -5\times 10^{-6} \times -4\left[\frac{x^4}{4}\right]_0^2$$

$$W = 80\ \mu\text{Joules} \qquad \text{...Ans.}$$

UEx. 4.2.4 MU - Dec. 10, 10 Marks

Find the amount of energy required to move a 4 C charge from B (1, 0, 0) to A (0, 2, 0) in the field.

$E = 5x\,\bar{a}_x + 5y\,\bar{a}_y$ V/m along the straight line path $y = 2 - 2x, z = 0$.

☑ **Soln. :**

Given : $\bar{E} = 5x\,\bar{a}_x + 5y\,\bar{a}_y$,

$$Q = 4\,C,\ y = 2 - 2x \quad \rightarrow dy = -2\,dx$$

$$W = -Q \int_{\text{initial}}^{\text{final}} \bar{E} \cdot d\bar{l}$$

$$= -4 \int (5x\,\bar{a}_x + 5y\,\bar{a}_y) \cdot (dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)$$

$$= -4 \int 5x\,dx + 5y\,dy = -4 \int 5x\,dx + 5(2-2x)(-2\,dx)$$

$$= -4 \int_1^0 -20\,dx + 25\,x\,dx = -4[-20\,(x)_1^0 + 25\left(\frac{x^2}{2}\right)_1^0]$$

$$W = -4\left[(20) - \frac{25}{2}\right] = -30\ (J)$$

Ex. 4.2.5 : Find the work done in moving a point charge Q = 4 μC from origin to (2 m, π / 4, π / 2) in spherical co-ordinates in the field. $\bar{E} = 5 \cdot e^{-r/4}\,\bar{a}_r + \dfrac{10}{r\sin\theta}\,\bar{a}_\phi$ (V / m).

☑ **Soln. :**

To move a charge from origin to (2, π/4, π/2) there are different paths available. One path is shown in Fig. Ex. 4.2.11. This path is divided into three segments. First segment is along x-axis, second segment is on the spherical surface in xy plane and third segment is on the same spherical surface but in the yz plane.

For segment 1

$$\theta = \pi/2, \phi = 0;\ \therefore\ d\theta = 0,\ d\phi = 0$$

Then, $$d\bar{l} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi = dr\,\bar{a}_r$$

Also $$\bar{E} = 5e^{-r/4}\,\bar{a}_r + \frac{10}{r\sin(\pi/2)}\,\bar{a}_\phi$$

$$= 5e^{-r/4}\,\bar{a}_r + \frac{10}{r}\,\bar{a}_\phi$$

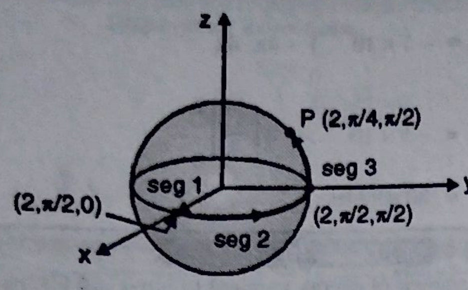


Fig. : Illustrating Ex. 4.2.5

Work done along segment 1 is obtained as,

$$W_1 = -Q \int_{\text{initial}}^{\text{final}} \bar{E} \cdot d\bar{l}$$

$$= -5\times 10^{-6} \int_0^2 \left(5 e^{-r/4} \bar{a}_r + \frac{10}{r} \bar{a}_\phi\right) \cdot dr\, \bar{a}_r$$

$$= -25\times 10^{-6} \int_0^2 e^{-r/4}\, dr \qquad \text{...(since, } \bar{a}_\phi \cdot \bar{a}_r = 0)$$

$$W_1 = -25\times 10^{-6} \left[\frac{e^{-r/4}}{-1/4}\right]_0^2 = -39.35\ \mu J \qquad \text{...Ans.}$$

For segment 2

$$r = 2,\quad \theta = \pi/4;\ \therefore\ dr = 0,\ d\theta = 0$$

Then $$d\bar{l} = r \sin\theta\, d\phi\, \bar{a}_\phi$$

$$= 2\sin(\pi/2)\, d\phi\, \bar{a}_\phi = 2 d\phi\, \bar{a}_\phi$$

and $$\bar{E} = 5 e^{-2/4} \bar{a}_r + \frac{10}{2\times \sin(\pi/2)} \bar{a}_\phi$$

$$= 3\bar{a}_r + 5\bar{a}_\phi$$

Work done along segment 2 is calculated as,

$$W_2 = -5\times 10^{-6} \int_0^{\pi/2} (3\bar{a}_r + 5\bar{a}_\phi)\cdot 2\, d\phi\, \bar{a}_\phi$$

$$W_2 = -5\times 10^{-6} \int_0^{\pi/2} 10\, d\phi = -78.54\ \mu J \qquad \text{...Ans.}$$

For segment 3

$$r = 2,\quad \phi = \pi/2;\quad \therefore\ dr = 0,\ d\phi = 0$$

Then $$d\bar{l} = r\, d\theta\, \bar{a}_\theta$$

and $$\bar{E} = 5 e^{-2/4} \bar{a}_r + \frac{10}{2\sin\theta} \bar{a}_\phi$$

Work done along this segment is,

$$W_3 = -5\times 10^{-6} \int_{\pi/2}^{\pi/4} \left(3\bar{a}_r + \frac{5}{\sin\theta} \bar{a}_\phi\right)\cdot r\, d\theta\, \bar{a}_\theta$$

$$= 0 \qquad \text{...(since, } \bar{a}_r \cdot \bar{a}_\theta = \bar{a}_\phi \cdot \bar{a}_\theta = 0)$$

The total work done is,

$$W = W_1 + W_2 + W_3 = -39.35 - 78.54 + 0$$

$$= -117.9\ \mu J \qquad \text{...Ans.}$$

4.3 ELECTRIC SCALAR POTENTIAL

We know that the work done by an external source in moving a charge Q from one point to another in an electric field $\bar{E}$ is,

$$W = -Q \int_{\text{initial}}^{\text{final}} \bar{E} \cdot d\bar{l}$$

Definition of potential difference

The potential of point A with respect to B is defined as the work done per unit charge.

$$\therefore \qquad V_{AB} = \frac{W}{Q} = -\int_B^A \bar{E} \cdot d\bar{l} \qquad \text{...(4.3.1)}$$

- This is also a potential difference between points A and B with point B as reference point.
- From Equation (4.3.1), the unit of potential function is joules per coulomb (J/C) or commonly referred to as volts (V).
- It is now evident from Equation (4.3.1) why we express the electric field intensity in terms of volts per meter (V/m). The electric field in (V/m) and dl in (m), the multiplication gives the unit volts.
- The potential difference is also expressed as,

$$V_{AB} = V_A - V_B \qquad \text{...(4.3.2)}$$

4.3.1 Potential due to Point Charge

Consider a point charge Q at the origin. The electric field intensity due to Q is

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2} \bar{a}_r$$

This field is radial and it is surrounding this point charge Q. Now take two points at radial distances r_1 and r_2, $r_2 > r_1$. We can find potential difference V_{12} as follows :

$$V_{12} = V_1 - V_2 = -\int_{r_2}^{r_1} \bar{E} \cdot d\bar{l}$$

In spherical coordinates, $d\bar{l} = dr\, \bar{a}_r + r\, d\theta\, \bar{a}_\theta + r\sin\theta\, d\phi\, \bar{a}_\phi$

Then $$V_{12} = -\int_{r_2}^{r_1} \frac{Q}{4\pi\epsilon_0 r^2} \bar{a}_r \cdot (dr\, \bar{a}_r + r\, d\theta\, \bar{a}_\theta + r\sin\theta\, d\phi\, \bar{a}_\phi)$$

$$= -\frac{Q}{4\pi\epsilon_0} \int_{r_2}^{r_1} \frac{dr}{r^2} = -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r}\right]_{r_2}^{r_1}$$

i.e. $V_{12} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$ ...(4.3.3)

As $(r_2 > r_1)$, V_{12} is positive that is $V_1 > V_2$.

From this analysis, it is clear that as test charge is moved against electric field there is a rise in potential.

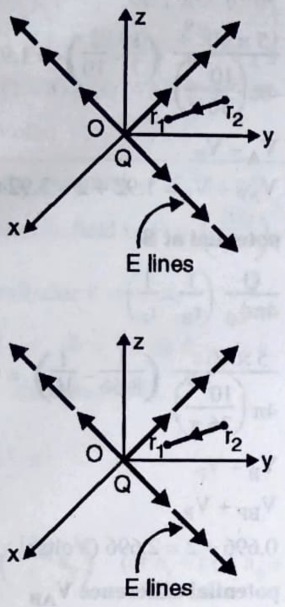


Fig. 4.3.1 : Finding potential difference V_{12}

4.3.2 Potential Difference due to Point Charge

Consider a point charge Q at the origin. The electric field intensity due to Q is,

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\bar{a}_r$$

This field is radial and it is surrounding this point charge Q. Now take two points at radial distances r_1 and r_2, $r_2 > r_1$. We can find potential difference V_{12} as follows :

$$V_{12} = V_1 - V_2 = -\int_{r_2}^{r_1} \bar{E}\cdot d\bar{l}$$

In spherical coordinates, $d\bar{l} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi$

$$\text{Then } V_{12} = -\int_{r_2}^{r_1} \frac{Q}{4\pi\epsilon_0 r^2}\bar{a}_r \cdot (dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi)$$

$$V_{12} = -\frac{Q}{4\pi\epsilon_0}\int_{r_2}^{r_1}\frac{dr}{r^2} = -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{r_2}^{r_1}$$

i.e. $V_{12} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$ ...(4.3.4)

As $(r_2 > r_1)$, V_{12} is positive that is $V_1 > V_2$.

From this analysis, it is clear that as test charge is moved against electric field there is a rise in potential.

4.3.3 Scalar (Absolute) Potential

The expression for V_{12} in Equation (4.3.4) gives the potential difference between two points. The absolute potential at a point is defined as :

> Scalar (Absolute) potential is the potential at a point taking reference point at infinity i.e. $r_2 \to \infty$.

The right hand side of above equation can be split into two terms.

$$V_1 = \frac{Q}{4\pi\epsilon_0 r_1} = \text{Potential at } r_1 \;;$$

$$V_2 = \frac{Q}{4\pi\epsilon_0 r_2} = \text{Potential at } r_2$$

If $r_2 = \infty$, $V_2 = 0$ and $V_{12} = V_1 = \frac{Q}{4\pi\epsilon_0 r_1}$ Volts.

This potential is called as **'Scalar or absolute potential'** at a point distance r_1 from charge Q. It is inversely proportional to the distance from Q to the point r_1. In general potential V at a point distance r from charge Q is,

$V = \frac{Q}{4\pi\epsilon_0 r}$ (Volts) ...(4.3.5(A))

4.3.4 Scalar Potential due to Charge not at Origin

If the charge Q is not at origin then potential expression is obtained by replacing $r \to R$. Thus,

$V = \frac{Q}{4\pi\epsilon_0 R}$ (Volts) ...(4.3.5(B))

Because it is a scalar quantity having only magnitude and no direction, it is often called as **scalar potential.**

4.3.5 Difference between Potential and Potential Difference

(i) Potential difference is the difference in potential between the two points at finite distances from the charge.

But potential is obtained at some point considering the other point (reference) at infinity.

(ii) Due to point charge :

$$\text{Potential difference} = V_{12} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\text{Potential} = V = \frac{Q}{4\pi\epsilon_0 R}$$

Module 1

4.3.6 Examples on Scalar Potential Formula

Important Formulae

$$V_{AB} = \frac{W}{Q} = -\int_B^A \bar{E} \cdot d\bar{l}$$

$$V_{AB} = V_A - V_B$$

Ex. 4.3.1 : Two point charges – 4 μC and 5 μC are located at (2, – 1, 3) and (0, 4, – 2) respectively. Find the potential at (1, 0, 1) assuming zero potential at infinity.

☑ **Soln. :**

Given : $Q_1 = -4\mu C$ and $Q_2 = 5\,\mu C$

In the Fig. Ex. 4.3.1 distances are,

$$R_1 = \sqrt{(2-1)^2 + (-1-0)^2 + (3-1)^2} = \sqrt{6}$$

and

$$R_2 = \sqrt{(0-1)^2 + (4-0)^2 + (-2-1)^2} = \sqrt{26}$$

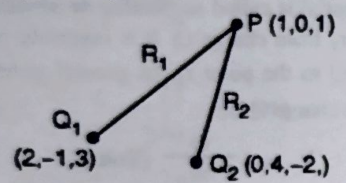


Fig. : Illustrating Ex. 4.3.1

Potential at point P due to Q_1 is,

$$V_1 = \frac{Q_1}{4\pi\epsilon_0 R_1} = \frac{-4 \times 10^{-6}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)\sqrt{6}} = -14.70 \text{ (kV)}$$

Potential at P due to Q_2 is,

$$V_2 = \frac{Q_2}{4\pi\epsilon_0 R_2} = \frac{5 \times 10^{-6}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)\sqrt{26}} = 8.83 \text{ (kV)}$$

Total potential at point P is obtained by Superposition, thus,

$$V = V_1 + V_2 = (-14.70 + 8.83)\,k = -5.87 \text{ (kV)} \quad \text{...Ans.}$$

Ex. 4.3.2 : A point charge of 5 nC is located at the origin, if V = 2 volts at P (0, 6, – 8), find

(i) The potential at A (– 3, 2, 6)

(ii) The potential at B (1, 5, 7)

(iii) The potential difference V_{AB}.

☑ **Soln. :Always the point whose voltage is known is considered as reference point, for example point P in the problem. Different distances involved in the problem from point charge Q at origin are,**

$$r_A = \sqrt{(-3)^2 + 2^2 + 6^2} = 7 \text{ m}$$

$$r_B = \sqrt{1^2 + 5^2 + 7^2} = 8.66 \text{ m}$$

$$r_P = \sqrt{0^2 + 6^2 + (-8)^2} = 7 \text{ m}$$

(i) To find the potential at A

The potential at A w.r.t. point P is,

$$V_{AP} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_P}\right)$$

$$= \frac{5 \times 10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left(\frac{1}{7} - \frac{1}{10}\right) = 1.92 \text{ V}$$

But $V_{AP} = V_A - V_P$

or $V_A = V_{AP} + V_P = 1.92 + 2 = 3.92$ (Volts) ...Ans.

(ii) To find the potential at B

$$V_{BP} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_P}\right)$$

$$= \frac{5 \times 10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left(\frac{1}{8.66} - \frac{1}{10}\right) = 0.696 \text{ V}$$

But $V_{BP} = V_B - V_P$

or $V_B = V_{BP} + V_P$

$= 0.696 + 2 = 2.696$ (Volts) ...Ans.

(iii) To find the potential difference V_{AB}

We have $V_{AB} = V_A - V_B$

$= 3.92 - 2.696 = 1.22$ Volts

The potential difference can also be obtained from,

$$V_{AB} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

$$= \frac{5 \times 10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left(\frac{1}{7} - \frac{1}{8.66}\right)$$

$= 1.22$ (Volts) ...Ans.

Ex. 4.3.3 : Given the field $E = 40\,xy\,\bar{a}_x + 20\,x^2\,\bar{a}_y + 5\,\bar{a}_z$ (v/m). Calculate V_{PQ} given P (1, – 1, 0) and Q (2, 2, 2).

☑ **Soln. :** In cartesian system

$$d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$\bar{E} \cdot d\bar{l} = 40\,xy\,dx + 20\,x^2\,dy + 5\,dz$$

$$V_{PQ} = -\int_Q^P \bar{E} \cdot d\bar{l}$$

$$= -\left[\int_2^1 40\,xy\,dx + \int_1^{-1} 20\,x^2\,dy + \int_2^0 5\,dz\right]$$

Consider the path of integration as,

Q (2, 2, 2) to (1, 2, 2) to (1, – 1, 2) to P (1, – 1, 0)

Between first and second point : y = 2, z = 2 and x → 2 to 1

Between second and third point : $x = 1, z = 2$ and $y \to 2$ to -1

Between third and fourth point : $x = 1, y = -1$ and $z \to 2$ to 0.

$$\therefore \quad V_{PQ} = -\left[\int_2^1 40\,(2)\,x\,dx + \int_2^{-1} 20\,(1^2)\,dy + \int_2^0 5\,dz\right]$$

$$= -\left[80\left(\frac{x^2}{2}\right)_2^1 + 20\,(y)_2^{-1} + 5\,(z)_2^0\right]$$

$$V_{PQ} = -[40\,(1-4) + 20\,(-1-2) + 5\,(0-2)]$$

$$= 190 \text{ (Volts)} \qquad \text{...Ans.}$$

Ex. 4.3.4 : Find the potential of P $(1, \pi/2, 3)$ with respect to Q $(3, \pi/4, 2)$ if the electric field is, $\bar{E} = \left(\frac{100}{r}\right)\bar{a}_r$ (V/m)

☑ **Soln. :** In cylindrical system,

$$d\bar{l} = dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z$$

The potential of P with respect to Q,

$$V_{PQ} = -\int_Q^P \bar{E}\cdot d\bar{l}$$

$$= -\int_Q^P \left(\frac{100}{r}\bar{a}_r\right)\cdot(dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z)$$

$$V_{PQ} = -\int_3^1 \frac{100}{r}\,dr = -100\,[\ln r]_3^1 = 109.86 \text{ Volts.} \qquad \text{...Ans.}$$

Ex. 4.3.5 : A uniform line charge of 0.6 nC / m lies along the z-axis in free space. Find the potential at P (3, 4, 2) if (a) V = 0 at A (2, – 9, 3) (b) V = 24 Volts at B (10, 24, 1).

☑ **Soln. :**

In the problem, electric field required to find potential is not given. But line charge is given. We can find the electric field due to line charge.

$$\bar{E} = \frac{\rho_l}{2\pi\epsilon_0 r}\bar{a}_r = \frac{0.6\times10^{-9}}{2\pi\times\left(\frac{10^{-9}}{36\pi}\right)\times r}\bar{a}_r$$

$$= \frac{10.8}{r}\bar{a}_r \text{ (V / m)}$$

Here, r is the length of perpendicular drawn from given point on the line charge.

For point P (3, 4, 2) : $r_P = \sqrt{3^2+4^2} = 5$

For point P (2 – 9, 3) : $r_A = \sqrt{2^2+9^2} = \sqrt{85}$

For point B (10, 24, 1) : $r_B = \sqrt{10^2+24^2} = 26$

(a) It is given that V = 0 at A (2, – 9, 3) means this a reference point.

$$\therefore \quad V_{PA} = -\int_A^P \left(\frac{10.8}{r}\bar{a}_r\right)\cdot(dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z)$$

$$\therefore \quad V_{PA} = -\int_{\sqrt{85}}^{5} \frac{10.8}{r}\,dr$$

$$= -10.8\,[\ln r]_{\sqrt{85}}^{5} = 6.60 \text{ Volts.} \qquad \text{...Ans.}$$

(b) The potential difference between P and B is,

$$V_{PB} = -\int_B^P \left(\frac{10.8}{r}\bar{a}_r\right)\cdot(dr\,\bar{a}_r + r\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z)$$

$$= -\int_{26}^{5} \frac{10.8}{r}\,dr$$

$$= -10.8\,[\ln r]_{26}^{5} = 17.80 \text{ Volts.}$$

i.e. $\quad V_P - V_B = 17.80$ Volts.

But point B is at 24 volts. That is reference point is not at zero volts.

Then, $\quad V_P = 17.80 + V_B$

i.e. $\quad V_P = 17.80 + 24 = 41.8$ Volts. ...Ans.

UEx. 4.3.6 MU - May 10, May 16, 8 Marks

If the zero potential reference is at r = 10 m, and a point charge Q = 0.5 nC is at the origin, find the potentials at r = 5 m, and r = 15 m. At what radius is the potential the same in magnitude as that at r = 5 m but opposite in sign ?

☑ **Soln. :**

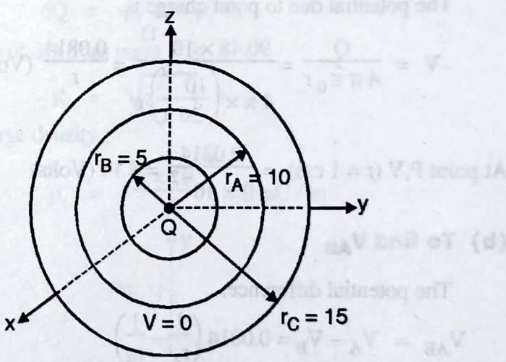


Fig. : Illustrating Ex. 4.3.6

Since the zero potential is at r = 10, we use this as reference. The potential at $r_B = 5$ with respect to reference is

$$V_{BA} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

$$V_{BA} = \frac{0.5\times10^{-9}}{4\pi\times(10^{-9}/36\pi)}\left(\frac{1}{5} - \frac{1}{10}\right)$$

$$= 0.45 \text{ (Volts)}$$

The potential at $r_C = 15$ with respect to reference is,

$$V_{CA} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_C} - \frac{1}{r_A}\right) = \frac{0.5\times10^{-9}}{4\pi\times(10^{-9}/36\pi)}\left(\frac{1}{15} - \frac{1}{10}\right)$$

$$V_{CA} = -0.15 \text{ (Volts)}$$

To find the radius where potential is – 0.45 Volts.

$$-0.45 = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{1}{r_A}\right) = \frac{0.5\times 10^{-9}}{4\pi\times(10^{-9}/36\pi)}\left(\frac{1}{r}-\frac{1}{10}\right)$$

$$= 4.5\left(\frac{1}{r}-\frac{1}{10}\right)$$

$$\frac{1}{r} = \frac{-0.45}{4.5}+\frac{1}{10} = 0 \text{ or } r = \infty$$

Ex. 4.3.7 : A uniform surface charge density of 20 nC / m^2 is present on the spherical surface r = 0.6 cm in free space.

(a) Find the absolute potential at P (r = 1 cm, θ = 25°, ϕ = 50°)

(b) Find V_{AB} given points A (2 cm, 30°, 60°) and B (3 cm, 45°, 90°)

☑ **Soln. :**

The problem seems like surface charge problem. The uniform distribution of charge will make the potential independent of θ and ϕ. The potential is only a function of r. Also the given points are outside the spherical surface, then the spherical charge can be treated as a point charge :

$$Q = 4\pi a^2 \rho_s = 4\pi (0.6\times 10^{-2})^2 \times 20\times 10^{-9}$$

$$= 90.48\times 10^{-13} \text{ (C)}$$

(a) To find potential at P

The potential due to point charge is,

$$V = \frac{Q}{4\pi\epsilon_0 r} = \frac{90.48\times 10^{-13}}{4\pi\times\left(\frac{10^{-9}}{36\pi}\right)r} = \frac{0.0814}{r} \text{ (Volts)}$$

$$\text{At point P, V (r = 1 cm)} = \frac{0.0814}{1\times 10^{-2}} = 8.14 \text{ (Volts)} \qquad \text{...Ans.}$$

(b) To find V_{AB}

The potential difference,

$$V_{AB} = V_A - V_B = 0.0814\left(\frac{1}{r_A}-\frac{1}{r_B}\right)$$

$$= 0.0814\left(\frac{1}{0.02}-\frac{1}{0.03}\right)$$

$$\therefore \quad V_{AB} = 1.357 \text{ (Volts)} \qquad \text{...Ans.}$$

4.4 ELECTRIC POTENTIAL DUE TO CHARGES DISTRIBUTIONS

Equation (4.3.5) is useful in finding potential at a distance R from point charge Q. But in case of line charge, surface charge or volume charge distributions it cannot be applied directly. First of all it is necessary to find potential or voltage dV at a given point due to differential charge dQ on this charge distribution. Then integrating dV we get voltage V.

$$dV = \frac{dQ}{4\pi\epsilon_0 R}$$

$$V = \int dV$$

Whether the integral is single, double or triple is decided by the nature of charge. For line, surface or volume charge it is single, double or triple integral respectively.

(i)	V due to line charge distribution
(ii)	V due to surface charge distribution
(iii)	V due to volume charge distribution
(iv)	V due to more than one point charges

4.4.1 V due to Line Charge Distribution

The differential charge on the line is $dQ = \rho_l \, dl$

The voltage at point P due to charge dQ is,

$$dV = \frac{dQ}{4\pi\epsilon_0 R} = \frac{\rho_l \, dl}{4\pi\epsilon_0 R}$$

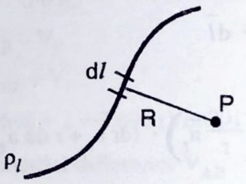


Fig. 4.4.1(i) : To find V due to line charge

The total voltage due to line charge is, $V_l = \int_l dV$

i.e. $$V_l = \frac{1}{4\pi\epsilon_0}\int_l \frac{\rho_l \, dl}{R} \qquad \text{...(4.4.1(a))}$$

4.4.2 V due to Surface Charge Distribution

The differential charge is, $dQ = \rho_s \, ds$

The voltage due to dQ is,

$$dV = \frac{dQ}{4\pi\epsilon_0 R} = \frac{\rho_s \, ds}{4\pi\epsilon_0 R}$$

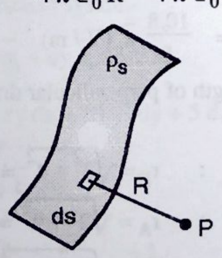


Fig. 4.4.1(ii) : To find V due to surface charge

The total voltage due to surface charge, $V_s = \int_s dV$

i.e. $$V_s = \frac{1}{4\pi\epsilon_0}\int_s \frac{\rho_s\, ds}{R} \qquad ...(4.4.1(b))$$

4.4.3 V due to Volume Charge Distribution

If the volume charge density is ρ_v then the differential charge is,

$$dQ = \rho_v\, dv$$

and $$dV = \frac{dQ}{4\pi\epsilon_0 R} = \frac{\rho_v\, dv}{4\pi\epsilon_0 R}$$

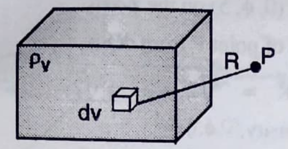


Fig. 4.4.1(iii) : To find V due to volume charge

The total voltage due to volume charge, $V_v = \int_v dv$

i.e. $$V_v = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho_v\, dv}{R} \qquad ...(4.4.1(c))$$

4.4.4 V due to more than One Point Charges

In this case find voltages due to each charge separately and the total voltage is sum of all. Let Q_1, Q_2, Q_3 be the point charges at R_1, R_2, R_3 respectively. Then the potential due to them is,

$$V_P = \frac{Q_1}{4\pi\epsilon_0 R_1} + \frac{Q_2}{4\pi\epsilon_0 R_2} + \frac{Q_3}{4\pi\epsilon_0 R_3}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{R_1} + \frac{Q_2}{R_2} + \frac{Q_3}{R_3}\right)$$

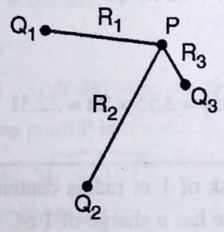


Fig. 4.4.1(iv) : To find V due to number of point charges

This can be extended to any number of charges. If N charges are present.

$$V_P = \frac{1}{4\pi\epsilon_0}\sum_{n=1}^{N}\frac{Q_n}{R_n} \qquad ...(4.4.1(d))$$

In a situation where all charge distributions are present the total voltage at point P is found by principle of superposition.

i.e. $$V = V_P + V_l + V_s + V_v \text{ (Volt)} \qquad ...(4.4.2)$$

4.4.5 Examples on Potential due to Charge Configurations

Important Formulae

$$V_l = \frac{1}{4\pi\epsilon_0}\int_l \frac{\rho_l\, dl}{R}$$

$$V_s = \frac{1}{4\pi\epsilon_0}\int_s \frac{\rho_s\, ds}{R}$$

$$V_v = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho_v\, dv}{R}$$

$$V_P = \frac{1}{4\pi\epsilon_0}\sum_{n=1}^{N}\frac{Q_n}{R_n}$$

$$V = V_P + V_l + V_s + V_v \text{ (Volt)}$$

Ex. 4.4.1 : A uniform line charge of 1 nC is situated along x-axis between the points (– 500, 0) mm and (500, 0) mm. Find the electric scalar potential V at (0, 1000) mm.

☑ **Soln. :**

The differential charge on the line charge along x-axis is,

$$dQ = \rho_l\, dl = \rho_l\, dx$$

The distance of dQ from point P is,

$$R = \sqrt{x^2 + 1}$$

The line charge density ,

$$\rho_l = \frac{1 \times 10^{-9}}{1} = 1 \text{ nC/m}$$

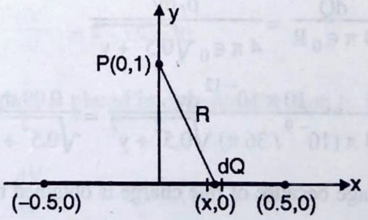


Fig. : Illustrating Ex. 4.4.1

The voltage because of dQ is,

$$dV = \frac{dQ}{4\pi\epsilon_0 R} = \frac{1 \times 10^{-9}\, dx}{4\pi(10^{-9}/36\pi)\sqrt{1+x^2}} = \frac{9\, dx}{\sqrt{1+x^2}}$$

The total voltage is obtained by integrating above result.

$$V = \int dV = 9\int_{-0.5}^{0.5}\frac{dx}{\sqrt{1+x^2}}$$

$$V = 9\left[\ln\left(x+\sqrt{1+x^2}\right)\right]_{-0.5}^{0.5} = 8.66 \text{ (Volts)} \quad \text{...Ans.}$$

Ex. 4.4.2 : A square of side 1 m has a point charge $Q_1 = 1$ pC at the upper right corner, a point charge $Q_2 = -10$ pC at the lower right corner and a line distribution of charge $\rho_l = 10$ pC / m along the left edge. Find V at point P at the center of the square.

☑ **Soln. :**

The distances of Q_1 and Q_2 from centre of square is,

The potential due to point charges Q_1 and Q_2 is,

$$V_P = \frac{Q_1}{4\pi\epsilon_0 R_1} + \frac{Q_2}{4\pi\epsilon_0 R_2}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{10^{-12}}{0.707} - \frac{10\times10^{-12}}{0.707}\right) = -0.115 \text{ V}$$

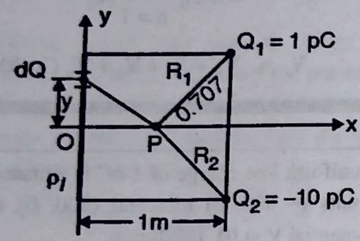


Fig. : Illustrating Ex. 4.4.2

To find potential due to line charge,

$$dQ = \rho_l \, dl = \rho_l \, dy$$

The distance of dQ from point P is,

$$R = \sqrt{0.5^2 + y^2}$$

The potential due to dQ is,

$$dV = \frac{dQ}{4\pi\epsilon_0 R} = \frac{\rho_l \, dy}{4\pi\epsilon_0\sqrt{0.5^2+y^2}}$$

$$= \frac{10\times10^{-12} \, dy}{4\pi(10^{-9}/36\pi)\sqrt{0.5^2+y^2}} = \frac{0.09 \, dy}{\sqrt{0.5^2+y^2}}$$

The voltage because of line charge is obtained by integrating dV as,

$$V_L = \int_{-0.5}^{0.5} \frac{0.09 \, dy}{\sqrt{0.5^2+y^2}}$$

$$= 0.09\left[\ln\left(y+\sqrt{0.5^2+y^2}\right)\right]_{-0.5}^{0.5}$$

$$= 0.09\,(0.188 + 1.57) = 0.158 \text{ V}$$

The total voltage

$$V = V_P + V_L = 43 \text{ mV}. \quad \text{...Ans.}$$

UEx. 4.4.3 **MU - May 09, Dec. 09, Dec. 12, 10 Marks**

A total charge of (40 / 3) nC is uniformly distributed around circular ring of radius 2 m, find the potential at a point on the axis 5 m from the plane of the ring.

☑ **Soln. :**

Consider the ring of charge is in xy plane with center at origin.

The differential charge on it is,

$$dQ = \rho_l \, dl = \rho_l \, \rho \, d\phi = 2\,\rho_l \, d\phi$$

Let this charge is at (2, ϕ, 0), and the point where we desire the electric field is (0, ϕ, 5) on the z-axis.

The distance of point P from dQ is,

$$R = \sqrt{2^2+5^2} = \sqrt{29}$$

The line charge density,

$$\rho_l = \frac{(40/3)\times10^{-9}}{2\pi\times2} = \frac{10}{3\pi} \text{ nC/m}$$

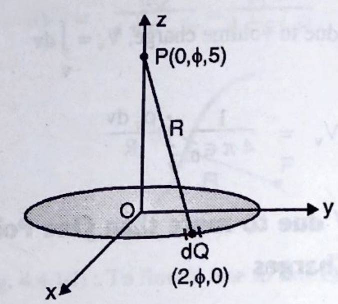


Fig. : Illustrating Ex. 4.4.3

The voltage at point P due to dQ is,

$$dV = \frac{dQ}{4\pi\epsilon_0 R} = \frac{2\,\rho_l \, d\phi}{4\pi\epsilon_0\sqrt{2^2+5^2}}$$

$$= \frac{2\times(10/3\pi)\times10^{-9} \, d\phi}{4\pi\,(10^{-9}/36\pi)\sqrt{29}} = 3.55 \, d\phi$$

The total voltage.

$$V = \int_0^{2\pi} 3.55 \, d\phi = 3.55\times2\pi = 22.31 \text{ Volts.} \quad \text{...Ans.}$$

Ex. 4.4.4 : A flat disk of 1 m radius centered at the origin and situated in the xy plane has a charge of 1 nC uniformly distributed over its surface. Find the electric scalar potential at a distance of 1 m on the disk axis.

☑ **Soln. :**

The differential charge $dQ = \rho_s \, ds = \rho_s \, r \, dr \, d\phi$

The distance of dQ form point P is,

$$R = \sqrt{r^2+1}$$

The surface charge density on the disc is,

$$\rho_s = \frac{1\times10^{-9}}{\pi\,(1)^2} = 0.318 \text{ nC/m}^2$$

$$dV = \frac{dQ}{4\pi\epsilon_0 R} = \frac{0.318 \times 10^{-9} \times r\,dr\,d\phi}{4\pi(10^{-9}/36\pi)\sqrt{r^2+1}}$$

$$dV = \frac{2.86\,r\,dr\,d\phi}{\sqrt{r^2+1}}$$

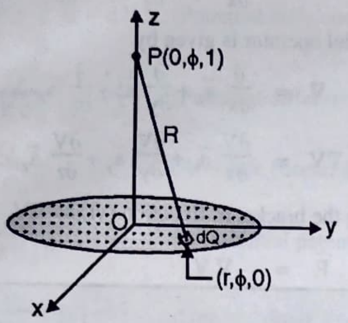


Fig. : Illustrating Ex. 4.4.4

The total voltage is obtained by integrating dV as,

$$V = \int_0^{2\pi}\int_0^{1} \frac{2.86\,r\,dr\,d\phi}{\sqrt{r^2+1}} = 2.86 \times 2\pi \int_0^{1} \frac{r\,dr}{\sqrt{r^2+1}}$$

To solve integral :

Put, $1 + r^2 = t$ | Limits : $r \to 0$; $t \to 1$

$2r\,dr = dt$ | $r \to 1$; $t \to 2$

Then $V = 18 \int_1^2 \frac{dt/2}{t^{1/2}} = 9\left[\frac{t^{1/2}}{1/2}\right]_1^2$

$V = 18\,[\sqrt{t}]_1^2 = 7.46$ Volts. ...Ans.

Ex. 4.4.5 : The angular surface 1 cm < r < 3 cm, z = 0, carries the uniform surface charge density $\rho_s = 5\ r\ (nC / m^2)$. Find V at P (0, 0, 2 cm) if V = 0 at infinity.

☑ **Soln. :** We have,

$dQ = \rho_s\,ds = 5\,r \times 10^{-9}\,(r\,dr\,d\phi) = 5r^2\,dr\,d\phi \times 10^{-9}$ (C)

The distance of dQ from point P is

$$R = \sqrt{r^2 + 0.02^2}$$

The differential voltage due to dQ is :

$$dV = \frac{dQ}{4\pi\epsilon_0 R}$$

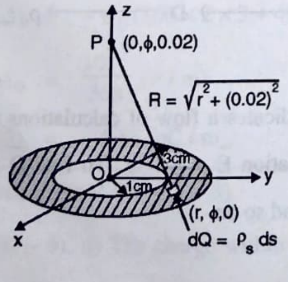


Fig. : Illustrating Ex. 4.4.5

Integrating we get total potential as,

$$V = \int dV = \int \frac{dQ}{4\pi\epsilon_0 R}$$

$$= \int_0^{2\pi}\int_{0.01}^{0.03} = \frac{5\,r^2\,dr\,d\phi \times 10^{-9}}{4\pi \times \left(\frac{10^{-9}}{36\pi}\right) \times \sqrt{r^2 + (0.02)^2}}$$

$$V = 5 \times 10^{-9} \times (18\pi) \int_{0.01}^{0.03} \frac{r^2\,dr}{\sqrt{r^2 + (0.02)^2}}$$

$$= 5 \times 10^{-9} \times (18\pi) \times \left[\frac{r}{2}\sqrt{r^2 + (0.02)^2} - \frac{(0.02)^2}{2}\ln\left(r + \sqrt{r^2 + (0.02)^2}\right)\right]_{0.01}^{0.03}$$

$V = 0.081$ (Volts) ...Ans.

4.5 POTENTIAL GRADIENT

4.5.1 The Concept of Potential Gradient

GQ. Explain the concept of potential gradient and the relation between electric field and potential. **(5 Marks)**

We know that, $V = -\int \bar{E} \cdot d\bar{l}$

When this equation is applied to a very short element of length $d\bar{l}$ along which $\bar{E}$ is a constant, then an incremental potential difference dV is given by,

$$dV = -\bar{E} \cdot d\bar{l} = -E\,dl\cos\theta$$

Where, θ is the angle between $d\bar{l}$ and $\bar{E}$.

Since V is function of coordinates of any point and it is a single valued function, we have

$$\frac{dV}{dl} = -E\cos\theta$$

If $d\bar{l}$ (= $dl\,d_l$) is placed in opposite direction to $\bar{E}$ (θ = 180°) then

$$\frac{dV}{dl} = -E\cos(180^\circ) = E$$

This is the maximum value of dV/dl.

Hence, $$\left.\frac{dV}{dl}\right|_{max} = E$$

This shows that :

1. The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.
2. This maximum value is obtained when the direction of the distance increment is opposite to $\bar{E}$ or in other words, the

direction of $\bar{E}$ is opposite to the direction in which the potential is increasing most rapidly.

Now $\bar{E}$ is perpendicular to the equipotential surface (Equipotential surface is the surface wherein every point is with same potential) and is directed towards the decreasing value of the potential. Let $\bar{a}_N$ is the unit vector towards the increasing value of the potential. Then

$$\bar{E} = \frac{-dV}{dl}\bigg|_{max} \bar{a}_N$$

Since, $\frac{dV}{dl}\Big|_{max}$ occurs when $d\bar{l}$ is in the direction of $\bar{a}_N$,

$$\frac{dV}{dl}\bigg|_{max} = \frac{dV}{dN} \quad \text{and} \quad \bar{E} = -\frac{dV}{dN}\bar{a}_N$$

The gradient of a scale field ϕ is defined as

$$\text{Gradient of } \phi = \text{grad } \phi = \frac{d\phi}{dN}\bar{a}_N \qquad ...(4.5.1)$$

Where $\bar{a}_N$ is a unit vector normal to the equipotential surfaces and the normal is chosen which points in the direction of increasing value of ϕ. So $\bar{E}$ is expressed in terms of gradient as,

$$\bar{E} = -\text{grad } V \qquad ...(i)$$

4.5.2 The Relation between Electric Field (E) and Potential (V)

> **UQ.** Show that $\bar{E} = -\nabla V$. **(MU - Q. 1(g), Dec. 18, 5 Marks)**
>
> **UQ.** What is the relation between electric potential and electric field intensity ? **MU - May 10, 2 Marks**
>
> **UQ.** Derive relationship between Electric field and voltage. **(MU - Q. 1(g), May 19, 5 Marks)**

Since, V is function of (x, y, z), we have

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \qquad ...(ii)$$

Let $\bar{E} = E_x\bar{a}_x + E_y\bar{a}_y + E_z\bar{a}_z \qquad ...(iii)$

and $d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z \qquad ...(iv)$

Then $dV = -\bar{E}\cdot d\bar{l} = -E_x\,dx - E_y\,dy - E_z\,dz \qquad ...(v)$

Comparing Equation (ii) with (v), we get

$$E_x = -\frac{\partial V}{\partial x};\ E_y = -\frac{\partial V}{\partial y};\ E_z = -\frac{\partial V}{\partial z}$$

Putting these values in the expression for $\bar{E}$ (Equation (iii))

$$\bar{E} = -\left(\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right) \qquad ...(vi)$$

As $\bar{E} = -\text{grad } V$

We have, $\text{grad } V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$

Now the del operator is given by

$$\nabla = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z$$

Hence, $\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$

Replacing the bracket in Equation (vi) by ∇V,

$$\bar{E} = -\nabla V \qquad ...(4.5.2)$$

> **Note :**
> 1. Electric field intensity is perpendicular to the equipotential surface (a surface on which potential is constant) and is directed from higher to lower potential i.e. towards the decreasing value of potential.
> 2. Maximum rate of change of potential with respect to distance gives the magnitude of electric field intensity.
> 3. In terms of mathematical relations, remember the following important formulae.

$$|\bar{E}| = \frac{dV}{dl}\bigg|_{max}$$

$$\bar{E} = -\text{grad } V = -\nabla V \qquad ...(4.5.3)$$

$$\bar{E} = \frac{-dV}{dl}\bigg|_{max} \bar{a}_N = -\frac{dV}{dN}\bar{a}_N$$

The expressions for ∇V in three coordinate systems are :

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z \quad \text{(Cartesian)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{\partial V}{\rho\partial\phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z \quad \text{(Cylindrical)} \qquad ...(4.5.4)$$

$$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{\partial V}{r\,\partial\theta}\bar{a}_\theta + \frac{\partial V}{r\sin\theta\,\partial\phi}\bar{a}_\phi \quad ...\text{(Spherical)}$$

The different relations between field quantities till now we have studied can be related as shown below.

$$V \xrightarrow{\bar{E} = -\nabla V} \bar{E} \xrightarrow{\bar{D} = \epsilon\bar{E}} \bar{D} \xrightarrow{\nabla\cdot\bar{D} = \rho_v} \rho_v \xrightarrow{Q = \int_v \rho_v\,dv} Q \qquad ...(4.5.5)$$

The arrow indicates a flow of calculations that is to find $\bar{E}$ from V use the relation $\bar{E} = -\nabla V$. To find $\bar{D}$ from $\bar{E}$ use the relation $\bar{D} = \epsilon\bar{E}$ and so on.

Module 1

4.5.3 Solved Examples on $\bar{E} = -\nabla V$

Important Formulas on scalar potential

1. $V_{AB} = -\int_B^A \bar{E} \cdot d\bar{l}$ ...(Potential difference)
2. $V_{AB} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$ ...(Potential difference)
3. $V = \frac{Q}{4\pi\epsilon_0 R}$ ...(Absolute potential)
4. $dV = \frac{dQ}{4\pi\epsilon_0 R}$ ...(Differential potential)
5. $V_{AB} = \frac{\rho_l}{2\pi\epsilon_0} \ln\left(\frac{r_A}{r_B}\right)$ ...(Due to infinite line charge)
6. $\bar{E} = -\nabla V$ ...(Relation between E and V)

Ex. 4.5.1 : Find volume charge density, ρ_v at (1, 2, 3) if in free space, $V = 50\, x^2 yz + 20y^2$ (V)

☑ Soln. :

We know,

$$\bar{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right)$$

Now $\frac{\partial V}{\partial x} = 100\, xyz\, ;\ \frac{\partial V}{\partial y} = 50\, x^2 z + 40y\, ;\ \frac{\partial V}{\partial z} = 50x^2 y$

then $\bar{E} = -(100\, xyz\, \bar{a}_x + [50\, x^2 z + 40\, y]\, \bar{a}_t + 50\, x^2 y\, \bar{a}_z)$ (V.m)

To find $\bar{D}$

$$\bar{D} = \epsilon_0 \bar{E} = -\epsilon_0 (100\, xyz\, \bar{a}_x + [50\, x^2 z + 40\, y]\, \bar{a}_y + 50\, x^2 y\, \bar{a}_z)$$

...Ans.

To find ρ_v : $\rho_v = \nabla \cdot \bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$

here $\frac{\partial D_x}{\partial x} = -\epsilon_0\, 100\, yz\, ;\ \frac{\partial D_y}{\partial y} = -40\,\epsilon_0\, ;\ \frac{\partial D_z}{\partial z} = 0$

then $\rho_v = -\epsilon_0 (100\, yz + 40)$

$\rho_v (1, 2, 3) = -\epsilon_0 (100 \times 2 \times 3 + 40) = -640\, \epsilon_0$

Putting, $\epsilon_0 = \frac{10^{-9}}{36\pi}$ (F / m)

$\therefore\ \rho_v (1, 2, 3) = -5.66\ (\text{nC} / \text{m}^3)$ ...Ans.

Ex. 4.5.2 : In free space $V = x^2 y\, (z + 3)$

Find i) $\bar{E}$ at (3, 4, – 6), ii) The charge within the cube $0 < x, y, z < 1$.

☑ Soln. :

(i) To find $\bar{E}$

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$= -[2\, xy\, (z + 3)\, \bar{a}_x + x^2 (z + 3)\, \bar{a}_y + x^2 y\, \bar{a}_z]$$

$$\bar{E}\,|_{(3, 4, -6)} = -[-72\, \bar{a}_x - 27\, \bar{a}_y + 36\, \bar{a}_z]$$

$$= 72\, \bar{a}_x + 27\, \bar{a}_y - 36\, \bar{a}_z\ (\text{V/m})$$ ...Ans.

(ii) To find charge

$$\rho_V = \nabla \cdot \bar{D} = \epsilon_0 (\nabla \cdot \bar{E}) = \epsilon_0 \left[\frac{\partial Ex}{\partial x} + \frac{\partial Ey}{\partial y} + \frac{\partial Ez}{\partial z}\right]$$

$$= -\epsilon_0 [2\, y\, (z + 3) + 0 + 0]$$

$$= -2\, \epsilon_0\, y\, (z + 3)\ (\text{C/m}^3)$$

The charge is obtained using ρ_V as

$$Q = \int_V \rho_V\, dV = \int_0^1 \int_0^1 \int_0^1 -2\, \epsilon_0\, y\, (z + 3)\, dx\, dy\, dz$$

$$= -2\, \epsilon_0\, (x)_0^1 \left[\left(\frac{y^2}{2}\right)_0^1 \left(\frac{z^2}{2}\right)_0^1 + 3\left(\frac{y^2}{2}\right)_0^1\right]$$

$$Q = -2\, \epsilon_0 \left[\frac{1}{4} + \frac{3}{2}\right]$$

$$Q = -3.5\, \epsilon_0\, (\text{C}) = -30.95\ (\text{pC})$$ ...Ans.

UEx. 4.5.3 MU - May 16, Q. 4(b), Dec. 19, 10 Marks

In free space, $V = 6\, xy^2\, z + 8$. At point P (1, 2, – 5) find E and volume charge density.

☑ Soln. :

$$V = 6\, xy^2\, z + 8$$

$$\bar{E} = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$= -\left[6\, y^2 z\, \bar{a}_x + 12\, xyz\, \bar{a}_y + 6xy^2\, \bar{a}_z\right]$$

$$\bar{D} = \epsilon_0\, \bar{E} = -\epsilon_0 \left[6\, y^2 z\, \bar{a}_x + 12\, xyz\, \bar{a}_y + 6xy^2\, \bar{a}_z\right]$$

$$\rho_v = \nabla \cdot \bar{D}$$

$$= \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}$$

$$= 0 - 12\, x\, z\, \epsilon_0 + 0 = -12\, x\, z\, \epsilon_0$$

At P(1, 2, – 5)

$$\bar{E} = -\left[6 \times 2^2 \times (-5)\, \bar{a}_x + 12 \times 1 \times 2 \times (-5)\, \bar{a}_y + 6\, (1) \times 2^2\, \bar{a}_z\right]$$

$$= +\ 120\, \bar{a}_x + 120\, \bar{a}_y - 24\, \bar{a}_z\ (\text{V/m})$$

$$\rho_v = -12\, x\, z\, \epsilon_0$$

$$\rho_v = -12 \times 1 \times -5 \times \frac{10^{-9}}{36\pi} = 0.53\ (\pi c/m^3)$$

Ex. 4.5.4 : If a scalar potential is given by $\phi = xyz$, determine the potential gradient and also prove that $\bar{F}$ = grad ϕ is irrotational.

Soln. :

Given : $\phi = xyz$,

then
$$\bar{F} = \text{grad}\ \phi = \nabla\phi$$
$$= \left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right)(xyz)$$
$$= yz\,\bar{a}_x + xz\,\bar{a}_y + xy\,\bar{a}_z$$

$\bar{F}$ is irrotational if $\nabla \times \bar{F} = 0$

$$\nabla \times \bar{F} = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{bmatrix}$$
$$= \bar{a}_x(x-x) - \bar{a}_y(y-y) + \bar{a}_z(z-z) = 0 \quad \text{...Ans.}$$

Thus $\bar{F}$ is irrotational.

Ex. 4.5.5 : If $V = r^2\phi - 2\theta$. Find $\bar{E}$.

Soln. :

Here V is a function of (r, θ, ϕ) i.e. spherical coordinates.

Using the relation in spherical coordinates
$$\bar{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\bar{a}_r + \frac{\partial V}{r\,\partial\theta}\bar{a}_\theta + \frac{\partial V}{r\sin\theta\,\partial\phi}\bar{a}_\phi\right)$$

For given V, $\frac{\partial V}{\partial r} = 2r\phi$, $\frac{\partial V}{\partial\theta} = -2$, $\frac{\partial V}{\partial\phi} = r^2$

Then,
$$\bar{E} = -\left(2r\phi\,\bar{a}_r - \frac{2}{r}\bar{a}_\theta + \frac{r}{\sin\theta}\bar{a}_\phi\right)$$
$$= -2r\phi\,\bar{a}_r + \frac{2}{r}\bar{a}_\theta - \frac{r}{\sin\theta}\bar{a}_\phi\ (V/m) \quad \text{...Ans.}$$

Ex. 4.5.6 : Given $v = 10\frac{\sin\theta\cos\Psi}{r^2}$, find the field intensity at (2.5 m, – 60°, 45°)

Soln. :

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{\partial V}{r\partial\theta}\bar{a}_\theta + \frac{\partial V}{r\sin\theta\partial\theta}\bar{a}_\phi\right]$$
$$\frac{\partial V}{\partial r} = 10\sin\theta\cos\phi\left(\frac{-2}{r^3}\right) = \frac{-20}{r^3}\sin\theta\cos\phi$$
$$\frac{\partial V}{r\,\partial\theta} = \frac{1}{r}\times\frac{10}{r^2}\cos\theta\cos\phi = \frac{10}{r^3}\cos\theta\cos\phi$$
$$\frac{\partial V}{r\sin\theta\,\partial\theta} = \frac{1}{r\sin\theta}\times -10\frac{\sin\theta\sin\phi}{r^2} = \frac{-10}{r^3}\sin\phi$$

At (2.5, – 60°, 45°)
$$\frac{\partial V}{\partial r} = 0.7838; \quad \frac{\partial V}{r\,\partial\theta} = 0.2262$$
$$\frac{\partial V}{r\sin\theta\,\partial\theta} = -0.4525$$

UEx. 4.5.7 **MU - Dec. 10, Dec. 16, 10/5 Marks**

Given the potential $V = 2x^2y - 5z$ and a point P (– 4, 3, 6), find E, D and ρ_v at P.

Soln. :

Given : $V = 2x^2y - 5z$, P(– 4, 3, 6)

To find V at P
$$V\Big|_{(-4,3,6)} = 2(-4)^2(3) - 5(6) = 66\ (V)$$

To find $\bar{E}$
$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$
$$= -[4xy\,\bar{a}_x + 2x^2\,\bar{a}_y - 5\,\bar{a}_z]$$
$$= -4xy\,\bar{a}_x - 2x^2\,\bar{a}_y + 5\,\bar{a}_z\ (V/m)$$
$$\bar{E}\Big|_{(-4,3,6)} = -4(-4)(3)\,\bar{a}_x - 2(-4)^2\,\bar{a}_y + 5\,\bar{a}_z$$
$$\bar{E}\Big|_{(-4,3,6)} = 48\,\bar{a}_x - 32\,\bar{a}_y + 5\,\bar{a}_z\ (V/m)$$

To find $\bar{D}$
$$\bar{D} = \epsilon_0\bar{E} = \epsilon_0[-4xy\,\bar{a}_x - 2x^2\,\bar{a}_y + 5\,\bar{a}_z]$$
$$\bar{D}\Big|_{(-4,3,6)} = \epsilon_0 \times \bar{E}\Big|_{(-4,3,6)}$$
$$= 48\epsilon_0\,\bar{a}_x - 32\epsilon_0\,\bar{a}_y + 5\epsilon_0\,\bar{a}_z\ (C/m^2)$$

To find ρ_v
$$\rho_v = \nabla\cdot\bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$
$$= \frac{\partial}{\partial x}[-4xy\,\epsilon_0] + \frac{\partial}{\partial y}[-2x^2\epsilon_0] + \frac{\partial}{\partial z}[5\epsilon_0]$$
$$= -4y\epsilon_0 + 0 + 0 = -4y\epsilon_0$$
$$\rho_v\Big|_{(-4,3,6)} = -4(3)\epsilon_0 = -12\epsilon_0$$
$$= 0.106\ (nC/m^3)$$

Chapter Ends...

Module 1

CHAPTER 5

Laplace's and Poisson's Equations

5.1 LAPLACE'S AND POISSON'S EQUATIONS

UQ. Derive the Poissions and Laplace equation. **(MU - Dec. 11, 5 Marks, Dec. 12, 10 Marks, Q. 1(c), Dec. 19, 5 Marks)**

UQ. Write short note on : Laplace and Poisson's equation. **MU - Dec. 10, May 11, 10 Marks**

UQ. Derive Laplacian's Equation for charge free dielectric region. **(MU - Q. 1(d), May 19, 5 Marks)**

Laplace's equation gives us method of finding potential function V when conducting materials in the form of planes, curved surfaces or lines are given and voltage on one is known with respect to some reference. Often the other conductor.

Laplace's equation is the special case of Poisson's equation. To obtain Poisson's equation from Gauss's law is very simple :

From the point form of Gauss's law,

$$\nabla \cdot \bar{D} = \rho_v$$

but $\bar{D} = \epsilon \bar{E}$

Putting the value of $\bar{D}$ in Gauss's law,

$$\nabla \cdot (\epsilon \bar{E}) = \rho_v$$

For homogenous medium for which ϵ is a constant, we write

$$\nabla \cdot \bar{E} = \rho_v / \epsilon$$

Also, $\bar{E} = -\nabla V$

Then equation previous to above equation becomes,

$$\therefore \nabla \cdot (-\nabla V) = \rho_v / \epsilon$$

or $\nabla \cdot (\nabla V) = -\rho_v / \epsilon$

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad ...(5.1.1)$$

which is a Poisson's equation.

If $\rho_v = 0$, indicating zero volume charge density, but allowing point charges, line charge and surface charge density to exist on the boundaries as sources of field then,

$$\nabla^2 V = 0 \quad ...(5.1.2)$$

which is Laplace's equation.

5.1.1 Laplace Equations in Three Coordinate Systems

Equations which we have derived in chapter 1 for Laplace are derived again with same shortcut.

In cartesian coordinates

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

and $\nabla \cdot \bar{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

knowing $\nabla^2 V = \nabla \cdot \nabla V$,

Hence, Laplace's equation is,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad ...(5.1.3)$$

In cylindrical coordinates

$$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z$$

and $\nabla \cdot \bar{A} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(r A_r) + \dfrac{1}{r}\dfrac{\partial A_\phi}{\partial \phi} + \dfrac{\partial A_z}{\partial z}$

so that Laplace's equation is,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad ...(5.1.4)$$

In spherical coordinates

$$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi$$

and $\nabla \cdot \bar{A} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_r) + \dfrac{1}{r \sin\theta}\dfrac{\partial}{\partial \theta}(A_\theta \sin\theta) + \dfrac{1}{r \sin\theta}\dfrac{\partial A_\phi}{\partial \phi}$

Then Laplace's equation is,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad ...(5.1.5)$$

5.1.2 General Procedure for Solving Poisson's or Laplace's Equations

Following steps are performed while solving problems of Poisson's or Laplace's equation.

- **Step 1 :** Solve Laplace's (if $\rho_v = 0$) or Poisson's (if $\rho_v \neq 0$) equation using either
  - (i) Direct integration when V is a function of one
  - (ii) Separation of variables if V is a function of more than one variable.

  The solution at this point is not unique but expressed in terms of unknown integration constants to be determined.
- **Step 2 :** Applying boundary conditions, determine the constants involved in the integration. Now the solution is a unique.

- **Step 3 :** From V, determine $\bar{E}$ and $\bar{D}$ using,

$$\bar{E} = -\nabla V \text{ and } \bar{D} = \epsilon \bar{E}$$

- **Step 4 :** The charge density on the capacitor plates is obtained by, $\rho_s = D_n$

Where D_n is the component of $\bar{D}$ normal to the conductor. This relation we shall study in article on boundary conditions.

- **Step 5 :** The charge on the capacitor plates can be found by

$$Q = \int_S \rho_s \, ds$$

- **Step 6 :** If desired, the capacitance between the plates is obtained using the relation,

$$C = \frac{Q}{V}$$

When V is a function of only one variable then the Poisson's or Laplace equations become one dimensional. The solution of such a equation is very easy to obtain.

5.1.3 Solved Examples on Poisson's and Laplace Equation

Important Formulae

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \nabla^2 V = 0$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Ex. 5.1.1 : Let $V = 2xy^2 z^3$ and $\epsilon = \epsilon_0$. Given point P (1, 3, – 1). Find V at point P. Also find if V satisfies Laplace equation.

☑ **Soln. :**

Given : $V = 2xy^2 z^3$

$$V\Big|_{P(1,3,-1)} = 2 \times 1 \times 3^2 (-1)^3 = -18 \text{ volts}$$

Laplace equation in cartesian system is,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Differentiating given V,

$$\frac{\partial V}{\partial x} = 2y^2 z^3 ;$$

$$\frac{\partial^2 V}{\partial x^2} = 0$$

$$\frac{\partial V}{\partial y} = 4\,xy\,z^3 ;$$

$$\frac{\partial^2 V}{\partial y^2} = 4\,x\,z^3$$

$$\frac{\partial V}{\partial z} = 6\,xy^2 z^2 ;$$

$$\frac{\partial^2 V}{\partial z^2} = 12\,x\,y^2\,z$$

Adding double differentiated terms,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 + 4\,x\,z^3 + 12\,x\,y^2\,z \neq 0$$

Thus given V does not satisfy Laplace equation.

Ex. 5.1.2 : Find the potential and the volume charge density at P (0.5, 1.5, 1) in free space given the potential field :

(i) $V = 2x^2 - y^2 - z^2$ (V)

(ii) $V = 6r\,\phi\,z$

☑ **Soln. :**

Given : (i) $V = 2x^2 - y^2 - z^2$ (volts)

$$V\Big|_{P(0.5,\,1.5,\,1)} = 2(0.5)^2 - (1.5)^2 - 1^2 = -2.75 \text{ (volts)}$$

To find ρ_v we use Poisson's equation,

$$\nabla^2 V = \frac{-\rho_v}{\epsilon_0}$$

In cartesian system,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho_v}{\epsilon_0} \quad \text{...(i)}$$

Differentiating given V,

$$\frac{\partial V}{\partial x} = 4x ;$$

$$\frac{\partial^2 V}{\partial x^2} = 4$$

$$\frac{\partial V}{\partial y} = -2y ;$$

$$\frac{\partial^2 V}{\partial y^2} = -2$$

$$\frac{\partial V}{\partial z} = -2z ;$$

$$\frac{\partial^2 V}{\partial z^2} = -2$$

Putting in Equation (i),

$$4-2-2 = \frac{-\rho_v}{\epsilon_0}$$

...Ans.

or $\rho_v = 0$

(ii) Given : $V = 6\, r\, \phi\, z$

This problem is in cylindrical system, so convert the given point in cylindrical.

$$r = \sqrt{x^2+y^2} = \sqrt{0.5^2+1.5^2} = 1.581 \text{ (m)}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1.5}{0.5}\right) = 71.56°$$

$$= 71.56 \times \frac{\pi}{180} = 1.249 \text{ rad.}$$

$$z = z = 1$$

The potential at point P is,

$$V\Big|_{P(1.581,\, 71.56°,\, 1)} = 6 \times 1.581 \times 1.249 \times 1 = 11.848 \text{ (Volts)}$$

To find ρ_v, use Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon_0}$$

In cylindrical system,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= -\frac{\rho_v}{\epsilon_0} \quad \text{...(ii)}$$

Differentiating given V to obtain each term separately,

$$\frac{\partial V}{\partial r} = 6\phi z;$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \frac{\partial}{\partial r}(6 r \phi z) = 6\phi z;$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \frac{6\phi z}{r}$$

$$\frac{\partial V}{\partial \phi} = 6 r z;$$

$$\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$\frac{\partial V}{\partial z} = 6 r \phi;$$

$$\frac{\partial^2 V}{\partial z^2} = 0$$

Putting these values in Equation (ii),

$$\frac{6\phi z}{r} + 0 + 0 = \frac{-\rho_v}{\epsilon_0}$$

or $$\rho_v = -\epsilon_0\left(\frac{6\phi z}{r}\right)$$

and $$\rho_v\Big|_{\text{at } P} = -\left(\frac{10^{-9}}{36\pi}\right) \times \frac{6 \times 1.249 \times 1}{1.581}$$

$$= -41.970 \text{ (pC/m}^3\text{)}$$

...Ans.

Ex. 5.1.3 : Determine whether the following potential fields satisfied Laplace equation or not ?

(i) $V = 4x^2 + 3y - 4z^2$ (ii) $V = r \sin\phi + z$

☑ **Soln. :**

(i) $V = 4x^2 + 3y - 4z^2$

$$\frac{\partial V}{\partial x} = 8x; \quad \frac{\partial V}{\partial y} = 3; \quad \frac{\partial V}{\partial z} = -8z$$

$$\frac{\partial^2 V}{\partial x^2} = 8; \quad \frac{\partial^2 V}{\partial y^2} = 0; \quad \frac{\partial^2 V}{\partial z^2} = -8$$

$$\nabla^2 V = 8 + 0 - 8 = 0$$

(ii) $V = r \sin\phi + z$

$$\frac{\partial V}{\partial r} = \sin\phi; \quad \frac{\partial V}{\partial \phi} = r\cos\phi; \quad \frac{\partial V}{\partial z} = 1$$

$$\nabla^2 V = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial V}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial V}{\partial z}\right)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r\sin\phi) + \frac{\partial}{\partial \phi}\left(\frac{1}{r}\cos\phi\right) + \frac{\partial}{\partial z}(r \cdot 1)\right]$$

$$= \frac{1}{r}[\sin\phi - \sin\phi + 0] = 0$$

Since both functions are having $\nabla^2 V = 0$, they are Laplace equations.

UEx. 5.1.4 MU - May 03, 10 Marks

It is known that $V = XY$ is a solution of Laplace equation where X is a function of x alone and Y is a function of y alone. Determine whether following functions are solutions of Laplace's equation :

(a) $V_a = 100\,XY$; (b) $V_b = 100\,XY + 2x$;

(c) $V_c = X + 3Y$ (d) $V_d = 2XY + y^2 - x^2$

☑ **Soln. :** The Laplace equation is :

$$\nabla^2 V = 0$$

In cartesian coordinates,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

As $V = XY$ is a solution of Laplace's equation we have,

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} + \frac{\partial^2 (XY)}{\partial z^2} = 0$$

i.e. $$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

...(∵ V is not a function of z) ...(A)

Note : A particular function is a solution of Laplace equation if it satisfies Laplace equation.

(a) $V_a = 100\,XY$

$$\nabla^2 V_a = 100\left[Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}\right]$$

$$= 100\,(0) = 0 \qquad \text{...(using Equation (A))}$$

Since V_a satisfies Laplace's equation, it is a solution of Laplace's equation.

(b) $V_b = 100\,XY + 2x$

$$\frac{\partial V_b}{\partial x} = 100\,Y\frac{\partial X}{\partial x}+2\;;\qquad \frac{\partial^2 V_b}{\partial x^2} = 100\,Y\frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial V_b}{\partial y} = 100\,X\frac{\partial Y}{\partial y}\;;\qquad \frac{\partial^2 V_b}{\partial y^2} = 100\,X\frac{\partial^2 Y}{\partial y^2}$$

$$\therefore\ \nabla^2 V_b = \frac{\partial^2 V_b}{\partial x^2}+\frac{\partial^2 V_b}{\partial y^2}$$

$$= 100\left[X\frac{\partial^2 Y}{\partial y^2}+Y\frac{\partial^2 X}{\partial x^2}\right] = 100\,(0) = 0$$

Hence V_b is a solution of Laplace's equation.

(c) $V_c = X + 3Y$

$$\frac{\partial V_c}{\partial x} = \frac{\partial X}{\partial x}\;;\qquad \frac{\partial^2 V_c}{\partial x^2} = \frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial V_c}{\partial y} = 3\frac{\partial Y}{\partial y}\;;\qquad \frac{\partial^2 V_c}{\partial y^2} = 3\frac{\partial^2 Y}{\partial y^2}$$

$$\therefore\ \nabla^2 V_c = \frac{\partial^2 V_c}{\partial x^2}+\frac{\partial^2 V_c}{\partial y^2}$$

$$= \frac{\partial^2 X}{\partial x^2}+3\frac{\partial^2 Y}{\partial y^2} \neq 0$$

Hence V_c is not a solution of Laplace's equation.

(d) $V_d = 2XY + y^2 - x^2$

$$\frac{\partial V_d}{\partial x} = 2Y\frac{\partial X}{\partial x}-2x\;;\qquad \frac{\partial^2 V_d}{\partial x^2} = 2Y\frac{\partial^2 X}{\partial x^2}-2$$

$$\frac{\partial V_d}{\partial y} = 2X\frac{\partial Y}{\partial y}+2y\;;\qquad \frac{\partial^2 V_d}{\partial y^2} = 2X\frac{\partial^2 Y}{\partial y^2}+2$$

$$\therefore\ \nabla^2 V_d = \frac{\partial^2 V_d}{\partial x^2}+\frac{\partial^2 V_d}{\partial y^2}$$

$$= 2Y\frac{\partial^2 X}{\partial x^2}-2+2X\frac{\partial^2 Y}{\partial y^2}+2 = 2$$

$$\left[Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}\right] = 2(0) = 0$$

Hence V_d is a solution of Laplace's equation.

Ex. 5.1.5 : Two plates of a parallel plate capacitors are separated by a distance 'd' and maintained at potential '0' and 'V_1' respectively. Assuming negligible fringing effect determine

(i) Potential at any point between the plates.

(ii) Surface charge densities on the plates.

(iii) Capacitance of the arrangement.

☑ **Soln. :**

(i) To find potential at any point between the plates

Let us place two conducting planes with voltages '0' and V_1 at z = 0 and z = d respectively. Thus voltage between the plates is a function of z only, i.e. $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ is equal to zero. We know that the Laplace's equation in cartesian system is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 0$$

In the given problem this becomes,

$$\text{i.e.}\quad \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{...}\left(\text{since, } \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0\right)$$

Converting the partial differential into normal, (we can do this because V is a function of only one variable) we get,

$$\frac{d^2 V}{dz^2} = 0$$

In order to separate V, integrate above expression twice,

$$\text{i.e.}\quad \frac{dV}{dz} = A$$

$$\text{and}\quad V = Az + B \qquad \text{...(i)}$$

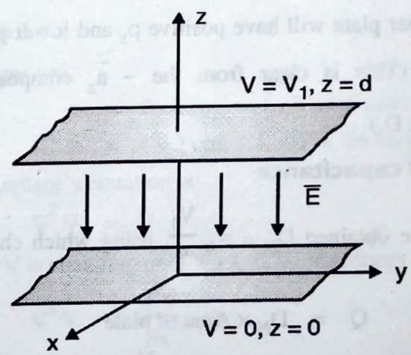


Fig. : Illustrating Ex. 5.1.5

Applying boundary conditions to Equation (i) gives values of A and B.

i.e. $V = 0$ at $z = 0$ gives

$0 = A(0) + B$

i.e. $B = 0$

Now $V = V_1$ at $z = d$ gives,

$V_1 = A(d) + B$

$$\therefore\ A = \frac{V_1}{d}$$

Putting these values in Equation (i) we get,

$$V = \frac{V_1}{d} z \quad \text{...Ans.}$$

(ii) To find surface charge densities on the plates

Steps are as follows :

- Given V, use $\bar{E} = -\nabla V$ to find $\bar{E}$.
- Use $\bar{D} = \epsilon \bar{E}$ to find $\bar{D}$.
- Evaluate $\bar{D}$ at either plate, $\bar{D} = \bar{D}_s = D_N \bar{a}_N$.
- Recognize that $\rho_s = D_N$.

To find $\bar{E}$

$$\bar{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right)$$

$$= -\frac{\partial V}{\partial z}\bar{a}_z = -\frac{V_1}{d}\bar{a}_z$$

$$\text{...}\left(\text{since, } \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0\right)$$

Now $\bar{D} = \epsilon_0 \bar{E} = -\epsilon_0 \frac{V_1}{d}\bar{a}_z$

Since, D is constant and normal to the plates, we get

$$D_N = \epsilon_0 \frac{V_1}{d}$$

$$\therefore \rho_s = \pm \epsilon_0 \frac{V_1}{d}$$

The upper plate will have positive ρ_s and lower plate having negative ρ_s. (This is clear from the $-\bar{a}_z$ component in the expression for $\bar{D}$.)

(iii) To find capacitance

We have obtained $D_N = \epsilon_0 \frac{V_1}{d}$, using which charge on the plate is,

$$Q = D_N \times \text{Area of plate}$$

$$Q = D_N \times A = \epsilon_0 \frac{V_1}{d} A$$

But $C = \frac{Q}{V_1} = \frac{\epsilon_0 A}{d}$. ...Ans.

Ex. 5.1.6 : In Cartesian co-ordinates a potential is a function of x only. At x = – 20 cm, V = 25 V and E = – 1.5 × 10^3 $\bar{a}_x$ V/m throughout the region. Find V at x = 3 cm.

☑ **Soln. :** When potential is a function of x only

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$$

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2} = 0$$

$$\therefore \frac{dV}{dx} = A \quad \text{and} \quad V = Ax + B \quad \text{...(1)}$$

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial x}\bar{a}_x = -A\bar{a}_x \quad \text{...(2)}$$

At $x = -20 \times 10^{-2}$, $V = 25$

and $\bar{E} = -1.5 \times 10^3 \bar{a}_x$

From Equation (1),

$$25 = A(-20 \times 10^{-2}) + B \quad \text{...(3)}$$

From Equation (2),

$$-1.5 \times 10^3 \bar{a}_x = -A\bar{a}_x \qquad \therefore A = 1.5 \times 10^3$$

Putting in Equation (3),

$$25 = 1.5 \times 10^3 (-20 \times 10^{-2}) + B$$

$$\therefore B = 325$$

Putting in Equation (1),

$$V = 1.5 \times 10^3 x + 325$$

At $x = 3$ cm,

$$V = 1.5 \times 10^3 \times 3 \times 10^{-2} + 325 = 370 \text{ (V).}$$

Ex. 5.1.7 : The one dimensional Laplace's equation is given as $\frac{\partial^2 V}{\partial x^2} = 0$

The boundary conditions are V = 9 at x = 1 and V = 0 at x = 10. Find the potential and also show the variation of V with respect to x.

☑ **Soln. :** Since V is a function of only x we change partial differentiation into normal thus we have,

$$\frac{d^2 V}{dx^2} = 0$$

In order to find V, integrate the given equation twice, we get

$$\frac{dV}{dx} = A$$

and $V = Ax + B$ ...(i)

Putting boundary conditions in Equation (i)

i.e. $V = 9$ at $x = 1$,

gives $9 = A \cdot 1 + B$

i.e. $A + B = 9$ ...(ii)

Putting second boundary condition in Equation (i) we have

i.e. $V = 0$ at $x = 10$,

gives $0 = 10 \cdot A + B$ ...(iii)

Solving Equations (ii) and (iii), we get

$$A = -1 \text{ and } B = +10.$$

Putting these values in Equation (i), we get the potential as

$$V = -x + 10. \quad \text{...Ans.}$$

The variation of V with respect to x is a straight line as shown in Fig. Ex. 5.1.7.

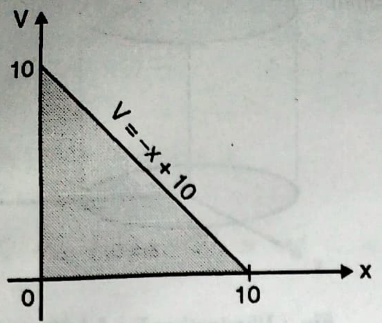


Fig. : Illustrating Ex. 5.1.7

UEx. 5.1.8 **MU - Q. 4(b), Dec. 18, 10 Marks**

Obtain Poisson's and Laplacian's Equation used to solve boundary problems for conducting plates described as V (z = 0) = 0 V and V (z = 2 mm) = 50 V. Determine :

$V, \bar{E}, \bar{D}$

☑ **Soln. :** Since the voltage is changing w.r.t. z,

$$\nabla^2 V = \frac{d^2 V}{dz^2} = 0$$

Solving $V = Az + B$ ...(1)

When $z = 0, V = 0 \rightarrow B = 0$

$z = d, V = V_0 \rightarrow V_0 = Ad$ or $A = \frac{V_0}{d}$

Putting in Equation (1)

$$V = \frac{V_0 z}{d} \quad ...(2)$$

We have $\bar{E} = -\nabla V = -\frac{\partial V}{\partial z}\bar{a}_z = -\frac{V_0}{d}\bar{a}_z$ ...(3)

$$\bar{D} = \epsilon\bar{E} = -\epsilon_0 \epsilon_r \frac{V_0}{d}\bar{a}_z \quad ...(4)$$

Given : $V_0 = 50$ V at d = 2 mm

∴ Using Equations (2), (3) and (4)

$$V = \frac{50}{2 \times 10^{-3}} z = 25\, z\ (kV)$$

$$\bar{E} = -25\,\bar{a}_z\ (kV/m)$$

$$\bar{D} = -\frac{10^{-9}}{36\pi} \times 1.5 \times 25 \times 10^3\,\bar{a}_z$$

$$\bar{D} = -332\,\bar{a}_z\ (nC/m^2)$$

UEx. 5.1.9 **MU - Dec. 05, 10 Marks**

Consider a parallel plate capacitor occupying planes x = 0 and x = d and is kept at a potential V = 0 and V = V_0 respectively. The medium consists of two dielectrics ϵ_1 for 0 < x < t and ϵ_2 for t < x < d. If d = 4 cm, t = 2 cm, ϵ_{r1} = 2, ϵ_{r2} = 4 and V_0 = 100 volts. Using Laplace's equation find the potential and electric field intensities in the two regions.

☑ **Soln. :** Two different dielectrics with ϵ_1 and ϵ_2 forms two different capacitors in series. Consider top and bottom area of each capacitor is A (m^2). Two capacitances can be obtained as :

$$C_1 = \frac{\epsilon_1 A}{t} = \frac{\epsilon_0 \epsilon_{r1} A}{t}$$

$$C_1 = \frac{2\epsilon_0 A}{2 \times 10^{-2}} = 100\,\epsilon_0 A\ (F)$$

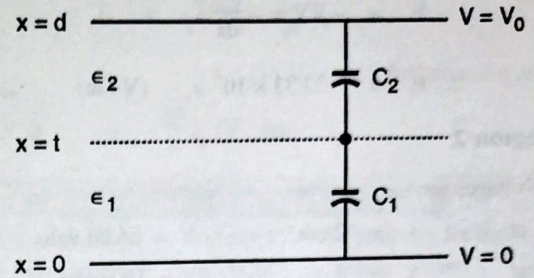


Fig. : Illustrating Ex. 5.1.9

and $C_2 = \frac{\epsilon_2 A}{d - t} = \frac{4\epsilon_0 A}{2 \times 10^{-2}} = 200\,\epsilon_0 A\ (F)$

Using voltage divider we obtain voltage across C_1 as :

$$V_{C1} = \frac{C_2}{C_1 + C_2} \times V_0$$

$$= \frac{200\,\epsilon_0 A}{300\,\epsilon_0 A} \times 100 = 66.66 \text{ Volts.}$$

Thus at x = t, voltage V = 66.66 volts.

Now to find potential variation we solve Laplace's equation separately for region 1 and region 2.

For region 1

Voltages are,

at x = 0, V = 0 volts

at x = t = 2 cm, V = 66.66 volts

The Laplace's equation is

$$\nabla^2 V = 0$$

Since V is changing only w.r.t. x, $\nabla^2 V$ reduces to :

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$$

or $\frac{d^2 V}{dx^2} = 0$

Integrating we get,

$$\frac{dV}{dx} = A$$

Integrating once again,

$$V = Ax + B \quad ...(i)$$

To find A and B apply known voltages to Equation (i)

V = 0 at x = 0

gives, 0 = A (0) + B

i.e. B = 0 and

$$V = 66.66 \text{ at } x = 2 \text{ cm}$$

gives, $66.66 = A(2 \times 10^{-2})$

i.e. $A = 33.33 \times 10^{-2}$

Putting in Equation (i),

$$V = 33.33 \times 10^{2} x \quad \text{...Ans.}$$

This is variation of voltage in region 1. To find E we have,

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial x}\bar{a}_x$$

$$\therefore \bar{E} = -33.33 \times 10^{2}\,\bar{a}_x \quad (V/m) \quad \text{...Ans.}$$

For region 2

Voltages are,

at $x = 2$ cm, $V = 66.66$ volts

at $x = 4$ cm, $V = 100$ volts

Since V is changing only w.r.t. x, the expression for V after solving Laplace equation will be similar to Equation (i). But the values of A and B will be different. To find A and B apply known voltages in region 2 to Equation (i).

$$V = 66.66 \text{ at } x = 2 \text{ cm}$$

gives, $66.66 = A(2 \times 10^{-2}) + B$ ...(ii)

and $V = 100$ at $x = 4$ cm gives,

$100 = A(4 \times 10^{-2}) + B$ ...(iii)

Solving Equations (ii) and (iii) we get,

$$A = 16.665 \times 10^{2} \text{ and } B = 33.33$$

Thus variation of voltage in region (2) is,

$$V = (16.665 \times 10^{2})\,x + 33.33 \quad \text{(Volts)...Ans.}$$

To find electric field,

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial x}\bar{a}_x$$

$$\therefore \bar{E} = -16.665 \times 10^{2}\,\bar{a}_x \quad (V/m) \quad \text{...Ans.}$$

Ex. 5.1.10 : Two parallel conducting disks are separated by a distance 5 mm at $z = 0$ and $z = 5$ mm. If $V = 0$ at $z = 0$ and $V = 100$ at $z = 5$ mm. Find charge densities on the disks.

☑ **Soln. :**

From the given conditions it is clear that V is a function of z only. Therefore putting $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial \phi}$ equal to zero in the Laplace's equation in cylindrical coordinate system, we get

$$\nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0; \text{ i.e. } \frac{d^2 V}{dz^2} = 0$$

Integrating twice we get,

$$\frac{dV}{dz} = A \text{ and } V = Az + B \quad \text{...(i)}$$

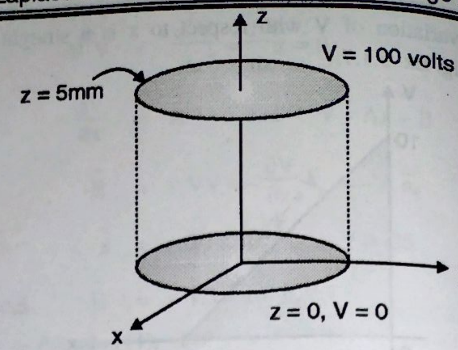


Fig. : Illustrating Ex. 5.1.10

where, A and B are constants of integration. To determine these constants apply the boundary conditions.

$V = 0$ at $z = 0$ and $V = 100$ Volts at $z = 5$ mm

Putting these in Equation (i),

$$0 = A(0) + B$$

and $100 = A(5 \times 10^{-3}) + B$

i.e. $B = 0$

i.e. $A = \frac{100}{5 \times 10^{-3}} = 20 \times 10^{3}$

Putting these values in Equation (i), the potential is obtained as,

$$V = 20 \times 10^{3} z$$

Now, the electric field between the circular disks is obtained using the relation,

$$\bar{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\bar{a}_r + \frac{\partial V}{r\,\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z\right)$$

$$\bar{E} = -\frac{\partial V}{\partial z}\bar{a}_z = -20 \times 10^{3}\,\bar{a}_z$$

...(since V is not a function of r and ϕ)

Consider, the medium between circular disks as free space, which gives the flux density as,

$$\bar{D} = \epsilon_0 \bar{E} = \frac{10^{-9}}{36\pi} \times (-20 \times 10^{3})\,\bar{a}_z$$

$$= -0.177 \times 10^{-6}\,\bar{a}_z\,(C/m^2)$$

Since D is constant between the plates and normal to the plates, we say that

$$\rho_s = D_n = \pm 0.177 \times 10^{-6} \quad (C/m^2)$$

Note the direction of D, it is the negative $\bar{a}_z$ direction, i.e. from upper plate to lower plate or in other words from high potential plate to low potential plate. As we know the electric flux D originate on positive charge and terminate on negative charge, the upper plate has positive charge density while lower has negative.

UEx. 5.1.11 MU - Dec. 06, 10 Marks

Find the potential function and the electric field intensity for the region between two concentric right circular cylinders, where V = 0 at r = 1 mm and V = 150 volts at r = 20 mm. Neglect fringing effect.

☑ **Soln. :** The Laplace's equation is,

$$\nabla^2 V = 0$$

Since V is not varying w.r.t. ϕ and z, in cylindrical coordinates $\nabla^2 V$ reduces to,

$$\nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

i.e. $$\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

Integrating we get,

$$r\frac{dV}{dr} = A$$

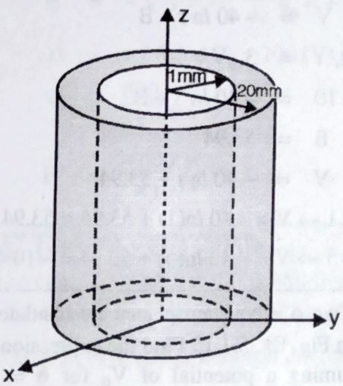


Fig. : Illustrating Ex. 5.1.11

or $$\frac{dV}{dr} = \frac{A}{r}$$

Integrating we get,

$$V = A\,ln\,(r) + B \qquad \text{...(a)}$$

To find A and B use the given voltages,

at $r = 1\text{ mm} = 1 \times 10^{-3}$ m,

$V = 0$ and

at $r = 20\text{ mm} = 20 \times 10^{-3}$ m,

$V = 150$ Volts

Putting in Equation (a) gives,

$$0 = A\,ln\,(1 \times 10^{-3}) + B$$

i.e. $$B = -A\,ln\,(10^{-3})$$

and $$150 = A\,ln\,(20 \times 10^{-3}) + B$$

$$= A\,ln\,(20 \times 10^{-3}) - A\,ln\,(10^{-3})$$

$$= A\,ln\,(20)$$

$$\therefore A = \frac{150}{ln\,(20)} = 50.0$$

Similarly, $$B = -A\,ln\,(10^{-3})$$

$$= -50.16\,ln\,(10^{-3}) = 345.4$$

Putting values of A and B in Equation (A) we get,

$$V = 50\,ln\,(r) + 345.4 \qquad \text{...Ans.}$$

The electric field can be obtained from V using,

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{a}_r$$

($\because$ V is not a function of ϕ and z)

$$\therefore \bar{E} = -\frac{50}{r}\bar{a}_r\,(V/m) \qquad \text{...Ans.}$$

UEx. 5.1.12 MU- Dec.11, Dec. 12, 5/10 Marks

Find the potential function and electric field intensity for the region between two concentric right circular cylinders, where $V = V_0$ at r = a and V = 0 at r = b. Neglect fringing.

☑ **Soln. :**

In the problem it is given that V = 0 Volts at r = b and $V = V_0$ at r = a, which reveals that the potential is a function of r only, therefore Laplace's equation in cylindrical system reduces to,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

Because V is a function of r only, the partial differentiation may be replaced by ordinary differentiation.

i.e. $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

Integrating we get,

$$r\frac{dV}{dr} = A$$

Integrating again, we get,

$$V = A\,lnr + B \qquad \text{...(i)}$$

Here, A and B are constants of integration. To determine these constants use boundary conditions.

$V = V_0$ at r = a

and $V = 0$ V at r = b, b > a

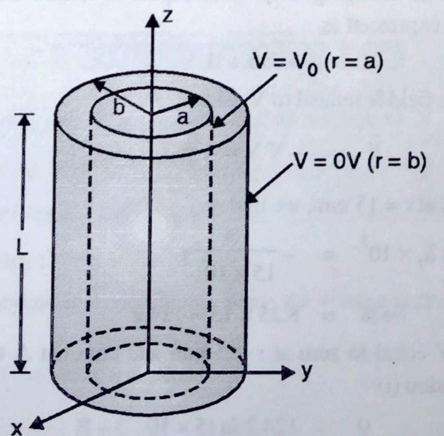


Fig. : Illustrating Ex. 5.1.12

Putting these in above equation, we get

$$V_0 = A\, ln\, a + B, 0 = A\, ln\, b + B$$

Solution of this gives

$$A = \frac{V_0}{ln\,(a/b)}, \quad B = \frac{V_0}{ln\,(b/a)}\, ln\, b$$

Using values of A and B, Equation (i) becomes

$$V = \frac{V_0}{ln\,(a/b)}\, ln\, r + \frac{V_0}{ln\,(b/a)}\, ln\, b$$

$$= \frac{V_0}{ln\,(b/a)}[ln\, b - ln\, r] = V_0 \times \frac{ln\,(b/r)}{ln\,(b/a)}$$

The electric field is now obtained by using the relation.

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{a}_r$$

$$\bar{E} = -\frac{d}{dr}V_0\frac{ln\,(b/r)}{ln\,(b/a)}(\bar{a}_r) = \frac{V_0}{r}\frac{1}{ln\,(b/a)}(\bar{a}_r)$$

We can find the capacitance as shown below.

The flux density at r = a is,

$$D = \in E = \frac{\in V_0}{a\, ln\,(b/a)}$$

Thus, the total charge on inner cylinder

$$Q = DA = \frac{\in V_0}{a\, ln\,(b/a)}2\pi aL$$

Therefore, capacitance

$$C = \frac{Q}{V_0} = \frac{2\pi \in L}{ln\,(b/a)}$$

UEx. 5.1.13 MU - May 10, 10 Marks

In cylindrical coordinates, $\rho = 5$ mm and $\rho = 25$ mm have voltages of zero and V_0 respectively. If $E = -8.28\,\bar{a}_r$ kV / m at $\rho = 15$ mm, find V_0 and the charge density on the outer conductor using Laplace's equation.

☑ **Soln. :**

The voltage is changing only with respect to r, then the voltage variation is expressed as

$$V = A\, ln\, r + B \quad \text{...(i)}$$

The electric field is related to V using

$$\bar{E} = -\nabla V = -\frac{A}{r}\bar{a}_r$$

Knowing E at r = 15 mm, we find A.

$$-8.28\,\bar{a}_r \times 10^3 = -\frac{A}{15 \times 10^{-3}}$$

$$\rightarrow A = 8.25 \times 15 = 124.2$$

Knowing V equal to zero at r = 5 mm and constant A we find B using Equation (i)

$$0 = 124.2\, ln\,(5 \times 10^{-3}) + B$$

$$\rightarrow B = 658$$

So the voltage variation in Equation (i) is

$$V = 124.2\, ln\,(r) + 658$$

The voltage V_0 is given at r = 25 mm, so

$$V_0 = 124.2\, ln\,(25 \times 10^{-3}) + 658$$

$$V_0 = -458 + 658 = 200 \text{ (Volts)}$$

Ex. 5.1.14 : If V = A ln r + B, find the potential at r = 1 and r = 5 when $\bar{E}_r = 20$ V/m at r = 2 and V = 10 volts at r = 3.

☑ **Soln. :**

Given :

$$V = A\, ln\, r + B$$

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{a}r = -\frac{A}{r}\bar{a}_r$$

Since at, r = 2, $E_r = 20$

$$\therefore -\frac{A}{2} = 20 \rightarrow A = -40$$

$$\therefore V = -40\, ln\, r + B$$

Since at, r = 3, V = 10

$$\therefore 10 = -40\, ln\, 3 + B$$

$$\therefore B = 53.94$$

$$\therefore V = -40\, ln\, r + 53.94$$

At $r = 1 \rightarrow V = -40\, ln(1) + 53.94 = 53.94$ (V)

At $r = 5 \rightarrow V = -40\, ln(5) + 53.94 = -10.438$ (V)

Ex. 5.1.15 : Two ϕ = constant planes are insulated along the z axis, as shown in Fig. Ex. 5.1.15 Find the expression for E between the planes, assuming a potential of V_0 for $\phi = \alpha$ and a zero reference at $\phi = 0$. Neglect fringing. Find V and E when, $\alpha = \pi/$ $V_0 = 100$ (V).

☑ **Soln. :**

The potential is a function of ϕ only therefore the Laplace equation in cylindrical system becomes

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

i.e. $$\frac{d^2V}{d\phi^2} = 0$$

Integrating once, $$\frac{dV}{d\phi} = A$$

Integrating again,

$$V = A\phi + B$$

To determine A and B apply boundary conditions,

$$0 = A\,(0) + B$$

and $$V_0 = A\,(\alpha) + B$$

Solving, $$A = \frac{V_0}{\alpha}; \quad B = 0$$

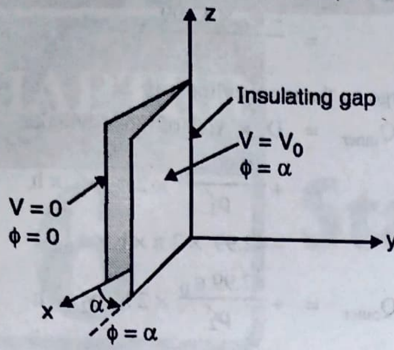


Fig. : Illustrating Ex. 5.1.15

Putting these values in Equation (i), the potential V becomes

$$V = \frac{V_0}{\alpha}\,\phi$$

The electric field can now be obtained using the relation

$$\bar{E} = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \phi}\bar{a}_\phi = -\frac{V_0}{r\alpha}\bar{a}_\phi \text{ (V/m)}$$

when $\alpha = \pi/6$ and $V_0 = 100$ (V),

then $V = \frac{600}{\pi}\phi$ (V)

and $\bar{E} = \frac{-600}{\pi r}\bar{a}_\phi$ (V/m) ...Ans.

UEx. 5.1.16 MU - May 12,10 Marks

Two conducting cones ($\theta = \pi/10$ and $\theta = \pi/6$) of infinite extent are separated by an infinitesimal gap at $r = 0$. If $V(\theta = \pi/10) = 0$ and $V(\theta = \pi/6) = 50$ V. Find V and $\bar{E}$ between the cones.

☑ **Soln. :**

The conducting cones in the problem are as shown in Fig. Ex. 5.1.16.

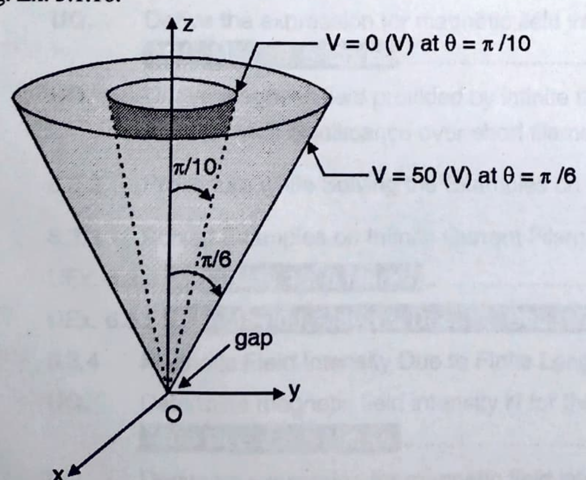


Fig. Illustrating Ex. 5.1.16

The Laplace's equation in spherical system is,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Since the potential given varies only w.r.t θ, all differentiations w.r.t r and φ are zero, resulting in Laplace's equation,

$$\nabla^2 V = \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

or $$\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

Integrating once gives,

$$\sin\theta\frac{dV}{d\theta} = A$$

or $$\frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

Integrating this gives,

$$V = A\int\frac{d\theta}{\sin\theta} = A\int\frac{d\theta}{2\cos(\theta/2)\sin(\theta/2)}$$

$$= A\int\frac{1/2\sec^2(\theta/2)}{\tan(\theta/2)}d\theta = A\int\frac{d(\tan\theta/2)}{\tan(\theta/2)}$$

$$V = A\,ln(\tan\theta/2) + B \quad ...(i)$$

To obtain A and B apply boundary conditions,

$$V = 0 \text{ at } \theta = \frac{\pi}{10} \text{ and } V = 50 \text{ at } \theta = \frac{\pi}{6}$$

Applying to Equation (i),

$$0 = A\,ln(\tan\pi/20) + B$$

and $$50 = A\,ln(\tan\pi/12) + B$$

Solving we get,

$$A = 95.1 \text{ and } B = 175.24$$

Putting these values in Equation (i),

$$V = 95.1\,ln(\tan\theta/2) + 175.24 \text{ (V)} \quad ...\text{Ans.}$$

We have, $$\bar{E} = -\nabla V$$

$$= -\frac{\partial V}{r\,\partial\theta}\bar{a}_\theta \quad (\because \partial V/\partial r = \partial V/\partial\phi = 0)$$

$$\bar{E} = -\frac{95.1}{r\sin\theta}\bar{a}_\theta \text{ (V/m).} \quad ...\text{Ans.}$$

UEx. 5.1.17 MU - Q. 5(a), May 19, 10 Marks

Two plates of cylindrical capacitor describe by their radius $\rho_1 = 1$ mm and $\rho_2 = 10$ mm holding voltage of $V_1 = 1$ V and $V_2 = 100$ V find out $\bar{E}$ in capacitor, also prove that dielectric of capacitor does not carries any charge.

☑ **Soln. :**

In cylindrical coordinates, when the voltage is changing w.r.t. ρ, then

$$V = A\,ln(\rho) + B \quad ...(A)$$

Given :

$$\rho = 1 \text{ mm} = 10^{-3} \text{ m} \rightarrow V = 1 \text{ V}$$

$$\rho = 10 \text{ mm} = 10^{-2} \rightarrow V = 100 \text{ V}$$

Putting in Equation (A)

$$1 = A \ln(10^{-3}) + B \quad \text{...(i)}$$

$$100 = A \ln(10^{-2}) + B \quad \text{...(ii)}$$

Solving, $A = 42.99$, $B = 297.98$

Putting in Equation (A)

$$V = 42.99 \ln(\rho) + 297.98$$

The electric field is,

$$\bar{E} = -\nabla V = -\frac{\partial}{\partial \rho} \bar{a}_\rho$$

$$= -42.99 \left(\frac{1}{\rho}\right) \bar{a}_\rho$$

$$\therefore \bar{D} = \epsilon_0 \bar{E}$$

$$= \frac{-\epsilon_0 \times 42.99}{\rho} \bar{a}_\rho$$

The charge on the inner cylinder is

$$Q_{inner} = D_1 \times \text{Area of inner cylinder}$$

$$= +\frac{42.99\, \epsilon_0}{\cancel{\rho_1}} \times 2\pi\, \cancel{\rho_1} \times h$$

$$= -42.99 \times 2\pi \times h \times \epsilon_0$$

$$Q_{outer} = +\frac{42.99\, \epsilon_0}{\cancel{\rho_2}} \times 2\pi\, \cancel{\rho_2} \times h$$

$$= -42.99 \times 2\pi \times h \times \epsilon_0$$

We find that the charges on both cylinders are equal, and so there is no charge inside the dielectric.

Chapter Ends...

□□□

Module 1

CHAPTER 6

Static Magnetic Field

Module 1

6.1 INTRODUCTION

- In chapters of electrostatics we observed that the source of electric field is charge. The other half of electromagnetics is ofcourse the magnetic field. And the source of magnetic field is current.
- In the study of magnetic field the current came into picture by a experiment performed by H.C. Oersted in 1820. He found that a magnetic needle was deflected by a current in a wire. Scientists then realized that electric currents are also sources of magnetic fields.
- There are other sources like permanent magnet and electric field changing with time.
- Immediately after Oersted's discovery, Biot-Savart experimentally formulated a equation to determine magnetic field at a point produced by a current carrying conductor.
- Biot-Savart law we are accepting without proof of it. We view Biot-Savart law as magnetic equivalent of Coulomb's law.
- This chapter is devoted to the study of the steady magnetic field, i.e. the magnetic fields produced by steady currents.
- We begin our discussion with the Biot-Savart law and using it as a basic tool, calculate the magnetic field set up by any given distribution of currents.
- Different types of current distributions are discussed initially.

6.1.1 Magnetic Effects of Current Flow

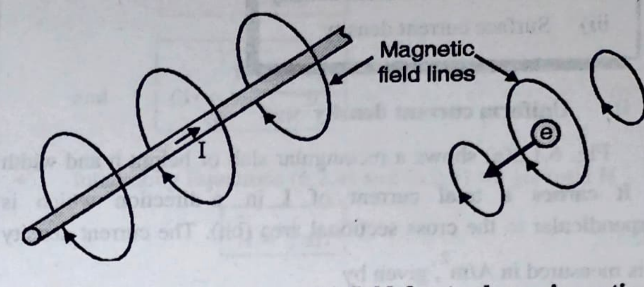
- In the electrostatics we find electric field lines originate on positive charge and terminate on negative charge.
- Such charges are absent in case of magnetic, due to which magnetic field lines can never have a beginning or an end. Magnetic field lines always form closed loops.

Magnetic field lines are always close in nature

- Without magnetic charges, magnetic fields can only arise indirectly.
- In fact, all magnetic fields are generated indirectly by moving electric charges.
- It is a fundamental fact of nature that moving electrons, as well as any other charges, produce a magnetic field in motion.

Any charge in motion produces magnetic field

- Then the question arises, how the current carrying filament produces magnetic field?
- The answer is, current is basically the collective movement of large number of electrons.
- A steady (DC) current through a wire produces a magnetic field that encircles the wire, as shown in Fig. 6.1.1(a).



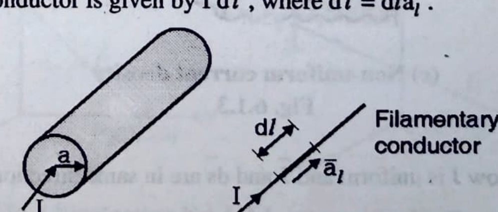
(a) Field due to current **(b) field due to charge in motion**

Fig. 6.1.1 : Production of magnetic field

- A single charge moving at constant velocity also produces a tubular magnetic field that encircles the charge, as shown in Fig. 6.1.1(b).
- However, the field of a single charge decays along the axis of propagation, with the maximum field occurring in the neighbourhood of the charge.
- Here note that when the velocity is constant then the acceleration is (a = dv/dt) zero. But if the charge has some acceleration it will not produce the magnetic field.
- The law that describes the field is called the Biot-Savart law, named after two French scientists who discovered it.
- Since one of the source of magnetic field is current, in the next article we discuss current and current distributions.

6.1.2 Current Distributions

- The most common type of conductor used in practice is circular in cross section of radius 'a'. It carries a total current I as shown in Fig. 6.1.2(a).
- If the radius of the conductors is very small then it is known as a filamentary, Fig. 6.1.2(b).
- Fig. 6.1.2(b) shows a very small section of length dl of a filamentary conductor carrying a current I in $\bar{a}_l$ direction.
- Then a differential current element for a filamentary conductor is given by $I\,d\bar{l}$, where $d\bar{l} = dl\bar{a}_l$.



(a) Circular conductor current I **(b) Filamentary conductor carrying current I**

Fig. 6.1.2

Now we discuss three types of current densities :

- i) Uniform current density
- ii) Non uniform current density
- iii) Surface current density

i) Uniform current density

Fig. 6.1.3(a) shows a rectangular slab of height h and width b. It carries a total current of I in $\bar{a}$ direction which is perpendicular to the cross sectional area (bh). The current density $\bar{J}$ is measured in A/m^2, given by

$$\bar{J} = \frac{I}{bh}\bar{a} \text{ A/m}^2 \quad \text{...(6.1.1)}$$

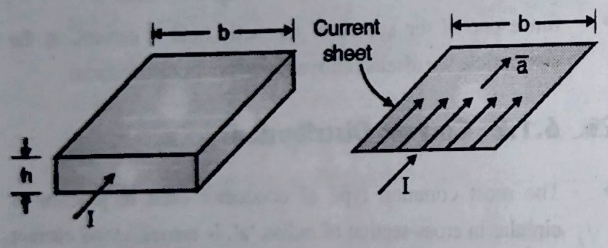
and the total current if J is uniform is given by

$$I = J(bh) \quad \text{...(6.1.2)}$$

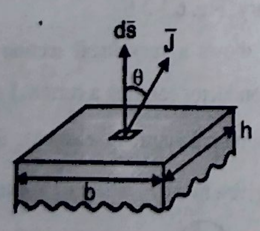
ii) Non uniform current density

For a non uniform current density $\bar{J}$, when the direction of current is not perpendicular to cross section then integration of dot product $\bar{J}\cdot d\bar{s}$ is necessary. (Fig. 6.1.3(c))

i.e.
$$\boxed{I = \int_s \bar{J}\cdot d\bar{s}} \quad \text{...(6.1.3)}$$



(a) Conductor, rectangular in cross section (b) Current sheet



(c) Non-uniform current density

Fig. 6.1.3

Consider now J is uniform and $\bar{J}$ and $d\bar{s}$ are in same direction, then

$$\bar{J}\cdot d\bar{s} = J\,ds$$

and
$$I = \int_s \bar{J}\,d\bar{s} = \int_s J\,ds = J\int_s ds = J\,(bh)$$

which is same as Equation (6.1.2).

iii) Surface current density

- Fig. 6.1.3(b) shows a rectangular slab with height h very small. Then, current I will flow on a surface called as surface current and the surface current density measured in amperes per meter width is designated by $\bar{K}$.
- If we assume that width 'b' is measured perpendicular to the direction in which the current is flowing, then surface current density $\bar{K}$ is given by,

$$\bar{K} = \frac{I}{b}\bar{a} \text{ A/m} \quad \text{...(6.1.4)}$$

- If the surface current density is uniform, the total current I in any width 'b' is,

$$I = Kb \quad \text{...(6.1.5)}$$

- Again for a non uniform surface current density, integration is necessary, therefore,

$$I = \int K dl \quad \text{...(6.1.6)}$$

where, dl is a differential element of the path across which the current is flowing. Thus, the differential current element $I\,d\bar{l}$, where $d\bar{l}$ is in the direction of the current, may be expressed in terms of surface current density $\bar{K}$ or current density as

$$I d\bar{l} = \bar{K}\,ds = \bar{J}\,dv \quad \text{...(6.1.7)}$$

6.2 BIOT-SAVART LAW

UQ. State and explain Biot-Savart's Law. **MU - May 11, 4 Marks**

UQ. Explain Biot--Savart law. **MU - Dec. 11, 5 Marks**

UQ. Express Biot-Savart's law in vector format. **(MU - Q. 1(c), Dec. 18, 5 Marks)**

- This law is also called as **Ampere's law** for the current element. It gives differential magnetic field intensity, $d\bar{H}$ due to differential current element $I\,d\bar{l}$.
- Consider a filament through which current of I amp is passing. We are interested in finding field intensity H at point P.

6.2.1 Statement of Biot Savart Law

GQ. Define magnetic field intensity. **(2 Marks)**

GQ. State and prove Biot-Savart law. **(8 Marks)**

Magnitude of dH at point P is proportional to

i) Product of current, differential length dl, the sine of the angle between the filament and line connecting differential length to the point of interest P.

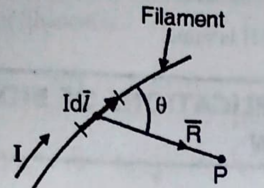


Fig. 6.2.1 : Biot-Savart law

ii) It is inversely proportional to the square of the distance from the filament to point P.

iii) The constant of proportionality is $1/4\pi$.

6.2.2 Mathematical Expression for Biot Savart Law

- Mathematically,

$$dH = \frac{(Idl)\sin\theta}{4\pi R^2} \quad ...(6.2.1)$$

- The direction of dH is normal to the plane containing the differential element and the line drawn from the filament to the point P.
- Of the two possible normals that normal is chosen which is in the direction of progress of a right handed screw turned from $d\bar{l}$ through the smaller angle to the line from the filament to P.
- In vector notations,

$$\boxed{\overline{dH} = \frac{I d\bar{l} \times \bar{a}_R}{4\pi R^2} \text{ (A/m)}} \quad ...(6.2.2)$$

where, $\times$ = this sign is a cross product.

a_R = unit vector from differential current element to point P.

$Id\bar{l}$ = differential current element.

R = distance of differential current element from point P.

- Differential current elements have no separate existence. All elements making up the complete current filament contribute to $\bar{H}$ and must be included. The summation leads to the integral form of the Biot-savart law as,

$$\boxed{\bar{H} = \oint \frac{I d\bar{l} \times \bar{a}_R}{4\pi R^2}} \quad ...(6.2.3)$$

- The above integral is a closed line integral, which is must to ensure that all current elements are included (the contour may close at ∞)
- Alternate forms of the Biot-savart law in terms of distributed sources like J and K are obtained using Equation (6.1.7) as,

$$\boxed{\overline{dH} = \frac{\bar{J}\, dv \times \bar{a}_R}{4\pi R^2}} \quad ...(6.2.4)$$

and

$$\boxed{\overline{dH} = \frac{\bar{K}\, ds \times \bar{a}_R}{4\pi R^2}} \quad ...(6.2.5)$$

- Integrating Equations (6.2.4) and (6.2.5) we get total $\bar{H}$.

i.e. $$\boxed{\bar{H} = \int \overline{dH}} \quad ...(6.2.6)$$

- Note that, $\overline{dH}$ inside the integral is from Equations (6.2.2), (6.2.4) or (6.2.5) then the integral is single integral, tripple integral or double integral, which is very clear from $d\bar{l}$, dv and ds in the expressions for dH.

6.2.3 Solved Examples on Biot Savart Law

Important Formulae

$$\overline{dH} = \frac{I d\bar{l} \times \bar{a}_R}{4\pi R^2} \text{ (A/m)}$$

$$\overline{dH} = \frac{\bar{J}\, dv \times \bar{a}_R}{4\pi R^2}$$

$$\overline{dH} = \frac{\bar{K}\, ds \times \bar{a}_R}{4\pi R^2}$$

$$\bar{H} = \int \overline{dH}$$

Ex. 6.2.1 : Estimate the incremental field $\overline{dH_2}$ at point 'P_2' caused by a source at 'P_1' of $I_1 \overline{dL_1}$:

(i) $2\pi\ \bar{a}_z$ μA - mt, given P_1 (4,0,0) and P_2 (0,3,0).

(ii) $2\pi \bar{a}_z$ μA - mt, given P_1 (4,– 2,3) and P_2 (0,3,0).

☑ Soln. :

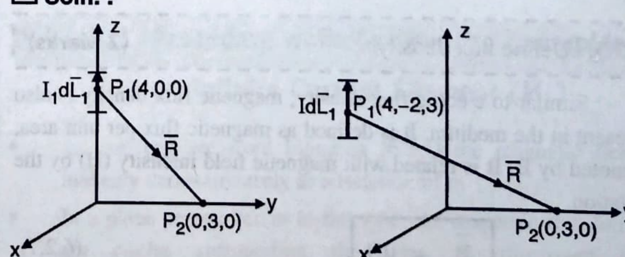


Fig. : Illustrating Ex. 6.2.1, (a) part (i), (b) part (ii)

(i) We have, $$\overline{dH_2} = \frac{I\, d\bar{L} \times \bar{a}_R}{4\pi R^2}$$

where, $\bar{R} = (0-4)\bar{a}_x + (3-0)\bar{a}_y + 0\,\bar{a}_z$

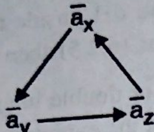
$= -4\,\bar{a}_x + 3\,\bar{a}_y$ and $\bar{a}_R = \dfrac{-4\,\bar{a}_x + 3\bar{a}_y}{5}$

Putting these in the expression for $\bar{dH}_2$ we get,

$$\bar{dH}_2 = \frac{2\pi \times 10^{-6}\,\bar{a}_z}{4\pi (5)^2} \times \left(\frac{-4\,\bar{a}_x + 3\bar{a}_y}{5}\right)$$

$$= 4 \times 10^{-9}\,\bar{a}_z \times (-4\,\bar{a}_x + 3\,\bar{a}_y)$$

Using the triangle of cross product we find : $\bar{a}_z \times \bar{a}_x = \bar{a}_y$ and $\bar{a}_z \times \bar{a}_y = -\bar{a}_x$



$\therefore \quad \bar{dH}_2 = -12\,\bar{a}_x - 16\,\bar{a}_y$ (nA/m) ...Ans.

(ii) In Fig. (b)

$\bar{R} = (0-4)\,\bar{a}_z + (3+2)\,\bar{a}_y + (0-3)\,\bar{a}_z$

$= -4\,\bar{a}_x + 5\,\bar{a}_y - 3\bar{a}_z$

and $\bar{a}_R = \dfrac{-4\bar{a}_x + 5\bar{a}_y - 3\bar{a}_z}{5\sqrt{2}}$

Then we get $\bar{dH}_2$ as

$$\bar{dH}_2 = \frac{2\pi \times 10^{-6}\,\bar{a}_z}{4\pi(5\sqrt{2})^2} \times \left(\frac{-4\,\bar{a}_x + 5\,\bar{a}_y - 3\,\bar{a}_z}{4\pi\, 5\sqrt{2}}\right)$$

$$= 1.414\,\bar{a}_z \times (-4\,\bar{a}_x + 5\,\bar{a}_y - 3\bar{a}_z)$$

Using the triangle of cross product we find : $\bar{a}_z \times \bar{a}_x = \bar{a}_y$;

$\bar{a}_z \times \bar{a}_y = -\bar{a}_x$ and we know $\bar{a}_z \times \bar{a}_z = 0$

$\therefore \quad \bar{dH}_2 = -7.071\,\bar{a}_x - 5.657\,\bar{a}_y$ (nA/m) ...Ans.

6.2.4 Magnetic Flux Density ($\bar{B}$)

GQ. Define flux density. **(2 Marks)**

Similar to electric flux density, magnetic flux density is also present in the medium. It is defined as magnetic flux per unit area, denoted by B. B is related with magnetic field intensity (H) by the relation.

$$\boxed{\bar{B} = \mu \bar{H}} \quad ...(6.2.7)$$

Where, $\bar{B}$ = flux density T ;

μ = permeability of medium, H/m ;

$\bar{H}$ = Magnetic field intensity, A/m.

Direction of $\bar{B}$ and $\bar{H}$ is same.

6.3 APPLICATIONS OF BIOT-SAVART LAW

The Biot savart law is basically applied to differential current element. Now in this section we shall study, how this law can be applied to some standard types of current filaments like

i) Infinite current filament
ii) Finite length filament
iii) Current filament circular in nature (circular loop)

6.3.1 Magnetic Field Intensity Due To Infinite Long Straight Filament

UQ. Derive formula to find Magnetic intensity due to infinite long straight conductor on z-axis by Biot-Savart's law.
MU - May 09, Q. 2(b), Dec. 19, 10 Marks

UQ. Derive the expression for magnetic field intensity due to infinite line conductor.
MU - May 16, 5 Marks

UQ. Derive magnetic field provided by infinite thin filament carrying current I suspended on 'z' axis. Also, provide significance over short filament.
(MU - Q. 2(a), May 19, 10 Marks)

Assumptions

i) Consider the infinite current filament is placed along z-axis.
ii) Since z-axis is the axis of the cylinder in cylindrical system use cylindrical coordinate system.
iii) Let the current of I Amp is passing through filament in positive z-direction.
iv) The point where the field is desired is in xy plane, the coordinates of it is P (ρ, ϕ, 0).

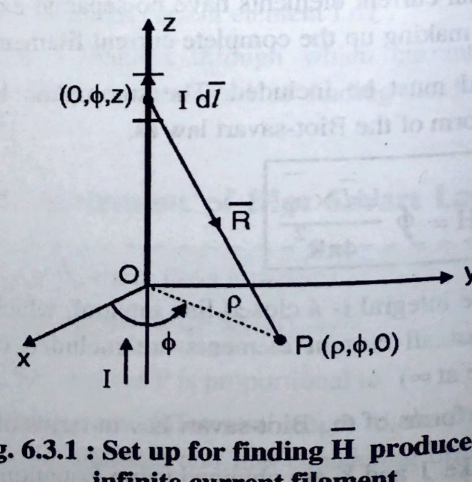


Fig. 6.3.1 : Set up for finding $\bar{H}$ produced by infinite current filament

The differential current filament is

$$I\,d\bar{l} = dz\,\bar{a}_z \qquad ...(i)$$

Vector from differential current element to point P is

$$\bar{R} = (\rho - 0)\,\bar{a}_\rho + (\phi - \phi)\,\bar{a}_\phi + (0 - z)\,\bar{a}_z$$

$$= \rho\,\bar{a}_\rho - z\bar{a}_z$$

$$\therefore \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{\rho\,\bar{a}_\rho - z\bar{a}_z}{\sqrt{\rho^2 + z^2}} \qquad ...(ii)$$

v) To apply Biot-savart law we should take differential current filament ($I\,d\bar{l}$) on z-axis at a height of 'z' from xy plane. The coordinates of differential current filaments is (0, ϕ, z). Using Biot- savart law :

$$\bar{dH} = \frac{I\,d\bar{l} \times \bar{a}_R}{4\pi R^2} = \frac{I\,dz\,\bar{a}_z}{4\pi(\rho^2 + z^2)} \times \left(\frac{\rho\bar{a}_\rho - z\bar{a}_z}{\sqrt{\rho^2 + z^2}}\right)$$

From the theory of cross product

$$\bar{a}_z \times \bar{a}_\rho = \bar{a}_\phi\,; \quad \text{and} \quad \bar{a}_z \times \bar{a}_z = 0$$

Hence, the above expression for differential field reduce to,

$$\bar{dH} = \frac{I\rho\,dz}{4\pi(\rho^2 + z^2)^{3/2}}\,\bar{a}_\phi$$

This is the field intensity due to differential current element. Hence the field intensity due to infinite long current filament is obtained by integrating $\bar{dH}$ over the limits z ranging from $-\infty$ to $+\infty$.

$$\therefore \bar{H} = \int_{-\infty}^{\infty} \bar{dH} = \int_{-\infty}^{\infty} \frac{I dz\,\rho}{4\pi(\rho^2 + z^2)^{3/2}}\,\bar{a}_\phi$$

$$= \frac{I\rho\,\bar{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{(\rho^2 + z^2)^{3/2}}$$

☞ **To solve integral**

Put, $z = \rho\tan\theta$ | limits $z \to \infty$, $\theta \to \pi/2$

$\therefore$ $dz = \rho\sec^2\theta\,d\theta$ | limits $z \to -\infty$, $\theta \to -\pi/2$

Also $(\rho^2 + z^2)^{3/2} = (\rho^2 + \rho^2\tan^2\theta)^{3/2} = \rho^3\sec^3\theta$

Putting these values in the above expression of $\bar{H}$, we get

$$\bar{H} = \frac{I\rho\,\bar{a}_\phi}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{\rho\sec^2\theta\,d\theta}{\rho^3\sec^3\theta\,d\theta}$$

$$= \frac{I\rho\,\bar{a}_\phi}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{\rho^2} = \frac{I\,\bar{a}_\phi}{4\pi\rho} \times [1 - (-1)]$$

$$\bar{H} = \frac{I}{2\pi\rho}\,\bar{a}_\phi \qquad ...(6.3.1)$$

The significance of each term in the expression (6.3.1) is as follows :

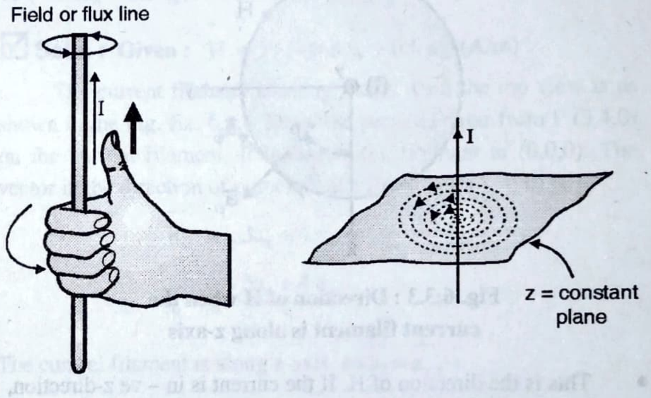
- ρ stands for the minimum distance of the point P where we desire field and the current filament i.e. it is the perpendicular distance of point P form current filament.
- The direction $\bar{a}_\phi$ is obtained by right hand rule. The rule says : **grip the current filament in your right hand with thumb in the direction of current, then the direction of fingers around the current filament gives direction of $\bar{H}$.**
- The unit vector $\bar{a}_\phi$ is perpendicular to the perpendicular from point P on the filament. In general for current filament in any arbitrary direction Equation (6.3.1) changes to :

$$\boxed{\bar{H} = \frac{I}{2\pi R}\,\bar{a}_\phi} \qquad ... (6.3.2)$$

The magnetic flux density ($\bar{B}$) can be obtained using $\bar{H}$ as

$$\boxed{\bar{B} = \mu\bar{H} = \frac{\mu I}{2\pi R}\,\bar{a}_\phi}$$

Note : The relation between $\bar{B}$ and $\bar{H}$ we will study in detail in section 6.6.



(a) Right hand rule (b) Magnetic field is concentric circles

Fig. 6.3.2

6.3.2 Procedure while Solving the Examples on Infinite Current Filament ($\bar{H}$)

- As you can see from Equation (6.3.1) the magnetic field intensity varies inversely as a function of ρ.
- In a plane perpendicular to the wire, the magnetic flux lines are circles surrounding the wire, as illustrated in Fig. 6.3.2(b). While solving the problem procedure to tackle the problem is given below :
- In the result Equation (6.3.1) note that when current is along z-axis (i.e. vertical) the direction of $\bar{H}$ is $\bar{a}_\phi$ (i.e. in horizontal plane).

- You will get perfect picture of H when you take top view of the coordinate system (Fig. 6.3.1).
- In the top view, z-axis (the current filament) is a point and H will be tangential to circle of radius ρ. At any point on this circle magnitude of H is same as it is only a function of ρ and not ϕ or z.
- Now in the problem always current is not necessarily be in the z-direction. If the current filament is along x-direction, take the front view of coordinate system so that x-axis is a point.
- If the current is along y axis, then take right view of the coordinate system so that y axis is a point.
- Remember while solving the problem consider a view such that the **filament is always perpendicular** to the plane of the page.
- In all three above cases, the direction of H is clockwise or anticlock-wise is found by right hand rule.
- In Fig. 6.3.3, current in the filament is along z-axis i.e. perpendicular to the page and in upward direction. When you grip it in your right hand with thumb in the direction of current fingers are anticlockwise.

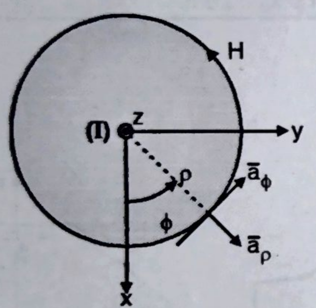


Fig. 6.3.3 : Direction of H when the current filament is along z-axis

- This is the direction of H. If the current is in – ve z-direction, then thumb is down ward and fingers are clockwise.
- Thus, H is clockwise if current is downward. The same logic can be applied to other orientations of current filament.
- Once you draw the correct figure and get the direction of H, the *steps for solving the problems are as follows :*

Step 1 : From the given point, draw a perpendicular on the current filament and find the base of perpendicular.

In chapter 2, we have seen how to find point of intersection of perpendicular on the line charge. In this chapter we have current filament instead of line charge.

Step 2 : Find the distance of perpendicular i.e. R in expression (6.3.2).

Step 3 : Find the unit vector from base of perpendicular to point of intersect (i.e. $\bar{a}_R$).

Step 4 : Find the unit vector in the direction of $\bar{a}_\phi$. This vector is always perpendicular to $\bar{a}_R$. To find this unit vector is always easy. The simple way to obtain $\bar{a}_\phi$ is, from Biot-Savart law, the direction of H is decided by cross product $\bar{a}_l \times \bar{a}_R$. Let the unit vector in the direction of d l (i.e. direction of current) be a_l, then

$$\bar{a}_\phi = \bar{a}_l \times \bar{a}_R \quad ...(6.3.$$

This vector gives direction of $\bar{H}$. The direction of H can be checked by using right hard rule.

Step 5 : The magnitude of H is obtained by $H = \frac{I}{2\pi R}$ where R is the length of perpendicular from P to the current filament.

Step 6 : Use the expression (6.3.2) to obtain $\bar{H}$,

$$\bar{H} = \frac{I}{2\pi R} \bar{a}_\phi \text{ (A / m)}$$

6.3.3 Solved Examples on Infinite Current Filament ($\bar{H}$)

Important Formula

$$\bar{H} = \frac{I}{2\pi R} a_\phi$$

Ex. 6.3.1 : Two infinitely long wires, separated by a distance 5*l* carry currents I in opposite directions as shown in Fig. Ex. 6.3.1(a). Obtain an expression for the magnetic field intensity at the point 'P' shown.

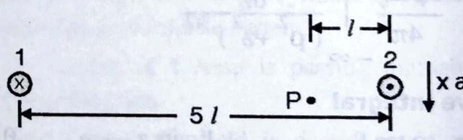


Fig. (a) : Illustrations Ex. 6.3.1

Soln. : Let us assume these two wires be parallel to z-axis. The direction of field intensities at P due to these two long wires are shown in Fig. Ex. 6.3.1(b) Which are obtained by using right hand rule. These two intensities are in the same direction and can be added to get total field at point P. The distance of P from first wire is 4*l*, while from second wire is *l*.

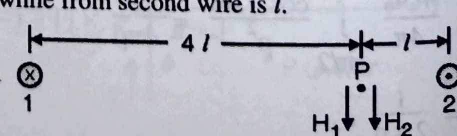


Fig. (b) : Illustrations Ex. 6.3.1

The field intensity $\bar{H}_1$ due to wire 1 is,

$$\bar{H}_1 = \frac{I}{2\pi R}\bar{a}_\phi = \frac{I}{2\pi(4l)}\bar{a}_\phi \text{ (A/m)}$$

The field intensity H_2 due to wire 2 is,

$$\bar{H}_2 = \frac{I}{2\pi R}\bar{a}_\phi = \frac{I}{2\pi(l)}\bar{a}_\phi \text{ (A/m)}$$

The total field $\bar{H}$ is,

$$\bar{H} = \bar{H}_1 + \bar{H}_2 = \frac{5I}{8\pi l}\bar{a}_\phi \text{ (A/m)} \qquad \text{...Ans.}$$

The direction $\bar{a}_\phi$ is in $\bar{a}_x$ direction.

Ex. 6.3.2 : A long straight wire carries a current I = 1 amp at what distance is the magnetic field H = 1 A/m.

☑ **Soln. : Given :** I = 1A H = 1 A/m

For a long straight wire magnetic field $\bar{H}$ is given by

$$\bar{H} = \frac{I}{2\pi R}\bar{a}_\phi \text{ A/m}$$

where R = distance

$$\therefore \quad |\bar{H}| = \frac{I}{2\pi R}$$

$$\therefore \quad 1 = \frac{I}{2\pi R}$$

$$R = \frac{I}{2\pi} = 0.1592 \text{ m}$$

$\therefore$ At 0.1592 m from the wire, H will be 1 A/m.

UEx. 6.3.3 **MU - May 10, 6 Marks**

A current filament of 5.0 A in $\bar{a}_y$ direction is parallel to y axis at x = 2, z = – 2. Find $\bar{H}$ at the origin.

☑ **Soln. :** The current filament is parallel to y-axis, then in the side view the figure is as shown. Draw perpendicular from (0,0,0) on the current filament, it intersects filament at (2,0,– 2). The vector in the direction of perpendicular towards (0,0,0) is

$$\bar{R} = -2\bar{a}_x + 2\bar{a}_z \quad \therefore \quad \bar{a}_R = \frac{-2\bar{a}_x + 2\bar{a}_z}{\sqrt{8}}$$

The current filament is parallel to y-axis, so $\bar{a}_l = \bar{a}_y$.

Then,

$$\bar{a}_\phi = \bar{a}_l \times \bar{a}_R$$

$$= \bar{a}_y \times \left(\frac{-2\bar{a}_x + 2\bar{a}_z}{\sqrt{8}}\right) = \frac{2\bar{a}_x + 2\bar{a}_z}{\sqrt{8}}$$

This talies with the direction of $\bar{a}_\phi$ in the Fig. Ex. 6.3.3.

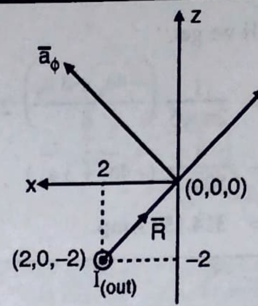


Fig. : Illustrations Ex. 6.3.3

The magnetic field $\bar{H}$ due to infinite current element is

$$\bar{H} = \frac{I}{2\pi R}\bar{a}_\phi$$

$$\bar{H} = \frac{5}{2\pi \times \sqrt{8}} \times \left(\frac{2\bar{a}_x + 2\bar{a}_z}{\sqrt{8}}\right) = 0.1989(\bar{a}_x + \bar{a}_z)$$

or $$\bar{H} = 0.1989\,\bar{a}_x + 0.1989\,\bar{a}_z \text{ (A/m)} \qquad \text{...Ans.}$$

Ex. 6.3.4 : An infinite long current filament is placed along z-axis. The magnetic field intensity at point P (3, 4, 0) is $10(-0.8\bar{a}_x + 0.6\,\bar{a}_y)$ A/m. Find the current through the filament.

☑ **Soln. : Given :** $\bar{H} = 10(-0.8\,\bar{a}_x + 0.6\,\bar{a}_y)$ (A/m)

The current filament is along z-axis, then the top view is as shown in the Fig. Ex. 6.3.4. Draw the perpendicular from P (3,4,0) on the current filament. It intersects the filament at (0,0,0). The vector in the direction of perpendicular towards P (3, 4, 0) is

$$\bar{R} = 3\bar{a}_x + 4\,\bar{a}_y$$

$$\therefore \quad \bar{a}_R = \frac{3\bar{a}_x + 4\,\bar{a}_y}{5}$$

The current filament is along z-axis, so $\bar{a}_l = \bar{a}_z$.

Then $$\bar{a}_\phi = \bar{a}_l \times \bar{a}_R$$

$$= \bar{a}_z \times \left(\frac{3\bar{a}_x + 4\,\bar{a}_y}{5}\right) = \frac{-4\bar{a}_x + 3\,\bar{a}_y}{5}$$

$\bar{H}$ due to infinite current filament is

$$\bar{H} = \frac{I}{2\pi R}\bar{a}_\phi = \frac{I}{2\pi(5)}\left(\frac{-4\bar{a}_x + 3\,\bar{a}_y}{5}\right)$$

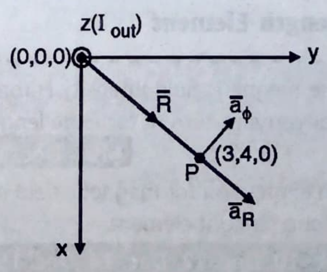


Fig. : Illustrating Ex. 6.3.4

Equating it with given $\overline{H}$ we get,

$$10(-0.8\overline{a_x}+0.6\overline{a_y}) = \frac{I}{2\pi\times 5}\left(\frac{-4\overline{a_x}+3\overline{a_y}}{8}\right)2(-4\overline{a_x}+3\overline{a_y})$$

$$= \frac{I}{50\times\pi}(-4\overline{a_x}+3\overline{a_y})$$

$\therefore$ $I = 100\,\pi = 314.159$ Amp. ...Ans.

UEx. 6.3.5

MU - May 09. May 12. Dec. 12. Dec. 15, 10 Marks

Prove that a static electric field is irrotational and the static magnetic field is solenoidal.

Soln. : Electric field is said to be irrotational if

$$\nabla\times\overline{E} = 0$$

Consider electric field due to infinite line charge

$$\overline{E} = \frac{\rho_l}{2\pi\varepsilon r}\overline{a_r} = E_r\overline{a_r}$$

In cylindrical curl is given by

$$\nabla\times\overline{E} = \frac{1}{r}\begin{vmatrix}\overline{a_r} & r\overline{a_\phi} & \overline{a_z}\\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z\\ E_r & rE_\phi & E_z\end{vmatrix} = \frac{1}{r}\begin{vmatrix}\overline{a_r} & r\overline{a_\phi} & \overline{a_z}\\ \partial/\partial r & 0 & 0\\ E_r & 0 & 0\end{vmatrix} = 0\text{... (Hence proved)}$$

Now the static magnetic field is solenoidal if

$$\nabla\cdot\overline{B} \text{ or } \nabla\cdot\overline{H} = 0$$

Consider $\overline{H}$ due to infinite current filament,

$$\overline{H} = \frac{I}{2\pi r}\overline{a_\phi} = H_\phi\overline{a_\phi}$$

In cylindrical the divergence is

$$\nabla\cdot\overline{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_r)+\frac{1}{r}\frac{\partial}{\partial\phi}(H_\phi)+\frac{\partial H_z}{\partial_z}$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial\phi}(H_\phi) = \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{I}{2\pi r}\right) = 0$$

...(Hence proved)

6.3.4 Magnetic Field Intensity Due to Finite Length Element

UQ. Determine magnetic field intensity H for the straight conductor carrying current for finite length.
MU - Dec. 11, 5 Marks

UQ. Derive an expression for magnetic field intensity due to finite long straight element.
MU - Dec. 10, Dec. 12, Dec. 14, Dec. 15, 10 Marks

Assumptions

i) Consider a current filament through which the current I amp is passing is placed along z-axis from $z = z_1$ to $z = z_2$.

ii) Our aim is to find the magnetic field intensity at any point. For simplicity consider point P in xy plane i.e. P $(\rho, \phi, 0)$.

- When we consider a differential length dz at distance z from origin then, its coordinates are $(0, \phi, z)$.
- The vector pointing from differential length dz to P is,

$$\overline{R} = \rho\,\overline{a_\rho} - z\,\overline{a_z}$$

- The unit vector in this direction is,

$$\overline{a_R} = \frac{\rho\overline{a_\rho}-z\,\overline{a_z}}{\sqrt{\rho^2+z^2}}$$

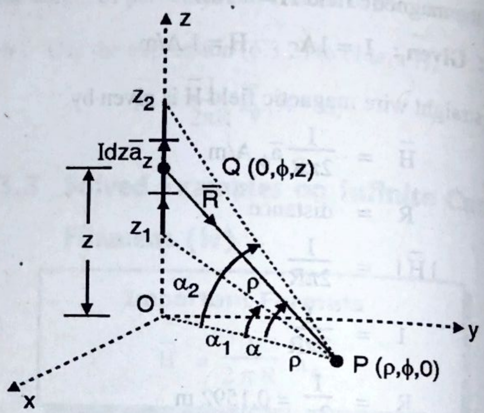


Fig. 6.3.4 : Finding H produced by finite length filament

- The differential current element at z is $I\,d\overline{l} = Idz\,\overline{a_z}$.
- Now according to Biot-savart law :

$$\overline{dH} = \frac{I\,d\overline{l}\times\overline{a_R}}{4\pi R^2} = \frac{Idz\,\overline{a_z}}{4\pi(\rho^2+z^2)}\times\left(\frac{\rho\overline{a_\rho}-z\,\overline{a_z}}{\sqrt{\rho^2+z^2}}\right)$$

- Using the cross product of unit vectors

$$\overline{a_z}\times\overline{a_\rho} = \overline{a_\phi} \quad\text{and}\quad \overline{a_z}\times\overline{a_z} = 0$$

We get, $$\overline{dH} = \frac{Idz}{(4\pi(\rho^2+z^2)^{3/2})}(\rho\overline{a_\phi})$$

- This is the differential field. To find total field intensity integrate the above result.

$$\overline{H} = \int\overline{dH} = \int_{z_1}^{z_2}\frac{I\rho dz}{4\pi(\rho^2+z^2)^{3/2}}\overline{a_\phi}$$

- Notice the triangle OPQ. The line PQ makes an angle α with base line OP, so,

$$\tan\alpha = \frac{OQ}{OP} = \frac{z}{\rho}$$

or $$z = \rho\tan\alpha$$

- When the differential filament at Q shifts to upper end (at $z = z_2$), the angle α will increase to α_2. But when differential filament is taken to lower end (at $z = z_1$), α decreases to α_1. To solve the integral,

Put $z = \rho \tan \alpha$ | limits: $z \rightarrow z_1,\ \alpha \rightarrow \alpha_1$

$\therefore$ $dz = \rho \sec^2 \alpha\, d\alpha$ | $z \rightarrow z_2,\ \alpha \rightarrow \alpha_2$

also, $(\rho^2 + z^2)^{3/2} = (\rho^2 + \rho^2 \tan^2 \alpha)^{3/2} = \rho^3 \sec^3 \alpha$

Putting in the expression for H

$$\bar{H} = \frac{I r}{4\pi} \int \frac{\rho \sec^2 \alpha\, d\alpha}{\rho^3 \sec^3 \alpha}\, \bar{a}_\phi$$

$$= \frac{I}{4\pi\rho} \int_{\alpha_1}^{\alpha_2} \cos \alpha\, d\alpha\, \bar{a}_\phi = \frac{I}{4\pi\rho} [\sin \alpha]_{\alpha_1}^{\alpha_2}$$

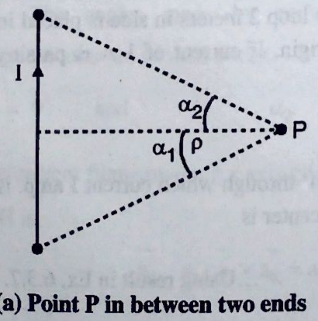
$$= \frac{I}{4\pi\rho}\, \bar{a}_\phi\, [\sin \alpha_2 - \sin \alpha_1]\, \bar{a}_\phi$$

i.e. $$\boxed{\bar{H} = \frac{I}{4\pi\rho} [\sin \alpha_2 - \sin \alpha_1]\, \bar{a}_\phi} \quad ...(6.3.4)$$

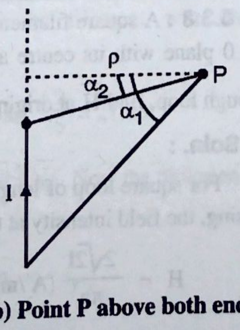
6.3.5 Procedure While Solving the Examples of Finite Length Filament

Procedure while solving the Examples of finite length filament is as follows :

- **Step 1 :** Draw perpendicular on the filament from the point where we desire a field.
- **Step 2 :** Connect upper and lower end of the filament with point P.
- **Step 3 :** Mark angles α_1 and α_2. Line connected from upper end to P makes angle α_2 with perpendicular and line connected from lower end to the point P makes angle α_1.
- **Step 4 :** Decide the signs of α_1 and α_2; while solving the problems following situations may occur.
  - (i) Point P between two ends (Fig. 6.3.5 (b)) here : α_1 is negative and α_2 is positive.
  - (ii) Point P above upper end Fig. 6.3.5. α_1 and α_2 both negative.
  - (iii) Point P below lower end (Fig. 6.3.5 (c)) both α_1 and α_2 positive.

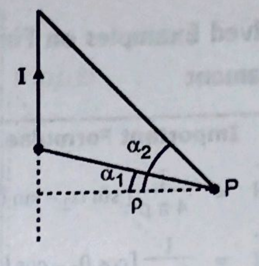


(a) Point P in between two ends



(b) Point P above both ends

Fig. 6.3.5(Contd...)



(c) Point P below both ends

Fig. 6.3.5

- **Step 5 :** Direction $\bar{a}_\phi$ is obtained using the formula (Equation 6.3.3).

$$\bar{a}_\phi = \bar{a}_l \times \bar{a}_R$$

where, $\bar{a}_R$ = unit vector in the direction of perpendicular towards, P and

$\bar{a}_l$ = unit vector in the direction of current.

- In Equation (6.3.4), the angles α_1 and α_2 are measured with respect to perpendicular on the filament. Instead if they are measured with the filament (θ_1 and θ_2 in Fig. 6.3.5(a)) then,

$$\alpha_2 = 90° - \theta_2 \quad \text{and} \quad \alpha_1 = 90° - \theta_1$$

Now the formula for H changes as,

$$\bar{H} = \frac{I}{4\pi\rho} [\sin (90 - \theta_2) - \sin (90 - \theta_1)]\, \bar{a}_\phi$$

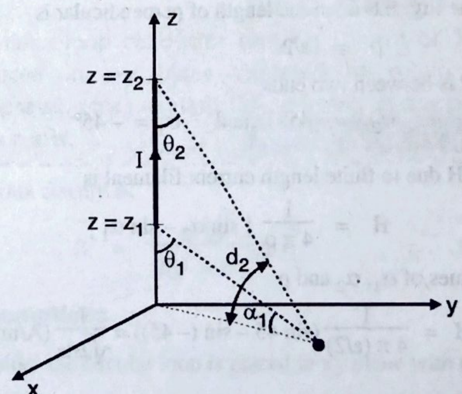


Fig. 6.3.5(d)

or $$\boxed{\bar{H} = \frac{I}{4\pi\rho} [\cos \theta_2 - \cos \theta_1]\, \bar{a}_\phi} \quad ...(6.3.4(a))$$

- When filament and point P are in the plane of the paper, $\bar{a}_\phi$ will be perpendicular to the page going into or coming out of the page. The inward or outward direction is decided by the current direction.

6.3.6 Solved Examples on Finite Length Filament

Important Formulae

$$\overline{H} = \frac{I}{4\pi\rho}[\sin\alpha_2 - \sin\alpha_1]\,\overline{a_\phi} \;;$$

$$\overline{H} = \frac{I}{4\pi\rho}[\cos\theta_2 - \cos\theta_1]\,\overline{a_\phi}$$

Ex. 6.3.6 : The finite length current element through which current of I amp. is passing placed from y = 0 to y = a. Find $\overline{H}$ at P (a/2, a/2).

OR

Find the field intensity at a point P due to a straight conductor carrying I as shown.

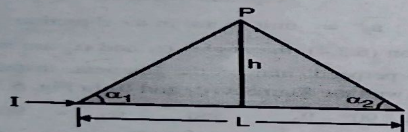


Fig. Ex. 6.3.6

☑ **Soln. :**

In the Fig. Ex. 6.3.6 the length of perpendicular is

$$\rho = a/2$$

As point P is between two ends

$$\alpha_2 = 45° \quad \text{and} \quad \alpha_1 = -45°$$

We know $\overline{H}$ due to finite length current filament is

$$\therefore \quad H = \frac{I}{4\pi\rho}[\sin\alpha_2 - \sin\alpha_1]$$

Putting values of α_1, α_2 and ρ

$$H = \frac{I}{4\pi(a/2)}(\sin 45 - \sin(-45)) = \frac{I}{\sqrt{2}\pi a} \text{ (A/m)}$$

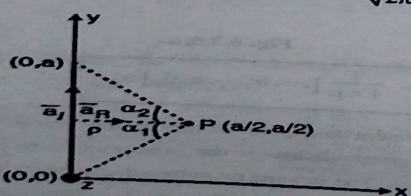


Fig.: Illustrating Ex. 6.3.6(a)

The current filament is along y-axis, so $\overline{a_l} = \overline{a_y}$. The unit vector in the direction of perpendicular towards point P is $\overline{a_R} = \overline{a_x}$.

Using the formula,

$$\overline{a_\phi} = \overline{a_l} \times \overline{a_R} = \overline{a_y} \times \overline{a_x} = -\overline{a_z}$$

Thus, the magnetic field intensity at point P is,

$$\overline{H} = \frac{-I}{\sqrt{2}\pi a}\,\overline{a_z} \text{ (into the page)}$$

...Ans.

Ex. 6.3.7 : Consider square loop of length 'a' through which current of I amp. is passing in clockwise direction. Find H at the centre of the square loop.

☑ **Soln. :** We know the magnetic field intensity due to finite length current filament H_{AD} is (See Ex. 6.3.6)

$$\overline{H_{AD}} = \frac{-I}{\sqrt{2}\pi a}\,\overline{a_z}$$

The square loop consists of 4 sides of length 'a' therefore the total magnetic field at the centre of the loop is obtained by adding field intensities due to 4 sides.

i.e. $$H = H_{AD} + H_{DC} + H_{CB} + H_{BA}$$

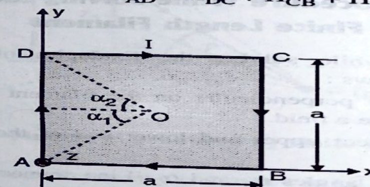


Fig. : Illustrating Ex. 6.3.7

We can easily check that field intensities due to all 4 elements are equal in magnitude and perpendicular to the page (going into)

$$\therefore \quad H = 4H_{AD} = \frac{4I}{\sqrt{2}\pi a} = \frac{2\sqrt{2}I}{\pi a} \text{ (A/m)}$$

Direction of H is into the page and perpendicular to it (i.e. $-\overline{a_z}$ direction)

...Ans.

Ex. 6.3.8 : A square filamentary loop 2 meters in side is placed in z = 0 plane with its centre at origin. If current of 1 A is passing through loop, find $\overline{H}$ at origin.

☑ **Soln. :**

For square loop of length 'a' through which current I amp. is passing, the field intensity at the center is

$$H = \frac{2\sqrt{2}I}{\pi a} \text{ (A /m)}$$

...Using result in Ex. 6.3.7.

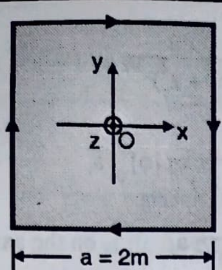


Fig. : Illustrating Ex. 6.3.8

Given : I = 1 A, a = 2m.

$$\therefore \quad H = \frac{2\sqrt{2} \times 1}{\pi \times 2} = 0.45 \text{ (A/m)} \qquad \text{...Ans.}$$

For the assumed direction of I through the loop, direction of H will be into the page (i.e. $-\bar{a}_z$ direction).

Ex. 6.3.9 : The conducting triangular loop in figure carries a current of 10 A. Find $\bar{H}$ at (0, 0, 5) due to side 1 of the loop.

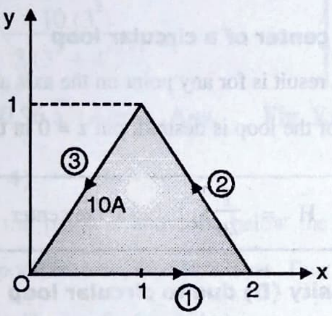


Fig. Ex. 6.3.9(a)

☑ Soln. :

Note that the point of interest is on z-axis and the side 1 of the triangle is on x-axis. No need to consider other sides.

The perpendicular from point P on the current filament is at (0, 0, 0). The vector and unit vector in the direction of perpendicular towards point P is,

$$\bar{R} = 5\,\bar{a}_z, \quad \text{and} \quad \bar{a}_R = \bar{a}_z$$

The angles made by ends of the filament with the perpendicular are (dotted lines),

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_2 = \tan^{-1}\frac{2}{5} = 21.80°$$

The current filament along x axis gives $\bar{a}_l = \bar{a}_x$. Now the direction of H is,

$$\bar{a}_\phi = \bar{a}_l \times \bar{a}_R = \bar{a}_x \times \bar{a}_z = -\bar{a}_y$$

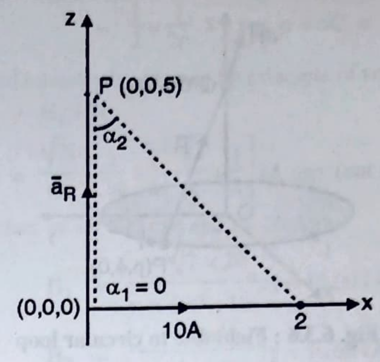


Fig. Ex. 6.3.9(b)

The field intensity due to finite filament is,

$$\bar{H} = \frac{I}{4\pi R}\,[\sin\alpha_2 - \sin\alpha_1]\,\bar{a}_\phi$$

$$\bar{H} = \frac{10}{4\pi(5)}\,[\sin(21.80) - \sin(0)]\;(-\bar{a}_y)$$

$$\bar{H} = -59.11\,\bar{a}_y \text{ (mA/m)} \qquad \text{...Ans.}$$

6.3.7 Magnetic Field Intensity on the Axis of a Circular Loop

UQ. Circular loop conductor carrying a current of 'I' A is placed in x-y plane centred at origin. Find expression for –H at any point P on z-axis. **MU - May 11, 10 Marks**

UQ. Circular loop conductor carrying current of 'I' A is placed in x-y plane centered at origin. Find expression for magnetic field intensity at any point P on z-axis. **MU - May 16, 10 Marks**

Magnetic flux density is,

$$\bar{B} = \frac{\mu I \rho^2}{2(\rho^2 + z^2)^{3/2}}\,\bar{a}_z \qquad \because \bar{B} = \mu\bar{H}$$

Assumptions

i) Consider the circular loop is placed in xy plane with centre at the origin.

ii) The z - axis is the axis of the loop and field is desired at a point on this axis i.e. point Q (0,ϕ, z).

iii) Let current I is passing through the loop in anticlockwise direction.

- The differential current filament at point P (ρ, ϕ, 0) is -

$$I\,d\bar{l} = I\,\rho\,d\phi\,\bar{a}_\phi$$

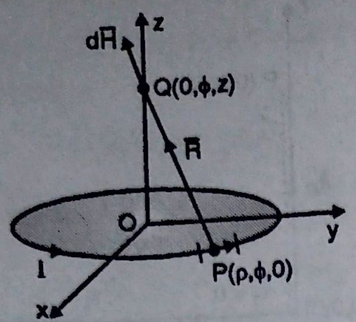


Fig. 6.3.6 : Field due to circular loop

- The vector joining P to Q is

$$\bar{R} = -\rho\bar{a}_\rho + z\,\bar{a}_z \; ; \quad \therefore \; \bar{a}_R = \frac{-\rho\bar{a}_\rho + z\bar{a}_z}{\sqrt{\rho^2 + z^2}}$$

- The field intensity at point Q due to differential current filament is obtained by using Biot-savart law as

$$d\bar{H} = \frac{I\, d\bar{l} \times \bar{a}_R}{4\pi R^2} = \frac{I\rho d\phi\, \bar{a}_\phi}{4\pi(\rho^2+z^2)} \times \left(\frac{-\rho\bar{a}_\rho + z\bar{a}_z}{\sqrt{\rho^2+z^2}}\right)$$

We have, $\bar{a}_\phi \times \bar{a}_\rho = -\bar{a}_z$ and $\bar{a}_\phi \times \bar{a}_z = \bar{a}_\rho$

- Then field intensity,

$$d\bar{H} = \frac{I\rho d\phi}{4\pi(\rho^2+z^2)^{3/2}} (\rho\,\bar{a}_z + z\,\bar{a}_\rho) \quad \text{...(A)}$$

- This is the differential field intensity. The total is obtained by integrating it.
- Equation (A), shows that $d\bar{H}$ has two components ($\bar{a}_r$ and $\bar{a}_z$). When we considered a filament at $(\rho, \phi, 0)$ in above discussion, there is one more small filament at exactly diametrically opposite side, point P′ in Fig. 6.3.7.
- The filed intensity due to differential filament at P′ also has two components.
- When these two field intensities (dH and dH′) added, the horizontal components of it get cancelled and only vertical components remain. Thus the summation or integration results in only vertical components, therefore

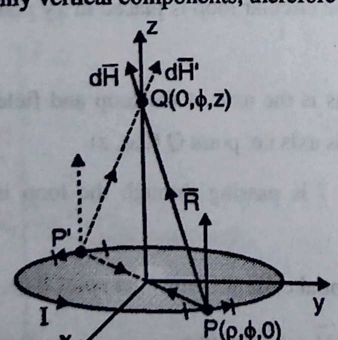


Fig. 6.3.7 : Showing cancellation of horizontal components

$$\bar{H} = \int_0^{2\pi} \frac{I\rho^2}{4\pi} \frac{d\phi}{(\rho^2+z^2)^{3/2}} \bar{a}_z$$

$$= \frac{I\rho^2}{4\pi(\rho^2+z^2)^{3/2}} [\phi]_0^{2\pi}\, \bar{a}_z$$

or $$\bar{H} = \frac{I\rho^2}{2(\rho^2+z^2)^{3/2}} \bar{a}_z \quad \text{... on the axis} \quad \text{...(6.3}$$

Note

- Even if we consider a point Q below the circular loop, the direction of $\bar{H}$ remains same. This is because if z replaced by – z, the z-component of dH remain the same while ρ component adds upto zero.
- One more important conclusion can be drawn from Equation (6.3.5) is that **H at any point on the axis is always perpendicular to plane of the circular loop.**
- The direction of H is upward or downward is obtained again by the right hand rule.

$\bar{H}$ at the center of a circular loop

The above result is for any point on the axis at a distance z. If $\bar{H}$ at the center of the loop is desired, put z = 0 in the above result then.

$$\boxed{\bar{H} = \frac{I}{2\rho}\bar{a}_z \quad \text{... at the center}} \quad \text{...(6.3.6}$$

Flux density (B) due to circular loop

Using $\bar{B} = \mu\bar{H}$, $$\bar{B} = \frac{\mu I\rho^2}{2(\rho^2+z^2)^{3/2}}\bar{a}_z \quad \text{...(on the axis}$$

$$= \mu\frac{I}{2\rho}\bar{a}_z \quad \text{...(at the center}$$

6.3.8 Solved Examples on Magnetic Field Intensity of a Circular Loop

Important Formulae

$$\bar{H} = \frac{I\rho^2}{2(\rho^2+z^2)^{3/2}}\bar{a}_z \; ;$$

$$\bar{H} = \frac{I}{2\rho}\bar{a}_z$$

Ex. 6.3.10 : A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries direct current of 10 A along $\bar{a}_\phi$. Determine $\bar{H}$ at (0, 0, 4) and (0, 0, – 4).

☑ Soln. :

The given circular loop is in xy plane (z = 0), with centre at origin. The standard equation for circle is

$$x^2 + y^2 = \rho^2$$

Comparing with given equation, the radius of the circle is $\rho = 3$. We have to find $\bar{H}$ at (0, 0, 4) and (0, 0, – 4), both these points are on z-axis, which is the axis of the loop.

$\bar{H}$ at (0, 0, 4)

The magnetic field intensity on the axis of the circular loop is,

$$\bar{H} = \frac{I\rho^2}{2(\rho^2 + z^2)^{3/2}}\, a_n$$

Substituting the values, r = 3, z = 4, and unit vector normal to plane of the loop, $\bar{a}_n = \bar{a}_z$ we get

$$\bar{H} = \frac{10\,(3^2)}{2\,(3^2 + 4^2)^{3/2}}\, \bar{a}_z$$

$$= 0.36\, \bar{a}_z \text{ (A/m)...Ans.}$$

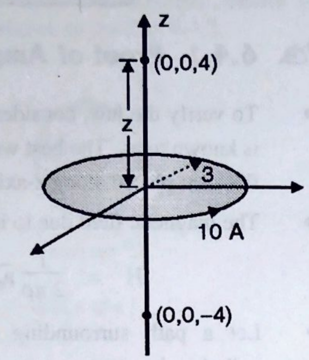


Fig. Ex. 6.3.10

$\bar{H}$ at (0, 0, – 4)

Whether the point is above or below the loop, when the distance of the point is same, then $\bar{H}$ is same. Thus

$$\bar{H} = 0.36\, \bar{a}_z \text{ (A/m).} \qquad \text{...Ans.}$$

Ex. 6.3.11 : The current is passing through loop as shown in Fig. Ex. 6.3.11, find H at P, if a = 600 mm and I = 20 Amp. Find $\bar{H}$.

☑ Soln. : For a half square loop the field intensity is :

$$H_1 = \frac{1}{2} \times \text{H due to square loop} = \frac{1}{2} \times \frac{2\sqrt{2}I}{\pi a} = \frac{\sqrt{2}I}{\pi a}$$

Due to half circular loop the field intensity is :

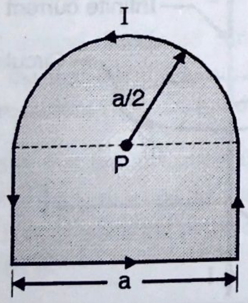


Fig. : Illustrating Ex. 6.3.11

$$H_2 = \frac{1}{2} \times \text{H due to circular loop}$$

$$= \frac{1}{2} \times \frac{I}{2\rho} \qquad \text{but } \rho = a/2 = \frac{I}{2a}$$

The total filed intensity is obtained by principle of superposition :

$$H = H_1 + H_2$$

$$= \frac{\sqrt{2}I}{\pi a} + \frac{I}{2a} = \frac{I}{a}\left[\frac{\sqrt{2}}{\pi} + \frac{1}{2}\right] \text{(A /m) (out of the page.)}$$

For given values of a = 600 mm and I = 20 Amp.

$$H_1 = \frac{\sqrt{2} \times 20}{\pi \times 600 \times 10^{-3}} = 15.00 \text{ (A /m)} \quad \text{...Ans.}$$

and

$$H_2 = \frac{20}{2 \times 600 \times 10^{-3}} = 16.66 \text{ (A /m)} \quad \text{...Ans.}$$

$$\therefore \quad H = H_1 + H_2 = 31.667 \text{ (A /m).} \quad \text{...Ans.}$$

Ex. 6.3.12 : An infinite long straight wire carries a current of 1 kA. A loop 20 mm in diameter is situated with its centre 1m from wire. How must the loop be turned and what current is required in it to reduce the field from the long wire to be zero at the centre of the loop.

☑ Soln. : Magnetic field intensity at the centre of loop due to infinite current element is,

$$\bar{H} = \frac{I}{2\pi R}\, \bar{a}_\phi = \frac{1 \times 10^3}{2\pi \times 1}\, \bar{a}_\phi$$

The direction is into the paper.

To nullify H, we must place the loop in the plane of the paper with its centre at P and the current in anticlockwise direction. The direction of $\bar{H}$ at the center of the loop is out of the paper.

H at the centre of loop due to loop current I′ is $= \frac{I'}{2\rho}$

Now to nullify H at the center

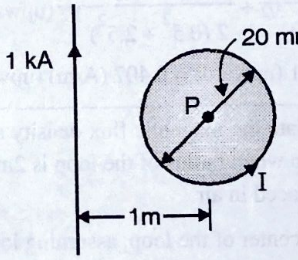


Fig . : Illustrating Ex. 6.3.12

$$\frac{1 \times 10^3}{2\pi} = \frac{I'}{2\rho}$$

$$\therefore \quad I' = \frac{1 \times 10^3 \times 10 \times 10^{-3}}{\pi} = 3.18 \text{ Amp....Ans.}$$

UEx. 6.3.13 MU - Dec. 10, 10 Marks

Two circular coils are located at the z = 0 plane and z = 5 m plane, cantered about the z-axis. The first coil having a radius of 1 m carries a current of 10 Amp. The second coil having a radius of 0.5 m carries a current of 20 Amp. Calculate the magnetic filed intensity H at (0, 0, 2.5 m).

☑ **Soln. :**

The magnetic field intensity on the axis of a circular coil having radius r and at a distance z from the plane of the coil is,

$$H = \frac{I\rho^2}{2(\rho^2 + z^2)^{3/2}}$$

Assume direction of currents in both the coils to be same then the direction of H at point M is same irrespective of the position of M above or below the coil. Hence, total H is

$$\bar{H} = \bar{H}_1 + \bar{H}_2$$

$$\bar{H} = \frac{I_1 \rho_1^2}{2(\rho_1^2 + z_1^2)^{3/2}} + \frac{I_2 \rho_2^2}{2(\rho_2^2 + z_2^2)^{3/2}} \text{ (upward)}$$

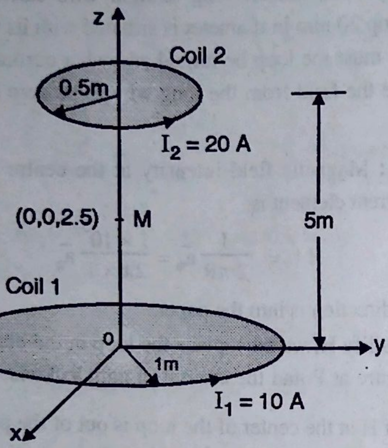


Fig. : Illustrating Ex. 6.3.13

$$= \frac{10 \times 1^2}{2(1^2 + 2.5^2)^{3/2}} + \frac{20 \times 0.5^2}{2(0.5^2 + 2.5^2)^{3/2}} \text{ (upward)}$$

$= 0.256 + 0.151$ (upward) $= 0.407$ (A/m) (upward) ...Ans.

Ex. 6.3.14 : Calculate the magnetic flux density at the center of a current carrying loop when radius of the loop is 2m, loop current of 1 mA, and loop is placed in air.

☑ **Soln. :** At the center of the loop, assuming loop is horizontal,

$$\bar{H} = \frac{I}{2\rho}\bar{a}_z$$

Then,

$$\bar{B} = \mu_0 \bar{H} = 4\pi \times 10^{-7} \times \frac{1 \times 10^{-3}}{2 \times 2}\bar{a}_z$$

$$\bar{B} = \pi \times 10^{-10}\bar{a}_z = 0.314\,\bar{a}_z \text{ (nWb / m}^2\text{)}$$

6.4 AMPERE'S CIRCUITAL LAW OR AMPERE'S WORK LAW

UQ. State and explain Amperes circuital law.
MU - Dec. 09, 5 Marks

UQ. Explain : Ampere's circuital law.
MU - May 12, 5 Marks

UQ. Write integral form of Ampere's Law and interpret the same.
MU - Dec. 14, 5 Marks

Statement

The line integral of $\bar{H}$ around a single closed path is equal to the current enclosed by that path.

Mathematically,
$$\boxed{\oint \bar{H} \cdot d\bar{l} = I} \quad ...(6.4.1)$$

6.4.1 Proof of Ampere's Circuital Law

- To verify the law, consider a source whose magnetic filed is known to us. The best well known source is infinite current filament placed along z-axis (Refer Fig. 6.4.1(a)).
- The magnetic field due to it is -

$$\bar{H} = \frac{I}{2\pi\rho}\bar{a}_\phi$$

- Let a path surrounding it is circular, for this path in cylindrical system,

$$d\bar{l} = d\rho\,\bar{a}_\rho + \rho d\phi\,\bar{a}_\phi + dz\,\bar{a}_z$$

- If the path has radius 'ρ' and it is horizontal then $d\rho = dz = 0$ and the reduced expression for $d\bar{l}$ is

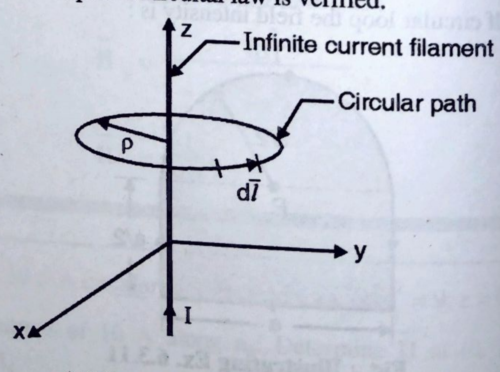
$$d\bar{l} = \rho d\phi\,\bar{a}_\phi$$

Now
$$\bar{H} \cdot d\bar{l} = \frac{I}{2\pi\rho}\bar{a}_\phi \cdot (\rho d\phi\,\bar{a}_\phi) = \frac{I}{2\pi}d\phi$$

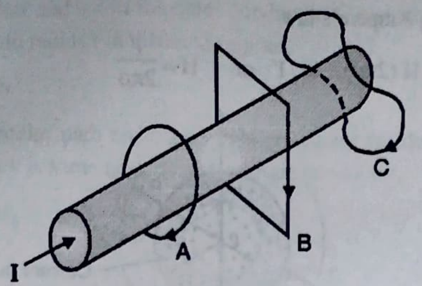
- The Amperes circuital law is $\oint \bar{H} \cdot d\bar{l} = I$

$$\text{L. H. S} = \oint \bar{H} \cdot d\bar{l} = \frac{I}{2\pi}\int_0^{2\pi} d\phi = \frac{I}{2\pi} \times (2\pi) = I = \text{RHS}$$

- Hence Ampere's circuital law is verified.



(a) Verifying amperes circuital law
Fig. 6.4.1(Contd...)



(b) Showing different closed path

Fig. 6.4.1

- Consider the conductor shown in Fig. 6.4.1(b), carries a current I. Different paths are shown in the Fig. 6.4.2.

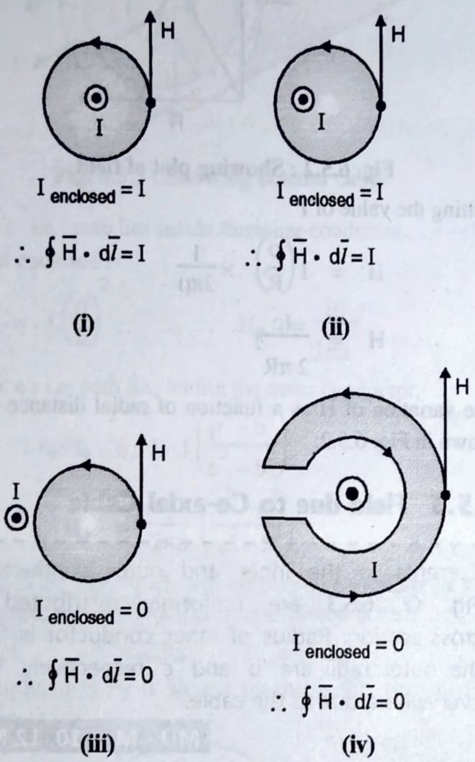


Fig. 6.4.2 : Showing determination of $\oint \bar{H} \cdot d\bar{l}$

- Path A and B enclose the conductor, therefore the line integral of $\bar{H}$ around the closed path A and B is equal to current I but the line integral around path C is less than I, since the entire current is not enclosed by the path.
- Fig. 6.4.2 shows that the line integral $\bar{H}$ around closed path equals current in the wire when path enclose the wire (Fig. (i) and (ii)), but is zero when the paths do not enclose the wire (Fig. (iii) and (iv)).

6.5 APPLICATIONS OF AMPERE'S LAW

GQ. Discuss applications of Ampere circuital law.(8 Marks)

- By Equation (6.4.1), one would think that, Ampere's law is used to determine current I. Instead, the current is usually known and using this law we find H. But for this two conditions must be satisfied :

1. At each point of the closed path $\bar{H}$ is either tangential or normal to the path.
2. H has the same value at all points of the path where $\bar{H}$ is tangential.

- In order to satisfy these conditions, proper selection of the path is required.

Applications

i) To find field due to thin current filament.

ii) To find field due to solid cylindrical conductor.

iii) To find field due to co-axial cable.

iv) To find field due to current sheet.

These applications are discussed next.

6.5.1 Field due to an Infinite Thin Line Current Filament

- In section 6.4, we have seen the magnetic field intensity due to infinite current filament is,

$$\bar{H} = \frac{I}{2\pi\rho}\bar{a}_\phi$$

- where, r is the radius of circular path about z axis with center on z-axis. At any point on this circular path, magnitude of $\bar{H}$ is constant (since ρ is constant).
- Similarly when we take any section of circular path $(d\bar{l} = \rho\, d\phi\, \bar{a}_\phi)$, $\bar{H}$ is tangential, to it (since direction of $\bar{H}$ and $d\bar{l}$ is same).
- Because $\bar{H}$ is tangential the dot product in Equation (6.4.1) can be replaced by multiplying the magnitudes.
- Also as $\bar{H}$ is constant it can be taken outside the integral sign in Equation (6.4.1). Then closed line integral of dl results in path length 2πr, in short

$$\oint \bar{H} \cdot d\bar{l} = H\,(2\pi\rho)$$

- According to Ampere's law this is equal to I.

$$\therefore \quad H\,(2\pi\rho) = I \quad \text{or} \quad H = \frac{I}{2\pi\rho}$$

- Since H has only φ component,

$$H_\phi = \frac{I}{2\pi\rho} \quad \text{...(6.5.1)}$$

- The above procedure can be applied to a single conductor or co-axial cable to find magnetic fields in different regions due to current in the conductor or cable.

6.5.2 Field due to a Solid Cylindrical Conductor

UQ. For infinite long conductor of radius 'a' carrying current I, determine Magnetic field everywhere. **(MU - Q. 2(b), Dec. 18, 10 Marks)**

- For $\rho > R$ (i.e. outside the wire) $\bar{H} = \frac{I}{2\pi\rho}$

$$B = \mu H = \frac{\mu I}{2\pi\rho} \text{ Magnetic flux density}$$

- For $\rho \leq R$ (i.e. inside the wire) $H = \frac{I\rho}{2\pi R^2}$
- Magnetic flux density is, $B = \mu H \therefore B = \frac{\mu I_\rho}{2\pi R^2}$
- A single circular conductor having radius R and uniform current density is shown in Fig. 6.5.1. There are different regions where we can find magnetic field intensities.
- The different regions are inside the conductor, and outside the conductor. To find H in these regions consider circular path with radius ρ.
- **For $\rho \geq R$** (i.e. outside the wire)

Since the current I is enclosed by the path as path is outside the wire

$$H(2\pi\rho) = I \text{ or } H = \frac{I}{2\pi\rho}$$

- **For $\rho \leq R$** (i.e. inside the wire)
- The circular path for $\rho < R$, enclose part of the conductor. The current enclosed will be less than I, let it be I′.

$$\therefore \quad I' = I\left(\frac{\rho}{R}\right)^2$$

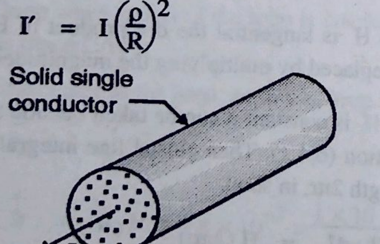


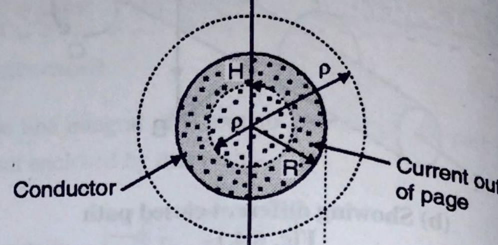
Fig. 6.5.1 : Showing a solid conductor

- This value is obtained by using the fact that current I flows through area (πR^2). If the area enclosed by the circular path is πr^2, the current enclosed will be

$$I' = I\left(\frac{\pi\rho^2}{\pi R^2}\right) = I\left(\frac{\rho}{R}\right)^2$$

- Using Ampere's law

$$H(2\pi\rho) = I' \text{ or } H = \frac{I'}{2\pi\rho}$$



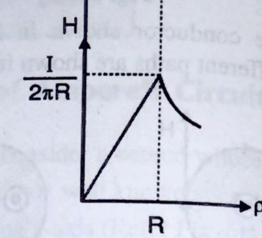


Fig. 6.5.2 : Showing plot of field

- Putting the value of I′

$$H = I\left(\frac{\rho}{R}\right)^2 \times \frac{1}{2\pi\rho}$$

Or

$$H = \frac{I\rho}{2\pi R^2}$$

- The variation of H as a function of radial distance (i.e.ρ) shown in Fig. 6.5.2.

6.5.3 Field due to Co-axial Cable

UQ. Currents in the inner and outer conductors of Fig. Q. 6.5.3 are uniformly distributed over cross-section. Radius of inner conductor is 'a' and the outer radii are 'b' and 'c' respectively. Find B everywhere due to the cable. **MU - May 10, 12 Marks**

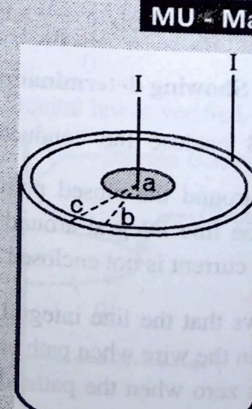


Fig. Q. 6.5.3

- Consider an infinitely long co-axial transmission line carrying a uniformly distributed current I in the centre conductor and – I in the outer conductor. Consider a circular path with radius r in different regions.

For a < ρ < b

- The circular path encloses single conductor inside therefore the result is same as derived for single conductor.

$$\therefore \; H_\phi = \frac{I}{2\pi r\rho}$$

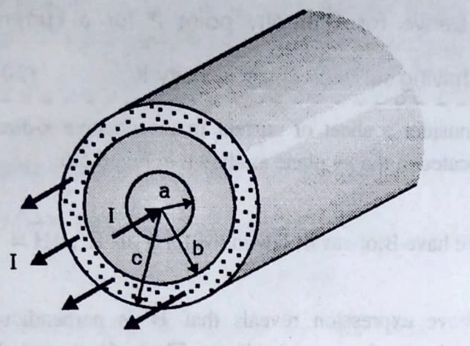


Fig. 6.5.3 : Showing coaxial cable

For ρ < a : i.e. path lies inside the inner conductor.

The current enclosed is

$$I' = I\left(\frac{\rho}{a}\right)^2 \; ; \qquad \therefore \; H_\phi = \frac{I\rho}{2\pi a^2}$$

For b< ρ < c : i.e. path lies within the outer conductor,

We have $2\,\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$

$$\therefore \quad H_\phi = \frac{I}{2\pi\rho}\left(\frac{c^2-\rho^2}{c^2-b^2}\right)$$

- For ρ > c : This path will enclose two currents in opposite direction, therefore total current enclosed is zero.

  Hence, $H_\phi(2\pi\rho) = 0$ or $H_\phi = 0$

- Multiplying H by μ we get magnetic flux density B in all regions.

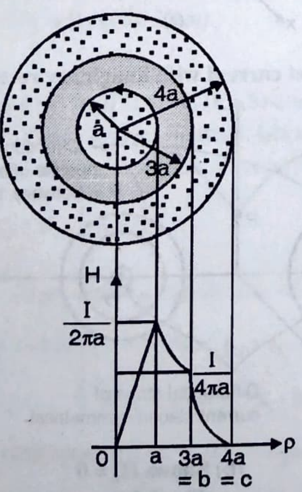


Fig. 6.5.4 : Coaxial cable and its field

- This gives a very important result; **no magnetic field exist outside the co-axial cable.** The variation of H versus distance ρ is plotted in Fig. 6.5.4.

6.5.4 Ampere's Law Applied to a Conducting Medium

- Suppose that the origin of the coordinates is situated inside a conducting medium of large extent.
- Let the current density in the medium be $\bar{J}$ (A / m^2) in the positive y direction as shown in Fig. 6.5.5.

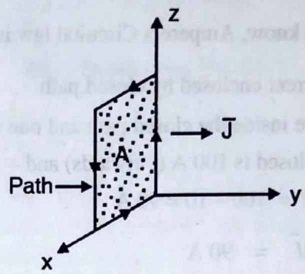


Fig. 6.5.5 : Rectangular area (A) in conducting medium

- According to Ampere's law, the line integral of $\bar{H}$ around the rectangular path enclosing the area A is equal to current enclosed. In this case, the current I enclosed by the path is given by the integral of the normal component of $\bar{J}$ over the surface A, or

$$\boxed{\oint \bar{H}\cdot d\bar{l} = \iint_S \bar{J}\cdot d\bar{s} = I} \qquad ...(6.5.2)$$

- Thus Ampere's circuital law now can be expressed in $\bar{J}$ as

$$\boxed{\oint \bar{H}\cdot d\bar{l} = I_{enclosed} = \int_S \bar{J}\cdot d\bar{s}}$$

6.5.5 Solved Examples on Applications of Ampere's Law

Important Formulae

$$\oint \bar{H}\cdot d\bar{l} = I_{enclosed} = \int_S \bar{J}\cdot d\bar{s}$$

$$H_\phi = \frac{I}{2\pi\rho} \; ; \qquad I = \int_S \bar{J}\cdot d\bar{s} \; ;$$

$$\oint \bar{H}\cdot d\bar{l} = \iint_S \bar{J}\cdot d\bar{s} = I$$

Ex. 6.5.1 : For the configuration shown calculate the value of $\oint \bar{H} \cdot d\bar{l}$.

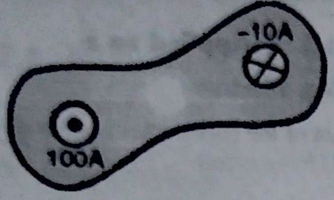


Fig. Ex. 6.5.1

Soln. : As we know, Ampere's Circuital law is

$\oint \bar{H} \cdot d\bar{l} = I$ = Current enclosed by closed path

Here two points are inside the closed path and one is outside.

$\therefore$ The current enclosed is 100 A (outwards) and – 10 A (inwards).

$\therefore$ Total current is = 100 – 10 = 90 A

$$\therefore \oint \bar{H} \cdot d\bar{l} = 90 \text{ A}$$

The current shown outside the closed path has no effect on the integral.

Ex. 6.5.2 : Given $\bar{J} = 10^3 \sin\theta\, \bar{a}_r$ A/m^2 in spherical coordinates, find the current crossing the spherical shell of r = 0.02 m, where r = radius of shell.

Soln. :

We have $I = \int_S \bar{J} \cdot \bar{ds}$

The differential area for the spherical shell is given by

$$\bar{ds} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r = (0.02)2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$$

Using this expression :

$$I = \int_0^{2\pi} \int_0^{\pi} (10^3 \sin\theta\, \bar{a}_r) \cdot (0.02^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r)$$

$$= 10^3 \times (0.02)^2 [\phi]_0^{2\pi} \times \int_0^{\pi} \sin^2\theta\, d\theta$$

$$= 2.513 \times \int_0^{\pi} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta$$

$$= 2.513 \left\{\frac{1}{2}[\theta]_0^{\pi} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi}\right\}$$

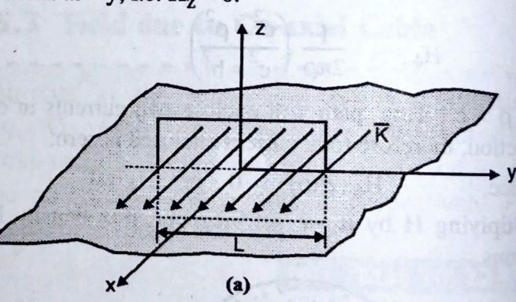
$$I = 1.257\pi = 3.947 \text{ Amp.}$$

...Ans.

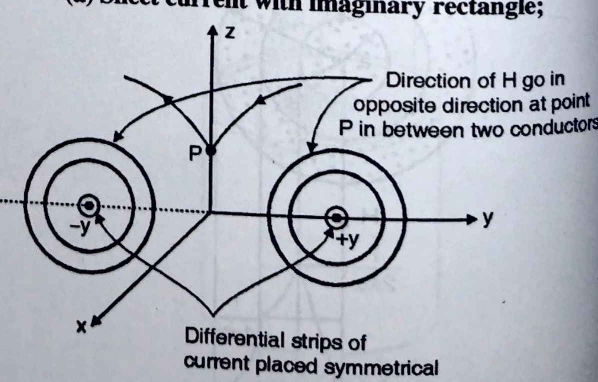
6.5.6 Field due to Uniform Current Sheet Density [K]

> **GQ.** What is the positive direction of current. Use this law to find magnetic field intensity due to an infinite current sheet. **(5 Marks)**
>
> **GQ.** Derive for $\bar{H}$ at any point P for a current sheet having surface current density $\bar{K}$. **(10 Marks)**

- Consider a sheet of current in the positive x-direction and located in the xy plane as shown in Fig. 6.5.6.
- We have Biot-savart law in the form of $\bar{K}$, $d\bar{H} = \dfrac{\bar{K} ds \times \bar{a}_R}{4\pi R^2}$
- Above expression reveals that $\bar{H}$ is perpendicular to $\bar{K}$ (because of cross product). Thus, H cannot have any x-component i.e. $H_x = 0$.
- Now let us check whether H has z-component ?
- The solution can be carried out by dividing the sheet into number of thin horizontal strips.
- Consider two such strips, which are on either side of the x-axis and equidistant from it (Fig. 6.5.6(b)).
- Applying right hand rule to these strips we find that a field due to filament at + y is cancelled by the corresponding filament at – y, i.e. $H_z = 0$.



(a) Sheet current with imaginary rectangle;



(b) Shows $H_z = 0$

Fig. 6.5.6

- Hence, H can only have a y-component i.e. H_y. Because the current sheet is infinite in extent in x and y direction, $\bar{H}$ cannot vary with x or y.
- Since $\bar{H}$ has only H_y component, we choose a rectangular loop composed of straight line segments which are either parallel or perpendicular to H_y as shown in Fig. 6.5.6(a).
- Let it be equal distance above and below the surface z = 0. We have Ampere's circuital law :

$$\oint \bar{H} \cdot d\bar{l} = I$$

- Applying it to rectangular path discussed above :

$$(H_y)(L) + 0 + (H_y)(L) + 0 = I = KL$$

i.e. $2H_y = K$ or $H_y = K/2$

- Thus, for all z > 0, $\bar{H} = (K/2)\ \bar{a}_y$ and for all z < 0, $\bar{H} = -(K/2)\ \bar{a}_y$.
- More generally, for an arbitrary orientation of the current sheet,

$$\boxed{\bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_n} \quad ...(6.5.3)$$

6.5.7 Solved Examples on Field Due to Uniform Current Sheet Density [K]

Important Formula

$$\bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_n$$

Ex. 6.5.3 : A current density $6\ \bar{a}_x$ A/m lies in the z = 0 plane and a current filament is located at y = 0, z = 4m. Determine the current I and its direction if $\bar{H} = 0$ at (0,0,1.5m).

Soln. :

The total magnetic field H at (0, 0, 1.5) consists of H due to current sheet and H due to current filament. Let us find these field intensities separately. Due to current sheet we have,

$$\bar{H}_s = \frac{1}{2}\bar{K} \times \bar{a}_n$$

where, $\bar{a}_n$ is a unit vector perpendicular to the sheet charge.

For the given sheet charge in z = 0 plane, $\bar{a}_n = \bar{a}_z$, then,

$$\bar{H}_s = \frac{1}{2}(6\ \bar{a}_x \times \bar{a}_z) = -3\ \bar{a}_y\ (A/m)$$

In order to make total field at (0,0,1.5) equal to zero, H due to current filament must by $+3\ \bar{a}_y$, i.e. magnitude equal to 3 (A/m) in the $\bar{a}_y$ direction.

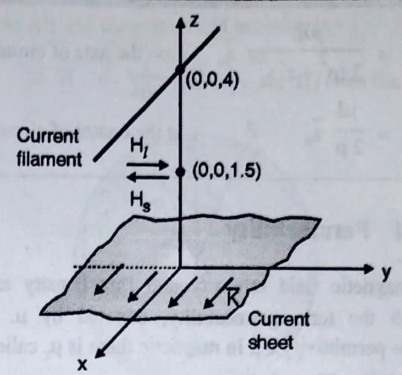


Fig. : Illustrating Ex. 6.5.3

i.e. $|H| = \dfrac{I}{2\pi R} = 3$

We know that R is the perpendicular distance between current filament and the point where we desire the field, in the Fig. Ex. 6.5.3 it is equal to 2.5 m.

$$\therefore \quad 3 = \frac{I}{2\pi \times 2.5}$$

or $I = 47.12$ Amp. ...Ans.

In order to have H due to this current $\bar{a}_y$ direction, using right hand thumb rule we get direction of I in the $\bar{a}_x$ direction. Thus, current of 47.12 amp in the $\bar{a}_x$ direction makes the field at (0, 0, 1.5) equal to zero.

6.6 MAGNETIC FLUX DENSITY

- Similar to electric flux density, magnetic flux density is also present in the medium. It is defined as magnetic flux per unit area, denoted by B. B is related with magnetic field intensity (H) by the relation.

$$\boxed{\bar{B} = \mu \bar{H}} \quad ...(6.6.1)$$

Where,

$\bar{B}$ = flux density T ;

μ = permeability of medium, H/m ;

$\bar{H}$ = Magnetic field intensity, A/m.

Direction of $\bar{B}$ and $\bar{H}$ is same.

- We have already derived expressions for H due to different types of current filaments. Using Equation (6.6.1), we can write expressions for B due to these current filaments directly.

$$\bar{B} = \frac{\mu I}{2\pi\rho}\bar{a}_\phi \quad \text{....due to infinite filament.}$$

$$\bar{B} = \frac{\mu I}{4\pi\rho}[\sin\alpha_2 - \sin\alpha_1]\ \bar{a}_\phi \quad \text{...due to finite filament.}$$

$$\bar{B} = \frac{\mu I \rho^2}{2(\rho^2+z^2)^{3/2}}\bar{a}_n \quad \text{....on the axis of circular loop.}$$

$$\bar{B} = \frac{\mu I}{2\rho}\bar{a}_n \quad \text{...at the center of circular loop.}$$

6.6.1 Permeability

- The magnetic field intensity and flux density are related through the term permeability, denoted by μ. Just like relative permittivity ϵ_r, in magnetic there is μ_r called relative permeability. The relation is,

$$\mu = \mu_0 \mu_r$$

Here, μ = absolute permeability or sometimes just permeability of the medium and is measured in H/m.

μ_r = relative permeability

μ_0 = permeability of free space.

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ (H/m)}}$$

- The term μ_r is also expressed as,

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

Where, χ_m = magnetic susceptibility (dimensionless)

- The term χ_m and therefore μ_r can be a function of space coordinates. For a simple medium-linear, isotropic, and homogeneous-χ_m and μ_r are constants.
- The permeability of most materials is very close to that of free space (μ_0).
- For ferromagnetic materials such as iron, nickel, and cobalt, μ_r could be very large (50-5000 and upto 10^6 or more for special alloys).
- The permeability depends not only on the magnitude of $\bar{H}$ but also on the previous history of the material.

6.6.2 Solved Examples on Magnetic Flux Density

Important Formulae

$$\bar{B} = \mu\bar{H}; \quad \bar{B} = \frac{\mu I}{2\pi\rho}\bar{a}_\phi;$$

$$\bar{B} = \frac{\mu I}{4\pi\rho}[\sin\alpha_2 - \sin\alpha_1]\,\bar{a}_\phi$$

$$\bar{B} = \frac{\mu I \rho^2}{2(\rho^2+z^2)^{3/2}}\bar{a}_n$$

$$\bar{B} = \frac{\mu I}{2\rho}\bar{a}_n$$

Ex. 6.6.1 : Two identical circular loops of 1m radius are situated side by side on a common axis. The distance between the loops is 1m. If both loops carry a current of 1A in the same direction, find B (a) at the center of one loop and (b) at a point midway between the loops on their common axis.

Soln. :

(a) **Flux density B at point P due to loop 1 is**

$$= \frac{\mu I}{2\rho} = \frac{4\pi \times 10^{-7} \times 1}{2}$$

$$= 6.28 \times 10^{-7} \text{ (Wb/m}^2\text{) or (T)}$$

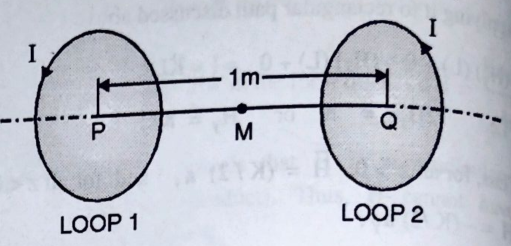


Fig. : Illustrating Ex. 6.6.1

Flux density B at point P due to loop 2,

$$= \frac{\mu I \rho^2}{2(\rho^2+z^2)^{3/2}} = \frac{4\times10^{-7}\times1\times1}{2(1+1)^{3/2}} = 2.22\times10^{-7} \text{ (T)}$$

∴ The Total B at point P is,

$$= 6.28\times10^{-7} + 2.22\times10^{-7} = 8.50\times10^{-7} \text{ (T) ...Ans.}$$

(b) B at M due to loop 1 $= \dfrac{4\pi\times10^{-7}\times1\times1}{2(1+0.5^2)^{3/2}}$

$$= 4.49\times10^{-7} = \text{B due to loop 2}$$

∴ B at M = B at M due to loop 1 + B at M due to loop 2

$$= 4.49\times10^{-7} + 4.49\times10^{-7} = 8.99\times10^{-7} \text{ T. ...Ans.}$$

Ex. 6.6.2 : Find the contribution to the magnetic flux density at P in the Fig. Ex. 6.6.2 caused by

(a) Semicircle section;

(b) Two horizontal conductors;

(c) Short vertical section.

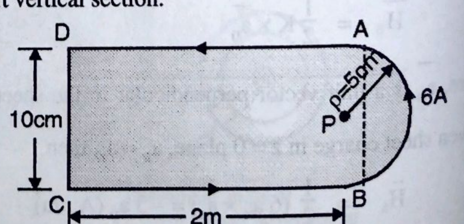


Fig. : Illustrations Ex. 6.6.2

☑ Soln. :

(a) $\bar{B}$ due to semicircle section : (Refer Fig. (a))

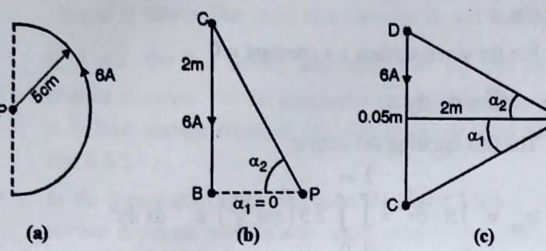


Fig. : Illustrating Ex. 6.6.2 ; Part (a), (b), (c)

Flux density at point P due to semicircle will be half of due to full circle.

$$B_{semi} = \frac{1}{2} \times \frac{\mu_0 I}{2\rho} = \frac{4\pi \times 10^{-7} \times 6}{4 \times 0.05} = 37.69\ (\mu Wb/m^2) \text{ ...Ans.}$$

(b) $\bar{B}$ due to horizontal conductor : Refer Fig. (b)

In the figure, $\alpha_2 = \tan^{-1} 2/0.05 = 88.87^\circ$ and $\alpha_1 = 0$

$$B_{CB} = \frac{\mu I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1) = \frac{4\pi \times 10^{-7} \times 6}{4\pi \times 0.05} \times 1 = 12\ (\mu Wb/m^2)$$

B because of two horizontal conductors,

$$B_{hori} = 2 \times B_{CB} = 24\ (\mu Wb/m^2) \text{ (out of the page) ...Ans.}$$

(c) $\bar{B}$ due to short vertical section : Refer Fig. (c).

$$B_{DC} = \frac{4\pi \times 10^{-7} \times 6}{4\pi \times 2} \times (\sin\alpha_2 - \sin\alpha_1)$$

Here $\alpha_2 = \tan^{-1}\dfrac{0.05}{2} = 1.43^\circ$ and $\alpha_1 = -\alpha_2 = -1.43^\circ$

$\therefore$ $B_{DC} = 0.15\ (\mu Wb/m^2)$(out of the page) ...Ans.

UEx. 6.6.3 MU - May 12, 10 Marks

A circuit carrying a current of I Amp form regular polygon of n side inscribed in circumscribing circle of radius R. Calculate the magnetic flux density $\bar{B}$ at the centre of the polygon and show that $\bar{B}$ approaches that for a circular loop if 'n' tends to ∞.

☑ Soln. : For a polygon having n sides

$$\theta = \frac{360}{n}$$

Then angles α_1 and α_2 shown in Fig. Ex. 6.6.3 are given by

$$\therefore \alpha_2 = \frac{\theta}{2} = \frac{180}{n} \text{ and } \alpha_1 = -\frac{\theta}{2} = -\frac{180}{n}$$

Sign of α_1 is negative due to the fact that, if you make side AB of the polygon vertical with current upward, the lower end will be below point P, hence angle is negative.

For side AB, the magnetic field intensity is :

$$\bar{H} = \frac{I}{4\pi P}(\sin\alpha_2 - \sin\alpha_1) \text{ (into the page)}$$

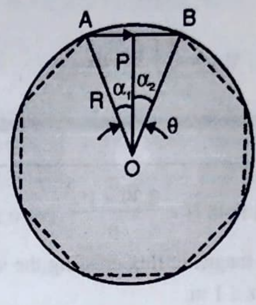


Fig. : Illustrating Ex. 6.6.3

$$= \frac{I}{4\pi P} 2\sin\alpha_2 = \frac{I}{2\pi P}\sin(180/n)$$

Put $P = R\cos\left(\frac{180}{n}\right)$ $\therefore$ $H = \frac{I}{2\pi R}\tan\left(\frac{180}{n}\right)$

For n sides, $\bar{H} = \frac{nI}{2\pi R}\tan\left(\frac{180}{n}\right)$

$$\therefore \bar{B} = \mu\bar{H} = \frac{\mu nI}{2\pi R}\tan\left(\frac{180}{n}\right)$$

as $n \to \infty$, $\frac{180}{n} \to 0$, $\tan\left(\frac{180}{n}\right) \to \left(\frac{180}{n}\right) = \frac{\pi}{n}$

$$\therefore B = \frac{\mu nI}{2\pi R}\left(\frac{\pi}{n}\right) = \frac{\mu I}{2R} \text{ (T) (into the page) ...Ans.}$$

6.6.3 Magnetic Flux (ψ_m)

GQ. Define magnetic flux. **(2 Marks)**

- The magnetic flux crossing any surface is found by multiplying magnetic flux density with area of the surface, provided that B is normal to the surface. If B makes some angle with surface not equal to 90°, then

$$\boxed{\psi_m = \iint_S \bar{B} \cdot d\bar{s}} \quad \text{...(6.6.2)}$$

Where, ψ_m = magnetic flux (Wb)

$\bar{B}$ = Magnetic flux density (Wb/m^2 or T)

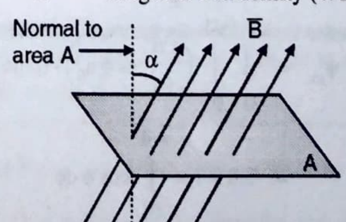


Fig. 6.6.1 : Flux density not in normal to the area

6.6.4 Solved Examples on Magnetic Flux

Important Formula

$$\Psi_m = \iint_S \bar{B} \cdot d\bar{s}$$

Ex. 6.6.4 : A radial field $\bar{H} = \dfrac{2.39 \times 10^6}{\rho} \cos\phi\, \bar{a}_\rho$ (A/m) exists in free space. Find the magnetic flux crossing the surface defined by $0 \le \phi \le \pi/4$ and $0 \le z \le 1$ m.

☑ Soln. :

Magnetic flux density is obtained from magnetic field intensity using the relation,

$$\bar{B} = \mu_0 \bar{H} = 4\pi \times 10^{-7} \times \frac{2.39 \times 10^6}{\rho} \cos\phi\, \bar{a}_\rho$$

$$\bar{B} = \frac{3.0}{\rho} \cos\phi\, \bar{a}_\rho \text{ (T)}$$

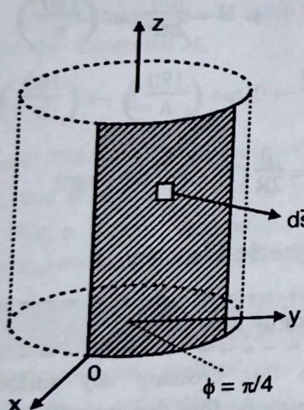


Fig. : Illustrating Ex. 6.6.4

Magnetic flux crossing the given surface is obtained by

$$\phi = \iint \bar{B} \cdot d\bar{s}$$

For the given surface ϕ and z are varying therefore the third co-ordinate is constant i.e. ρ = constant. Then,

$$d\bar{s} = \rho d\phi\, dz\, \bar{a}_\rho$$

Thus,

$$\Psi_m = \int_0^1 \int_0^{\pi/4} \left(\frac{3.0}{\rho} \cos\phi\, \bar{a}_\rho\right) \cdot (\rho\, d\phi\, dz\, \bar{a}_\rho)$$

$$= 3.0 \times 1 \times \int_0^{\pi/4} \cos\phi\, d\phi$$

$$= 3.0\, [\sin\phi]_0^{\pi/4} = 2.12 \text{ (Wb)} \qquad \text{...Ans.}$$

Ex. 6.6.5 : Given that $\bar{B} = 2.5 \left(\sin \dfrac{\pi x}{2}\right) e^{-2y}\, \bar{a}_z$ Telsa, find the total magnetic flux crossing the strip $z = 0,\ y \ge 0,\ 0 \le x \le 2$m.

☑ Soln. :

For the given surface z = constant = 0

☑ $\therefore \quad d\bar{s} = dx\, dy\, \bar{a}_z$

The flux crossing the strip is

$$\Psi_m = \int_S \bar{B} \cdot d\bar{s} = \int_0^2 \int_0^\infty 2.5 \left(\sin \frac{\pi x}{2}\right) e^{-2y}\, dx\, dy$$

$$= 2.5 \times \frac{[-\cos(\pi x/2)]_0^2}{\pi/2} \times \left[\frac{e^{-2y}}{-2}\right]_0^\infty$$

$$= \frac{2.5}{\pi/2} (-1 - 1) \times -\frac{1}{2} = 1.59 \text{ Wb} \qquad \text{...Ans.}$$

Ex. 6.6.6 : Find the flux crossing the portion of the plane $\phi = \pi/4$ defined by $0.01 < \rho < 0.05$ m and $0 < z < 2$m. A current filament of 2.50 A along the z-axis is in the $\bar{a}_z$ direction.

☑ Soln. :

Magnetic flux density is obtained by using the relation

$$\bar{B} = \mu_0 \bar{H} = \frac{\pi_0 I}{2\pi\rho} \bar{a}_\phi$$

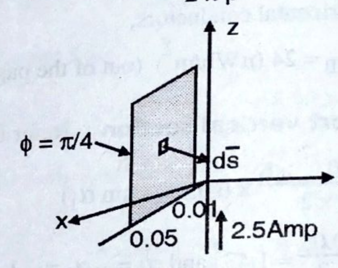


Fig. : Illustrating Ex. 6.6.6

For the given surface ϕ must be constant,

$$\therefore \quad d\bar{s} = d\rho\, dz\, \bar{a}_\phi$$

and
$$\Psi_m = \int_S \bar{B} \cdot d\bar{s} = \int_0^2 \int_{0.01}^{0.5} \left(\frac{\mu_0 I}{2\pi\rho} \bar{a}_\phi\right) \cdot (d\rho\, dz\, \bar{a}_\phi)$$

$$= \frac{2 \times 4\pi \times 10^{-7} \times 2.5}{2\pi} [\ln \rho]_{0.01}^{0.05}$$

$$\Psi_m = 1.61 \times 10^{-6} \text{ Wb} = 1.61\ \mu\text{Wb}. \qquad \text{...Ans.}$$

6.6.5 Gauss's Law for Magnetic Field

- The magnetic flux is

$$\Psi_m = \iint_S \bar{B} \cdot d\bar{s}$$

- This gives a flux crossing the open surface. What happens when the surface is closed ?
- We know that the magnetic field intensity H is always circular in nature. Due to relation between $\bar{B}$ and $\bar{H}$ which is, $\bar{B} = \mu\bar{H}$, the flux density and hence flux ψ_m will be also circular in nature. For the standard source of magnetic, which is infinite current filament, the flux lines are as shown in Fig. 6.6.2.
- In the presence of these magnetic flux lines when a closed surface is placed, one can very easily observe that number of incoming and outgoing flux lines are equal.

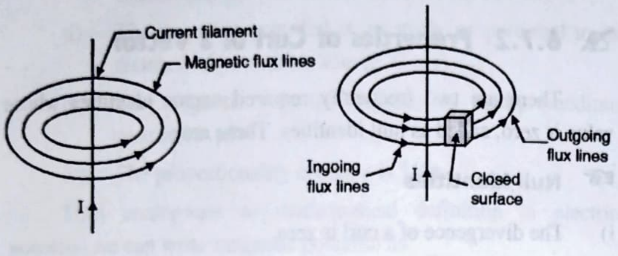


Fig. 6.6.2 : Showing magnetic flux lines

- If incoming flux is considered positive then the outgoing flux will be negative. This gives **total flux crossing the closed surface as zero.** Mathematically it is expressed as

$$\oint_S \bar{B} \cdot d\bar{s} = 0 \qquad ...(6.6.3)$$

- This is called as Gauss's law for magnetic, also called as **Maxwell's equation in integral form** derived from Gauss's law for static field. In the chapter on time varying field.
- We shall see that this law is used as it is for time varying field. To obtain point form of this equation, apply divergence theorem to above equation.

$$\oint \bar{B} \cdot d\bar{s} = \int_V (\nabla \cdot \bar{B})\, dv = 0$$

i.e. $$\boxed{\nabla \cdot \bar{B} = 0} \qquad ...(6.6.4)$$

- This is called as **Maxwell's equation in point form** derived from Gauss's law for static field.

6.7 AMPERE'S CIRCUITAL LAW IN DIFFERENTIAL VECTOR FORM ($\nabla \times \bar{H} = \bar{J}$)

GQ. Derive the point form of Ampere's circuit law. **(10 Marks)**

GQ. State and explain Ampere's law in differential form as used in magnetic fields. **(5 Marks)**

Module 1

The integral form of Ampere's circuital law is

$$\oint \bar{H} \cdot d\bar{l} = I = \int \bar{J} \cdot d\bar{s} \qquad ...(6.7.1)$$

The H and J terms cannot be compared directly as H is associated with line integral and J with surface integral.

It can be done by converting closed line integral into surface integral using Stoke's theorem.

$$\oint \bar{A} \cdot d\bar{l} = \int (\nabla \times \bar{A}) \cdot d\bar{s}$$

Appling this to Equation (1)

$$\oint \bar{H} \cdot d\bar{l} = \int (\nabla \times \bar{H}) \cdot d\bar{s} = \int \bar{J} \cdot d\bar{s}$$

i.e. $$\int (\nabla \times \bar{H}) \cdot d\bar{s} = \int \bar{J} \cdot d\bar{s}$$

or $$\nabla \times \bar{H} = \bar{J}$$

This is called as point form of Ampere's law.

6.7.1 Solved Examples on Ampere's Circuital Law

Important Formulae

$$\nabla \times \bar{H} = \bar{J}$$

Expressions for curl in three coordinate systems is given below :

Cartesian system : $\nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix}$

Cylindrical system : $\nabla \times \bar{H} = \begin{vmatrix} \frac{1}{\rho}\bar{a}_\rho & \bar{a}_\phi & \frac{1}{\rho}\bar{a}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$

Spherical system : $\nabla \times \bar{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ H_r & rH_\theta & r\sin\theta H_\phi \end{vmatrix}$

If you find these three equations difficult to remember, you can use method explained in Chapter 1.

Ex. 6.7.1 : An $\bar{H}$ due to a current source is given by

$\bar{H} = y\cos(2x)\,\bar{a}_x + (y + e^x)\bar{a}_z$. Describe the current density at origin.

Soln. :

We have $\bar{J} = \nabla \times \bar{H}$

In cartesian co-ordinate system

$$\bar{J} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\cos 2x & 0 & y+e^x \end{vmatrix}$$

$$= \bar{a}_x (1) - \bar{a}_y (e^x) + \bar{a}_z (-\cos 2x)$$

$$= \bar{a}_x - e^x \bar{a}_y - \cos 2x\, \bar{a}_z$$

At origin (0,0,0)

$$\bar{J} = \bar{a}_x - e^0 \bar{a}_y - \cos(0)\, \bar{a}_z = \bar{a}_x - \bar{a}_y - \bar{a}_z \ (A/m^2) \quad \text{...Ans.}$$

Ex. 6.7.2 : A flat perfectly conducting surface in xy plane is situated in a magnetic field

$\bar{H} = \bar{a}_x 3\cos x + \bar{a}_y z\cos x$ (A/m) for $z \geq 0 = 0$ for $z < 0$

Find the current density on the conductor surface.

Soln. : The current is given by

$$\bar{J} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3\cos x & z\cos x & 0 \end{vmatrix}$$

$$= \left[0 - \frac{\partial}{\partial z}(z\cos x)\right]\bar{a}_x - \left[0 - \frac{\partial}{\partial z}(3\cos x)\right]\bar{a}_y + \left[\frac{\partial}{\partial x}(z\cos x) - \frac{\partial}{\partial y}(3\cos x)\right]\bar{a}_z$$

$$= [-\cos x]\,\bar{a}_x - [0-0]\,\bar{a}_y + [-z\sin x - 0]\,\bar{a}_z$$

$$\bar{J} = -\cos x\, \bar{a}_x - z\sin x\, \bar{a}_z \ \text{A/m}^2$$

The conducting surface is at z = 0, thus

$$\bar{J}\Big|_{z=0} = -\cos x\, \bar{a}_x \ (A/m^2)$$

Ex. 6.7.3 : If magnetic field intensity in a region is

$\bar{H} = x^2 \bar{a}_x + 2yz\, \bar{a}_y + (-x^2)\, \bar{a}_z$

Find the current density at point (i) (2, 3, 4) (ii) (1, 1, –1) and (0, 0, 0).

Soln. :

We know that, point form of Ampere's law is

$$\bar{J} = \nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 2yz & -x^2 \end{vmatrix}$$

So, $\nabla \times \bar{H} = (0 - 2y)\, \bar{a}_x - (-2x - 0)\, \bar{a}_y + (0 - 0)\, \bar{a}_z$

$$\therefore \quad \bar{J} = -2y\, \bar{a}_x + 2x\, \bar{a}_y \ (A/m^2)$$

Current density $\bar{J}$ at

(i) (2,3,4) $\bar{J} = -2 \times (3)\, \bar{a}_x + 2\,(2)\, \bar{a}_y$ i.e. $\bar{J}$

$$= -6\, \bar{a}_x + 4\, \bar{a}_y \ (A/m^2) \quad \text{...Ans.}$$

(ii) (1,1, – 1) $= -2 \times (1)\, \bar{a}_x + 2\,(1)\, \bar{a}_y$

$$= -2\, \bar{a}_x + 2\bar{a}_y \ (A/m^2) \quad \text{...Ans.}$$

(iii) (0,0,0) $= -2 \times (0)\, \bar{a}_x + 2(0)\, \bar{a}_y = 0$ ...Ans.

6.7.2 Properties of Curl of a Vector

There are two frequently required vector identities whose value is zero, called as null identities. These are :

Null identities

i) The divergence of a curl is zero.

$$\nabla \cdot (\nabla \times \bar{A}) = 0$$

ii) The curl of a gradient is zero.

$$\nabla \times (\nabla \cdot V) = 0$$

The other properties are ,

iii) $\nabla \times (\bar{A} + \bar{B}) = \nabla \times \bar{A} + \nabla \times \bar{B}$

iv) $\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$

v) $\nabla \times \bar{A} \times \bar{B} = \bar{A}\,(\nabla \cdot \bar{B}) - \bar{B}\,(\nabla \cdot \bar{A}) + (\bar{B} \cdot \nabla)\,\bar{A} - (\bar{A} \cdot \nabla)\,\bar{B}$

vi) $\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$

6.8 CONCEPT OF VECTOR MAGNETIC POTENTIAL

UQ. Write short note on Magnetic vector potential. **MU - May 12, 5 Marks**

UQ. Define and explain Vector Magnetic Potential. **MU - May 16, 5 Marks**

UQ. Define and explain the significance of vector magnetic potential. **(MU - Q. 1(f), Dec. 19, 5 Marks)**

- The concept of vector magnetic potential can be developed by considering the analogy between the electric field and the magnetic field. Thus, as for the point charge configuration in electric field the electric potential at any point is defined as

$$dV = \frac{dq}{4\pi\epsilon_0 r} \quad \text{...(A)}$$

From this equation we have following conclusion :

- i) The electric potential is directly proportional to its source.
  ii) The electric potential is inversely proportional to the distance of point from source.
  iii) The electric potential depends on media.
  iv) The proportionality constant is $1/4\pi$.
- Analogues to these lines we can have, the concept of magnetic potential based on the following points.
  i) The magnetic potential is directly proportional to its source. (i.e. differential current element $I\,d\bar{l}$).
  ii) The magnetic potential is inversely proportional to its distance of point from source. (i.e. r).
  iii) The magnetic potential depends on media (medium constant is μ)
  iv) The proportionality constant is $1/4\pi$.

Thus analogoues to mathematical definition of electric potential we can write magnetic potential as

$$d\bar{A} = \frac{\mu I\, d\bar{l}}{4\pi r} \quad \text{...(B)}$$

It is measured in amperes.

6.8.1 Relation between $\bar{B}$ and $\bar{A}$

- Secondly, we know that $\bar{E} = -\nabla V$, this means that the space derivative of electric potential is equal to one of the electric field quantities which depends on media i.e. electric field intensity.
- But for the magnetic potential which is defined in above Equation (B) we can easily observe that the magnetic potential is a vector quantity (due to presence of $d\bar{l}$ which is a vector).
- Thus if we want to equate the space derivative of magnetic vector potential with some magnetic quantity which is a vector then the possible space derivative is curl operation and we can write either.

$$\nabla\times\bar{A} = \bar{H} \quad \text{or} \quad \nabla\times\bar{A} = \bar{B}$$

- But as we have accepted that the magnetic potential depends on media then its space derivative i.e. $\nabla\times\bar{A}$ must depend on media. Out of the two magnetic quantities, i.e. $\bar{B}$ and $\bar{H}$ we have studied that $\bar{H}$ is independent of media and $\bar{B}$ depend on media, thus we can define.

$$\boxed{\nabla\times\bar{A} = \bar{B}} \quad \text{...(6.8.1)}$$

- This is the **relation between $\bar{B}$ and $\bar{A}$.**
- So using Equation (B) we can obtain vector magnetic potential and then using Equation (6.8.1), by taking curl of $\bar{A}$, we obtain $\bar{B}$.

Module 1

6.8.2 Relation between ψ_m and $\bar{A}$

- This concept of magnetic vector potential is practically useless but many times for deriving the expression of magnetic field intensity for complex fields the concept is widely used.
- Vector potential $\bar{A}$ relates to the magnetic flux ψ through a given area s that is bounded by contour C in a simple way.

$$\psi_m = \int_s \bar{B}\cdot d\bar{s}$$

- Using Equation (6.8.1) and applying Stoke's theorem

$$\psi_m = \int_s (\nabla\times\bar{A})\cdot d\bar{s} = \oint \bar{A}\cdot d\bar{l} \text{ (Wb)}$$

i.e. $$\boxed{\psi_m = \oint \bar{A}\cdot d\bar{l}} \quad \text{...(6.8.2)}$$

6.8.3 Poisson's Equations for Magnetostatic Field

- We have $\bar{B} = \nabla\times\bar{A}$ ...(i)
- For the sake of simplicity we choose $\nabla\cdot\bar{A} = 0$ ...(ii)
- Taking the curl of Equation (i) we get,

$$\nabla\times\nabla\times\bar{A} = \nabla\times\bar{B} = \nabla\times\mu\bar{H} = \mu\nabla\times\bar{H} = \mu\bar{J} \quad \text{...(iii)}$$

- But the vector identity curl of curl of a vector is,

$$\nabla\times\nabla\times\bar{A} = \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} \quad \text{...(iv)}$$

- Using Equations (ii) and (iv), Equation (iii) becomes,

$$\nabla\times\nabla\times\bar{A} = -\nabla^2\bar{A} = \mu\bar{J}$$

i.e. $$\boxed{\nabla^2\bar{A} = -\mu\bar{J}} \quad \text{...(6.8.2(a))}$$

- This is **vector Poisson's equation.** In terms of rectangular components of $\bar{A}$ and $\bar{J}$.

$$\nabla^2\bar{A} = \bar{a}_x\nabla^2 A_x + \bar{a}_y\nabla^2 A_y + \bar{a}_z\nabla^2 A_z$$
$$= -\mu(J_x\bar{a}_x + J_y\bar{a}_y + J_z\bar{a}_z)$$

- Comparing coefficients of $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$, we get

$$\nabla^2 A_x = -\mu J_x \quad \text{...(v)}$$
$$\nabla^2 A_y = -\mu J_y \quad \text{...(vi)}$$
$$\nabla^2 A_z = -\mu J_z \quad \text{...(vii)}$$

- Each of these three equations has the same form as the Poisson's equation in electrostatics. These equations are called as Poisson's equations for magnetostatic field.
- In free space the equation $\nabla^2 V = -\rho_v/\epsilon_0$, has a particular solution,

$$V = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_v}{R}\, dv$$

- Hence, the solution for Equations (v), (vi) and (vii) are

$$A_x = \frac{\mu}{4\pi} \iiint \frac{J_x\, dv}{r} ;$$

$$A_y = \frac{\mu}{4\pi} \iiint \frac{J_y\, dv}{r} ; A_z = \frac{\mu}{4\pi} \iiint \frac{J_z\, dv}{r}$$

- Taking the vector sum of these components for $\bar{A}$ gives,

$$\bar{A} = \frac{\mu}{4\pi} \iiint \frac{1}{r} (J_x \bar{a}_x + J_y \bar{a}_y + J_z \bar{a}_z)\, dv$$

i.e. $$\boxed{\bar{A} = \int_{vol.} \frac{\mu \bar{J}\, dv}{4\pi r}} \qquad ...(6.8.3)$$

6.8.4 Solved Examples on Vector Magnetic Potential

Important Formulae

$$\nabla \times \bar{A} = \bar{B}$$

$$\psi_m = \oint \bar{A} \cdot d\bar{l}$$

Ex. 6.8.1 : A direct current I flows in a straight wire of length 2L. At a point located at a distance r from the wire in the bisecting plane determine

(i) the vector magnetic potential and

(ii) the magnetic flux density.

Soln. : Consider the current carrying element placed along z-axis with center of the filament at origin as shown in Fig. Ex. 6.8.1 The distance vector $\bar{R}$ of point P from the current element I dz $\bar{a}_z$ is

$$\bar{R} = \rho \bar{a}_\rho - z \bar{a}_z \text{ and } \bar{a}_R = \frac{\rho \bar{a}_\rho - z \bar{a}_z}{\sqrt{\rho^2 + z^2}}$$

(i) To find vector magnetic potential

The magnetic vector potential at point P is

$$\bar{A} = \frac{\mu}{4\pi} \int \frac{I\, d\bar{l}}{R}$$

Since $d\bar{l} = dz\, \bar{a}_z$ and assuming free space $\mu = \mu_0$, we have

$$\bar{A} = \frac{\mu_0 I}{4\pi} \bar{a}_z \int_{-L}^{L} \frac{dz}{\sqrt{\rho^2 + z^2}} = \frac{\mu_0 I}{4\pi} \left[ln\, z + \sqrt{z^2 + \rho^2} \right]_{-L}^{L} \bar{a}_z$$

$$= \frac{\mu_0 I}{4\pi} ln \left(\frac{\sqrt{L^2 + \rho^2} + L}{\sqrt{L^2 + \rho^2} - L} \right) \bar{a}_z = A_z \bar{a}_z$$

...Ans.

(ii) To find $\bar{B}$: To find $\bar{B}$ form $\bar{A}$ we have,

$$\bar{B} = \nabla \times \bar{A} = \begin{vmatrix} 1/\rho \bar{a}_\rho & \bar{a}_\phi & 1/\rho \bar{a}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ 0 & 0 & A_z \end{vmatrix}$$

$$= \frac{1}{\rho} \bar{a}_\rho \frac{\partial A_z}{\partial \phi} - \bar{a}_\phi \frac{\partial A_z}{\partial \rho}$$

$$= -\frac{\partial A_z}{\partial \rho} \bar{a}_\phi \qquad \text{...Since } A_z \text{ is not a function of } \phi$$

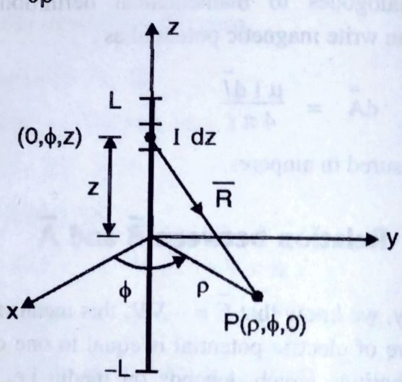


Fig. : Illustrating Ex. 6.8.1

Thus, $$B = -\bar{a}_\phi \frac{\partial}{\partial \rho} \left[\frac{\mu_0 I}{4\pi} ln \frac{\sqrt{L^2 + \rho^2} + L}{\sqrt{L^2 + \rho^2} - L} \right]$$

$$= \bar{a}_\phi \frac{\mu_0 IL}{2\pi r \sqrt{L^2 + \rho^2}} \qquad ...(A)$$

When $\rho << L$, above equation reduces to $\bar{B} = \frac{\mu_0 I}{2\pi\rho} \bar{a}_\phi$

which is the expression for $\bar{B}$ at a point located at a distance ρ from an infinitely long, straight wire carrying current I.

Ex. 6.8.2 : Using same data in above example find $\bar{B}$ using Biot savart law.

Soln. : Applying Biot-savart law :

We have, $$\bar{B} = \mu_0 \oint \frac{I\, d\bar{l} \times \bar{a}_R}{4\pi R^2} = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{dz\, \bar{a}_z}{(\rho^2 + z^2)} \times \frac{\rho \bar{a}_\rho - z \bar{a}_z}{\sqrt{\rho^2 + z^2}}$$

$$= \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{\rho\, dz}{(\rho^2 + z^2)^{3/2}}$$

Solving we get, $\bar{B} = \bar{a}_\phi \dfrac{\mu_0 IL}{2\pi\rho\sqrt{L^2+\rho^2}}$ ...(B)

Ex. 6.8.3 : Obtain the vector magnetic potential $\bar{A}$ in the region surrounding an infinitely long, straight filamentary current I.

☑ **Soln. :** Solution to this problem is similar to last problem, in which $\bar{A}$ is obtained in terms of L. The only change in this problem is the filament is infinite in extent. Thus, for the filament in the problem L >> ρ. In the last problem this approximation is used while finding B, which is not asked in this problem, so approximation is to be applied to the expression of $\bar{A}$. In order to do this, the integral from – L to L is solved by slightly different method. The last problem could be solved by this method also.

We have, $\bar{R} = \rho\,\bar{a}_\rho - z\,\bar{a}_z$ and $\bar{a}_R = \dfrac{\rho\,\bar{a}_\rho - z\,\bar{a}_z}{\sqrt{\rho^2+z^2}}$

The vector magnetic potential is :

$$\bar{A} = \frac{\mu}{4\pi}\int \frac{I\,d\bar{l}}{R}$$

$$\text{i.e. } \bar{A} = \frac{\mu_0 I}{4\pi}\bar{a}_z \int_{-L}^{L} \frac{dz}{\sqrt{\rho^2+z^2}} = \frac{\mu_0 I}{4\pi}\bar{a}_z \times 2\int_{0}^{L}\frac{dz}{\sqrt{\rho^2+z^2}}$$

$$= \frac{\mu_0 I}{2\pi}\bar{a}_z\left[\ln\left(z+\sqrt{\rho^2+z^2}\right)\right]_0^L$$

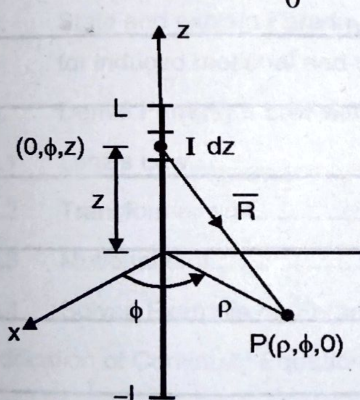


Fig. : Illustrating Ex. 6.8.3

$$= \frac{\mu_0 I}{2\pi}\left[\ln\left(L+\sqrt{\rho^2+L^2}\right) - \ln\rho\right]\bar{a}_z$$

for L >> ρ

$$\bar{A} = \frac{\mu_0 I}{2\pi}(\ln 2L - \ln\rho)\,\bar{a}_z = \frac{\mu_0 I}{2\pi}\ln\left(\frac{2L}{\rho}\right)\bar{a}_z$$

This is vector magnetic potential due to straight, long filament. We could have used this expression to find $\bar{B}$ in the last problem.

Ex. 6.8.4 : Given the magnetic vector potential.

$\bar{A} = -\rho^2/4\,\bar{a}_z$ (Wb/m) Calculate the total magnetic flux crossing the surface $\phi = \pi/2$, $1 \le \rho \le 2$m, $0 \le z \le 5$ m.

☑ **Soln. :** First we obtain magnetic flux density $\bar{B}$

$$\bar{B} = \nabla\times\bar{A} = \begin{vmatrix} \frac{1}{\rho}\bar{a}_\rho & \bar{a}_\phi & \frac{1}{\rho}\bar{a}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ 0 & 0 & -\rho^2/4 \end{vmatrix}$$

$$= -\bar{a}_\phi\frac{\partial}{\partial\rho}(-\rho^2/4) = \frac{\rho}{2}\bar{a}_\phi$$

For the $\phi = \pi/2$ = constant surface,

$$d\bar{s} = d\rho\,dz\,\bar{a}_\phi$$

Flux through given surface is

$$\Psi_m = \int_s \bar{B}\cdot d\bar{s} = \int_s \frac{\rho}{2}\bar{a}_\phi\cdot d\rho\,dz\,\bar{a}_\phi$$

$$= \int_0^5\int_1^2 \frac{\rho}{2}\,d\rho\,dz = \frac{1}{2}\left(\frac{\rho^2}{2}\right)_1^2, (z)_0^5 = 3.75\text{ (Wb)}$$

Ex. 6.8.5 : Given magnetic vector potential $A = 10\sin\theta\,\bar{a}_\theta$ in spherical co-ordinates. Find the magnetic flux density at (2, π/2, 0).

☑ **Soln. : Given :** $\bar{A} = 10\sin\theta\,\bar{a}_\theta = A_\theta\,\bar{a}_\theta$

The magnetic flux density is given by

$$\bar{B} = \nabla\times\bar{A}$$

In spherical system

$$\bar{B} = \nabla\times\bar{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & .\,r A_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

$$\bar{B} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 10\,r\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\times r\sin\theta\,\bar{a}_\phi\frac{\partial}{\partial r}(10r\sin\theta)$$

$$= \frac{10\sin\theta}{r}\bar{a}_\phi\text{ (Wb/m}^2\text{)}$$

At (2, π/2, 0) :

$$\bar{B} = \frac{10\sin(90)}{2} = 5\text{ (Wb/m}^2\text{)}$$

Chapter Ends...

□□□