Ambiguity in Context Free Languages

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Abstract. Four principal results about ambiguity in languages (i.e., context free languages) are proved. It is first shown that the problem of determining whether an arbitrary language is inherently ambiguous is recursively unsolvable. Then a decision procedure is presented for determining whether an arbitrary bounded grammar is ambiguous. Next, a necessary and sufficient algebraic condition is given for a bounded language to be inherently ambiguous. Finally, it is shown that no language contained in $w_1^*w_2^*$, each w_i a word, is inherently ambiguous.

Introduction

One especially desirable feature in a grammar for a programming language is freedom from ambiguity. There is extensive interest in questions of ambiguity for natural languages as well, and context free languages, the constituent parts of the ALGOL-like languages, have been employed to sharpen them [4, p. 118]. These considerations motivate the study of ambiguity in context free languages (= languages). To date very few results are known. The three most significant are: (i) it is recursively unsolvable whether an arbitrary grammar for a language is ambiguous [2; 5, p. 154]; (ii) there exist inherently ambiguous languages, for example, $\{a^ib^ic^jd^k/i, j, k \geq 1\} \cup \{a^ib^jc^kd^j/i, j, k \geq 1\}$ [12, pp. 205ff.]; and (iii) no regular set is inherently ambiguous [5, p. 153]. In this paper we obtain additional results about ambiguity in languages.

In Section 2 it is shown that it is unsolvable whether an arbitrary language is inherently ambiguous. In other words, there is no algorithm for determining of an arbitrary language whether it is generated by some unambiguous grammar. The remaining sections deal with ambiguity in a special family of languages introduced in [9], the so-called "bounded" languages. (Apart from their own intrinsic interest, these special languages are valuable because they enable one to obtain results about languages in general. For example, the first pair of languages whose intersection was proved not to be a language was a pair of bounded languages [15]. The first known inherently ambiguous languages were bounded [12, pp. 205ff.; 5, p. 153]. Also, the recursive unsolvability of determining whether an arbitrary language is inherently ambiguous is proved with the aid of certain bounded languages.) Sections 3 and 4 are concerned with establishing that, in contrast with the full family of languages, it is decidable whether an arbitrary grammar for a bounded language is ambiguous. Sections 5 and 6 provide a necessary and sufficient algebraic condition for a bounded language to be inherently ambiguous. This condition is useful in showing certain specific languages to be inherently ambiguous. (A decision procedure for this condition is still unknown.) Section 7 gives a proof of the fact that for all words w_1, w_2, no language contained in $w_1^*w_2^*$ is inherently ambiguous. This result is interesting in view of the existence of an inherently ambiguous language contained in $w_1^*w_2^*w_3^*$, with each w_i a word [5, p. 153].

1. *Preliminaries*

Let Σ be a finite nonempty set and $\theta(\Sigma)$ the free semi-group, with identity ϵ generated by Σ. Thus $\theta(\Sigma)$ is the set of all finite sequences, or words over Σ, and ϵ is the empty word. Certain subsets of $\theta(\Sigma)$ which are called "context free languages" are to be considered. These sets are known [7, p. 355] to be identical with the constituent parts in the ALGOL-like programming languages, i.e., those languages defined by Backus normal form.

A *grammar* G is a 4-tuple (V, Σ, P, σ), where V is a finite set, Σ is a subset of V, σ is an element of $V - \Sigma$, and P is a finite set of ordered pairs of the form (ξ, w) with ξ in $V - \Sigma$ and w in $\theta(V)$. P is called the set of *productions* of G. We write $\xi \rightarrow w$ when (ξ, w) is in P. For x, y in $\theta(V)$ we write $x \Rightarrow y$ if there exist w_1, w_2, w_3 in $\theta(V)$ and ξ in V, such that $x = w_1\xi w_3$, $y = w_1 w_2 w_3$, and $\xi \rightarrow w_2$. For x, y in $\theta(V)$ we write $z \overset{\bullet}{\Rightarrow} y$ if either $x = y$, or there exist $x_1, \cdots, x_k$ such that $x = x_1, x_k = y$, and $x_i \Rightarrow x_{i+1}$ for each i. In this case we write

$$(*) \qquad x_1 \Rightarrow x_2 \Rightarrow \cdots \Rightarrow x_k$$

and call $(*)$ a *derivation* or *generation* of x_k (from x_1). The *language generated by* G, denoted by $L(G)$, is the set of words $\{w/\sigma \overset{\bullet}{\Rightarrow} w, w \text{ in } \theta(\Sigma)\}$. A *context free language* (over Σ) is a language $L(G)$ generated by some grammar $G = (V, \Sigma, P, \sigma)$. Unless otherwise stated, "language" always means a context free language.

The reader is referred to [1, 3] for motivation and additional details about languages. It is assumed that the reader is familiar with the elementary properties of languages, such as the fact that if A and B are languages then so are AB,[1] $A \cup B$ and A^*.[2]

Let $G = (V, \Sigma, P, \sigma)$ be a grammar. We shall define a *generation tree* for each derivation

$$w_1 \Rightarrow w_2 \Rightarrow \cdots \Rightarrow w_r,$$

where w_1 is a variable, i.e., an element of $V - \Sigma$. This generation tree is to be a rooted, directed tree with an element of $V \cup \{\epsilon\}$, called the *node name*, associated at each node.

The nodes of the tree are certain tuples of the form $(i_1, \cdots, i_k)$, where $k \leq r$ and i_j is a positive integer. The directed lines of the tree are all the ordered pairs $((i_1, \cdots, i_k), (i_1, \cdots, i_k, i_{k+1}))$ of nodes. Let the 1-tuple (1) be the root and w_1 the node name of (1). The root is said to be of *level* 1. If $w_2 = \epsilon$, let $(1, 1)$ be a node in the tree and ϵ the node name of $(1, 1)$. If $w_2 = x_1^{(2)} \cdots x_{n(2)}^{(2)}$, with each $x_i^{(2)}$ in V, let $(1, i)$, $1 \leq i \leq n(2)$, be a node and $x_i^{(2)}$ its node name. Continuing by induction, suppose that for all $t \leq k$, every occurrence in w_t of an element of V serves as node name of some node. Let

$$(**) \qquad u_k y_k v_k = w_k \Rightarrow w_{k+1} = u_k z_k v_k$$

with y_k a variable. Let $(i_1, \cdots, i_s)$ be the node whose node name is the occurrence of y_k indicated in $(**)$. If $z_k = \epsilon$, let $(i_1, \cdots, i_s, 1)$ be a node, said to be of level $s + 1$, and ϵ its node name. If $z_k = x_1^{(k)} \cdots x_{n(k)}^{(k)}$, with each $x_i^{(k)}$ in V, let

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[1] If A and B are subsets of $\theta(\Sigma)$, then the *product* of A and B, written AB, is the set of words $\{ab/a \text{ in } A, b \text{ in } B\}$. If A (or B) consists of a single word, say $A = \{a\} (B = \{b\})$, then aB (Ab) is written instead of AB.

[2] If A is a set of words, then $A^* = \bigcup_0^\infty A^n$, where $A^0 = \{\epsilon\}$ and $A^{i+1} = A^i A$ for $i \geq 0$.

$(i_1, \cdots, i_s, 1)$, with $1 \leq i \leq n(k)$, be a node, said to be of *level* $s + 1$, and $x_i^{(k)}$ its node name. This procedure is repeated through $k = r-1$. The resulting entity is the generation tree.

A node $(j_1, \cdots, j_t)$ is said to be an *extension* of the node $(i_1, \cdots, i_s)$ if $s \leq t$ and $i_k = j_k$, for all $k \leq s$.

A node q is called *maximal* if there is no node distinct from q which is an extension of q. Otherwise it is called *nonmaximal*.

Familiarity with the elementary notions and facts about generation trees is assumed. The reader is referred to [1] for a more detailed exposition. Among the results to be used are the following (where T is a generation tree associated with $\xi \overset{*}{\Rightarrow} w$):

(a)   If N is a nonmaximal node, then the node name x of N is a variable and $\xi \overset{*}{\Rightarrow} uxv$ for some u, v in $\theta(V)$.

(b)   Let N be a nonmaximal node in a generation tree and x its node name. Then the "subtree" of T, formed by using as nodes all extensions of N, is a generation tree.

(c)   Let $w = u\gamma v$ and let T_1 be a generation tree of $\gamma \overset{*}{\Rightarrow} w_1$. If T_1 is placed (in the obvious way) with its root on the node whose node name is γ in $u\gamma v$, then a generation tree of $\xi \overset{*}{\Rightarrow} uw_1v$ is obtained.

To be considered are the concept of an "ambiguous grammar" and the problem of the existence of an unambiguous grammar for a language.

Given a grammar $G = (V, \Sigma, P, \sigma)$, a word w is said to be *ambiguously derivable* if there exist two derivations of w from σ whose associated generation trees are different. A grammar G is said to be *ambiguous* if there exists an ambiguously derivable word in $L(G)$, and is otherwise *unambiguous*. A language L is said to be *inherently ambiguous* if there is no unambiguous grammar for L.

For each generation tree T of $\xi \overset{*}{\Rightarrow} w$, there is a derivation

$$\xi = w_1 \Rightarrow w_2 \Rightarrow \cdots \Rightarrow w_r = w$$

of w, whose generation tree is T, satisfying the following: For $0 \leq i \leq r-1$, if $w_i = u_i\xi_iv_i$, $w_{i+1} = u_iy_iv_i$ and $\xi_i \to y_i$, then u_i is in $\theta(\Sigma)$ [6, pp. 2-27]. (Such a derivation is said to be a *leftmost* derivation.)

If T_1 and T_2 are distinct generation trees, then their associated leftmost derivations are distinct; the converse also holds. Thus a word is ambiguously derivable if and only if it has two leftmost derivations.

It is well known that there is no decision procedure for determining whether an arbitrary grammar is ambiguous [2; 5, p. 154]. It is shown here that there is no decision procedure for determining if an arbitrary language is inherently ambiguous.

The following result, needed in later sections, is established by the proof of Theorem 8.1 of [1].

THEOREM 1.1.   *If L is a language (generated by an unambiguous grammar) and R is a regular set,[3] then $L \cap R$ is a language (generated by some unambiguous grammar). Furthermore, if an unambiguous grammar for L is known, then an unambiguous grammar for $L \cap R$ can be effectively found.[4]*

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[3] The family of *regular* sets is characterized as the smallest family of subsets of $\theta(\Sigma)$ containing the finite sets and closed under the operations of union, product and $*$ [14, p. 122]. The regular sets form a Boolean algebra of sets [14, p. 118], and each regular set is a language [1, p. 146].

[4] All the proofs in the paper are effective. This is implicit except in Theorem 1.1 and Theorem 4.1.

Some concepts and results about "bounded" languages and "semi-linear sets" are now reviewed. Additional details and information are in [9].

Let $w_1, \cdots, w_n$ be words in $\theta(\Sigma)$. If $X \subseteq w_1^* \cdots w_n^*$, then X is said to be *bounded with respect to* $\langle w_1, \cdots, w_n \rangle$, or simply *bounded* when $\langle w_1, \cdots, w_n \rangle$ is understood. If G is a grammar for a bounded language, then G is said to be a *bounded grammar*.

It is recursively solvable whether an arbitrary grammar is bounded.

The finite product and finite union of bounded sets are bounded. If X is bounded and Y is a set of subwords of words in X, then Y is bounded. In particular, a subset of a bounded set is bounded.

Let N denote the non-negative integers and let N^n be the Cartesian product of N with itself n times. For $x = (x_1, \cdots, x_m)$ in N^m and $y = (y_1, \cdots, y_n)$ in N^n, $x \times y$ is the element $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ of N^{m+n}. For $X \subseteq N^m$ and $Y \subseteq N^n$, $X \times Y = \{x \times y / x$ in X, y in $Y\}$. For elements $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in N^n, let $x + y = (x_1 + y_1, \cdots, x_n + y_n)$ and $k(x_1, \cdots, x_n) = (kx_1, \cdots, kx_n)$, k in N.

A subset A of N^n is said to be *linear* if there exist elements $c, p_1, \cdots, p_m$ in N^n such that

$$A = \{x / x = c + k_1 p_1 + \cdots + k_m p_m, \text{ each } k_i \text{ in } N\}.$$

c is called the *constant* and $p_1, \cdots, p_m$ the *periods*. When the constant and the periods are to be indicated, A is written as $L(c; p_1, \cdots, p_m)$. A subset A of N^n is said to be *semi-linear* if it is a finite union of linear sets.

The union, intersection, product and difference of semi-linear sets are semi-linear.

A basic relation between languages and semi-linear sets is expressed in the following result due to Parikh [12, p. 203].

THEOREM 1.2. *Let* $\Sigma = \{a_i / 1 \le i \le n\}$.[5] *Let* τ *be the mapping of* $\theta(\Sigma)$ *into* N^n *defined by*

$$\tau(z) = \langle \#_z(a_1), \cdots, \#_z(a_n) \rangle,$$

where $\#_z(a_i)$ *is the number of occurrences of the symbol* a_i *in the word* z. *If* M *is a language, then* $\tau(M) = \{\tau(z) / z$ *in* $M\}$ *is semi-linear.*

A subset X of N^n is said to be *stratified* if (a) each element in X has at most two nonzero coordinates, and (b) there are no integers i, j, k, m with $1 \le i < j < k < m \le n$ and $x = (x_1, \cdots, x_n)$, $x' = (x_1', \cdots, x_n')$ in X such that $x_i x_j' x_k x_m' \ne 0$, that is, no two elements of X have nonzero coordinates which "interlace."

THEOREM 1.3. [10] *Let* $\Sigma = \{a_i / 1 \le i \le n\}$. *If* L *is a subset of* N^n, *then* $\{a_1^{x_1} \cdots a_n^{x_n} / (x_1, \cdots, x_n)$ *in* $L\}$ *is a language if and only if* L *can be represented as a finite union of linear sets, each of whose set of periods is stratified and contains at most* n *elements.*

2. Inherent Ambiguity for Languages

Consider the problem of finding an unambiguous grammar for an arbitrary language. Since there are languages which have no unambiguous grammar, there arises the problem of determining when an arbitrary language is inherently ambiguous. It is now shown that this problem is, in general, recursively unsolvable.

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[5] Whenever we write $\Sigma = \{a_i / 1 \le i \le n\}$, it is to be understood that Σ consists of n distinct letters.

THEOREM 2.1. *It is recursively unsolvable whether an arbitrary language over a two-letter alphabet is inherently ambiguous.*

PROOF. Let a, b, c, d, e be distinct symbols. For each positive integer i let $\bar{\imath} = de^i$. Let f be the mapping of $\theta(c, d, e)$ into $\theta(a, b)$ defined by $f(\epsilon) = \epsilon$, $f(c) = bab$, $f(d) = ba^2b$, $f(e) = ba^3b$ and $f(u_1 \cdots u_j) = f(u_1) \cdots f(u_j)$, with each u_i in $\{c, d, e\}$. For each n-tuple $x = (x_1, \cdots, x_n)$ of non-ϵ words in $\theta(d, e)$, let

$$L(x) = \{f(\bar{\imath}_1) \cdots f(\bar{\imath}_k)f(c)f(x_{i_k}) \cdots f(x_{i_1})/k \geq 1, \ 1 \leq i_j \leq n\}.$$

$L(x)$ is a language over $\{a, b\}$ generated by the unambiguous grammar $G_x = (\{\sigma_x, a, b\}, \{a, b\}, P_x, \sigma_x)$, where P_x consists of all productions of the form $\sigma_x \rightarrow f(\bar{\imath})f(c)f(x_i)$ and $\sigma_x \rightarrow f(\bar{\imath})\sigma_x f(x_i)$, $1 \leq i \leq n$. For arbitrary n-tuples x and y of non-ϵ words in $\theta(d, e)$, it is easily seen that $L(x) \cap L(y) \neq \varphi$ if and only if there exists a sequence $i_1, \cdots, i_k$ of integers such that $x_{i_1} \cdots x_{i_k} = y_{i_1} \cdots y_{i_k}$. By the Post Correspondence Theorem [13], it is recursively unsolvable whether there exists such a sequence of integers. To complete the proof it therefore suffices to show that

$$L(x, y) = \bigcup_{i,j \geq 1}^{\infty} a^i b L(x) ba^i ba^j \ \cup \ \bigcup_{i,j \geq 1}^{\infty} a^i b L(y) ba^i ba^j$$

is a language over $\{a, b\}$ which is inherently ambiguous if and only if $L(x) \cap L(y) \neq \varphi$.

Let $G_{x,y} = (\{\sigma, \sigma_x, \sigma_y, \xi_1, \xi_2, \xi_3, a, b\}, \{a, b\}, P, \sigma)$, where $P = P_x \cup P_y \cup \{\sigma \rightarrow a\xi_1 ab\xi_2, \ \sigma \rightarrow \xi_2 b\sigma_y ba\xi_3 a, \ \xi_2 \rightarrow a\xi_2, \ \xi_2 \rightarrow a, \ \xi_1 \rightarrow b\sigma_x b, \ \xi_1 \rightarrow a\xi_1 a, \ \xi_3 \rightarrow b, \ \xi_3 \rightarrow a\xi_3 a\}$. The grammar $G_{x,y}$ generates $L(x, y)$, so that $L(x, y)$ is a language. If $L(x) \cap L(y) = \varphi$, then clearly $G_{x,y}$ is unambiguous. Suppose that $L(x) \cap L(y) \neq \varphi$. Then it must be shown that $L(x, y)$ is inherently ambiguous. Suppose $L(x, y)$ is generated by some unambiguous grammar. Let w be a word in $L(x) \cap L(y)$. By construction, $w \neq \epsilon$. Since $a^* bwba^* ba^*$ is regular, it follows from Theorem 1.1 that

$$A = L(x, y) \cap a^* bwba^* ba^*$$

$$= \{a^i bwba^i ba^j/i, j \geq 1\} \cup \{a^i bwba^j ba^i/i, j \geq 1\}$$

is a language generated by some unambiguous grammar. It is shown below in Theorem 6.2, however, that A is inherently ambiguous. Hence $L(x, y)$ is inherently ambiguous if $L(x) \cap L(y) \neq \varphi$, and the theorem follows.

3. *Semi-linear Sets of Words*

As noted earlier, there is no decision procedure for determining of an arbitrary grammar whether or not it is ambiguous. In the next section it is seen that the analogous problem for a bounded grammar is recursively solvable. First, however, the concept of a "semi-linear set of words" is introduced and a number of basic results about it are established.

Notation. For p in N^n, let $p(k)$, $1 \leq k \leq n$, denote the kth coordinate of p.

Notation. For each n-tuple of words $w = \langle w_1, \cdots, w_n \rangle$ let f_w denote the function defined on N^n by $f_w(p) = w_1^{p(1)} \cdots w_n^{p(n)}$.

Definition. If $M \subseteq w_1^* \cdots w_n^*$, $w = \langle w_1, \cdots, w_n \rangle$ and $f_w^{-1}(M)$ is semi-linear,[6] then M is said to be semi-linear in w.

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[6] For any function f, $f^{-1}(M) = \{x/f(x) \text{ in } M\}$.

In the remainder of this section $w = \langle w_1, \cdots, w_n \rangle$ is an n-tuple of words.

LEMMA 3.1. $\{p \times q/p, q \text{ in } N^n, f_w(p) = f_w(q)\}$ *is a semi-linear subset of* N^{2n}.

PROOF. Let Δ be the alphabet of w, and $\Sigma = \{a_1, \cdots, a_n, b_1, \cdots, b_n, c\}$ a set of $2n+1$ symbols not in Δ. Let S be the one state gsm[7] $(\{p\}, \Sigma, \Delta, \delta, \lambda, p)$, where $\lambda(p, a_i) = w_i$, $\lambda(p, b_i) = w_i^R$,[8] and $\lambda(p, c) = c$. Now the set $M = \{wcw^R/w$ in $\theta(\Delta)\}$ is a language, by Theorem 2.1 of [1]. By Theorem 3.4 of [8], $S^{-1}(M) = \{v/S(v)$ in $M\}$ is a language.[9] Since $a_1^* \cdots a_n^* c b_n^* \cdots b_1^*$ is regular, the set

$$H = S^{-1}(M) \cap a_1^* \cdots a_n^* c b_n^* \cdots b_1^*$$

is a language by Theorem 1.1. Clearly H consists of all words

$$a_1^{p(1)} \cdots a_n^{p(n)} c b_n^{q(n)} \cdots b_1^{q(1)}$$

such that $f_w(p) = f_w(q)$. Let τ be the mapping of $\theta(\Sigma)$ into N^{2n+1} defined by

$$\tau(z) = \langle \#_z(a_1), \cdots, \#_z(a_n), \#_z(b_1), \cdots, \#_z(b_n), \#_z(c) \rangle,$$

where $\#_z(d)$ is the number of occurrences of the symbol d in the word z. Since H is a language, $\tau(H)$ is semi-linear by Theorem 1.2. Let Π be the mapping of N^{2n+1} onto N^{2n} defined by $\Pi(\langle x(1), \cdots, x(2n+1) \rangle) = \langle x(1), \cdots, x(2n) \rangle$. By Lemma 6.3 of [9], Π preserves semi-linearity. Thus

$$\Pi\tau(H) = \{p \times q/p, q \text{ in } N^n, f_w(p) = f_w(q)\}$$

is semi-linear.

LEMMA 3.2. *If* $H \subseteq N^n$ *is semi-linear, then* $f_w(H)$ *is semi-linear in* w.

PROOF. Let $Q = \{p \times q/p, q \text{ in } N^n, f_w(p) = f_w(q)\}$. By Lemma 3.1, Q is semi-linear. Since $H \times N^n$ is semi-linear, $Q \cap (H \times N^n)$ is semi-linear. Let Π be the mapping of N^{2n} onto N^n defined by $\Pi(\langle p(1), \cdots, p(2n) \rangle) = \langle p(n+1), \cdots, p(2n) \rangle$. By Lemma 6.3 of [9], Π preserves semi-linearity. Thus $\Pi(Q \cap (H \times N^n))$ is semi-linear. To prove the lemma it thus suffices to show that $f_w^{-1}[f_w(H)] = \Pi(Q \cap (H \times N^n))$.

Suppose q is in $f_w^{-1}[f_w(H)]$. Then there exists p in H such that $f_w(p) = f_w(q)$. Thus $p \times q$ is in $Q \cap (H \times N^n)$, and q is in $\Pi(Q \cap (H \times N^n))$. Conversely, suppose q is in $\Pi(Q \cap (H \times N^n))$. Then there exists p in N^n such that $p \times q$ is in $Q \cap (H \times N^n)$. Thus p is in H and $f_w(p) = f_w(q)$. Therefore q is in $f_w^{-1}[f_w(H)]$, and $f_w^{-1}[f_w(H)] = \Pi(Q \cap (H \times N^n))$.

COROLLARY. $L \subseteq w_1^* \cdots w_n^*$ *is semi-linear in* w *if and only if there is a semi-linear set* $H \subseteq N^n$ *such that* $f_w(H) = L$.

LEMMA 3.3. *If* $L \subseteq u_1^* \cdots u_m^*$ *is semi-linear in* w, *then* L *is semi-linear in* $u = \langle u_1, \cdots, u_m \rangle$.

PROOF. Since L is semi-linear in w, $f_w^{-1}(L) \subseteq N^n$ is semi-linear. Hence, $f_w^{-1}(L) \times$

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[7] A *generalized sequential machine* (gsm) S is a 6-tuple $(K, \Sigma, \Delta, \delta, \lambda, p_1)$ where (i) K is a finite nonempty set (of "states"); (ii) Σ is a finite nonempty set (of "inputs"); (iii) Δ is a finite nonempty set (of "outputs"); (iv) δ is a mapping of $K \times \Sigma$ into K (the "next state" function); (v) λ is a mapping of $K \times \Sigma$ into $\theta(\Delta)$ (the "output" function); and (vi) p_1 is a distinguished element of K (the "start" state).

[8] Let Σ be an alphabet. Then $\epsilon^R = \epsilon$ and $(x_1 \cdots x_k)^R = x_k \cdots x_1$, each x_i in Σ.

[9] For a gsm $S = (K, \Sigma, \Delta, \delta, \lambda, p_1)$, extend δ and λ to $K \times \theta(\Sigma)$ as follows. Let $\delta(q, \epsilon) = q$ and $\lambda(q, \epsilon) = \epsilon$. For each word $x_1 \cdots x_{k+1}$, each x_i in Σ, let $\delta(q, x_1 \cdots x_{k+r}) = \delta[\delta(q, x_1 \cdots x_k), x_{k+1}]$ and $\lambda(q, x_1 \cdots x_{k+1}) = \lambda(q, x_1 \cdots x_k)\lambda[\delta(q, x_1 \cdots x_k), x_{k+1}]$. For each word v in $\theta(\Sigma)$, let $S(v) = \lambda(p_1, v)$.

$O^m \subseteq N^{n+m}$ is semi-linear. Let $wu = \langle w_1, \cdots, w_n, u_1, \cdots, u_m \rangle$. Since $f_{wu}(f_w^{-1}(L) \times O^m) = L$, $L \subseteq w_1^* \cdots w_n^* u_1^* \cdots u_m^*$ is semi-linear in wu by Lemma 3.2. Thus $f_{wu}^{-1}(L) \subseteq N^{n+m}$ is semi-linear. Let Π be the mapping of N^{n+m} onto N^m defined by $\Pi(\langle x(1), \cdots, x(n+m) \rangle) = \langle x(n+1), \cdots, x(n+m) \rangle$. Then Π preserves semi-linearity. Since $f_{wu}^{-1}(L) \cap (O^n \times N^m)$ is semi-linear, $\Pi(f_{wu}^{-1}(L) \cap (O^n \times N^m))$ is semi-linear. The lemma then follows from the fact that $f_u^{-1}(L) = \Pi(f_{wu}^{-1}(L) \cap (O^n \times N^m))$.

In view of Lemma 3.3, a set may be called semi-linear if it is semi-linear in some m-tuple $\langle u_1, \cdots, u_m \rangle$.

An important connection between bounded languages and semi-linear sets of words is provided by the next result.

THEOREM 3.1. *Every bounded language is semi-linear.*[10]

PROOF. Suppose $L \subseteq w_1^* \cdots w_n^*$ is a language. Let $a = \langle a_1, \cdots, a_n \rangle$ be an n-tuple of distinct symbols. Let S be the one state gsm mapping a_i into w_i for each i. Then $S^{-1}(L)$ is a language by Theorem 3.4 of [8]. By Theorem 1.1, $S^{-1}(L) \cap a_1^* \cdots a_n^*$ is then a language. By Theorem 1.2, $A = f_a^{-1}(S^{-1}(L) \cap a_1^* \cdots a_n^*)$ is a semi-linear subset of N^n. Since $A = f_w^{-1}(L)$, $f_w(A) = L$. By Lemma 3.2, $f_w(A) = L$ is semi-linear.

LEMMA 3.4. *If L_1 and L_2 are semi-linear sets of words, then so are (a) $L_1 \cup L_2$, (b) $L_1 \cap L_2$, (c) $L_1 - L_2$ and (d) $L_1 L_2$.*

PROOF. Let L_1 and L_2 be semi-linear in w and $u = \langle u_1, \cdots, u_m \rangle$ respectively. Let $wu = \langle w_1, \cdots, w_n, u_1, \cdots, u_m \rangle$. By Lemma 3.3, L_1 and L_2 are both semi-linear in wu.

(a) $f_{wu}^{-1}(L_1 \cup L_2) = f_{wu}^{-1}(L_1) \cup f_{wu}^{-1}(L_2)$ is semi-linear. Thus $L_1 \cup L_2$ is semi-linear. The proofs of (b) and (c) are similar to that of (a).

(d) $f_w^{-1}(L_1) \times f_u^{-1}(L_2)$ is semi-linear, and $f_{wu}(f_w^{-1}(L_1) \times f_u^{-1}(L_2)) = L_1 L_2$. By Lemma 3.2, $L_1 L_2$ is semi-linear.

LEMMA 3.5. *Let L be a semi-linear set of words, each of which contains exactly one occurrence of the symbol c. If $M \subseteq w_1^* \cdots w_n^*$ is semi-linear, then $\{uMv/ucv \text{ in } L\}$ is semi-linear.*

PROOF. Let $L \subseteq u_1^* \cdots u_r^* c^* v_1^* \cdots v_s^*$, where c occurs in no u_i or v_j. Let

$$y = \langle u_1, \cdots, u_r, c, \epsilon_1, \cdots, \epsilon_n, v_1, \cdots, v_s \rangle,$$

$$z = \langle \epsilon_1, \cdots, \epsilon_{r+1}, w_1, \cdots, w_n, \epsilon_1, \cdots, \epsilon_s \rangle,$$

$$x = \langle u_1, \cdots, u_r, \epsilon, w_1, \cdots, w_n, v_1, \cdots, v_s \rangle,$$

with each $\epsilon_i = \epsilon$. Since $f_y^{-1}(L)$ and $f_z^{-1}(M)$ are semi-linear, so is $f_y^{-1}(L) \cap f_z^{-1}(M)$. By Lemma 3.2, $f_x(f_y^{-1}(L) \cap f_z^{-1}(M))$ is semi-linear. Thus,

$$\{uMv/ucv \text{ in } L\} = f_x(f_y^{-1}(L) \cap f_z^{-1}(M))$$

is semi-linear.

LEMMA 3.6. *Let $S = (K, \Sigma, \Delta, \delta, \lambda, q_1)$ be a gsm and c a symbol not in $\Sigma \cup \Delta$. Then $\{xcS(x)/x \text{ in } w_1^* \cdots w_n^*\}$ is semi-linear.*

PROOF. By Lemma 4.3 of [9], $S(w_1^* \cdots w_n^*) \subseteq y_1^* \cdots y_r^*$ for some $y_1, \cdots, y_r$. Since $w_1^* \cdots w_n^*$ is regular, $A = \{x^k cx/x \text{ in } w_1^* \cdots w_n^*\}$ is a language by Theorem 2.1 of [1]. Clearly there exists a gsm T whose output is ϵ until the first input of c after which T duplicates S. Then

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[10] This result is comparable to Theorem 1.2.

$$T(A) = \{x^R c S(x)/x \text{ in } w_1^* \cdots w_n^*\}$$

is a language by Theorem 3.1 of [8]. Furthermore,

$$T(A) \subseteq (w_n^R)^* \cdots (w_1^R)^* c^* y_1^* \cdots y_r^*.$$

Let $\Gamma = \{a_1, \cdots, a_n, b_1, \cdots, b_r, c\}$ be a set of $n+r+1$ symbols. Let S_1 be the one state gsm which maps each a_i into w_i^R, c into c, and b_j into y_j. By Theorem 3.4 of [8], $S_1^{-1}(T(A))$ is a language. Thus

$$S_1^{-1}(T(A)) \cap a_n^* \cdots a_1^* c b_1^* \cdots b_r^* = B$$

is a language (by Theorem 1.1) and a subset of $a_n^* \cdots a_1^* c^* b_1^* \cdots b_r^*$. Let τ be the mapping of $\theta(\Gamma)$ into N^{n+r+1} defined by

$$\tau(z) = \langle \#_z(a_1), \cdots, \#_z(a_n), \#_z(c), \#_z(b_1), \cdots, \#_z(b_r) \rangle,$$

where $\#_z(x)$ is the number of occurrences of x in z. By Theorem 1.2, $\tau(B)$ is semi-linear. Let $C = \{xcS(x)/x \text{ in } w_1^* \cdots w_n^*\}$. Since $\tau(B) = f_d^{-1}(C)$, where $d = \langle w_1, \cdots, w_n, c, y_1, \cdots, y_r \rangle$, C is semi-linear.

COROLLARY. $\{xcx/x \text{ in } w_1^* \cdots w_n^*\}$ is semi-linear.

THEOREM 3.2. If $L \subseteq w_1^* \cdots w_n^*$ is semi-linear and S is a gsm, then $S(L)$ is semi-linear.

PROOF. By Lemma 4.3 of [9], $S(w_1^* \cdots w_n^*) \subseteq y_1^* \cdots y_r^*$ for some words $y_1, \cdots, y_r$. Let $S = (K, \Sigma, \Delta, \delta, \lambda, q_1)$ and let c be a symbol not in $\Sigma \cup \Delta$. By Lemma 3.6, $\{xcS(x)/x \text{ in } w_1^* \cdots w_n^*\}$ is semi-linear. By Lemma 3.4, $Lcy_1^* \cdots y_r^*$ is semi-linear. Then

$$A = \{xcS(x)/x \text{ in } L\} = \{xcS(x)/x \text{ in } w_1^* \cdots w_n^*\} \cap Lcy_1^* \cdots y_r^*$$

is semi-linear. Let Π be the mapping of N^{n+1+r} onto N^r defined by

$$\Pi(\langle x(1), \cdots, x(n + 1 + r) \rangle) = \langle x(n + 2), \cdots, x(n + 1 + r) \rangle.$$

Let

$$wcy = \langle w_1, \cdots, w_n, c, y_1, \cdots, y_r \rangle.$$

Since Π preserves semi-linearity, $S(L) = f_y \Pi f_{wcy}^{-1}(A)$ is semi-linear.

The inverse of the machine mapping does not preserve semi-linearity. For example, let S be the one state gsm mapping a and b into ϵ. Then $\{\epsilon\}$ is semi-linear, but $S^{-1}(\{\epsilon\}) = \theta(a, b)$ is not bounded by Corollary 2 to Theorem 3.1 of [9], and thus is not semi-linear. However there is the following result:

THEOREM 3.3. Let $L \subseteq w_1^* \cdots w_n^*$ and M be semi-linear. If S is a gsm, then $S^{-1}(M) \cap L$ is semi-linear.

PROOF. By Lemma 4.3 of [9], $S(w_1^* \cdots w_n^*) \subseteq y_1^* \cdots y_r^*$ for some words $y_1, \cdots, y_r$. Let $S = (K, \Sigma, \Delta, \delta, \lambda, q_1)$ and let c be a letter not in $\Sigma \cup \Delta$. Then $\{xcS(x)/x \text{ in } w_1^* \cdots w_n^*\}$, $Lcy_1^* \cdots y_r^*$, and $w_1^* \cdots w_n^* cM$ are semi-linear sets, so that

$$\{xcS(x)/x \text{ in } L \text{ and } S(x) \text{ in } M\}$$
$$= \{xcS(x)/x \text{ in } w_1^* \cdots w_n^*\} \cap Lcy_1^* \cdots y_r^* \cap w_1^* \cdots w_n^* cM$$

is semi-linear. The function Π defined by

$$\Pi(\langle x(1), \cdots, x(n + m + 1) \rangle) = \langle x(1), \cdots, x(n) \rangle$$

preserves semi-linearity. Let $wcy = \langle w_1, \cdots, w_n, c, y_1, \cdots, y_r \rangle$. Then

$$S^{-1}(M) \cap L = f_w \Pi f_{wcy}^{-1}(\{xcS(x)/x \text{ in } L \text{ and } S(x) \text{ in } M\}),$$

and thus is semi-linear.

THEOREM 3.4.   *Let S be a gsm and $L \subseteq w_1^* \cdots w_n^*$ be semi-linear. Then*

$$\alpha(S, L) = \{x \text{ in } S(L)/x = S(y) = S(z) \text{ for some } y, z \text{ in } L, y \neq z\}$$

is semi-linear.

PROOF.   Let $S = (K, \Sigma, \Delta, \delta, \lambda, q_1)$ and let c be a symbol not in $\Sigma \cup \Delta$. By the corollary to Lemma 3.6, $\{xcx/x \text{ in } w_1^* \cdots w_n^*\}$ is semi-linear. Since LcL is semi-linear, $A = LcL - \{xcx/x \text{ in } w_1^* \cdots w_n^*\}$ is semi-linear. Let T be the gsm $(K, \Sigma \cup \{c\}, \Delta, \delta', \lambda', q_1)$, where $\delta'(q, x) = \delta(q, x)$ and $\lambda'(q, x) = \lambda(q, x)$, for (q, x) in $K \times \Sigma$, and $\delta'(q, c) = q_1$ and $\lambda'(q, c) = c$, for each q in K. By Theorem 3.2, $T(A) = \{S(y)cS(z)/y, z \text{ in } L, y \neq z\}$ is semi-linear. Since $S(L)$ is bounded, $S(L) \subseteq y_1^* \cdots y_r^*$ for some $y_1, \cdots, y_r$ in $\theta(\Delta)$. Since $\{xcx/x \text{ in } y_1^* \cdots y_r^*\}$ is semi-linear, $B = T(A) \cap \{xcx/x \text{ in } y_1^* \cdots y_r^*\}$ is semi-linear. Let $y = \langle y_1, \cdots, y_r \rangle$ and $ycy = \langle y_1, \cdots, y_r, c, y_1, \cdots, y_r \rangle$. The function Π defined by

$$\Pi(\langle x(1), \cdots, x(2r + 1) \rangle) = \langle x(1), \cdots, x(r) \rangle$$

preserves semi-linearity. Then

$$f_y \Pi f_{ycy}^{-1}(B) = \{x/xcx \text{ in } B\} = \alpha(S, L)$$

is semi-linear.

From Theorems 3.3 and 3.4 there immediately follows:

COROLLARY.   *If $L \subseteq w_1^* \cdots w_n^*$ is semi-linear, S is a gsm, and $\alpha(S, L)$ is as in Theorem 3.4, then $S^{-1}(\alpha(S, L)) \cap L$ is semi-linear.*

4. *Ambiguity of a Bounded Grammar*

As noted earlier there is no decision procedure for determining whether an arbitrary grammar is ambiguous. It is now shown that the same problem for an arbitrary bounded grammar does have a decision procedure.

Definition.   Let $L_1, \cdots, L_r$ be sets of words. $\langle u_1, \cdots, u_r \rangle$ *represents* u in $\langle L_1, \cdots, L_r \rangle$ if each u_i is in L_i and $u = u_1 \cdots u_r$. If two r-tuples represent u in $\langle L_1, \cdots, L_r \rangle$, then u is said to be $\langle L_1, \cdots, L_r \rangle$-*ambiguous.*

LEMMA 4.1.   *If $L_1, \cdots, L_r$ $(r \geq 2)$ are semi-linear, then the set of $\langle L_1, \cdots, L_r \rangle$-ambiguous words is semi-linear.*

PROOF.   Let d be a symbol not in Σ. Then $L = L_1 d \cdots d L_r$ is semi-linear. Let $S = (\{p_1\}, \Sigma \cup \{d\}, \Sigma, \delta, \lambda, p_1)$ be the one-state gsm such that $S(x) = x$ for x in Σ and $S(d) = \epsilon$. Then the set of $\langle L_1, \cdots, L_r \rangle$-ambiguous words is

$$\alpha(S, L) = \{x \text{ in } S(L)/x = S(y) = S(z) \text{ for some } y, z \text{ in } L, y \neq z\}.$$

By Theorem 3.4, $\alpha(S, L)$ is semi-linear.

From Theorem 3.1 and Lemma 4.1 there follows

COROLLARY.   *If $L_1, \cdots, L_r$ $(r \geq 2)$ are bounded languages, then the set of $\langle L_1, \cdots, L_r \rangle$-ambiguous words is semi-linear.*

THEOREM 4.1.   *Given a bounded grammar G, the set of words H, ambiguously derivable in G, is semi-linear and effectively calculable. (That is, from G the constants and periods of linear sets whose union is H are effectively calculable.)*

PROOF. Let $G = (V, \Sigma, P, \sigma)$ be a grammar for a bounded language $L \subseteq w_1^* \cdots w_r^*$. By Theorem 5.2 of [1], we can test to see if $L = \varphi$. If $L = \varphi$, the theorem is trivial. Suppose $L \neq \varphi$. From Lemma 5.1 of [1] we may assume that

(1) each variable in G generates at least one word in $\theta(\Sigma)$, and

(2) for each variable ξ, σ generates a word in $\theta(V)$ containing ξ.

Let $\xi_1, \cdots, \xi_n$ be the distinct variables of G. For $1 \leq i \leq n$, let $w_{i,1}, \cdots, w_{i,m(i)}$ be all the distinct words w in $\theta(V)$ such that $\xi_i \to w$. For each word ψ in $\theta(V)$, let

$$L(\psi) = \{x/\psi \overset{*}{\Rightarrow} x, x \text{ in } \theta(\Sigma)\}.$$

By (1), (2) and Lemma 1.1 of [9], $L(\alpha)$ is a bounded language for each α in V. Since $L(\psi) = L(\epsilon) = \{\epsilon\}$, if $\psi = \epsilon$ and $L(\psi) = L(\alpha_1) \cdots L(\alpha_t)$ if $\psi = \alpha_1 \cdots \alpha_t$, $L(\psi)$ is a bounded language and, by Theorem 3.1, is semi-linear for each ψ in $\theta(V)$. Hence

$$\Gamma_i = \bigcup_{j \neq k} (L(w_{i,j}) \cap L(w_{i,k}))$$

is semi-linear for each i. For each (i, j) let $w_{i,j} = x_{i,j,1} \cdots x_{i,j,s(i,j)}$, where the $x_{i,j,k}$ are in $V[s(i, j) = 0, \text{ if } w_{i,j} = \epsilon]$. Let

$$\Delta_{i,j} = \{x/x \text{ is } \langle L(x_{i,j,1}), \cdots, L(x_{i,j,s(i,j)}) \rangle \text{-ambiguous}\}.$$

By the corollary to Lemma 4.1, $\Delta_{i,j}$ is semi-linear. Thus $\Delta_i = \bigcup_{j=1}^{m(i)} \Delta_{i,j}$ and $\Gamma_i \cup \Delta_i$ are semi-linear for each i.

For $1 \leq i \leq n$, let L_i be the set of all words x in $\theta(V)$ such that

(3) $\sigma \overset{*}{\Rightarrow} x$ and

(4) x has exactly one occurrence of ξ_i and no occurrence of any other variable.

By Theorem 1.1, it is easily seen that each L_i is a language. By Lemma 1.1 of [9], $\text{Init}(L) = \{w \neq \epsilon/wv \text{ in } L \text{ for some } v\}$ and $\text{Fin}(L) = \{w \neq \epsilon/vw \text{ in } L \text{ for some } v\}$ are bounded. Moreover,

$$L_i \subseteq [\text{Init}(L) \cup \{\epsilon\}]\xi_i[\text{Fin}(L) \cup \{\epsilon\}].$$

Thus L_i is a bounded language, and is semi-linear by Theorem 3.1. Let Q_i be the set of all words uwv with w in $\Gamma_i \cup \Delta_i$ and u, v in $\theta(\Sigma)$, such that $u\xi_i v$ is in L_i. By Lemma 3.5, Q_i is semi-linear. Thus $\bigcup_{i=1}^n Q_i$ is semi-linear. It is next shown that $\bigcup_{i=1}^n Q_i$ is the set of words ambiguously derivable in G. The theorem will then follow, since $\bigcup_1^n Q_i$ is effectively calculable.

Suppose that x is in Q_i for some i. Then there exist w in $\Gamma_i \cup \Delta_i$, and u, v in $\theta(\Sigma)$, such that $u\xi_i v$ is in L_i and $x = uwv$. Since $u\xi_i v$ is in L_i, $\sigma \overset{*}{\Rightarrow} u\xi_i v$ by a leftmost derivation. First assume w is in Γ_i. Then there are distinct j and k such that w is in $L(w_{i,j}) \cap L(w_{i,k})$. Then

$$\sigma \overset{*}{\Rightarrow} u\xi_i v \Rightarrow uw_{i,j}v \overset{*}{\Rightarrow} uwv \quad \text{and} \quad \sigma \overset{*}{\Rightarrow} u\xi_i v \Rightarrow uw_{i,k}v \overset{*}{\Rightarrow} uwv,$$

so that there are two distinct leftmost derivations of x in G. Now assume w is in Δ_i. Then w is in $\Delta_{i,j}$ for some j. Hence there are two distinct leftmost derivations of w from $w_{i,j}$, i.e., $w_{i,j} \overset{*}{\Rightarrow}^{(1)} w$ and $w_{i,j} \overset{*}{\Rightarrow}^{(2)} w$. Then

$$\sigma \overset{*}{\Rightarrow} u\xi_i v \Rightarrow uw_{i,j}v \overset{*}{\Rightarrow}^{(1)} uwv \quad \text{and} \quad \sigma \overset{*}{\Rightarrow} u\xi_i v \Rightarrow uw_{i,j}v \overset{*}{\Rightarrow}^{(2)} uwv,$$

so that there again are two distinct leftmost derivations of x in G. Thus x is ambiguously derivable in G.

The following terminology is used in the remainder of the proof. If T_1 and T_2 are generation trees, then T_1 and T_2 are said to *diverge* at level k if k is the largest in-

teger with the following property: The subgraph of T_1 which consists of all nodes and node names in T_1 of level $<k$ is identical with the subgraph of T_2 which consists of all nodes and node names in T_2 of level $<k$.

Suppose that x is an ambiguously derivable word in G. Then there exist distinct generation trees T_1 and T_2 of x from σ. Hence there is a smallest integer $p \geq 2$ with the property that there exist a variable ξ_i, a word w in $\theta(\Sigma)$, and subtrees S_1 of T_1 and S_2 of T_2 which diverge at level p and which both generate w from ξ_i. Two cases arise.

(α) $p = 2$. Then there exist distinct j and k such that $\xi_i \rightarrow w_{i,j}$ is the production used at level 1 in S_1 and $\xi_i \rightarrow w_{i,k}$ is the production used at level 1 in S_2. Clearly w is in $L(w_{i,j}) \cap L(w_{i,k})$ and hence in Γ_i.

(β) $p > 2$. Then the production used at level 1 in S_1 and level 1 in S_2 is the same, say

$$\xi_i \rightarrow w_{i,j} = x_{i,j,1} \cdots x_{i,j,s(i,j)} .$$

Let $\alpha_1, \cdots, \alpha_{s(i,j)}$ and $\beta_1, \cdots, \beta_{s(i,j)}$ be the occurrences (in the obvious order) of $x_{i,j,1}, \cdots, x_{i,j,s(i,j)}$ at level 2 in S_1 and S_2, respectively. For each k let $w(\alpha_k)$ be α_k if α_k is in Σ, and let $w(\alpha_k)$ be the word generated by the subtree of α_k otherwise. Similarly for $w(\beta_k)$. Then

$$w = w(\alpha_1) \cdots w(\alpha_k) = w(\beta_1) \cdots w(\beta_k)$$

by hypothesis. Since S_1 and S_2 diverge at level $p > 2$, there exists an integer k such that the subtree of α_k diverges at level $p-1$ from the subtree of β_k. By the minimality of p, $w(\alpha_k) \neq w(\beta_k)$. This implies that w is $\langle L(x_{i,j,1}), \cdots, L(x_{i,j,s(i,j)})\rangle$-ambiguous. Then w is in $\Delta_{i,j}$ and thus in Δ_i.

In either case, w is in $\Gamma_i \cup \Delta_i$. If the occurrence of S_1 in T_1 is replaced with ξ_i, then a generation tree from σ of $u\xi_i v$ is obtained for some u, v in $\theta(\Sigma)$. Then $u\xi_i v$ is in L_i. From this it follows that $x = uwv$ is in Q_i. Thus $\bigcup_i Q_i$ is the set of ambiguously derivable words in G.

COROLLARY.   *It is solvable whether an arbitrary bounded grammar is ambiguous.*

5. *Inherently Ambiguous Languages in* $a_1^* \cdots a_n^*$

An algebraic characterization (Theorem 6.1) of the inherently ambiguous bounded languages is now considered. In this section a characterization is provided of inherently ambiguous languages contained in $a_1^* \cdots a_n^*$, where the a_i are distinct symbols. This result is used in Section 6 to derive an algebraic criterion for the general case, that is, a characterization of inherently ambiguous languages contained in $w_1^* \cdots w_n^*$, where the w_i are words.

LEMMA 5.1.   *Let $\Sigma = \{a_i/1 \leq i \leq n\}$ and $a = \langle a_1, \cdots, a_n\rangle$. If M is a language contained in $a_1^* \cdots a_n^*$ and $f_a^{-1}(M)$ is the finite union of disjoint linear sets L_i, each with stratified and independent[11] periods, then M has an unambiguous grammar.*

PROOF.   By Theorem 1.3 and the fact that f_a is one to one, M is the finite union of the disjoint languages $f_a(L_i)$. It obviously suffices to show that each $f_a(L_i)$ has an unambiguous grammar. Hence without loss of generality it may be assumed that $f_a^{-1}(M)$ is a linear set $L = L(c; p_1, \cdots, p_t)$ with stratified and independent periods.

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[11] A set of elements in N^n is *independent* (*dependent*) if it is independent (dependent) as a set of vectors over the rational numbers.

It may further be supposed that $p_1, \cdots, p_s$ $(0 \le s \le t)$ are the periods with two nonzero coordinates. For $1 \le i \le t$, let $p_i = (p_{i1}, \cdots, p_{in})$.[12] For $1 \le i \le s$, let $p_{ij(i)}$ and $p_{ik(i)}$, $j(i) < k(i)$, be the nonzero coordinates of p_i. If $j(i) = j(h)$ and $k(i) = k(h)$, then p_i is said to *match* p_h $(i, h \le s)$. By independence, each p_i $(i \le s)$ matches at most one period other than itself. It may be supposed that $r \le s$ is such that (i) if $i, h \le r$ and p_i matches p_h, then $i = h$; and (ii) if $r < h \le s$, then there exists $i \le r$ such that p_i matches p_h. A period p_m *dominates* a period p_q $(m \ne q; m, q \le r)$ if $j(m) \le j(q) < k(q) \le k(m)$ and there is no $u \le r$ such that $u \ne q$, $u \ne m$, and $j(m) \le j(u) \le j(q) < k(q) \le k(u) \le k(m)$. Again it may be assumed that $p_1, \cdots, p_q$ $(q \le r)$ are the periods dominated by no p_i $(i \le r)$ and that $j(1) < \cdots < j(q)$. Thus $j(1) < k(1) \le j(2) < k(2) \le \cdots \le j(q) < k(q)$. For each $i \le r$, let $f(i, 1), \cdots, f(i, m(i))$ be the indices of those periods, if any, dominated by p_i, with $j(f(i, 1)) < \cdots < j(f(i, m(i)))$. If p_i $(i \le r)$ dominates no period, p_i is called *final*.

Let $G = (V, \Sigma, P, \sigma)$, where $V = \Sigma \cup \{\sigma, \xi_1, \cdots, \xi_s, \gamma_1, \cdots, \gamma_n\}$ and P consists of the following productions:

(i) $\sigma \to \gamma_1 \cdots \gamma_{j(1)} \xi_1 \gamma_{k(1)} \gamma_{k(1)+1} \cdots \gamma_{j(2)} \xi_2 \gamma_{k(2)} \cdots \gamma_{j(q)} \xi_q \gamma_{k(q)} \cdots \gamma_n$.

(If $s = q = 0$ this is to be interpreted as $\sigma \to \gamma_1 \cdots \gamma_n$.)

(ii)   (a) $\xi_h \to \gamma_{j(i)+1} \cdots \gamma_{j(f(i,1))} \xi_{f(i,1)} \gamma_{k(f(i,1))} \gamma_{k(f(i,1))+1} \cdots \gamma_{j(f(i,2))}$
$\xi_{f(i,2)} \gamma_{k(f(i,2))} \cdots \gamma_{j(f(i,m(i)))} \xi_{f(i,m(i))} \gamma_{k(f(i,m(i)))} \cdots \gamma_{k(i)-1}$

if p_h matches p_i, $i \le r$, and p_i is not final. (If $j(f(i, 1)) = j(i)$ then the terms preceding $\xi_{f(i,1)}$ vanish. Similarly if $k(f(i, m(i))) = k(i)$ then the terms following $\xi_{f(i,m(i))}$ vanish.)

(b) $\xi_h \to \gamma_{j(i)+1} \cdots \gamma_{k(i)-1}$ if p_h matches p_i, $i \le r$, and p_i is final. (If $j(i) + 1 = k(i)$ this is to be interpreted as $\xi_h \to \epsilon$.)

(c) $\xi_i \to a_{j(h)}^{p_{hj(h)}} \xi_h a_{k(h)}^{p_{hk(h)}}$ if $i \le r < h \le s$ and p_i matches p_h .

(iii) $\xi_i \to a_{j(i)}^{p_{ij(i)}} \xi_i a_{k(i)}^{p_{ik(i)}}$ for $1 \le i \le s$.

(iv) $\gamma_{g(i)} \to \gamma_{g(i)} a_{g(i)}^{p_{ig(i)}}$ for $s < i \le t$, where $p_{ig(i)} > 0$.

(v) $\gamma_i \to a_i^{c_i}$ for $1 \le i \le n$.

We complete the proof of the lemma by showing that G is an unambiguous grammar generating M.

Call a production a *μ-production* if it is of the form $\mu \to w$. Now suppose that

$$(*) \qquad \sigma = w_0 \Rightarrow \cdots \Rightarrow w_d = w$$

is a derivation of a word w in $L(G)$. Obviously there is only one application of production (i) used in $(*)$. For each $i \le s$ there is at most one application of a ξ_i-production of type (ii). (For after application of a ξ_i-production of type (ii), there is no way for ξ_i to occur again.) And for each $i \le s$ there is at least one application of a ξ_i-production of type (ii) if ξ_i occurs in $(*)$. (For the only productions of the form $\xi_i \to v$ with ξ_i not in v are of type (ii).) For $1 \le i \le r$, ξ_i must occur in $(*)$. (If $i \le q$, then ξ_i occurs to the right of $\to$ of the σ-production. If $q < i \le r$, then p_i is dominated by some period, so that ξ_i occurs in $(*)$ by an application of a production of type (a).) For $r < i \le s$, ξ_i occurs in $(*)$ if and only if there is an application of a production of type (c) with ξ_i to the right of $\to$. For $1 \le i \le t$, let $y_i = 1$, if $r < i \le s$ and ξ_i occurs in $(*)$, and $y_i = 0$ otherwise. For $1 \le i \le n$, there is at least one occurrence of γ_i in $(*)$. (This is because σ and each ξ_i, $1 \le$

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[12] The form $(p_{i1}, \cdots, p_{in})$ is used in this proof instead of $(p_i(1), \cdots, p_i(n))$ to simplify the notation.

$i \leq r$, occurs in $(*)$.) Thus for $1 \leq i \leq n$, there is exactly one application of the γ_i-production of type (v). For $1 \leq i \leq s$, let z_i be the number of applications of the ξ_i-production of type (iii) in $(*)$; and for $s < i \leq t$, let z_i be the number of applications of the $\gamma_{g(i)}$-production of type (iv) in $(*)$. Then for $1 \leq i \leq t$, z_i is a nonnegative integer. Elements of Σ occur in $(*)$ only by application of productions of types (c), (iii), (iv) and (v). Those from (v) form the coordinates of the constant c, and the others correspond to occurrences of periods.

From the above discussion it is clear that $w = f_a(c + \sum_{i=1}^{t} (y_i + z_i)p_i)$. Thus $L(G) \subseteq M$.

Suppose that w is in M. Then there exist $x_1, \cdots, x_t \geq 0$ such that $f_a^{-1}(w) = c + \sum_{i=1}^{t} x_i p_i$. Since the periods are linearly independent, the x_i are uniquely determined. Using the notation above, a derivation D_w, from σ, can be constructed in which (α) $z_i = x_i - 1$ and $y_i = 1$, if $r < i \leq s$ and $x_i > 0$, (β) $z_i = x_i$ and $y_i = 0$ otherwise. Then $x_i = y_i + z_i$ for each i, so that D_w is a derivation of w and w is in $L(G)$. Thus $M \subseteq L(G)$, whence $L(G) = M$.

Now from the uniqueness of the x_i, it follows that any derivation D, from σ, of w satisfies (α) and (β). Then D uses every production in P exactly as often as D_w does. From this it follows that D and D_w have the same generation tree. Thus G is unambiguous.

LEMMA 5.2.   Let $\Sigma = \{a_j / 1 \leq j \leq n\}$ and $a = \langle a_1, \cdots, a_n \rangle$. If a language M contained in $a_1^* \cdots a_n^*$ has an unambiguous grammar, then $f_a^{-1}(M)$ is the finite union of disjoint linear sets L_i, each with stratified and independent periods.

PROOF.   Let $G = (V, \Sigma, P, \sigma)$ be an unambiguous grammar for M. Clearly it may be assumed that M is infinite since the lemma is trivially true otherwise. It may also be assumed that no production in P is of the form $\xi \rightarrow \epsilon$. For otherwise, by Lemma 4.1 of [1],[13] an unambiguous grammar G' can be constructed in which no production is of the form $\xi \rightarrow \epsilon$ such that $L(G') = M - \{\epsilon\}$, and then $L(G')$ can be considered. Furthermore, it may be supposed that for every ξ in $V - \Sigma$, (i) there exists a word u in $\theta(\Sigma)$ such that $\xi \overset{*}{\Rightarrow} u$, and (ii) there exist words v_1, v_2 in $\theta(V)$ such that $\sigma \overset{*}{\Rightarrow} v_1 \xi v_2$. For otherwise, by Lemma 5.1 of [1],[13] an unambiguous grammar G'' can be found satisfying (i) and (ii) such that $L(G'') = M$. Thus,

(1) There is no ξ in $V - \Sigma$ and w in $\theta(V)$ such that w is ambiguously derivable from ξ in G.

Finally, it may be supposed that for every ξ in $V - \Sigma$ there are distinct words u_1, u_2 in $\theta(\Sigma)$ such that $\xi \overset{*}{\Rightarrow} u_1$ and $\xi \overset{*}{\Rightarrow} u_2$. For if G contains a set Γ of variables such that each ξ in Γ derives only one word w_ξ in $\theta(\Sigma)$, P' can easily be constructed such that $G''' = (V - \Gamma, \Sigma, P', \sigma)$ is unambiguous, $L(G''') = M$, and each ξ in $(V - \Gamma) - \Sigma$ derives (in G''') at least two words in $\theta(\Sigma)$. The proof of Lemma 5.2 is a modification of an argument due to Parikh [12, pp. 203–205].

(2) For ξ in $V - \Sigma$, $\xi \overset{*}{\Rightarrow} \xi$ has no nontrivial derivation, that is, no derivation using a production.

For if there were a nontrivial derivation, then there would be two of them, by iteration, contradicting (1).

(3) No generation tree in G has incomparable occurrences of the same variable.

For otherwise, $\sigma \overset{*}{\Rightarrow} w_1 \xi w_2 \xi w_3$ for some ξ in $V - \Sigma$ and w_1, w_2, w_3 in $\theta(\Sigma)$. Let u and v be distinct words in $\theta(\Sigma)$ such that $\xi \overset{*}{\Rightarrow} u$ and $\xi \overset{*}{\Rightarrow} v$. Then (i) $\sigma \overset{*}{\Rightarrow} w_1 u w_2 v w_3$,

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[13] The statement of this result says nothing about unambiguous grammars. However the proof shows that the final grammar is unambiguous if the initial one is.

and (ii) $\sigma \overset{*}{\Rightarrow} w_1 v w_2 u w_3$. Both $u w_2 v$ and $v w_2 u$ are subwords of words in M. Since neither u nor v can be ϵ, it follows that $u w_2 v$ is in $a_j{}^*$ for some j ($1 \leq j \leq n$). Thus $u w_2 v = v w_2 u$. Then (i) and (ii) provide two distinct derivations in G of $w_1 u w_2 v w_3$, contradicting (1).

Let ξ be in $V - \Sigma$. If $\xi \overset{*}{\Rightarrow} u \xi v (u, v$ in $\theta(\Sigma); uv \neq \epsilon)$, then we call (u, v) a *period* of ξ.

(4) If (u_1, v_1) and (u_2, v_2) are periods of ξ, then there are $i, j \leq n$ such that u_1, u_2 are in $a_i{}^*$ and v_1, v_2 are in $a_j{}^*$.

For let (u_1, v_1) and (u_2, v_2) be periods of ξ. Then

$$\xi \overset{*}{\Rightarrow} u_2 u_1 u_2 u_1 \xi v_1 v_2 v_1 v_2 \overset{*}{\Rightarrow} u_2 u_1 u_2 u_1 w v_1 v_2 v_1 v_2$$

for some w in $\theta(\Sigma)$. Then $u_2 u_1 u_2 u_1$ and $v_1 v_2 v_1 v_2$ are subwords of words in M. Thus $u_1 u_2$ is in $a_i{}^*$ and $v_2 v_1$ is in $a_j{}^*$, for some i, j, whence (4).

If $\xi \overset{*}{\Rightarrow} u \xi v$, $uv \neq \epsilon$, by a generation tree with exactly two occurrences of ξ, then (u, v) is called a *minimal* period of ξ.

(5) ξ has at most one minimal period (hence exactly one if it has any periods at all).

For suppose the contrary. By (4), there are $i, j \leq n$ such that $(a_i{}^{p_1}, a_j{}^{q_1})$ and $(a_i{}^{p_2}, a_j{}^{q_2})$ are minimal periods of ξ, with $(p_1, q_1) \neq (p_2, q_2)$. Let T_1 and T_2 be the generation trees from ξ of $a_i{}^{p_1} \xi a_j{}^{q_1}$ and $a_i{}^{p_2} \xi a_j{}^{q_2}$, respectively. Both T_1 and T_2 contain exactly two occurrences of ξ. Let T' (T'') be the result of replacing the lower occurrence of ξ in $T_1(T_2)$ by $T_2(T_1)$. Both T' and T'' are generation trees from ξ of $a_i{}^{p_1+p_2} \xi a_j{}^{q_1+q_2}$, and each has three occurrences of ξ. However, T' and T'' are distinct. This is because in T'' the subtree of the middle occurrence of ξ is T_1, while in T' the subtree of the middle occurrence of ξ is T_2. This contradicts (1).

If $(a_i{}^p, a_j{}^q)$ is the minimal period of ξ, then there is a unique generation tree T of $a_i{}^p \xi a_j{}^q$ from ξ. No production used in T can have two occurrences of variables on its right side. For suppose that one did. Then $\xi \overset{*}{\Rightarrow} w_1 \xi_1 w_2 \xi_2 w_3$ for some w_1, w_2, w_3 in $\theta(V)$, and ξ_1, ξ_2 in $V - \Sigma$. Then either

$$w_1 \xi_1 w_2 \xi_2 w_3 \overset{*}{\Rightarrow} w_4 \xi_1 w_5 \xi_2 w_3 \quad \text{or} \quad w_1 \xi_1 w_2 \xi_2 w_3 \overset{*}{\Rightarrow} w_1 \xi_1 w_6 \xi_2 w_7$$

for some w_4, w_5, w_6, w_7 in $\theta(V)$. If the former is true, then

$$w_4 \xi_1 w_5 \xi_2 w_3 \overset{*}{\Rightarrow} w_4{}^2 \xi_1 w_5 \xi_2 w_3 w_5 \xi_2 w_3 .$$

If the latter is true, then

$$w_1 \xi_1 w_6 \xi_2 w_7 \overset{*}{\Rightarrow} w_1 \xi_1 w_6 w_1 \xi_1 w_6 \xi_2 w_7{}^2 .$$

Thus either $\xi \overset{*}{\Rightarrow} w_4{}^2 \xi_1 w_5 \xi_2 w_3 w_5 \xi_2 w_3$ or $\xi \overset{*}{\Rightarrow} w_1 \xi_1 w_6 w_1 \xi_1 w_6 \xi_2 w_7{}^2$, a contradiction of (3). Hence for some $k \geq 0$, T represents a derivation

$$\xi \Rightarrow a_i{}^{p_1} \xi_1 a_j{}^{q_1} \Rightarrow \cdots \Rightarrow a_i{}^{p_k} \xi_k a_j{}^{q_k} \Rightarrow a_i{}^p \xi a_j{}^q .$$

Since T is the unique tree representing a derivation from ξ of $a_i{}^p \xi a_j{}^q$ and $(a_i{}^p, a_j{}^q)$ is ξ's minimal period, $\xi_1, \cdots, \xi_k$ are all distinct from ξ. Moreover, $\xi_1, \cdots, \xi_k$ are distinct variables. For suppose that x is the smallest integer such that $\xi_x = \xi_v$ for some $v > x$. Let y and z be the smallest and largest integers q such that $x < q \leq k$ and $\xi_x = \xi_q$. Then there are two distinct derivations of the minimal period of ξ_x. One derivation involves those ξ_m such that $x \leq m < y$. The other involves both x and those ξ, such that $z < m \leq k$ or $1 \leq m < x$. This contradicts (1), since one

derivation involves ξ and the other does not. Furthermore, each ξ_u $(1 \leq u \leq k)$ has (a_i^p, a_j^q) as its minimal period. For the derivation

$$\xi_u \Rightarrow a_i^{p_{u+1} - p_u} \xi_{u+1} a_j^{q_{u+1} - q_u} \Rightarrow \cdots \Rightarrow a_i^{p_k - p_u} \xi_k a_j^{q_k - q_u}$$

$$\Rightarrow a_i^{p - p_u} \xi a_j^{q - q_u} \Rightarrow a_i^{p + p_1 - p_u} \xi_1 a_j^{q + q_1 - q_u}$$

$$\Rightarrow \cdots \Rightarrow a_i^p \xi_u a_j^q$$

is represented by a tree which contains exactly two occurrences of ξ_j. Let $S_\xi = \{\xi, \xi_1, \cdots, \xi_k\}$. Then

(6)   every two variables in S_ξ have the same minimal period.

Let V' be any subset of V containing $\Sigma \cup \{\sigma\}$. Let M' be the set of all words x in M such that in some generation tree from σ of x the variables used are exactly all the elements of $V' - \Sigma$. Clearly there are only finitely many such M', and M is their union. Since G is unambiguous, the M' are pairwise disjoint. It thus suffices to show that $f_a^{-1}(M')$ is the finite union of disjoint linear sets, each with stratified and independent periods.

Let τ be the extension of f_a^{-1} defined on $\theta(V')$ by $\tau(\xi) = (0, \cdots, 0)$ for each ξ in $V' - \Sigma$ and $\tau(z_1 \cdots z_k) = \sum_{i=1}^k \tau(z_i)$, each z_i in V'. Let $(a_{f(1)}^{p_1}, a_{g(1)}^{q_1})$, $\cdots$, $(a_{f(r)}^{p_r}, a_{g(r)}^{q_r})$ be all the distinct minimal periods which belong to variables ξ such that $S_\xi \subseteq V'$. (There may be no such pairs.) For $1 \leq i \leq r$, let $v_i = \tau(a_{f(i)}^{p_i} a_{g(i)}^{q_i})$ and let ξ_i be a variable with minimal period $(a_{f(i)}^{p_i}, a_{g(i)}^{q_i})$, $S_{\xi_i} \subseteq V'$. By definition of minimal period, no v_i is $(0, \cdots, 0)$. Let J be the set of all words in $\theta(\Sigma)$ for which there is a generation tree from σ containing only occurrences of variables in V' and containing either one or two occurrences of each of the variables in V'. J is finite (possibly empty). Let $z_1, \cdots, z_s$ be all those words z in J for which there is no w in J and v_i $(1 \leq i \leq r)$ such that $\tau(z) = \tau(w) + v_i$. Finally, let $u_i = \tau(z_i)$ and $A_i = \{u_i + \sum_{j=1}^r k_j v_j / k_j \geq 0\}$, $1 \leq i \leq s$. To complete the proof of the lemma it suffices to show that

(a) $\tau(M') = \bigcup_{i=1}^s A_i$,

(b)   each A_i has stratified and independent periods,

(c)   the A_i are pairwise disjoint.

If ξ in $V' - \Sigma$ has minimal period (x, y) and $S_\xi \subseteq V'$, for $k \geq 0$ let T_ξ^k be the generation tree from ξ of $x^k \xi y^k$. (T_ξ^0 is the single node ξ.)

Now let x be in some A_i. If $x = u_i$, then $x = \tau(z_i)$, and z_i is in M' by definition of J. Suppose that $x = y + v_j$ for y in $A_i \cap \tau(M')$ and $1 \leq j \leq r$. There is a generation tree T from σ of a word w in M' such that $\tau(w) = y$. Since w is in M', T contains an occurrence of ξ_j. Let T' be the tree obtained by replacing one occurrence of ξ_j in T by $T_{\xi_j}^1$. Then T' (i) is a generation tree from σ of a word z such that $\tau(z) = y + v_j = x$, and, (ii) contains occurrences of exactly the variables in V'. Thus z is in M'. Then x is in $\tau(M')$ and $A_i \subseteq \tau(M')$ by induction.

Let w be in M' with T a generation tree from σ of w. Let k be the largest integer such that some variable occurs k times in T. If $k \leq 2$ then w is in J. From the definition of $z_1, \cdots, z_s$ it follows that there exist u_i $(1 \leq i \leq s)$ and non-negative integers $k_1, \cdots, k_r$ such that $\tau(w) = u_i + \sum_{j=1}^r k_j v_j$. Thus $\tau(w)$ is in A_i. Suppose that $k > 2$ and that $\tau(z)$ is in some A_i $(1 \leq i \leq s)$ whenever z is in M', and the generation tree of z from σ contains less than k occurrences of each variable of V'. Since T contains $k > 2$ occurrences of some variable γ_1 and contains no incomparable occurrences of γ_1 by (3), T contains at least two occurrences of $T_{\gamma_1}^1$. For some

number $h(1)$, $1 \leq h(1) \leq r$, γ_1 has the same minimal period as $\xi_{h(1)}$. Let T_1 be the tree obtained by replacing one occurrence of $T^1_{\gamma_1}$ by an occurrence of γ_1. Then T_1 generates a word t_1 in $\theta(\Sigma)$ from σ such that $\tau(w) = \tau(t_1) + v_{h(1)}$. T_1 contains every variable contained in T since there were at least two occurrences of $T^1_{\gamma_1}$ in T. Also T_1 has $k-1$ occurrences of γ_1. If T_1 contains k occurrences of some variable γ_2, a tree T_2 may be obtained by the same procedure. After a finite number of such operations a tree T_d is obtained which generates a word z from σ such that $\tau(w) = \tau(z) + \sum_{j=1}^{d} v_{h(j)}$, where T_d contains occurrences of each variable in V' and has at most $k-1$ occurrences of any variable. Then z is in M', $\tau(z)$ is in some A_i by the induction hypothesis, and $\tau(w)$ is in the same A_i. Therefore $\tau(M') \subseteq \bigcup_1^s A_i$, completing the proof of (a).

It follows from (4) that no A_i has a period with more that two nonzero co-ordinates. To establish that the periods of A_i are stratified it therefore suffices to show that v_i and v_j do not interlace $(i, j \leq r)$. Let T be the generation tree from σ of z_1. Let v_i have nonzero coordinates at its αth and βth places, v_j at its μth and νth places, and suppose that $\alpha < \mu < \beta < \nu$. Let T' be the tree which is obtained by replacing some occurrence of ξ_i and some occurrence of ξ_j in T by $T^1_{\xi_i}$ and $T^1_{\xi_j}$ respectively. Then T' generates a word w in M' from σ. In w either some a_β occurs before some a_μ, or some a_μ before some a_α, or some a_ν before some a_β. Each contradicts the fact that w is in $a_1^* \cdots a_n^*$. Hence the periods are stratified.

Suppose that the periods $v_1, \cdots, v_r$ are dependent. Then there exist non-negative integers $c_1, \cdots, c_r, k_1, \cdots, k_r$ and m $(1 \leq m \leq r)$ such that $\sum_{i=1}^{r} c_i v_i = \sum_{i=1}^{r} k_i v_i$ and $c_m \neq k_m$. Again let T be the generation tree from σ of z_1, and let z be the word in M' such that $\tau(z) = \tau(z_1) + \sum_{i=1}^{r} c_i v_i$. Let T' be the tree obtained from T by replacing simultaneously for each i $(1 \leq i \leq r)$ one occurrence of ξ_i by $T^{c_i}_{\xi_i}$. Then T' is a generation tree from σ of a word z' in M'. Since $\tau(z') = \tau(z_1) + \sum_{i=1}^{r} \tau(a_{f(i)}^{c_i p_i} a_{g(i)}^{c_i q_i}) = \tau(z_1) + \sum_1^r c_i v_i$, $z' = z$. Now T' contains c_m more occurrences of ξ_m than does T. For $T^{c_m}_{\xi_m}$ contains $c_m + 1$ occurrences of ξ_m while $T^{c_j}_{\xi_j}$ contains none if $j \neq m$ by (6). Similarly a generation tree T'' of z from σ is constructed in which there are k_m more occurrences of ξ_m than in T. Since G is unambiguous, T' and T'' are identical. This is a contradiction. Hence $v_1, \cdots, v_r$ are independent and (b) is established.

Suppose that the A_i are not pairwise disjoint. Then there is some n-tuple ψ, minimal with respect to the sum of its coordinates, such that ψ is in two different A_i, say A_j and A_m. Hence there exist non-negative integers $c_1, \cdots, c_r, k_1, \cdots, k_r$ such that

(7) $\psi = u_j + \sum_{i=1}^{r} c_i v_i = u_m + \sum_{i=1}^{r} k_i v_i$.

Since u_j is not in A_m and u_m is not in A_j, there are numbers p and q such that c_p and k_q are positive. Thus $\psi - v_p$ is in A_j and $\psi - v_q$ is in A_m. Let T be the generation tree from σ of the word z in M' such that $\tau(z) = \psi$. Now $T^1_{\xi_p}$ and $T^1_{\xi_q}$ have no variable in common. For otherwise $S_{\xi_p} \cap S_{\xi_q} \neq \varphi$, whence $v_p = v_q$ by (6). Then $\psi - v_p$ is in $A_j \cap A_m$, which contradicts the minimality of ψ. By the construction procedure of the previous paragraph, T contains an occurrence β_p of $T^1_{\xi_p}$ and an occurrence β_q of $T^1_{\xi_q}$. Let T' be the generation tree obtained from T by replacing β_p by an occurrence of ξ_p and β_q by ξ_q. T' is a generation tree from σ of a word w such that $\tau(w) = \psi - v_p - v_q$. Consider T'. $S_{\xi_p} \cap S_{\xi_q} = \varphi$. Since $\psi - v_p$ is in A_j and $\psi - v_q$ is in A_m, each variable in $S_{\xi_p} \cup S_{\xi_q}$ occurs at least twice in T. Exactly one occurrence of each variable in $S_{\xi_p} \cup S_{\xi_q}$ is deleted in going from T to T'

and no occurrence of any other variable is deleted. Thus T' contains an occurrence of each variable in V'. Therefore w is in M'. Thus $\tau(w) = \psi - v_p - v_q$ is in A_h for some h. Then $\psi - v_p$, $\psi - v_q$ are in A_h, $\psi - v_p$ is in A_j, and $\psi - v_q$ is in A_m. Since $j \neq m$, either $h \neq j$ or $h \neq m$, say the former. Then $\psi - v_p$ is in $A_j \cap A_h$, contradicting the minimality property of ψ. Another contradiction arises if $h \neq m$. Thus (c) is established and the lemma is proved.

Combining Lemmas 5.1 and 5.2 we have

THEOREM 5.1. *Let* $\Sigma = \{a_j / 1 \leq j \leq n\}$ *and* $a = \langle a_1, \cdots, a_n \rangle$. *A language* M *contained in* $a_1^* \cdots a_n^*$ *is inherently ambiguous if and only if* $f_a^{-1}(M)$ *is not a finite union of disjoint linear sets, each with stratified and independent periods.*

6. *Inherent Ambiguity for* $M \subseteq w_1^* \cdots w_n^*$

We now generalize Theorem 5.1 to the case where M is an arbitrary bounded language. The main result (Theorem 6.1), obtained as the culmination of a sequence of lemmas, is that a language M contained in $w_1^* \cdots w_n^*$ is inherently ambiguous if and only if $f_w^{-1}(M)$, $w = \langle w_1, \cdots, w_n \rangle$, is not a finite union of disjoint linear sets, each with stratified and independent periods. Using this characterization, the inherent ambiguity is then established of the languages involved in the proof of the unsolvability of the inherent ambiguity problem for arbitrary languages. The problem of determining for an arbitrary bounded language whether it satisfies this condition is unresolved. The authors believe that a decision procedure does exist.

LEMMA 6.1. *Let* $w = \langle w_1, \cdots, w_n \rangle$ *be an* n-*tuple of words and* M *a language contained in* $w_1^* \cdots w_n^*$. *If* M *has an unambiguous grammar, then* $f_w^{-1}(M)$ *is a finite union of disjoint linear sets, each with stratified and independent periods.*

PROOF. Let $a_1, \cdots, a_n$ be n letters not in the alphabet of w and $a = \langle a_1, \cdots, a_n \rangle$. Then f_a is a one to one function. Thus by Theorem 5.1, it suffices to show that $f_a f_w^{-1}(M)$ has an unambiguous grammar.

Let $G = (V, \Sigma, P, \sigma)$ be an unambiguous grammar for M. Each production π of P is of the form $\xi \rightarrow \gamma_0 c_1 \gamma_1 \cdots c_m \gamma_m$, where ξ is in $V - \Sigma$, $m \geq 0$, each c_i is in Σ, and each γ_i is in $\theta(V - \Sigma)$. For each such production π of P let

$$\Pi^+ = \{\xi \rightarrow \gamma_0 \mu_1 \gamma_1 \cdots \mu_m \gamma_m / \mu_i = c_i a_j \text{ or } \mu_i = c_i, 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}.$$

Clearly $\Pi_i^+ \cap \Pi_j^+ = \varphi$ if $\pi_i \neq \pi_j$. Let $P' = \bigcup_{\pi \text{ in } P} \Pi^+$. Consider the grammar $G' = (V \cup \{a_1, \cdots, a_n\}, \Sigma \cup \{a_1, \cdots, a_n\}, P', \sigma)$. Suppose that G' is ambiguous. Then there exist two leftmost derivations in G' of a word w' in $L(G')$. For each of these derivations consider the leftmost derivation in G obtained by replacing each application of a production in a set Π^+ by an application of π. The resulting derivations both generate the word obtained by deleting all occurrences of $a_1, \cdots, a_n$ in w'. Since G is unambiguous, the associated leftmost derivations in G are the same. Thus the two leftmost derivations in G' differ only in the use of different productions from the same sets Π^+. This is impossible since the use of different productions from a set Π^+ results in one of the generated words having an occurrence of some a_i which is not matched in the other. Thus,

(1) G' is unambiguous.

The set $(w_1 a_1)^* \cdots (w_n a_n)^*$ is regular. By Theorem 1.1, there exists an unambiguous grammar $G'' = (V'', \Sigma \cup \{a_1, \cdots, a_n\}, P'', \sigma)$ for $L(G') \cap (w_1 a_1)^* \cdots (w_n a_n)^*$. Let $G''' = (V'', \{a_1, \cdots, a_n\}, P''', \sigma)$, where $P''' = P'' \cup \{\xi \rightarrow \epsilon / \xi \text{ in } \Sigma\}$.

Suppose that G''' is ambiguous. Let $\sigma = y_0 \Rightarrow \cdots \Rightarrow y_r = y$ and $\sigma = z_0 \Rightarrow \cdots \Rightarrow z_s = y$ be two derivations in G''', with different generation trees, of a word $y = a_1^{x(1)} \cdots a_n^{x(n)} = f_a(x)$. There is no loss in assuming that in each derivation no production in P'' is used after an application of a production of the form $\xi \rightarrow \epsilon$, ξ in Σ. Let i be the smallest integer such that $y_i \Rightarrow y_{i+1}$ is obtained from a production $\xi \rightarrow \epsilon$, ξ in Σ. Let j be the smallest integer such that $z_j \Rightarrow z_{j+1}$ involves a production $\xi \rightarrow \epsilon$, ξ in Σ. Then $y_i = z_j = f_{wa}(x)$, where $wa = \langle w_1 a_1, \cdots, w_n a_n \rangle$. Since G'' is unambiguous, $\sigma = y_0 \Rightarrow \cdots \Rightarrow y_i$ and $\sigma = z_0 \Rightarrow \cdots \Rightarrow z_j$ yield the same generation tree. From this it readily follows that $y_0 \Rightarrow y_1 \Rightarrow \cdots \Rightarrow y_r$ and $z_0 \Rightarrow \cdots \Rightarrow z_s$ yield the same generation tree. Thus

(2) G''' is an unambiguous grammar for a language $L(G''') \subseteq a_1^* \cdots a_n^*$.

In view of (2), to complete the proof it suffices to show that $f_a f_w^{-1}(M) = L(G''')$. Suppose that $f_a(p)$ is in $L(G''')$. Then $f_{wa}(p)$ is in $L(G'')$, hence in $L(G')$. Since $f_w(p)$ is the result of deleting all occurrences of $a_1, \cdots, a_n$ from $f_{wa}(p)$, $f_w(p)$ is in $L(G')$. Since $M = L(G') \cap w_1^* \cdots w_n^*$, it follows that $f_w(p)$ is in M. Conversely, suppose that $f_w(p)$ is in M. Then $f_{wa}(p)$ is in $L(G')$, hence in $L(G'')$. Thus $f_a(p)$ is in $L(G''')$. Therefore $f_a(p)$ is in $L(G''')$ if and only if $f_w(p)$ is in M, that is, $f_a f_w^{-1}(M) = L(G''')$.

To prove the converse of Lemma 6.1 the notion of a regular subset of N^n is needed, together with a sequence of lemmas.

Definition.   Let $a_1, \cdots, a_n$ be n distinct symbols and $a = \langle a_1, \cdots, a_n \rangle$. $H \subseteq N^n$ is said to be *regular* if $f_a(H)$ is a regular subset of $a_1^* \cdots a_n^*$.

From the definition there immediately follows

LEMMA 6.2.   *Let $i_1 < \cdots < i_r \leq n$ be r integers and Π the mapping of N^n onto N^r defined by $\Pi(\langle x(1), \cdots, x(n) \rangle) = \langle x(i_1), \cdots, x(i_r) \rangle$. If $A \subseteq N^n$, $B \subseteq N^n$, and $C \subseteq N^q$ are regular, then $A \cup B$, $A \cap B$, $A - B$, $A \times C$ and $\Pi(A)$ are regular.*

LEMMA 6.3.   *Let $H_1 \subseteq N^n$ be a finite union of disjoint linear sets, each with stratified and independent periods. If $H_2 \subseteq N^n$ is regular, then $H_1 \cap H_2$ is a finite union of disjoint linear sets, each with stratified and independent periods.*

PROOF.   By Theorem 1.3, $f_a(H_1)$ is a bounded language. Since $f_a^{-1} f_a(H_1) = H_1$, $f_a(H_1)$ has an unambiguous grammar by Theorem 5.1. Since $f_a(H_2)$ is regular, $f_a(H_1) \cap f_a(H_2)$ is a bounded language with an unambiguous grammar by Theorem 1.1. By Theorem 5.1 again, $f_a^{-1}(f_a(H_1) \cap f(H_2)) = f_a^{-1}[f_a(H_1)] \cap f_a^{-1}[f_a(H_2)] = H_1 \cap H_2$ is a finite union of disjoint linear sets, each with stratified and independent periods.

LEMMA 6.4.   *Let $H \subseteq N^n$ be regular and $w = \langle w_1, \cdots, w_n \rangle$ an n-tuple of words. Then $f_w^{-1}[f_w(H)]$ is regular.*

PROOF.   Let S be the one state gsm which maps a_i into w_i $(1 \leq i \leq n)$. Since $f_a(H)$ is regular and, by Theorem 3.1 of [8], S preserves regularity, $S(f_a(H))$ is regular. Since S^{-1} also preserves regularity [8, Theorem 3.4], $S^{-1}(S(f_a(H)))$ is regular. Thus

$$f_w^{-1}[f_w(H)] = f_a^{-1}[S^{-1}(S(f_a(H))) \cap a_1^* \cdots a_n^*]$$

is regular.

The next lemma generalizes Lemma 5.1.1 in [11] and is an analogue of Theorem 3.4.

LEMMA 6.5.   *Let S be a gsm and U a regular set. Then*

$$\alpha(S, U) = \{x \text{ in } S(U)/x = S(y) = S(z) \text{ for some } y, z \text{ in } U, y \neq z\}$$

is regular.

PROOF. Let $S = (K, \Sigma, \Delta, \delta, \lambda, p_0)$. Let $B = (K_B, \Sigma, \delta_B, q_0, F_B)$ be an automaton[14] such that $T(B) = U$.[15] Consider the following finite, directed, labelled graph H. Each element (p, q) in $K \times K_B$ is a node in H. For each (p, q) in $K \times K_B$, y in Σ, and $\lambda(p, y) = z_1 \cdots z_k$, z_i in Δ, $k \geq 2$, let the abstract symbols $(p, q)_1{}^y, \cdots, (p, q)_{k-1}{}^y$ be nodes in H. For (p, q) in $K \times K_B$ and y in Σ, if $\lambda(p, y)$ is in $\Delta \cup \{\epsilon\}$, let $((p, q), (\delta(p, y), \delta_B(q, y)))$ be a directed line in H with label $\lambda(p, y)$, with a different directed line for each such y. Thus H may have several directed lines, even with the same label, connecting the same vertices. For (p, q) in $K \times K_B$ and y in Σ such that $\lambda(p, y) = z_1 \cdots z_k$, z_i in Δ, $k \geq 2$, let $((p, q), (p, q)_1{}^y), ((p, q)_1{}^y, (p, q)_2{}^y), \cdots, ((p, q)_{k-1}{}^y, (\delta(p, y), \delta_B(q, y)))$ be directed lines in H with labels $z_1, \cdots, z_k$ respectively. Clearly $\alpha(S, U)$ consists of possibly ϵ together with all non-ϵ words x such that there exist at least two paths in H, with label x, from (p_0, q_0) to elements in $K \times F$.

To prove the lemma it suffices to show the following: "Let H be a finite, directed, labelled graph, with a distinguished node s_0 and a distinguished set R of nodes, having the property that any vertices p and q may be connected by many directed lines with the same label, and with all labels in $\Delta \cup \{\epsilon\}$. Then the set $\alpha(H)$, consisting of all words u for which there exist at least two paths, with label u, from s_0 to nodes in R, is regular." To do this an automaton $A = (K_A, \Delta, \delta_A, r_0, F_A)$ is constructed such that $\{\epsilon\} \cup \alpha(H) = \{\epsilon\} \cup T(A)$.

Let V be the set of nodes of the graph H. Let $K_A = \{(M, N)/N \subseteq M \subseteq V\}$ and let $r_0 = (\{s_0\}, \varphi)$. A state (M, N) is to be in F_A if and only if either

(1) $M \cap R$ contains at least two elements, or

(2) $\varphi \neq M \cap R \subseteq N$.

Let $N \subseteq M \subseteq V$ and y be in Δ. Let $\delta_1(M, y)$ be the set of those nodes to which there exists a path, with label y, from an element of M. Let $\delta_2((M, N), y)$ be the union of $\delta_1(N, y)$ and the set of nodes with the following property: There exist nodes q_1, q_2 in M (possibly the same) and paths Π_1 from q_1 to q, Π_2 from q_2 to q, such that each path has label y and neither is a terminal subpath of the other. Let $\delta_A((M, N), y) = (\delta_1(M, y), \delta_2((M, N), y))$. The following is readily seen:

(3)   If $\delta_A(r_0, y_1 \cdots y_k) = (M, N)$, each y_i in Δ, $k \geq 1$, then N consists of exactly those nodes to which there exist at least two paths from s_0 with label $y_1 \cdots y_k$.

Let $y_1 \cdots y_k$ be in $\theta(\Delta)$, each y_i in Δ, $k \geq 1$, and let $(M, N) = \delta_A(r_0, y_1 \cdots y_k)$. To complete the proof of the lemma it suffices to show that $y_1 \cdots y_k$ is in $\alpha(H)$ if and only if it is in $T(A)$.

(α)   Suppose that $y_1 \cdots y_k$ is in $\alpha(H)$. Then there are two paths from s_0 to nodes q_1, q_2 in R with label $y_1 \cdots y_k$. Thus $\{q_1, q_2\} \subseteq M \cap R$. If $M \cap R$ contains at least two elements, then (M, N) is in F_A by (1). If $M \cap R$ contains just one element, then $q_1 = q_2$ and $M \cap R = \{q_1\}$. By (3), q_1 is in N. Thus $\varphi \neq M \cap R \subseteq N$,

[14] An *automaton* is a 5-tuple $(K, \Sigma, \delta, q_0, F)$, where K is a finite nonempty set (of "states"), Σ is a finite nonempty set (of "inputs"), δ is a ("next state") function of $K \times \Sigma$ into K, q_0 is a ("start") state in K, and F is a subset (the "final states") of K. A word is said to be *accepted* by the automaton if the word takes (by successive application of δ) the automaton from q_0 to one of the states in F. A set is regular if and only if it is the set of words accepted by some automaton [14, p. 122].

[15] For each automaton B, $T(B)$ is the set of words accepted by B.

and (M, N) is in F_A by (2). In either case (M, N) is in F_A, so that $y_1 \cdots y_k$ is in $T(A)$.

(β) Suppose that $y_1 \cdots y_k$ is in $T(A)$. Then (M, N) is in F_A. If $M \cap R$ contains at least two elements q_1, q_2, then there exist paths from s_0 to q_1, q_2 with label $y_1 \cdots y_k$. If $\varphi \neq M \cap R \subseteq N$, let q be in $M \cap R$. By (3), there exist two paths from s_0 to q with label $y_1 \cdots y_k$. In either case, $y_1 \cdots y_k$ is in $\alpha(H)$.

From (α) and (β) it follows that $\{\epsilon\} \cup \alpha(H) = \{\epsilon\} \cup T(A)$.

COROLLARY 1. *Let S be a gsm and U a regular set. Then each of the following sets is regular.*

(1)   *The set of all words z such that $S(y) = z$ for exactly one word y in U.*

(2)   *The set of all words y in U such that $S(y) \neq S(z)$ for all z in $U - \{y\}$.*

(3)   *The set of all words y in U such that $S(y) = S(z)$ for some z in U, $y \neq z$.*

The next corollary is analogous to Lemma 4.1.

COROLLARY 2. *If $R_1, \cdots, R_r$ ($r \geq 2$) are regular sets, then the set of $\langle R_1, \cdots, R_r \rangle$-ambiguous words is regular.*

PROOF. Let b by a symbol not in Σ. Since each R_i is regular, so is $R_1 b R_2 b \cdots b R_r$. Let S be the one state gsm $(\{p_0\}, \Sigma \cup \{b\}, \Sigma, \delta, \lambda, p_0)$ such that $\lambda(p_0, x) = x$ for x in Σ and $\lambda(p_0, b) = \epsilon$. A word u is $\langle R_1, \cdots, R_r \rangle$-ambiguous if and only if there exist two r-tuples representing u, which happens if and only if there exist two words z_1 and z_2 in $R_1 b \cdots b R_r$ such that $S(z_1) = S(z_2) = u$. By Lemma 6.5, the set of $\langle R_1, \cdots, R_r \rangle$-ambiguous words is thus regular.

Some notions to be used in Lemma 6.6 only are now introduced.

Notation. For p and q in N^n, write $p \leq' q$ if $p = q$ or if $p \neq q$ and $p(j) < q(j)$ for the smallest integer j such that $p(j) \neq q(j)$.

The relation $\leq'$ is a simple order on N^n.

Let $w = \langle w_1, \cdots, w_n \rangle$ be an n-tuple of non-ϵ words. Then for each z in $w_1^* \cdots w_n^*$, $f_w^{-1}(z)$ is finite and has a (unique) maximal (under $\leq'$) element.

Definition. Let $w = \langle w_1, \cdots, w_n \rangle$ be an n-tuple of non-ϵ words. Then p in N^n is said to be w-*maximal* if $p = \max (f_w^{-1}(z))$ for some z in $w_1^* \cdots w_n^*$.

LEMMA 6.6. *If $w = \langle w_1, \cdots, w_n \rangle$ is an n-tuple of non-ϵ words, then $\{p$ in N^n/p is w-maximal$\}$ is regular.*

PROOF. For $A, B \subseteq N^n$ let $A \oplus B$ denote the set $\{p + q/p$ in A, q in $B\}$. From Lemma 6.2 it follows that

(1)   if A, B are regular and $A \subseteq N^j \times O^{n-j}$, $B \subseteq O^j \times N^{n-j}$ for some j, then $A \oplus B$ is regular.

For $A \subseteq N^n$ let

$$\alpha(A) = \{p \text{ in } A / f_w(p) = f_w(q) \text{ for some } q \neq p \text{ in } A\}.$$

Suppose A is regular. Then $f_a(A)$ is regular. By Corollary 1 to Lemma 6.5 the set

$$Z_S = \{y \text{ in } f_a(A) / S(y) = S(z) \text{ for some } z \text{ in } f_a(A) - \{y\}\}$$

is regular for each gsm S. Let S be the one-state gsm which maps each a_i into w_i, $1 \leq i \leq n$. Then $\alpha(A) = f_a^{-1}(Z_S)$. Thus

(2) $\alpha(A)$ is regular if A is.

Let $R_n = N^n$. For $j = n-1, \cdots, 1$, define R_j inductively as $R_{j+1} - T_j$, where

$$S_j = (O^j \times N^{n-j}) \cap \alpha((O^{j-1} \times N^{n-j+1}) \cap R_{j+1})$$

and

$$T_j = (N^j \times O^{n-j}) \oplus S_j .$$

Suppose R_{j+1} is regular. Then $(O^{j-1} \times N^{n-j+1}) \cap R_{j+1}$ is regular. By (2), $\alpha((O^{j-1} \times N^{n-j+1}) \cap R_{j+1})$ is regular. Then S_j is regular, and thus T_j is regular by)1). Since R_n is regular, it follows by induction that R_j is regular for each j. To complete the proof it suffices to show that $R_1 = \{p \text{ in } N^n / p \text{ is } w\text{-maximal}\}$.

If p in N^n is not w-maximal then there is a largest integer h ($1 \leq h \leq n$) such that, for some q in N^n, $f_w(q) = f_w(p)$, $q(i) = p(i)$ for $1 \leq i < h$, and $p(h) < q(h)$. In this case we say that p *fails* at h. Then p in N^n is w-maximal if and only if there is no h ($1 \leq h \leq n$) at which p fails. It thus suffices to show that for each i ($1 \leq i \leq n$).

(3) $R_i = \{p \text{ in } N^n / p \text{ fails at } h \text{ for no } h \geq i\}$.

Since no n-tuple fails at n, (3) is satisfied for $i = n$. Continuing by induction assume that (3) holds for $i = j+1, \cdots, n$ and

$$T_i = \{p \text{ in } N^n / p \text{ fails at } i\}$$

for $j < i < n$. Since $R_j = R_{j+1} - T_j$, (3) will follow for $i = j$ if we show that

(4) $T_j = \{p \text{ in } N^n / p \text{ fails at } j\}$.

To prove (4), let p be in T_j. Then there exist r in $N^j \times O^{n-j}$ and s in $S_j = (O^j \times N^{n-j}) \cap \alpha((O^{j-1} \times N^{n-j+1}) \cap R_{j+1})$ such that $p = r+s$. Since s is in $\alpha((O^{j-1} \times N^{n-j+1}) \cap R_{j+1})$, there exists $t \neq s$ in $(O^{j-1} \times N^{n-j+1}) \cap R_{j+1}$ such that $f_w(t) = f_w(s)$. Since s and t are in R_{j+1}, it follows by the induction hypothesis that neither s nor t fails at h for any $h \geq j+1$. Since t is in $O^{j-1} \times N^{n-j+1}$ and s is in $O^j \times N^{n-j}$, $t(i) = 0 = s(i)$ for $i < j$ and $t(j) \geq 0 = s(j)$. If $t(j) = s(j)$, then there exists $k \geq j+1$ which is the smallest integer h such that $t(h) \neq s(h)$. Now neither t nor s can fail at h for any $h > k$, since $k \geq j+1$. Thus s fails at k if $t(k) > s(k)$, and t fails at k if $s(k) > t(k)$. Both are impossible. Therefore $t(j) > s(j)$, so that

(5) s fails at j.

Now r is in $N^j \times O^{n-j}$ while s and t are in $O^{j-1} \times N^{n-j+1}$. Thus $f_w(r+s) = f_w(r)f_w(s)$ and $f_w(r+t) = f_w(r)f_w(t)$. Since $f_w(s) = f_w(t)$, $f_w(r+s) = f_w(r+t)$. Clearly $(r+t)(i) = (r+s)(i)$ for $i < j$, but $(r+t)(j) > (r+s)(j)$ since $t(j) > s(j)$. To establish that $p = r+s$ fails at j it remains only to show that p fails at h for no $h > j$. Therefore assume that p fails at h for some $h > j$. Then there exists q in N^n such that $f_w(q) = f_w(p)$, $q(i) = p(i)$ for $1 \leq i < h$, and $q(h) > p(h)$. Since $p = r+s$, with r in $N^j \times O^{n-j}$ and s in $O^j \times N^{n-j}$, $p(i) = s(i)$ for $j+1 \leq i \leq n$. Define y in $O^j \times N^{n-j}$ by $y(i) = 0$ for $1 \leq i \leq j$, and $y(i) = q(i)$ for $j+1 \leq i \leq n$. Then $q = r+y$, since $r(i) = p(i)$ for $1 \leq i \leq j$ and $p(i) = q(i)$ for $1 \leq i \leq j < h$, while $r(i) = 0$ for $j+1 \leq i \leq n$. Moreover, $f_w(r)f_w(s) = f_w(r+s) = f_w(p) = f_w(q) = f_w(r+y) = f_w(r)f_w(y)$, whence $f_w(s) = f_w(y)$. Now $y(h) = q(h) > p(h) = s(h)$, while $y(i) = s(i) = 0$ for $1 \leq i \leq j$ and $y(i) = q(i) = p(i) = s(i)$ for $j + 1 \leq i < h$. Then s fails at k for some $k \geq h$, contradicting (5). Therefore p fails at j.

Finally assume that p in N^n fails at j. Then the set $A = \{z \text{ in } N^n / f_w(z) = f_w(p), z(i) = p(i) \text{ for } 1 \leq i < j \text{ and } z(j) > p(j)\}$ is nonempty. Since $A \subseteq f_w^{-1}(f_w(p))$, A is finite. Let $q = \max \{y/y \text{ in } A\}$. Let t in $O^{j-1} \times N^{n-j+1}$ be defined by $t(i) = 0$ for $1 \leq i < j$, $t(j) = q(j) - p(j)$, $t(i) = q(i)$ for $j < i \leq n$. Define r in $N^j \times O^{n-j}$ by $r(i) = q(i)$ for $1 \leq i < j$, $r(j) = p(j)$, $r(i) = 0$ for $j < i \leq n$. Then $q = r+t$

and $f_w(q) = f_w(r)f_w(t)$. Define s in $O^j \times N^{n-j}$ by $s(i) = 0$ for $1 \leq i \leq j$, $s(i) = p(i)$ for $j < i \leq n$. Then $p = r+s$, since $p(i) = q(i)$ for $1 \leq i < j$, and $f_w(p) = f_w(r)f_w(s)$. From $f_w(p) = f_w(q)$ it follows that $f_w(s) = f_w(t)$. Since $t(j) = q(j) - p(j) > 0 = s(j)$, $t \neq s$.

To prove that $p = r+s$ is in T_j, it suffices to show that s is in S_j. By construction, s is in $O^j \times N^{n-j}$. Both s and t are in $O^{j-1} \times N^{n-j+1}$. It remains to show that s and t are in R_{j+1}. For then s will be in $\alpha((O^{j-1} \times N^{n-j+1}) \cap R_{j+1})$ and so in S_j.

Suppose that either s or t is not in R_{j+1}. By the induction hypothesis, s or t fails at h for some $h \geq j+1$, so by the induction hypothesis either s or t is in T_h. Since $h \geq j+1$, r is in $N^h \times O^{n-h}$. Since $T_h = (N^h \times O^{n-h}) \oplus S_h$, if x is in T_h and y in $N^h \times O^{n-h}$ then $x+y$ is in T_h. Thus either $q = r+t$ or $p = r+s$ is in T_h. By assumption, then, p or q fails at h. But p fails at j, $j < h$. Therefore q fails at h. Hence there exists x in N^n such that $f_w(x) = f_w(q) = f_w(p)$, $x(i) = q(i)$ for $1 \leq i < h$, and $x(h) > q(h)$. Then $x(i) = q(i) = p(i)$ for $1 \leq i < j < h$, and $x(j) = q(j) > p(j)$. Thus $x \neq q$ is in A and $q = \max\{y/y \text{ in } A\}$ implies $x <' q$, a contradiction. Hence s and t are in R_{j+1} and the proof is complete.

LEMMA 6.7. *Let* $w = \langle w_1, \cdots, w_n \rangle$ *be an n-tuple of words. There is a regular set* $R \subseteq N^n$ *such that* $f_w(R) = w_1^* \cdots w_n^*$ *and* f_w *is one-to-one on* R.

PROOF. Let $i(1) < \cdots < i(r)$ be all the integers h such that $w_h \neq \epsilon$. If $r = 0$, that is, each $w_i = \epsilon$, the conclusion is obviously true. Suppose $r > 0$. Let $y = \langle w_{i(1)}, \cdots, w_{i(r)} \rangle$. By Lemma 6.6 there is a regular set $R' \subseteq N^r$ such that $f_y(R') = w_{i(1)}^* \cdots w_{i(r)}^* = w_1^* \cdots w_n^*$ and f_y is one-to-one on R'. Let $R \subseteq N^n$ be the set $\{p \text{ in } N^n/p(i) = 0 \text{ for each } i \text{ distinct from } i(1), \cdots, i(r) \text{ and } \langle p(i(1)), \cdots, p(i(r)) \rangle \text{ in } R' \}$. Clearly R is regular. Let f be the mapping of N^n onto N^r defined by $f(p) = \langle p(i(1)), \cdots, p(i(r)) \rangle$. Then f is one-to-one on R, $f(R) = R'$, and $f_w = f_y f$. Then $f_w(R) = f_y f(R) = f_y(R') = w_1^* \cdots w_n^*$ and f_w is one-to-one on R.

LEMMA 6.8. *Let* $w = \langle w_1, \cdots, w_n \rangle$ *be an n-tuple of words and* $f_w^{-1}(M)$ *a finite union of disjoint linear sets, each with stratified and independent periods. Then there exists a set* $F \subseteq N^n$ *which is a finite union of disjoint linear sets, each with stratified and independent periods, such that* $f_w(F) = M$ *and* f_w *is one-to-one on* F.

PROOF. Let R be as in Lemma 6.7. By Lemma 6.3, $F = f_w^{-1}(M) \cap R$ is a finite union of disjoint linear sets, each with stratified and independent periods. Furthermore, $f_w(F) = f_w(f_w^{-1}(M) \cap R) = M$. Since $F \subseteq R$ and f_w is one-to-one on R, f_w is one-to-one on F.

LEMMA 6.9. *Let* $w = \langle w_1, \cdots, w_n \rangle$ *be an n-tuple of words. Let* F *be a finite union of disjoint linear sets with stratified and independent periods,* $f_w(F) = M$, *and* f_w *be one-to-one on* F. *Then* M *is a language which has an unambiguous grammar.*

PROOF. Let $a = \langle a_1, \cdots, a_n \rangle$ be an n-tuple of n letters not in Σ. By Theorem 5.1, $f_a(F)$ is a language with an unambiguous grammar $G' = (V, \{a_1, \cdots, a_n\}, P', \sigma)$. Let $P = P' \cup \{a_i \rightarrow w_i/1 \leq i \leq n\}$. Let g be the mapping defined by $g(\epsilon) = \epsilon$, $g(a_i) = w_i$ for $1 \leq i \leq n$, and $g(b_1 \cdots b_r) = g(b_1) \cdots g(b_r)$, each b_j in $\{a_1, \cdots, a_n\}$. Thus $gf_a = f_w$ and $G = (V \cup \Sigma, \Sigma, P, \sigma)$ is a grammar for $gf_a(F) = f_w(F) = M$. Since $gf_a = f_w$ is one-to-one on F, g is one-to-one on $f_a(F)$. Since G' is unambiguous and g is one-to-one on $f_a(F) = L(G')$, it readily follows that G is unambiguous.

Combining Lemmas 6.1, 6.8 and 6.9 we obtain the following characterization.

THEOREM 6.1. *Let* $w = \langle w_1, \cdots, w_n \rangle$ *be an n-tuple of words. A language* M *contained in* $w_1^* \cdots w_n^*$ *is inherently ambiguous if and only if* $f_w^{-1}(M)$ *is not a finite union of disjoint linear sets, each with stratified and independent periods.*

COROLLARY. *Every bounded language is a finite union of languages with unambiguous grammars.*

PROOF. Let L be a language contained in $w_1^* \cdots w_n^*$, each w_i a word. Let $w = \langle w_1, \cdots, w_n \rangle$ and $a = \langle a_1, \cdots, a_n \rangle$, where the a_i are distinct letters. By Lemma 2.6 of [9], $f_a(f_w^{-1}(L))$ is a language. By Theorem 1.3, $f_w^{-1}(L) = f_a^{-1}(f_a(f_w^{-1}(L)))$ is a finite union of linear sets, each with stratified periods. From the proof of Lemma 6.6 of [9], it further follows that $f_w^{-1}(L)$ is a finite union of linear sets $M_1, \cdots, M_k$, each with stratified and independent periods. Thus $L = f_w(f_w^{-1}(L)) = f_w(\bigcup_1^k M_i) = \bigcup_1^k f_w(M_i)$. By Theorem 6.1, each $f_w(M_i)$ has an unambiguous grammar.

Using Theorem 6.1 certain languages are now shown to be inherently ambiguous.

THEOREM 6.2. *For each non-ϵ word w in $\theta(a, b)$, the set of words*

$$M_w = \{a^i bwba^i ba^j/i, \ j \geq 1\} \cup \{a^i bwba^j ba^j/i, j \geq 1\}$$

is an inherently ambiguous language.

PROOF. M_w was shown to be a language in Theorem 2.1. Suppose M_w is not inherently ambiguous for some w. Let $\alpha = \langle a, bwb, a, b, a \rangle$. Since M_w is not inherently ambiguous, by Theorem 6.1, $f_\alpha^{-1}(M_w)$ is a finite union of disjoint linear sets $L_1, \cdots, L_k$, each with stratified and independent periods.

Consider an arbitrary L_j with constant c. Let z be a finite sum of periods of L_j. Obviously $z(1)$, $z(3)$ and $z(5)$ are the only possible nonzero coordinates of z. We may therefore denote z by $(z(1)/z(3)/z(5))$. Now suppose that $z(1) \neq z(3)$. For every $n \geq 1$, $c+nz$ is in L_j. Therefore, for each n, either $(c+nz)(1) = (c+nz)(3)$ or $(c+nz)(3) = (c+nz)(5)$. For those n for which the first equality holds, $z(3) = z(1) + [c(1) - c(3)]/n$. Since $z(1) \neq z(3)$, there is at most one such n. Hence for infinitely many n, $z(3) = z(5) + [c(5) - c(3)]/n$. This implies that $c(3) = c(5)$. Thus

(∗)   if z is a sum of periods of some L_j, $1 \leq j \leq k$, then either $z(1) = z(3)$ or $z(3) = z(5)$.

If y is a period of an L_j then, by stratification, either $y(1) = 0$, $y(3) = 0$, or $y(5) = 0$. Therefore every period of every L_j is of the form $(d/d/0)$, $(0/0/d)$, $(0/d/d)$, or $(d/0/0)$.

Let m be the largest integer which is a coordinate of a constant of an L_j. Let H be the set of those j ($1 \leq j \leq k$) such that L_j has periods y_1, y_2, y_3 with $y_1(1) > 0$, $y_2(3) > 0$, and $y_3(5) > 0$. Then if q is in $f_\alpha^{-1}(M_w)$ and $q(1) > m$, $q(3) > m$, and $q(5) > m$; q is in an L_j with j in H. Let H_1 be the set of all j in H such that L_j has a period $(d/d/0)$ for some $d > 0$. Let H_2 be the set of all j in H such that L_j has a period $(0/e/e)$ for some $e > 0$. Then $H = H_1 \cup H_2$. If j is in $H_1 \cap H_2$, then L_j contains $(d/d/0)$ and $(0/e/e)$ as periods for some $d > 0$ and $e > 0$. This leads to a contradiction of (∗), so that $H_1 \cap H_2 = \varphi$.

Let R be the set of those $d \neq 0$ such that $(d/d/0)$ or $(0/d/d)$ is a period of some L_j, j in H. Let p be the product of all numbers d in R. Now $f_\alpha^{-1}(M_w)$ contains $x = (m + p + 1, 1, m + p + 1, 1, m + p + 1)$. Since $m + p + 1 > m$, x is in L_r for some r in H. Either r is in H_1 or r is in H_2. Suppose r is in H_1. Since $f_\alpha^{-1}(M_w)$ contains $u = (m + p + 1, 1, m + 1, 1, m + 1)$, u is in L_s for some s in H. Either s is in H_1 or s is in H_2. If s is in H_1 then L_s has a period $y = (d/d/0)$ for some $d > 0$. Thus L_s contains $u + y = (m + p + d + 1, 1, m + d + 1, 1, m + 1)$. Since $u + y$ is not in $f_\alpha^{-1}(M_w)$, L_s is not a subset of $f_\alpha^{-1}(M_w)$, a contradiction. If s is in H_2 then L_s has a period $v = (0/e/e)$, where $e > 0$ and e divides p. Then L_s con-

tains $u + (p/e)v = x$, so that $L_r \cap L_s \neq \varphi$. Since the L_i are pairwise disjoint, $r = s$. Thus $H_1 \cap H_2 \neq \varphi$, a contradiction. Consequently r cannot be in H_1. An analogous argument shows that r cannot be in H_2, whence the theorem.

7. *Languages Contained in $w_1^* w_2^*$*

The set $\{a^i b^j c^j / i, j \geq 1\} \cup \{a^i b^i c^j / i, j \geq 1\}$ is an example of an inherently ambiguous language contained in $w_1^* w_2^* w_3^*$ for some words w_1, w_2, w_3 [5, p. 153]. In contrast, it is shown that for all words w_1, w_2, no language contained in $w_1^* w_2^*$ is inherently ambiguous.

Definition. A subset of N^2 is called a *FUDLIP* if it is a finite union of disjoint linear sets, each with independent periods.

LEMMA 7.1. *Let* $c_0, \cdots, c_r (r \geq 1)$ *be in* N^2 *and* $P \subseteq N^2$ *an independent set. If* c_j *is in* $L(c_i ; P)$ *if and only if* $j = i$ $(0 \leq i, j \leq r)$,[16] *then* $A = L(c_0 ; P) - \bigcup_1^r L(c_i ; P)$ *is a FUDLIP.*

PROOF. If $L(c_0 ; P) \cap \bigcup_1^r L(c_i ; P) = \varphi$ then $A = L(c_0 ; P)$, trivially a FUDLIP. Suppose $L(c_0 ; P) \cap \bigcup_1^r L(c_i ; P) \neq \varphi$. Then for some t $(1 \leq t \leq r)$, $L(c_0 ; P) \cap L(c_t ; P) \neq \varphi$. Since $P \subseteq N^2$ and is independent, P contains at most two elements. Hence either (a) there exist p_1 in P and non-negative integers s_1, s_2 such that $c_0 + s_1 p_1 = c_t + s_2 p_1$, or (b) there exist p_1 and p_2 in P, $p_1 \neq p_2$, and integers s_1, s_2, s_3, s_4 (all ≥ 0) such that $c_0 + s_1 p_1 + s_2 p_2 = c_t + s_3 p_1 + s_4 p_2$. If (a) holds and $s_1 \geq s_2$, so that $c_0 + (s_1 - s_2) p_1 = c_t$, then c_t is in $L(c_0 ; P)$, a contradiction. If $s_1 < s_2$, then a similar argument shows c_0 in $L(c_t ; P)$, another contradiction. Thus (b) holds. Now if $s_1 \geq s_3$ and $s_2 \geq s_4$, so that $c_0 + (s_1 - s_3) p_1 + (s_2 - s_4) p_2 = c_t$, then c_t is in $L(c_0 ; P)$, a contradiction. Similarly $s_3 \geq s_1$ and $s_4 \geq s_2$ is impossible. Therefore either $s_1 > s_3$ and $s_4 > s_2$, or $s_3 > s_1$ and $s_2 > s_4$. If the former, then

$$c_0 + (s_1 - s_3) p_1 = c_t + (s_4 - s_2) p_2 ,$$

that is, there is a possible integer h_1 such that $c_0 + h_1 p_1$ is in $L(c_t ; P)$. If the latter, then similarly there is a positive integer h_2 such that $c_0 + h_2 p_2$ is in $L(c_t ; P)$. Thus either (i) there is a smallest positive integer h_3 such that $c_0 + h_3 p_1$ is in $\bigcup_1^r L(c_i ; P)$, or (ii) there is a smallest positive integer h_4 such that $c_0 + h_4 p_2$ is in $\bigcup_1^r L(c_i ; P)$.

Three possibilities arise.

(a) Suppose (i) holds but (ii) does not. We shall show that $A = \bigcup_{h=0}^{h_3-1} L(c_0 + h p_1 ; p_2)$. Then A will be a FUDLIP, the disjointness following from the independence of P.

Let $0 \leq k \leq h_3 - 1$. Clearly $L(c_0 + k p_1 ; p_2) \subseteq L(c_0 ; P)$. Suppose y is in $L(c_0 + k p_1 ; p_2) \cap \bigcup_1^r L(c_i ; P)$. Then $y = c_0 + k p_1 + t_3 p_2 = c_t + t_1 p_1 + t_2 p_2$ for some t, $1 \leq t \leq r$, and some t_1, t_2, $t_3 \geq 0$. By the argument given above, either $k > t_1$ and $t_2 > t_3$, or $t_1 > k$ and $t_3 > t_2$. If the former, then $c_0 + (k - t_1) p_1 = c_t + (t_2 - t_3) p_2$. This contradicts the minimality of h_3 since $k - t_1 \leq k < h_3$. If the latter, then $c_0 + (t_3 - t_2) p_2 = c_t + (t_1 - k) p_1$, that is, $c_0 + (t_3 - t_2) p_2$ is in $\bigcup_{i=1}^r L(c_i ; P)$ with $t_3 - t_2 > 0$. Then (ii) holds, a contradiction. Hence $L(c_0 + k p_1 ; p_2) \cap \bigcup_1^r L(c_i ; P) = \varphi$, so that $\bigcup_{h=0}^{h_3-1} L(c_0 + h p_1 ; p_2) \subseteq A$.

Now let y be in A. Then $y = c_0 + k_1 p_1 + k_2 p_2$ for some k_1, $k_2 \geq 0$. If $k_1 < h_3$ then y is in $\bigcup_{h=0}^{h_3-1} L(c_0 + h p_1 ; p_2)$. If $k_1 \geq h_3$, then $y = (c_0 + h_3 p_1) + (k_1 - h_3) p_1$

[16] $L(c_i ; P)$ is the linear set with constant c and periods the elements of P.

$+ k_2 p_2$. Then $y = z + (k_1 - h_3)p_1 + k_2 p_2$, with z in $\bigcup_{i=1}^{r} L(c_i; P)$ by (i), so that y is in $\bigcup_{i=1}^{r} L(c_i; P)$. Thus y is not in A, a contradiction.

(b)   Suppose (ii) holds but (i) does not. By an argument analogous to (a) we see that $A = \bigcup_{h=0}^{h_4-1} L(c_0 + hp_2; p_1)$, so that A is a FUDLIP.

(c)   Suppose (i) and (ii) both hold. Then

$$A \subseteq \{c_0 + hp_1 + kp_2 / h < h_3, k < h_4\}.$$

Thus A is a finite set and therefore a FUDLIP.

LEMMA 7.2.   *If* $B = \bigcup_1^t L(c_i; P) \subseteq N^2$, *with* P *an independent set, then* B *is a FUDLIP.*

PROOF.   Let R be the relation on $\{c_1, \cdots, c_t\}$ defined by $c_i R c_j$, if and only if c_j is in $L(c_i; P)$. Clearly R is a partial order. Let $d_1, \cdots, d_s$ be the distinct R-minimal elements of $\{c_1, \cdots, c_t\}$. Then $s \geq 1$, d_i is in $L(d_j; P)$ if and only if $i = j$, and

$$B = \overset{s}{\underset{1}{\bigcup}} L(d_i; P) = \overset{s}{\underset{1}{\bigcup}} [L(d_i; P) - \overset{s}{\underset{j=i+1}{\bigcup}} L(d_j; P)].$$

By Lemma 7.1 each set $C_i = L(d_i; P) - \bigcup_{j=i+1}^{s} L(d_j; P)$ is a FUDLIP, and $C_i \cap C_k = \varphi$, for $i \neq k$. Thus B is a finite union of disjoint FUDLIPS, and hence a FUDLIP.

LEMMA 7.3.   *If* A *and* B *are linear subsets of* N^2, *each with independent periods, then* $A \cap B$ *is a FUDLIP.*

PROOF.   Let $A = L(c; P)$ and $B = L(d; Q)$, with P and Q independent. Let $A' = L((0, 0); P)$, $B' = L((0, 0); Q)$ and $C = A' \cap B' - \{(0, 0)\}$. Note that C contains the sum of each pair of elements in it. Let

$$\Pi = \{x \text{ in } C / x \text{ is not the sum of a pair of elements in } C\}.$$

By Lemma 6.5 of [9], with $w = (0, 0)$, Π is finite. Then $C = \bigcup_{x \text{ in } \Pi} L(x; \Pi)$. Now let Z be the set of all elements in $A \cap B$ which are not the sum of an element of $A \cap B$ and an element of Π. Since $x+y$ is in $A \cap B$ for x in $A \cap B$ and y in Π, $A \cap B = \bigcup_{z \text{ in } Z} L(z; \Pi)$ and Z is the set of all elements in $A \cap B$ which are not the sum of an element of $A \cap B$ and an element of C.

It is now shown that Z is finite. To see this observe that by independence neither P nor Q contains more than two elements.

(a)   Suppose P contains two elements p_1, p_2 and Q contains two elements q_1, q_2. Let

$$J = \{(k_1, k_2, k_3, k_4) \text{ in } N^4 / c + k_1 p_1 + k_2 p_2 = d + k_3 q_1 + k_4 q_2\}.$$

By Lemma 6.5 of [9], with $w = c-d$, J has only a finite number of minimal elements.[17] Suppose that (k_1, k_2, k_3, k_4) and (m_1, m_2, m_3, m_4) are distinct elements of J, with $k_i \leq m_i$. Then $y = c + m_1 p_1 + m_2 p_2$ is not in Z. For let $x = c + k_1 p_1 + k_2 p_2$. Then

$$y - x = (m_1 - k_1)p_1 + (m_2 - k_2)p_2 = (m_3 - k_3)q_1 + (m_4 - k_4)q_2$$

and is not $(0, 0)$ since $\{p_1, p_2\}$ are independent. Thus $y-x$ is in C so that $y = x + (y-x)$ is the sum of an element in $A \cap B$ and an element of C. Hence y is not in Z. Therefore

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[17] For u and v in N^4, $u \leq v$ if $u(i) \leq v(i)$ for each i.

$$Z \subseteq \{c + k_1 p_1 + k_2 p_2/(k_1, k_2, k_3, k_4) \text{ is a minimal element of } J\},$$

so that Z is finite.

(b) Suppose P or Q (or both) contains just a single element. By an argument similar to that in A, using subsets of N^3 or N^2, it can be shown that Z is finite.

Let $\Pi = \{\pi_1, \cdots, \pi_k\}$. Now every three vectors in N^2 are dependent. Thus for vectors v_1, v_2, v_3 in N^2 there exist positive integers h_2, h_4, h_6 and integers h_1, h_3, h_5 such that $(h_1/h_2)v_1 + (h_3/h_4)v_2 + (h_5/h_6)v_3 = 0$. Thus one of the v_i has some positive multiple which is the sum of non-negative multiples of the remaining v_j. From this it readily follows that there is an independent subset R such that each element of Π has some positive multiple which is the sum of non-negative multiples of elements of R. For $1 \leq i \leq k$, let g_i be the smallest positive integer such that $g_i \pi_i$ is the sum of non-negative multiples of elements of R.

For each element z in N^2, let $W(z)$ be the set of all elements in $L(z; \Pi)$ which are not the sum of an element of $L(z; \Pi)$ and an element of R. If x is in $L(z; \Pi)$ and u is in R, then $x+u$ is in $L(z; \Pi)$ (since $R \subseteq \Pi$). Therefore $L(z; \Pi) = \bigcup_{y \text{ in } W(z)} L(y; R)$ for each z in N^2. Then

$$A \cap B = \bigcup_{z \text{ in } Z} L(z; \Pi) = \bigcup_{z \text{ in } Z} [\bigcup_{x \text{ in } W(z)} L(x; R)].$$

Suppose that for some z there exists an element $z + \sum_{j=1}^{k} h_j \pi_j$ in $W(z)$ with $h_i \geq g_i$ for some i. Let $g_i \pi_i = \pi_s + q$, where π_s is in R and q is the sum of (zero or more) elements in R. Then

$$z + \sum_{j=1}^{k} h_j \pi_j = z + \sum_{j \neq i} h_j \pi_j + (h_i - g_i)\pi_i + g_i \pi_i$$

$$= z + \sum_{j \neq i} h_j \pi_j + (h_i - g_i)\pi_i + q + \pi_s.$$

Then $z + \sum_{j=1}^{k} h_j \pi_j$ is the sum of an element of $L(z; \Pi)$ with an element of R and thus is not in $W(z)$, a contradiction. Thus, for each z,

$$W(z) \subseteq \{z + \sum_{i=1}^{k} h_i \pi_i/h_i < g_i, 1 \leq i \leq k\}.$$

Then $W(z)$ is finite, so that $A \cap B$ is the finite union of linear sets, each with the same independent set of periods. By Lemma 7.2, $A \cap B$ is a FUDLIP.

LEMMA 7.4. *If* $A = L((0, 0); P) \subseteq N^2$, *with P independent, then $N^2 - A$ is a FUDLIP.*

PROOF. If $P = \varphi$ then $N^2 - A$ is the FUDLIP

$$L((1, 0); (0, 1), (1, 0)) \cup L((0, 1); (0, 1)).$$

Suppose $P \neq \varphi$. Let $\leq_s$ be the simple ordering on $N^2 - \{(0, 0)\}$ defined by $x \leq_s y$ if and only if $x(2)y(1) \leq x(1)y(2)$. Geometrically this means that the slope of the vector x is less than or equal to that of y. Let p be the minimal and q the maximal element of P. Since P is independent, p and q are uniquely determined. If P contains just one element, then $p = q$. In any case, $P = \{p, q\}$.

Let $B = \{x \text{ in } N^2 - \{(0, 0)\}/x <_s p\}$ and $C = \{x \text{ in } N^2 - \{(0, 0)\}/q <_s x\}$. Clearly B and C are disjoint subsets of $N^2 - A$. If $p(2) = 0$, then $B = \varphi$. Suppose $p(2) > 0$. For each j in N, let $f(j)$ be the smallest positive integer such that $(f(j), j) <_s p$. Obviously $f(j)$ exists for each j and

$$B = \bigcup_{j=0}^{p(2)-1} L((f(j), j); p, (1, 0)).$$

Since

$$L((f(j), j); p, (1, 0)) \cap L((f(i), i); p, (1, 0)) = \varphi$$

for $i \neq j$, $i, j \leq p(2) - 1$, B is a FUDLIP. Similarly C is a FUDLIP.

Let $D = \{y \text{ in } N^2 - A/p \leq_s y \leq_s q\}$. Since $D \cap (B \cup C) = \varphi$ and $N^2 - A = B \cup C \cup D$, it only remains to show that D is a FUDLIP. To this end let E be the set of those elements in N^2 which either (i) lie in the interior of the parallelogram whose vertices are $(0, 0)$, p, $p+q$ and q (if $p \neq q$); (ii) are in the interior of the line segment whose endpoints are $(0, 0)$ and p; or (iii) are in the interior of the line segment whose endpoints are $(0, 0)$ and q. E is clearly finite. It is left to the reader as a straightforward exercise about vectors in the plane, to verify that $D = \bigcup_{x \text{ in } E} L(x; P)$. By Lemma 7.2, therefore, D is a FUDLIP.

LEMMA 7.5. *If $A = L(c; P) \subseteq N^2$, with P independent, then $N^2 - A$ is a FUDLIP.*

PROOF. Let $B = L((0, 0); P)$. By Lemma 7.4, $N^2 - B$ is a FUDLIP, say $N^2 - B = \bigcup_1^r L(c_i; P_i)(r \geq 0)$. Then $C = \bigcup_1^r L(c + c_i; P_i) = \{x + c/x \text{ in } N^2 - B\}$ is a FUDLIP. Let $D = \bigcup_{i=0}^{c(1)-1} L((i, 0); (0, 1))$ and $E = \bigcup_{j=0}^{c(2)-1} L((c(1), j); (1, 0))$. Then D and E are FUDLIPS, $D \cap E = \varphi$, $C \cap (D \cup E) = \varphi$, and $N^2 - A = C \cup D \cup E$. Therefore $N^2 - A$ is a FUDLIP.

LEMMA 7.6. *Every semi-linear subset of N^2 is a FUDLIP.*

PROOF. By Lemma 6.6 of [9] every semi-linear subset L of N^2 is a finite union $\bigcup_1^n L_i$ of linear sets, each with independent periods. For $n = 1$ the lemma holds trivially. Continuing by induction suppose the lemma holds for $n = k$. Consider $n = k+1$. Now

$$L = (\bigcup_{i=1}^k L_i) \cup L_{k+1} = ((\bigcup_{i=1}^k L_i) \cap (N^2 - L_{k+1})) \cup L_{k+1}.$$

Also $\bigcup_{i=1}^k L_i$ is a FUDLIP, say $\bigcup_1^r J_i$, by the induction hypothesis, and $N^2 - L_{k+1}$ is a FUDLIP, say $\bigcup_1^s K_j$, by Lemma 7.5. Then

$$L = ((\bigcup_1^r J_i) \cap (\bigcup_1^s K_j)) \cup L_{k+1} = \bigcup_{1 \leq i \leq r; 1 \leq j \leq s} (J_i \cap K_j) \cup L_{k+1},$$

which is a disjoint union of $rs+1$ sets. By Lemma 7.3, each of the first rs sets is a FUDLIP. Thus L is a disjoint union of FUDLIPS and hence is a FUDLIP. By induction, the lemma is therefore true.

The main result of this section can now be proved.

THEOREM 7.1. *For all words w_1, w_2, no language contained in $w_1^* w_2^*$ is inherently ambiguous.*

PROOF. Let $w = \langle w_1, w_2 \rangle$ and let $L \subseteq w_1^* w_2^*$ be a language. By Theorem 3.1, L is semi-linear, i.e., $f_w^{-1}(L)$ is a semi-linear subset of N^2. By Lemma 7.6, $f_w^{-1}(L)$ is a FUDLIP. Since every set of periods in N^2 is stratified, it follows from Theorem 6.1 that L has an unambiguous grammar.

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