Algebraic Number Theory

Script

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Winter semester 2020/21

Contents

This script does not represent any replacement for the lectures given by professor Mihăilescu and will not be proof-read by him or anyone else in charge, these are basically my personal notes. Therefore I can not guarantee for its completeness and I will probably not write down any proofs given for theorems (because that's simply no fun in LaTeX.) glhf

III. Valuations and completions

1. Equivalent valuations and the theorem of Ostrowski

Definition 3.1.1 (Valuation)
For $a \in \mathbb{Z}_{\geq 0}$ we define the *valuation of a* $v_p(a)$ as the largest power of p dividing a, that is

$$a = p^m \cdot n, \ (n,p) = 1 \implies v_p(a) = m.$$

From that we conclude $a \in \mathbb{Z} \implies v_p(a) = v_p(|a|)$ and $a = \frac{a_1}{a_2} \in \mathbb{Q} \implies v_p(a) = v_p(a_1) - v_p(a_2)$. We also define $v_p(0) = \infty$.

Definition 3.1.2 (p-adic absolute value)
We define the *p-adic absolute value* to be$|a|_p = p^{-v_p(a)}$. From this, it follows that

1. $|a|_p = 0 \iff a = 0$,

2. $|ab|_p = |a|_p |b|_p$ and

3. $|a+b|_p \leq |a|_p + |b|_p$, making $|\cdot|_p$ a metric. Since it also also satisfies $|a+b|_p \leq \max\{|a|_p, |b|_p\}$, it is an ultrametric.

We can endow $\mathbb{Q}$ with the p-adic metric and build Cauchy sequences. Let $\mathcal{C}$ be the space of Cauchy sequences on $\mathbb{Q}$ with respect to $|\cdot|_p$ and let $\mathcal{N} = \{z = (z_n)_{n\in\mathbb{N}} \mid z \in \mathcal{C}, \ \lim_{n\to\infty}(z_n) = 0\}$. Then $\mathcal{C}$ ist an integral ring and $\mathcal{N}$ is a maximal ideal therein. Therefore $\mathcal{C}/\mathcal{N}$ is a field called $\mathbb{Q}_p$.

Example 3.1.3: $z = (1, p, p^2, \dots, p^n, \dots) \in \mathcal{N}, \ |p^n|_p = p^{-n} \to 0$. Note that a power series $f(z) = \sum_{n\in\mathbb{N}} a_n z^n, \ |a_n + a_{n+1}| \leq \max\{a_n, a_{n+1}\}$ verifies $|\sum_{n\in\mathbb{N}} a_n z^n|_p \leq |a_n z^n|$ if $a_n z^n$ are falling to 0.

Definition 3.1.4 (Limits in $\mathbb{Q}_p$)
For $x \in \mathbb{Q}_p$ we define $|x|_p = \lim |x_n|_p$ with $|x_n| \to x$
$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid v_p(x) \geq 0\} = \mathcal{C}(\mathbb{Z})/\mathcal{N}(\mathbb{Z})$ (valuation ring of $\mathbb{Q}_p$)

$(\mathbb{Z}_p/p^n\mathbb{Z}_p) \cong (\mathbb{Z}/p^n\mathbb{Z})$

Let $S \subset \mathbb{Z}$ be representatives of $\mathbb{F}_p$. Then $\mathbb{Z}_p = \left\{ x = \sum\limits_{\substack{n=0 \\ x_n \in S}}^{\infty} x_n p^n \right\}$.

Definitions