Algebraic Number Theory

Script

Prof. Dr. Preda Mihăilescu

LaTeX-version by Niklas Sennewald

Mathematisches Institut
Georg-August-Universität Göttingen

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Contents

This script does not represent any replacement for the lectures given by professor Mihăilescu and will not be proof-read by him or anyone else in charge, these are basically my personal notes. Therefore I can not guarantee for its completeness and I will probably not write down any proofs given for theorems (because that's simply no fun in LaTeX.) glhf

0.1 Non-Archimedean valuations and p-adic numbers

Let a, b be 'large' positive integers, $a > b$ and p a prime

Definition 0.1.1 (Valuation)
For $a \in \mathbb{Z}_{\geq 0}$ we define the *valuation of a* $v_p(a)$ as the largest power of p dividing a, that is

$$a = p^m \cdot n,\ (n,p) = 1 \implies v_p(a) = m.$$

From that we conclude $a \in \mathbb{Z} \implies v_p(a) = v_p(|a|)$ and $a = \frac{a_1}{a_2} \in \mathbb{Q} \implies v_p(a) = v_p(a_1) - v_p(a_2)$. We also define $v_p(0) = \infty$.

Definition 0.1.2 (p-adic metric)
We define the *p-adic metric* to be$|a|_p = p^{-v_p(a)}$. From this, it follows that

1. $|a|_p = 0 \iff a = 0$
2. $|ab|_p = |a|_p |b|_p$
3. $|a+b|_p \leq |a|_p + |b|_p$, $|a+b|_p \leq \max\{|a|_p, |b|_p\}$

We can endow $\mathbb{Q}$ with the p-adic metric and build Cauchy sequences. Let $\mathcal{C}$ be the space of Cauchy sequences on $\mathbb{Q}$ with respect to $|\cdot|_p$ and let $\mathcal{N} = \{z = (z_n)_{n\in\mathbb{N}} \mid z \in \mathcal{C},\ \lim_{n\to\infty}(z_n) = 0\}$. Then $\mathcal{C}$ ist an integral ring and $\mathcal{N}$ is a maximal ideal therein. So $\mathcal{C}/\mathcal{N}$ is a field called $\mathbb{Q}_p$.

Example 0.1.3: $z = (1, p, p^2, \ldots, p^n, \ldots) \in \mathcal{N},\ |p^n|_p = p^{-n} \to 0.$ Note that a power series $f(z) = \sum_{n\in\mathbb{N}} a_n z^n,\ |a_n + a_{n+1}| \leq \max\{a_n, a_{n+1}\}$ verifies $|\sum_{n\in\mathbb{N}} a_n z^n|_p \leq |a_n z^n|$ of falling $a_n z^n$.

Definition 0.1.4
For $x \in \mathbb{Q}_p$ we define $|x|_p = \lim |x_n|_p$ with $|x_n| \to x$
$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid v_p(x) \geq 0\} = \mathcal{C}(\mathbb{Z})/\mathcal{N}/(\mathbb{Z})$ (valuation ring of $\mathbb{Q}_p$)

$(\mathbb{Z}_p/p^n\mathbb{Z}_p) \cong (\mathbb{Z}/p^n\mathbb{Z})$