IMONST1 2023 - Senior Category

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Section A

Determine whether the statements below are **TRUE** or **FALSE**.

1. $ABCD$ is a square if and only if $ABCD$ is a cyclic quadrilateral and a parallelogram.
   #solution **FALSE**. By recalling the properties of a cyclic quadrilateral, opposite angles add up to 180°. In a parallelogram, opposite angles are equal. Hence, we have

   $$\angle A = \angle C = 180^\circ - \angle A \Leftrightarrow \angle A = \angle C = 90^\circ.$$

   Similarly, $\angle B = \angle D = 90^\circ$. This means if $ABCD$ is a cyclic quadrilateral and a parallelogram, $ABCD$ is either a square or a rectangle (which is not the same as saying "if $ABCD$ is a cyclic quadrilateral and a parallelogram, $ABCD$ is a square").

2. There are infinitely many rational numbers between any two different real numbers.
   #solution **TRUE**. See here.

3. The set $A = \{\varnothing\}$ has one element and two subsets.
   #solution **TRUE**. The only one element in A is the empty set. The subsets of A are $\{\varnothing\}$ and $\varnothing$.

4. $A = \{x \text{ is a perfect square and } x < 0\}$ is a valid set.
   #solution **TRUE**. By definition, a perfect square is a positive integer that is obtained by multiplying an integer by itself. No perfect squares satisfy the constraint $x < 0$, hence A is an empty set, but it is still a valid set since there are no repeated elements in A.

5. The equation $x = \sqrt{x+6}$ has two roots.
   #solution **FALSE**. $x = \sqrt{x+6} \implies x = 3$ only since $x = \sqrt{x+6} > 0$. The equation has only one root. (If we also considered $x = -2$, we would obtain $-2 = \sqrt{-2+6} = 2$ which is false.)

6. The product of a rational number and an irrational number is always irrational.
   #solution **TRUE**. Suppose otherwise that the product is rational. Then, we would have

   $$\frac{p}{q} \times N = \frac{r}{s},$$

   where N is the irrational number and $\gcd(p, q) = \gcd(r, s) = 1$. This equation gives $N = qr/ps$, which means N is rational, which contradicts the fact that N is supposed to be irrational.

7. 0 is divisible by -3.
   #solution **TRUE**. 0 divided by -3 does not leave any remainder.

8. $\sqrt{a^2} = a$ for any real number a.
#solution **FALSE**. $\sqrt{a^2} = a \geq 0$. It only applies to nonnegative real a, not ANY real value of a.

9. 2023 is a complex number.
#solution **TRUE**. In fact, $2023 \in \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

10. The number $0.1234567891011121314\ldots979899$ is irrational.
#solution **FALSE**. The number terminates itself, which means it can be written as a rational number. In fact,

$$0.1234567891011121314\ldots979899 = \frac{1234567891011121314\ldots979899}{10^{190}}$$

as a simplified fraction.

Section B

11. What is the largest possible area of an isosceles triangle with two sides of length 10?
#solution The area of an isosceles triangle is given by $A = \frac{1}{2}x^2 \sin\theta$. Differentiating once gives $\dfrac{dA}{d\theta} = \frac{1}{2}x^2\cos\theta$. When $\dfrac{dA}{d\theta} = 0$, $\dfrac{1}{2}x^2\cos\theta = 0 \Leftrightarrow \theta = \pi/2$. Surprisingly, the maximum area is formed when we have a right isosceles triangle, so $A = \dfrac{1}{2}(10^2)\sin\dfrac{\pi}{2} = \boxed{50}$.

12. Among 100 students in a class, 93 students know French, 73 students know Japanese, 69 students know Arabic and 65 students know Swahili. What is the least possible number of students who know all four languages?
#solution We can use a dichotomous approach to tackle the problem.
7 don't know French.
93 students know French.

Maximising those who don't know both French and Japanese, we get
$7 + (100 - 73) = 34$ that don't know both French and Japanese.
$100 - 34 = 66$ that know both French and Japanese.

Maximising those who don't know French, Japanese and Arabic, we get
$34 + (100 - 69) = 65$ that don't know French, Japanese and Arabic.
$100 - 65 = 35$ that know all French, Japanese and Arabic.

Maximising those who don't know French, Japanese, Arabic and Swahili, we get
$65 + (100 - 65) = 100$ that don't know French, Japanese, Arabic and Swahili.
$100 - 100 = \boxed{0}$ that know all French, Japanese, Arabic and Swahili.

13. A solid sphere is placed inside a closed box, touching all six inside faces of the box. The sphere filled $k\%$ of the space inside the box. Determine the nearest integer to k.
#solution Suppose the sphere has a radius r, then the side lengths of the cube that contains the sphere is $2r$. The percentage of volume of the sphere is given by

$$k = \frac{\frac{4}{3}\pi r^3}{(2r)^3} \times 100 = \frac{50\pi}{3}.$$

To the nearest integer, $\boxed{k = 52}$.

14. Given real numbers a and b such that

$$\frac{1}{1+a} + \frac{1}{1+b} = 1.$$

How many possible values of ab are there?
#solution Rearranging the terms yield

$$\begin{aligned} \frac{1}{1+a} + \frac{1}{1+b} &= 1 \\ 2 + a + b &= (1+a)(1+b) \\ &= 1 + a + b + ab \\ ab &= 1 \end{aligned}$$

Only $\boxed{1}$ possible value of ab exists.

15. Let α and β be the two roots of the quadratic equation $x^2 - 6x + 3 = 0$. Find the value of

$$\frac{\alpha}{1 - \frac{\beta}{\alpha}} + \frac{\beta}{1 - \frac{\alpha}{\beta}}.$$

#solution Rearranging the original expression gives

$$\begin{aligned} \frac{\alpha}{1 - \frac{\beta}{\alpha}} + \frac{\beta}{1 - \frac{\alpha}{\beta}} &= \frac{\alpha^2 - \beta^2}{\alpha - \beta} \\ &= \alpha + \beta \end{aligned}$$

By Vieta's theorem, $\alpha + \beta = \boxed{6}$.

16. Find the smallest positive integer n such that the following statement is true: For every prime p, the number $p^2 + n$ is not prime.
#solution All primes $p \neq 2$ are odd numbers. To make $p^2 + n$ a composite number (even), n has to be odd. To make $2^2 + n$ a composite number, we can try $n = 1$ but it gives 5, then try $n = 3$ but it gives 7, finally we try $n = 5$, which gives 9. Therefore, the least value of n is $\boxed{5}$.

17. Find the sum of the digits of the number $10^{2023} - 20 \times 23$.
#solution Evaluating the simple arithmetic operations:

$$
\begin{aligned}
10^{2023} - 20 \times 23 &= 10^{2023} - 460 \\
&= 10^{2023} - 1 - 459 \\
&= \underbrace{99\ldots99}_{2023} - 459 \\
&= \underbrace{99\ldots99}_{2020}540
\end{aligned}
$$

The sum of digits is $9 \times 2020 + 5 + 4 = \boxed{18189}$.

18. Given integers M and N such that $4^4 \cdot 21^{21} \cdot M^M \cdot N^N = 3^3 \cdot 7^7 \cdot 14^{14} \cdot 18^{18}$. Find $M + N$.

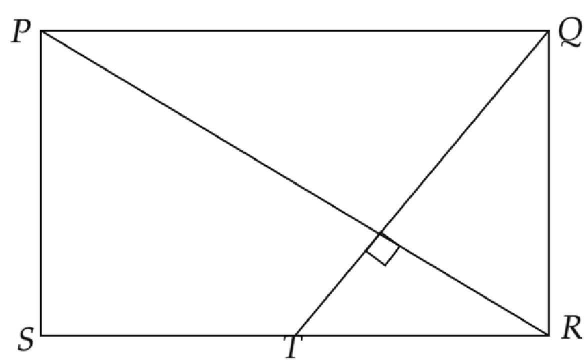
#solution We can solve by isolating M and N on one side.

$$
\begin{aligned}
4^4 \cdot 21^{21} \cdot M^M \cdot N^N &= 3^3 \cdot 7^7 \cdot 14^{14} \cdot 18^{18} \\
2^8 \cdot 7^{21}3^{21} \cdot M^M \cdot N^N &= 3^3 \cdot 7^7 \cdot 7^{14}2^{14} \cdot 3^{36}2^{18} \\
2^8 3^{21} 7^{21} \cdot M^M \cdot N^N &= 2^{32}3^{39}7^{21} \\
M^M \cdot N^N &= 2^{24}3^{18} \\
&= 8^8 \cdot 9^9
\end{aligned}
$$

Therefore, $M + N = 8 + 9 = \boxed{17}$.

19. Given a rectangle $PQRS$, with T the midpoint of side RS. If QT is perpendicular to PR, find the value of $\left(\frac{PQ}{QR}\right)^2$.

#solution



By angle chasing, $\angle RQT = 90^\circ - \angle PQT = \angle QPR$.

Notice that $\triangle PQR \sim \triangle QRT$ by AAA similarity. Then, $\dfrac{PQ}{QR} = \dfrac{QR}{RT} = \dfrac{QR}{PQ/2} \Leftrightarrow \dfrac{PQ^2}{QR^2} = \boxed{2}$.

20. Two boys and two girls are randomly arranged in a straight line. The probability that the two girls are between the two boys is $\frac{1}{k}$. What is k?

#solution There is only one possible configuration in which both girls are between the boys. In total, there are $\binom{4}{2} = 6$ arrangements without restrictions, so the probability of having both girls between the boys is $1/6$, i.e. $k = 6$.

Section C

21. We are given 10 line segments, each with integer length. No three segments can form a triangle. What is the minimum length of the longest line segment?
#solution To obtain the smallest lengths of the line segments, we can begin by picking two line segments of length 1. To ensure that no three segments form a triangle is equivalent to ensuring no three lengths satisfy the Triangle Inequality. Hence, if any three line segments of side lengths a, b, c were picked such that $a \leq b \leq c$, we need to have $a + b \leq c$ instead of $a + b > c$.

When $(a, b) = (1, 1)$, the smallest c that can be chosen is $c = 2$.
When $(a, b) = (1, 2)$, the smallest c that can be chosen is $c = 3$ and so on...

Interestingly, the sequence of the 10 line segments is equivalent to the first 10 terms of the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55.$$

Therefore, the answer is $\boxed{55}$.

22. Determine how many integers N satisfy the equation $N + s(N) = 2023$, where $s(N)$ is the sum of the digits of N.
#solution The maximum value of $s(N)$ that can be formed is 28 when $N = 1999$. Substituting values of $s(N)$ from 1 to 28 to the original equating and checking the corresponding values of N consequently yield only $\boxed{2}$ values of N ($N = 1997$ and $N = 2015$).

23. Given a square $ABCD$ with the side length s, and a circle Γ that passes through all vertices of the square. Let P be a point on Γ. Find the value of $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{s^2}$.
#solution **Fast solution**: We can pick one of the vertices of the squares (such as A) as P to simplify calculations. Then, $PA = 0, PB = PD = s$. Then, we can obtain PC using the Pythagorean theorem:

$$PC = \sqrt{s^2 + s^2} = s\sqrt{2}.$$

Hence,

$$\frac{PA^2 + PB^2 + PC^2 + PD^2}{s^2} = \frac{0 + s^2 + s^2 + 2s^2}{s^2} = \boxed{4}.$$

Rigorous solution: WLOG let Γ be a unit circle, so the parametrized coordinates of P is $(\cos\theta, \sin\theta)$.

Using Pythagoras's theorem,

$$\begin{aligned}
PA^2 &= \left(\cos\theta + \frac{1}{\sqrt{2}}\right)^2 + \left(\sin\theta - \frac{1}{\sqrt{2}}\right)^2 \\
PB^2 &= \left(\cos\theta - \frac{1}{\sqrt{2}}\right)^2 + \left(\sin\theta - \frac{1}{\sqrt{2}}\right)^2 \\
PC^2 &= \left(\cos\theta - \frac{1}{\sqrt{2}}\right)^2 + \left(\sin\theta + \frac{1}{\sqrt{2}}\right)^2 \\
PD^2 &= \left(\cos\theta + \frac{1}{\sqrt{2}}\right)^2 + \left(\sin\theta + \frac{1}{\sqrt{2}}\right)^2
\end{aligned}$$

Adding all of them up gives
$PA^2 + PB^2 + PC^2 + PD^2$
$= 2\left(\left(\cos\theta - \frac{1}{\sqrt{2}}\right)^2 + \left(\cos\theta + \frac{1}{\sqrt{2}}\right)^2 + \left(\sin\theta - \frac{1}{\sqrt{2}}\right)^2 + \left(\cos\theta + \frac{1}{\sqrt{2}}\right)^2\right)$
$= 2(2\cos^2\theta + 2\sin^2\theta + 2)$
$= 8$

Finally,

$$\begin{aligned}
\therefore \frac{PA^2 + PB^2 + PC^2 + PD^2}{s^2} &= \frac{8}{(\sqrt{2})^2} \\
&= \boxed{4}
\end{aligned}$$

24. Find the number of positive integers N such that $N < 1000$ and N has exactly 12 even factors and 6 odd factors (factors of N include 1 and N).
#solution For N to have 18 factors, it needs to have 9 positive factors and 9 negative factors. Then, consider the following possible products:

$$\begin{aligned}
9 &= 1 \times 9 \\
9 &= 3 \times 3
\end{aligned}$$

We also make the observation that since N has an odd number of positive factors, N has to be a perfect square.
By recalling the formula for number of divisors of N, if $N = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots$, where p_i is a prime number, then

$$\#\text{ divisors} = (1 + e_1)(1 + e_2)(1 + e_3)\cdots$$

Likewise, we can obtain the number of odd divisors as follows:
1. Separate even and odd prime divisors of N by expressing $N = 2^{e_1} p_2^{e_2} p_3^{e_3} \cdots$, where p_i are odd primes for $i \geq 2$.
2. The number of odd divisors is given by

$$\#\text{ odd divisors} = (1 + e_2)(1 + e_3)(1 + e_4)\cdots$$

3. Then, the number of even divisors can be obtained with

$$\# \text{ even divisors} = \# \text{ divisors} - \# \text{ odd divisors}.$$

Since we have evenly divided the number of factors according to their sign, this means N would have 3 odd factors and 6 even factors only. The following numbers have exactly 9 positive divisors:
- 2^8 has 1 odd and 8 even $\boxed{\times}$
- $2^2 \cdot 3^2$ has 3 odd and 6 even $\boxed{\checkmark}$
- $2^2 \cdot 5^2$ has 3 odd and 6 even $\boxed{\checkmark}$
- $3^2 \cdot 5^2$ has 9 odd and 0 even $\boxed{\times}$
- $2^2 \cdot 7^2$ has 3 odd and 6 even $\boxed{\checkmark}$
- $3^2 \cdot 7^2$ has 9 odd and 0 even $\boxed{\times}$
- $5^2 \cdot 7^2$ has 9 odd and 0 even $\boxed{\times}$
- $2^2 \cdot 11^2$ has 3 odd and 6 even $\boxed{\checkmark}$

Therefore, the answer is $\boxed{4}$.

25. Determine the number of solutions to the equation $x \cdot \lfloor x \rfloor = \pi$. [Note: For any real number x, $\lfloor x \rfloor$ is defined as the largest integer which is less than or equal to x. For example, $\lfloor 7.2 \rfloor = 7$, $\lfloor 7.8 \rfloor = 7$, $\lfloor 7 \rfloor = 7$, $\lfloor -7.2 \rfloor = -8$]

#solution Note that $\lfloor x \rfloor$ can only take up integral values. We can perform trial and error on positive integers $\lfloor x \rfloor$.

n	$x = \pi/n$	$\lfloor x \rfloor$
1	π	3
2	$\pi/2$	1
3	$\pi/3$	1
4	$\pi/4$	0
$\vdots$	$\vdots$	$\vdots$

For negative integers,

n	$x = \pi/n$	$\lfloor x \rfloor$
-1	$-\pi$	-4
-2	$-\pi/2$	-2
-3	$-\pi/3$	-2
-4	$-\pi/4$	-1
$\vdots$	$\vdots$	$\vdots$

We want to accept only values of x such that $n = \lfloor x \rfloor$. Only 1 value satisfies this condition, which is $x = -\dfrac{\pi}{2}$.

26. Find the number of integers k with $1 \leq k \leq 100$ such that $\frac{1}{k}$ has a finite decimal representation.

[Note: The number $\frac{1}{4} = 0.25$ has a finite decimal representation, but $\frac{1}{3} = 0.3333\ldots$ does not.]

#solution For $1/k$ to be a terminating decimal (finite decimal representation), k can only have prime factors 2 and/or 5. In other words, $k = 2^a 5^b$ where $a, b \geq 0$. By counting the number of possible pairs (a, b), we obtain a total of $3 + 5 + 7 = \boxed{15}$ pairs.

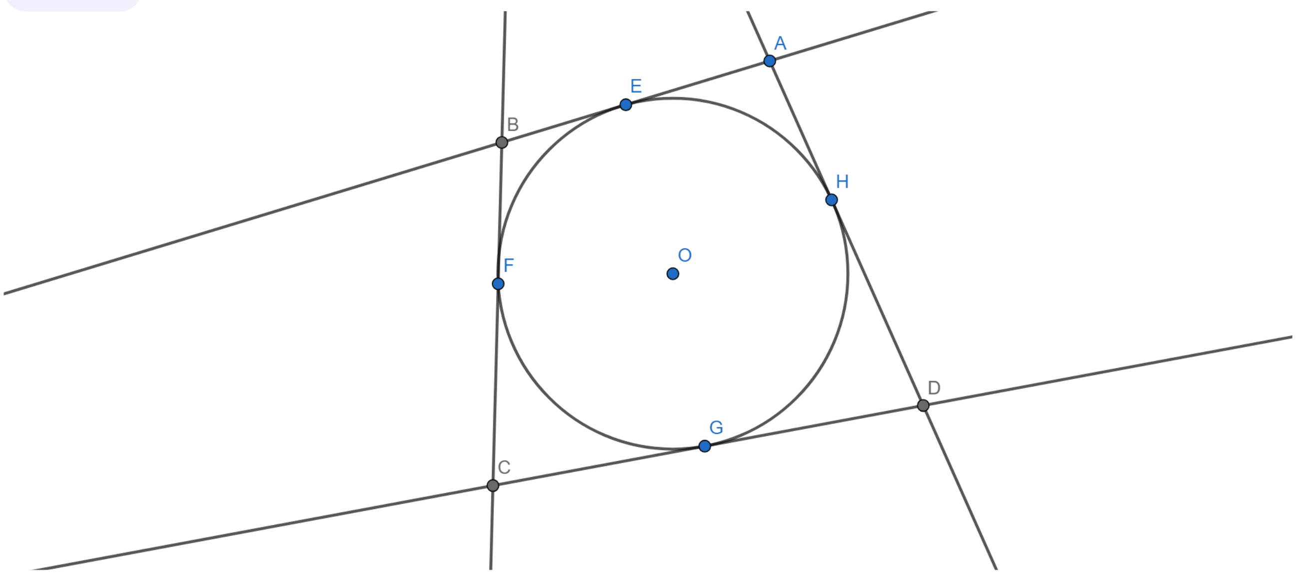
- When $b = 2$, $a \leq 2$. (3 pairs)
- When $b = 1$, $a \leq 4$. (5 pairs)
- When $b = 0$, $a \leq 6$. (7 pairs)

27. Given a number sequence $x_1, x_2, x_3, \ldots$ defined by $x_1 = 2$ and $x_{n+1} = x_n^{x_n}$ for all integers $n \geq 1$. If $x_4 = 2^{2^c}$, find c.

#solution We can compute x_2 and x_3 directly as follows: $x_2 = 2^2 = 4$, $x_3 = 4^4 = (2^2)^4 = 2^8$. Then, $x_4 = (2^8)^{2^8} = 2^{2^3 \cdot 2^8} = 2^{2^{11}} \implies c = \boxed{11}$.

28. A circle is tangent to the four sides of quadrilateral $ABCD$. It is known that $BC = 20, DA = 22$ and $CD = 2AB$. Find the length of AB.

#solution



Let $EA = AH = a, FB = BE = b, GC = CF = c, HD = DG = d$. With these variables, we can construct a system of equations:

$$\begin{aligned} a + d &= 22 && (1) \\ b + c &= 20 && (2) \\ c + d &= 2(a + b) && (3) \end{aligned}$$

Adding equations (1) and (2) gives $a + b + c + d = 42 \implies 3(a + b) = 42 \Leftrightarrow AB = a + b = \boxed{14}$.

29. Find the smallest positive integer k such that $11! \times k$ is a perfect square.

#solution By performing prime factorization on $11!$, we obtain $11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$. Since $2^8 \cdot 3^4 \cdot 5^2$ already makes a square, we only need to multiply $11!$ by $7 \cdot 11 = 77$ to "complete the square", so $\boxed{k = 77}$.

30. How many four-digit numbers are there such that at least one digit appears more than once? Example: $2020, 2021, 2022$ are counted, but 2019 is not counted.
#solution We can approach the problem through casework:
Case #1: One digit appears four times
- Trivial case $(1111, 2222, \ldots, 9999)$
- Hence, $\binom{10}{1} - 1 = 9$ ways

Case #2: One digit appears three times
- There are 10C2 ways to choose 2 distinct digits, then note that one of the digits can appear either one or three times, so multiply by 2, then we can arrange in 4C3 ways.
- We eliminate cases where 0 is the leading digit. When 0 is the leading digit, there are 9 ways to choose the 2nd digit. If 0 appears 3 times, there are 3C2 ways of arranging; if 0 appears once only, there is only 1 arrangement.
- Hence, $\binom{10}{2} \times 2 \times \frac{4!}{3!} - 9 \times \binom{3}{2} - 9 = 324$ ways

Case #3: One digit appears twice
- There are 10C3 ways to choose 3 distinct digits, then note that we can choose one of the 3 digits to appear twice (3 ways to choose), then we can arrange in 4!/2! ways.
- We eliminate cases where 0 is the leading digit. When 0 is the leading digit, there are 9C2 ways to choose the 2nd and 3rd digits. If 0 appears twice, there are 3! ways of arranging; if 0 appears once only, there are 3C2 ways of arranging.
- Hence, $\binom{10}{3} \times 3 \times \frac{4!}{2!} - \binom{9}{2} \times 3! - \binom{9}{2} \times 3 \times \frac{3!}{2!} = 3888$

Case 4: Two digits appears twice
- There are 10C2 ways to choose 2 distinct digits, then we can arrange them in 4C2 ways.
- We eliminate cases where 0 is the leading digit. When 0 is the leading digit, there are 9 ways to choose the 2nd digit and 3C2 ways to arrange them.
- Hence, $\binom{10}{2} \times \frac{4!}{2!2!} - 3(9) = 243$

Total: $9 + 324 + 3888 + 243 = \boxed{4464}$