CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS AND MATRICES

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1.1.1 LINEAR EQUATIONS

Any straight line in the *xy*-plane is represented by a ***linear equation*** of the form

$$a_1x + a_2y = b$$

where a_1, a_2, and b are real constants and a_1, a_2 are not simultaneously zeros

More generally, a ***linear equation*** in the variables (***unknowns***) $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real constants

Example. The equations

$$x + 3y = 7 \quad y = \tfrac{1}{2}x + 3z + 1 \quad x_1 - 2x_2 - 3x_3 + x_4 = 7$$

are linear, while the equations

$$x + 3\sqrt{y} = 5 \quad 3x + 2y - z + xz = 4 \quad y = \sin x$$

are not linear

A ***solution*** of a linear equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

is an n-tuple of real numbers $s_1, s_2, \ldots, s_n$ such that the equation is satisfied when we substitute $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$.

The set of all solutions is called the ***solution set***

Example. Finding the solution set of

(a) $4x - 2y = 1$ (b) $x_1 - 4x_2 + 7x_3 = 5$

Solution. (a) We assign to x an arbitrary value t

$$x = t$$

$$4t - 2y = 1$$

$$2y = 4t - 1$$

$$y = 2t - \tfrac{1}{2}$$

The arbitrary value t is called a ***parameter***. Substituting t by any specific value will give a particular solution.

For example with $t = 3$ we get

$$x = 3 \quad \text{and} \qquad y = \tfrac{11}{2}$$

We can also assign to y an arbitrary value t, then

$$x = \tfrac{1}{2}t + \tfrac{1}{4} \quad y = t$$

Note. The two sets of solutions are the same.

For example with $t = 3$ in the first set we have

$$x = 3 \quad y = \tfrac{11}{2}$$

this solution corresponds to the solution in the second set with $t = \tfrac{11}{2}$

Solution. (b) We assign arbitrary values to any two variables and solve for the third one. For example

$$x_1 = 5 + 4s - 7t \quad x_2 = s \qquad x_3 = t$$

1.1.2 LINEAR SYSTEMS

A ***system of linear equations*** (or a ***linear system***) is a finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$

A ***solution*** of a linear system is an n-tuple of real numbers $s_1, s_2, \ldots, s_n$ such that each equation is satisfied when we substitute $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$.

Example. Consider the system

$$\begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned}$$

Then $x_1 = 1, x_2 = 2, x_3 = -1$ is a solution.

However, $x_1 = 1, x_2 = 8, x_3 = 1$ is not a solution since these values satisfy only the first equation.

Note. Not all systems of linear equations have solutions. For example the system

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 6 \end{aligned}$$

has no solution. We say that it is ***inconsistent***

A system with at least one solution is called ***consistent***

A general system of two linear equations in two unknowns has the form:

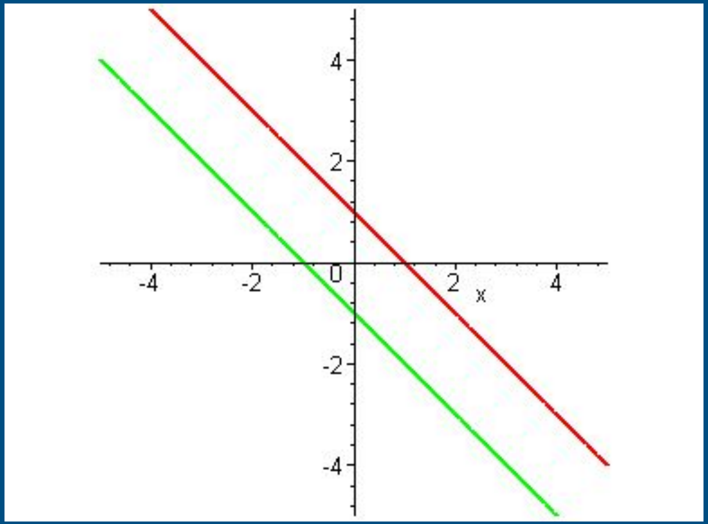
$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

The graph of these equations is two lines l_1 and l_2.

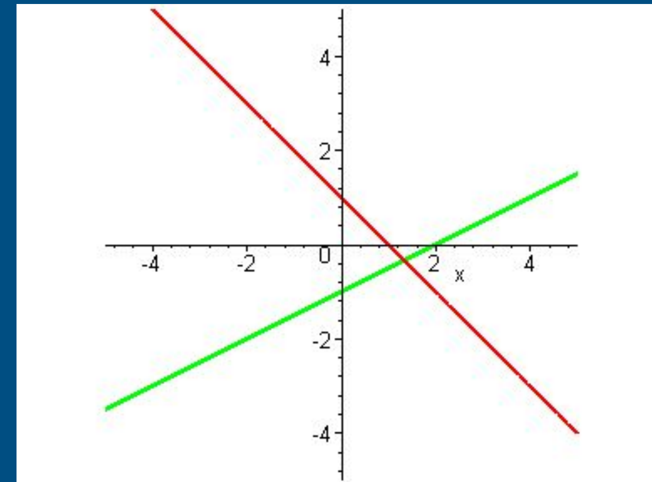
A pair x, y is a solution of the system if the point with coordinates (x,y) is a point of intersection of l_1 and l_2.

There are 3 cases

✔ The lines l_1 and l_2 are parallel: there is ***no*** intersection and the system is inconsistent

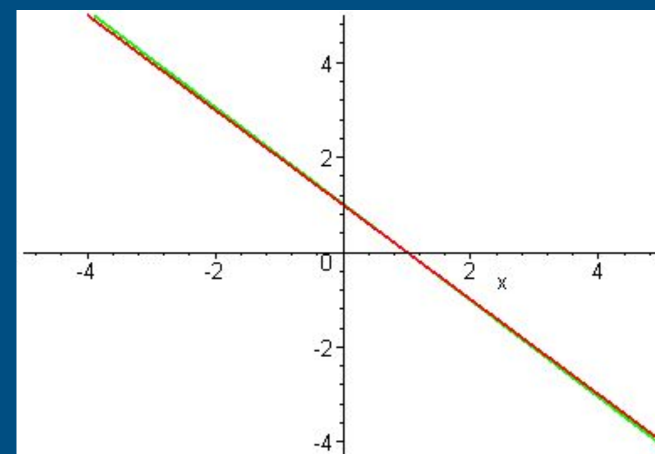


✔The lines l_1 and l_2 intersect at only ***one*** point:



The system has a unique solution

✔The lines l_1 and l_2 coincide:



The system has ***infinitely*** many solutions

In general

A system of linear equations may have no solution, exactly one solution or infinitely many solutions

An arbitrary system of m linear equations in n unknowns has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

The double subscripting on the coefficients of the unknowns allows us to specify their location:

- The first subscript on the coefficient $\boldsymbol{a_{ij}}$ indicates the equation in which this coefficient occurs (***i***)

- The second subscript indicates which unknown this coefficient multiplies (***j***)

1.1.3 AUGMENTED MATRICES

The general system of *m* linear equations in *n* unknowns may be abbreviated by using the two-dimensional array

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

which is called the ***augmented matrix*** for the system

Example. The augmented matrix for the system

$$\begin{aligned} x_1 + \quad x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

is

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Note. The unknowns must be written in the ***same order***, and the constant terms must be ***on the right***

The basic method for solving a system of linear equations is to replace it by a new system with the same solution set (the ***equivalent system***) but which is easier to be solved.

This is done in a series of steps by

1. Multiplying an equation through by a nonzero constant
2. Interchanging two equations
3. Adding a multiple of one equation to another

This results in a series of operations on the augmented matrix called ***elementary row operations***

Elementary row operations

1. Multiply a row through by a nonzero constant
2. Interchange two rows
3. Add a multiple of one row to another

Example. Solve the following system by carrying out elementary row operations on the augmented matrix

$$\begin{aligned} x + \quad y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad aug := \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Solution. Add –2 times the 1st equation to the 2nd

Add –2 times the 1st equation to the 2nd

$$\begin{aligned} x + \quad y + 2z &= \quad 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= \quad 0 \end{aligned}$$

$$aug := \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add –3 times the 1st equation to the 3rd

$$\begin{aligned} x + \quad y + 2z &= \quad 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

$$aug := \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply 2nd equation by $\frac{1}{2}$

$$\begin{aligned} x + \quad y + 2z &= \quad 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

$$aug := \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add –3 times the 2^{nd} equation to the 3^{rd}

$$\begin{aligned} x + \ y + 2z &= \ 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

$$aug := \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{-3}{2} \end{bmatrix}$$

Multiply 3^{rd} equation by –2

$$\begin{aligned} x + \ y + 2z &= \ 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= \ 3 \end{aligned}$$

$$aug := \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add –1 times the 2^{nd} equation to the 1^{st}

$$\begin{aligned} x \qquad + \tfrac{11}{2}z &= \ \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= \ 3 \end{aligned}$$

$$aug := \begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$
\begin{aligned}
x \qquad + \tfrac{11}{2} z &= \tfrac{35}{2} \\
y - \tfrac{7}{2} z &= -\tfrac{17}{2} \\
z &= 3
\end{aligned}
\qquad
aug := \begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}
$$

Add $-\frac{11}{2}$ times the 3rd equation to the 1st, and $\frac{7}{2}$ times the 3rd equation to the 2nd

$$
\begin{aligned}
x \qquad &= 1 \\
y \quad &= 2 \\
z &= 3
\end{aligned}
\qquad
aug := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}
$$

The unique solution $x = 1$, $y = 2$, $z = 3$ is now clear

1.2. GAUSSIAN ELIMINATION

1.2.1. Echelon Forms

1.2.2. Elimination Methods

1.2.3. Back Substitution

1.2.4. Homogeneous Linear Systems

1.2.1 ECHELON FORMS

In the last section a linear system in the unknowns x, y, z is solved by reducing the augmented matrix to the form

$$aug := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

from which the solution $x = 1$, $y = 2$, $z = 3$ becomes clear

This is an example of a matrix in ***reduced row-echelon form***

A row is said to be ***zero row*** if all of its entries are zero. Otherwise it is said to be a ***nonzero row***.

The first nonzero element in a nonzero row is called a ***leading entry***.

Definition. A matrix is said to be in ***row-echelon form*** if it satisfies the following two properties:

1. The zero rows are grouped together at the bottom,
2. The leading entry of each nonzero row occurs further to the right than the leading entry in the row above it.

A row-echelon matrix has the form of a staircase

$$\begin{bmatrix} \oplus & * & * & * \\ 0 & \oplus & * & * \\ 0 & 0 & \oplus & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \oplus & * & * & * & * & * & * \\ 0 & 0 & 0 & \oplus & * & * & * & * \\ 0 & 0 & 0 & 0 & \oplus & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \oplus & * \end{bmatrix}$$

Where the entry $\oplus$ may be any nonzero number (the leading entry), and * may be any number

Definition. A row-echelon matrix is said to be in ***reduced row-echelon form*** if it also satisfies:

3. The leading entry in each nonzero row is 1 (**leading 1**)

4. Each leading 1 is the only nonzero entry in its column.

Example. The following matrices are in reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row-echelon form but not in reduced row-echelon form

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example. Solve the following systems where the augmented matrices are in reduced row-echelon form.

(a) $$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Solution. The corresponding system of equations is

$$\begin{aligned} x_1 \qquad &= 5 \\ x_2 \quad &= -2 \\ x_3 &= 4 \end{aligned}$$

From this we can read out the solution immediately

$$x_1 = 5, x_2 = -2, x_3 = 4$$

(b)
$$\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

Solution. The corresponding system of equations is

$$\begin{aligned} x_1 \qquad\qquad + 4x_4 &= -1 \\ x_2 \qquad + 2x_4 &= 6 \\ x_3 + 3x_4 &= 2 \end{aligned}$$

Since x_1, x_2, x_3 correspond to the leading entries in the augmented matrix, we call them the ***leading variables***, the other variables (x_4) are ***free variables***

Solving for the leading variables in terms of the free variables we obtain

$$
\begin{aligned}
x_1 &= -1-4x_4 \\
x_2 &= 6-2x_4 \\
x_3 &= 2-3x_4
\end{aligned}
$$

We can assign an arbitrary value, say t to the free variable x_4 and get the corresponding values for the leading variables x_1, x_2, x_3

$$x_1 = -1 - 4t\ ,\ x_2 = 6 - 2t\ ,\ x_3 = 2 - 3t\ ,\ x_4 = t$$

There are infinitely many solutions.

(c) $\begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution. The last row doesn't contribute to find the solutions of the system. We can write the system as

$$\begin{aligned} x_1 + 6x_2 \qquad\qquad + 4x_5 &= -2 \\ x_3 \qquad + 3x_5 &= 1 \\ x_4 + 5x_5 &= 2 \end{aligned}$$

We can solve for the leading variables x_1, x_3, x_4 in terms of the free variables x_2 and x_5

$$\begin{aligned} x_1 &= -2 - 6x_2 - 4x_5 \\ x_3 &= 1 - 3x_5 \\ x_4 &= 2 - 5x_5 \end{aligned}$$

We can assign arbitrary values, say s and t to the free variable x_2 and x_5. The general solution is

$x_1 = -2 - 6s - 4t$, $x_2 = s$, $x_3 = 1 - 3t$, $x_4 = 2 - 5t$, $x_5 = t$

(d) $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 = 1$$

which has no solution. So the system has no solution

1.2.2 ELIMINATION METHODS

We have seen that a linear system is easily solved if its augmented matrix is in reduced row-echelon form

The ***Gaussian Elimination*** procedure (also called ***row-reduction***) allows us to put a matrix in a row-echelon form. It consists of ***five*** steps.

We will present these steps via an example

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost nonzero column: *column 1*

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 2. Interchange the 1st row with another row, if necessary, (row 2) so that the top of the column found in Step 1 is nonzero

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 3. Multiply the 1^{st} row with a suitable scalar (1/2) to introduce a leading 1

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 4. Add a suitable multiple of the 1^{st} row (-2) to the rows below (row 3) so that all entries below the leading 1 becomes zeros

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Step 5. Cover the 1st row and begin again with Step 1 applied to the remaining submatrix. Continue Step 5 until the matrix is in row-echelon form

Multiply new row 1 by –1/2 to introduce leading 1

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 \quad 0 \quad -2 \quad 0 \quad 7 \quad 12 \\ 0 \quad 0 \quad 5 \quad 0 \quad -17 \quad -29 \end{bmatrix}$$

Add –5 times new row 1 to new row 2

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 \quad 0 \quad 1 \quad 0 \quad \frac{-7}{2} \quad -6 \\ 0 \quad 0 \quad 5 \quad 0 \quad -17 \quad -29 \end{bmatrix}$$

Cover the new row 1 and begin again with Step1

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 \quad 0 \quad 1 \quad 0 \quad \frac{-7}{2} \quad -6 \\ 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{2} \quad 1 \end{bmatrix}$$

Multiply new row 1 by 2 to introduce leading 1

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The matrix is now in row-echelon form.

Note. Gaussian Elimination procedure as presented is convenient for hand computations
Traditionally, Step 3 is not included since it causes more computations

Step 3. Multiply the 1st row with a suitable scalar to introduce a leading 1

Without this step the result is slightly different.
A call of MAPLE's command **gausselim** produces

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Compare this with the result from Gaussian Elimination procedure in which Step 3 is included

Note. The position of the leading entries is the same. They are called ***pivot positions***.
The nonzero number at a pivot position is called a ***pivot***

The ***Gauss-Jordan Elimination*** procedure allows to find the reduced row-echelon form. It includes another Step

Step 6. Working upward from the last nonzero row. Add suitable multiples of each row to the rows above it so that all entries above the pivot becomes zeros

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{(2)+7/2*(3)} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{(1)-6*(3)} \begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{(1)+5*(2)} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The matrix is now in reduced row-echelon form

Example. Solve by Gauss-Jordan elimination.

$$\begin{array}{ccccccccc} x_1 & +3x_2 & -2x_3 & & +2x_5 & & = & 0 \\ 2x_1 & +6x_2 & -5x_3 & -2x_4 & +4x_5 & -3x_6 & = & -1 \\ & & 5x_3 & +10x_4 & & +15x_6 & = & 5 \\ 2x_1 & +6x_2 & & +8x_4 & +4x_5 & +18x_6 & = & 6 \end{array}$$

Solution. The augmented matrix of the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Apply the row-reduction to the augmented matrix

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is now in row-echelon form.
Apply step 6 to put it in reduced row-echelon form

Apply step 6 to put it in reduced row-echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is now in reduced row-echelon form

The corresponding system of equations is.

$$\begin{array}{cccccccc} x_1 & +3x_2 & & +4x_4 & +2x_5 & & = & 0 \\ & & x_3 & +2x_4 & & & = & 0 \\ & & & & & x_6 & = & \frac{1}{3} \end{array}$$

Solving for the leading variables

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

assigning arbitrary values to the free variables

$$x_1 = -3r - 4s - 2t \quad x_2 = r \quad x_3 = -2s$$

$$x_4 = s \quad x_5 = t \quad x_6 = \tfrac{1}{3}$$

1.2.3 BACK-SUBSITUTION

In the last example, we can stop at the row-echelon form with Gaussian elimination

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$\begin{array}{rrrrrrcr} x_1 & +3x_2 & -2x_3 & & +2x_5 & & = & 0 \\ & & -x_3 & -2x_4 & & -3x_6 & = & -1 \\ & & & & & 6x_6 & = & 2 \end{array}$$

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\qquad &= 0 \\
- x_3 - 2x_4 \qquad\qquad - 3x_6 &= -1 \\
6x_6 &= 2
\end{aligned}
$$

Solving backward (upward) for the leading variables

and assigning arbitrary values to the free variables

$$x_6 = \tfrac{1}{3} \quad x_5 = t \quad x_4 = s$$

$$x_3 = 1 - 2x_4 - 3x_6 = 1 - 2s - 3(\tfrac{1}{3}) = -2s \quad x_2 = r$$

$$x_1 = -3x_2 + 2x_3 - 2x_5 = -3r + 2(-2s) - 2t = -3r - 4s - 2t$$

This is the same solution as before.

The arbitrary values *r*, *s*, *t* are the *parameters*

The Gauss-Jordan elimination is convenient for hand computations

However, when programmed to run on computers, Gaussian elimination is **faster** than Gauss-Jordan elimination (about 50%)

1.2.4 HOMOGENEOUS LINEAR SYSTEMS

A system of linear equation is said to be ***homogeneous*** if all the constant terms are zero

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

$x_1 = 0$, $x_2 = 0$, …, $x_n = 0$ is always a solution called the ***trivial solution***.

All other solutions, if exist, are ***non trivial***

Thus a homogeneous linear system either has

- ✔ only the trivial solution, or
- ✔ infinitely many solutions

A general homogeneous linear system of two equations in two unknowns has the form:

$$a_1x + b_1y = 0$$
$$a_2x + b_2y = 0$$

The graphs of the two equations are two lines passing through the origin: the trivial solution corresponds to the origin (0, 0)

Example. Solve by Gauss-Jordan elimination.

$$\begin{array}{rrrrrcl} 2x_1 & +2x_2 & -x_3 & & +x_5 & = & 0 \\ -x_1 & -x_2 & +2x_3 & -3x_4 & +x_5 & = & 0 \\ x_1 & +x_2 & -2x_3 & & -x_5 & = & 0 \\ & & x_3 & +x_4 & +x_5 & = & 0 \end{array}$$

Solution. The augmented matrix of the system and its reduced row-echelon form are

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{array}{ccccccc} x_1 & +x_2 & & & +x_5 & = & 0 \\ & & x_3 & & +x_5 & = & 0 \\ & & & x_4 & & = & 0 \end{array}$$

The general solution is

$$x_1 = -s - t \quad x_2 = s \quad x_3 = -t \quad x_4 = 0 \quad x_5 = t$$

The trivial solution is obtained by putting $s = t = 0$

Note. For a general homogeneous system, the system corresponding to the reduced row-echelon augmented matrix is also homogeneous

In particular if $m < n$ then $r < n$,

where r is the number of nonzero equations in the system corresponding to the reduced row-echelon augmented matrix

Then we can solve for the r leading variables in terms of the free variables and get the general nontrivial solution

Therefore we have

Theorem. A homogeneous linear system with more unknowns than equations has infinitely many solutions

$$\begin{array}{rcrcrcrcr}
-x_1 & + & 2x_2 & - & x_3 & - & x_4 & = & -1; \\
3x_1 & - & 2x_2 & & & + & x_4 & = & 4; \\
4x_1 & - & 5x_2 & + & 2x_3 & + & x_4 & = & -1; \\
3x_1 & - & 3x_2 & + & x_3 & + & x_4 & = & 2.
\end{array}$$

$$\begin{array}{rcrcrcrcr}
3x_1 & + & 6x_2 & & & + & 9x_4 & = & 3; \\
-x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & = & 1; \\
-3x_1 & - & 6x_2 & + & 2x_3 & - & 11x_4 & = & 1.
\end{array}$$

$$\begin{array}{rcrcrcrcr}
x_1 & + & x_2 & + & 3x_3 & + & 4x_4 & = & 3; \\
2x_1 & + & 3x_2 & + & x_3 & + & 3x_4 & = & 2; \\
-x_1 & - & 2x_2 & + & 2x_3 & + & x_4 & = & 4.
\end{array}$$

1.3. MATRICES AND MATRIX OPERATIONS

1.3.1. Matrix Notation & Terminology

1.3.2. Operations on Matrices

1.3.3. Matrix Products as Linear Combinations

1.3.4. Matrix Form of a Linear System

1.3.5. Transpose of a Matrix

1.3.1 MATRIX NOTATION & TERMINOLOGY

Definition. A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix

Example.

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \quad [2 \quad 1 \quad 0 \quad -3] \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad [\ 4]$$

These are matrices of sizes 3×2, 1×4, 3×3, 2×1, 1×1 respectively

A matrix with only one column is called a ***column matrix***.
A matrix with only one row is called a ***row matrix***

The entries of a matrix are real numbers or complex numbers. We say that they are ***real scalars*** or ***complex scalars***

The entry that occurs in row i and column j of matrix A is denoted by a_{ij}. Thus a 3 × 4 matrix may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

And a general $m \times n$ matrix may be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It may also be written briefly as

$[a_{ij}]_{m \times n}$ or simply $[a_{ij}]$

The entry in row i and column j of a matrix A is also denoted by $(A)_{ij}$.

$$\mathbf{a} = [a_1 \ a_2 \ \ \ldots \ \ a_n] \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Row and column matrices are denoted as vectors

An $n \times n$ matrix A is called a square matrix of order n.

The entries a_{11}, a_{22}, … , a_{nn} are said to be on the ***main diagonal*** of A

1.3.2 OPERATIONS ON MATRICES

Definition. Two matrices are ***equal*** if they have the same size and their corresponding entries are equal.

$$A = B \quad \Leftrightarrow \quad (A)_{ij} = (B)_{ij}$$

Example. Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

Then $A = B$ if $x = 5$ and $A \neq B$ if $x \neq 5$.

There is no value of x such that $A = C$ since they are of different sizes

Definition. Let A and B be two matrices of the same size, then the ***sum*** $A + B$ and the ***difference*** $A - B$ are matrices defined by

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} \text{ and } \qquad (A-B)_{ij} = (A)_{ij} - (B)_{ij}$$

Example. Let

$$A := \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix} \qquad B := \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \qquad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

Definition. Let A be any matrix and c is a scalar, then the ***product*** cA is the matrix defined by

$$(cA)_{ij} := c(A)_{ij} \quad \text{for all } i, j$$

cA is also called a ***scalar multiple*** of A

Example. Let

$$A := \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}$$

Then

$$2A = \begin{bmatrix} 4 & 2 & 0 & 6 \\ -2 & 0 & 4 & 8 \\ 8 & -4 & 14 & 0 \end{bmatrix} \qquad (-1)A = -A = \begin{bmatrix} -2 & -1 & 0 & -3 \\ 1 & 0 & -2 & -4 \\ -4 & 2 & -7 & 0 \end{bmatrix}$$

Combining these operations we can define the ***linear combination*** of matrices $A_1, A_2, \ldots, A_n$ of the same size with ***coefficients*** (scalars) $c_1, c_2, \ldots, c_n$

$$\sum_{i=1}^{n} c_i A_i = c_1 A_1 + c_2 A_2 + \cdots + c_n A_n$$

Example.

$$A := \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix} \qquad B := \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix} \qquad C := \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

$$2A - B + \tfrac{1}{3}C = 2A + (-1)B + \tfrac{1}{3}C = \begin{bmatrix} 7 & 2 & 2 \\ 4 & 3 & 11 \end{bmatrix}$$

Definition. Let A be an $m \times r$ matrix and B an $r \times n$ matrix, then the ***product*** AB is the $m \times n$ matrix whose entries are defined as

$$(AB)_{ij} = (A)_{i1}(B)_{1j} + (A)_{i2}(B)_{2j} + \ldots + (A)_{ir}(B)_{rj}$$

$$= \sum_{k=1}^{r} (A)_{ik}(B)_{kj}$$

Example. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

To get $(AB)_{23}$ we will multiply the corresponding entries in row 2 and column 3 and sum up the products

To get $(AB)_{23}$ we will multiply the corresponding entries in row 2 and column 3 and sum up the products

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$(AB)_{23} = 2 \cdot 4 + 6 \cdot 3 + 0 \cdot 5 = 26$$

The other entries are calculated in a similar manner. The product is

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

In general

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

Then $$(AB)_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ir} b_{rj}$$

In particular, the j^th column matrix of *AB* is the product of *A* and the j^th column matrix of *B*

The i^th row matrix of *AB* is the product of the i^th row matrix of *A* with *B*

In general

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{ir}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & \boxed{b_{1j}} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & \boxed{b_{2j}} & \cdots & b_{2n} \\ \vdots & \vdots & & \boxed{\vdots} & & \vdots \\ b_{r1} & b_{r2} & \cdots & \boxed{b_{rj}} & \cdots & b_{rn} \end{bmatrix}$$

Then
$$(AB)_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

Example. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Then the 2nd column matrix of AB is

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

This is precisely the 2nd column matrix of the product AB already computed

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ be the column matrix of B so that B may be written as $B = [\mathbf{b}_1 | \mathbf{b}_2 | \ldots | \mathbf{b}_n]$, and

$$AB = A[\mathbf{b}_1 | \mathbf{b}_2 | \ldots | \mathbf{b}_n] = [A\mathbf{b}_1 | A\mathbf{b}_2 | \ldots | A\mathbf{b}_n]$$

Similarly let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ be the row matrix of A so that A may be written as

$$A = \left[\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right] \qquad \text{and} \qquad AB = \left[\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right] B = \left[\begin{array}{c} \mathbf{a}_1 B \\ \hline \mathbf{a}_2 B \\ \hline \vdots \\ \hline \mathbf{a}_m B \end{array}\right]$$

1.3.3 MATRIX PRODUCTS AS LINEAR COMBINATIONS

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

i.e. $A\mathbf{x}$ is a linear combination of the column matrices of A with coefficients the entries of $\mathbf{x}$

Example. The product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as a linear combination of the column matrices

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Similarly

If $\mathbf{y}$ is a $1 \times m$ matrix then $\mathbf{y}A$ is a linear combination of the row matrices of A with coefficients the entries of $\mathbf{y}$

and

The jth column (ith row) of the product AB is a linear combination of the column (row) matrices of A (B) with coefficients the entries of the jth column (ith row) of B (A)

Example. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Then the 2nd column matrix of AB is

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

which may be written as a linear combination of column matrices of A with coefficients from the 2nd column matrix of B

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

1.3.4 MATRIX FORM OF A LINEAR SYSTEM

Consider the system of m equations in n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

We can view this as an equality of the corresponding entries of two $m \times 1$ matrices

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The left hand side is the product of A with the column matrix **x** Therefore

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

This is a matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

1.3.5 TRANSPOSE OF A MATRIX

Definition. Let A be an $m \times n$ matrix, then the ***transpose of A*** denoted by $\boldsymbol{A^T}$ is the $n \times m$ matrix that results from interchanging the rows and columns of A

$$(A^T)_{ij} := (A)_{ji}$$

Note. The 1st row of A becomes the 1st column of A^T
The 2nd row of A becomes the 2nd column of A^T, …

Example. The transposes of some matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

$$C = [1 \quad 3 \quad 5] \qquad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad \mathrm{D} = [\;4] \qquad \mathrm{D}^T = [\;4]$$

Definition. Let A be a ***square*** matrix of order n, then the ***trace of A*** denoted by **tr(A)** is defined to be the sum of the entries on the main diagonal of A

$$\mathrm{tr}(A) = (A)_{11} + (A)_{22} + \ldots + (A)_{nn}$$

Example. The traces of some matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\mathrm{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \mathrm{tr}(B) = -1 + 5 + 7 + 0 = 11$$

1.4. INVERSES; RULES OF MATRIX ARITHMETIC

1.4.1. Properties of Matrix Operations

1.4.2. Zero and Identity Matrices

1.4.3. Inverse of a Matrix

1.4.4. Polynomial Expressions involving Matrices

1.4.5. Properties of the Transpose

1.4.1 PROPERTIES OF MATRIX OPERATIONS

Theorem. Assuming the following matrix operations are defined, then we have

(a) $A + B = B + A$ (Commutativity of $+$)

(b) $A + (B + C) = (A + B) + C$ (Associativity of $+$)

(c) $A(BC) = (AB)C$ (Associativity of $\times$)

(d) $A(B + C) = AC + BC$ (Left Distributivity)

(e) $(B + C)A = BA + CA$ (Right Distributivity)

Note. If a, b are scalars, we have similar rules

$$\boxed{\begin{aligned}&\text{(f) } a(B + C) = aB + aC\\&\text{(g) } (a + b)\, C = aC + bC\\&\text{(h) } a(bC) = (ab)C\\&\text{(i) } a(BC) = (aB)C\end{aligned}}$$

Recall that $-B = (-1)B$. Hence we also have

$$A(B - C) = AC - BC$$

$$(B - C)A = BA - CA$$

$$a(B - C) = aB - aC$$

$$(a - b)\, C = aC - bC$$

Example. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix} = (AB)C$$

Note. Due to the associativity of the addition and the multiplication we may write

$$A + (B + C) = (A + B) + C = A + B + C$$
$$A(BC) = (AB)C = ABC$$

Thus we can operate on the expressions involving matrices almost as in the case of real numbers

except interchanging the order of the factors in a product because the ***product of two matrices is not commutative*** as shown in the next example

Example. Let $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Then

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \neq BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

- It may happen that $AB \neq BA$ because one product is defined but the other is not, e.g. A is a 2×3 matrix and B is a 3×4 matrix.

- In the case both products are defined but having different sizes, e.g. A is a 2×3 matrix and B is a 3×2 matrix, then $AB \neq BA$

1.4.2 ZERO AND IDENTITY MATRICES

In the addition of real numbers, the number 0 plays a neutral role. The ***zero matrix*** plays a similar role

The ***zero matrix*** of size $m \times n$, denoted by $0_{m \times n}$ or simply 0, has all of its entries equal to 0

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The ***zero matrix*** is the neutral element for addition:

$$A + 0 = 0 + A = A$$

The zero matrix also satisfies:

$$A0 = 0 \qquad 0A = 0$$

Similarly the neutral element for multiplication is the ***identity matrix I***:

$$AI = A \quad \text{and} \quad IA = A$$

where the ***identity matrix*** I_n or simply I is the square matrix of order n with 1's on the main diagonal and 0's everywhere else

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

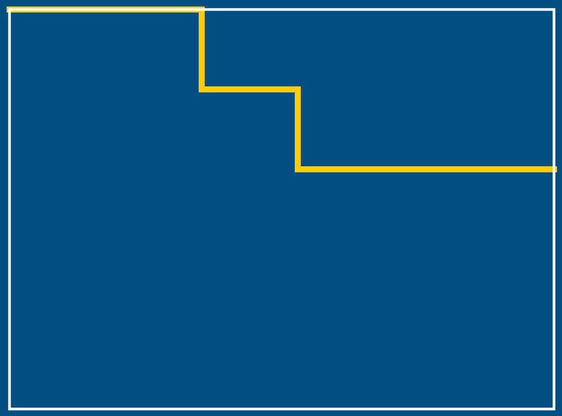
Theorem. Let A be a square matrix of order n. If A is in reduced row-echelon form and has no zero row, then A is the identity matrix I_n

Proof. Since a row-echelon matrix has the shape of a staircase, if it has no zero row then

the height is $\leq$ the length

Furthermore, the inequality is strict if there is one step longer than 1.

This shows that all steps are equal to 1 since A is a square matrix,



i.e. the pivots are in the main diagonal.

If A is also in reduced row-echelon form, then the diagonal entries are equal to 1, and all other entries are 0.

Therefore A is the identity matrix I_n

1.4.3 INVERSE OF A MATRIX

Recall that the multiplication of real numbers satisfies the Cancellation Law, namely if $a \neq 0$, then

$$ab = ac \Rightarrow b = c$$

or equivalently

$$ab = 0 \Rightarrow b = 0$$

However the multiplication of matrices does not satisfy the Cancellation Law as shown in the next example

Example. $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \neq 0$ and $B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} \neq 0$

but

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} = 0$$

- We can look at the Cancellation Law for multiplication of real numbers in another way,

  namely if a is invertible (has an inverse), then

$$ab = 0 \Rightarrow b = 0$$

The same result also holds for matrix multiplication

Let's define what is an invertible matrix first

Definition. A square matrix A is said to be ***invertible*** if there is a matrix of same size B such that

$$AB = BA = I$$

B is called the ***inverse*** of A, and denoted by A^{-1}.
A square matrix without ***inverse*** is called a ***singular*** matrix

Example. $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

since $$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and $$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now if A is an invertible matrix, then

$$AB = 0 \Rightarrow B = 0$$

Let's assume indeed $AB = 0$

Then $$A^{-1}(AB) = A^{-1}\,0 = 0$$

On the other hand

$$A^{-1}(AB) = (A^{-1}A)B = IB = B$$

Hence $$B = 0$$

The above result may also be rephrased as follows:

Assume that A is a square matrix, and there is a nonzero matrix B such that $AB = 0$. Then A is a singular matrix.

Example. The matrix $$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

is singular since

$$AB = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \quad \text{while} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq 0$$

Theorem. If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Similarly

$$(B^{-1}A^{-1})(AB) = I$$

Corollary. If $A_1, A_2, \ldots, A_n$ are invertible matrices of the same size, then

$$(A_1A_2\ldots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\ldots A_2^{-1}A_1^{-1}$$

1.4.4. POLYNOMIAL EXPRESSIONS INVOLVING MATRICES

$$A^0 := I$$

If n is a positive integer, and A a square matrix, then

$$A^n := \underbrace{A\, A\, \ldots\, A}_{n}$$

If r, s are non-negative integers, then

$$A^r A^s = A^{r+s}$$

If A is also invertible and n is a positive integer, then

$$A^{-n} := \underbrace{A^{-1}\, A^{-1}\, \ldots\, A^{-1}}_{n} = (A^n)^{-1}$$

If A is an invertible matrix and k a nonzero scalar, then kA is also invertible and

$$(kA)^{-1} = \frac{1}{k} A^{-1}$$

Example. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

$$A^3 A^{-3} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = I_2$$

Let A be a square matrix of order m, and

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$

is a polynomial of degree n, then we define

$$p(A) = a_0 I + a_1 A + \dots + a_n A^n$$

where I is the identity matrix of order m

Example. Let $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$ and $p(x) = 2x^2 - 3x + 4$

then $p(A) = 2A^2 - 3A + 4I$

$$= 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix}$$

1.4.5. PROPERTIES OF THE TRANSPOSE

Theorem. Assuming the following matrix operations are defined, then we have

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(A - B)^T = A^T - B^T$

(d) $(kA)^T = kA^T$

(e) $(AB)^T = B^T A^T$

More generally

$$(A_1 A_2 \ldots A_n)^T = A_n{}^T A_{n-1}{}^T \ldots A_2{}^T A_1{}^T$$

Theorem. If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Example. Let $A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix}$

then $A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}$

We have

$$\left(A^{-1}\right)^T = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix} = \left(A^T\right)^{-1}$$

1.5. FINDING INVERSES BY ELEMENTARY MATRICES

1.5.1. Elementary Matrices

1.5.2. A Method for Inverting Matrices

1.5.1 ELEMENTARY MATRICES

Definition. A square matrix of order n is ***elementary*** if it can be obtained from the identity matrix I_n by performing a single elementary row operation

Example. The following matrices are elementary

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem. If an elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then EA also results from performing the same row operation on A.

Example. Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

the elementary matrix that results from adding three times the 1st row of I_3 to the 3rd row, then

by adding three times the 1st row of A to the 3rd row, we obtain $EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$

Remark. If an elementary row operation is applied to the identity matrix I to produce an elementary matrix E, then there is a second row operation that, when applied to E, produces I back again

This is the ***inverse operation*** of the given operation

Row operation $I \to E$	Inverse Row operation $E \to I$
Multiply row *i* by $c \neq 0$	**Multiply row *i* by $1/c$**
Interchange rows *i* and *j*	**Interchange rows *i* and *j***
Add *c* times row *i* to row *j*	**Add $-c$ times row *i* to row *j***

Example. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

multiply row 2 by 7   multiply row 2 by $\frac{1}{7}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

interchange rows 1, 2   interchange rows 1, 2

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add 5 times row 2 to row 1   Add –5 times row 2 to row 1

Theorem. Every elementary matrix is invertible, and the inverse is also an elementary matrix

Proof. Let E be an elementary matrix corresponding to the row operation T, and T^{-1} the inverse row operation of T. Let E_0 be the elementary matrix obtained by performing T^{-1} on the identity matrix I

It is clear that E_0E is obtained by performing T on I then performing T^{-1}. The result is, of course I, i.e.

$$E_0E = I$$

Similarly $$EE_0 = I$$

Therefore E is invertible and $E^{-1} = E_0$ is also an elementary matrix

- From the Theorem, we see that a product of elementary matrices is invertible

- Conversely, let A be an invertible matrix, we will prove that A is a product of elementary matrices indirectly through several steps.

✔ Recall that if B is any matrix such that $AB = 0$ then $B = 0$. In particular if $\mathbf{x}$ is any column matrix such that $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$

In other words, the linear homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

✔ Now let A be any square matrix such that the linear homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

✔The linear homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$

Then the reduced row-echelon form of A has no zero row and hence is the identity matrix

✔Finally, let A be any square matrix such that the reduced row-echelon form of A is I_n.

In other words, there exist elementary matrices $E_1, E_2, \ldots, E_m$ such that

$$E_m E_{m-1} \ldots E_2 E_1 A = I_n$$

Then

$$A = E_1^{-1} E_2^{-1} \ldots E_m^{-1}$$

In fact, we have proved

Theorem. Let A be a square matrix, then the following statements are equivalent

(a) A is invertible

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(c) the reduced row-echelon form of A is I

(d) A is a product of elementary matrices

The matrices A and $B = E_m E_{m-1} \ldots E_1 A$ are said to be ***row equivalent***. Hence (a) and (c) show that

A matrix is invertible if and only if it is row equivalent to the identity matrix I

1.5.2. A METHOD FOR INVERTING MATRICES

Let A be invertible, then the above Theorem says that by performing a sequence of elementary row operations T_1, T_2, … , T_m we can reduce A to the identity matrix I. Let E_1, E_2, … , E_m be the corresponding elementary matrices, then we have

$$E_m E_{m-1} \ldots E_1 A = I$$

The above equality may be rewritten as

$$A^{-1} = E_m E_{m-1} \ldots E_1 I$$

1.5.2. A METHOD FOR INVERTING MATRICES

Let A be invertible, then the proof of the above Theorem says that by performing a sequence of elementary row operations $T_1, T_2, \ldots, T_m$ we can reduce A to the identity matrix I. Let $E_1, E_2, \ldots, E_m$ be the corresponding elementary matrices, then we have

$$E_m E_{m-1} \ldots E_1 A = I$$

The above equality may be rewritten as

$$A^{-1} = E_m E_{m-1} \ldots E_1 I$$

Thus we have proved

> **Method.** To find the inverse of an invertible matrix A,
>
> we find a sequence of elementary row operations that reduces A to the identity matrix I, then
>
> perform this sequence of operations to I to obtain A^{-1}

Example. Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution. To find the inverse by the above method, we will adjoin I to the right of A to obtain the matrix

$$[\, A \mid I \,]$$

Now applying elementary row operations to $[\, A \mid I \,]$ to reduce the left side to the identity matrix I, then the matrix will become

$$[\, I \mid A^{-1} \,]$$

The steps of the reduction are as follows

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 2 & 5 & 3 & | & 0 & 1 & 0 \\ 1 & 0 & 8 & | & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & -2 & 5 & | & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & -5 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 5 & -2 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Note. If we do not know in advance that A is invertible, then we can still carry out the foregoing reduction steps on $[\ A \mid I\ ]$ until:

- The left side is I so that A is invertible and the right side is A^{-1}
- The left side contains a zero row and we conclude that A is not invertible

Example. Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Apply the reduction steps on $[\ A \mid I\ ]$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Hence A is not invertible

Application. We have already seen that the following matrix A is invertible

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Hence the corresponding homogeneous linear system has only the trivial solution

$$\begin{aligned} x_1 \quad + 2x_2 \quad + 3x_3 &= 0 \\ 2x_1 \quad + 5x_2 \quad + 3x_3 &= 0 \\ x_1 \qquad\qquad\quad + 8x_3 &= 0 \end{aligned}$$

1.6. SYSTEMS OF EQUATIONS AND INVERTIBILITY

1.6.1. Solving Linear Systems by Matrix Inversion

1.6.2. Properties of Invertible Matrices

1.6.3. Conditions for Consistency

1.6.1. SOLVING LINEAR SYSTEMS BY MATRIX INVERSION

Theorem. Every system of linear equations either has no solution, exactly one solution, or infinitely many solutions.

Proof. Assume that the system $A\mathbf{x} = \mathbf{b}$ has at least two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. Then $A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$
Hence $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$ is a nonzero solution of $A\mathbf{x} = \mathbf{0}$
Let k be any scalar, then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A(\mathbf{x}_1) + kA(\mathbf{x}_0) = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

When k takes all real values, then the solutions $\mathbf{x}_1 + k\mathbf{x}_0$ are all distinct and form an infinite set

Theorem. If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of linear equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$

Example. Let

$$\begin{array}{llll} x_1 & +2x_2 & +3x_3 & = & 5 \\ 2x_1 & +5x_2 & +3x_3 & = & 3 \\ x_1 & & +8x_3 & = & 17 \end{array}$$

Or putting in matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

We already compute the inverse of A

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Therefore

$$\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1$, $x_2 = -1$, and $x_3 = 2$

- Consider now k systems of linear equations

$$\begin{aligned} A\mathbf{x} &= \mathbf{b}_1 \\ A\mathbf{x} &= \mathbf{b}_2 \\ &\cdots \\ A\mathbf{x} &= \mathbf{b}_k \end{aligned}$$

with the common invertible coefficients matrix A. The solutions are

$$\begin{aligned} \mathbf{x}_1 &= A^{-1}\mathbf{b}_1 \\ \mathbf{x}_2 &= A^{-1}\mathbf{b}_2 \\ &\cdots \\ \mathbf{x}_k &= A^{-1}\mathbf{b}_k \end{aligned}$$

These solutions may be computed using one matrix inversion and k matrix multiplications

However this method is not efficient

An efficient method is to augment the coefficient matrix A by k columns that are the right hand sides of the systems

$$[\, A \mid \mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_k \,]$$

Next, we reduce this matrix to the reduced row-echelon form and separate it into k reduced systems to obtain the solution for each system.

Another advantage of this method is that it also works even if A is not invertible

Example. Solve the systems

$$\begin{array}{llll} x_1 & +2x_2 & +3x_3 & = 4 \\ 2x_1 & +5x_2 & +3x_3 & = 5 \\ x_1 & & +8x_3 & = 9 \end{array} \quad \textit{and} \quad \begin{array}{llll} x_1 & +2x_2 & +3x_3 & = 1 \\ 2x_1 & +5x_2 & +3x_3 & = 6 \\ x_1 & & +8x_3 & = -6 \end{array}$$

Solution. Augment the common coefficient matrix by the right hand sides of two systems

$$A = \left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

Reduce this augment matrix to reduced row-echelon form

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

This leads to two systems

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 0 \\ x_3 &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} x_1 \qquad\qquad &= 2 \\ x_2 \qquad &= 1 \\ x_3 &= -1 \end{aligned}$$

The solutions are $x_1 = 1$, $x_2 = 0$, $x_3 = 1$; and $x_1 = 2$, $x_2 = 1$, $x_3 = -1$ respectively

1.6.2. PROPERTIES OF INVERTIBLE MATRICES

Theorem. Let A be a square matrix, then A is invertible if one of the following conditions holds

(a) There exists a matrix B such that $BA = I$

(b) There exists a matrix B such that $AB = I$

Moreover the matrix B in (a) or (b) is precisely A^{-1}

Proof. Assume that (a) holds and $\mathbf{x}$ is any solution of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Then

$$\mathbf{x} = BA\mathbf{x} = B\mathbf{0} = \mathbf{0}$$

Therefore A is invertible and $B = BAA^{-1} = IA^{-1} = A^{-1}$

Similarly if (b) holds then $A = B^{-1}$. Hence $B = A^{-1}$

Corollary. If A is an $n \times n$ matrix, then A is invertible if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

Proof. Let's assume that the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

Applying this for $\mathbf{b} = \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, the columns of I_n we obtain n column matrices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ such that

$$A\mathbf{x}_1 = \mathbf{e}_1,\ A\mathbf{x}_2 = \mathbf{e}_2,\ \ldots,\ A\mathbf{x}_n = \mathbf{e}_n$$

Let B be the matrix with columns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then

$$AB = [A\mathbf{x}_1 | A\mathbf{x}_2 | \ldots | A\mathbf{x}_n] = [\mathbf{e}_1 | \mathbf{e}_2 | \ldots | \mathbf{e}_n] = I$$

Hence A is invertible

1.6.3. CONDITIONS FOR CONSISTENCY

Problem. Let A be a fixed $m \times n$ matrix, find all $n \times 1$ matrix $\mathbf{b}$ such that the system $A\mathbf{x} = \mathbf{b}$ is consistent.

If A is invertible then all $n \times 1$ matrix $\mathbf{b}$ satisfy the problem.

However if A is not invertible, or A is not a square matrix, then the $n \times 1$ matrix $\mathbf{b}$ must satisfy certain conditions.

Example. Find conditions on b_1, b_1, …, b_n such that

$$\begin{aligned} x_1 \quad + x_2 \quad + 2x_3 &= b_1 \\ x_1 \qquad\qquad + x_3 &= b_2 \\ 2x_1 + x_2 \quad + 3x_3 &= b_3 \end{aligned}$$

is consistent

Solution. The augmented matrix is

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

The 3rd row may be written as

$$0x_1 + 0x_2 + 0x_3 = b_3 - b_2 - b_1$$

i.e. $$b_3 = b_1 + b_2$$

In other words, the column matrix **b** has the form

$$\begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}$$

Example. Find conditions on b_1, b_1, …, b_n such that

$$\begin{aligned} x_1 \quad &+2x_2 &+3x_3 &= b_1 \\ 2x_1 \quad &+5x_2 &+3x_3 &= b_2 \\ x_1 \quad & &+8x_3 &= b_3 \end{aligned}$$

is consistent

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -40b_1+16b_2+9b_3 \\ 0 & 1 & 0 & 13b_1-5b_2-3b_3 \\ 0 & 0 & 1 & 5b_1-2b_2-b_3 \end{array}\right]$$

There is no restriction on b_1, b_2, b_3. The given system has a unique solution for all **b**

$$\begin{aligned} x_1 &= -40b_1 + 16b_2 + 9b_3 \\ x_2 &= 13b_1 - 5b_2 - 3b_3 \\ x_1 &= 5b_1 - 2b_2 - b_3 \end{aligned}$$

1.7. DIAGONAL, TRIANGULAR & SYMMETRIC MATRICES

1.7.1. Diagonal Matrices

1.7.2. Triangular Matrices

1.7.3. Symmetric Matrices

1.7.1 DIAGONAL MATRICES

Definition. A square matrix is said to be a ***diagonal matrix*** if all entries off the main diagonal are zero

Example.

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

A general $n \times n$ diagonal matrix may be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

This matrix is invertible if and only all diagonal entries are non zero. Moreover

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

Powers of a diagonal matrix are simple to compute

$$D^k = \begin{bmatrix} d_1^{\,k} & 0 & \cdots & 0 \\ 0 & d_2^{\,k} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^{\,k} \end{bmatrix}$$

The product of a matrix with a diagonal matrix is also simple to compute by writing

$$D = [d_1\mathbf{e}_1 |\, d_2\mathbf{e}_2 |\, \ldots \,|\, d_n\mathbf{e}_n]$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$ are column vectors of I_n

$$\begin{aligned} AD &= [d_1A\mathbf{e}_1 |\, d_2A\mathbf{e}_2 |\, \ldots \,|\, d_nA\mathbf{e}_n] \\ &= [d_1\mathbf{c}_1 |\, d_2\mathbf{c}_2 |\, \ldots \,|\, d_n\mathbf{c}_n] \end{aligned}$$

$$AD = [d_1\mathbf{c}_1 \mid d_2\mathbf{c}_2 \mid \ldots \mid d_n\mathbf{c}_n]$$

where $\mathbf{c}_1 = A\mathbf{e}_1$, $\mathbf{c}_2 = A\mathbf{e}_2$, …, $\mathbf{c}_n = A\mathbf{e}_n$ are column vectors of A

Similarly $$DA = \begin{bmatrix} d_1\mathbf{r}_1 \\ d_2\mathbf{r}_2 \\ \vdots \\ d_n\mathbf{r}_n \end{bmatrix}$$

where $\mathbf{r}_1$, $\mathbf{r}_2$, …, $\mathbf{r}_n$ are row vectors of A

Example.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 12 & 14 & 16 \\ 27 & 3 & 6 & 9 \end{bmatrix}$$

Powers of a diagonal matrix are simple to compute

$$D^k = \begin{bmatrix} {d_1}^k & 0 & \cdots & 0 \\ 0 & {d_2}^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & {d_n}^k \end{bmatrix}$$

The product of a matrix with a diagonal matrix is also simple to compute

$$AD = [d_1\mathbf{c}_1 |\ d_2\mathbf{c}_2 |\ \ldots\ |\ d_n\mathbf{c}_n]$$

where $\mathbf{c}_1$, $\mathbf{c}_2$, …, $\mathbf{c}_n$ are column vectors of A

And $DA = \begin{bmatrix} d_1\mathbf{r}_1 \\ d_2\mathbf{r}_2 \\ \vdots \\ d_n\mathbf{r}_n \end{bmatrix}$ where $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ are row vectors of A

Example. $$\begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 1 \\ 3 & 7 & 2 \\ 4 & 8 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 10 & 27 \\ 2 & 12 & 3 \\ 3 & 14 & 6 \\ 4 & 16 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 12 & 14 & 16 \\ 27 & 3 & 6 & 9 \end{bmatrix}$$

1.7.2 TRIANGULAR MATRICES

Definition. A square matrix is a ***lower (upper) triangular*** matrix if all entries above (below) the main diagonal are zero. A matrix is a ***triangular*** if it is either lower triangular or upper triangular

Example.

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ 0 & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{ir} \end{bmatrix}$$

Note. Let $A = [a_{ij}]$ be a square matrix, then

- A is lower triangular if $a_{ij} = 0$ for $i < j$
- A is upper triangular if $a_{ij} = 0$ for $i > j$

Theorem.

(a) The transpose of a lower (upper) triangular matrix is an upper (lower) triangular matrix

(b) The product of lower (upper) triangular matrices is a lower (upper) triangular matrix

(c) The inverse of a lower (upper) triangular matrix is a lower (upper) triangular matrix

(d) A triangular matrix is invertible if and only if its diagonal entries are all nonzero

Example. Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

1.7.3 SYMMETRIC MATRICES

Definition. A square matrix is ***symmetric*** if $A^T = A$

Example. The following matrices are symmetric

$$\begin{bmatrix} 7 & 4 \\ 4 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix} \qquad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

Note. A matrix is symmetric if any pair of entries symmetric with respect to the main diagonal are equal to each other

Note. Let $A = [a_{ij}]$ be a square matrix, then A is symmetric if and only if $a_{ij} = a_{ji}$ for all pair (i, j)

Theorem. If A and B are symmetric matrices with the same size, and k is any scalar, then

(a) A^T is symmetric

(b) $A + B$ and $A - B$ are symmetric

(c) kA is symmetric

Note. The product of two symmetric matrices is not necessary symmetric

Let indeed A and B be symmetric matrices, then

$$(AB)^T = B^T A^T = BA$$

Therefore AB is symmetric if and only if $AB = BA$ (we say that A and B ***commute***)

Example. Consider the product of two symmetric matrices that is not symmetric

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

The reason is that these two matrices do not commute

$$\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}$$

On the other hand the product of the following two symmetric matrices is symmetric since they commute

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Theorem. If A is an invertible symmetric matrix, then A^{-1} is also symmetric

Proof. Since $A = A^T$ we have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

Note. Many matrices that arise in applications have the form AA^T or A^TA where A is an $m \times n$ matrix.

It is clear that these products are square matrices of size $m \times m$ and $n \times n$. Moreover, they are symmetric

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

and

$$(A^TA)^T = A^T (A^T)^T = A^TA$$

Note that both products are invertible if A is invertible

Example. The transpose of the matrix

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix}$$

We have

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

Note that both products are symmetric