FIN 971: Problem Set 3*

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11/24/2021

Exercise 3.15 of Tirole (project riskiness and credit rationing).

Consider the basic, fixed-investment model covered in Section 3.2 of Tirole (2006). In particular, investment is a fixed size I, the entrepreneur borrows $I - A$, the probability of success is either p_H (which yields no private benefit) or p_L (which yields private benefit B), success yields verifiable revenue R while failure yields 0. There are two types, "A" and "B", of the projects, which differ only with respect to "riskiness" defined by $p_H^A R^A = p_H^B R^B$, but $p_H^A > p_H^B$ so that project B is "riskier". The investment cost I is the same for both variants and furthermore, $\Delta p = p_H^A - p_L^A = p_H^A - p_L^A$. Which type of project is less prone to credit rationing?

Solution: Following the logic outlined in Section 3.2 of Tirole (2006), we can derive the $\bar{A}^i$ for each project i. First, assume that both projects have positive NPV if the entrepreneur the behaves (i.e. $p_H^i R^i > I$) and negative NPV if the entrepreneur does not behave (i.e., $p_L^i R^i + B < I$), which means that the lending contract will require the entrepreneur to behave. The borrower incentive compatibility constraint to behave is

$$p_H^i R_b^i \geq p_L^i R_b^i + B \iff (\Delta p) R_b^i \geq B$$

Thus, the most that can be pledged to the lender without violating the borrower IC constraint is

$$R_l^i = R - \frac{B}{\Delta p}$$

The participation constraint of the lender is

$$p_H^i R_l^i \geq I - A \iff p_H^i \left(R - \frac{B}{\Delta p} \right) \geq I - A$$

Where the borrower IC holds with equality. Thus, the net worth level at which financing is possible is:

$$\bar{A}^i = p_H^i \frac{B}{\Delta p} - (p_H^i R^i - I)$$

Entrepreneurs with net worth $A \geq \bar{A}^i$ get funded; otherwise, not. Thus, the project that is less prone to credit rationing is the project with the lower $\bar{A}^i$:

$$\bar{A}^A > \bar{A}^B \iff p_H^A \frac{B}{\Delta p} - (p_H^A R^A - I) > p_H^B \frac{B}{\Delta p} - (p_H^B R^B - I) \iff p_H^A > p_H^B$$

Thus, project B is less prone to credit rationing.

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Exercise 3.13 of Tirole (lender market power with fixed investment).

The environment is similar to Section 3.2 of Tirole with one exception. An enterpreneur has internal wealth A (which could be negative because of previous debt) and wants to undertake non-negative investment $I > A$ into a fixed size project. The project yields $R > 0$ with probability p and 0 with probability $1 - p$. The probability of success is p_H if the entrepreneur works and $p_L < p_H$ if he shirks. The entrepreneur obtains private benefit B if she shirks and 0 otherwise. The borrower is protected by limited liability and everyone is risk neutral. The project is worthwhile only if the entrepreneur behaves.

The exception is that there is a single lender. This lender has access to funds that command an expected rate of return equal to 0 (so the lender would content himself with a 0 rate of return, but will use his market power to obtain a superior rate of return). Assume $V \equiv p_H R - I > 0$ and let $\bar{A}$ and $\hat{A}$ be defined by

$$\bar{A} \equiv I - p_H \left[R - \frac{B}{\Delta p} \right]$$
$$\hat{A} \equiv p_H \frac{B}{\Delta p}$$

where $\Delta p = p_H - p_L$. Assume that $\bar{A} > 0$ and that the lender makes a take-it-or-leave-it offer to the borrower (i.e. the lender chooses R_b, the borrower's compensation in the case of success).

(i) What contract is optimal for the lender? Be sure to state the programming problem explicitly.

Solution: The lender maximizes their profit subject to some constraints. First, we require that the borrower behaves, so their IC constraint is that expected payoff from behaving $p_H R_b$ exceeds the expected payoff from not behaving $p_L R_l + B$. Subject to this IC, the lender's expected profit is the expected return on the loan (given the behaving borrower) $p_H R_l$ minus the initial loan size $I - A$. We also should consider the participation constraints of the lender and the borrower. The borrower could eat their net worth A (if $A > 0$) instead of their expected payoff $p_H R_b$ (given the borrower behaves). The lender could eat the loan amount $I - A$ instead of their expected payoff $p_H R_l$ (given the borrower behaves). Finally, by definition $R_l + R_b = R$. Thus, the programming problem is:

$$\begin{aligned}
&\max_{R_l, R_b} p_H R_l - I + A && \\
\text{s.t. } & p_H R_b \geq p_L R_b + B && [IC_b] \\
& p_H R_b \geq A && [PC_b] \\
& p_H R_l \geq I - A && [PC_l] \\
& R_l + R_b = R && \\
\implies & \max_{R_b} p_H (R - R_b) - I + A && \\
\text{s.t. } & R_b \geq \frac{B}{\Delta p} && [IC_b] \\
& R_b \geq \frac{A}{p_H} && [PC_b] \\
& R_b \leq R - \frac{I - A}{p_H} && [PC_l]
\end{aligned}$$

Thus, the lender will choose the lowest R_b such that all constraints hold. First let $\hat{A}$ be the net worth where IC_b and PC_b both bind:

$$\frac{B}{\Delta p} = \frac{\hat{A}}{p_H} \implies \hat{A} = p_H \frac{B}{\Delta p}$$

Second, observe that the PC_b and PC_l cannot both bind. Suppose not then

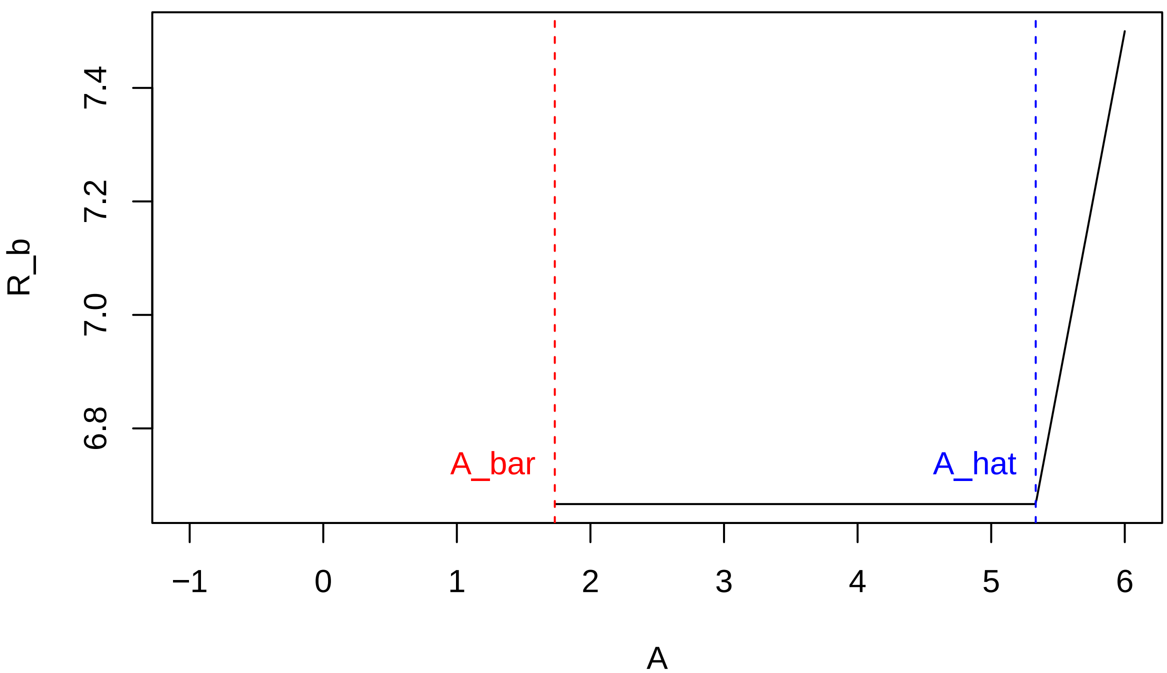
$$\frac{A}{p_H} = R - \frac{I - A}{p_H} \implies 0 = p_H R - I = V > 0 \Rightarrow\Leftarrow$$

Third, let $\bar{A}$ be the net worth where the IC_b and PC_l both bind:

$$\frac{B}{\Delta p} = R - \frac{I - \bar{A}}{p_H} \implies \bar{A} = I - p_H \left(R + \frac{B}{\Delta p} \right)$$

Thus, the optimal lending contract for the lender depends on A. If $A < \bar{A}$, then the lenders participation constraint does not hold, so there's no contract (i.e. credit rationing). If $\bar{A} \leq A < \hat{A}$, the borrowers incentive compatibility constraint binds and the borrowers participation constraint is slack, so $R_b = \frac{B}{\Delta p}$. At $A = \hat{A}$, both the borrowers incentive compatibility and participation constraints bind. If $I > A > \hat{A}$, the borrowers incentive compatibility constraint is slack and the borrowers participation constraint binds, so $R_b = \frac{A}{p_H}$.

An example where $R = 12, I = 6, p_H = .8, p_L = .2, B = 4$ is below:



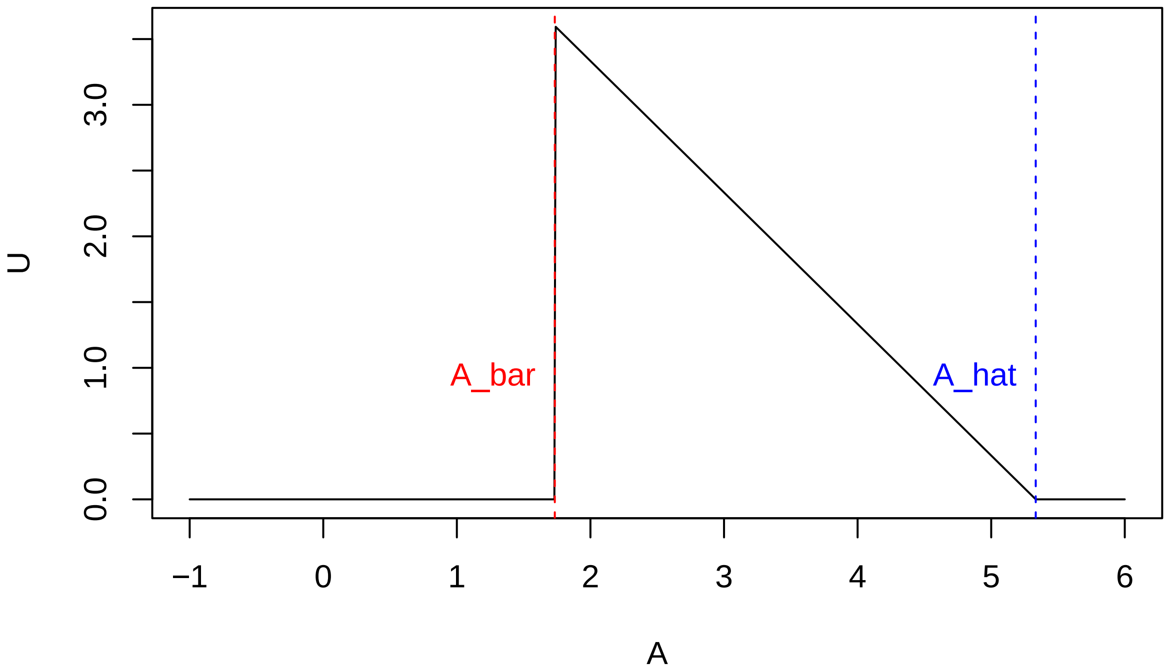
(ii) Is the financing decision affected by lender market power (i.e. compared to the case of competitive lenders in Section 3.2)?

Solution: The lender market power does not affect the credit rationing threshold. With or without market power, financing does not happen if $A < \bar{A}$. Lender market power affects the distribution of the the project return; it allows the lender to extract the entire surplus when the IC constraint is slack. In the competitive lending case, the borrower extracts the entire surplus.

(iii) Draw the borrower's net utility (i.e. net of A) as a function of A. Note that unlike the monotonic case in Section 3.2, it is nonmonotonic among the regions $(-\infty, \bar{A}), [\bar{A}, \hat{A}), [\hat{A}, I)$. Explain.

Solution: If $A < \bar{A}$, the lender participation constraint is not satisfied, so there's no lending and the borrower eats A. Net of A, the borrowers utility is zero.[1] If $\bar{A} \leq A < \hat{A}$, lending happens at $R_b = B/\Delta p$. Thus, the borrowers utility net of A is $p_H B/\Delta p - A$. In this range, the borrowers IC constraint is binding, so the borrower collects agency rent. If $I > A \geq \hat{A}$, then lending happens at $R_b = A/p_H$, so the borrower utility net of A is $p_H A/p_H - A = 0$. Here, the borrower IC is slack so there is no agency rents and the participation constraint is binding.

An example where $R = 12, I = 6, p_H = .8, p_L = .2, B = 4$ is below:



[1]Here, I assume that consumption can be negative. If not, then if $A < 0 < \bar{A}$, lending does not happen and the borrower consumes nothing, so net of A, their utility is $-A$. I suppose an equivalent alternative assumption is that consumption must be nonnegative and the borrower has some other baseline level of consumption $C > -A$, which we're also netting out.

Exercise 3.5 of Tirole (continuous investment and decreasing returns to scale).

Consider the continuous investment model of Section 3.4 of Tirole (2006) with one modification; investment I yields return $R(I)$ in the case of success and 0 in the case of failure, where $R' > 0$ and $R'' < 0, R'(0) > 1/p_H, R'(\infty) < 1/p_H$. The rest of the model is unchanged. That is, the entrepreneur starts with cash A, the probability of success is either p_H if he behaves or p_L if he misbehaves. The entrepreneur obtains private benefit BI if he misbehaves and 0 otherwise. Only the final outcome is observable. Let I^* denote the level of investment that maximizes total surplus (i.e. $p_H R'(I^*) = 1$).

(i) How does investment $I(A)$ vary with the level of cash?

Solution: The programming problem is

$$\begin{aligned} &\max_{I,R_b,R_l} p_H R_b \\ \text{s.t. } & p_H R_b \geq p_L R_b + BI && [IC_b] \\ & p_H R_l \geq I - A && [PC_l] \\ & R(I) = R_b + R_l \\ \implies & \max_{I,R_b} p_H R_b \\ \text{s.t. } & R_b \geq \frac{BI}{\Delta p} && [IC_b] \\ & R_b \leq R(I) - \frac{I-A}{p_H} && [PC_l] \end{aligned}$$

First, notice that PC_l binds. Suppose that PC_l does not bind at the optimum. Then R_b can be increased with violating any constraint, increasing the objective function $\Rightarrow\Leftarrow$. Thus, PC_l binds:

$$R_b = R(I) - \frac{I-A}{p_H}$$

Differentiating with respect to I, we get:

$$\frac{dR_b}{dI} = R'(I) - \frac{1}{p_H}$$

Notice that based on the assumptions about $R'(I)$, R_b is increasing in I for all $I < I^*$. So the borrower would choose the highest I as possible. Thus, IC_b binds for all $I \leq I^*$. Suppose not, then the borrower would go to a different lender. Taking IC_b and PC_l together:

$$R(I) - \frac{I-A}{p_H} = \frac{BI}{\Delta p}$$

This holds at all $I \leq I^*$, in particular at I^*. Applying total differentiation and using $R'(I^*) - \frac{1}{p_H} = 0$,

$$R'(I^*)dI - \frac{dI - dA}{p_H} = \frac{BdI}{\Delta p} \implies \left[R'(I^*) - \frac{1}{p_H}\right]dI + \frac{dA}{p_H} = \frac{BdI}{\Delta p} \implies \frac{dI}{dA} = \frac{\Delta p}{Bp_H} > 0$$

because all terms are positive. So I is increasing in A.

(ii) How does the shadow value v of cash (the derivative of the borrower's gross utility with respect to cash) vary with the level of cash?

The borrowers gross utility is

$$U_b = p_H R_b = p_H \left(R(I^*) - \frac{I - A}{p_H} \right) = p_H R(I^*) - I + A$$

So, $v = \frac{dU_b}{dA} = 1$. Thus, the shadow value of cash is constant in the level of cash.