ECON 712 - PS 7

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10/22/2020

1 Overlapping generations with housing

Consider the following 2-period OG model. Agents earn y when young and 0 when old. There is a fixed supply of housing $H^s = 1$. Agents utility function is given by

$$U(c_t^t, h_t, c_{t+1}^t) = \ln(c_t^t) + \alpha h_t + \beta c_{t+1}^t$$

where c_t^t is the period t consumption, h_t is the period t housing choice, and c_{t+1}^t is the period $t+1$ consumption of a person born in period t. The initial old hold the stock of housing. Assume that $1 + \alpha > \beta y$.

1. Write down and solve the planner's problem.

$$\max_{c_1^0, \{h_t, c_t^t, c_t^{t-1}\}_{\forall t} \geq 0} \beta c_1^0 + \sum_{t=1}^{\infty} U(c_t^t, h_t, c_t^{t-1})$$
$$\text{s.t. } c_t^t + c_t^{t-1} \leq y \text{ and } h_t \leq H^s$$

Since U in increasing in c_t^t, c_t^{t-1}, we know that the consumption resource constraint and housing resource constraint will hold at equality $h_t = H^s = 1$. Substituting in the utility function,

$$\max_{c_1^0, \{c_t^t, c_t^{t-1}\}_{\forall t} \geq 0} \beta c_1^0 + \sum_{t=1}^{\infty} \ln(c_t^t) + \alpha + \beta c_t^{t-1}$$
$$\text{s.t. } c_t^t + c_t^{t-1} = y$$

The lagrangian is

$$\mathcal{L} = \beta c_1^0 + \sum_{t=1}^{\infty} \ln(c_t^t) + \alpha + \beta c_{t+1}^t + \lambda(y - c_t^t - c_t^{t-1})$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

The first order conditions imply:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t^t} = 0 &\implies \frac{1}{c_t^t} - \lambda = 0 \implies \lambda = \frac{1}{c_t^t} \\
\frac{\partial \mathcal{L}}{\partial c_t^{t-1}} = 0 &\implies \beta - \lambda = 0 \implies \beta = \lambda \implies c_t^t = \frac{1}{\beta} \\
\frac{\partial \mathcal{L}}{\partial \lambda} = 0 &\implies y - c_t^t - c_t^{t-1} = 0 \implies c_t^{t-1} = y - \frac{1}{\beta}
\end{aligned}
$$

The social planner's solution is $\{c_t^t, h_t, c_{t+1}^t\} = \left\{\frac{1}{\beta}, 1, y - \frac{1}{\beta}\right\}$.

2. If p_t is the period t price of a house, solve for a competitive equilibrium with housing in the following parts:

(a) What is the optimization problem facing a young agent?

$$
\begin{aligned}
\max_{c_t^t, h_t, c_{t+1}^t} \quad & \ln(c_t^t) + \alpha h_t + \beta c_{t+1}^t \\
\text{s.t. } \quad & c_t^t + p_t h_t \leq y \\
& c_{t+1}^t \leq p_{t+1} h_t
\end{aligned}
$$

(b) What are the market clearing conditions?

- Goods Market: $c_t^t + c_t^{t-1} = y$
- Housing Market: $h_t = H_s = 1$

(c) Define a competitive general equilibrium.

A competitive general equilibrium is an allocation of goods where agents optimize and markets clear.

(d) Solve for an agent's optimal housing and consumption decision rules. How does housing depend on the current and future price of houses?[1]

Since U is increasing in c_t^t, h_t, and c_{t+1}^t, the household will consume until budget constraints hold at equality. So we can rewrite the household problem:

$$
\max_{h_t} \ln(y - p_t h_t) + \alpha h_t + \beta p_{t+1} h_t
$$

FOCs with respect to h_t imply:

$$
\begin{aligned}
\frac{-p_t}{y - p_t h_t} + \alpha + \beta p_{t+1} = 0 \implies \frac{y - p_t h_t}{p_t} = \frac{1}{\alpha + \beta p_{t+1}} &\implies h_t = \frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}} \\
c_t^t = y - p_t \left(\frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}} \right) &= \frac{1}{p_t(\alpha + \beta p_{t+1})} \\
c_{t+1}^t = p_{t+1} \left(\frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}} \right) &= \frac{y p_{t+1}}{p_t} - \frac{p_{t+1}}{\alpha + \beta p_{t+1}}
\end{aligned}
$$

If p_t, p_{t+1}, and $\frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}}$ are nonnegative, c_t^t, c_{t+1}^t, and h_t are nonnegative.

[1] If you choose to solve the problem without imposing non-negativity constraints, you should verify the conditions under which consumption and housing choices are non-negative.

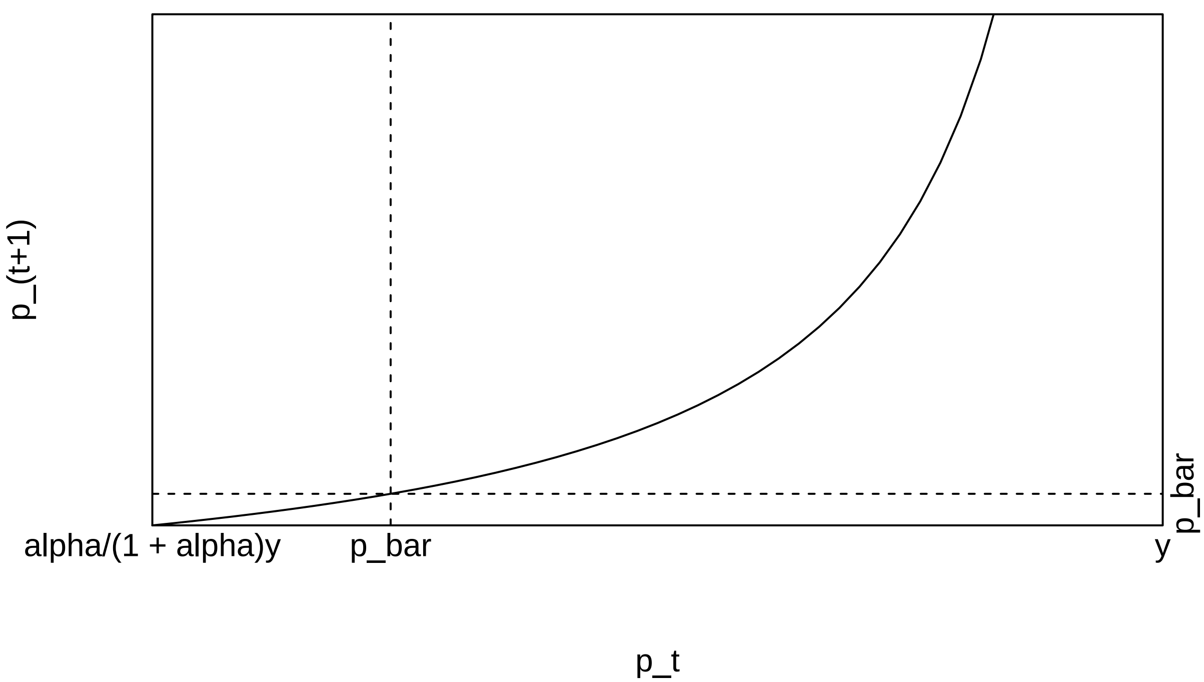
(e) Solve for the law of motion for house price in equilibrium and graph it in (p_t, p_{t+1}) space. Assume that $p_t < y$.

From housing market clearing condition ($h_t = H_s = 1$):

$$\begin{aligned}
\frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}} &= 1 \\
y(\alpha + \beta p_{t+1}) &= p_t(\alpha + \beta p_{t+1} + 1) \\
y\alpha + y\beta p_{t+1} &= p_t\alpha + p_t\beta p_{t+1} + p_t \\
y\beta p_{t+1} - p_t\beta p_{t+1} &= p_t\alpha + p_t - y\alpha \\
p_{t+1} &= \frac{p_t\alpha + p_t - y\alpha}{y\beta - p_t\beta} \\
p_{t+1} &= \frac{p_t - (y - p_t)\alpha}{(y - p_t)\beta} \\
p_{t+1} &= \frac{p_t}{\beta(y - p_t)} - \frac{\alpha}{\beta}
\end{aligned}$$

The non-negativity constraint on p_{t+1} implies a lower bound on p_t:

$$p_{t+1} \geq 0 \implies \frac{p_t}{\beta(y - p_t)} - \frac{\alpha}{\beta} \geq 0 \implies p_t \geq \frac{\alpha}{1+\alpha} y$$



(f) Solve for a steady state house price level.

Let $\bar{p} = p_{t+1} = p_t$:

$$\begin{aligned}
\bar{p} &= \frac{\bar{p}}{\beta(y-\bar{p})} - \frac{\alpha}{\beta} \\
\bar{p} &= \frac{\bar{p} - \alpha(y-\bar{p})}{\beta(y-\bar{p})} \\
\beta(y-\bar{p})\bar{p} &= \bar{p} - \alpha(y-\bar{p}) \\
\beta y\bar{p} - \beta\bar{p}^2 &= \bar{p} - \alpha y + \alpha\bar{p} \\
0 &= \beta\bar{p}^2 + \bar{p} - \beta y\bar{p} + \alpha\bar{p} - \alpha y \\
0 &= \beta\bar{p}^2 + (1-\beta y+\alpha)\bar{p} - \alpha y \\
\bar{p} &= \frac{-(1-\beta y+\alpha) \pm \sqrt{(1-\beta y+\alpha)^2 - 4\beta(-\alpha y)}}{2\beta} \\
\bar{p} &= \frac{\beta y - 1 - \alpha \pm \sqrt{(1-\beta y+\alpha)^2 + 4\beta\alpha y}}{2\beta}
\end{aligned}$$

Notice that since $1+\alpha > \beta y$, $\frac{\beta y-1-\alpha-\sqrt{(1-\beta y+\alpha)^2+4\beta\alpha y}}{2\beta}$ is negative. Thus, since $\bar{p}$ is nonnegative,

$$\bar{p} = \frac{\beta y - 1 - \alpha + \sqrt{(1-\beta y+\alpha)^2 + 4\beta\alpha y}}{2\beta}$$

(g) Does the competitive equilibrium implement the planner's allocation in a steady state?

Let us consider the steady state level of housing consumption $\bar{h}$, consumption of the young $\bar{c}_0$, and consumption of the old $\bar{c}_1$. For $\bar{h}$, the competitive equilibrium achieves the planner's allocation because the housing market clears, so $\bar{h} = 1$. Since budget constraints hold at equality,

$$\bar{h} = 1 \implies \bar{c}_1 = \bar{p} = \frac{\beta y - 1 - \alpha + \sqrt{(1-\beta y+\alpha)^2 + 4\beta\alpha y}}{2\beta} \text{ and } \bar{c}_0 = y - \bar{c}_1.$$

So the consumption of the young and old in the competitive equilibrium do not match the planner's allocation in the steady state.