

# ECON 709 - PS 5

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1. For the following sequences, show  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ :

(a)  $a_n = 1/n$

Fix  $\varepsilon > 0$ . Choose  $\bar{n} > \frac{1}{\varepsilon}$ . For all  $n \geq \bar{n}$ ,

$$|1/n - 0| = |1/n| = \varepsilon.$$

Thus,  $a_n = 1/n \rightarrow 0$  as  $n \rightarrow \infty$ .

(b)  $a_n = \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$

Fix  $\varepsilon > 0$ . Notice that  $|\sin(x)| \leq 1 \forall x$ . Choose  $\bar{n} > \frac{1}{\varepsilon}$ . For all  $n \geq \bar{n}$ ,

$$\left| \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) - 0 \right| = \left| \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \right| \leq |1| \left| \frac{1}{n} \right| = \left| \frac{1}{n} \right| \leq \varepsilon$$

Thus,  $a_n = \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \rightarrow 0$  as  $n \rightarrow \infty$ .

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2. Consider a random variable  $X^n$  with the probability function

$$X_n = \begin{cases} -n, & \text{with probability } 1/n \\ 0, & \text{with probability } 1 - 2/n \\ n, & \text{with probability } 1/n \end{cases}$$

(a) Does  $X_n \rightarrow_p 0$  as  $n \rightarrow \infty$ ?

Fix  $\varepsilon > 0$ . Choose  $\bar{n} > \varepsilon$ . For  $n \geq \bar{n}$ ,

$$P(|X_n| \geq \varepsilon) \leq P(|X_n| \geq n) = P(X_n = -n) + P(X_n = n) = 1/n + 1/n = 2/n$$

Since  $1/n \rightarrow 0$ ,  $2/n \rightarrow 0$ . Thus,  $X_n \rightarrow_p 0$  as  $n \rightarrow \infty$ .

(b) Calculate  $E(X_n)$ .

$$E(X_n) = \sum_{x \in \text{Supp}(X)} \pi(x)x = (1/n) * (-n) + (1 - 2/n)(0) + (1/n)(n) = -1 + 1 = 0.$$

(c) Calculate  $\text{Var}(X_n)$ .

$$\text{Var}(X_n) = E(X_n^2) - E(X_n)^2 = E(X_n^2) = \sum_{x \in \text{Supp}(X)} \pi(x)x^2 = (1/n)*(-n)^2 + (1 - 2/n)(0)^2 + (1/n)(n)^2 = n + n = 2n.$$

(d) Now suppose the distribution is

$$X_n = \begin{cases} 0, & \text{with probability } 1 - 1/n \\ n, & \text{with probability } 1/n \end{cases}$$

Calculate  $E(X_n)$ .

$$E(X_n) = \sum_{x \in \text{Supp}(X)} \pi(x)x = (1 - 1/n)(0) + (1/n)(n) = 0 + 1 = 1$$

(e) Conclude that  $X_n \rightarrow_p 0$  is not sufficient for  $E(X_n) \rightarrow 0$ .

Fix  $\varepsilon > 0$ . Choose  $\bar{n} > \varepsilon$ . For  $n > \bar{n}$

$$P(|X_n| \geq \varepsilon) \leq P(|X_n| \geq n) = P(X_n = n) = 1/n$$

Since  $1/n \rightarrow 0$ ,  $X_n \rightarrow_p 0$  as  $n \rightarrow \infty$ . Thus,  $X_n \rightarrow_p 0$  is not sufficient for  $E(X_n) \rightarrow 0$ .

3. A weighted sample mean takes the form  $\bar{Y}^* = \frac{1}{n} \sum_{i=1}^n w_i Y_i$  for some non-negative constants  $w_i$  satisfying  $\frac{1}{n} \sum_{i=1}^n w_i = 1$ . Assume that  $Y_i : i = 1, \dots, n$  are i.i.d.

(a) Show that  $\bar{Y}^*$  is unbiased for  $\mu = E(Y_i)$ .

$$E(\bar{Y}^*) = E\left(\frac{1}{n} \sum_{i=1}^n w_i Y_i\right) = \frac{1}{n} \sum_{i=1}^n w_i E(Y_i) = \frac{1}{n} \sum_{i=1}^n w_i \mu = (1)\mu = \mu$$

(b) Calculate  $Var(\bar{Y}^*)$ .

$$Var(\bar{Y}^*) = Var\left(\frac{1}{n} \sum_{i=1}^n w_i Y_i\right) = \frac{1}{n^2} \sum_{i=1}^n w_i^2 Var(Y_i)$$

(c) Show that a sufficient condition for  $\bar{Y}^* \rightarrow_p \mu$  is that  $\frac{1}{n^2} \sum_{i=1}^n w_i^2 \rightarrow 0$ . (Hint: use the Markov's or Chebyshev's Inequality).

Fix  $\varepsilon > 0$ . Because  $\frac{1}{n^2} \sum_{i=1}^n w_i^2 \rightarrow 0$ , there exists  $\bar{n}$  such that for  $n \geq \bar{n}$ ,

$$\left| \frac{1}{n^2} \sum_{i=1}^n w_i^2 \right| \leq \varepsilon$$

From (a) we know that  $E(\bar{Y}^*) = \mu$  and from (b) we know that  $Var(\bar{Y}^*) = \frac{1}{n^2} \sum_{i=1}^n w_i^2 Var(Y_i)$ , so by Chebyshev's Inequality,

$$P(|\bar{Y}^* - \mu| \geq \lambda) \leq \frac{Var(\bar{Y}^*)}{\lambda^2} = \frac{\frac{1}{n^2} \sum_{i=1}^n w_i^2 Var(Y_i)}{\lambda^2} \leq \frac{\varepsilon Var(Y_i)}{\lambda^2} = \frac{\varepsilon Var(Y_i)}{\left(\sqrt{Var(Y_i)}\right)^2} = \varepsilon$$

where  $\lambda = \sqrt{Var(Y_i)}$ . Thus  $\bar{Y}^* \rightarrow_p \mu$ .

(d) Show that the sufficient condition for the condition in part (c) is  $\max_{i \leq n} w_i/n \rightarrow 0$ .

Fix  $\varepsilon > 0$ . Let  $\delta = \sqrt{\frac{\varepsilon}{n}}$ . Because  $\max_{i \leq n} w_i/n \rightarrow 0$ , there exists a  $\bar{n}$  such that for  $n \geq \bar{n}$ ,

$$\begin{aligned} \left| \max_{i \leq n} w_i/n \right| \leq \delta &\implies |w_i/n| \leq \delta \quad \forall i \in \{1, \dots, n\} \\ &\implies (w_i/n)^2 \leq \delta^2 \quad \forall i \in \{1, \dots, n\} \\ &\implies \sum_{i=1}^n \frac{w_i^2}{n^2} \leq n\delta^2 \\ &\implies \sum_{i=1}^n \frac{w_i^2}{n^2} \leq n \left( \sqrt{\frac{\varepsilon}{n}} \right)^2 \\ &\implies \sum_{i=1}^n \frac{w_i^2}{n^2} \leq \varepsilon \end{aligned}$$

Thus,  $\frac{1}{n^2} \sum_{i=1}^n w_i^2 \rightarrow 0$ .

4. Take a random sample  $\{X_1, \dots, X_n\}$ . Which statistic converges in probability by the weak law of large numbers and continuous mapping theorem, assuming the moment exists?

(a)  $\frac{1}{n} \sum_{i=1}^n X_i^2$

Transform  $\{X_1, \dots, X_n\}$  to  $\{Y_1, \dots, Y_n\}$  such that  $Y_i = X_i^2$ . Thus,  $\{Y_1, \dots, Y_n\}$  is an i.i.d. sequence with  $E(|Y_i|) = E(X_i^2) = \mu_2 < \infty$ . By the weak law of large numbers  $\bar{Y}_N \rightarrow_p \mu_2$  as  $n \rightarrow \infty$ . Thus,  $\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow_p \mu_2$  as  $n \rightarrow \infty$ .

(b)  $\frac{1}{n} \sum_{i=1}^n X_i^3$

Transform  $\{X_1, \dots, X_n\}$  to  $\{Y_1, \dots, Y_n\}$  such that  $Y_i = X_i^3$ . Thus,  $\{Y_1, \dots, Y_n\}$  is an i.i.d. sequence with  $E(|Y_i|) = E(X_i^3) = \mu_3 < \infty$ . By the weak law of large numbers  $\bar{Y}_N \rightarrow_p \mu_3$  as  $n \rightarrow \infty$ . Thus,  $\frac{1}{n} \sum_{i=1}^n X_i^3 \rightarrow_p \mu_3$  as  $n \rightarrow \infty$ .

(c)  $\max_{i \leq n} X_i$

This statistic does not converge in probability by the weak law of large numbers and continuous mapping theorem. Instead we could apply the Fisher-Tippett-Gnedenko theorem, which can characterize the asymptotic distribution of extreme order statistics.

(d)  $\frac{1}{n} \sum_{i=1}^n X_i^2 - (\frac{1}{n} \sum_{i=1}^n X_i)^2$

From (a), we know that  $\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow_p \mu_2$ . An immediate result of the weak law of large numbers is  $\frac{1}{n} \sum_{i=1}^n X_i \rightarrow_p \mu$ . By the continuous mapping theorem,  $(\frac{1}{n} \sum_{i=1}^n X_i)^2 \rightarrow_p \mu^2$ . Thus,  $\frac{1}{n} \sum_{i=1}^n X_i^2 - (\frac{1}{n} \sum_{i=1}^n X_i)^2 \rightarrow_p \mu_2 - \mu^2$ .

(e)  $\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i}$  assuming  $\mu = E(X_i) > 0$ .

From (a), we know that  $\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow_p \mu_2$ . An immediate result of the weak law of large numbers is  $\frac{1}{n} \sum_{i=1}^n X_i \rightarrow_p \mu$ . Thus, by the Continuous Mapping Theorem,  $\frac{\frac{1}{n} \sum_{i=1}^n X_i^2}{\frac{1}{n} \sum_{i=1}^n X_i} = \frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i} \rightarrow_p \frac{\mu_2}{\mu}$ .

(f)  $1(\frac{1}{n} \sum_{i=1}^n X_i > 0)$  where

$$1(a) = \begin{cases} 1 & \text{if } a \text{ is true} \\ 0 & \text{if } a \text{ is not true} \end{cases}$$

is called the indicator function of event  $a$ .

Notice that  $1(\frac{1}{n} \sum_{i=1}^n X_i > 0) \sim \text{Bernoulli}(P(\frac{1}{n} \sum_{i=1}^n X_i > 0))$ . By the weak law of large numbers,  $\frac{1}{n} \sum_{i=1}^n X_i = \bar{X}_n \rightarrow_p \mu$ . So, if  $\mu > 0$ ,  $1(\frac{1}{n} \sum_{i=1}^n X_i > 0) \rightarrow_p 1$ . if  $\mu \leq 0$ ,  $1(\frac{1}{n} \sum_{i=1}^n X_i > 0) \rightarrow_p 0$ .

5. Take a random sample  $\{X_1, \dots, X_n\}$  where the support  $X_i$  is a subset of  $(0, \infty)$ . Consider the sample geometric mean  $\hat{\mu} = (\prod_{i=1}^n X_i)^{1/n}$  and population geometric mean  $\mu = \exp(E(\log(X)))$ . Assuming that  $\mu$  is finite, show that  $\hat{\mu} \rightarrow_p \mu$  as  $n \rightarrow \infty$ .

Assuming that  $\mu$  is finite,

$$\log(\hat{\mu}) = \log((\prod_{i=1}^n X_i)^{1/n}) = \frac{1}{n} \log(\prod_{i=1}^n X_i) = \frac{1}{n} \sum_{i=1}^n \log(X_i)$$

By the weak law of large numbers,  $\log(\hat{\mu}) = \frac{1}{n} \sum_{i=1}^n \log(X_i) \rightarrow_p E(\log(X))$ . By the continuous mapping theorem with  $g(x) = \exp(x)$ , we know that  $\hat{\mu} \rightarrow_p \exp(E(\log(X))) = \mu$ .

6. Let  $\mu_k = E(X^k)$  for some integer  $k \geq 1$ .

(a) Write down the natural moment estimator  $\hat{\mu}_k$  of  $\mu_k$ .

For i.i.d. sample  $X_i : i = 1, \dots, n$ , the “plug-in” estimator is the sample moment:

$$\hat{\mu}_k = \frac{1}{n} \sum_{i=1}^n X_i^k$$

(b) Find the asymptotic distribution of  $\sqrt{n}(\hat{\mu}_k - \mu_k)$  as  $n \rightarrow \infty$ , assuming that  $E(X^{2k}) < \infty$ .

Notice that the  $E(X_i^k) = \mu_k$  and  $Var(X_i^k) = E(X_i^{2k}) - (\mu_k)^2 = \mu_{2k} - \mu_k^2$ . Thus, by the central limit theorem,  $\sqrt{n}(\hat{\mu}_k - \mu_k) \rightarrow_d N(0, \mu_{2k} - \mu_k^2)$ .

7. Let  $m_k = (E(X_k))^{1/k}$  for some integer  $k \geq 1$ .

(a) Write down the natural moment estimator  $\hat{m}_k$  of  $m_k$ .

For i.i.d. sample  $X_i : i = 1, \dots, n$ , the “plug-in” estimator is:

$$\hat{m}_k = \left( \frac{1}{n} \sum_{i=1}^n X_i^k \right)^{1/k}$$

(b) Find the asymptotic distribution of  $\sqrt{n}(\hat{m}_k - m_k)$  as  $n \rightarrow \infty$ , assuming that  $E(X^{2k}) < \infty$ .

From 6(b), we know that  $\sqrt{n}(\hat{\mu}_k - \mu_k) \rightarrow_d N(0, \mu_{2k} - \mu_k^2)$ . Define  $g(x) = x^{1/k}$  for some  $k \in \mathbb{N}$ . Notice that  $g$  is continuous for all values of  $k$ . Furthermore,  $g(\hat{\mu}_k) = \hat{m}_k$ ,  $g(\mu_k) = m_k$ , and  $g'(\mu_k) = (1/k)\mu_k^{(1-k)/k} = (1/k)m_k^{1-k}$ . Thus, by the Delta Method,

$$\begin{aligned} \sqrt{n}(g(\hat{\mu}_k) - g(\mu_k)) &\rightarrow_d N(0, (g'(\mu_k))^2(\mu_{2k} - \mu_k^2)) \\ \implies \sqrt{n}(\hat{m}_k - m_k) &\rightarrow_d N(0, ((1/k)m_k^{1-k})^2(\mu_{2k}^k - m_k^k)) \\ \implies \sqrt{n}(\hat{m}_k - m_k) &\rightarrow_d N\left(0, \frac{m_k^2(m_{2k}^k - m_k^k)}{k^2 m_k^{2k}}\right) \end{aligned}$$

8. Suppose  $\sqrt{n}(\hat{\mu} - \mu) \rightarrow_d N(0, v^2)$  and set  $\beta = \mu^2$  and  $\hat{\beta} = \hat{\mu}^2$ .

(a) Use the Delta Method to obtain an asymptotic distribution for  $\sqrt{n}(\hat{\beta} - \beta)$ .

Define  $g(x) = x^2$ . Notice that  $g$  is continuous,  $\beta = g(\mu)$ , and  $\hat{\beta} = g(\hat{\mu})$ . Thus, by the Delta Method,

$$\begin{aligned}\sqrt{n}(\hat{\beta} - \beta) &\rightarrow_d N(0, (2\mu)^2 v^2) \\ \implies \sqrt{n}(\hat{\beta} - \beta) &\rightarrow_d N(0, 4\beta v^2)\end{aligned}$$

(b) Now suppose  $\mu = 0$ . Describe what happens to the asymptotic distribution from the previous part.

If  $\mu = 0$ , then the variance of the asymptotic distribution is zero, so it becomes a degenerate distribution at its mean:

$$\sqrt{n}(\hat{\beta} - \beta) = 0 \implies \hat{\beta} = \beta$$

(c) Improve on the previous answer. Under the assumption  $\mu = 0$ , find the asymptotic distribution of  $n\hat{\beta}$ .

(d) Comment on the differences between the answers in parts (a) and (c).