ECON 710A - Problem Set 4

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1. Let X be generated by the following random coefficients discrete choice model $X = 1\{-U_0 + ZU_1 > 0\}$ where $U = (U_0, U_1)'$ is independent of Z and $Z \in \{0, 1\}$. Provide conditions on U such that $Pr(Defying) = 0$ and $Pr(Complying) > 0$.

$Pr(Defying) = 0$ iff $X_U(1) = 0 \implies X_U(0) = 0$ and $X_U(0) = 1 \implies X_U(1) = 1$. $X_U(1) = 0 \implies X_U(0) = 0$ iff $-U_0 + (1)U_1 = -U_0 + U_1 < 0 \implies -U_0 + (0)U_1 = -U_0 < 0$. $X_U(0) = 1 \implies X_U(1) = 1$ iff $-U_0 + (0)U_1 = -U_0 > 0 \implies -U_0 + (1)U_1 = -U_0 + U_1 > 0$. Thus, $U_1 \geq 0$.

$Pr(Complying) > 0 \iff Pr(X_U(1) = 1$ and $X_U(0) = 0) > 0$. Since $U_1 \geq 0$, this implies that $U_1 > U_0 \geq 0$.

2. Let $\{Y_t\}_{t=1}^T$ be generated by the following MA(q) model, i.e., $Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$ where $\{\varepsilon_t\}_{t=0}^T$ are i.i.d. random variables with mean zero and variance σ^2.

(i) Find the autocovariance function $\gamma(k)$.

For $k = 0$:

$$
\begin{aligned}
\gamma(0) &= Cov(Y_t, Y_t) \\
&= Var(Y_t) \\
&= Var(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}) \\
&= Var(\varepsilon_t) + \theta_1^2 Var(\varepsilon_{t-1} + ... + \theta_q^2 Var(\varepsilon_{t-q}) \\
&= \sigma^2 + \theta_1^2 \sigma^2 + ... + \theta_q^2 \sigma^2 \\
&= \sigma^2 (1 + \theta_1^2 + ... + \theta_q^2)
\end{aligned}
$$

For $k = 1$:

$$
\begin{aligned}
\gamma(1) &= Cov(Y_t, Y_{t+1}) \\
&= Cov(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}, \mu + \varepsilon_{t+1} + \theta_1 \varepsilon_{t+1-1} + ... + \theta_q \varepsilon_{t+1-q}) \\
&= Cov(\varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}, \varepsilon_{t+1} + \theta_1 \varepsilon_t + ... + \theta_q \varepsilon_{t+1-q}) \\
&= Cov(\varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_{q-1} \varepsilon_{t+1-q}, \theta_1 \varepsilon_t + ... + \theta_q \varepsilon_{t+1-q}) \\
&= \theta_1 Var(\varepsilon_t) + \theta_1 \theta_2 Var(\varepsilon_{t-1}) + ... + \theta_{q-1} \theta_q Var(\varepsilon_{t+1-q}) \\
&= \sigma^2 (\theta_1 + \theta_1 \theta_2 + ... + \theta_{q-1} \theta_q)
\end{aligned}
$$

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For general k:

$$\gamma(k) = \begin{cases} \sigma^2(1+\theta_1^2+...+\theta_q^2) & \text{if } k=0 \\ \sigma^2(\theta_k+\theta_1\theta_{k+1}+\theta_2\theta_{k+1}+...+\theta_{q-k}\theta_q) & \text{if } 0<k\leq q \\ 0, & \text{if } k>q \end{cases}$$

(ii) Suppose that $q=1$ and find the autocorrelation function, $\rho(k)=\frac{\gamma(k)}{\gamma(0)}$.

If $q=1$:

$$Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1}$$

$\gamma(k)$ simplies to

$$\gamma(k) = \begin{cases} \sigma^2(1+\theta_1^2), & \text{if } k=0 \\ \sigma^2\theta_1, & \text{if } k=1 \\ 0, & \text{if } k>1 \end{cases}$$

The autocorrelation function:

$$\begin{aligned} \rho(k) &= \frac{\gamma(k)}{\gamma(0)} \\ &= \begin{cases} \frac{\sigma^2(1+\theta_1^2)}{\sigma^2(1+\theta_1^2)}, & \text{if } k=0 \\ \frac{\sigma^2\theta_1}{\sigma^2(1+\theta_1^2)}, & \text{if } k=1 \\ \frac{0}{\sigma^2(1+\theta_1)}, & \text{if } k>1 \end{cases} \\ &= \begin{cases} 1, & \text{if } k=0 \\ \frac{\theta_1}{1+\theta_1^2}, & \text{if } k=1 \\ 0, & \text{if } k>1 \end{cases} \end{aligned}$$

(iii) Is θ_1 identified from the autocorrelation function?

No. First, notice that θ_1 only appears in autocorrelation function when $k=1$. If $\theta_1=x$, $\theta_1=1/x$ yields the same value from the autocorrelation function:

$$\begin{aligned} \rho(1|\theta_1=x) &= \frac{x}{1+x^2} \\ \rho(1|\theta_1=x^{-1}) &= \frac{x^{-1}}{1+(x^{-1})^2} \\ &= \frac{x^{-1}}{1+x^{-2}}\frac{x^2}{x^2} \\ &= \frac{x}{1+x^2} \end{aligned}$$

(iv) Suppose $\theta_1\in[-1,1]$. Does you answer to (iii) change?

Yes, θ_1 is identified by the autocorrelation function because if $\theta_1=x\in[-1,1] \implies 1/x\notin[-1,1]$.

3. Consider an ARMA(1,1) model: $Y_t = \alpha_0 + Y_{t-1}\rho + U_t$ and $U_t = \varepsilon_t + \theta\varepsilon_{t-1}$ for all $t = 1, ..., T$; $Y_0 = \mu + \varepsilon_0 + \nu$ where $|\rho| < 1$, $|\theta| \leq 1$, $\varepsilon_0, ..., \varepsilon_T$ are idd $N(0, \sigma^2)$ and independent of $\nu \sim N(0, \tau)$.

(i) Find μ and τ (as functions of α_0, ρ, θ, and/or σ^2) such that $E[Y_t]$ and $Var(Y_t)$ does not depend on t.

If $E[Y_t]$ does not depend on $t \implies E[Y_0] = E[Y_1]$:

$$\begin{aligned} E[Y_0] &= E[\mu + \varepsilon_0 + \nu] \\ &= \mu \end{aligned}$$

$$\begin{aligned} E[Y_1] &= E[\alpha_0 + Y_0\rho + U_1] \\ &= E[\alpha_0 + Y_0\rho + \varepsilon_1 + \theta\varepsilon_0] \\ &= \alpha_0 + E[Y_0]\rho + E[\varepsilon_1] + \theta E[\varepsilon_0] \\ &= \alpha_0 + \mu\rho \end{aligned}$$

$$\begin{aligned} & E[Y_0] = E[Y_1] \\ \implies & \mu = \alpha_0 + \mu\rho \\ \implies & \mu = \frac{\alpha_0}{1-\rho} \end{aligned}$$

If $Var[Y_t]$ does not depend on $t \implies Var[Y_0] = Var[Y_1]$:

$$\begin{aligned} Var[Y_0] &= Var[\mu + \varepsilon_0 + \nu] \\ &= Var[\varepsilon_0] + Var[\nu] \\ &= \sigma^2 + \tau \end{aligned}$$

$$\begin{aligned} Var[Y_1] &= Var[\alpha_0 + Y_0\rho + U_1] \\ &= Var[Y_0\rho + \varepsilon_1 + \theta\varepsilon_0] \\ &= \rho^2 Var[Y_0] + Var[\varepsilon_1] + \theta^2 Var[\varepsilon_0] + 2\rho\theta Cov[Y_0, \varepsilon_0] \\ &= \rho^2(\sigma^2 + \tau) + \sigma^2 + \theta^2\sigma^2 + 2\rho\theta\sigma^2 \end{aligned}$$

$$\begin{aligned} Var[Y_0] &= Var[Y_1] \\ \implies \sigma^2 + \tau &= \rho^2\sigma^2 + \rho^2\tau + \sigma^2 + \theta^2\sigma^2 + 2\rho\theta\sigma^2 \\ \tau - \rho^2\tau &= \rho^2\sigma^2 + \theta^2\sigma^2 + 2\rho\theta\sigma^2 \\ \tau &= \frac{\sigma^2(\theta + \rho)^2}{1 - \rho^2} \end{aligned}$$

(ii) For the μ and τ found above, you may use without proof that $\{Y_t\}_{t=1}^T$ is covariance stationary. Under what conditions on α_0, ρ, θ, and/or σ^2 is $(1, Y_{t-2})$ a valid instrument for $(1, Y_{t-1})$.

A valid instrument is relevant and exogeneous.

Exogeneity requires

$$\begin{aligned} E[U_t|Y_{t-2}] &= 0 \\ \implies E[\varepsilon_t + \theta\varepsilon_{t-1}|Y_{t-2}] &= 0 \\ \implies E[\varepsilon_t|Y_{t-2}] + \theta E[\varepsilon_{t-1}|Y_{t-2}] &= 0 \\ \implies 0 + \theta(0) &= 0 \end{aligned}$$

Thus, exogeneity places no restrictions on the parameters.

Relevance requires

$$\begin{aligned} Cov(Y_{t-1}, Y_{t-2}) &\neq 0 \\ \implies Cov(\alpha_0 + Y_{t-2}\rho + U_{t-1}, Y_{t-2}) &\neq 0 \\ \implies Cov(Y_{t-2}\rho + \varepsilon_{t-1} + \theta\varepsilon_{t-2}, Y_{t-2}) &\neq 0 \\ \implies \rho Var(Y_{t-2}) + \theta Cov(\varepsilon_{t-2}, Y_{t-2}) &\neq 0 \\ \implies \rho(\sigma^2 + \tau) + \theta\sigma^2 &\neq 0 \\ \implies \rho\left(\sigma^2 + \frac{\sigma^2(\theta+\rho)^2}{1-\rho^2}\right) + \theta\sigma^2 &\neq 0 \\ \implies \left(\rho + \theta + \frac{\rho(\theta+\rho)^2}{1-\rho^2}\right) &\neq 0 \\ \implies \rho + \theta &\neq 0 \end{aligned}$$

These restrictions make sense for an ARMA(1, 1). For exogeniety, with only one autoregressive term and one moving average term, we already get that the twice lagged value has no direct effect on the current value. For relevance, we need that the twice lagged value has an effecton the once lagged value either through the autoregressive term or the moving average term (or both).