Sovereign Debt Ratchets and Welfare Destruction[*]

Peter DeMarzo[†]   Zhiguo He[‡]   Fabrice Tourre[§]

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Abstract

We study an impatient, risk-neutral government that cannot commit to a particular debt path, financed by competitive lenders. In equilibrium, debt adjusts slowly towards a debt-to-income target, exacerbating booms and busts. Strikingly, gains from trade dissipate when trading is continuous, leaving the government no better off than in financial autarky, due to a "sovereign debt ratchet effect." Moreover, citizens who are more patient than their government are strictly harmed. We characterize equilibrium debt dynamics, ergodics, and comparative statics when income follows a geometric Brownian motion, and analyze devices that allow the sovereign to recapture gains from trade.

Keywords: Sovereign Default, Commitment Problems, Time Consistency.

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[†]Stanford University and NBER; pdemarzo@stanford.edu

[‡]University of Chicago and NBER; zhiguo.he@chicagobooth.edu

[§]Copenhagen Business School; ft.fi@cbs.dk

1 Introduction

Since the seminal work of Eaton and Gersovitz (1981), a large number of articles have studied small open economies issuing defaultable sovereign debt. The theoretical building blocks of this literature include a government that makes financing and default decisions without being able to commit, and competitive creditors that price the sovereign debt rationally. Natural questions to consider include predictions for average debt-to-income ratios, the level of sovereign credit spreads, the behavior of the current account, and the ultimate consequence of external finance on citizen welfare.

While recent work has acknowledged that a sovereign's inability to commit to its future financing policy and indebtedness entails welfare losses,[1] a sharp theoretical characterization of the determinants and magnitude of these losses has not been available. One challenge is that there are limited theoretical results establishing existence and uniqueness of equilibria in an analytically tractable context.[2] A related debate has emerged over the optimality of short-term vs. long-term debt, with several articles suggesting that the use of short-term debt provides welfare gains to an optimizing government over the use of long-term debt.

In this paper, we make progress on these questions by taking a standard model of sovereign default and modifying it along two dimensions. First, we assume the government's motive to take on debt purely stems from its impatience relative to international creditors: defaultable debt is used for consumption "tilting" but not for consumption "smoothing" purposes.[3] This setting with a relatively impatient but risk-neutral government is one in which the potential gains from trade are significant and straightforward. Second, we analyze an environment where the time-step between financing decisions – effectively the length of time during which the government can commit – becomes arbitrarily small, by studying a continuous-time model of trading. When doing so, we hold the debt maturity constant. We therefore eliminate the implicit (one-period) commitment that arises with discrete trade.

Though these assumptions are admittedly idealized, many important results emerge. First, in the class of Markov perfect equilibria ("MPE") we focus on – in which the government issues debt "smoothly," what we refer to as "Smooth MPEs" – debt adjusts slowly in response to income shocks, exacerbating (and extending) booms and busts. We characterize the bond issuance policy of the government, and show that it is equal to (a) the wedge between the government's rate of time preference and debt investors' required rate of return, divided by (b) the marginal price impact of bond issuance. Furthermore, sovereign debt is mean reverting; in the special case of geometric Brownian motion income dynamics, the government targets a specific debt-to-income attraction level that we compute analytically.

We demonstrate that supply-side shocks in debt capital markets – whether they are interest rate shocks or risk-price shocks – lead to an adjustment of the government financing policy and corresponding current account adjustments that are qualitatively consistent with empirical studies.[4] In the partic-

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[1]See for example Hatchondo, Martinez and Sosa-Padilla (2016) or Arellano and Ramanarayanan (2012).

[2]Notable exceptions include Chatterjee and Eyigungor (2012), who establish the existence of a Markov perfect equilibrium in the presence of long-term debt, and Auclert and Rognlie (2016) and Aguiar and Amador (2019), who provide a uniqueness result in a model with one-period debt. In connection with long term debt, Lorenzoni and Werning (2019) and Aguiar and Amador (2020) build environments that feature multiple equilibria.

[3]The "smoothing" purpose seems at odds with the striking empirical regularity that external borrowing raises consumption growth volatility for emerging market economies relative to their output growth volatility (Aguiar and Gopinath (2004); Neumeyer and Perri (2005).

[4]As shown by Mendoza (2010) or Edwards (2004), during periods of international capital market turbulences, small open economies tend to revert to running current account surpluses.

ular case where capital markets' investors are risk-neutral, we recover an old result from the sovereign default literature, first established by Bulow and Rogoff (1988) in the context of a static model: a government should never buy back its own debt. This result is over-turned in the presence of risk-averse investors: when bond risk-premia demanded by credit market investors are sufficiently high, it becomes optimal for the government to proactively reduce its outstanding debt through bond buybacks.

Strikingly, our model produces a sharp welfare result: in our Smooth MPE, despite the potential gains from trade, and despite active and consistent borrowing in equilibrium, the risk-neutral government is unable to capture any welfare benefit from external financing. The government's option to borrow from more patient lenders is fully undermined by its inability to commit to a particular debt path, and thus, if not initially indebted, it does no better than the autarky benchmark. Moreover, despite the extensive borrowing that occurs in equilibrium, its welfare at any point is equal to the present value of future consumption flows computed *as if it never trades again in international capital markets*. This result echoes the conjecture made by Coase (1972) in the context of a durable goods' monopolist.

Section 3 develops the economic intuition for these results in a discrete-time setting, starting from a traditional sovereign default model featuring a government with concave preferences. There we show that equilibrium debt issuances are proportional to the wedge between (i) the government effective time preference rate, inclusive of any desire for consumption smoothing, and (ii) creditors' discount rate. Moreover, the trading gains accrued by the government over a given time interval, when the interval becomes small, are proportional to the product of (i) the curvature of the utility function, (ii) the squared debt issuance proceeds per unit of time, and (iii) the trading time interval dt. Thus, with linear preferences (so the curvature is zero) and a relatively impatient government, positive amounts of debt are systematically issued—the "sovereign debt ratchet effect"—yet the per-period welfare gains become vanishingly small in comparison with the length of the time period. Lifetime welfare gains then converge to zero in the continuous-time limit.

With a risk-neutral government, this welfare result holds independently of the maturity of any long-term amortizing debt, and thus applies even for relatively short-term debt, as long as trading opportunities are sufficiently frequent.[5] Our analysis emphasizes the three crucial conditions necessary for this welfare neutrality result to arise: (1) a government with linear preferences, (2) continuous time, and (3) unrestricted trade.

In Section 4, we maintain these three crucial assumptions and show that our welfare neutrality result is valid in continuous time for a wide range of income processes for the sovereign as well as various international credit market specifications and default settlement mechanisms. An immediate implication is that equilibrium welfare does not depend on international risk-free rates nor on risk-prices. In equilibrium, the anticipation of future debt issuance by the government drives down the current debt price to the point that the marginal benefit from additional borrowing just equals the marginal cost of the future debt burden. On the other hand, changes in capital market conditions directly affect the issuance strategy of the government – more benign market environments result in a higher rate of debt issuance.

We also consider the impact of this economic environment on citizens' welfare. Here we make the

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[5]Our model thus highlights an important distinction between discrete-time and continuous-time models. With discrete-time and single opportunity to trade, one-period debt automatically implies commitment, as there is no opportunity to trade again before the debt matures. With continuous trading, we are able to compare different debt maturities while holding the lack of commitment constant.

natural assumption that citizens are more patient than the current government regime.[6] In that case, while the government does not capture any welfare gains from being able to trade with more patient investors, we show that citizens will be strictly worse off with open capital markets. Indeed, since the government balances exactly the benefit of borrowing to support current consumption against future default costs, more patient citizens will put higher weight on default costs, making their welfare lower than the no-trade benchmark. Hence, for the citizens of the small open economy, financial autarky is better than trade.

While our environment can possibly feature multiple MPEs, we show that at most one of them can be a Smooth MPE. An additional contribution of our paper is to show that such Smooth MPE, in the special case of lognormal income dynamics, is unique within the broader class of MPEs (i.e. there does not exist a non-smooth MPE).

Our model provides new insight into the role of debt maturity and other features of debt contracts. As long as debt maturity is strictly positive (that is, has a finite amortization rate), the welfare of a risk-neutral government is independent of debt maturity. This result even extends to certain types of state contingent bonds (such as GDP-linked debt). In all cases, the inability of the government to commit not to issue additional bonds – even over short time periods – prevents it from realizing gains from trade. Although the government is indifferent, debt maturity does affect the level and dynamics of sovereign debt, and hence can significantly impact more patient citizen's welfare. Gains from trade only reemerge in our setting if the debt maturity is "infinitesimally short" – so that debt maturity and trading frequency coincide.[7]

To provide a concrete illustration of the Smooth MPE we focus on, we derive in Section 5 a complete analytical characterization of the government value function, debt prices, issuance policy and default policy in the particular case where the small open economy's income process follows geometric Brownian motion dynamics. This analytical characterization allows us to derive explicit comparative statics that are unavailable in most of this literature.[8] We show that the government financing policy systematically targets a particular debt-to-income attraction point, and that the speed at which debt adjustments occur following income shocks (a) increases with the discount rate wedge between creditors and the government, and (b) decreases with debt maturity. Finally, we provide an analytic formula for the ergodic debt-to-income density, which facilitates computing the macroeconomic and asset pricing moments of interest to economists. In the paper, we focus specifically on the ergodic default rate, bond credit spread and average debt-to-income ratio.

In the final section of the paper, by leveraging our analysis of the three key conditions required for our welfare neutrality result to hold, we study different devices that may enable the government to recapture some of the welfare gains from trade. Our analysis demonstrates that the details of these devices matter crucially. For example, trading frictions that allow the government to trade only at stochastically determined *points* in time does lead to limited welfare gains for the government (while the gains from commitment grow with the expected trading interval, they are ultimately offset by the

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[6]For example, the current government may only care about consumption while in power, and anticipate the stochastic arrival of regime change.

[7]The magnitude of the trading gains with infinitely short debt depends on the nature of the small open economy's income process; with a continuous income process the first-best may be obtained.

[8]See for example Hatchondo, Martinez and Sapriza (2010), Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012), which rely on numerical methods.

cost of delaying trade). On the other hand, if trading is possible only in stochastically determined *windows* of time – such as an environment with sudden stops and starts to international credit markets – the government cannot capture any gains. The contrast between these settings shows that it is critical whether "stops" to trade are perfectly anticipated (and so can provide commitment) or not.

We also investigate commitment devices that have been prevalent in international institutional arrangements. Restrictions on debt issuances once a country is too highly indebted (sometimes called "debt-ceiling" policies) allow the country to recapture some of the welfare gains from trade, but only if the restriction is sufficiently tight. Instead, restrictions on the rate of debt issuance – a close analog to budget deficit rules or current account deficit limits – always lead to welfare improvements. Our model thus has the policy implication that international organizations should consider more flow-based rather than balance-based interventions.

Finally, we re-introduce government risk-aversion in our setting, and study the extent to which preferences for consumption smoothing discipline the country's debt path. As the degree of risk-aversion gets small, the sequence of MPEs converges numerically to the Smooth MPE we characterize analytically in the linear preference case. With risk aversion, debt issuances are dampened by the consumption smoothing motive, and the government defaults at lower levels of indebtedness. We compute the welfare gains achieved by the risk-averse government and those of a risk-neutral government that could imitate the policies of the risk-averse government; in both cases, the degree of welfare gains are similar (for low levels of risk-aversion), suggesting that a large component of the overall welfare gains that arise with risk-aversion stems from its ability to discipline the government's financial policies.

Our paper is organized as follows. After reviewing the existing literature, we introduce a canonical model of sovereign default in discrete time, and show heuristically the intuition for the results we obtain in the continuous-time limit. We then present our general theory and welfare neutrality result. We apply these results to a particular income process for which we can obtain an analytical characterization of all equilibrium objects of interest. We then discuss alternative devices that can allow the small open economy to recapture some of the gains from trade.

2   Related Literature

Our paper relates to the vast literature on sovereign credit risk, which includes the seminal papers of Eaton and Gersovitz (1981) and Cole and Kehoe (1996), and more recently Aguiar and Gopinath (2004) and Arellano (2008). These original articles focus on discrete-time economies, one-period debt contracts, income risk, and feature impatient governments with finite intertemporal elasticities of substitution. Most of the literature focuses on the quantitative implications of this class of models for consumption, default probabilities, the behavior of the current account, but the authors rarely analyze the welfare costs incurred by a government that lacks a commitment technology. More recently, Rebelo, Wang and Yang (2021) recast the sovereign debt problem in continuous time but continue to assume short-term debt. They focus on a small open economy whose income follows jump diffusive dynamics, and study how the type of debt contract (local currency vs. foreign currency) and the ability to hedge macroeconomic risks influence welfare.

A related literature has analyzed the properties of sovereign default models in the presence of long term debt. Chatterjee and Eyigungor (2012) establish the existence of an MPE of their model using fixed-

point arguments, while our proof is constructive. They perform a numerical welfare comparison using different types of bond maturities, concluding that short term debt leads to a greater ergodic welfare than long term debt. Arellano and Ramanarayanan (2012) analyze a government that has the option to issue both short-term and long-term debt, and argue that the government policy balances the "hedging benefits" of long-term debt vs. the "incentive benefits" of short-term debt. Those hedging benefits are absent from our paper, given the linear preferences we assume.

Aguiar et al. (2019) study a model of sovereign default without income risk but with "outside option" risk. They cast their model in a discrete-time setting, allow for any arbitrary debt maturity structure, and show that debt with long-term maturity—i.e., any maturity that is longer than the smallest time interval in their discrete-time setting—is never traded by the government in equilibrium. The very nature of discrete-time models is that the sovereign's commitment length coincides with the shortest maturity debt contract. In our paper with continuous trade, we decouple the debt maturity structure and commitment length, and argue instead that the lack of ability to commit over any possible time period renders the maturity structure of debt irrelevant for welfare purposes when the sovereign is risk-neutral.

We show that the Smooth MPE we focus on, when it exists, is unique within the class of Smooth MPEs; moreover, it is unique within the broader class of MPEs when income follows geometric Brownian motion dynamics. Our results are however consistent with Aguiar et al. (2022), who study a setting with "outside option" risk, linear preference and consumption constraints, and who establish the existence of at least two equilibria (namely a "borrowing" and a "savings" equilibria). Absent those consumption constraints, none of these equilibria satisfy our definition of "smoothness", as both of them feature jumps in the debt process.

Our focus on citizen welfare vs. government welfare is motivated by considerations similar to those in Aguiar, Amador and Fourakis (2020). In that article, welfare losses are incurred by citizens since the country spends a large amount of time in the default state, in which consumption fluctuations are costly for private citizens. In our work, all agents have linear preferences, and citizens are better off in autarky than with trade as soon as they are more patient than their government.

Our result that a government facing risk-neutral lenders never buys back its own bonds echoes a result obtained long ago by Bulow and Rogoff (1988) in the context of a one period model. Our study sheds light on this question under a more general environment where the sovereign and creditors do not necessarily share the same probability measure. This could be related to debt investors having a pricing kernel that correlates with the small open economy's income process (see Borri and Verdelhan (2011) for an example with short-term debt, or Tourre (2017) for a continuous-time example with long-term debt), or disagreements between investors and the government over the future income prospects of the small open economy (see Harrison and Kreps (1978) for the foundational article for the "agree to disagree" framework). In contrast to Bulow and Rogoff (1988), in our environment, the government might find it optimal to buy back its own debt. This occurs when debt risk-premia required by international creditors are sufficiently high, or when lenders are sufficiently more pessimistic than the government about the small economy's future income growth rate.

A separate literature in corporate finance analyzes bond issuances and default in the presence of long term debt and a lack of commitment. Whereas the incentive to take on debt in the sovereign credit risk literature stems from impatience and consumption smoothing motives, firms' desire to issue debt

in capital markets is typically linked to the tax benefits of debt. Our paper is closest to Admati et al. (2013) and DeMarzo and He (2021), who study the leverage dynamics when shareholders cannot commit to any debt policies and derive a similar welfare destruction effect.[9] Dangl and Zechner (2021) study the dynamic capital structure decision of a firm that faces issuance costs and covenants that limit the issuance rate of new debt. He and Milbradt (2016) instead study a firm that can commit to keeping a constant amount of debt outstanding but has flexibility to issue short-term or long-term bonds; they show that "shortening" equilibria – equilibria in which the firm close to default chooses to issue short-term as opposed to long-term bonds – can be Pareto dominated by "lengthening" equilibria.

3 Model Overview and Intuition

In this section, we study a standard sovereign default model, develop notation that will be useful in later sections, and establish several results intentionally heuristically, in order to provide some intuition behind the economic mechanisms at play in our environment.

3.1 The Environment

We consider a government that seeks to maximize

$$\mathbb{E}_t \left[\int_t^{+\infty} e^{-\delta(s-t)} u\left(C_s\right) ds \right],$$

where C_t is the consumption *rate* at date t, δ is the government's rate of time preference, the utility function $u\left(\cdot\right)$ has constant relative risk-aversion coefficient γ, and the subscript on the expectation indicates it is conditional on the information available at time t. The government earns stochastic income at rate Y_t (per unit of time). Absent financial markets, the government's flow utility would equal $u\left(Y_t\right)$.[10]

Suppose, however, that the government has the ability to borrow or lend with outside investors with discount rate $r < \delta$. In this case, the government has an incentive to take on debt from (1) an impatience motive and (2) a consumption smoothing motive. To better understand the role of each, much of our analysis considers the limiting case where u is linear and only the first motive remains relevant.

Specifically, the government can issue or repurchase long-term, exponentially amortizing bonds with coupon rate κ and amortization rate m. Hence, given aggregate debt F outstanding, the government must pay interest $\kappa F dt$ and retire bonds by repaying principal $mF dt$ during the time interval $[t, t + dt)$. We can interpret $1/m$ as the average maturity of the debt.[11]

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[9]DeMarzo and He (2021) point out that their constructed smooth MPE produces the lowest possible equilibrium payoff for the sovereign (Markov or not); as a consequence, all non-MPE equilibria can be supported by using their equilibrium as "grim trigger" punishment in response to an off-equilibrium deviation, as in the folk theorem literature. In the corporate finance literature, Malenko and Tsoy (2020) have used these results to support non-Markov strategies in which the firm is "punished" under certain circumstances by reverting to our MPE.

[10]Consumption can be interpreted as government expenditure, and income as tax revenue. Alternatively, once can interpret consumption as the citizen's consumption, under the assumption that the government has sufficient policy instruments to directly control such consumption. In this setting, income is interpreted as the country's GDP. The discount rate δ may represent an implicit rate of return generated on investment in public goods; alternatively it may correspond to the desire of current politicians to accelerate spending during their time in office.

[11]To describe the government liability, two parameters are sufficient: the current flow liability, and the rate at which this flow liability shrinks over time. In our notation, we introduce the coupon rate κ, amortization rate m and debt face value F_t to more easily map the model objects to readily observable data counterparts; we could have instead abstracted from the coupon rate

Figure 1: Model Timeline
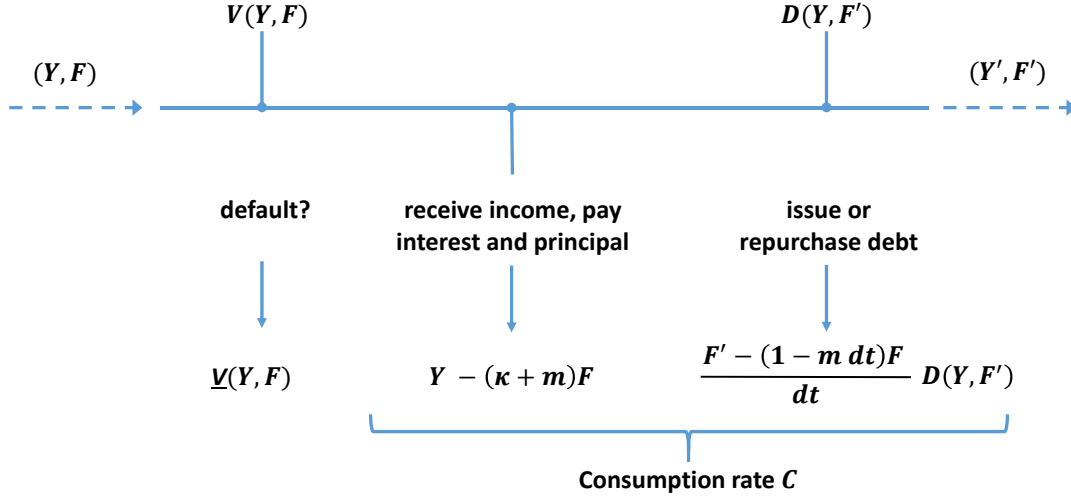
We focus on Markov perfect equilibria ("MPE") in which the two payoff-relevant state variables are the small open economy's income Y_t, and the outstanding aggregate debt face value, F_t, which is an endogenous state variable. The bonds issued by the government are traded in a competitive market at price $D(Y_t, F_t)$ (per unit of face value) that will be determined in equilibrium. This price D equals the present value of the coupon and principal payments received up until the time τ of default, plus some final recovery value $\underline{D}(Y_\tau, F_\tau)$:[12]

$$D(Y_t, F_t) := \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^\tau e^{-(r+m)(s-t)} (\kappa + m) ds + e^{-(r+m)(\tau-t)} \underline{D}(Y_\tau, F_\tau) \right].$$

The debt price will depend on the government's current income and debt level (which influence the expected time to default) and an appropriate pricing measure $\mathbb{Q}$, with related expectation operator $\mathbb{E}^{\mathbb{Q}}$. The pricing measure $\mathbb{Q}$ might differ from the probability measure under which the government discounts consumption streams because either (i) creditors are risk-averse and their marginal utility co-moves with the country's income, or (ii) creditors and the government have different perceptions of the country's expected income growth rate (and they agree to disagree).

3.2 Discrete Time Analysis

To provide intuition, we first study the discrete time counterpart to our model, in which the government makes decisions at $dt > 0$ time intervals. The timing of events in our model is depicted in Figure 1.

Suppose the government has current income (per unit of time) Y and existing debt F. If the government chooses not to default, the government will earn income, pay interest and principal on its current outstanding debt, and may issue or repurchase debt to adjust its total debt to F'. In that case, the

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κ, using a modified state variable $Z_t := (\kappa + m)F_t$ along with the debt amortization rate m.

[12] The recovery value might be the exogenously specified result of a final settlement agreed with creditors, or endogenously determined from a renegotiation that leads to a sovereign debt haircut. We will consider both, but for now the specific settlement mechanism upon default is not essential.

government's consumption over this dt interval of time is given by

$$Cdt := \underbrace{Ydt}_{\text{income}} - \underbrace{(\kappa + m)Fdt}_{\text{interest and principal due}} + \underbrace{D\left(Y, F'\right)\left(F' - (1 - mdt)F\right)}_{\text{proceeds from net debt issuance}}.$$

The debt price is determined by the current income Y and the *new* debt level F'. The government has two interrelated commitment problems which affect this price. First, it cannot commit to repay its debt, forcing creditors to bear default risk. If the government elects to default, it achieves a default value given by $\underline{V}(Y, F)$ (which we will endogenize in our model, and entails some degree of loss). Second, it cannot commit to a future financing policy – in other words, when issuing bonds at time t, the government cannot credibly promise to follow a future path of debt. Thus, creditors must anticipate the government's equilibrium debt issuance behavior and set the debt price $D(Y, F')$ accordingly.

Letting $V(Y, F)$ be the government's expected payoff given initial income Y and debt F if it chooses not to default immediately, we can write the discrete-time government Bellman equation as:

$$V(Y, F) = \max_{F'} \left\{ u\left(Y - (\kappa + m)F + D\left(Y, F'\right)\frac{F' - (1 - mdt)F}{dt}\right)dt \right.$$
$$\left. + e^{-\delta dt}\mathbb{E}_Y\left[\max\left(V(Y', F'), \underline{V}(Y', F')\right)\right] \right\}. \quad (1)$$

This maximization problem results in an optimal next period debt level $\mathcal{F}^*(Y, F)$, an optimal debt issuance $d\Gamma^*(Y, F) := \mathcal{F}^*(Y, F) - (1 - mdt)F$, a decision to repay versus default which we can represent by the repayment indicator function $\mathcal{R}^*(Y, F) := \mathbb{1}_{\{V(Y, F) \geq \underline{V}(Y, F)\}}$, and a consumption function $\mathcal{C}^*(Y, F)$.

For simplicity (we will relax all these assumptions later), consider for now the case with risk-neutral creditors and common priors, and upon a sovereign default, zero recovery rate for bond holders ($\underline{D} = 0$) and a government value $\underline{V}(Y)$ that is a fraction of its autarky value. Given the government's optimal decision rules, and denoting by D^+ the cum-coupon value of the debt at the start of the period before the government's default or issuance decision is made, the debt price can be written recursively as

$$D\left(Y, F'\right) = e^{-rdt}\mathbb{E}_Y\left[D^+\left(Y', F'\right)\right], \text{where} \quad (2)$$
$$D^+(Y, F) := \mathcal{R}^*\left(Y, F\right)\left[(\kappa + m)dt + (1 - mdt)D\left(Y, \mathcal{F}^*(Y, F)\right)\right].$$

When $dt = 1$, equations (1) and (2) are the two canonical equations of most discrete time sovereign default models. An MPE is defined as a pair of functions (V, D) that satisfies these equations. For the remainder of this section, we assume that an MPE exists and that the corresponding government value and debt pricing functions are differentiable and strictly decreasing in the level of indebtedness. Later in the paper we verify conditions for both existence and uniqueness of such MPE in specific settings.

3.3 Optimal Issuance

For intuition regarding debt dynamics in this context, consider the government's optimal issuance policy. The first-order condition for the issuance decision is

$$\left[D(Y, \mathcal{F}^*) + d\Gamma^*\partial_F D(Y, \mathcal{F}^*)\right]u'(\mathcal{C}^*) = -e^{-\delta dt}\mathbb{E}_Y\left[\partial_F V(Y', \mathcal{F}^*)\mathcal{R}^*(Y', \mathcal{F}^*)\right], \quad (3)$$

where the functions $\mathcal{F}^*, \mathcal{C}^*, d\Gamma^*$ are evaluated at the point (Y, F). Using the envelope condition for (1),

$$\partial_F V(Y, F) = -\left[(\kappa + m)dt + (1 - mdt)D(Y, \mathcal{F}^*)\right] u'(\mathcal{C}^*), \tag{4}$$

together with the debt-pricing equation (2), we can rearrange the first-order condition to obtain

$$\begin{aligned}
D(Y, \mathcal{F}^*) + d\Gamma^* \partial_F D(Y, \mathcal{F}^*) &= e^{-\delta dt} \mathbb{E}_Y \left[\frac{u'(\mathcal{C}^*(Y', \mathcal{F}^*))}{u'(\mathcal{C}^*(Y, F))} D^+(Y', \mathcal{F}^*) \right] \\
&= e^{(r-\delta)dt} \mathbb{E}_Y \left[\frac{D^+(Y', \mathcal{F}^*)}{\mathbb{E}_Y[D^+(Y', \mathcal{F}^*)]} \frac{u'(\mathcal{C}^*(Y', \mathcal{F}^*))}{u'(\mathcal{C}^*(Y, F))} \right] D(Y, \mathcal{F}^*).
\end{aligned} \tag{5}$$

The left-hand side of equation (5) is the marginal revenue from issuing one extra unit of debt today, and the right-hand side is the standard asset pricing equation for the discounted marginal cost of government's future liability. To interpret the second line of (5), define the government's certainty-equivalent consumption growth $\hat{\mu}_c(Y, F)$ as follows:

$$e^{-\gamma \hat{\mu}_c dt} := \mathbb{E}_Y \left[\frac{D^+(Y', \mathcal{F}^*)}{\mathbb{E}_Y[D^+(Y', \mathcal{F}^*)]} \frac{u'(\mathcal{C}^*(Y', \mathcal{F}^*))}{u'(\mathcal{C}^*(Y, F))} \right] \tag{6}$$

Note that if the consumption growth rate were deterministic with mean $\bar{\mu}_c$, then $\hat{\mu}_c = \bar{\mu}_c$. Otherwise, $\hat{\mu}_c$ will depart from the mean to reflect an appropriate risk-premium.[13]

Given our assumption that the equilibrium debt price is strictly decreasing in the debt level, we can rearrange equation (5) and use $\hat{\mu}_c$ to obtain a simple expression for the optimal issuance $d\Gamma^*$:

$$d\Gamma^* = \frac{1 - \exp\left((r - \delta - \gamma \hat{\mu}_c) dt\right)}{-\partial_F \ln D}. \tag{7}$$

Equation (7) reveals that equilibrium debt issuance increases with the gap between (i) the government's "effective" time preference inclusive of any desire for smoothing, $\delta + \gamma \hat{\mu}_c$, and (ii) the creditor's discount rate, r. This impatience gap is tempered by the percentage decline in the debt price that results from new issuance, $-\partial_F \ln D > 0$.

Our characterization of the optimal issuance policy provides several useful observations regarding the evolution of debt. The first result is pathwise monotonicity:

Claim 1 *(Policy Monotonicity) $\mathcal{F}^*$ is increasing in F.*

This result implies that if the sovereign starts with a higher level of current debt, it will have higher path of debt at all future dates. Claim 1 follows from the supermodularity of the Bellman equation in (F, F');[14] intuitively, for a given F', higher current debt raises marginal revenue from issuance (by lowering $d\Gamma$) and lowers marginal cost (by lowering current consumption and thereby raising $u'(C)$).[15]

Equation (7) also reveals that because $\delta > r$, there will be a bias toward issuance. The impatience wedge $\delta - r$ pushes the government to issue more debt; this effect will be reversed only if the certainty-equivalent consumption growth is sufficiently negative and the desire for intertemporal smoothing is

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[13]For example, if the consumption and debt values were jointly locally lognormal, with consumption-growth volatility σ_c and consumption-debt covariance σ_{cd}, then $\hat{\mu}_c = \bar{\mu}_c + \sigma_{cd} - \frac{1}{2}\gamma\sigma_c^2$. More generally, $\frac{D^+(Y', \mathcal{F}^*)}{\mathbb{E}_Y[D^+(Y', \mathcal{F}^*)]}$ represents the Radon–Nikodym derivative associated with a change in measure that puts more weight on states where the debt price is high.

[14]See for example Milgrom and Shannon (1994).

[15]This result is not unique to our setting. See for example Chatterjee and Eyigungor (2012) for a similar discussion.

sufficiently high so that $\gamma \hat{\mu}_c$ offsets the impatience wedge. With risk-neutrality, the smoothing term disappears and the government always issues new debt, independent of its current indebtedness:[16]

Claim 2 *(Positive Issuance) If the government and investors are risk-neutral, $d\Gamma^* > 0$.*

This debt ratchet effect, highlighted by Bulow and Rogoff (1988) (see also Admati et al. (2013) in the context of corporate debt), demonstrates that without commitment there is no direct incentive for a risk-neutral government to actively repurchase its outstanding debt. Repurchases must come from a desire for smoothing by the government (or a risk premium charged by creditors, as in Section 4).

Even in the risk-neutral case, however, the level of outstanding debt may still decline due to required principal repayments on maturing debt. While the ratchet effect always pushes the government to issue more debt, when the debt price is sufficiently sensitive to the debt level (i.e. debt demand is inelastic), the rate of debt issuance will be below the rate of amortization and total indebtedness will decline. In the continuous-time limit $dt \to 0$ for example, we will have

$$-\frac{\partial \ln D}{\partial \ln F} > \frac{\delta - r}{m} \quad \text{implies} \quad \frac{d\Gamma^*}{dt} < mF \quad \text{and thus} \quad \frac{dF}{dt} < 0.$$

Moreover, the debt-to-income ratio of the sovereign can also decline due to income growth. In Section 5, we will show that with i.i.d. income growth the government's debt-to-income ratio F/Y mean-reverts toward a target level.

The final important implication of equation (7) is that, as we increase the frequency of trade and approach the continuous-time limit, the optimal issuance policy becomes smooth.

Claim 3 *(Smooth Issuance) At the continuous time limit $dt \to 0$, $d\Gamma^* = G^* dt$, where*

$$G^* = \frac{\delta + \gamma \hat{\mu}_c - r}{-\partial_F \ln D}. \tag{8}$$

For $\gamma > 0$, smooth issuance is to be expected, as the risk-averse government never finds it optimal to consume in a lumpy fashion, and as the lack of storage technology tightly links consumption to debt issuance.[17] Perhaps more surprising is the fact that issuance remains smooth even absent risk-aversion. Indeed, when risk-aversion is zero, the intertemporal elasticity of substitution is infinite, and nothing conceptually prevents the government from issuing a strictly positive measure of debt and enjoying a strictly positive measure of consumption at a given instant.[18]

Equation (8) shows that, as long as the equilibrium debt price is strictly decreasing in F, debt adjusts smoothly and the debt process F will be absolutely continuous (in time) even if $\gamma = 0$. Because income may fluctuate much faster (for example due to Brownian income shocks or Poisson jumps), this result implies that the equilibrium debt-to-income ratio must be (locally) counter-cyclical – falling after large upward income shocks (booms), and rising after large income declines (busts).

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[16]If investors are risk-averse (as we will see in Section 4) a risk premium π will be added to the debt cost r, and so buybacks will occur if $\delta < r + \pi$.

[17]Here we rely on $\hat{\mu}_c$ being finite and well defined at the limit $dt \to 0$, which is the case since flow consumption cannot have predictable jumps when the intertemporal elasticity of substitution is finite.

[18]In fact, when we study commitment devices in Section 6, we will see that debt-ceiling policies lead to equilibria in which the government optimally issues "lumpy" amounts of debt.

3.4 Welfare Gains

We now consider the incremental gains from trade the government can capture by selling debt to external investors. Recall that there are two motives for trade in our setting. First, the government is more impatient than external investors, and so would prefer to front-load consumption. Second, a risk-averse government may gain from consumption-smoothing. We argue next that when trade is frequent, the first motive alone is insufficient for welfare gains.

Specifically, consider the government's welfare following the optimal policy compared with the welfare it would achieve without trade. If we denote by $\mathcal{F}_0 = (1 - mdt)F$ the next-period no-trade debt balance, and $\mathcal{C}_0 = Y - (\kappa + m)F$ the related no-trade consumption rate, then the incremental welfare gains from being able to issue sovereign debt (between t and $t + dt$) can be computed as follows:

$$\text{Gain}_{[t,t+dt)} = [u(\mathcal{C}^*) - u(\mathcal{C}_0)] dt + e^{-\delta dt}\mathbb{E}_Y \left[\max\left(\underline{V}(Y'), V(Y', \mathcal{F}^*)\right) - \max\left(\underline{V}(Y'), V(Y', \mathcal{F}_0)\right) \right] \quad (9)$$

By the usual Taylor expansion, the first term in equation (9) can be written

$$[u(\mathcal{C}^*) - u(\mathcal{C}_0)] dt = u'(\mathcal{C}^*) G^*D(Y, \mathcal{F}^*) dt - \frac{1}{2}u''(\tilde{\mathcal{C}}) (G^*D(Y, \mathcal{F}^*))^2 dt,$$

for some appropriate intermediate level of consumption $\tilde{\mathcal{C}}$ between $\mathcal{C}_0$ and $\mathcal{C}^*$. In Appendix A, we expand the second term in (9) and show, using the envelope condition from the Bellman equation (1), that the change in the government's future value is

$$e^{-\delta dt}\mathbb{E}_Y \left[\max\left(\underline{V}(Y'), V(Y', \mathcal{F}^*)\right) - \max\left(\underline{V}(Y'), V(Y', \mathcal{F}_0)\right) \right] \underset{dt\to 0}{=} -u'(\mathcal{C}^*) G^*D(Y, \mathcal{F}^*) dt + O(dt^2).$$

Combining these terms leads to our next key observation:

Claim 4 *(Gains from Trade) The incremental gains from trade, on the interval $[t, t+dt)$, are*

$$Gain_{[t,t+dt)} \underset{dt\to 0}{=} -\frac{1}{2}u''(\tilde{\mathcal{C}}) (G^*D)^2 dt + O(dt^2) \quad (10)$$

When $u''(\cdot) = 0$, as the frequency of trade increases ($dt \to 0$), cumulative trading gains vanish.

When the government is risk-neutral, incremental gains from trade are of order dt^2, and their accumulation of order dt. In that case, the combined effect of frequent trading and lack of commitment entirely destroys welfare. Although creditors are more patient, they anticipate that the uncommitted risk-neutral government will continue to issue new debt and raise the future risk of default. In equilibrium the debt price today falls to the point that it just equals the government's marginal cost of repaying the debt. In the end, the potential gains from trade are fully dissipated by the higher default costs resulting from continued debt issuance. This result only arises at the limit $dt \to 0$; in discrete time, the government has some commitment power, as it implicitly (and credibly) promises not to default and not to issue new debt over a positive measure of time $dt > 0$.[19]

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[19]An alternative approach to remove commitment power while maintaining the discrete time setting would have been to give the government an infinite number of trading opportunities within each time period, as in Bizer and DeMarzo (1992); such approach would reduce – but not entirely eliminate – the surplus achieved by the borrower, resulting in an equilibrium outcome that is in-between the "traditional" discrete time setting and continuous time.

Moreover, as will be discussed more precisely in Section 4 (see Proposition 2), this "no gain from trade" result provides for a simple strategy for constructing an MPE in our model with a risk neutral government: we can compute the no-trade welfare, back out debt prices from the envelope condition (4) (which becomes $\partial_F V + D = 0$ when $u'' = 0$ and $dt \to 0$) and then verify that such debt prices are indeed strictly decreasing in F.

Coase Conjecture Connection. Our "no gain from trade" result is related to the conjecture made in Coase (1972), and formally proven by Stokey (1981) and Gul, Sonnenschein and Wilson (1986), who show that a monopolist with constant marginal costs selling a durable good to a continuum of consumers will actually behave competitively in the continuous-time limit, failing to extract any monopoly rent. In the context of our model, the government acts as a monopolist over a durable good – its own sovereign debt. Default risk embedded in the sovereign debt creates a downward sloping bond price schedule, analogous to the downward sloping demand curve arising from the distribution of consumer's private valuations in Coase's model. Without commitment, no matter how many bonds the government sold in the past, the government will sell more bonds if there are marginal gains from doing so. In equilibrium the bond price adjusts until the government is indifferent to any amount of bond issuances, stripping away the potential welfare gain that the government may extract from patient financiers.[20]

3.5 Recovering Gains from Trade

Our "no gain from trade" result depends on three key assumptions: (i) continuous time, (ii) risk-neutral government, and (iii) unconstrained trade. After describing our benchmark model and deriving the above results formally in Section 4, we investigate in Section 6 the consequences of relaxing each of these assumptions in turn.

First, as illustrated in Section 6.1.1, the discrete-time counterpart to our model yields strictly positive welfare gains for the risk-neutral government. The benefit of discrete trade is that the government effectively commits not to issue bonds and not to default during a strictly positive measure of time. Stokey (1981) highlights the importance of this assumption in the context of durable goods monopoly.

Second, the risk neutrality of the sovereign implies that its flow payoff is linear in the debt proceeds. If instead the government exhibits some degree of risk aversion, gains from trade can be restored as we can see from Claim 4: when $u'' < 0$, the first term of expression (10) is always positive and of order dt, meaning that the accumulation of these dt-order gains over the infinite time horizon are $O(1)$. This outcome is then analogous to the results in Kahn (1986) in the context of the durable goods monopoly problem: rents can be extracted by the monopolist if its marginal production costs are increasing. We investigate this channel in Section 6.3 when income follows geometric Brownian motion dynamics. Importantly, we will show that the bulk of these trading gains arises because risk aversion effectively restrains the rate of issuance, rather than from a direct benefit of consumption smoothing.

Finally, we also consider mechanisms that can constrain trade and help restore welfare gains. Consider for example a risk-neutral government that may face a flow issuance constraint, limiting the amount

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[20]Investors in our model are competitive and thus do not extract any welfare gains either, despite the fact that trades do occur in equilibrium; the available surplus is offset in equilibrium by expected future default costs. Also, unlike in Coase (1972) setting, marginal cost (which is the equilibrium debt price) is endogenous in our model, and investors share a common valuation for the asset. Similar "no-gain-from-trade" results can be found in DeMarzo and Urošević (2006) in the context of trading by a large shareholder, in Daley and Green (2020) where a monopolistic buyer makes frequent offers to a privately informed seller, and more recently in Hu and Varas (2021) where a bank keeps selling its loans ex post.

of new debt issued each period. In Appendix B, we compute the incremental gains from trade provided by this type of constraint and establish heuristically the following result.

Claim 5 *Consider a risk-neutral government whose debt issuance rate must satisfy $d\Gamma \leq \bar{G}dt$. Suppose an MPE exists in which the debt price is strictly decreasing in the level of government debt F. The incremental gains from trade, on the interval $[t, t + dt)$, are*

$$Gain_{[t,t+dt)} \underset{dt \to 0}{=} \lambda_t \bar{G}dt + O(dt^2), \tag{11}$$

where $\lambda_t \geq 0$ is the Lagrange multiplier on the flow debt issuance constraint.

With such a constraint, gains from trade appear so long as the constraint binds in some states of the world. The magnitude of those gains is then increasing in (a) the issuance rate cap $\bar{G}$ and (b) the tightness of the constraint (encoded via the Lagrange multiplier λ). The optimal policy $\bar{G}$ will thus involve a subtle trade-off: a higher value means higher gains when the constraint binds, but also a lower shadow marginal value of relaxing the constraint. We investigate this commitment device in Section 6.2.2 when income follows geometric Brownian motion dynamics.

Our key takeaway is that despite the attractiveness of external debt, a risk-neutral government that can borrow freely and frequently is unable to capture the gains from trade. Furthermore, as we will show, if citizens are more patient than the government itself, their welfare may be destroyed. Thus, in order to benefit from external debt, either (a) some constraints on the government's ability to borrow, or (b) a desire by the government to smooth income over time, must be introduced. We examine both types of devices in the paper and look at their consequences for the country's (and citizens') welfare.

4 The General Result

We now restrict ourselves to a risk-neutral government, solve our problem in continuous time, and generalize the results established heuristically in the previous section. We consider a broad class of income processes for our small open economy of interest, introduce risk-averse creditors whose marginal utility process might co-vary with the small open economy's income process, and consider general default settlement mechanisms.

4.1 Small Open Economy's Income Process

We assume that the small open economy's real income Y_t (per unit of time) is an Itô process that takes positive values only. International capital market conditions, which might affect the dynamics of the small open economy's income process, are described by the state variable $s_t \in \mathcal{E} \subset \mathbb{R}$. Y_t and s_t satisfy

$$dY_t = \mu_Y(Y_t, s_t)dt + \sigma_Y(Y_t, s_t) \cdot d\boldsymbol{B_t}, \tag{12}$$

$$ds_t = \mu_s(s_t)dt + \sigma_s(s_t) \cdot d\boldsymbol{B_t}. \tag{13}$$

Here, $\{B_t\}_{t \geq 0}$ is a multi-dimensional Brownian motion on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.[21] The multi-dimensional nature of the Brownian shocks could capture country-specific shocks and aggregate shocks, which might also affect the marginal utility of international financial market participants.[22] We assume that the government autarky value is finite.

Assumption 1 *The impatience parameter δ, income drift rate $\mu_Y(\cdot, \cdot)$, income volatility $\sigma_Y(\cdot, \cdot)$, and the stochastic process $\{s_t\}_{t \geq 0}$ are such that for all values of $(Y, s) \in \mathbb{R}_+ \times \mathcal{E}$, the autarky value satisfies*

$$W(Y, s) := \mathbb{E}_{Y,s}\left[\int_0^{+\infty} e^{-\delta t} Y_t dt\right] < +\infty.$$

4.2 Default Resolution

Upon a default at time τ,[23] we set the creditors' recovery rate to be a (state dependent) constant $\underline{d}(s)$ and the government's payoff to be a fraction $\alpha \in (0, 1)$ of its autarky value net of the payment to creditors:[24]

$$\underline{D}(Y, F, s) = \underline{d}(s) < \frac{\kappa + m}{\delta + m},$$
$$\underline{V}(Y, F, s) = \alpha W(Y, s) - \underline{d}(s) F.$$

One can view α as a reduced form representation of disruptions to trade and financial flows inducing a GDP drop at the time of a sovereign default, and $\underline{d}(s)$ as the minimum recovery value that can be imposed on creditors. While these outcomes are the result of some bargaining game between the sovereign and its creditors that occurs in default, these strategic interactions are not a focus of this paper and so for now we treat the determination of α and $\underline{d}(s)$ as exogenous. Later, we will endogenize the recovery rate by assuming the debt face value is renegotiated upon default (see Section 4.9).

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[21]Here, $\mathbb{P}$ is the physical probability measure, and $\mathcal{F}_t$ the σ-algebra generated by the Brownian motion B_t. Our notation will use bold letters for vectors. By convention, we assume that the entries of the volatility vector σ_Y are positive.

[22]Note that with our small open economy assumption, the income level Y_t does not affect the dynamics of the international capital market conditions s_t. The model specified in (12) can capture a feedback loop between financial market conditions and the country's income, by allowing the growth rate μ_Y and volatility σ_Y to depend on s_t. Our results are also robust to more general specifications of the process s_t; we could have assumed that s_t is a multi-dimensional Itô process, or that its dynamics include a jump component, with a jump measure only dependent on s_t, without changing any of the results of our paper.

[23]In traditional sovereign default models, Cole and Kehoe (1996) and Aguiar et al. (2022) (amongst others) all assume that the bond auction happens before the default decision is made by the government, while Aguiar and Gopinath (2006), Arellano (2008) and many other quantitative models of sovereign debt assume that the government makes its default decision before the bond auction takes place. The former timing convention allows, when consumption is constrained to be positive, for the existence of potentially multiple equilibria even with one-period debt; this is induced by the creditor's self-fulfilling belief that the government will default immediately after debt has been issued, leading to a failed auction that forces the government default. These considerations are not crucial in our environment with a risk-neutral government who issues long-term amortizing debt because the no-trade consumption $Y - (\kappa + m)F$ (even when it is negative) is always feasible. Hence, the government is never "forced" to default. Even if negative consumption is not possible, if the income process Y is continuous, the government will never be forced to default if default occurs before F/Y climbs above the critical value $1/(\kappa + m)$.

[24]Note that the upper bound $(\kappa + m)/(\delta + m)$ on the recovery rate assures that the cost to the government in default is less than the cost of simply repaying the debt in perpetuity.

4.3 Creditors

International investors purchase the debt issued by the government. We model their marginal utility process M_t (which we will also refer to as the stochastic discount factor, or "SDF") as follows:

$$\frac{dM_t}{M_t} = -r(s_t)dt - \boldsymbol{v}(s_t) \cdot d\boldsymbol{B_t}. \tag{14}$$

The international investors' risk free rate is $r(s)$, while $\boldsymbol{v}(s)$ is the international risk price vector in state s. The jth coordinate of $\boldsymbol{v}(s)$ represents the expected excess return compensation per unit of jth Brownian shock earned by investors in state s.[25] Given our assumed investor pricing kernel, any $\mathcal{F}_T$-measurable amount A_T received at time $T > t$ will be valued by investors at

$$\text{Price}_t\left(A_T\right) = \mathbb{E}_t\left[\frac{M_T}{M_t}A_T\right] := \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s_u)du}A_T\right],$$

with $\mathbb{E}^{\mathbb{Q}}$ being the risk-neutral expectation operator, and $\mathbb{Q}$ the associated measure, under which $\boldsymbol{B}_t^{\mathbb{Q}} := \boldsymbol{B_t} + \int_0^t \boldsymbol{v}(s_u)du$ is a standard multi-dimensional Brownian motion. Alternatively, one can also interpret investors' behavior through the lens of heterogeneous beliefs: investors and the government have different perceptions about the drift of the driving processes (Y_t, s_t), they are aware of each other's beliefs and simply agree to disagree.

We end this section by introducing the assumption that guarantees potential gains from trade, so that the government has incentive to borrow from international lenders.

Assumption 2 *The international risk-free rate $r(\cdot)$ satisfies:*

$$r(s) < \delta \qquad \forall s \in \mathcal{E}.$$

4.4 Government Problem and Debt Valuation in Continuous Time

In our model, the payoff-relevant variables for the government and creditors are Y_t, F_t and s_t. The state space will be $\mathbb{R}_+^2 \times \mathcal{E}$, or a subset thereof. Consistent with Maskin and Tirole (2001), we focus on MPEs, where the government's strategy (including both its debt issuance and default timing), and therefore the debt price, only depend on (Y_t, F_t, s_t).

The government controls its outstanding debt F_t through an endogenous issuance policy $d\Gamma_t \in \mathbb{R}$ (repurchase if $d\Gamma_t < 0$) where Γ_t represents the cumulative debt issuance from time 0 to time t. An admissible debt issuance policy Γ_t is a right-continuous-left-limit process, so that its increment (i.e., the issuance/repurchase) $d\Gamma_t : \mathbb{R}_+^2 \times \mathcal{E} \to \mathbb{R}$ is measurable with respect to the Markov state vector (Y_t, F_t, s_t). We will denote $\mathcal{G}$ the set of admissible debt issuance policies.

The government can also default on its debt. An admissible default policy is a stopping time τ, with respect to the filtration $\mathcal{F}_t$, that can be written as the first hitting time of a subset of the state space:

$$\tau = \inf\{t \geq 0 : (Y_t, F_t) \in \mathcal{O}\left(s_t\right)\}.$$

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[25] $|\boldsymbol{v}|$ will determine the Sharpe ratio associated with government debt. More general specifications of the stochastic discount factor, for example including jumps dependent on the state variable s_t, would not alter our main results.

Here, $\{\mathcal{O}(s)\}_{s \in \mathcal{E}}$ is a family of open sets representing the default regions of the state space. We let $\mathcal{T}$ be the set of admissible default policies. Our formulation of admissible policies leads to a controlled face value process $F_t^{(\Gamma)}$:

$$F_t^{(\Gamma)} = F_0 + \underbrace{\int_0^t d\Gamma\left(Y_u, F_u^{(\Gamma)}, s_u\right)}_{\text{debt issuances}} - \underbrace{\int_0^t mF_u^{(\Gamma)} du}_{\text{principal payments}} ,$$

with the superscript notation $F_t^{(\Gamma)}$ highlighting that the face value process F_t depends on the issuance policy Γ. The consumption increment of the sovereign between t and $t + dt$ is:

$$dC_t^{(\Gamma;D)} := \left(Y_t - (\kappa + m)F_t^{(\Gamma)}\right) dt + D\left(Y_t, F_t^{(\Gamma)}, s_t\right) d\Gamma\left(Y_t, F_t^{(\Gamma)}, s_t\right).$$

Creditors price the newly issued or repurchased debt competitively at a price D (per unit of promised face value) that reflects creditors' expectations regarding future financing and default decisions. Since they realize a recovery rate $\underline{d}(s_\tau)$ in default, they value one unit of sovereign debt as follows:

$$D(Y, F, s; (\Gamma, \tau)) := \mathbb{E}_{Y,F,s}^{\mathbb{Q}}\left[\int_0^\tau e^{-\int_0^t (r(s_u) + m) du}(\kappa + m) dt + e^{-\int_0^\tau (r(s_u) + m) du}\underline{d}(s_\tau)\right]. \tag{15}$$

The government takes as given the debt price function D and chooses its issuance and default policies (Γ, τ) to solve the following problem:

$$V(Y, F, s; D) := \sup_{(\Gamma, \tau) \in \mathcal{G} \times \mathcal{T}} \mathbb{E}_{Y,F,s}\left[\int_0^\tau e^{-\delta t} dC_t^{(\Gamma;D)} + e^{-\delta \tau}\underline{V}\left(Y_\tau, F_\tau^{(\Gamma)}, s_\tau\right)\right]. \tag{16}$$

When choosing its issuance policy, the government takes into account its potential price impact on its bond proceeds. In equilibrium, creditors price the debt in equation (15) by correctly anticipating the government's equilibrium policies.

4.5 Smooth Equilibrium

Following our discussion of Section 3.3, we focus on a special class of MPEs, in which the government adjusts its outstanding debt in a "smooth" fashion, i.e., of order "dt." In this special class of MPEs, the debt policy Γ can be characterized by a measurable function $G : \mathbb{R}_+^2 \times \mathcal{E} \to \mathbb{R}$ such that:

$$d\Gamma_t = G(Y_t, F_t, s_t)\, dt.$$

The resulting face value and cumulative consumption processes are absolutely continuous. We formally define a "Smooth MPE" as follows.

Definition 1 *A Smooth MPE is a set of Markovian issuance and default policies $(G^*, \tau^*) \in \mathcal{G} \times \mathcal{T}$, a default region $\mathcal{O}^*$, and a value function V that is twice continuously differentiable everywhere except when $(Y, F) \in \partial\mathcal{O}^*(Y, F)$ (where it is once continuously differentiable), such that for any initial state (Y, F, s),*

$$(G^*, \tau^*) = \arg\max_{(G, \tau) \in \mathcal{G} \times \mathcal{T}} \mathbb{E}_{Y,F,s}\left[\int_0^\tau e^{-\delta t} dC_t^{(G;D(\cdot,\cdot,\cdot;(G^*,\tau^*)))} + e^{-\delta \tau}\underline{V}\left(Y_\tau, F_\tau^{(G)}, s_\tau\right)\right].$$

4.6 Optimality Conditions and No Gain from Trade

We now derive necessary conditions for the Smooth MPE to exist, and show that in equilibrium the government will just earn its no-trade value despite the potential gains from trade. This echoes the key insight from Section 3. For notational convenience, denote by $X_t := (Y_t, s_t)'$ the state vector of driving variables, $\mu_X := (\mu_Y, \mu_s)'$ its related drift vector, and $\sigma_X := (\sigma_Y', \sigma_s')'$ its related diffusion matrix.

In the continuation region (i.e. when the government is servicing its debt), the government value function satisfies the following Hamilton-Jacobi-Bellman ("HJB") equation:

$$\delta V = \sup_G \left[Y + GD - (\kappa + m) F + (G - mF) \partial_F V + \mu_X \cdot \partial_X V + \frac{1}{2} \text{tr} \left(\sigma_X' \partial_{XX'} V \sigma_X \right) \right]. \tag{17}$$

When the government defaults, we have

$$V(Y, F, s) = \underline{V}(Y, F, s), \quad (Y, F) \in \mathcal{O}(s). \tag{18}$$

Default optimality gives a smooth pasting condition imposed on the boundaries of the default region:

$$\partial_X [V(Y, F, s)] = \partial_X [\underline{V}(Y, F, s)], \quad (Y, F) \in \partial \mathcal{O}(s). \tag{19}$$

Loosely speaking, this condition imposes a minimum amount of "smoothness" of the value function at the boundaries of the default region, and is essential when using verification theorems that establish the optimality of the government decisions. For a given debt price D, we note that the existence of a (classical) solution to the parabolic partial differential equation ("PDE") (17) with boundary data (18) is not guaranteed, and only a limited number of existence results are available in the literature.[26] As will be clear shortly, our approach in Section 5.1 will be to find, for a given income process, an analytic solution to this PDE.

For a solution to equation (17) to exist, we must have:

$$D(Y, F, s) + \partial_F V(Y, F, s) = 0. \tag{20}$$

Equation (20) is a necessary condition that needs to hold in any Smooth MPE; it is the continuous-time counterpart to the envelop condition (4) in the discrete time model of Section 3. Reinjecting this optimality condition (20) into equation (17) leads to:

$$\delta V = Y - (\kappa + m) F - mF \cdot \partial_F V + \mu_X \cdot \partial_X V + \frac{1}{2} \text{tr} \left(\sigma_X' \partial_{XX'} V \sigma_X \right). \tag{21}$$

This immediately allows us to derive the following lemma:

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[26] See for instance theorem 4.1 in Fleming and Soner (2006) for the stochastic control case, when the controls must live on a compact set; or Strulovici and Szydlowski (2015) in the one-dimensional case for stochastic control and optimal stopping problems. In both articles, the HJB must be uniformly parabolic and the drift and diffusion coefficient must be Lipschitz, amongst other technical conditions.

Lemma 1 *In any Smooth MPE, the debt price must satisfy $\partial_F D + V = 0$. When facing equilibrium debt prices, the government is indifferent to any absolutely continuous debt policy.*

Denote by $G_0 := 0$ the financing policy with no debt issuances or repurchases, and notice that under this policy, the consumption rate no longer depends on the pricing of debt. Let $V_0(Y, F, s)$ be the corresponding "no-trade" value, which is achieved if the government never issues new debt or buys back existing debt (but keeps the option to default); that is,

$$V_0 (Y, F, s) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_{Y, F, s} \left[\int_0^\tau e^{-\delta t} dC_t^{(G_0)} + e^{-\delta \tau} \underline{V} \left(Y_\tau, F_\tau^{(G_0)}, s_\tau \right) \right]. \tag{22}$$

Lemma (1), applied to the issuance policy G_0, and equations (18-19), give us the following proposition:

Proposition 1 *In any Smooth MPE, the government welfare is identical to its no-trade value: $V (Y, F, s) = V_0 (Y, F, s)$. In addition, the welfare of a government without debt outstanding is equal to its autarky value: $V (Y, 0, s) = W(Y, s)$.*

Proposition 1 is a formal way to state the no-gain-from-trade result illustrated in Section 3. While there should be gains from trade in this economic environment (since the government is more impatient than its creditors), those gains are entirely dissipated by default costs. Interestingly, the equilibrium welfare of the government is entirely independent of the characteristics of investors—say interest rates and risk premia—in international debt markets.

The above discussion also suggests a method to construct an MPE for our model, by solving the government problem under the assumption that it does not trade in the future, but retains the option to default on its outstanding debt. The next proposition formalizes this insight.

Proposition 2 *If the no-trade value function V_0 (defined via (22)) is twice continuously differentiable in F and strictly convex in F, then a Smooth MPE exists, with debt price $D (Y, F, s) = -\partial_F V_0 (Y, F, s)$ that is strictly decreasing in F.*

4.7 The Optimal Financing Policy

We now characterize the financing policy of the government, which naturally depends upon the risk premium required by international creditors.

4.7.1 Risk Premia

In equilibrium creditors earn a risk-premium as long as the country's income process exhibits non-zero local correlation with investors' pricing kernel. Denote by $\pi (Y_t, F_t, s_t)$ the instantaneous expected sovereign bond excess return earned by creditors:

$$\pi (Y_t, F_t, s_t) \, dt := \mathbb{E}_t \left[\frac{dD_t + (m + \kappa) \, dt - mD_t dt}{D_t} - r(s_t) dt \right].$$

We show in the Online Appendix A.1 that

$$\pi (Y, F, s) = (\sigma_X (Y, s) \, \boldsymbol{\nu} (s)) \cdot \partial_X \ln D (Y, F, s). \tag{23}$$

Equation (23) states that the debt investors' required risk premium is proportional to the risk premium of the economy's income process ($\sigma_X \nu$) and the beta of the debt ($\partial_X \ln D$) with respect to the state. In particular, the debt beta and thus the required risk premium will vary with the riskiness of the debt.

4.7.2 Equilibrium Debt Issuance Policy

We use equation (20) and differentiate equation (21) with respect to F to obtain:

$$(\delta + m)D = \kappa + m + \mu_X \cdot \partial_X D + \frac{1}{2}\text{tr}\left(\sigma'_X \partial_{XX'} D\sigma_X\right) - mF\partial_F D.$$

On the other hand, the valuation equation of creditors in (15) implies that

$$(r + \pi + m)\, D = \kappa + m + (G(Y, F, s) - mF)\, \partial_F D + \mu_X \cdot \partial_X D + \frac{1}{2}\text{tr}\left(\sigma'_X \partial_{XX'} D\sigma_X\right).$$

Subtracting one equation from the other allows us to obtain a formula for the optimal issuance policy, as a function of the debt price (which itself can be computed via equation (20)).

Proposition 3 *In equilibrium the optimal bond issuance policy of the government is:*

$$G^*(Y, F, s) = \frac{\delta - (r(s) + \pi(Y, F, s))}{-\partial_F \ln D(Y, F, s)}. \tag{24}$$

This equation generalizes the issuance formula (7) obtained in the discrete time model of Section 3 to the case of a non-trivial investors' SDF; the bond issuance rate is the wedge between (i) the rate of impatience of the government and (ii) the required rate of return of international bond investors, divided by the semi-elasticity of the bond price with respect to the total face value F. Upward shocks to risk-free rates $r(s_t)$ or risk-premia $\pi(Y_t, F_t, s_t)$ cause the government to adjust its bond issuance policy downwards, creating an upward adjustment in the sovereign country's current account, which is defined as $Y_t - C_t$. This mechanism is consistent with empirical evidence presented in Neumeyer and Perri (2005), who document a positive correlation for various emerging market economies between their net exports and real interest rates.

The optimal Markov policy uncovered in equation (24) delivers additional insights. First, when the probability measure $\mathbb{Q}$ of investors and the probability measure $\mathbb{P}$ of the government correspond to each other (in other words, when $\nu(s) = 0$ for all s), under Assumption 2 the bond issuance rate is always positive: in this particular case, it is never efficient for the government to buy back its own debt, as was already hinted at in Section 3. This result echoes Bulow and Rogoff (1988) who show, in the context of a one-period model of sovereign default with a risk-neutral government and risk-neutral lenders, that it is never welfare-improving for a country to buy back its own debt.

This no-buyback result breaks down in the presence of risk-averse lenders, whose price of risk has a positive local correlation with the country's endowment process: in such case, equation (24) shows that when sovereign bond risk premia are sufficiently high – either due to high investor risk prices, or high bond risk exposure – the country might find it optimal to buy back its own debt.[27]

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[27]Linking this to the interpretation of belief divergence between creditors and the sovereign, the endogenous risk premium

4.8 Citizen Welfare: When Autarky is Better Than Trade

Our welfare-neutrality result is applicable when the preferences of the government correspond to those of the citizens of the small open economy. One could instead imagine, for political economy reasons, that the government has an effective discount rate greater than the discount rate of the citizens of the small open economy; this assumption could be rationalized for example by observing that government officials are elected for short durations. Cuadra and Sapriza (2008) or Hatchondo, Martinez and Sapriza (2009), amongst others, have studied similar economic environments in which policy makers behave more impatiently than their citizens.[28] We show that even if the country's citizens are more impatient than international investors, the government's lack of commitment brings a worse welfare for citizens than under autarky.

Specifically, suppose that government officials have an effective discount rate δ, while the citizens of the small open economy are more "patient," with a discount rate $\hat{\delta} < \delta$. Consider a simplified economic environment in which, upon default, the entire small open economy's income stream is lost (i.e. $\alpha = 0$) and creditors lose their entire investment; and there is no risk premium channel (i.e., $\nu(s) = 0$). Let $\hat{V}(Y, F, s)$ be the present value of life-time utility for the citizens of the small open economy under the impatient government's financial polices studied in Section 4.7, and $\hat{V}_0(Y, F, s)$ the present value of life-time utility of citizens if the government refrains from future trading.

Proposition 4 *The citizens are strictly worse off when their country has access to international debt markets (and the government makes borrowing and default decisions) than when their country is restricted from future trading, i.e., $\hat{V}(Y, F, s) < \hat{V}_0(Y, F, s)$.*

This result holds even if citizens discount the future at a rate $\hat{\delta}$ that exceeds the expected return $r(s)$ of the debt. As the proof in Online Appendix A.2 shows, for $\hat{\delta} < \delta$, the marginal cost of debt to citizens exceeds the debt price (i.e. $\partial_F \hat{V} + D < 0$). Thus, at the margin, citizens would prefer to buy back bonds and reduce the country's indebtedness. While the government is exactly indifferent between financial autarky and having access to international debt markets, Proposition 4 shows that citizens are strictly worse off, suggesting that autarky is better than trade. This result, while surprising, is rather intuitive. The risk-neutral government, with a higher discount rate δ, balances the upside of borrowing to front-load consumption (by issuing debt today) versus its downside of future default (and the resulting income losses). Since the benefits of debt issuances are incurred today, whereas the costs are in the future, citizens of the country with a lower discount rate weigh those benefits less than their related cost, and will thus incur a welfare loss compared to the autarky benchmark.[29]

4.9 Default Settlement Mechanisms

Thus far, we have assumed for simplicity that the payoffs to creditors and the government in the event of default are exogenously given. We could instead endogenize these payoffs by assuming that default

(potentially connected to the endogenous sovereign default policy) can be also viewed as endogenous pessimism of international creditors, and hence the sovereign might buy back the debt from pessimistic creditors.

[28]See also Alesina and Tabellini (1990) and Persson and Svensson (1989) for political economy models that produce endogenous impatience when parties alternate power. The stylized model we use here is a simplified setting in which regime change occurs with a constant hazard rate and each regime only cares about consumption while it is in power.

[29]The argument is more subtle when the country can "restart" after a default. We generalize this result to a range of default regimes in Section 5.

involves a restructuring in which the government (i) suffers a permanent loss to its future income, but (ii) is able to write down some portion of the debt. Specifically, for $\alpha, \theta \in (0, 1)$,

$$\underline{V}(Y, F, s) = V(\alpha Y, \alpha \theta F, s), \tag{25}$$

$$\underline{D}(Y, F, s) = \alpha \theta D(\alpha Y, \alpha \theta F, s). \tag{26}$$

Here, α represents the decline in income that the government suffers in exchange for forgiveness of a fraction $1 - \alpha\theta$ of its outstanding debt; the debt recovery value is thus endogenously determined. We consider this default settlement mechanism in Section 5.

More generally, we could allow the government to face a state-dependent menu of alternative default regimes, each with different (and possibly random) default parameters (α, θ). The only requirement we impose is that in default, the recovery value of creditors is equal to marginal value of the debt to the government:

$$\underline{D}(Y, F, s) = -\partial_F \underline{V}(Y, F, s), \tag{27}$$

a condition that holds in equation (26). This condition, which is a natural extension to the framework used throughout Section 4, implies that there are no inherent gains from trade introduced in the default process itself, as any value recovered by the marginal creditor is "paid for" by the government. Absent (27), the government could capture gains from trade simply by issuing (or repurchasing) debt at the moment of default.

4.10 Discussion

4.10.1 Debt Maturity

In our risk-neutral setting, the "no gain from trade" result holds irrespective of the debt maturity $1/m > 0$. This maturity irrelevance crucially depends on our risk-neutrality assumption; as we illustrate in Online Appendix C.5.4, a risk-averse government can capture some gains from trade, but longer maturity reduces these gains, consistent with several existing studies (see for instance Chatterjee and Eyigungor (2012)).

We stress that for any given $m < +\infty$, the class of debt contracts considered in this paper can be viewed as "long-term" debt. Indeed, by taking the limit $dt \to 0$ while fixing any bond maturity $1/m$ (however small), the opportunities to trade are always (infinitely) more frequent than the bonds' maturity. This choice is empirically motivated – the majority of hard-currency emerging market sovereign debt is long-term, with new debt issuance occurring at a much higher frequency than maturity.

An alternative approach is to consider *infinitesimally short-term* debt, by taking the limit of a discrete-time model with one-period debt as *both* the debt maturity and the trading interval shrink to zero (see e.g. Bornstein (2020) for such a model). In that case, the lack of commitment that is central to our setting is avoided, since at each moment of issuance the government restarts from zero debt. If the income process has continuous sample paths (as is the case in Section 5.1), the government is then able to capture all of the gains from trade (as if it could fully commit to an issuance policy). If instead income is subject to downward jumps, some dead-weight loss due to default cannot be avoided, as in Abel

(2018). In a corporate debt setting, Hu, Varas and Ying (2021) extend DeMarzo and He (2021) and show that the ability to issue infinitesimally short-term debt by firms enhances enterprise value.

Intuitively, infinitesimally short-term debt restores the sovereign's commitment power and hence serves as a superior financing instrument; this is consistent with Aguiar et al. (2019), who show that a government having access to a menu of debt contracts should only actively manage one period debt (while allowing all long-term debt to run off). Mapping back to the Coase (1972) model, infinitesimally short-term debt is analogous to leasing – rather than selling – the durable good.

4.10.2   Transaction Costs and Non-Pecuniary Benefits

Our analysis so far assumes away transaction costs in debt issuance. Using a similar modeling device, one could also consider the case of transaction costs that are proportional to either (i) proceeds raised, or (ii) the notional of bonds sold (per unit of time).[30] Similarly, the government may also enjoy additional (proportional) non-pecuniary benefits when selling bonds in international capital markets, which can be motivated by political economy considerations. In both situations, we show in Online Appendix A.3 that our welfare-neutral result is robust to these alternative assumptions. Transaction costs over debt issuance tend to decrease bond issuance and increase debt prices; in this case the potential gains from trade are dissipated by the joint effect of transactions and default costs. Conversely, non-pecuniary benefits cause debt issuance to increase to the point that incremental default costs offset these benefits. Since patient citizens weigh future default costs more than immediate consumption benefits, a proportional debt issuance tax – whether rebated to citizens or a pure dead-weight loss – would thus unconditionally improve citizens' welfare.

4.10.3   Equilibrium Multiplicity

The sovereign default literature has investigated in various environments the possible existence of multiple MPEs. In discrete time, equilibrium uniqueness obtains with one-period defaultable debt.[31] In the presence of long-term debt, Lorenzoni and Werning (2019) and Aguiar and Amador (2020) (amongst others) study economic environments where multiple equilibria emerge. Loosely speaking, if creditors price the sovereign bonds issued at a low (high) level, it will be optimal for the government to default "early" ("late"), i.e., at debt levels that are relatively low (high); this "expectation" channel conceptually opens the door to equilibrium multiplicity. While our environment might also feature multiple MPEs, at most one can be a Smooth MPE, as the next proposition discusses.

Corollary 2 *If a Smooth MPE exists, then it must be unique within the class of Smooth MPEs.*[32]

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[30]See Benzoni et al. (2020) for a model of the firm making dynamic debt issuance decisions in the presence of fixed costs.

[31]This result was proven in Auclert and Rognlie (2016) and Aguiar and Amador (2019).

[32]When $F = 0$, there also exists a "trivial" MPE due to the government's indifference between financial autarky and debt issuance. In the "trivial" equilibrium, the government never borrows and debt prices are identical to those in our Smooth MPE, except at $F = 0$, where the debt would be priced as risk-free (this debt price is only relevant off-equilibrium if the government has debt and buys it all back). The nature of this equilibrium multiplicity is different from the one documented in Passadore and Xandri (2015); there, given the lack of exogenous default costs and the inability to save, autarky does not penalize a government that decides to default, thereby supporting a no-borrowing equilibrium with zero debt prices everywhere. This no-debt / zero-debt price equilibrium does not arise in our framework: indeed, when the government has a small outstanding amount of debt, defaulting immediately would incur a real loss that might make the government prefer to continue paying its debt, even if facing a zero debt price.

The proof is immediate: using Proposition 1, in a Smooth MPE, the problem solved by the government is equivalent to a single-agent default problem that is independent of the pricing of debt – neither the HJB equation in the continuation region, nor the value matching condition at the default boundary or the smooth-pasting default optimality condition contain the debt price function. Note that Corollary 2 is not inconsistent with articles documenting the existence of multiple equilibria; in Aguiar and Amador (2020) for example, neither the "saving" nor "borrowing" equilibrium constructed are Smooth MPEs, as the government either (a) faces constraints on consumption, or (b) uses debt policies that are not absolutely continuous.

That said, in the setting with geometric Browning motion considered in Section 5, we will go further and show that the Smooth MPE is indeed the unique MPE (smooth or not). As in many economic models, adding sufficient "noise" can be helpful in isolating a unique equilibrium.

5 Geometric Brownian Motion Income Process

The results discussed until now are general to an entire class of income and stochastic discount factor processes. In this section, we focus on a particular income process, characterize the Smooth MPE, show in this particular case that it is the unique MPE, and provide comparative static results. We assume that our small open economy's income has i.i.d. growth rate innovations:

$$\frac{dY_t}{Y_t} = \mu dt + \sigma \cdot d\boldsymbol{B_t}. \tag{28}$$

To insure that Assumption 1 is satisfied, we impose that:

$$\delta > \mu. \tag{29}$$

Upon default, we use the resolution mechanism discussed in Section 4.9: the small open economy's income drops by a factor $\alpha \in (0,1)$, the debt face value suffers a haircut $1 - \alpha\theta$ (with $\theta \in (0,1)$), and the government immediately regains access to credit markets after default. These assumptions imply that the debt-to-income is reduced upon default by a factor θ. Lastly, we assume that the creditor's SDF features a constant risk-free rate r and a constant risk price vector $\boldsymbol{v}$. For interpretation, $\sigma \cdot \boldsymbol{v}$ is the risk premium that would be earned by international investors for holding a claim on the income process Y. Alternatively, $\boldsymbol{v}$ determines the Sharpe ratio associated with investments in government debt.

5.1 Equilibrium with GBM Income Process

In this setting, the flow payoff and the state dynamics are linear in the state variables (Y, F). This scale invariance property implies that the economically relevant measure of sovereign indebtedness is $x := F/Y$, i.e. the debt-to-income ratio of our small open economy. Because all subgames with the same initial debt-to-income ratio x_t are strategically equivalent, we will look for an MPE in this unidimensional state variable, and show that it must be a Smooth MPE.[33]

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[33]Maskin and Tirole (2001) define Markov Perfect Equilibria in terms of the coarsest partition such that equivalent subgames are "strategically equivalent." A sufficient condition for strategic equivalence is that the payoffs are equivalent up to an affine transformation, which is satisfied by this model. DeMarzo and He (2021) follow the same approach.

Under this observation, the value function V and the optimal issuance policy G are homogeneous of degree one in (Y, F), the debt price function is homogeneous of degree 0 in (Y, F), and the default policy will follow a cutoff strategy (in terms of x). More specifically,

$$V(Y, F) := v(x)Y, \qquad D(Y, F) := d(x), \qquad G(Y, F) = g(x)Y, \qquad \tau_{k+1} = \inf\{t \geq \tau_k : x_t > \bar{x}\}.$$

The debt-to-income ratio $x_t^{(g,\tau)}$ is a controlled stochastic process that evolves as follows on $(0, \bar{x})$:

$$dx_t^{(g,\tau)} = \left(g(x_t^{(g,\tau)}) - \left(m + \mu - |\sigma|^2\right) x_t^{(g,\tau)}\right) dt - x_t^{(g,\tau)}\sigma \cdot dB_t + (\theta - 1)\, x_t^{(g,\tau)} dN_{d,t}^{(\tau)}.$$

The debt-to-income ratio grows with debt issuances $g(x)$, and otherwise declines at a rate equal to the sum of (i) the debt amortization rate m, (ii) the income growth rate μ, corrected by (iii) a convexity adjustment term $|\sigma|^2$. The government chooses to default as soon as the debt-to-income ratio exceeds an endogenous threshold $\bar{x}$, at which point x_t resets to $\theta\bar{x} < \bar{x}$; $N_{d,t}^{(\tau)}$ denotes the counting process associated with default decisions of the government.

5.1.1 Equilibrium Characterization in GBM Case

To characterize the equilibrium, first define ξ as the positive root of the characteristic polynomial of the HJB equation for the value function:

$$\frac{1}{2}|\sigma|^2\xi^2 - \left(m + \mu + \frac{1}{2}|\sigma|^2\right)\xi - (\delta - \mu) = 0. \tag{30}$$

Lemma 3 *The positive root of (30) satisfies $\xi > 1$. Moreover, $\xi > 2$ if and only if $|\sigma| < \sqrt{\delta + 2m + \mu}$.*

In our context, we expect income volatility to be sufficiently low so that $\xi > 2$ is the natural case we will illustrate throughout the main text.

The next proposition offers a complete characterization of the Smooth MPE; later in Section 5.1.3 we show that this is the unique MPE in this economy.

Proposition 5 *When the small open economy's income process follows a geometric Brownian motion as in (28), there exists a Smooth MPE where the government's optimal default cutoff $\bar{x}$ is given by*

$$\bar{x} = \frac{\xi}{\xi - 1}\left(\frac{1 - \alpha}{1 - \alpha\theta}\right)\left(\frac{\delta + m}{\kappa + m}\right)\frac{1}{\delta - \mu}. \tag{31}$$

For $x \in (0, \bar{x}]$, the debt price $d(x)$ and the (scaled) government value function $v(x)$ are

$$d(x) = \left(\frac{\kappa + m}{\delta + m}\right)\left[1 - \underbrace{\left(\frac{1 - \alpha\theta}{1 - \alpha\theta^\xi}\right)\left(\frac{x}{\bar{x}}\right)^{\xi - 1}}_{\text{default risk}}\right], \tag{32}$$

$$v(x) = \underbrace{\frac{1}{\delta - \mu}\left(1 - \left(\frac{1 - \alpha}{1 - \alpha\theta^\xi}\right)\left(\frac{x}{\bar{x}}\right)^\xi\right)}_{\text{value of income adjusted for default}} - \underbrace{xd(x)}_{\text{debt market value}}. \tag{33}$$

24

Let $\epsilon(x)$ be the semi-elasticity of the debt price w.r.t. the debt-to-income ratio:

$$\epsilon(x) := \frac{-d'(x)}{d(x)} = \frac{(\xi - 1)/x}{\left(\frac{1-\alpha\theta^\xi}{1-\alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi-1} - 1}. \tag{34}$$

For $x \in (0, \bar{x}]$, the bond risk-premium is increasing in x and equal to:

$$\pi(x) = x\epsilon(x)\sigma \cdot \nu. \tag{35}$$

The scaled optimal issuance policy $g(x)$ has the following expression:

$$g(x) = \frac{\delta - (r + \pi(x))}{\epsilon(x)} = \frac{\delta - r}{\epsilon(x)} - x\sigma \cdot \nu. \tag{36}$$

Our proof, detailed in Online Appendix B.1, relies on calculating analytically the no-trade value by solving the HJB equation with appropriate boundary conditions, and then verifying that the resulting debt price $d = -v'$ is strictly decreasing (and thus that $v = v_0$ is strictly convex). In Proposition 5, the first two terms of the default threshold in equation (31) capture the default option value and the cost of default, while the last two terms reflect the ratio of the perpetuity value of income ($\frac{1}{\delta-\mu}$) versus that of the debt ($\frac{\kappa+m}{\delta+m}$), both from the perspective of sovereign. For the debt price in equation (32), the second term in the bracket captures default risk, which increases with the distance to default ($x/\bar{x}$). The (scaled) government value in equation (33) equals the present value of income adjusted for future default costs, minus the debt market value. Note that $d(\bar{x}) = v(\bar{x}) = 0$ if and only if $\alpha = 0$, so that all income is lost in default, or $\theta = 1$, so that there is no debt forgiveness; in the latter case, once default occurs the government remains in default ($x = \bar{x} = \theta x$), leading to an infinite sequence of defaults until all income is destroyed. Finally, the issuance policy in equation (36) is a special case of the general formula in either the discrete (see equation (8)) or continuous-time settings (see equation (24)).

In Figure 2, we provide an illustrative example of the scaled value function $v(x)$ and the issuance policy $g(x)$. Alongside the value function $v(x)$, we also show the "reservation value" $\alpha v(\theta x)$, in other words the value (per unit of output) realized by the government if it chose to default at that point. At the default boundary $\bar{x}$, the continuation and default values are equal to each other, and both share the same sensitivity to the debt-to-income ratio (a requirement for default optimality).

Figure 2 also shows the issuance policy $g(x)$. When $\xi > 2$ (see Lemma 3), g is a decreasing and convex function of x, with limit $+\infty$ as $x \to 0$, given that the debt price semi-elasticity ϵ converges to zero at that point. Instead, when $\xi \in (1,2)$, it is a concave and hump-shaped function of x, with $g(0) = 0$, since the debt price semi-elasticity ϵ diverges to $+\infty$ at that point (see Online Appendix B.2 for illustrative plots). In all cases, $g(x) > 0$ in the right neighborhood of $x = 0$. Due to a positive risk price $\nu > 0$, $g(x)$ is not always positive: for high debt-to-income ratios, the bond risk-premium is sufficiently high to push the government to buy back its own debt – this is the region of the state space where $\pi(x) > \delta - r$ (see Figure 4). The drift rate $g(x) - (m + \mu - \sigma^2)x$ of the state variable x_t is downward sloping and intersects zero, implying that x_t is mean-reverting and that the government's debt balance systematically "targets" a debt-to-income attraction level x_a, as the next proposition highlights.

Figure 2: Value Function and Issuance Policy

(a): Value Function $v(x)$ **(b):** Issuance Rate $g(x)$
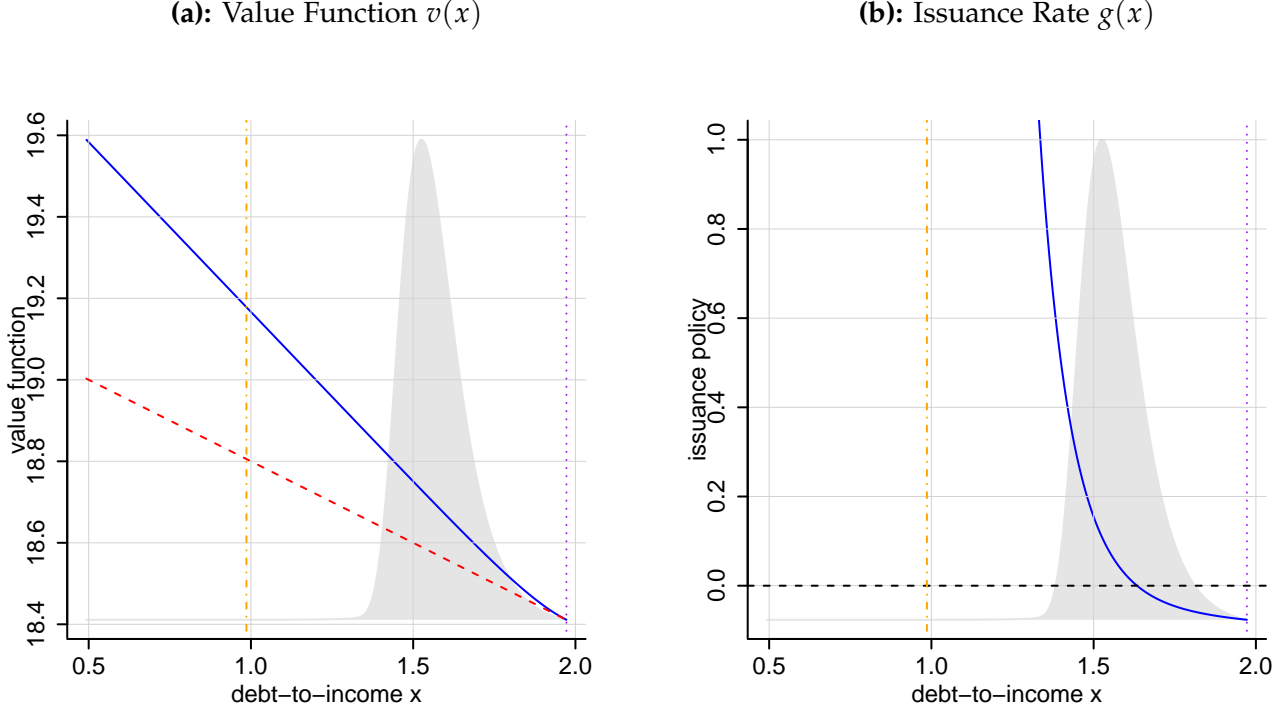
In plot (a) the value function $v(x)$ is depicted in solid blue while the default value $\alpha v(\theta x)$ is depicted in dashed red. In both plots, the dotted purple vertical line is the default boundary, the dot-dash orange vertical line is the reinjection point, and the shaded grey area represents the stationary distribution of the debt-to-income ratio. The plots were computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

Proposition 6 *The debt face value F_t satisfies dynamics*

$$dF_t^{\xi-1} = -\eta \left[F_t^{\xi-1} - (x_a Y_t)^{\xi-1} \right] dt - \left(1 - (\alpha\theta)^{\xi-1} \right) (\bar{x} Y_{t-})^{\xi-1} dN_{d,t}, \tag{37}$$

where η represents the speed of reversion of the debt face value towards its target $x_a Y_t$, and where x_a, the debt-to-income attraction point, satisfies $g(x_a) = m x_a$. Those constants are equal to

$$\eta := \delta - r + (\xi - 1)(m + \nu \cdot \sigma), \tag{38}$$

$$x_a := \bar{x} \left[\left(\frac{1 - \alpha\theta}{1 - \alpha\theta^\xi} \right) \left(\frac{\eta}{\delta - r} \right) \right]^{-\frac{1}{\xi-1}}. \tag{39}$$

Integrating equation (37) yields

$$F_t = \left[e^{-\eta t} F_0^{\xi-1} + \eta \int_0^t e^{\eta(u-t)} (x_a Y_u)^{\xi-1} du - \left(1 - (\alpha\theta)^{\xi-1} \right) \int_0^t e^{\eta(u-t)} (\bar{x} Y_{u-})^{\xi-1} dN_{d,u} \right]^{1/(\xi-1)}. \tag{40}$$

Proposition 6 is proven in Online Appendix B.3. The first two terms in (40) illustrate the fact that the sovereign debt level at time t can be thought of as a geometric average of past debt targets $\{x_a Y_s\}_{s \le t}$, weighted with an exponential decay with rate η. The final term in (40) takes into account debt adjustments related to past sovereign defaults, which occur when the debt-to-income ratio reaches $\bar{x}$.

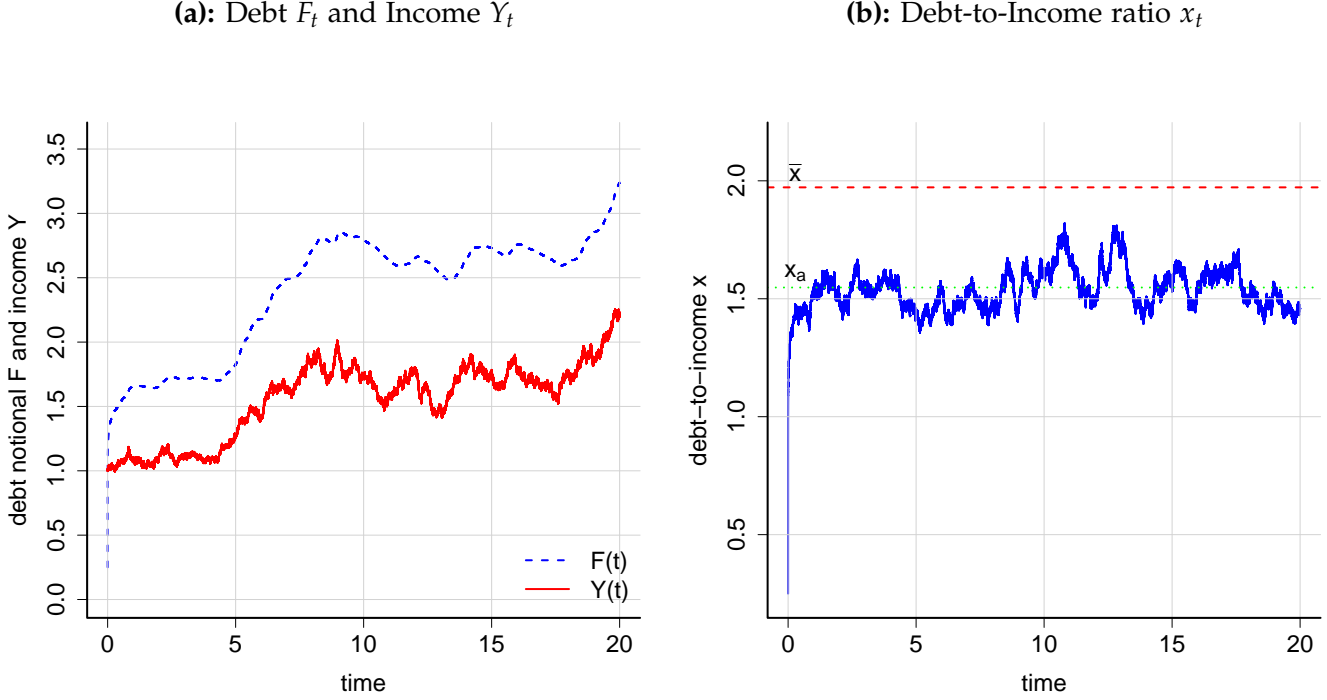**(a):** Debt F_t and Income Y_t **(b):** Debt-to-Income ratio x_t

Figure (a) shows the time path of the debt balance F_t (in dashed blue) and of income Y_t (in solid red) for one realization of the Brownian motion B_t. Figure (b) shows the resulting debt-to-income ratio, as well as the default boundary $\bar{x}$ and the attraction point x_a. The plot was computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

Figure 3 shows one sample path for output Y together with the debt face value F (left panel), and the debt-to-income ratio x (right panel). We also indicate the default boundary $\bar{x}$ as well as the attraction point x_a. The no-debt government ramps up its debt very quickly at time zero, since the drift rate of x_t turns unbounded when $x \to 0$. While the country's income Y fluctuates with Brownian income shocks, the aggregate debt face value F evolves smoothly.

Given the debt price $d(x)$, we can define implicitly the sovereign bond spread $\varsigma(x)$ as the yield spread over the risk-free benchmark that is needed to discount the sovereign bond's cash flow stream

$$d(x) := \int_0^\infty e^{-(r+m+\varsigma(x))t} (\kappa + m) \, dt.$$

The bond price and bond credit spreads are showed in Figure 4. The bond price is decreasing in the debt-to-income ratio, a necessary condition for the optimality of sovereign's smooth debt issuance policy (see Proposition 2). The bond credit spread is increasing in x and bounded away from zero: even without debt ($x \to 0$), credit spreads are strictly positive and equal to $\delta - r$, as creditors anticipate the dilution risk via future debt issuances – a phenomenon extensively studied in the literature, see for instance Hatchondo, Martinez and Sosa-Padilla (2016) or Chatterjee and Eyigungor (2015). In the same figure we also show the debt price and corresponding credit spread for the hypothetical bond price if the government commits to never issue any future debt (i.e., no-trade). Compared to the no-commitment equilibrium bond prices, the hypothetical bond prices implied by the no-trade policy are higher at low debt-to-income ratios, but could be lower at high debt-to-income ratios – due to the fact that in our

Figure 4: Debt Price and Credit Spreads

(a): Debt price $d(x)$ | **(b):** Credit spread $\varsigma(x)$
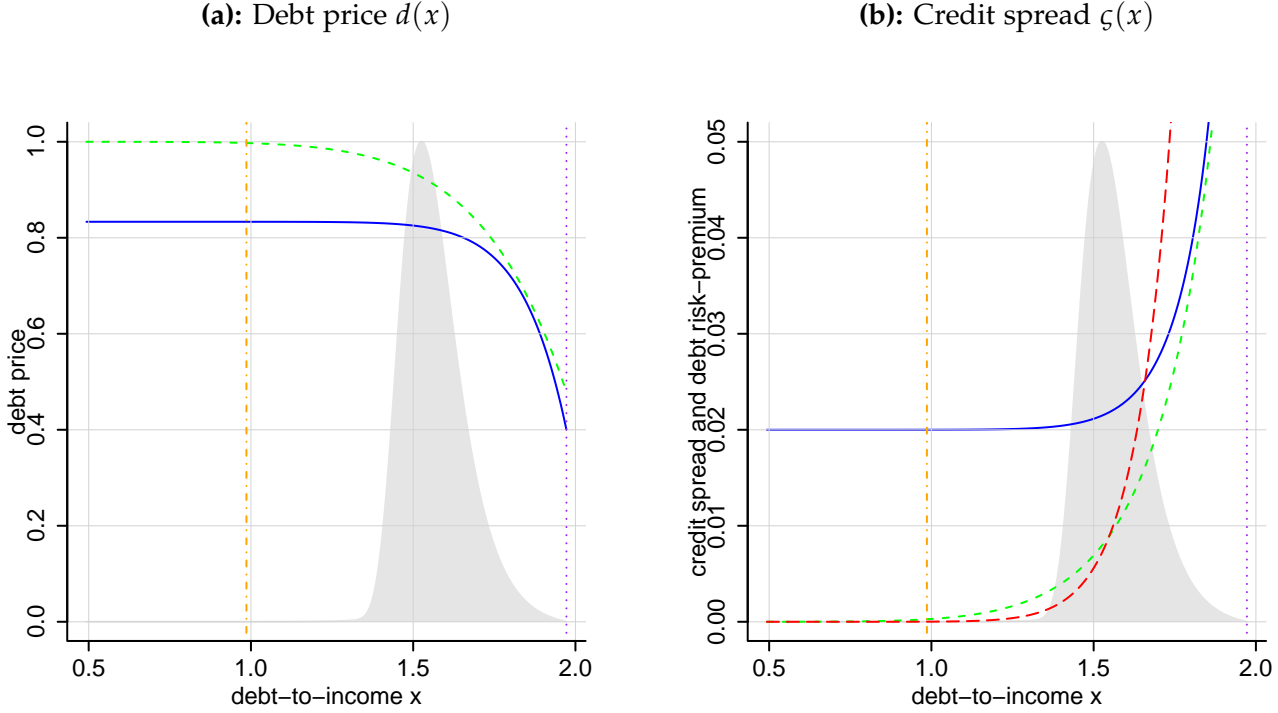
Solid blue line shows the no-commitment bond price and credit spread. Dash green line shows the bond price and credit spread in the corresponding model with no future issuance (or buybacks) of government debt. Long dash red line on the right plot shows the bond risk-premium; buybacks are optimal when the risk premium exceeds $\delta - r$ (the bond's initial credit spread). In both plots, the dotted purple vertical line is the default boundary, the dot-dash orange vertical line is the reinjection point, and the shaded grey area represents the stationary distribution of the debt-to-income ratio in our Smooth MPE. Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

equilibrium, the sovereign is actually buying back its own bonds in the states with high x_t.[34]

5.1.2 Comparative Statics: Analytical Results

Our model with closed-form expressions allows us to derive comparative static results; we summarize them in Corollary 4 with details provided in Online Appendix B.4.

Corollary 4 *The default boundary $\bar{x}$, the speed of reversion η, attraction point x_a and the government value function $v(x)$ (keeping fixed the debt-to-income ratio $x > 0$) admit comparative statics with respect to model parameters as given in Table 1.*

More impatient governments will default at lower debt-to-income ratios; indeed, they will front-load consumption via large bond issuances, and will adjust their debt level at a faster pace. Since they discount utility flow at higher levels, this leads to lower welfare.

When we reduce the punishment upon default (i.e. when $1 - \alpha$ decreases or θ decreases), the incentive for the government to default increases (so a lower default boundary in debt-to-income). This also makes the government unconditionally better off.

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[34]If $\nu = 0$, the government never buys back debt in the no-commitment equilibrium, and the bond price in that case is always strictly lower than the corresponding hypothetical "no-trade" bond price.

Table 1: Comparative Static Table

	Parameter	$\bar{x}$	η	x_a	$v(x)$
volatility	$|\sigma|$	$+$			$+$
income growth rate	μ	$+$	$+$		$+$
impatience	δ	$-$	$+$		$-$
coupon rate	κ	$-$	$0^{(1)}$		$-$
debt maturity	$1/m$	$+^{(2)}$	$-$		$+^{(3)}$
dead-weight costs at default	$1-\alpha$	$+$	0		$-$
debt restructuring at default	θ	$+$	0	$+$	$-$
risk-free rate	r	0	$-$	$-$	0
risk price	ν	0	$+$	$-$	0

(1) 0 indicates that the equilibrium object is independent of the parameter.
(2) If and only if $\kappa < \bar{\kappa} \in (\delta, +\infty]$, where $\bar{\kappa}$ depends on all other model parameters.
(3) If and only if $\kappa < \delta$.

An increase in debt interest payments κ leads to a lower default boundary as well as a lower welfare. To see the effect of m, a higher rate of principal amortization m (corresponding to a shorter debt maturity) increases the speed of debt-to-income mean-reversion. Moreover, shorter debt cuts the government's flow utility today but alleviates the sovereign's future debt burden (the drift of x_t decreases in m); we show that for low values of the coupon rate κ, the former dominates the latter, pushing the government to default earlier (i.e., a lower $\bar{x}$) and to enjoy lower welfare.[35]

The default boundary is increasing in the income volatility of the country, suggesting that the government is delaying its default. One can think of the indebted government as being long an American put option, whose value increases with volatility. This makes both the default boundary and the value of the government increasing functions of the small open economy's income volatility.

While capital markets' parameters r and ν do not influence government welfare or the default boundary, they play an important role in debt dynamics: an increase in r reduces the wedge $\delta - r$, leading to a decrease in the intensity of debt issuance and hence the speed of reversion η of the debt-to-income ratio and the attraction point x_a. When we increase the price of risk ν, the slope of the issuance intensity g gets steeper, increasing the adjustment speed η. At the same time, a higher price of risk tends to move bond prices lower, decreasing the incentive for the government to issue debt and ultimately lowering the debt-to-income attraction point x_a.

5.1.3 Equilibrium Uniqueness

Proposition 5 establishes the existence of the Smooth MPE, by construction. One might still wonder whether any other MPE might exist – in which for example the debt face value process was not absolutely continuous everywhere. The next proposition gives a negative answer.

Proposition 7 *When the small open economy's income process follows a geometric Brownian motion as in (28), the MPE in the state variable $x = F/Y$ is unique, which is the Smooth MPE described in Proposition 5.*

The uniqueness of our MPE (in the state variable x) is a consequence of the geometric Brownian motion assumption we made for the income process Y; for other income processes, MPEs other than

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[35]If one were to consider, as a state variable, the "debt coverage ratio" $z := (\kappa + m)x$, one can show that the default boundary $\bar{z} := (\kappa + m)\bar{x}$ and value function $\tilde{v}(z)$, keeping constant $z > 0$, are decreasing with the average debt maturity $1/m$.

the Smooth MPE might exist. Our proof, detailed in Online Appendix B.5, relies on several building blocks. We first show that in any equilibrium, the value function v must be convex, by observing that at any point in time the government is free to issue debt. This allows us to show that v is differentiable almost everywhere, and to establish the relationship $v'(x) + d(x) = 0$ in any MPE. We then partition the state space into three regions, depending on the size of sovereign credit risk-premia – (i) a low bond risk-premium region where the government always issues debt, (ii) an intermediate bond risk-premium region where the government does not trade, and (iii) a high bond risk-premium region where the government always buys back debt. This allows us to express the cumulative issuance policy in any MPE as the difference between two monotone processes. We then leverage Lebesgue decomposition theorem to show that in any MPE, it is never optimal for the government to use either an impulse or singular control strategy. This then allows us to focus on environments where the debt face value process is absolutely continuous, and prove that all MPEs must be Smooth MPEs.

5.1.4 Alternative Default Options

We have solved our model under a particular set of assumptions following a default: the small open economy's income Y drops by a constant fraction to αY, while the debt face value F is renegotiated to $\alpha \theta F$. We could alternatively calibrate the model based on an observed default threshold, $\bar{x}$, and creditor recovery value $\underline{d}$ estimated empirically. Below we characterize the Smooth MPE consistent with these outcomes and the given income process Y.

Lemma 5 *Consider any default threshold $\bar{x} > 0$ and debt recovery value $\underline{d} \in \left[0, \frac{\kappa + m}{\delta + m}\right)$. There exists a Smooth MPE with these outcomes, with scaled government welfare and equilibrium debt price given by*

$$v(x) = \frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m}x + \left(\frac{x}{\bar{x}}\right)^{\xi}\left(\frac{\kappa + m}{\delta + m} - \underline{d}\right)\frac{\bar{x}}{\xi}, \tag{41}$$

$$d(x) = \frac{\kappa + m}{\delta + m} - \left(\frac{x}{\bar{x}}\right)^{\xi - 1}\left[\frac{\kappa + m}{\delta + m} - \underline{d}\right]. \tag{42}$$

The debt issuance policy follows from the debt price according to (36). This equilibrium is supported by any government and creditor default payoff functions $\underline{v}(x)$ and $\underline{d}(x)$ satisfying $\underline{v}(\bar{x}) = v(\bar{x})$, $\underline{d}(\bar{x}) = \underline{d}$, and, for all x,

$$\underline{v}(x) \text{ is weakly convex, } \underline{v}(x) \leq v(x), \text{ and } \underline{d}(x) = -\underline{v}'(x) \geq 0. \tag{43}$$

This equilibrium is identical to that of Proposition 5 if $\bar{x}$ satisfies (31) and

$$\underline{d} = \left(\frac{\kappa + m}{\delta + m}\right)\left(1 - \frac{1 - \alpha\theta}{1 - \alpha\theta^{\xi}}\right). \tag{44}$$

The proof of Lemma 5 is almost identical to the proof of Proposition 5, and is thus omitted. Equations (41) and (42) highlight the impact of the government's default option on equilibrium welfare and debt prices. Condition (43) assures that there are no gains from trade embedded in the default process itself.[36]

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[36]Condition (43) is sufficient but not necessary. More generally, to assure it is never optimal to jump to default requires that for all x and x', $\underline{v}(x') + (x' - x)\underline{d}(x) \leq v(x)$. Also, if $\underline{d}(\bar{x}) < -\underline{v}'(\bar{x})$ then it will be optimal to stochastically delay default by

Fundamentally, Lemma 5 allows us to describe the family of solutions either through the efficiency and debt haircut parameters (α, θ) or through the default boundary and debt recovery values $(\bar{x}, \underline{d})$.[37] Finally, given equation (36), we note that conditional on $\underline{d}$, both the debt price $d(x)$ and the proportional debt issuance rate, $g(x)/x = G/F$, depend only on the ratio $x/\bar{x}$. Therefore, because the debt price is strictly monotone in x, the debt growth rate expressed as a function of the debt price (or equivalently, its yield) will be identical for all default regimes with the same debt recovery $\underline{d}$ in default.

5.2 Model Implications and Extensions

We next consider a variety of economic implications based on our model. The tractability of our setting allows us to analyze these issues analytically, as opposed to relying on numerical analysis. We then conclude with a number of empirically relevant extensions to the model.

5.2.1 Citizens vs. Government

We first investigate the case discussed in Section 4.8, in which citizens have a discount rate $\hat{\delta}$ that is lower than the government discount rate δ. We previously showed, for general income processes, that the citizens' indirect utility function is strictly lower than the indirect utility function if the country was stuck in financial autarky.

In Figure 5, we plot the percentage loss in welfare for the citizens of the country assuming the country has no initial debt. The left panel illustrates citizens' welfare loss as a function of the discount rate $\hat{\delta}$. Not surprisingly, the closer citizens' discount rate is to the government's, the lower the welfare loss versus the autarky benchmark.

In the right panel, we assume citizens are as patient as international investors ($\hat{\delta} = r$) and show that citizen welfare declines as debt maturity shortens. With shorter-term debt, the debt price is less sensitive to borrowing, causing the government to borrow more aggressively and consume at a higher rate than with longer-term debt. For the government, the welfare gains from consumption front-loading are perfectly offset by the future losses from default, irrespective of debt maturity. Citizens who are as patient as external credit markets would be indifferent between borrowing or staying in financial autarky, if they were able to choose the fiscal path. Steeper consumption profiles instead hurt them, and the shorter the debt, the steeper the consumption profile, and the greater the welfare losses.

5.2.2 Current Account and Consumption Growth Volatility

Our analytical expressions for the debt price d and the issuance policy g allow us to derive the consumption-to-output ratio $c(x) := 1 + g(x)d(x) - (\kappa + m)x$. The current account-to-income ratio is then simply $1 - c(x)$. Some algebra allows us to compute the derivative of the consumption-to-output ratio $c(x)$:

$$c'(x) = -(\kappa + m) - 2(\delta - r)d(x) + (\delta - r)\left(\frac{d(x)}{d'(x)}\right)^2 d''(x) - \nu \cdot \sigma \left(d(x) + xd'(x)\right).$$

repurchasing debt at $\bar{x}$ until the debt price at $\bar{x}$ just equals marginal cost.

[37]While for any given (α, θ) we can solve for the equivalent $(\bar{x}, \underline{d})$ from (31) and (44), the converse need not be true. For example, sufficiently high values of $\bar{x}$ may imply a negative default value for the government (implying that the government incurs cost beyond losing Y in default), which cannot occur in our renegotiation specification.

Figure 5: Welfare Losses $\hat{v}(0)/\hat{v}_0(0) - 1$

(a): As a function of impatience $\hat{\delta}$ **(b):** As a function of maturity $1/m$
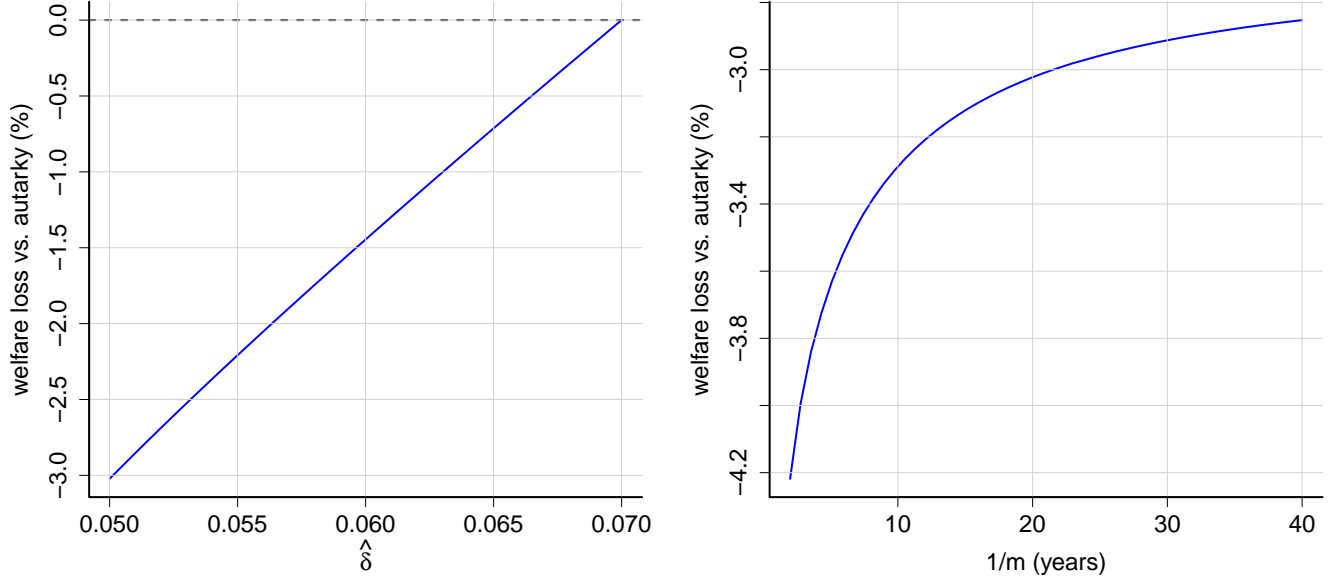
Solid blue line shows the percentage loss in welfare $\hat{v}(0)/\hat{v}_0(0) - 1$ evaluated when the government has no debt outstanding. Plots assume $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $\nu = 40\%$, $r = \hat{\delta} = \kappa = 5\%$ and $\delta = 7\%$.

When $\xi > 2$ (see Lemma 3), the debt price $d(x)$ is a concave function of the debt-to-income ratio. Thus, so long as $\nu \cdot \sigma$ is not "too large", the consumption-to-output ratio is strictly decreasing in the debt-to-income ratio. In other words, consistent with the empirical evidence presented in Arellano (2008) or Aguiar and Gopinath (2006), the current-account is counter-cyclical– a sequence of good income shocks will push the debt-to-income ratio downwards, causing the small open economy to run a current account deficit. One can also compute the ratio of consumption growth volatility divided by output growth volatility:

$$\frac{\text{stdev}_t\left(d \ln C_t\right)}{\text{stdev}_t\left(d \ln Y_t\right)} = 1 - \frac{x_t c'(x_t)}{c(x_t)}.$$

Since $c'(x) < 0$, the consumption growth volatility is greater than the output growth volatility across debt-to-income ratios. Our model is thus qualitatively consistent with the empirical evidence of Neumeyer and Perri (2005) or Aguiar and Gopinath (2004), who show that consumption growth volatility for emerging market economies is systematically greater than output growth volatility.[38]

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[38]Neumeyer and Perri (2005) for example compute ratios of standard deviations of total consumption over GDP for Argentina (1.17), Brazil (1.24), Korea (2.05), Mexico (1.29) and the Philippines (2.78).

5.2.3 Ergodic Distribution and Average Default Rate

Our model admits a stationary distribution f that is relatively straightforward to characterize. Recall that the debt-to-income ratio satisfies the stochastic differential equation

$$dx_t = \left(g(x_t) - \left(m + \mu - |\sigma|^2\right) x_t\right) dt - x_t \sigma \cdot dB_t + (\theta - 1) \bar{x} dN_{d,t}$$
$$:= \mu_x(x_t)dt + \sigma_x(x_t)dB_t + (\theta - 1)\bar{x}dN_{d,t},$$

with $\mu_x(x)$ the drift of x_t and $\sigma_x(x)$ its volatility. Let $J(x)$ be the probability flux associated with $f(x)$:

$$J(x) := \mu_x(x)f(x) - \frac{d}{dx}\left[\frac{\sigma_x^2(x)}{2}f(x)\right], \tag{45}$$

which can be interpreted as the "mass of particles per unit of time" that crosses at x.[39] $J(x)$ satisfies the Kolmogorov-Forward equation $\frac{dJ}{dx} = 0$ for $x \in (0, \theta\bar{x})$ and for $x \in (\theta\bar{x}, \bar{x})$, with $J(\bar{x})$ yielding the ergodic average default rate of the sovereign. Given that $\bar{x}$ is an exit point of the system, we have $f(\bar{x}) = 0$, implying an ergodic default rate $\lambda_d = J(\bar{x}) = -\frac{\sigma^2 \bar{x}^2}{2}f'(\bar{x})$. Let $a > 0$ and b be two constants given by:

$$a := \frac{2x_a^{\xi-1}}{|\sigma|^2}\left(\frac{\eta}{\xi-1}\right), \qquad b := \frac{2}{|\sigma|^2}\left(\frac{\eta}{\xi-1} + \mu - |\sigma|^2\right).$$

In Online Appendix B.6, we solve the Kolmogorov-Forward equation, leading to our next proposition.

Proposition 8 *In our Smooth MPE, the boundary $x = 0$ is a natural boundary, meaning that it is not attainable.[40] The ergodic density is equal to*

$$f(x) = \lambda_d \int_{\max(\theta\bar{x},x)}^{\bar{x}} \left(\frac{u}{x}\right)^b \exp\left(\frac{-a}{1-\xi}\left(u^{1-\xi} - x^{1-\xi}\right)\right) du.$$

The ergodic default rate λ_d is the unique positive real number satisfying $\int_0^{\bar{x}} f(u)du = 1$.

This expression allows us to calculate various model-implied moments of interest to macro-economists – in Online Appendix B.8, we focus on the ergodic default rate, bond credit spread, and debt-to-income ratio (as a function of various model parameters) for illustration. Several comparative statics are worth mentioning. For instance, income volatility σ has an ambiguous impact on ergodic average default rates. On the one hand, keeping the government financing policy and the default boundary fixed, the higher the income volatility, the more likely to hit the default barrier, and hence the greater the ergodic default rate. On the other hand, with a higher income volatility, the government defaults "later", pushing the default frequency downwards. For our parameter choice, these two effects almost offset each-other. Instead, ergodic average credit spreads increase with σ, since on average, the government spends more time at higher debt-to-income levels, closer to the default boundary. For our parameter choice, we also notice that a shorter maturity $1/m$ leads to higher default frequencies, but lower average credit

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[39]One can interpret our stochastic differential equation for x_t as describing the movement of particles getting hit by idiosyncratic Brownian shocks.

[40]The boundary $x = 0$ would have been "attainable" if the process x_t, started at an arbitrary point $x > 0$, can reach $x = 0$ in finite time almost surely.

spreads.[41] Our analytical characterization of the ergodic density is an additional tool that improves macro-economists' ability to quickly calibrate sovereign default models, by avoiding time-consuming and sometimes inaccurate simulation methods when computing model-implied moments of interest.

5.2.4 Time-Varying International Capital Market Conditions

For simplicity, in this section we have so far assumed constant risk-free rates and risk-prices for international creditors. Suppose instead that risk-free rates and risk-prices are time-varying; in particular, they depend on international capital market conditions s_t as in equation (14) so that $r_t = r(s_t)$ and $v_t = v(s_t)$. Using our remark at the end of Section 4.6, the government value function must be independent of the international capital market state s_t, and thus equal to formula (33). Given the optimality condition (20), the bond price must also be independent of the state s_t, and thus equal to formula (32). The international capital market state matters only for the country's debt issuance policy, as shown in Section 4.7.2.

Corollary 6 *With time-varying risk-free rates and risk-prices, the government financing policy becomes*

$$g(x,s) = \frac{\delta - r(s)}{\epsilon(x)} - v(s) \cdot \sigma x.$$

Upward shocks to risk-free rates or risk-prices lead the government to adjust its financing policy downwards.

This observation is qualitatively consistent with various sovereign debt crisis episodes and large current account adjustments. As an example, the Latin American sovereign debt crisis, that began in 1982 with the default of Mexico, occurred at a time US interest rates had been increased significantly by Paul Volker. In response, Latin American economies reduced their hard currency bond issuances, with certain countries even buying back their own debt at deep discounts.[42] Similarly, the Asia Tiger crisis of 1997 could be interpreted, through the lens of our model, as a sudden rise in risk-prices, which led Indonesia, Malaysia and the Philippines to go through a rapid current account adjustment.

5.2.5 GDP-Linked bonds

We finally extend our analysis to study an environment where the debt contract has some state-contingency. Consider GDP-linked bonds, whose principal balance goes up and down with income shocks – and denote ϕ the sensitivity of the debt face value to such shocks. In this environment, F_t evolves as follows:

$$dF_t = (G(Y_t, F_t, s_t) - mF_t)\, dt + F_t \phi \cdot d\boldsymbol{B}_t. \tag{46}$$

For example, a government could issue bonds that "share" some of the small open economy's income risk with international investors, so that $\phi \propto \sigma$. Our treatment of GDP-linked bonds is covered in Online Appendix B.7. We summarize here the main result.

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[41] We do not claim that these numerical comparative statics are global; in fact, we have found alternative parameter specifications with sensitivities of the opposite sign.

[42] See for example Bulow and Rogoff (1988) for a detailed discussion on such buy-backs and debt-for-equity swaps by Mexico, Chile, Brazil and Bolivia.

Proposition 9 *The economy where the government issues GDP-linked bonds whose face value follows (46) is isomorphic to an economy in which (a) the government issues non-state-contingent bonds, (b) the country's income growth risk has volatility vector $\sigma - \phi$, and (c) creditor's interest rate is $r(s) + \phi \cdot v(s)$.*

This isomorphism is intuitive. A country with income risk σ, that hedges a component ϕ of that risk via GDP-linked bonds, is identical to the one whose income risk is "only" $\sigma - \phi$; since investors pricing kernel might be correlated to the aggregate risk they are now assuming, their cost of capital is distorted upwards by $\phi \cdot v(s)$.

Because the Smooth MPE remains, our welfare conclusions are unchanged: there is no welfare gain from issuing GDP-linked bonds. That said, the implied volatility reduction and higher implied cost of capital will influence equilibrium policies, such as the target debt-to-income level, speed of adjustment, and default rate, all of which can be solved for using the methods of this section.

6 Restoring Gains from Trade

As discussed in Section 3, one might be able to restore gains from trade for the small open economy by introducing trading frictions, constraints, or curvature in the government utility function. In this section, we consider several of these devices and study whether they enable the government to re-capture any of these potential gains. To simplify the exposition, we assume going forward that the international risk-free rate is constant equal to r, and that aggregate risk is not priced – i.e. $v = 0$.

6.1 Trading Frictions

We begin with the study of two trading frictions; in one setting, the government has the ability to issue lump amounts of debt at random *points in time*; in the other, the government can issue debt continuously, but might be locked out of capital markets during exponentially distributed times. For simplicity, we only consider the case $\alpha = \theta = 0$.

6.1.1 Infrequent Debt Issuance Opportunities

A natural place to start our study of commitment devices is an environment almost identical to the heuristic "discrete time" discussion of Section 3. For tractability, and perhaps more realistically, rather than modeling trading times as discrete, suppose the government receives opportunities to trade at Poisson arrival times with intensity $1/\Delta$ (where Δ represents the average time between trading dates). Between trades, debt continues to mature and income evolves, so that the country's debt-to-income ratio shrinks at rate $m + \mu$. But at each debt adjustment opportunity, the government can issue (or buy back) "lump" amounts of debt that depend on the level of indebtedness at that time. This trading constraint is thus similar to the "Calvo" friction in the sticky price literature (see Calvo, 1983) or the model of inattentive consumers in the rational inattention literature (see Reis, 2006). The inability of the government to rebalance its debt at any time introduces certain commitment power, and thus welfare gains from trade. In this section, we study the magnitude of those gains from trade as the average time between trades Δ changes.

The MPE is characterized by two delayed differential equations discussed in Online Appendix C.1.1. The first is the government's HJB equation solved by its value function $v_\Delta(\cdot)$ in the "continuation region," together with its optimal financing strategy specifying the debt-to-income ratio $n_\Delta(x)$ that the government optimally "jumps to" upon a debt adjustment opportunity. The second is an asset pricing equation pinning down the debt price function d_Δ, given the sovereign's debt issuance behavior.

Figure 6 illustrates how government's initial welfare, starting with no debt, varies with the average trading time interval Δ, for three different debt maturities. When $\Delta \to +\infty$, the government almost never has opportunities to issue bonds, and its no-debt welfare is close to its autarky value. Similarly, as $\Delta \to 0$, the speed of trading opportunities becomes arbitrarily large, and the lack of commitment again causes government welfare to approach its autarky value. Thus, welfare $v_\Delta(0)$ is a hump-shaped function of Δ, and is maximized at a trading frequency that depends upon the debt maturity. As the average debt maturity decreases, the optimal trading frequency and maximum "achievable" welfare both increase. Intuitively, shorter trading interval means less "externally-imposed" commitment; short-term debt, which forces the government to pay down its principal balance faster, has a relatively stronger commitment power, thus requiring less "externally-imposed" commitment to restore trading gains.

To better understand the effect of commitment on welfare, we also show in Figure 6 the welfare that can be achieved by a government with *full commitment* to a target debt-to-income ratio x^* but that still retains the option to default. Online Appendix C.1.2 characterizes this commitment equilibrium and the target x^* which is chosen optimally ex ante. As in our main model, the government can choose to default at any time, but absent default it must rebalance to x^* at each trading date.[43] As Figure 6 highlights, as the frequency of trade increases, commitment leads to substantially higher welfare for the government in this alternative setting. In the limit as $\Delta \to 0$, all gains from trade are extracted as the optimal debt-to-income ratio converges to $1/(r - \mu)$, the first-best value of the government's income stream. The comparison between this benchmark and our equilibrium also emphasizes that our results are driven by the lack of commitment over future debt issuances, rather than over future defaults.

### 6.1.2 Markov-Switching Restricted Issuance

The previous analysis shows that welfare gains can be reintroduced when the government is only allowed to issue debt at infrequent times – as is the case in discrete time models. We now show that the institutional arrangement of such trading restrictions is important in obtaining welfare gains.

To illustrate our point, consider the following 2-state Markov-switching economy. In the unrestricted state (state "u"), the government is entirely free to issue debt, while in the constrained state (state "c"), the government is prohibited from issuing any debt. These two Markov states switch back-and-forth at specific Poisson arrival times, with arrival intensities λ_u and λ_c respectively. One can also think about this economic environment from a credit supply standpoint: state c can be thought of as a sudden stop state, during which international capital markets are shut for the small open economy, while state u can be thought of as a normal state, during which international capital markets are functioning normally.

Different from Section 6.1.1 in which the government can issue bonds only at Poisson events (which occur in a discrete fashion), in this setting once international capital markets are in the "u" state, the

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[43] In this model, there will be two default boundaries: one that applies between trading dates, and a tighter one that applies on the trading date when the government, who is forced to rebalance by repurchasing outstanding debt back to x^*, finds it optimal to default.

Figure 6: Welfare sensitivity to trading time and debt maturity
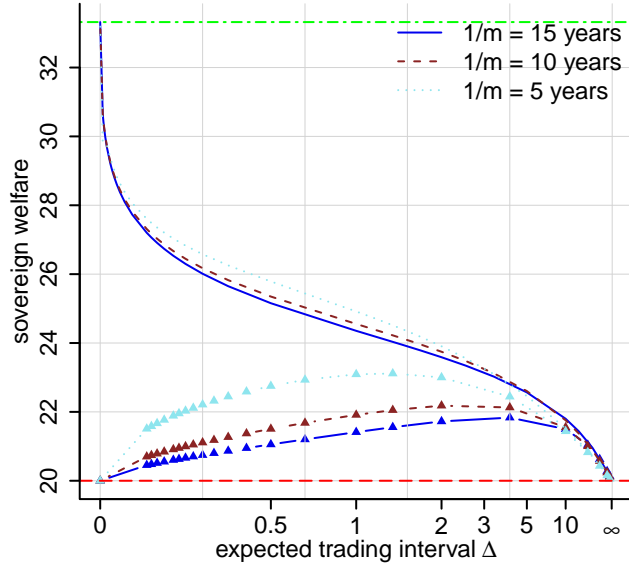
Plot shows sovereign welfare assuming no initial debt and infrequent trading without commitment (lines with triangles) and with commitment (lines without triangle), as a function of the expected trading interval Δ, for different average debt maturity parameters. Plots computed assuming $\mu = 2\%$ p.a., $\sigma = 20\%$ p.a., $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$, $\delta = 7\%$, and $1/m$ equal to 5 years, 10 years and 15 years. For these parameters, the first-best value per dollar of government income is $1/(r - \mu) = 33.3$.

government can issue sovereign bonds at *any time* before international capital markets revert back to the "c" state. The next proposition shows that the government receives no gain from trade in equilibrium, just like our no-commitment benchmark, while creditors benefit (i.e. their investment market value increases) whenever the government is constrained.

Proposition 10 *There exists a Smooth MPE of this economy, where the government value function v_c in state "c" and v_u in state "u" are equal to the no-commitment value function v:*

$$v_u(x) = v_c(x) = v(x).$$

In such economy, the debt price satisfies:

$$d_c(x) > d_u(x) = d(x).$$

Issuances $g_u(x)$ in the unconstrained state are greater than issuances $g(x)$ in the no-commitment MPE:

$$g_u(x) = g(x) + \lambda_u \frac{d_c(x) - d_u(x)}{-d'_u(x)} \qquad \forall x \leq \bar{x}.$$

Closed form expressions are provided in Online Appendix C.2.

Proposition 10 is proven in Online Appendix C.2. Intuitively, in this environment, whenever capital markets are open, they will stay open in the next instant almost surely, prompting the government to use a smooth issuance strategy that wipes out gains from trade. This aspect crucially differs from our

Figure 7: Debt price and issuance policy in Markov-switching economy

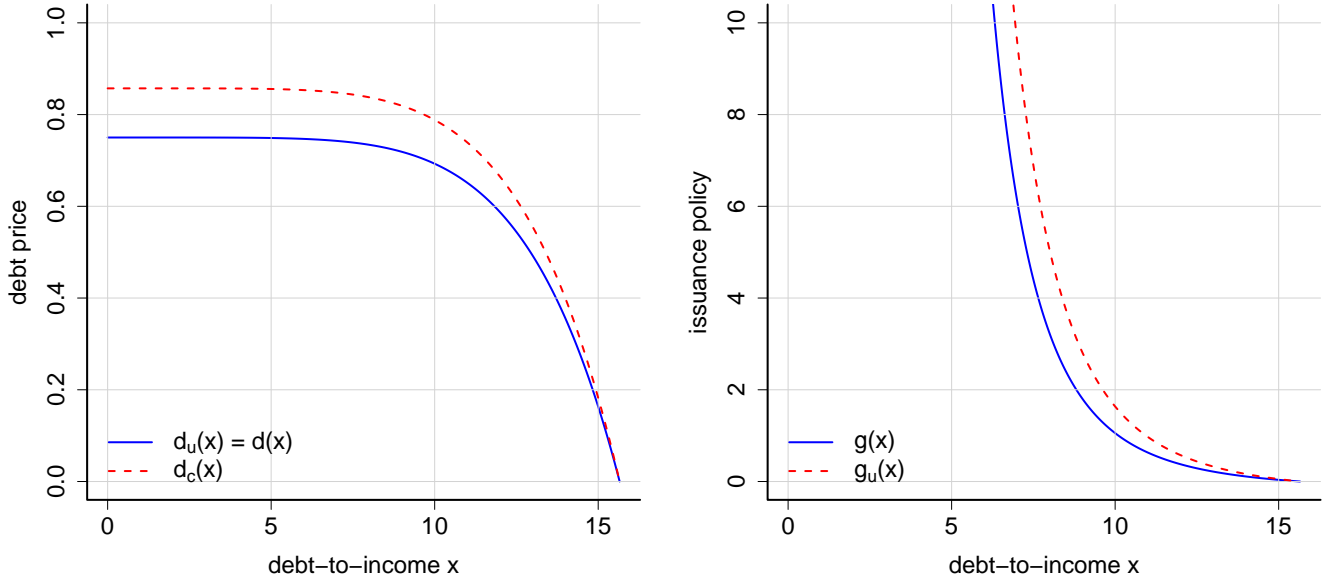**(a):** Debt Price   **(b):** Issuance Policy

Figure (a) shows the debt price d_u (resp. d_c) in the unrestricted (resp. constrained) state. Figure (b) shows the issuance policy g_u in the unrestricted state, compared to the issuance policy in the no-commitment equilibrium. The plot was computed assuming $\mu = 0\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 10\%$.

previous case of infrequent trading opportunities of Section 6.1.1, in which whenever a Poisson trading time arrives, the government – knowing that it will afterwards be locked out of capital markets for a positive measure of time – issues a positive measure of debt.

Figure 7 illustrates that the equilibrium debt price d_u in the unconstrained state is strictly below the debt price d_c—creditors understand that in such state c, no bond issuances occur, meaning that indebtedness grows slower than in the unconstrained state u. From the government's perspective, knowing that it will be limited in its ability to issue bonds in state c, the government adjusts its issuance policy $g_u(x)$ *upward* in the unrestricted state compared to the no-commitment economy (i.e., $g_u(x) > g(x)$). Proposition 10 formally shows that debt issuances in state u are so large that they completely offset the gains from commitment (of no debt issuance) in state c.

One implication of Proposition 10 might seem counter-factual, as the debt price in the sudden-stop state "c" is *higher* than in the unconstrained state "u." This stems from the assumption that the market price of risk is constant (and equal to zero) across capital market states. One could instead make the more empirically reasonable assumption that the market price of risk jumps when the economy transitions into the sudden-stop constrained state c. In Online Appendix C.2, we show that if the market price of risk in the constrained state "c" is strictly positive, then the debt price d_c in such state is strictly less than the debt price d_u in the unconstrained state u for sufficiently high debt to income ratios. As a result, if sudden stops coincide with jumps in risk premia, we expect the debt price of highly indebted countries to fall, while those of less indebted countries rise.

6.2 Financing Constraints

So far, we have considered exogenous constraints on trade. In this section, we consider financing restrictions that may be adopted as matters of policy, and which the government may help to control or design, such as debt ceilings or issuance constraints. Similar commitment devices have been studied previously in the sovereign debt literature (see for example Hatchondo, Martinez and Roch, 2012), as well as the literature focused on fiscal rules and commitment problems (see Halac and Yared, 2014 or Halac and Yared, 2018). We continue to assume $\alpha = \theta = 0$.

6.2.1 Debt Ceiling Policies

Imagine that the government alters its constitution, such that it limits future governments from being able to issue additional bonds once the debt-to-income ratio x_t exceeds a certain limit x^*. When $x_t < x^*$, the government has full flexibility to issue additional bonds in international capital markets. But when $x_t > x^*$, no new debt can be issued (but there is no requirement to immediately reduce debt). Such restrictions can be found in many advanced economies. In the U.S. for example, the debt ceiling is a legislative limit on the notional amount of national debt that can be incurred by the U.S. Treasury. While such limit is not expressed as a percentage of GDP (as it is in our model in order to preserve its homogeneity properties), it achieves a purpose similar to the one we intend to study here. Similarly, under the Maastricht Treaty, member states must maintain their debt-to-GDP ratio below 60%.

We first show that a "debt ceiling" commitment technology does not always provide welfare gains for the government. If the debt-to-income limit x^* is too high, then the improvement in debt prices is not sufficient to allow the government to benefit. In that case, a Smooth MPE continues to exist, and the value function v_c of the government *across the entire state space* is identical to the value function v in the no-commitment economy. Creditors, however, do benefit from higher debt prices when the debt-to-income ratio is above the cutoff x^*.

If the debt ceiling x^* is low enough, the government can monetize the welfare benefits from being able to borrow from patient lenders. In this case, once the debt-to-income ratio exceeds x^*, it varies only with income shocks, except for being reflected — by immediate new debt issuances — at the lower bound x^*. In that case, the sovereign benefits by being able to issue debt that restores the debt-to-income to x^* at a higher debt price.

Proposition 11 *There exists two cutoffs $\underline{x}^*, \bar{x}^*$, with $0 < \underline{x}^* \leq \bar{x}^* < \bar{x}$ (with possibly $\underline{x}^* = \bar{x}^*$), such that:*

1. Loose debt-ceiling: *if $x^* > \bar{x}^*$, there exists a Smooth MPE in which the government value function $v_c(x) = v(x)$ without gain from trade, in which the debt price satisfies $d_c(x) = d(x)$ when $x < x^*$ and $d_c(x) > d(x)$ when $x > x^*$. In such MPE, the debt price $d_c(x)$ is continuously differentiable on $[0, \bar{x}]$ except at $x = x^*$, where it is continuous but exhibits a convex kink;*

2. Moderate debt-ceiling: *if $x^* \in [\underline{x}^*, \bar{x}^*]$, there exists a regulated equilibrium of this economy. In such equilibrium, the government uses a singular issuance strategy to maintain its debt-to-income ratio above x^*. There exists a second endogenous cutoff $\hat{x} \in (0, x^*)$, such that if $x \in (0, \hat{x})$, the government follows a smooth debt issuance strategy, while if $x \in (\hat{x}, x^*)$, the government jumps to x^* by issuing a lump amount of debt; the value function $v_c(x) > v(x)$ except at $x = 0$, where both values coincide;*

3. **Tight debt-ceiling**: *if $x^* < \underline{x}^*$, there exists a regulated equilibrium of this economy. In such equilibrium, the government uses a singular issuance strategy to maintain its debt-to-income ratio above x^*. The government jumps to the debt-to-income level x^* whenever $x \in (0, x^*)$; the government value function $v_c(x) > v(x)$ everywhere, achieving a welfare gain compared to its no-trade value.*

Closed form expressions are provided in Online Appendix C.3 for all cases.

Figure 8: Debt price in loose and moderate debt ceiling policies

(a): Loose debt ceiling   **(b):** Moderate debt ceiling
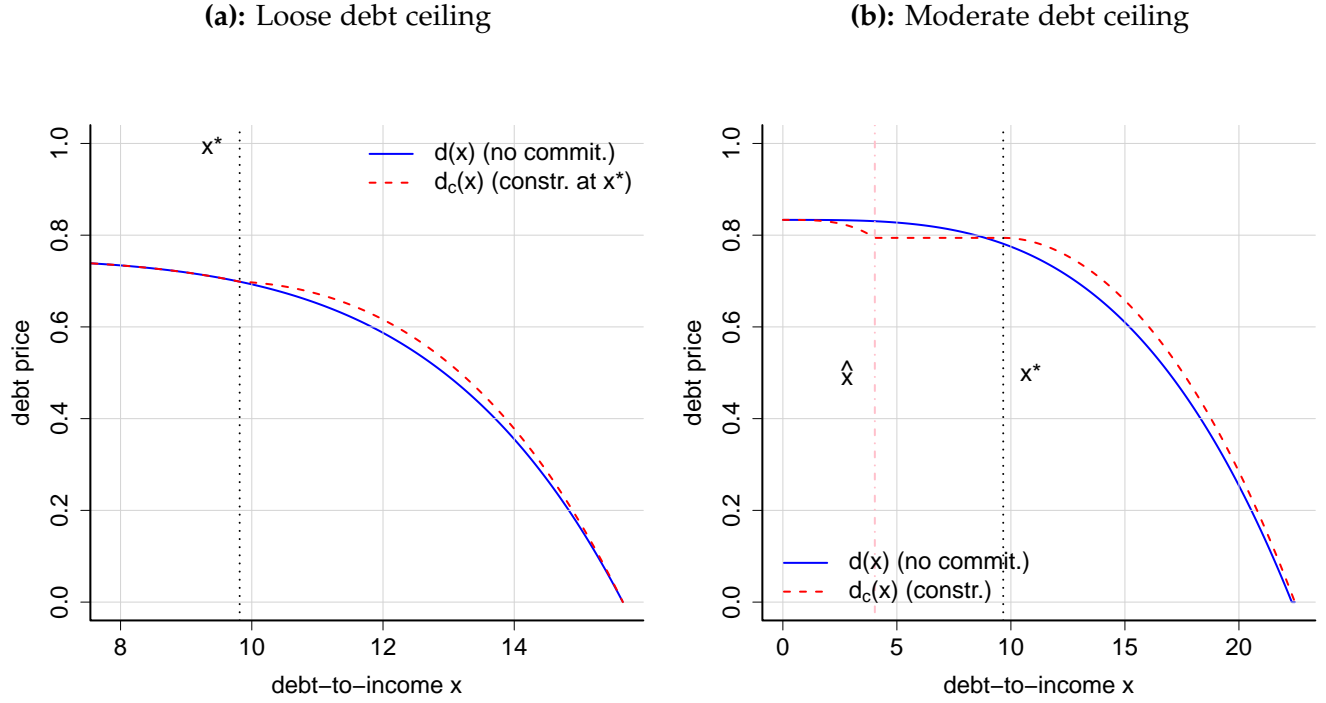
Both figures show the debt price in the debt-ceiling equilibrium (dash red line) and in the no-commitment equilibrium (solid blue line). In both cases, we set $\mu = 0\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, and $r = \kappa = 5\%$. Figure (a) illustrates a loose debt ceiling policy, which is the relevant equilibrium when we set $\delta = 10\%$ and a debt ceiling threshold $x^* > \bar{x}^*$. Figure (b) illustrates a moderate debt ceiling policy, which is the relevant equilibrium when we set $\delta = 6\%$ and a debt ceiling threshold $x^* \in (\underline{x}^*, \bar{x}^*)$.

Figure 8a illustrates the shape of the debt price function $d_c(x)$ for a loose debt-ceiling policy (i.e. $x^* > \bar{x}^*$, the case where the Smooth MPE still exists). The debt price $d_c(x)$ is identical to the no-commitment debt price $d(x)$ for $x \leq x^*$, while $d_c(x) > d(x)$ holds for $x > x^*$. As discussed in Online Appendix C.3, the debt price exhibits a convex kink, and the right-derivative of d_c at $x = x^*$ is strictly greater than its left-derivative.[44]

As we reduce the policy parameter x^*, the kink in the debt price at $x = x^*$ becomes so severe that past the limiting value $\bar{x}^*$, the debt price is no longer monotone decreasing in x, and our Smooth MPE no longer exists. For $x^* < \bar{x}^*$, we necessarily have to look for equilibria featuring jumps. In Figure 8b, we give an illustration of the debt price in the MPE for a moderate debt ceiling policy, i.e. $x^* \in (\underline{x}^*, \bar{x}^*)$. On the interval $x \in (\hat{x}, x^*)$, the debt price function is flat (and the value function is linear), with a level $d(x^*)$

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[44]The no-arbitrage condition puts a restriction on the issuance policy at the point x^*, so that the resulting debt-to-income process needs to be a skew-Brownian motion. The debt issuance strategy is such that the excursions of the state variable x_t at x^* are asymmetric, with probability $\frac{1+\chi}{2}$ to the left and with probability $\frac{1-\chi}{2}$ to the right, where $\chi \in (0,1]$. Note that when $\chi = 1$ this boundary corresponds to a reflection boundary, while $\chi = 0$ corresponds to standard Brownian motion. For more details, see in Harrison and Shepp (1981).

that is strictly below the critical value $\frac{\kappa+m}{\delta+m}$, as the government finds it optimal to jump immediately to the debt-to-income level x^*. On $x \in (0, \hat{x})$, the government follows a smooth debt financing policy, with a value function $v_c(x)$ that is weakly greater than $v(x)$ – both functions are equal when the government is not indebted (i.e. $v(0) = v_c(0)$).

Figure 9: Reflecting equilibrium with "tight" constraint x^*

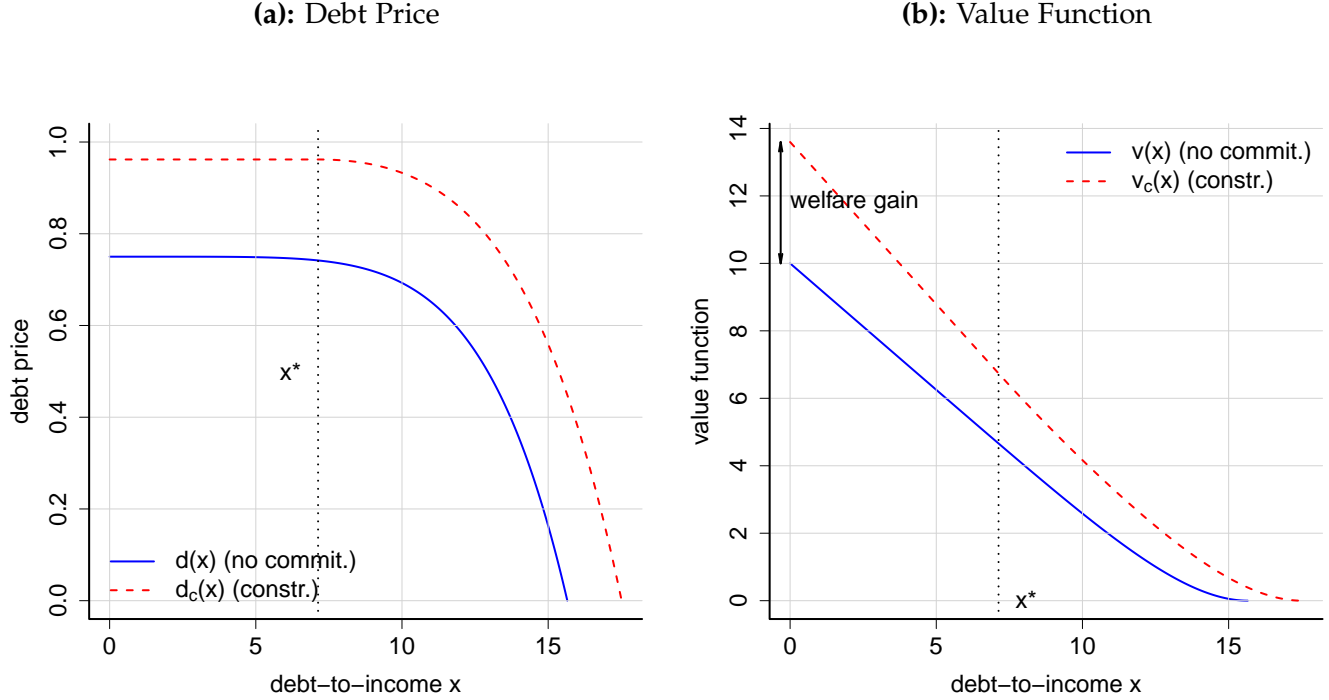**(a):** Debt Price   **(b):** Value Function

Figure (a) shows the debt price in the constrained equilibrium (dash red line) and in the no-commitment equilibrium (solid blue line). Figure (b) shows the scaled value function in the constrained equilibrium (dash red line) and in the no-commitment equilibrium (solid blue line). The plot was computed assuming $\mu = 0\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 10\%$. The threshold x^* is set to satisfy $x^* = 0.80\bar{x}^*$, in other words $x^* < \underline{x}^*$.

Finally, Figure 9 gives an illustration of the MPE given a tight debt ceiling policy $x^* < \underline{x}^*$. The debt price d_c and the government value function v_c are uniformly higher than in the no-commitment equilibrium, and one can see the government welfare gain (without debt outstanding) graphically as the distance between $v_c(0)$ and $v(0)$. For $x < x^*$, the debt price satisfies $d_c(x) = d_c(x^*)$, since creditors anticipate that the government will issue a lump amount of debt in order achieve a debt-to-income ratio equal to x^*, while the scaled value function satisfies $v_c(x) = v_c(x^*) + (x^* - x)d_c(x^*)$ – in other words the value function is linear in the debt-to-income ratio when $x < x^*$.

Figure 10 shows the welfare gains vs. the no-commitment MPE for a range of choices of constraints x^*, for the government and citizens discounting cashflows at two possible rates: $\hat{\delta}_1$ that is equidistant from r and δ, and $\hat{\delta}_2 = r$.

Several insights can be drawn from this analysis. First, when the debt ceiling $x^* = 0$ and the government commits to never issuing any debt, or $x^* \geq \bar{x}^*$, i.e. above the limiting case beyond which a Smooth MPE exists, the welfare of the government always equals its no-trade value. While the government does not gain whenever the policy $x^* > \bar{x}^*$, citizens instead always increase their welfare, since they are more patient than the government and since those policies end up curbing the government's debt issuance behavior. The more patient citizens are, the more they gain. While there exists an optimal policy x^* at

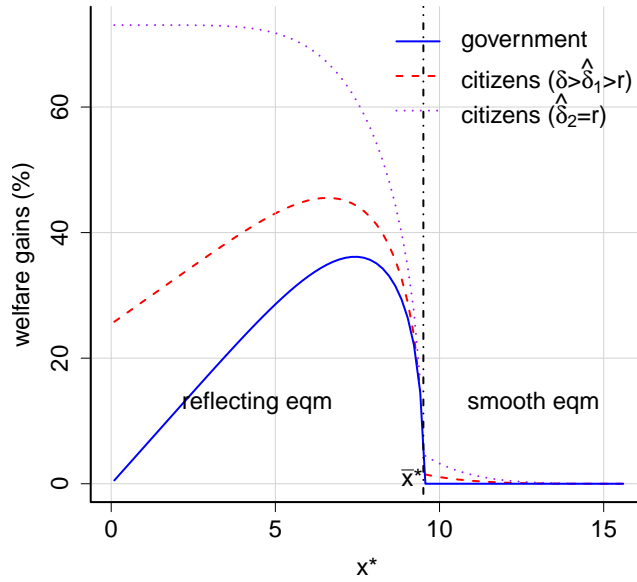**Figure 10:** Debt ceiling policies' impact on welfare

Figure shows the percentage gain in welfare (for the government and for citizens with different discount rates). The plot is computed for a range of values for the threshold x^*, with $\mu = 0\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 10\%$.

which the government welfare improvement is highest, such optimal policy does not correspond to the policy that would be chosen if instead the objective was to maximize citizens' welfare.

6.2.2 Issuance Rate Cap

In Section 6.2.1, we analyzed a constraint targeting the stock of government debt. In this section, we instead discuss a commitment device that prevents the government from issuing bonds at an intensity greater than an exogenously specified cap. We discussed this device briefly in Section 3.5 and showed that its ability to deliver welfare gains depends on a subtle trade-off between (a) the tightness of the constraint and (b) the size of the issuance cap. To accommodate the scale invariance of our model, we focus on caps of the form $G_t \leq \bar{g} Y_t$, i.e., the bond issuance rate (per unit of income) $g(x)$ is capped by $\bar{g} > 0$. Policies of the type $g(x) \leq \bar{g}$, targeting the government flow budget constraint rather than the stock of outstanding government debt, have empirical relevance. In a mapping of our model where C is government spending and Y is government tax revenue, a cap on the rate of bond issuances is a close analog to a cap on budget deficits, as is the case under the Maastricht Treaty (where member states must maintain their deficit below 3% of GDP). In a mapping of our model where C is domestic consumption and Y is GDP, a cap on the rate of bond issuances can be thought of as a limit on the current account deficit (as a fraction of GDP), a measure of economic imbalance frequently targeted by the IMF in connection with emergency lending programs.[45]

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[45]In both model interpretations, a cap on budget deficit to GDP (or current account deficit to GDP) is not exactly identical to a cap on the bond issuance intensity. A cap on budget deficit as in the Maastrich treaty would be of the form $C - Y \leq bY$, with $b = 0.03$. This constraint is equivalent to a constraint of the form $g_t \leq (b - (\kappa + m)x_t)/d_t$, which is similar to the constraint studied in this paper when the debt price d_t is close to par and when debt service payments to GDP are small vs. b.

When $\xi > 2$ (see Lemma 3), the issuance rate in our no-commitment benchmark is decreasing in x, and unbounded as $x \to 0$. This implies that in the economy with bond issuance caps, the constraint is binding for low debt-to-income states (i.e. for $x \in [0, x^*]$, for some endogenously determined debt-to-income hurdle x^*), but slack for high debt-to-income states (i.e. for $x \in [x^*, \bar{x}_c]$, for some endogenously determined default boundary $\bar{x}_c$). We summarize below our key result.

Figure 11: Cap on bond issuance rate – debt price and value function

(a): Debt Price $d_c(x)$

(b): Value Function $v_c(x)$
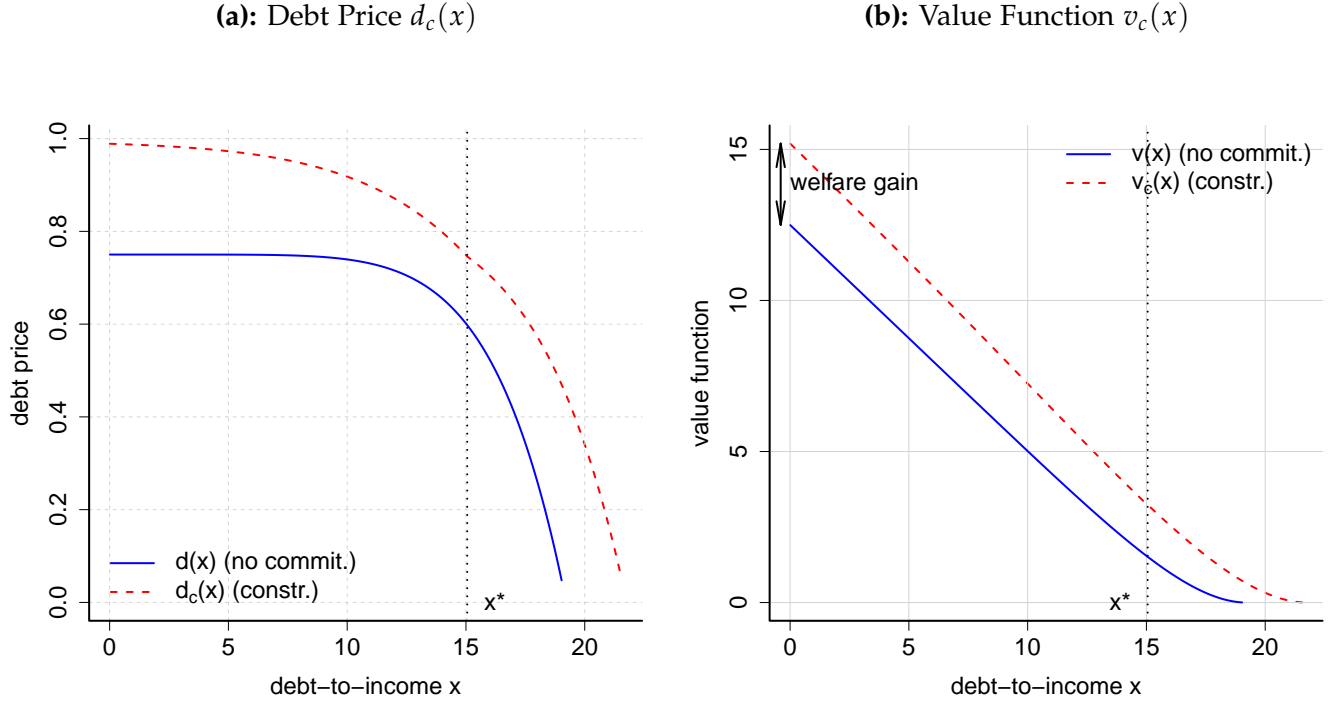
Debt price in (a) and value function in (b), for constrained (dash red line) and unconstrained equilibrium (solid blue line). Plot computed with $\mu = 2\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$, $\delta = 10\%$ and $\bar{g} = 100\%$.

Proposition 12 *Subject to the existence of a solution to a set of 2 algebraic equations in 2 unknowns disclosed in Online Appendix C.4, there exists two endogenous cutoffs $x^*, \bar{x}_c$, with $0 < x^* < \bar{x}_c$ such that:*

1. *When $x \in (0, x^*)$, the government financing is constrained at $g_c(x) = \bar{g}$;*

2. *When $x \in (x^*, \bar{x}_c)$, the government is unconstrained, uses a smooth issuance policy $g_c(x) < \bar{g}$, and defaults optimally when $x = \bar{x}_c$. On this interval, the debt price satisfies $d_c(x) = -v_c'(x)$.*

In both intervals, the debt price and value function are analytic, with expressions given in Online Appendix C.4. The no-debt government welfare is strictly greater than the autarky welfare for any $\bar{g} \in (0, +\infty)$.

In our proof of Online Appendix C.4, we construct the constrained MPE explicitly, by solving the HJB equation satisfied by v_c and d_c on the constrained and unconstrained domains. In Figure 11, we plot the debt price and value function together with those in the no-commitment equilibrium. Both the debt price and value function in the constrained equilibrium are uniformly higher than in the no-commitment case, and default occurs at a higher debt-to-income ratio. While the value function is $\mathcal{C}^1$ at $x = x^*$, the debt price features a kink. In Figure 12, we show the resulting consumption-to-income and issuance-to-income policies. The issuance rate is capped at $\bar{g}$ when $x < x^*$, and is unconstrained, decreasing (as a
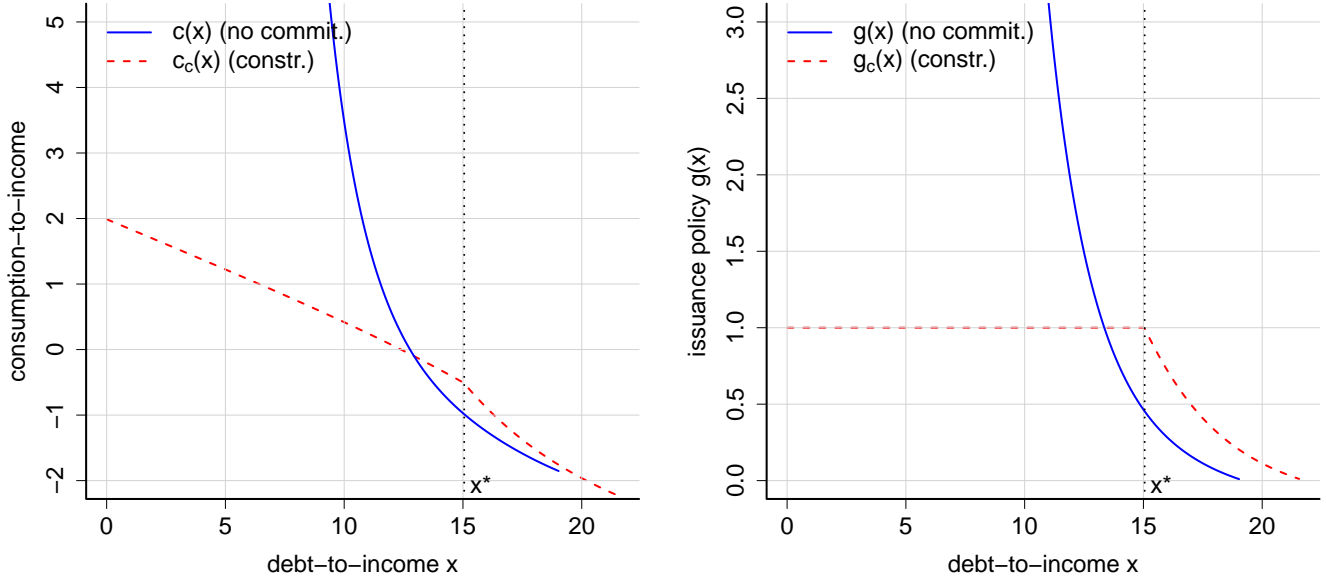
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function of x) when $x > x^*$. At the default boundary, since the debt price is zero and the gains from trade are proportional to the debt price, there are no incentives to issue debt, which means $g(\bar{x}_c) = 0$.

Figure 12: Cap on bond issuance rate – consumption and financing policy

(a): Consumption-to-income $c(x)$ **(b):** Issuance Rate $g(x)$

Consumption-to-income in (a) and issuance rate in (b), for constrained (dash red line) and no-commitment equilibrium (solid blue line). Plot computed with $\mu = 2\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$, $\delta = 10\%$ and $\bar{g} = 100\%$.

Figure 13 shows the welfare gains vs. the no-commitment MPE as we vary the policy $\bar{g}$, for the government (discounting cashflows at δ) as well as for citizens (discounting cashflows either at rate $\hat{\delta}_1$ that is equidistant from δ and r, or at rate $\hat{\delta}_2 = r$). When $\bar{g}$ is small, the government is allowed to issue little debt, implying a government value $v_c(0)$ that is close to the no-trade benchmark. Welfare gains initially increase as $\bar{g}$ increases, as the government is gradually allowed to issue debt and benefit from constrained access to external credit. Welfare gains eventually peak for the government, at an optimal $\bar{g}$ that differs from that of more patient citizens. For high levels of $\bar{g}$, the constraint becomes more slack, and the government welfare gains drop, to reach zero as $\bar{g} \to +\infty$, and as the equilibrium approaches the no-commitment benchmark.

Our analysis of debt ceiling policies and caps on debt issuance rates have uncovered strikingly different patterns w.r.t. their efficiency at restoring gains from trade for the government. In particular, debt issuance caps, which only constrain low debt borrowers, are more "reliable" at preserving welfare gains than debt ceiling policies. This insight should push international institutions, such as the IMF, to consider flow-based, rather than stock-based interventions.

6.3 Risk Aversion

Motivated by our discussion in Section 3.4, we now study risk aversion as an alternative device to help restore gains from trade. Specifically, given the income dynamics specified in (28), we now assume that

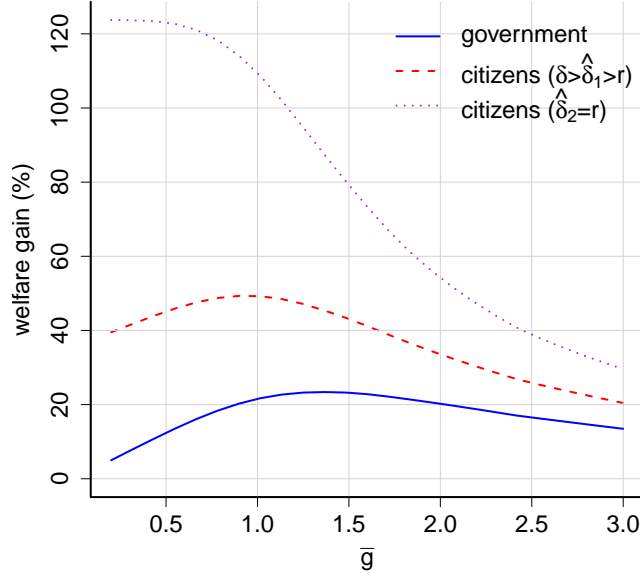**Figure 13:** Issuance cap policies' impact on welfare

Figure shows the percentage gain in welfare (for the government and for citizens with different discount rates). Plot computed with $\mu = 2\%$ p.a., $\sigma = 20\%$ p.a., $1/m = 10$ years, $\theta = 0\%$, $\alpha = 0\%$, $\nu = 0\%$, $r = \kappa = 5\%$ and $\delta = 10\%$.

the government has constant relative risk-aversion γ,

$$V_\gamma := \sup_{(G,\tau)} \mathbb{E}\left[\int_0^{+\infty} e^{-\delta s} u(C_s)ds\right] \text{ with } u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

As before, the consumption *rate* is $C_t = Y_t + G_t D_t - (\kappa + m)F_t$. At default, the small open economy's income suffers a haircut and debt is renegotiated, as in Section 5. Without trade, the autarky government welfare V_{aut} is equal to

$$V_{aut}(Y_t) = \frac{Y_t^{1-\gamma}}{\rho(1-\gamma)} := Y_t^{1-\gamma} v_{aut}, \qquad \rho := \delta - (1-\gamma)\left(\mu - \frac{\gamma\sigma^2}{2}\right)$$

We consider a parameter set $(\mu, \sigma, \delta, r, \theta, \alpha, m, \kappa, \gamma)$ so that (i) $\rho > 0$, and (ii) equilibrium consumption is strictly positive when $\gamma = 0$. We look for an MPE where the government value function V_γ, debt prices D_γ, issuance policy G_γ, and default policy satisfy:

$$V_\gamma(Y, F) := Y^{1-\gamma} v_\gamma(x) \qquad D_\gamma(Y, F) := d_\gamma(x) \qquad G_\gamma(Y, F) := Y g_\gamma(x) \qquad \tau_{k+1} = \inf\{t \geq \tau_k : x_t > \bar{x}_\gamma\}$$

In Online Appendix C.5, we write the HJB equation satisfied by v_γ, as well as the Feynman-Kac equation satisfied by d_γ. We solve the model numerically (using a finite difference scheme), and plot the value function, debt price, credit spread, debt issuance policy and the stationary debt-to-income distribution for the particular case $\gamma = 2$.

Figure 14 depicts a number of scaled value functions and optimal issuance policies for a sequence of equilibria, taking risk-aversion $\gamma \to 0$. The sequence of value functions and debt prices converge

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numerically (and point-wise) towards our risk-neutral benchmark. The risk-neutral and risk-averse policy functions differ however on several interesting dimensions.

First, in most of the state space, greater risk aversion tilts the optimal debt issuance rate downwards. This effect can be best understood by returning to equation (8) and the smoothing term $\gamma \hat{\mu}_c$; since $\delta > r$, the government tilts consumption into the present, and thus the certainty-equivalent consumption growth $\hat{\mu}_c < 0$ in most states of the world, dampening the debt issuance incentive. Intuitively, the desire to smooth consumption limits the degree of front-loading that is optimal.

Second, for the parameter configuration we focus on ($\xi > 2$ – see Lemma 3), the debt issuance policy g_0 in the risk-neutral case diverges to $+\infty$ as the government is near debt-free (i.e. as $x \to 0$). Naturally, the optimal issuance policy must remain finite for any risk-aversion coefficient $\gamma > 0$.

Third, in the risk-neutral case consumption (as a proportion of income) falls as debt increases and default looms, and then jumps once default occurs and some debt is forgiven. But when $\gamma > 0$, given the finite intertemporal elasticity of substitution, equilibrium consumption must feature continuous sample paths. As a result, in order to insure this condition, near the default boundary the government increases its debt issuance rate so as to insure that consumption $C_{\tau-} = C_\tau$, where τ is a default time. Effectively, the government smooths consumption in anticipation of the debt renegotiation. This region of increasing debt issuances and consumption gets smaller and disappears as $\gamma \to 0$.

Despite these differences, the sequence of equilibrium issuance and consumption policies appear to converge point-wise to their risk-neutral counterparts, except at the boundaries $x = 0$ and $x = \bar{x}$.

Figure 14: Value functions and issuance policies

(a): Value functions $\{v_\gamma\}_{\gamma \geq 0}$ **(b):** Issuance policies $\{g_\gamma\}_{\gamma \geq 0}$
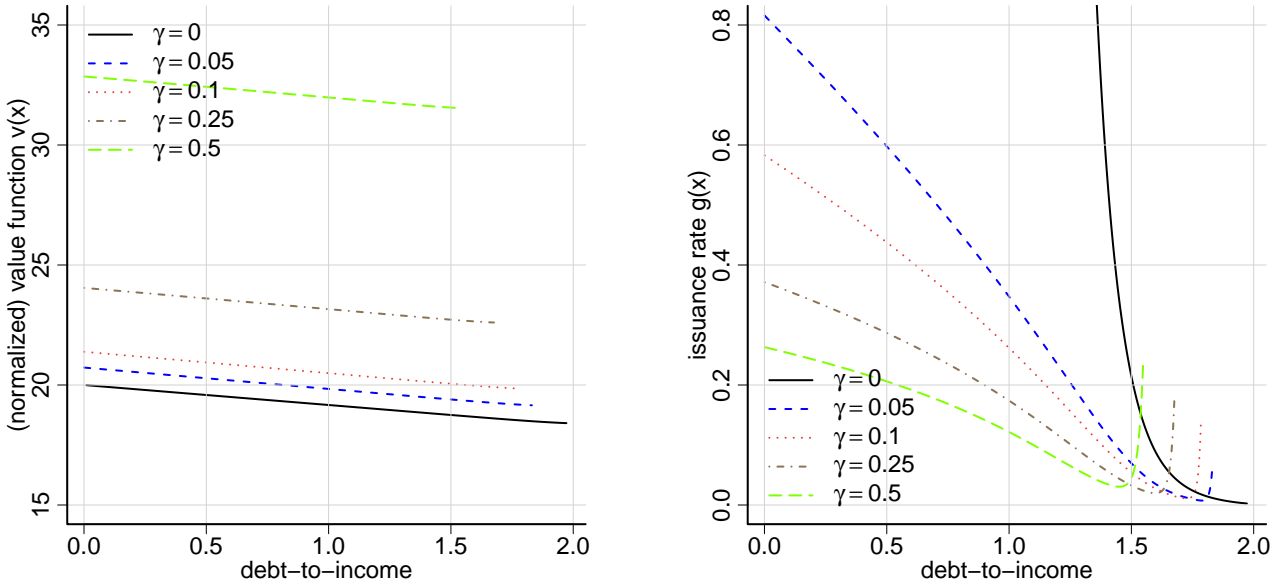
Figure (a) shows a set of scaled value functions v_γ, while Figure (b) shows a set of issuance policies g_γ, for varying levels of risk aversion γ. Plot computed with $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

In Online Appendix C.4, we also compute and plot various model-implied moments as a function of γ. The default boundary $\bar{x}_\gamma$, as well as the ergodic mean debt-to-income ratio, are decreasing functions

of the risk-aversion coefficient γ; as risk-aversion increases, the government slows down its intensity of debt issuances (at most debt-to-income levels), and uses default as consumption-smoothing device at lower debt-to-income ratios. This reduction in debt issuance intensity with risk aversion was one of the qualitative conclusions from our decomposition of the optimal debt issuance in equation (8). This economic mechanism also impacts the ergodic default rate and ergodic average credit spreads, which are (at least locally around $\gamma = 0$) decreasing functions of the risk-aversion parameter γ.

Finally, we measure the welfare gains from trade as the difference between $v_\gamma(0)$ and v_{aut}, measured in their consumption equivalent multiple – i.e., the proportional increase in income that a country in autarky would need in order to be indifferent with the environment where the country has access to international borrowing. We denote this multiplier by k_γ which satisfies

$$V_{aut}(k_\gamma Y) := V_\gamma(Y, 0) \Rightarrow k_\gamma = \left(\frac{v_\gamma(0)}{v_{aut}} \right)^{\frac{1}{1-\gamma}}. \tag{47}$$

As shown in solid blue line in Figure 15, welfare gains measured via k_γ are a hump-shaped function of γ, with a global optimum.

Risk aversion introduces two potential sources of gains from trade. One is the ability of the sovereign to share risk with outside creditors. But our paper offers a fresh perspective on a second important source of gains arising because risk aversion restrains the government's debt issuance policy, thereby allowing it to capture gains associated with its relative impatience.

To evaluate this source of gains, we compute the welfare of a *risk-neutral* government who can commit to using the debt issuance and default policies of a *risk-averse* government, and compare such welfare to the welfare of such risk-neutral government without commitment. To that end, let $\hat{v}_\gamma$ be defined via

$$Y\hat{v}_\gamma(x) := \mathbb{E}_{Y,F} \left[\int_0^{+\infty} e^{-\delta t} C_\gamma(Y_t, F_t) dt \right].$$

Here, $C_\gamma(Y_t, F_t)$ is the equilibrium consumption rate in the economy with risk-aversion γ. We compute the consumption-equivalent multiple $\hat{k}_\gamma$ in the same manor as in equation (47), and plot the result as the dashed purple line in Figure 15. We observe that a risk-neutral borrower who uses a risk-averse issuance and default policy enjoys similar welfare gains when γ is small, and about half when $\gamma \in (1, 2)$. Put differently, risk aversion is a device that disciplines the government's fiscal path by slowing down debt issuances, reducing welfare losses arising from default and allowing it to capture gains from trade due to relative impatience.

We conclude this section with a final remark on the alternative specifications of the government's payoff function. We view our risk-neutral objective as most relevant when the government is borrowing to fund internal investment opportunities (e.g. education, healthcare, etc.), whereas the risk-averse utility function corresponds to borrowing for immediate (perishable) consumption. A more realistic model should include a real consumption/investment tradeoff for the government, with potential adjustment costs, irreversibilities, and feedback to the income process. We leave such extensions for future work.

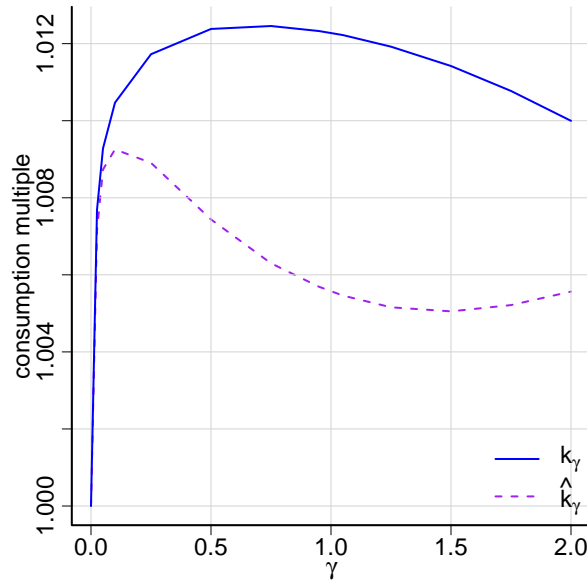**Figure 15:** Welfare gains with risk-aversion

Figure shows the consumption equivalent multipliers k_γ (solid blue line) and $\hat{k}_\gamma$ (dash purple line) as a function of risk-aversion γ. k_γ is the multiple needed to make the risk-averse government indifferent between (a) having access to external credit markets with starting income Y, and (b) being in financial autarky with starting income $k_\gamma Y$; $\hat{k}_\gamma$ is the corresponding multiple for a risk-neutral government that can commit to using the policies of a risk-averse government, thereby isolating the gains that arise solely from commitment (versus risk-sharing). Plot computed with $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $1/m = 20$ years, $\theta = 50\%$, $\alpha = 96\%$, $r = \kappa = 5\%$ and $\delta = 7\%$.

7 Conclusion

Lack of commitment is a powerful force that can dissipate entirely the gains from trade. We apply this original insight of Coase (1972) to the sovereign default framework to show that a risk-neutral government who can borrow from more patient lenders but cannot commit to a financing or default policy, does not gain from trade – a welfare neutrality result familiar from the durable goods' monopoly literature. This insight allows us to build an analytically tractable framework for sovereign debt models. As long as we restrict our focus to a risk-neutral government, a continuous-time setting, and unrestricted trade, then for a very large class of income processes, and for any (long-term) debt maturity, our methodology allows us to directly construct and analyze the Smooth MPE. Furthermore, not only does lack of commitment destroy gains from trade for the government, but it also makes more patient citizens worse off than financial autarky, since the future costs of default exceed, from citizens' perspective, the immediate benefits of current consumption.

Along with these stark welfare results, the model generates realistic and tractable debt-to-income dynamics: debt adjusts slowly towards a target multiple of income, exacerbating consumption booms and busts. With lognormal income, we prove that our Smooth MPE is unique within the broader class of MPEs, and provide a complete analytical characterization of equilibrium valuations, the speed of adjustment, credit spreads, default rates, and related comparative statics – results that have previously only been obtained numerically in the sovereign default literature. Moreover, our general methodology will hold, and remain tractable, in many other settings.

We also use our model to study alternative devices that could restore gains from trade. While unpredictable "sudden stops" to trade provide no commitment benefit, we show that predictable delays in trade can serve as commitment, but have a limited ability to restore welfare since the gains from commitment are offset by a loss in flexibility. Similarly, a debt-ceiling policy can enhance welfare, but only if the ceiling is low enough that it binds immediately. Instead, a policy akin to a limit on public deficits consistently produces welfare gains, though the optimal policy for the government will be "too loose" from citizens' perspective. Finally, risk-aversion – the counterpart to increasing marginal costs in the Coase (1972) model – allows the government to reap gains from trading in international credit markets; the desire to intertemporally smooth consumption slows the pace of debt issuance and thereby disciplines the government's debt path.

Other approaches can be pursued in order to reintroduce gains from trade. Benzoni et al. (2020) for example introduce fixed bond issuance costs, while Malenko and Tsoy (2020) allow for non-Markov equilibria – they study "grim-trigger" strategies from creditors. In these non-Markov equilibrium models, the borrower is punished for past misbehavior by reversion to our Smooth MPE. Infinitessimally short-term debt allows the government to make new debt issuance decisions after having repaid its previously issued debt, thus resolving the commitment problem; gains from trade can be fully restored when income has continuous sample paths, while they are only partially restored when income is subject to downward jumps that can lead to default (Hu, Varas and Ying, 2021). Alternatively, when the small open economy's income admits a strictly positive lower bound, and if the government is able to pledge such income lower bound to a separate class of secured creditors, it could then realize gains from trade, by borrowing immediately against it; in that case, the natural borrowing limit is strictly positive, a departure from the geometric Brownian motion case we study analytically in this article.

Our work raises new questions for the sovereign debt literature. It implies that welfare gains from external financing can only arise from the imposition of binding contractual constraints on future trade, or from a consumption smoothing motive. With respect to constraints on the debt path, we only study environments where such commitment devices are assumed to be kept in place by successive governments; however, those devices are not time-consistent, in the sense that future governments might find it beneficial to renegotiate them. With respect to risk-sharing, empirically sovereign borrowers do not appear to be successful at actually "smoothing" consumption; moreover, even when the smoothing motive is at play, the welfare gains realized are quite modest. What other methods could then be put in place in order for emerging market economies to fully benefit from external capital markets' access?

DeMarzo (2019) hints at an alternative solution, in the context of corporate capital structure decisions: collateral can act as a powerful commitment device. For emerging market economies, it begs the question of the optimal security design for sovereign bonds when the issuing country is able to pledge assets to international creditors, a practice that has recently become more relevant.[46] In any case, our work highlights the need for a more detailed understanding of the frictions and contracting devices that may constrain sovereign borrowing in order to predict the potential welfare gains that can be achieved and optimal policies that we may observe.

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[46]See January 2020 IMF report and September 2020 IMF report for example.

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A Incremental Welfare Gains with Risk-Aversion

Note $\mathcal{F}_0 = (1 - mdt)F$ the next-period no-trade debt balance, and $\mathcal{C}_0 = Y - (\kappa + m)F$ the related no-trade consumption rate. The incremental welfare gains from being able to issue sovereign debt (between t and $t + dt$) can be computed as follows

$$\text{Gain}_{[t,t+dt)} = [u(\mathcal{C}^*) - u(\mathcal{C}_0)]\, dt + e^{-\delta dt}\mathbb{E}_Y\left[\max\left(\underline{V}(Y'), V(Y', \mathcal{F}^*)\right) - \max\left(\underline{V}(Y'), V(Y', \mathcal{F}_0)\right)\right] \quad (48)$$

Since u is twice differentiable, we can use the Taylor theorem to find a consumption level $\tilde{\mathcal{C}}$, between autarky consumption $\mathcal{C}_0$ and equilibrium consumption $\mathcal{C}^*$, such that

$$u(\mathcal{C}_0) = u(\mathcal{C}^*) - u'(\mathcal{C}^*)\frac{d\Gamma^*}{dt}D(Y, \mathcal{F}^*) + \frac{1}{2}u''(\tilde{\mathcal{C}})\left(\frac{d\Gamma^*}{dt}D(Y, \mathcal{F}^*)\right)^2$$

Consider then the expectation in equation (48), and re-write it as follows:

$$e^{-\delta dt}\mathbb{E}_Y\left[\max\left(\underline{V}(Y'), V(Y', \mathcal{F}^*)\right) - \max\left(\underline{V}(Y'), V(Y', \mathcal{F}_0)\right)\right] =$$
$$e^{-\delta dt}\mathbb{E}_Y\left[\mathcal{R}^*(Y', \mathcal{F}^*)V(Y', \mathcal{F}^*) + (1 - \mathcal{R}^*(Y', \mathcal{F}^*))\underline{V}(Y')\right.$$
$$\left. - \mathcal{R}^*(Y', \mathcal{F}_0)V(Y', \mathcal{F}_0) - (1 - \mathcal{R}^*(Y', \mathcal{F}_0))\underline{V}(Y')\right],$$

where, as a reminder, we have noted $\mathcal{R}^*$ the repayment indicator. Some manipulation yields:

$$e^{-\delta dt}\mathbb{E}_Y\left[\max\left(\underline{V}(Y'), V(Y', \mathcal{F}^*)\right) - \max\left(\underline{V}(Y'), V(Y', \mathcal{F}_0)\right)\right] =$$
$$e^{-\delta dt}\mathbb{E}_Y\left[\mathcal{R}^*(Y', \mathcal{F}^*)\left(V(Y', \mathcal{F}^*) - V(Y', \mathcal{F}_0)\right)\right.$$
$$\left. + \left(\mathcal{R}^*(Y', \mathcal{F}_0) - \mathcal{R}^*(Y', \mathcal{F}^*)\right)\left(\underline{V}(Y') - V(Y', \mathcal{F}_0)\right)\right] \quad (49)$$

The right-hand-side of this equation is the sum of two terms. The first term can be written, using Taylor's theorem, the first order condition (3) and the fact that $d\Gamma^* = O(dt)$ as $dt \to 0$:

$$e^{-\delta dt}\mathbb{E}_Y\left[\mathcal{R}^*(Y', \mathcal{F}^*)\left(V(Y', \mathcal{F}^*) - V(Y', \mathcal{F}_0)\right)\right] \underset{dt \to 0}{=} e^{-\delta dt}\mathbb{E}_Y\left[\mathcal{R}^*(Y', \mathcal{F}^*)\partial_F V(Y', \mathcal{F}^*)\right]d\Gamma^* + O\left(dt^2\right)$$
$$\underset{dt \to 0}{=} -D(Y, \mathcal{F}^*)u'(\mathcal{C}^*)d\Gamma^* + O\left(dt^2\right)$$

Let $H(\cdot|Y)$ be the conditional distribution of Y' given Y. Let $\mathcal{Y}_d(\cdot)$ be the default boundary, which satisfies $V(\mathcal{Y}_d(F), F) = \underline{V}(\mathcal{Y}_d(F))$, and $V(y, F) > \underline{V}(y)$ for $y > \mathcal{Y}_d(F)$. Consider the inverse function $\mathcal{F}_d(Y)$, which satisfies $\mathcal{F}_d(\mathcal{Y}_d(F)) = F$. Note that by definition, for any y, $V(y, \mathcal{F}_d(y)) = \underline{V}(y)$. The second term of interest can then be written:

$$\mathbb{E}_Y\left[\left(\mathcal{R}^*(Y', \mathcal{F}_0) - \mathcal{R}^*(Y', \mathcal{F}^*)\right)\left(\underline{V}(Y') - V(Y', \mathcal{F}_0)\right)\right] = \int_{\mathcal{Y}_d(\mathcal{F}_0)}^{\mathcal{Y}_d(\mathcal{F}^*)} \left(\underline{V}(z) - V(z, \mathcal{F}_0)\right)dH(z|Y)$$
$$= \int_{\mathcal{Y}_d(\mathcal{F}_0)}^{\mathcal{Y}_d(\mathcal{F}^*)} \left(V(z, \mathcal{F}_d(z)) - V(z, \mathcal{F}_0)\right)dH(z|Y)$$
$$= O(dt^2),$$

where the last equality follows from the fact that since $\mathcal{F}^* = \mathcal{F}_0 + d\Gamma^* \underset{dt \to 0}{=} \mathcal{F}_0 + O(dt)$, we have

$$|\mathcal{Y}_d(\mathcal{F}^*) - \mathcal{Y}_d(\mathcal{F}_0)| \underset{dt \to 0}{=} O(dt)$$
$$\max_{z \in [\mathcal{Y}_d(\mathcal{F}_0), \mathcal{Y}_d(\mathcal{F}^*)]} |V(z, \mathcal{F}_d(z)) - V(z, \mathcal{F}_0)| \underset{dt \to 0}{=} O(dt)$$

Thus, putting all the pieces together, we obtain

$$\text{Gain}_{[t,t+dt]} \underset{dt \to 0}{=} -\frac{1}{2} u'' (\tilde{C}) \left(\frac{d\Gamma^*}{dt} D (Y, \mathcal{F}^*) \right)^2 dt + O(dt^2)$$

$\square$

B Incremental Welfare Gains with Constraints

We now consider the risk-neutral case $u'' = 0$. The incremental welfare gains (vs. no-trade) from being able to credibly commit to not issuing debt at an intensity greater than a certain level $\bar{G}$ between t and $t + dt$ can be computed as follows

$$\text{Gain}_{[t,t+dt]} = [\mathcal{C}_c^* - \mathcal{C}_0] dt + e^{-\delta dt} \mathbb{E}_Y \left[\max \left(\underline{V}(Y'), V (Y', \mathcal{F}_c^*) \right) - \max \left(\underline{V}(Y'), V (Y', \mathcal{F}_0) \right) \right], \quad (50)$$

where $\mathcal{C}_c^*$ is the constrained consumption policy, and $\mathcal{F}_c^*$ is the constrained next-period debt policy, which satisfy the first order condition

$$D (Y, \mathcal{F}_c^*) + d\Gamma_c^* \partial_F D (Y, \mathcal{F}_c^*) + e^{-\delta dt} \int_{\mathcal{Y}_d(\mathcal{F}_c^*)}^{+\infty} \partial_F V(z, \mathcal{F}_c^*) dH(z, Y) = \lambda(Y, F) \geq 0 \quad (51)$$

In the above, $\lambda(Y, F)$ is the Lagrange multiplier on the constraint $d\Gamma \leq \bar{G} dt$, and we have $\lambda(Y, F) \geq 0$, with a strict inequality if and only if $d\Gamma = \bar{G} dt$ (i.e. when the constraint binds). Consider then the expectation in equation (50), and use a simplification similar to that in equation (49) to obtain:

$$e^{-\delta dt} \mathbb{E}_Y \left[\max \left(\underline{V}(Y'), V (Y', \mathcal{F}_c^*) \right) - \max \left(\underline{V}(Y'), V (Y', \mathcal{F}_0) \right) \right] =$$
$$e^{-\delta dt} \mathbb{E}_Y \left[\mathcal{R}^* (Y', \mathcal{F}_c^*) \left(V (Y', \mathcal{F}_c^*) - V (Y', \mathcal{F}_0) \right) \right.$$
$$\left. + \left(\mathcal{R}^* (Y', \mathcal{F}_0) - \mathcal{R}^* (Y', \mathcal{F}_c^*) \right) \left(\underline{V} (Y') - V (Y', \mathcal{F}_0) \right) \right]$$

The right-hand-side of this equation is the sum of two terms. The first term can be written, using Taylor's theorem, the first order condition (51) and the fact that $d\Gamma_c^* = O(dt)$ as $dt \to 0$:

$$e^{-\delta dt} \mathbb{E}_Y \left[\mathcal{R}^* (Y', \mathcal{F}_c^*) \left(V (Y', \mathcal{F}_c^*) - V (Y', \mathcal{F}_0) \right) \right] \underset{dt \to 0}{=} e^{-\delta dt} \mathbb{E}_Y \left[\mathcal{R}^* (Y', \mathcal{F}_c^*) \partial_F V (Y', \mathcal{F}_c^*) \right] d\Gamma_c^* + O \left(dt^2 \right)$$
$$\underset{dt \to 0}{=} -D (Y, \mathcal{F}_c^*) d\Gamma_c^* + \lambda(Y, F) d\Gamma_c^* + O \left(dt^2 \right)$$

Using the same notation as in Appendix A and a similar logic, the second term of interest can then be written:

$$\mathbb{E}_Y \left[\left(\mathcal{R}^* (Y', \mathcal{F}_0) - \mathcal{R}^* (Y', \mathcal{F}_c^*) \right) \left(\underline{V} (Y') - V (Y', \mathcal{F}_0) \right) \right] = \int_{\mathcal{Y}_d(\mathcal{F}_0)}^{\mathcal{Y}_d(\mathcal{F}_c^*)} \left(\underline{V}(z) - V (z, \mathcal{F}_0) \right) dH(z|Y) = O(dt^2)$$

Thus, putting all the pieces together, we obtain

$$\text{Gain}_{[t,t+dt]} \underset{dt \to 0}{=} \lambda(Y, F) d\Gamma_c^* + O(dt^2)$$

$\square$