University of Wisconsin Madison
FNCE-970
Prof. Ivan Shaliastovich
Spring 2022

Final Exam
Due by May 13th

You need to solve 4 problems. Choose 1a or 1b; one from 2a - 2h; and one between 3a and 3b; and problem 4. If you are preparing for the qualifier, you may want to go through all the problems at some stage. To complete the exam you can consult any written materials such as textbooks or academic papers, but you cannot discuss these problems with each other, your peers, or other faculty. If you do use external resources, please cite them appropriately.

Problem 1a: Term Structure, No Arbitrage Models.

There are two equivalent ways to specify a no-arbitrage Gaussian term structure model. They both start with the following assumptions:

1. Let X_t denote the vector of state variables in the economy. Factors X_t follow Gaussian VAR(1):
$$X_{t+1} = \mu + \Phi X_t + \Sigma \epsilon_{t+1}, \qquad \epsilon_{t+1} \sim N(0, I)$$
2. One-period risk-free rate is linear in the states:
$$r_t = \delta_0 + \delta_1' X_t.$$

Under one approach, which I'll call "the SDF approach," we further proceed in the following manner:

3. The stochastic discount factor (SDF) is log-normal:
$$M_{t+1} = exp\left(-r_t - \frac{1}{2}\lambda_t' \lambda_t - \lambda_t \epsilon_{t+1}\right)$$
where the market prices of risks are affine in X_t :
$$\lambda_t = \lambda_0 + \lambda_1 X_t.$$
As always, we use the discount factor to price any asset in the economy,
$$P_t = E_t M_{t+1} Z_{t+1}, \tag{1}$$
where P_t is the price of the asset and Z_{t+1} is the asset payoff tomorrow.

Alternatively, we can use "the risk-neutral density" approach, which replaces the above assumption 3 with the following one:

3' Under the *risk-neutral* measure, the factors X_t follow Gaussian VAR(1):

$$X_{t+1} = \mu^Q + \Phi^Q X_t + \Sigma \epsilon^Q_{t+1}, \quad \epsilon^Q_{t+1} \sim N(0, I).$$

(Σ is the same under both measures). The risk-neutral measure is used to price any asset in the economy in the following way:

$$P_t = e^{-r_t} E^Q_t Z_{t+1}. \tag{2}$$

That is, to figure out the price of the asset, we just take an expectation of its payoff tomorrow using the risk-neutral law of motion above, and discounting this risk-neutral expectation by the risk-free rate.

- Show that the two approaches are indeed equivalent, using their term structure implications. Specifically:

  1. Derive the price of an n-period bond under the SDF approach using the pricing condition 3. The bond price will depend on the VAR model parameters (μ, Φ, Σ), the one-period rate parameters δ_0, δ_1, and the market prices of risks (λ_0, λ_1).
  2. Derive the price of an n-period bond under the risk-neutral valuation approach using the pricing condition (2). The bond price will depend on the risk-neutral model parameters (μ^Q, Φ^Q, Σ), and the one-period rate parameters δ_0, δ_1.
  3. Show that there is a unique one-to-one mapping between the risk-neutral parameters (μ^Q, Π^Q) and the market prices of risks (λ_0, λ_1) which results in identical bond prices, for any maturity, under the two approaches. That is, the two model approaches are equivalent: you can start with the SDF and solve for the corresponding risk-neutral density which support these valuations, or you can write down the risk-neutral density and solve for the appropriate SDF.

Problem 1b: Foreign Exchange Risk

Let s_t stand for a real spot exchange rate, in logs, per unit of foreign currency (e.g. dollars spot price of one pound). Superscript * will denote the corresponding variable in a foreign country, e.g. let $y_{t,1}$ stand for a one-period risk-free rate at home, and $y^*_{t,1}$ for the foreign risk-free rate. Define a one-period excess log return in foreign bonds:

$$rx^{FX}_{t+1} = s_{t+1} - s_t + y^*_{t,1} - y_{t,1}. \tag{3}$$

This corresponds to an excess return on buying foreign currency today, investing the money into the foreign risk-free asset and converting the proceeds back using the spot rate next period.

We would like to investigate the predictability of excess foreign bond returns by the interest rate spread, i.e. the magnitude of the slope coefficient β^{FX} in the projection

$$rx^{FX}_{t+1} = const + \beta^{FX}(y_{t,1} - y^*_{t,1}) + error_{t+1}. \tag{4}$$

In what follows, assume complete markets, and the log-normality of the stochastic discount factor (SDF) at home $M_{t+1} = e^{m_{t+1}}$ and abroad $M^*_{t+1} = e^{m^*_{t+1}}$.

I. General Case: Complete Markets and Log-Normality

1. Show that in a homoscedastic environment where all the conditional volatilities are constant, the population value of the FX slope coefficient in regression (4) is equal to 0.

II. Power Utility

1. Now consider a power utility case. The SDFs at home and abroad are given by,

   $$m_{t+1} = \log\beta - \gamma\Delta c_{t+1}, \quad m^*_{t+1} = \log\beta - \gamma\Delta c^*_{t+1},$$

   and the consumption growth rates in the two countries are,

   $$\Delta c_{t+1} = \mu + \sigma_t\eta_{t+1}, \quad \Delta c^*_{t+1} = \mu + \sigma^*_t\eta^*_{t+1}.$$

   In the above I assume that the preference parameters β, γ are the same across the two countries. For simplicity, mean consumption growth rates μ are also the same, however, the consumption shocks η_{t+1} and η^*_{t+1} are country-specific and are i.i.d. standard Normal. σ_t and σ^*_t are the processes which drive consumption volatilities at home and abroad, respectively. Maintain the complete markets assumption.

   Calculate the population value of FX slope coefficient in this economy. Are excess foreign bond returns predictable? How does the magnitude of this coefficient compare to the data?

III. Habits

1. Now consider a Campbell-Cochrane external habits economy, in the extension of Wachter (2006) to time-varying risk-free rates — that is, where the $\bar{S}$ is redefined as

   $$\bar{S} = \sigma_v\sqrt{\frac{\gamma}{1-\phi-b/\gamma}},$$

where b is the preference parameter.

Consider i.i.d. consumption growth rates across the two countries,

$$g_{t+1} = g + v_{t+1}, \quad g^*_{t+1} = g + v^*_{t+1}, \quad Var(v_{t+1}) = Var(v^*_{t+1}) = \sigma_v^2.$$

Now calculate the FX slope in this economy. Show that the sign of this slope is pinned down by the preference parameter b. How is it possible that excess returns are predictable even though consumption growth rates are i.i.d. – didn't we establish in Part I that in a homoscedastic economy the slope coefficient is always 0?

2. In the data, the FX slope coefficient is negative — what does this FX evidence imply about the parameter b? Now, what does the term-structure evidence in Wachter (2006) imply about this preference parameter? Is the FX and Term-structure evidence conflicting — if so, what are some possible ways to reconcile the two approaches and explain both the FX and term structure evidence (remaining in habits setup)?

IV. Recursive Utility

1. Now consider the economy where agents have recursive Epstein-Zin utility,

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

Let the consumption dynamics in each country be the following:

$$\begin{aligned} \Delta c_{t+1} &= \mu + \sigma_t \eta_{t+1}, \\ \sigma^2_{t+1} &= \sigma_0^2 + \nu(\sigma_t^2 - \sigma_0^2) + \sigma_w w_{t+1}, \end{aligned}$$

where all the shocks are i.i.d. uncorrelated standard Normal. Same as before, the economy in the other country has the same model and preference parameters, but is driven by its own shocks η^*_{t+1} and w^*_{t+1}.

Compute the FX slope coefficient in this environment. Show that it's negative when $\gamma > 1$. Under what conditions β^{FX} will be below -1?

Problem 2a: 2019 Finance qualifier at UW Madison

Investors have recursive Epstein-Zin preferences over consumption,

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

The log stochastic discount factor is given by,

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

where $\theta \equiv \frac{1-\gamma}{1-1/\psi}$, Δc_{t+1} is the log real consumption growth, and $r_{c,t+1}$ is the log return on the consumption asset.

Real consumption growth Δc_t and inflation rate π_t follow the dynamics:

$$\Delta c_{t+1} = \mu_c + \rho_c \Delta c_t + \rho_{c\pi} \pi_t + \sigma_c \epsilon_{t+1},$$
$$\pi_{t+1} = \mu_\pi + \rho_\pi \pi_t + \sigma_\pi u_{t+1},$$

where consumption and inflation shocks ϵ and u are i.i.d. Normal (0,1), homoscedastic, and are uncorrelated with each other.

Note that real consumption growth and inflation are persistent processes, and furthermore, inflation has a non-neutral effect on future real growth through a parameter $\rho_{c\pi}$. In particular, when $\rho_{c\pi} < 0$ high inflation today indicates lower growth in the future.

1. Conjecture that the equilibrium price to consumption ratio is a linear combination of real consumption growth and inflation rate. Using standard log linearization, derive the loadings of the price-to-consumption ratio on the relevant economic states. (To save time, you can ignore the solutions to the constant coefficients and log-linearization parameters.) Briefly explain how these risk exposures depend on the preference, and consumption and inflation dynamics parameters.

2. Solve for the equilibrium stochastic discount factor. What are the aggregate risks which demand a risk compensation? What are the market prices of these risks? Briefly explain how the market prices of risks depend on the preference and model dynamics parameters.

3. Do price-consumption valuations and real bond prices depend on inflation rate? If so, why: they are claims to real economy, why are they exposed to inflation risks?

4. Conjecture that the equilibrium *nominal* bond prices are linear combinations of real consumption growth and inflation rate. Derive the loadings of nominal bond prices to economic states. (You can ignore the constant coefficients, and you can specify the solutions to bond coefficients recursively.)

5. Consider the risk-premium on buying a 2 period nominal bond, $\approx -Cov_t(m^{\$}_{t+1}, p_{1,t+1})$, where $m^{\$}_{t+1}$ is the nominal pricing kernel, and $p_{1,t+1}$ is the price of the remaining 1-period nominal bond next period. How does the inflation non-neutrality parameter $\rho_{c\pi}$ affect the sign of the risk premium (assume a standard preference configuration $\psi, \gamma > 1$). What is the economics behind getting a positive risk premium on nominal bonds?

6. Can the model explain the violations of the Expectations Hypothesis? If so, explain what its implications are. If not, discuss how the model can be enriched to do so.

Problem 2b: 2018 Finance qualifier at UW Madison

Investors have recursive Epstein-Zin preferences over consumption,

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

The log stochastic discount factor is given by,

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

where $\theta \equiv \frac{1-\gamma}{1-1/\psi}$, Δc_{t+1} is the log real consumption growth, and $r_{c,t+1}$ is the log return on the consumption asset.

Consumption growth follows simplified dynamics of Bansal-Yaron 2004:

$$\begin{aligned} \Delta c_{t+1} &= \mu + \sigma_t \epsilon_{t+1}, \\ \sigma_{t+1}^2 &= \sigma_0^2 + \nu(\sigma_t^2 - \sigma_0^2) + \sigma_w w_{t+1}. \end{aligned}$$

The expected consumption growth is constant, but consumption shocks are heteroscedastic.

All innovations are i.i.d. Normal (0,1).

1. Using standard log linearization, solve for the equilibrium price-to-consumption ratio. Derive the loadings of the price-to-consumption ratio on the relevant economic states. (To save time, you can ignore the solutions to the constant coefficients and log-linearization parameters.) Briefly explain how the loadings depend on the preference and consumption dynamics parameters.

2. Solve for the equilibrium stochastic discount factor. What are the aggregate risks which demand a risk compensation? What are the market prices of these risks? Briefly explain how the market prices of risks depend on the preference and consumption dynamics parameters.

3. We would like to price an equity claim to future dividends. Dividend dynamics are specified exogenously as:

$$\begin{aligned} \Delta d_{t+1} &= \mu_d + \phi \sigma_t \epsilon_{t+1} + \tilde{\sigma}_t u_{t+1}, \\ \tilde{\sigma}_{t+1}^2 &= \tilde{\sigma}_0^2 + \tilde{\nu}(\tilde{\sigma}_t^2 - \tilde{\sigma}_0^2) + \tilde{\sigma}_w \tilde{w}_{t+1}. \end{aligned}$$

   Expected dividend growth is constant μ_d. Dividends are exposed to aggregate consumption innovations with a loading of ϕ. Additionally, dividends feature an idiosyncratic shock u_{t+1} whose volatility is stochastic and given by $\tilde{\sigma}_t$. The shocks u_t and $\tilde{\omega}_t$ are i.i.d. Normal (0,1), and specifically, they are uncorrelated with aggregate consumption shocks.

   Conjecture that the price-dividend ratio is a linear function of the two volatility states: $pd_t = const + H\sigma_t^2 + \tilde{H}\tilde{\sigma}_t^2$. Using standard log-linearization approach, solve for the equilibrium loadings H and $\tilde{H}$.

4. A common wisdom is that equity volatility is not a very good predictor of future excess equity returns. Compute equity volatility as the conditional variance of the return on equity claim in part c, $Var_t r_{d,t+1}$. Compute the expected excess returns (the equity risk premium): $rp_t = E_t(r_{d,t+1} - r_{f,t}) + 1/2 Var_t r_{d,t+1} \approx -Cov_t(m_{t+1}, r_{d,t+1})$. Can the model rationalize that variations in the equity premium are not directly related to the movements in the market variance?

5. You may have noticed that the price-dividend loadings on the aggregate and idiosyncratic volatilities are quite different. While we would expect the loading on the aggregate volatility (H) to be negative, the loading on idiosyncratic volatility ($\tilde{H}$) appears to be positive. What is the economics of that - intuitively, why the equity price responses to aggregate versus idiosyncratic volatility are so different?

Problem 2c: From 2014 Finance qualifier at Wharton

3. Let C_t denote consumption, W_t denote wealth and $R_{w,t}$ the return of the wealth portfolio (the asset that pays consumption as its dividend). The representative agent has recursive Epstein-Zin-Weil (EZW) preferences. The intertemporal Euler equation is

$$E_t\left[\delta^{\theta}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{w,t+1}^{\theta-1} R_{i,t+1}\right] = 1,$$

where $\theta = (1-\gamma)/(1-1/\psi)$, and γ is the coefficient of relative risk aversion, and ψ is the intertemporal elasticity of substitution. The budget constraint is given by

$$W_{t+1} = R_{w,t+1}(W_t - C_t).$$

Log consumption growth follows the heteroskedastic process below:

$$\Delta c_{t+1} = g + \pi\sigma_t^2 + \sigma_t e_{t+1}, \tag{16}$$
$$\sigma_{t+1}^2 = \sigma^2 + \phi(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}, \tag{17}$$
$$e_{t+1}, w_{t+1} \sim Ni.i.d.(0,1), \tag{18}$$
$$Corr(e_{t+1}, w_{t+1}) = 0 \tag{19}$$

(a) Using the Campbell-Shiller approximation, solve for the log price-consumption ratio for the asset that pays consumption as a dividend.

(b) Using the approximate solution above, express the log IMRS in terms of the exogenous shocks.

(c) Solve for the risk-free rate and the risk premia on the return on the consumption claim, $rp_t \equiv E_t[r_{w,t+1} - r_{f,t}] + 0.5var_t(r_{w,t+1})$

(d) How does the risk premium and risk free rate depend on the sign and magnitude of the parameter π? Briefly explain the intuition for the results.

(e) Derive the R^2 for predicting the risk premium rp_{t+1} by the period t price-consumption ratio.

(f) Now assume that for all periods the volatility deterministically fluctuate from a high to low on even/odd periods. That is, $\sigma_t = \sigma_h$ for even periods while $\sigma_t = \sigma_l$ for odd periods.

   i. How would you solve for the price-consumption ratio (no need to actually solve)–what are the relevant state variables?

   ii. What, if any, is the market price of volatility risk? What is the market price of consumption risk (no lengthy calculations are required to answer this question)

   iii. What is the expected excess return on the consumption paying asset?

   iv. Is the expected excess return (rp_{t+1}) on the return on wealth predictable by the time t price-consumption ratio? Explain.

Problem 2d: From 2016 Finance qualifier at Wharton

Consider an economy with Epstein and Zin preferences and complete frictionless markets such that:

$$U_t = \left\{(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right\}^{\frac{\theta}{1-\gamma}}$$

and C_t denotes consumption, γ is risk aversion, and $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$ and ψ is the IES. The consumption growth and log dividend dynamics are given by

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma\eta_{t+1} \\
x_{t+1} &= \rho x_t + \varphi_x \sigma e_{t+1} \\
d_{t+1} - c_{t+1} &= \mu_{cd} + s_{t+1} + \varphi_u \sigma u_{t+1} \\
s_{t+1} &= \rho_s s_t + \phi_{sx} x_t + \varphi_s \sigma \epsilon_{t+1}
\end{aligned}$$

with all shock innovations being iidN(0,1). The representative agent maximizes his utility function subject to the budget constraint $W_{t+1} = (W_t - C_t)R_{c,t+1}$ where $R_{c,t+1}$ is the return on wealth. It can be shown the log SDF in this economy follows:

$$m_{t+1} = \theta\log(\delta) - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1)r_{c,t+1}$$

1. Write the growth rate of dividends Δd_{t+1} in terms of state variables of the economy and their respective shocks
2. The price consumption ratio for a claim on aggregate consumption is $p_t - c_t = A_0 + A_1 x_t$ with $A_1 = \frac{1-\frac{1}{\psi}}{1-\kappa_c\rho}$. Use the Campbell-Shiller approximation to solve for the price-dividend ratio:
   (a) Show what are the relevant state variable(s) for the solution of $p_t - d_t$
   (b) Solve for elasticity coefficient(s)
   (c) Does a positive shock to s_t raises or lower $p_t - d_t$ ratio? explain the economic intuition.

3. How much of the $p_t - d_t$ variation is due to discount rates and how much is due to cashflow?

4. Compute the risk premia on the dividend asset. What is the contribution of the dividend shock u_{t+1}? Would this model be able to explain return predictability–explain?

5. Compute the innovation to the return on dividends in terms of the shocks (that is $r_{t+1} - E_t(r_{t+1})$. Assume a monthly model and show what are the effects of dividend shock u_{t+1} on the return in the presence of cointegration when $\rho_s \approx 0.9$ and $\kappa_d \approx 0.99$. What happens when cointegration is not imposed, that is when $\rho_s = 1$?

Problem 2e: From 2015 Finance qualifier at Wharton

The representative agent has an Epstein-Zin utility function with the following specification:

$$U_t = \left((1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(E_t[U_{t+1}^{1-\gamma}] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right)^{\frac{1}{1-\frac{1}{\psi}}}$$

where C_t is consumption, and $\psi = IES$ and γ=risk aversion. The log nominal SDF is equal to $m_{t+1} - \pi_{t+1}$ where π is the inflation rate. Assume the process for consumption and inflation follows:

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \eta_{c,t+1} + \omega \eta_{c,t} \\
x_{t+1} &= \rho x_{t+1} + \varphi_x e_{t+1} \\
\pi_{t+1} &= \mu_\pi + \eta_{\pi,t+1} \\
\rho_{\pi,c} &= corr(\eta_{\pi,t+1}, \eta_{c,t+1})
\end{aligned}$$

1. Solve for the price-consumption ratio (of the valuation ratio for the consumption paying asset).

2. Derive the risk premium for the consumption paying asset. Is it time varying?

3. Derive the excess return on a 2-period nominal bond. Is the nominal term structure upward/downward sloping? What role, if any, do $\rho_{\pi,c}$ and ω play in determining your answer?

4. Suggest a way to empirically test this model. That is which data would you use and how would you evaluate model fit?

5. Does the Expectation Hypothesis hold in this model? If so, how would you extend the model to capture violations of **EH** in the data?

Problem 2f: 2017 UW Madison qualifier

Investors have recursive Epstein-Zin preferences over consumption,

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

It can be shown that the log stochastic discount factor in this economy is given by,

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

where $\theta \equiv \frac{1-\gamma}{1-1/\psi}$, Δc_{t+1} is the log real consumption growth, and $r_{c,t+1}$ is the log return on wealth.

The consumption growth dynamics are given by,

$$\begin{aligned} \Delta c_{t+1} &= \mu + \sigma_t \epsilon_{t+1}, \\ \sigma_{t+1}^2 &= \sigma_0^2 + \nu(\sigma_t^2 - \sigma_0^2) + \tau(\Delta c_t - \mu) + \sigma_w w_{t+1}. \end{aligned}$$

Note the extra component in the variance equation: consumption volatility now depends on the realized consumption growth. For $\tau < 0$, low realized consumption growth increases future aggregate volatility.

All innovations are i.i.d. Normal (0,1).

1. Conjecture that the price-consumption ratio is linear in the economic states: $pc_t = A_0 + A_c \Delta c_t + A_\sigma \sigma_t^2$. Solve for the equilibrium loadings A_c and A_σ. (Ignore the constants. Further, you do not need to provide exact closed form solutions for the coefficients. It is sufficient to derive the two equations which pin them down as functions of each other and other model parameters.)

2. What are the signs of the price-consumption loadings as a function of the model and preference parameters. How does the sign of τ affect the loadings – what is the economic intuition?

3. Campbell-Shiller decompose price variation into the cash-flow and the discount rate news. In our economy, how much of the movements in pc_t are due to the revisions in expected cash flows and expected discount rates?

4. What economic risks determine the risk compensation? Compute the equilibrium market prices of risks. What are their signs in a benchmark parameter configuration of $\gamma, \psi > 1$, $\tau < 0$.

5. Solve for the risk premium on the wealth portfolio, $rp_t = E_t(r_{c,t+1} - r_{f,t}) + 1/2 Var_t r_{c,t+1} \approx -Cov_t(m_{t+1}, r_{c,t+1})$. What is the sign of the risk premium? A common wisdom is that the risk premium is high in recessions and low in expansions. Can the model account for this fact?

6. Are the future cash flows and future excess returns predictable by the price-consumption ratio in the model? Can the model explain the predictability patterns of future returns and future cash flows by the price-dividend ratio in the data?

7. In some ways, the model has similar predictions to the Campbell-Cochrane habits model. List some of these similarities. Also, list some of the key differences which could help distinguish the two models in the data.

Problem 2g: 2020 Finance qualifier at UW Madison

Investors have recursive Epstein-Zin preferences over consumption,

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

The log stochastic discount factor is given by,

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

where $\theta \equiv \frac{1-\gamma}{1-1/\psi}$, Δc_{t+1} is the log real consumption growth, and $r_{c,t+1}$ is the log return on the consumption asset.

Real consumption growth Δc_t follows *simplified* Bansal-Yaron 2004 dynamics:

$$\begin{aligned}
\Delta c_{t+1} &= \mu + x_t + +\sigma_c \epsilon_{t+1}, \\
x_{t+1} &= \rho x_t + \sigma_t e_{t+1}, \\
\sigma_{t+1}^2 &= \sigma_0^2 + \nu(\sigma_t^2 - \sigma_0^2) + \sigma_w w_{t+1},
\end{aligned}$$

where ϵ, e and w are i.i.d. Normal (0,1). Note that for simplicity, only the long-run news in x_t are subject to stochastic volatility, whereas the short run and volatility news are homoscedastic.

1. Conjecture that the equilibrium price to consumption ratio is linear in the economic state variables. Use standard log linearization to determine the loadings of the price-to-consumption ratio on the relevant economic states. (To save time, throughout the problem ignore solutions to the constant drift coefficients and log-linearization parameters.) Briefly explain how the risk exposures depend on the preference and consumption dynamics parameters.

2. Solve for the equilibrium stochastic discount factor. What are the aggregate risks which demand a risk compensation? What are the market prices of these risks? Briefly explain how the market prices of risks depend on the preference and consumption dynamics parameters.

3. Now let's solve for the equilibrium value of *real perpetuity* (a.k.a. consol) S_t which makes a constant payment (say, 1) each period to infinity.

   Note that we can treat the perpetuity as an equity claim whose price today is S_t, price tomorrow is S_{t+1}, and tomorrow's dividend payment is fixed at 1. Use standard log-linearization to express the log return on perpetuity as a linear combination of its log prices today and tomorrow and the log growth of its payments. To simplify the math, you can assume that the log-linearization parameters κ_1 for the consumption claim in a) and the perpetuity are the same.

   Conjecture that the log price of perpetuity $s_t = \log S_t$ is linear in the economic states, and find the equilibrium loadings.

4. What are signs of the equilibrium loadings of the perpetuity to expected growth and volatility state variables? [If you get bogged down with the algebra, take them to be: negative to expected growth, and positive to consumption volatility.]

   What is the economics behind this: why would the value of perpetuity drop at times of high expected growth or low consumption volatility?

5. You collected data on market prices of equity and perpetuity. Interestingly, the difference between the log perpetuity price and the log (proxy for) price-consumption ratio is steadily increasing over the last 10 years. How can you explain this behavior from the perspective of our model?

6. We would like to solve for the optimal portfolio allocation between the consumption claim and the perpetuity. Can our model say anything interesting about the investor's portfolio choice? Explain.

Problem 2h: 2021 Finance qualifier at UW Madison

Investors have recursive Epstein-Zin preferences over consumption,

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta \left(E_t U_{t+1}^{1-\gamma} \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

The log stochastic discount factor is given by,

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1},$$

where $\theta \equiv \frac{1-\gamma}{1-1/\psi}$, Δc_{t+1} is the log real consumption growth, and $r_{c,t+1}$ is the log return on the consumption asset.

Real consumption growth Δc_t follows simplified Bansal-Yaron 2004 dynamics:

$$\Delta c_{t+1} = \mu + x_t + \sigma \epsilon_{t+1},$$
$$x_{t+1} = \rho x_t + \varphi_e \sigma e_{t+1},$$

where ϵ, e are i.i.d. Normal (0,1). For simplicity the volatilities of shocks are constant.

1. Conjecture that the equilibrium log price to consumption ratio ($pc_t = \log P_t/C_t$) is linear in the economic state variables. Use standard log linearization to solve for the price-to-consumption ratio, and the return on consumption claim $r_{c,t+1}$. Briefly explain how the risk exposures depend on the preference and consumption dynamics parameters.

   (To save time, throughout the problem ignore solutions to constant drift coefficients.)

2. Solve for the equilibrium stochastic discount factor M_{t+1}. What are the aggregate risks which demand a risk compensation? What are the market prices of these risks? Briefly explain how the market prices of risks depend on the preference and consumption dynamics parameters.

3. "Consumption strip at maturity n " is an asset which pays consumption C_{t+n} in period $t+n$ and nothing in other periods, so its price is time-t discounted value of $t+n$ consumption:

   $$P_{t,n} = E_t(M_{t+1} M_{t+2} \dots M_{t+n}) C_{t+n}.$$

   Note that the (total) consumption claim in part a) is a portfolio of consumption strips, so the price of consumption claim is the sum of prices of consumption strips at all the maturities:

   $$P_t = E_t M_{t+1} C_{t+1} + E_t(M_{t+1} M_{t+2}) C_{t+2} + E_t(M_{t+1} M_{t+2} M_{t+3}) C_{t+3} + \dots = \sum_{n=1}^{\infty} P_{t,n}$$

   Let's consider the consumption strip at maturity 1. Its payoff is C_{t+1} next period and nothing in other periods. Its price $P_{t,1}$ is

   $$P_{t,1} = E_t M_{t+1} C_{t+1}.$$

   Compute the equilibrium price of this consumption strip, given the dynamics of the economy and the stochastic discount factor. (Hint: divide both sides of the Euler equation above by C_t, and solve for $pc_{t,1} = \log P_{t,1}/C_t$. As before, you can ignore the constants.)

4. Solve for the return on the consumption strip, $\log r_{t+1,1} = \log C_{t+1}/P_{t,1} = \Delta c_{t+1} - pc_{t,1}$. The risk premium on any asset is approximately $-Cov_t(r_{t+1}, m_{t+1})$. Compute the risk premium on the consumption strip, and compare it to the risk premium on (total) consumption claim in part a).

   Some of the recent literature claims that short-term strips (1-period consumption strip) have higher average excess returns in the data than the claim on all the future cash-flows (the consumption asset). Is the model consistent with this evidence?

5. The literature also finds that the risk premium on consumption strips varies over time. Can the model explain this evidence? If not, how it can be extended to introduce variation in the risk premium?

Problem 3a: Adapted from 2015 Wharton qualifier

Let z_t be a factor which predicts equity returns. The autocorrelation of z is $0 < \rho < 1$. Assume the following relationship holds between the log market return and the factor:

$$r_{t+1} = \beta_1 z_t + \epsilon_{t+1},$$

where for simplicity ϵ_t is i.i.d. over time and independent from z_t, and $\beta_1 > 0$. Consider the k-period return predictability relationship:

$$r_{t+1} + r_{t+2} + \ldots + r_{t+k} = \beta_k z_t + \epsilon_{t,t+k}. \tag{5}$$

1. You want to test the predictability relation (5) empirically, so you regress $k-$period cumulative returns on z. How would you assess the economic and statistical significance of the predictor? What standard errors would you use and why?

   Now let's consider theoretical predictions for slope coefficients and the R^2s

2. Show that β_k coefficients increase with the horizon k (if you are short on time, you can compare the results for $k = 2$ and $k = 1$.)

3. Suppose we modified the left-hand side to be the *average* cumulative return $\frac{1}{k}\left(r_{t+1} + r_{t+2} + \ldots + r_{t+k}\right)$. How would this change the slope coefficients, the R^2s, and your answer to 2).

4. Will the R^2 in regressions 5 grow with k? (again, you may want to compare the R^2 in $k = 2$ case to the $k = 1$). What is the intuition behind the results? For example, contrast two cases: $Var(\epsilon_t)$ is very small ≈ 0, and $Var(\epsilon_t)$ is very large $\to \infty$.

5. The assumption that z shocks are independent from the return shocks may not be fully innocuous. What would you expect to be the sign of their correlation under a standard present-value decomposition? Given that it is assumed to be zero, what is the implied relationship between the factor and the cash flow news?

Problem 3b: From 2015 Wharton qualifier

3. Ignore the notation from above and now consider the following system of returns and dividend growth for the S& P 500:

$$\begin{aligned} r_{t+1} &= x_{r,t} + \eta_{r,t+1}, \qquad & x_{r,t+1} = \rho_r x_{r,t} + \epsilon_{r,t+1} \\ \Delta d_{t+1} &= x_{d,t} + \eta_{d,t+1}, \qquad & x_{d,t+1} = \rho_d x_{d,t} + \epsilon_{d,t+1} \end{aligned}$$

where r_{t+1} is log return on the S&P 500, and $x_{r,t}$ and $x_{d,t}$ are the expected return and dividend growth respectively. All ηs and ϵs shocks are mean zero and are independent of each other, and $0 < \rho_d, \rho_r < 1$.

Derive the (log) price-dividend ratio (z_t) in terms of state variables $x_{r,t}$ and $x_{d,t}$.

4. Derive a formula for the volatility of the price-dividend ratio? How does the autocorrelation of returns affect this volatility?

5. In the annual data the $\sigma(\Delta d_{t+1})^2 \approx 0.12^2$, $\sigma(\eta_{d,t+1})^2 \approx 0.06^2$, while $\sigma(r_{t+1})^2 \approx 0.16^2$, $\sigma(\eta_{r,t+1})^2 \approx 0.06^2$, and $\rho_d = 0.3$, $\rho_r = 0.3$, and the steady-state level of the log price-dividend ratio is 3.5. What is the volatility of the log price-dividend ratio?

6. Alternatively, suppose ~~in section (c) above~~ you learned that $\sum_{j=1}^{\infty} \kappa^j \beta_j \approx -0.6$, where β_j is the projection of r_{t+j} on the current price-dividend ratio. What does that information imply about the ability of the dividend yield to predict future dividend growth?

Problem 4: Jumps

There are lots of ways we can incorporate jumps (non-Gaussian shocks) into our economic models. Can we empirically distinguish their implications in the data?

Let's consider the following specification of consumption growth:

$$\Delta c_{t+1} = \mu + x_t + \sigma \eta_{t+1} + Q_{c,t+1}, \tag{6}$$

$$x_{t+1} = \rho x_t + \varphi_e \sigma \epsilon_{t+1} + Q_{x,t+1}. \tag{7}$$

As usual, η and ϵ are i.i.d. standard Normal innovations. Q_c and Q_x represent independent jump shocks which affect the realized and expected consumption, respectively. We model them as compensated compound Poisson processes:

$$Q_{c,t+1} = \sum_{i=1}^{N_{c,t+1}} J_{i,c,t+1} - \lambda_c \mu_{jc}, \quad Q_{x,t+1} = \sum_{i=1}^{N_{x,t+1}} J_{i,x,t+1} - \lambda_x \mu_{jx}.$$

N_c and N_x are independent Poisson processes with *constant* jump intensities λ_c and λ_x, respectively (i.e., $E_t N_{i,t+1} = \lambda_i$, for $i = c, x$). J are independent random variables which capture realized jumps in consumption (expected consumption). We specify the jump distributions through the moment-generating functions:

$$E_t e^{uJ_{i,c,t+1}} = \alpha_c(u), \quad E_t e^{uJ_{i,x,t+1}} = \alpha_x(u).$$

Finally, μ_{jc} and μ_{jx} are the mean jumps, e.g. $\mu_{jc} = E(J_{i,c,t}) = \alpha_c'(0)$. From an economic perspective, we assume that jumps are negative on average: $\mu_{jc} < 0, \mu_{jx} < 0$. We don't need to make any further parametric assumptions on jump size distributions.

While the jump terms may look complicated, they actually are very compatible with our usual exponentially-affine models. Indeed, under the assumed independence of jump components, one can show that:

$$\log E_t e^{u_1 Q_{c,t+1} + u_2 Q_{x,t+1}} = \log \left(E_t e^{u_1 Q_{c,t+1}} E_t e^{u_2 Q_{x,t+1}} \right) = \log E_t e^{u_1 Q_{c,t+1}} + \log E_t e^{u_2 Q_{x,t+1}}$$
$$= \lambda_c(\alpha_c(u_1) - u_1 \mu_{jc} - 1) + \lambda_x(\alpha_x(u_2) - u_2 \mu_{jx} - 1),$$

for all u_1 and u_2 for which these expressions exist.

1. Show that the jump components Q_c and Q_x are mean-zero innovations: $E_t Q_{c,t+1} = E_t Q_{x,t+1} = 0$. Sketch a typical time-series of Q shocks: how do they behave at jump times, and at times of no jumps?

2. Now let's consider a recursive utility environment. Solve for the equilibrium stochastic discount factor. Show that the jumps in realized and expected consumption growth are priced. Specifically, show that under the typical assumptions for the preference parameters ($\psi > 1, \gamma > 1$) the market prices of both jump risks are positive. Show that the market price of expected consumption jumps increases in the persistence of the expected consumption growth.

   Do these model implications make economic sense?

3. Consider the return on consumption asset – we can think about it as an equity claim. Show that it is exposed to both types of jumps, and that its jump betas are both positive (under usual preference configuration).

4. Let's consider the impact of jumps on the components of returns, such as the price-consumption ratio and the consumption growth itself. Show that when the economy gets hit by the realized consumption jumps Q_c, they immediately affect cash flows (consumption) but not asset valuations (price-dividend ratio). On the contrary, the expected consumption jumps Q_x affect asset valuations, but have no immediate impact on cash flows. Why is this the case?

5. How do these jumps affect equity risk premium? That is, does the conditional equity premium increase/decrease/stay the same when the economy gets hit by these jumps? Explain why.

6. In our setup, market volatility is constant. However, it is well known that return volatility fluctuates over time, and it can also jump upwards. Further, a lot of empirical evidence suggests that positive jumps in market volatility happen at times of large negative jumps in market returns themselves. How would you extend the model to capture this effect?