ECON 711 - PS 4

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Question 1. Choice rules from preferences

Let X be a choice set and $\succsim$ a complete and transitive preference relation on X. Show that the choice rule induced by $\succsim$, $C(A, \succsim) = \{x \in A : x \succsim y \; \forall y \in A\}$, must satisfy the Weak Axiom of Revealed Preference (WARP).

Proof: $C(\cdot)$ satisfies WARP if for any sets $A, B \subset X$ and any $x, y \in A \cap B$, if $x \in C(A)$ and $y \in C(B)$, then $x \in C(B)$ and $y \in C(A)$. Since $x \in C(A)$ and $y \in C(B)$, $x \succsim y$ and $y \succsim x$. For an arbitrary $w \in B$, $y \succsim w$ because $y \in C(B)$. By transitivity, $x \succsim w$, so $x \in C(B)$. For arbitrary $z \in A$, $x \succsim z$ because $x \in C(A)$. By transitivity, $y \succsim z$, so $y \in C(A)$. $\square$

Question 2. Preferences from choice rules

Let X be a choice set and $C : \mathcal{P}(X) \to \mathcal{P}(X)$ a nonempty choice rule. Show that if C satisfies WARP, then the preference relation $\succsim_C$ defined on X by "$x \succsim_C y$ iff there exists a set $A \subseteq X$ such that $x, y \in A$ and $x \in C(A)$" is complete and transitive, and that the choice rule it induces, $C(\cdot, \succsim_C)$, is equal to C.

Proof: For completeness, choose $x, y \in X$. Construct $A := \{x, y\}$. Since C is nonempty, we know that $x \in C(A)$ and/or $y \in C(A)$. If $x \in C(A)$, then $x \succsim_C y$. If $y \in C(A)$, then $y \succsim_C x$. Thus, $\succsim_C$ is complete.

For transitivity, choose $x, y, z \in X$ such that $x \succsim_C y$ and $y \succsim_C z$. This setup implies that there exists $A, B \subset X$ such that $x, y \in A$, $y, z \in B$, $x \in C(A)$, and $y \in C(B)$. Assume for sake of a contradiction that $x \notin C(A \cup B)$ and $z \in C(A \cup B)$. By WARP, $z \in C(A \cup B)$ and $y \in C(B)$ implies that $y \in C(A \cup B)$, By WARP, $y \in C(A \cup B)$ and $x \in C(A)$ implies that $x \in C(A \cup B) \Rightarrow\Leftarrow$. This is a contradiction, so $x \in C(A \cup B) \implies x \succsim_C z$.

For equality of $C(\cdot, \succsim_C)$ and C, fix nonempty $A \subset X$. Choose $x \in C(A)$. For an arbitrary $y \in A$, $x \succsim_C y$. Thus, $x \in C(A, \succsim)$. Choose $x \in C(A, \succsim_C)$, then $x \succsim_C y$ for all $y \in A$. Thus, $x \in C(A)$. Therefore, $C(\cdot, \succsim_C)$ is equal to C. $\square$

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Question 3. Choice over finite sets

Let X be a finite set, and $\succsim$ a complete and transitive preference relation on X. (Hint: for (a), fix X finite, and prove by induction. For (b) use induction on $|X|$ to prove the stronger result that when X is finite, a utility representation exists with range $\{1, 2, ..., |X|\}$)

(a) Show that the induced choice rule $C(\cdot, \succsim)$ is nonempty - that is $C(A, \succsim) \neq \emptyset$ if $A = \emptyset$.

Proof (by induction): Let nonempty $A, B \subset X$ such that $A := \{x\}$ for some $x \notin B$ and $|B| = n$ for some $n \in \mathbb{N}$. Notice that $|A| = 1$. Because $\succsim$ is complete, $x \succsim x$. Thus, x is weakly preferred to all elements of A. Thus, $x \in C(A, \succsim) \neq \emptyset$. Assume $C(B, \succsim) \neq \emptyset$. Notice that $|A \cup B| = n + 1$. Choose arbitrary y from $C(B, \succsim)$, so by definition $y \succsim z$ for all $z \in B$. By completeness, $x \succsim y$ and/or $y \succsim x$. If $x \succsim y$, x is weakly preferred to all elements in B by transitivity, so $x \in C(A \cup B, \succsim)$. If $y \succsim x$, then y is weakly preferred to all elements in $A \cup B$, so $y \in C(A \cup B, \succsim)$. Thus, $C(A \cup B, \succsim) \neq \emptyset$. $\square$

(b) Show that a utility representation exists.

Proof (by induction):