ECON 711 - PS 7

Alex von Hafften*

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A Risky Investment

You have wealth $w > 0$ and preferences over lotteries represented by a von Neumann-Morgenstern expected utility function with Bernoulli utility u which is strictly increasing, twice differentiable, and weakly concave. Your friend wants you to invest in his startup; you can choose any amount $a \leq w$ to invest, and your investment will either triple in value (with probability p) or become worthless (with probability $1-p$). Your expected utility if you invest a is therefore

$$U(a) = pu(w - a + 3a) + (1-p)u(w-a) = pu(w+2a) + (1-p)u(w-a)$$

(a) Show that if u is linear, then you invest all your wealth if $p > \frac{1}{2}$ and nothing if $p < \frac{1}{3}$.

From here on, assume $p > \frac{1}{3}$, so the expected value of the investment is positive; and assume that you are strictly risk-averse ($u'' < 0$).

(b) Show that it's optimal to invest a strictly positive amount. (You can do this by showing that $U'(0) > 0$ - the marginal expected utility of increasing a is positive when $a = 0$.)

(c) Show that $U(a)$ is strictly concave in a, so that expect at a corner solution, the first-order condition is necessary and sufficient to find a^*.

(d) Show that if $u'(0)$ is infinite, it's not optimal to invest all your wealth; and that if $u'(0)$ is finite, then there's a cutoff $\bar{p}$ such that it's optimal to invest all of your wealth if $p \geq \bar{p}$.

From here on, assume that either $u'(0)$ is infinite or $p \in (\frac{1}{3}, \bar{p})$, so the optimal level of investment a^* is strictly positive but below w.

(e) Show that if $u(x) = 1 - e^{-cx}$ (the Constant Absolute Risk Aversion or CARA utility function), your optimal investment a^* does not depend on w.

(f) For general u, show that if your Coefficient of Absolute Risk Aversion $A(x) = -\frac{u''(x)}{u'(x)}$ is decreasing, you invest more as w increases.

Now reframe the question as deciding what fraction t of your wealth to invest; writing $a = tw$,

$$U(t) = pu(w(1+2t)) + (1-p)u(w(1-t))$$

(g) Show that if $u(x) = \frac{1}{1-\rho}x^{1-\rho}$, with $\rho \leq 1$ and $\rho \neq 0$ (the Constant Relative Risk Aversion or CRRA utility function), you invest the same fraction of your wealth regardless of w.

(h) For general u, show that if your Coefficient of Relative Risk Aversion $R(x) = -\frac{xu''(x)}{xu'(x)}$ is increasing, you invest a smaller fraction of your wealth as w increases.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, Tyler Welch, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.