# ECON 709B - Problem Set 1

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1. 2.1 - 2.2[1]

2.1 Find $E[E[E[Y|X_1, X_2, X_3]|X_1, X_2]|X_1]$.

Using the law of iterated expectations,

$$E[E[E[Y|X_1, X_2, X_3]|X_1, X_2]|X_1] = E[E[Y|X_1, X_2]|X_1] = E[Y|X_1]$$

2.2 If $E[Y|X] = a + bX$, find $E[YX]$ as a function of moments of $X$.

Using the law of iterated expectations,

$$E[YX] = E[E[YX|X]] = E[XE[Y|X]] = E[X(a + bX)] = E[aX + bX^2] = aE[X] + bE[X^2]$$

2. 2.3 Prove conclusion (4) of Theorem 2.4.

If $E|Y| < \infty$ then for any function $h(x)$ such that $E|h(X)e| < \infty$ then $E[h(X)e] = 0$.

Proof: Using the law of iterated expectations, Theorem 2.3, and conclusion (1) (i.e., $E[e|X] = 0$),

$$E[h(X)e] = E[E[h(X)e|X]] = E[h(X)E[e|X]] = E[h(X)(0)] = E[0] = 0$$

$\square$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

[1] These problems come from *Econometrics* by Bruce Hansen, revised on October 23, 2020.

3. 2.4 Suppose that the random variables $Y$ and $X$ only take the values 0 and 1, and have the following joint probability distribution

| | $X=0$ | $X=1$ |
|---|---|---|
| $Y=0$ | .1 | .2 |
| $Y=1$ | .4 | .3 |

Find $E[Y|X]$, $E[Y^2|X]$ and $\text{var}[Y|X]$ for $X=0$, $X=1$.

$$E[Y|X=0]=(1)P[Y=1|X=0]+(0)P[Y=0|X=0]=(1)(.4)/(.5)=.8$$
$$E[Y|X=1]=(1)P[Y=1|X=1]+(0)P[Y=0|X=1]=(1)(.3)/(.5)=.6$$
$$E[Y^2|X=0]=(1)^2P[Y=1|X=0]+(0)^2P[Y=0|X=0]=(1)^2(.4)/(.5)=.8$$
$$E[Y^2|X=1]=(1)^2P[Y=1|X=1]+(0)^2P[Y=0|X=1]=(1)^2(.3)/(.5)=.6$$
$$\text{var}[Y|X=0]=E[Y^2|X=0]-(E[Y|X=0])^2=(.8)-(.8)^2=0.16$$
$$\text{var}[Y|X=1]=E[Y^2|X=1]-(E[Y|X=1])^2=(.6)-(.6)^2=0.24$$

4. 2.5 (c) Show that $\sigma^2(X)$ is the best predictor of $e^2$ given $X$. Show that $\sigma^2(X)$ minimizes the mean-squared error and is thus the best predictor.

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5. 2.8 Suppose that $Y$ is discrete-valued, taking values only on the non-negative integers, and the conditional distribution of $Y$ given $X=x$ is Poisson: $P[Y=j|X=x]=\frac{\exp(-x'\beta)(x'\beta)^j}{j!}, j=0,1,2,....$ Compute $E[Y|X]$ and $\text{var}[Y|X]$. Does this justify a linear regression model of the form $Y=X'\beta+e$?[2]

Using the hint, we know that $E[Y|X]=x'\beta$ and $\text{var}[Y|X]=x'\beta$.

Yes, this justifies a linear regression model because $E[e|X]=E[Y-X'\beta|X]=E[Y|X]-E[X'\beta|X]=x'\beta-x'\beta=0$.

6. 2.10 - 2.14 Explain your answers.

2.10 If $Y=X\beta+e, X\in\mathbb{R}$, and $E[e|X]=0$, then $E[X^2e]=0$.

True, based on the law of iterated expectation:

$$E[X^2e]=E[E[X^2e|X]]=E[X^2E[e|X]]=E[X^2(0)]=E[0]=0$$

2.11 If $Y=X\beta+e, X\in\mathbb{R}$, and $E[Xe]=0$, then $E[X^2e]=0$.

False, for a counter example, assume $X\sim N(0,1)$ and $e$ is a degenerate random variable equal to 1. Notice that $E[Xe]=E[X]=0$ and $E[X^2e]=E[X^2]=1$.

2.12 If $Y=X'\beta+e$, and $E[e|X]=0$, then $e$ is independent of $X$.

False, for a counter example, assume $X=(X_1,...,X_k)'$ and $U=(U_1,...,U_k)'$ where $X_1,...,X_k$ and $U_1,...,U_k$ are independently distributed standard normal. Let $e=X'U$. Thus, conditional on $X$, $e$ is distributed $N(0,X'X)$, so $E[e|X]=0$, but $X$ and $e$ are not independent.

2.13 If $Y=X'\beta+e$, and $E[Xe]=0$, then $E[e|X]=0$.

False, for a counter example, assume $X=(X_1,...,X_k)'$ where $X_1,...,X_k\sim N(0,1)$ and $e$ is a degenerate random variable equal to 1. Notice that $E[Xe]=E[X]=0$ and $E[e|X]=E[e]=1$.

[2] Hint: $P[Y=j]=\frac{\exp(-\lambda)(\lambda)^j}{j!}$, then $E[Y]=\lambda$ and $\text{var}[Y]=\lambda$.

2.14 If $Y = X'\beta + e$, and $E[e|X] = 0$, and $E[e^2|X] = \sigma^2$, then $e$ is independent of $X$.

False, $\ldots$

7. 2.16 Let $X$ and $Y$ have the joint density $f(x,y) = \frac{3}{2}(x^2+y^2)$ on $0 \le x \le 1, 0 \le y \le 1$. Compute the coefficients of the best linear predictor $Y = \alpha + \beta X + e$. Compute the conditional expectation $m(x) = E[Y|X=x]$. Are the best linear predictor and conditional expectation different?

Best Linear Predictor (BLP):

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \frac{3}{2}(x^2+y^2)dy = \frac{3}{2}\left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{1} = \frac{3}{2}x^2 + \frac{1}{2}$$

$$E[X] = \int_0^1 x\left(\frac{3}{2}x^2 + \frac{1}{2}\right)dx = \left[\frac{3}{8}x^4 + \frac{1}{4}x^2\right]_0^1 = \frac{5}{8}$$

$$E[X^2] = \int_0^1 x^2\left(\frac{3}{2}x^2 + \frac{1}{2}\right)dx = \left[\frac{3}{10}x^5 + \frac{1}{6}x^3\right]_0^1 = \frac{7}{15}$$

Since the joint distribution is symmetric, $E[Y] = \frac{5}{8}$ and $E[Y^2] = \frac{7}{15}$.

$$\begin{aligned} E[XY] &= \int_0^1\int_0^1 xy\frac{3}{2}(x^2+y^2)dxdy \\ &= \frac{3}{2}\int_0^1\int_0^1 x^3y + xy^3 dxdy \\ &= \frac{3}{2}\int_0^1 \left[\frac{x^4y}{4} + \frac{x^2y^3}{2}\right]_{x=0}^{1} dy \\ &= \frac{3}{2}\int_0^1 \frac{y}{4} + \frac{y^3}{2}dy \\ &= \frac{3}{2}\left[\frac{y^2}{8} + \frac{y^4}{8}\right]_0^1 \\ &= \frac{3}{8} \end{aligned}$$

$$\begin{aligned} S(\alpha,\beta) &= E[(Y-\alpha-\beta X)^2] \\ &= E[Y^2 - \alpha Y - \beta XY - \alpha Y + \alpha^2 + \alpha\beta X - \beta XY + \alpha\beta X + \beta^2 X^2] \\ &= E[Y^2] - \alpha E[Y] - \beta E[XY] - \alpha E[Y] + \alpha^2 + \alpha\beta E[X] - \beta E[XY] + \alpha\beta E[X] + \beta^2 E[X^2] \\ &= \frac{7}{15} - \frac{5}{8}\alpha - \frac{3}{8}\beta - \frac{5}{8}\alpha + \alpha^2 + \frac{5}{8}\alpha\beta - \frac{3}{8}\beta + \frac{5}{8}\alpha\beta + \frac{7}{15}\beta^2 \\ &= \frac{7}{15} - \frac{5}{4}\alpha - \frac{3}{4}\beta + \alpha^2 + \frac{5}{4}\alpha\beta + \frac{7}{15}\beta^2 \end{aligned}$$

FOC [$\alpha$]:

$$0 = -\frac{5}{4} + 2\alpha + \frac{5}{4}\beta \implies \beta = 1 - \frac{8}{5}\alpha$$

FOC [$\beta$]:

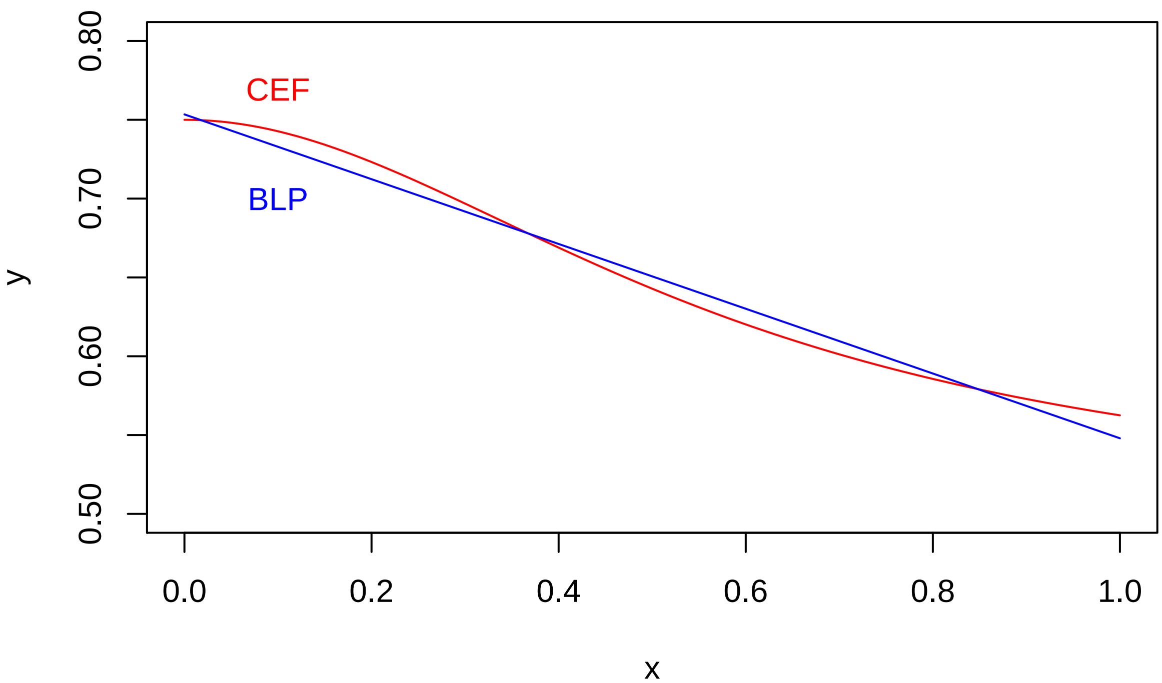
$$0 = -\frac{3}{4} + \frac{5}{4}\alpha + \frac{14}{15}\beta \implies 45 = 75\alpha + 56\beta$$

$$45 = 75\alpha + 56\left(1 - \frac{8}{5}\alpha\right) \implies \alpha = \frac{55}{73} \implies \beta = \frac{-15}{73}$$

Conditional Expectation Function (CEF):

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}x^2 + \frac{1}{2}} = \frac{3x^2+3y^2}{3x^2+1}$$

$$m(x) = E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_0^1 \frac{3x^2y+3y^3}{3x^2+1}dy = \left[\frac{\frac{3}{2}x^2y^2 + \frac{3}{4}y^4}{3x^2+1}\right]_{y=0}^{1} = \frac{6x^2+3}{12x^2+4}$$



8. 4.1 - 4.6

4.1 For some integer $k$, set $\mu_k = E[Y^k]$.

(a) Construct an estimator $\hat{\mu}_k$ for $\mu_k$.

$$\hat{\mu}_k = \frac{1}{n}\sum_{i=0}^{n} Y_i^k$$

(b) Show that $\hat{\mu}_k$ is unbiased for $\mu_k$.

$$E[\hat{\mu}_k] = E\left[\frac{1}{n}\sum_{i=0}^{n} Y_i^k\right] = \frac{1}{n}\sum_{i=0}^{n} E[Y_i^k] = \frac{1}{n}\sum_{i=0}^{n} \mu_k = \mu_k$$

(c) Calculate the variance of $\hat{\mu}_k$, say $\text{var}[\hat{\mu}_k]$. What assumption is needed for $\text{var}[\hat{\mu}_k]$ to be finite?

We need to assume that $\mu_{2k} < \infty$ for $\text{var}[\hat{\mu}_k]$ to be finite:

$$\text{var}[\hat{\mu}_k] = \text{var}\left[\frac{1}{n}\sum_{i=0}^{n} Y_i^k\right] = \frac{1}{n^2}\sum_{i=0}^{n} \text{var}[Y_i^k] = \frac{1}{n^2}\sum_{i=0}^{n}(E[Y_i^{2k}] - E[Y_i^k]^2) = \frac{1}{n^2}\sum_{i=0}^{n}(\mu_{2k} - \mu_k^2) = \frac{\mu_{2k} - \mu_k^2}{n}$$

(d) Propose an estimator of $\text{var}[\hat{\mu}_k]$.

$$\frac{\hat{\mu}_{2k} - \hat{\mu}_k^2}{n} = \frac{\sum_{i=0}^{n} Y_i^{2k} - (\sum_{i=0}^{n} Y_i^k)^2}{n}$$

4.2 Calculate $E[(\bar{Y} - \mu)^3]$, the skewness of $\bar{y}$. Under what conditions is it zero?

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4.3 Explain the difference between $\bar{Y}$ and $\mu$. Explain the difference between $n^{-1}\sum_{i=1}^{n} X_i X_i'$ and $EX_iX_i'$.

...

4.4 True or False. If $Y_i = X_i\beta + e_i, X_i \in \mathbb{R}, E[x_i|X_i] = 0$, and $\hat{e}_i$ is the OLS residual from the regression of $Y_i$ on $X_i$, then $\sum_{i=1}^{n} X_i^2\hat{e}_i = 0$.

...

4.5 Prove (4.15) and (4.16).

(4.15) $E[\hat{\beta}|X] = \beta$

...

(4.16) $\text{var}[\hat{\beta}|X] = (X'X)^{-1}(X'\Omega X)(X'X)^{-1}$

...

4.6 Prove Theorem 4.5.

Theorem 4.5 Generalized Gauss-Markov

In the linear regression model (Assumption 4.2) and $\Omega > 0$, if $\tilde{\beta}$ is a linear unbiased estimator of $\beta$ then $\text{var}[\tilde{\beta}|X] \geq (X'\Omega^{-1}X)^{-1}$

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