ECON 736A: Problem Set 2

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1 Acharya and Dogra (2020)

1. *Carefully prove Proposition 1 in the paper*

   Proposition 1. *Individual decision problem: Given a sequence of real interest rates, aggregate output, and idiosyncratic risk $\{r_t, y_t, \sigma_{y,t}\}$, household i's consumption decision can be expressed as*

   $$c_t^i = \mathcal{C}_t + \mu_t(a_t^i + y_t^i) \tag{1}$$

   where $a_t^i = A_t^i/P_t$ is real net worth at the state of date t and $\mathcal{C}_t$ and μ_t solve the following recursions:

   $$\mathcal{C}_t[1 + \mu_{t+1}(1 + r_t)] = -\frac{1}{\gamma}\ln\beta(1 + r_t) + \mathcal{C}_{t+1} + \mu_{t+1}\bar{y}_{t+1} - \frac{\gamma\mu_{t+1}^2\sigma_{y,t+1}^2}{2} \tag{2}$$

   $$\mu_t = \frac{\mu_{t+1}(1 + r_t)}{1 + \mu_{t+1}(1 + r_t)} \tag{3}$$

   We prove with guess and verify. With CARA utility, the household Euler equation is

   $$e^{-\gamma c_t^i} = \beta(1 + r_t)E_t e^{-\gamma c_{t+1}^i}$$

   Taking logs of both sides

   $$-\gamma c_t^i = \ln\beta(1 + r_t) + \ln E_t e^{-\gamma c_{t+1}^i}$$

   Guess consumption decision rule is

   $$c_t^i = \mathcal{C}_t + \mu_t(a_t^i + y_t^i)$$

   where $\mathcal{C}_t$ and μ_t are deterministic. Using the HH budget constraint and the guessed consumption decision rule, we can get c_{t+1}^i:

$$
\begin{aligned}
a^i_{t+1} &= (1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t \\
\implies c^i_{t+1} &= \mathcal{C}_{t+1} + \mu_{t+1}[a^i_{t+1} + y^i_{t+1}] \\
&= \mathcal{C}_{t+1} + \mu_{t+1}[(1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t + y^i_{t+1}]
\end{aligned}
$$

All terms in c^i_{t+1} are known at t except for y^i_{t+1} which is normal, so c^i_{t+1} is normal with

$$
\begin{aligned}
E_t[c^i_{t+1}] &= \mathcal{C}_{t+1} + \mu_{t+1}[(1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t + \bar{y}_{t+1}] \\
Var_t[c^i_{t+1}] &= \mu^2_{t+1}\sigma^2_{y,t+1}
\end{aligned}
$$

Applying MGF of normal distributions,

$$
\begin{aligned}
\ln E_t[e^{-\gamma c^i_{t+1}}] &= E_t[-\gamma c^i_{t+1}] + \frac{Var_t[-\gamma c^i_{t+1}]}{2} \\
&= -\gamma E_t[c^i_{t+1}] + \frac{\gamma^2 Var_t[c^i_{t+1}]}{2} \\
&= -\gamma \mathcal{C}_{t+1} - \gamma\mu_{t+1}[(1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t + \bar{y}_{t+1}] + \frac{\gamma^2\mu^2_{t+1}\sigma^2_{y,t+1}}{2}
\end{aligned}
$$

Substituting into the logged Euler equation,

$$
-\gamma[\mathcal{C}_t + \mu_t(a^i_t + y^i_t)] = \ln\beta(1+r_t) - \gamma\mathcal{C}_{t+1} - \gamma\mu_{t+1}[(1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t + \bar{y}_{t+1}] + \frac{\gamma^2\mu^2_{t+1}\sigma^2_{y,t+1}}{2}
$$

Matching coefficients on $(a^i_t + y^i_t)$:

$$
\begin{aligned}
-\gamma\mu_t &= -\gamma\mu_{t+1}(1+r_t)(1-\mu_t) \\
\mu_t(1 + \mu_{t+1}(1+r_t)) &= \mu_{t+1}(1+r_t) \\
\mu_t &= \frac{\mu_{t+1}(1+r_t)}{1+\mu_{t+1}(1+r_t)}
\end{aligned}
$$

Matching constant coefficients:

$$
\begin{aligned}
-\gamma\mathcal{C}_t &= \ln\beta(1+r_t) - \gamma\mathcal{C}_{t+1} + \gamma\mu_{t+1}(1+r_t)\mathcal{C}_t + \gamma\bar{y}_{t+1} + \frac{\gamma^2\mu^2_{t+1}\sigma^2_{y,t+1}}{2} \\
\mathcal{C}_t[1 + \mu_{t+1}(1+r_t)] &= -\frac{1}{\gamma}\ln\beta(1+r_t) + \mathcal{C}_{t+1} + \mu_{t+1}\bar{y}_{t+1} - \frac{\gamma\mu^2_{t+1}\sigma^2_{y,t+1}}{2}
\end{aligned}
$$

2. *Carefully derive equations 4.1-4.4 in the paper.*

Linearizing (3.3), (3.5), (3.6), and (2.2)

$$\mu_t = \frac{\mu_{t+1}(1+r_t)}{1+\mu_{t+1}(1+r_t)} \tag{4}$$

$$y_t = y_{t+1} - \frac{\ln\beta(1+r_t)}{\gamma} - \frac{\gamma\mu_{t+1}^2\sigma_{y,t+1}^2}{2} + g_t - g_{t+1} \tag{5}$$

$$\Psi\Pi_t(\Pi_t - 1) = 1 - \theta(1 - mc_t) + \Psi(\Pi_{t+1} - 1)\Pi_{t+1}\left[\frac{1}{1+r_t}\frac{x_{t+1}}{x_t}\right] \tag{6}$$

$$1 + i_t = (1+r)\Pi_t^{\Phi_\pi} \geq 1 \tag{7}$$

around the flexible price level of output $y^* = 1$, $\mu = \frac{r}{1+r}$, and $\Pi = 1$, we get

$$\hat{y}_t = \Theta\hat{y}_{t+1} - \frac{1}{\gamma}(i_t - \pi_{t+1}) - \Lambda\hat{\mu}_{t+1} \tag{8}$$

$$\hat{\mu}_t = \tilde{\beta}\hat{\mu}_{t+1} + \tilde{\beta}(i_t - \pi_{t+1}) \tag{9}$$

$$\pi_t = \tilde{\beta}\pi_{t+1} + \kappa\hat{y}_t \tag{10}$$

$$i_t = \Phi_\pi\pi_t \tag{11}$$

where

$$\Theta = 1 - \frac{\gamma\mu^2}{2}\frac{d\boldsymbol{\sigma}^2(y^*)}{dy}$$

$$\Lambda = \gamma\mu^2\boldsymbol{\sigma}_y^2(y^*)$$

$$\tilde{\beta} = \frac{1}{1+r}$$

κ denotes the slope of the linearized Phillips curve, and $\hat{y}_t$, $\hat{\mu}_t$, i_t, and π_t denote the log deviation of y_t, μ_t, $1 + i_t$, and Π_t from their steady state values.

2 Werning (2015)

1. *Carefully derive Proposition 5 in the paper.*

Proposition 5. *Suppose utilities satisfy are log utility, household income satisfies $\gamma_t^i(s,Y) = \tilde{\gamma}_t^i(s)Y$ (i.e., proportional to aggregate income) and borrowing constraints satisfy $B_t^i(s,Y) = \tilde{B}_t^i(s)Y$ (i.e., also proportional to aggregate income. In addition, suppose initial bond holdings are zero $b_0^i = 0$ for all households. Then $\{C_t, R_t\}$ is part of an equilibrium if and only if*

$$U'(C_t) = \beta_t R_t U'(C_{t+1}) \tag{12}$$

for some sequence of discount factors $\{\beta_t\}$, independent of both $\{R_t\}$ and $\{C_t\}$.

I follow Werning's discussion following the proposition.

First, we can construct the discount factors β_t and household allocation by considering a "reference" equilibrium where aggregate income is constant $\tilde{Y}_t = 1$. This equilibrium includes interest rates $\{\tilde{R}_t\}$, household consumption and wealth $\{\tilde{c}(s^t;a_0), \tilde{a}(s^t;a_0)\}$, and asset prices are the discounted value of future dividends $\tilde{q}_t = \sum_{s=0}^{\infty}(\tilde{R}_t\tilde{R}_{t+1}...\tilde{R}_{t+1})^{-1}\tilde{d}_{t+1+s}$. Define $\beta_t \equiv \frac{1}{\tilde{R}_t}$. The aggregate euler equation (12) holds trivially:

$$\begin{aligned}
U'(\tilde{C}_t) &= \beta_t \tilde{R}_t U'(\tilde{C}_{t+1}) \\
\iff U'(\tilde{Y}_t) &= \frac{1}{\tilde{R}_t}\tilde{R}_t U'(\tilde{Y}_{t+1}) \\
\iff U'(1) &= U'(1)
\end{aligned}$$

We can now consider another sequence $\{C_t, R_t\}$ that satisfies (12) and guess the equilibrium objects and verify that equilibrium conditions hold. We can guess that household i consumption and wealth, interest rates, and asset prices are the following:

$$\begin{aligned}
c^i(s^t;a_0) &\equiv \tilde{c}^i(s^t;a_0)C_t \\
a^i(s^t;a_0) &\equiv \tilde{a}^i(s^t;a_0)C_t \\
R_t &\equiv \tilde{R}_t \frac{C_{t+1}}{C_t} \\
q_t &\equiv \tilde{q}_t C_t
\end{aligned}$$

For the household Euler equation, start with the household Euler equation in the reference equilibrium:

$$
\begin{aligned}
U'(\tilde{c}^i(s^t;a_0)) &\geq \beta \tilde{R}_t E_t[U'(\tilde{c}^i(s^{t+1};a_0))] \\
\iff U'(\tilde{c}^i(s^t;a_0))\frac{C_t}{C_t} &\geq \beta \tilde{R}_t E_t\left[U'(\tilde{c}^i(s^{t+1};a_0))\frac{C_{t+1}}{C_{t+1}}\right] \\
\iff U'(c^i(s^t;a_0)) &\geq \beta \frac{\tilde{R}_t C_{t+1}}{C_t} E_t\left[U'(c^i(s^{t+1};a_0))\right] \\
\iff U'(c^i(s^t;a_0)) &\geq \beta R_t E_t[U'(c^i(s^{t+1};a_0))]
\end{aligned}
$$