ECON 703 - PS 3

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(1) Let (X, d) be a nonempty complete metric space. Suppose an operator $T : X \to X$ satisfies $d(T(x), T(y)) < d(x, y)$ for all $x \neq y, x, y \in X$. Prove or disprove that T has a fixed point. Compare with the Contraction Mapping Theorem.

I prove that T has a fixed point.

Proof: Define $\beta_{xy} := \frac{d(T(x),T(y))}{d(x,y)}$ and $\beta := \max \beta_{xy}$ for all $x \neq y, x, y \in X$. Notice that $d(T(x), T(y)) < d(x, y) \implies \beta_{xy} < 1$. Thus, $d(T(x), T(y)) = \beta_{xy} d(x, y) \leq \beta d(x, y)$. By the convergence mapping theorem, T has a fixed point. $\square$

(2) Does there exist a countable set, which is compact?

Yes.

Proof: Consider $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. A is a countable set because the union of countable sets is countable. Using the Heine-Borel theorem, I show that $A \subset \mathbb{R}$ is compact because it is bounded and closed. First, A is bounded between 0 and 1. Second, let $\{a_{n_k}\}$ be an arbitrary convergent subsequence. $B^{\varepsilon}(0)$ contains an infinite number of a_{n_k}. There is a finite number of a_{n_k} to get to ε.

(3) Prove that the function $f(x) = \cos^2(x)e^{5-x-x^2}$ has a maximum on $\mathbb{R}$.

Proof: Using the extreme value theorem, I show that f attains a maximum on $\mathbb{R}$ by (i) finding a compact set $C \subset \mathbb{R}$ where $\exists x \in C$ such that $f(x) > f(y) \forall y \in C^c$ and (ii) showing that f is continuous on $\mathbb{R}$. Define $g(x) = \cos^2(x)$, $h_1(x) = e^x$, $h_2(x) = 5 - x - x^2$, so that $f(x) = g(x)h_1(h_2(x))$.

For (i), define $C = [(-\sqrt{21} - 1)/2, (\sqrt{21} - 1)/2]$. As a closed interval of $\mathbb{R}$, C is compact by the Heine-Borel theorem. For $0 \in C$, $f(0) = \cos^2(0)e^{5-0-0^2} = e^5 > 1$. Since $0 \leq \cos^2(x) \leq 1$, $f(x) \leq h_1(h_2(x))$ for all $x \in \mathbb{R}$. Furthermore, if $h_2(x) < 0 \implies h_1(h_2(x)) < 1$. Because $h_2(0) = 5 - 0 - 0^2 = 5$ and the quadratic roots of h_2 are $x = (\sqrt{21} - 1)/2$ and $x = (-\sqrt{21} - 1)/2$, $f(x) \leq h_1(h_2(x)) < 1$ for all $x \in C^c$. Thus, the maximum of $f(x)$ on $\mathbb{R}$ must occur on C.

For (ii), since g, h_1, and h_2 are continuous, f is continuous.

Thus, by the extreme value theorem, f attains a maximum on C and therefore on $\mathbb{R}$. $\square$

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(4) Suppose you have two maps of Wisconsin: one large and one small. You put the large one on top of the small one, so that the small one is completely covered by the large one. Prove that it is possible to pierce the stack of those two maps in a way that the needle will go through exactly the same (geographical) points on both maps.

Proof: Draw Cartesian planes on both maps such that the Wisconsin Capital Building in Madison is at the origin. For the larger map, scale the x coordinate such that moving one unit right cooresponds to the point on the map that is one inch east of the Capital Building on the map and scale the y coordinate such that moving one unit up cooresponds to the point on the map that is one inch north of the Capital Building on the map. Scale the Cartesian plane on the smaller map similarly.

Define β as the the ratio of the miles between geographical points per inch on the larger map to the miles between geographical points per inch on the smaller map. Notice that $\beta < 1$. Define $A \subset \mathbb{R} \times \mathbb{R}$ as the closed set of points on the larger map that are on or within Wisconsin's borders. Notice that the metric space (A, d_E) is complete because A is closed. Define operator $T : A \to \mathbb{R} \times \mathbb{R}$ such that, for $x \in A$, (x_1, x_2) on the larger map and $(T(x_1), T(x_2))$ on the smaller map represent the same geographical point. Notice that, for all $x, y \in A, x \neq y$, $d_E(x, y) = \beta d_E(T(x), T(y))$. Thus, T is a contraction with modulus $\beta < 1$. Thus, T has a fixed point where $x^* = T(x^*)$, where you can pierce the stack of maps at the same geographical point. $\square$

(6) Consider the following system of linear equations

$$
\begin{align}
x_1 + x_2 + 2x_3 + x_4 = 0 \tag{1}\\
3x_1 - x_2 + x_3 - x_4 = 0 \tag{2}\\
5x_1 - 3x_2 - 3x_4 = 0 \tag{3}
\end{align}
$$

Let X be the set of $\{x_1, x_2, x_3, x_4\}$ which satisfy the system of equations.

(a) Show that X is a vector space.

I first find a basis for X:

$$
\begin{aligned}
(1) + (2) \implies 4x_1 + 3x_3 &= 0\\
x_1 &= (-3/4)x_3\\
(1) \implies x_2 + (5/4)x_3 + x_4 &= 0\\
(3) - 3 * (1) \implies 5(-3/4x_3) + 3x_2 + (3/4)x_3 &= 0\\
x_2 &= x_3\\
(2) \implies -x_3 - (5/4)x_3 - x_4 &= 0\\
x_4 &= (-9/4)x_3
\end{aligned}
$$

Letting $x_3 = \alpha$, all elements of X can be represented by $(-3/4\alpha, \alpha, \alpha, -9/4\alpha)$ for some $\alpha \in \mathbb{R}$. Thus, $\mathfrak{B} = \{(-3, 4, 4, -9)\}$ forms a basis for X because it spans X and it (having one element) is linearly independent, so any $x \in X$ can be written as $x = a * v$ where $v = (-3, 4, 4, -9)$ and $a \in \mathbb{R}$.

For $x, y, z \in X$, we know that $x = a * v$, $y = b * v$, $z = c * v$ for some $a, b, c \in \mathbb{R}$.

Associativity of $+$ in X:

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(b) Calculate dim X.

From (a), we found that $\mathfrak{B} = \{(-3, 4, 4, -9)\}$. The dimension of X is the cardinality of $\mathfrak{B}$, so $\dim X = 1$.