ECON 899B - PS2

Alex von Hafften

11/22/2021

Part 1 - Quadrature Integration

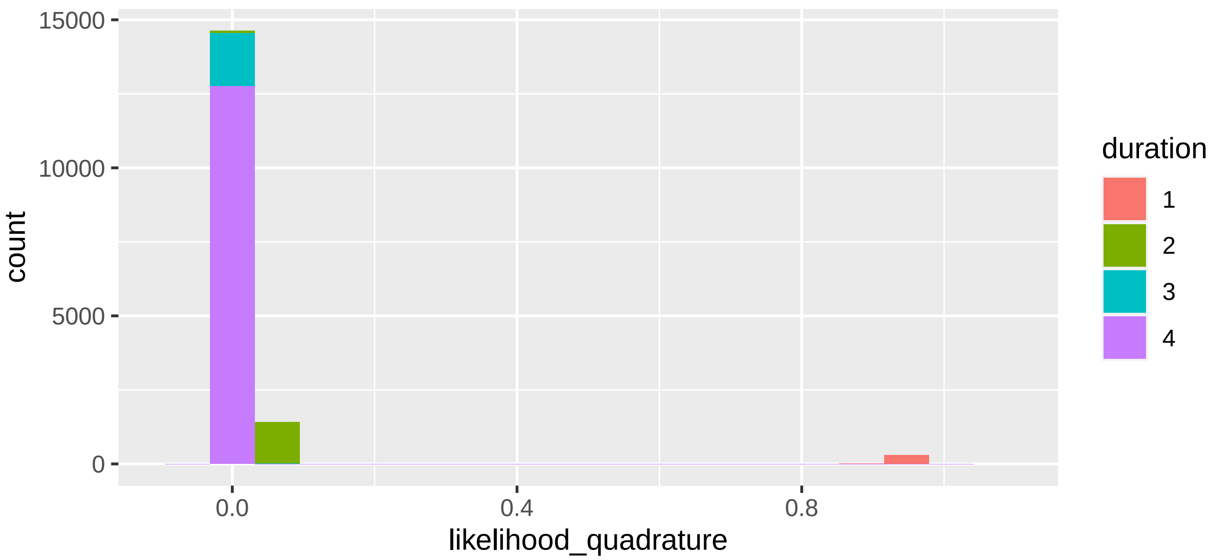
To implement the quadrature integration, I did not understand the provided equations from the problem set, so I derived the equations myself:

$$
\begin{aligned}
&P(T_i|X_i, Z_{it}, \theta) \\
&= \begin{cases} P(\alpha_0 + X_i\beta + Z_{i0}\gamma + \varepsilon_{i0} > 0) & \text{if } T_i = 1 \\ P(\alpha_0 + X_i\beta + Z_{i0}\gamma + \varepsilon_{i0} < 0, \alpha_0 + X_i\beta + Z_{i1}\gamma + \varepsilon_{i1} > 0) & \text{if } T_i = 2 \\ P(\alpha_0 + X_i\beta + Z_{i0}\gamma + \varepsilon_{i0} < 0, \alpha_0 + X_i\beta + Z_{i1}\gamma + \varepsilon_{i1} < 0, \alpha_0 + X_i\beta + Z_{i2}\gamma + \varepsilon_{i2} > 0) & \text{if } T_i = 3 \\ P(\alpha_0 + X_i\beta + Z_{i0}\gamma + \varepsilon_{i0} < 0, \alpha_0 + X_i\beta + Z_{i1}\gamma + \varepsilon_{i1} < 0, \alpha_0 + X_i\beta + Z_{i2}\gamma + \varepsilon_{i2} < 0) & \text{if } T_i = 4 \end{cases} \\
&= \begin{cases} \Phi((-\alpha_0 - X_i\beta - Z_{i0}\gamma)/\sigma_0) & \text{if } T_i = 1 \\ \int_{-\infty}^{-\alpha_0 - X_i\beta - Z_{i0}\gamma} \phi(\frac{\varepsilon_{i0}}{\sigma_0})\frac{1}{\sigma_0}[1 - \Phi(-\alpha_1 - X_i\beta - Z_{i1}\gamma - \rho\varepsilon_{i0})]d\varepsilon_{i0} & \text{if } T_i = 2 \\ \int_{-\infty}^{-\alpha_0 - X_i\beta - Z_{i0}\gamma} \int_{-\infty}^{-\alpha_1 - X_i\beta - Z_{1i}\gamma} \phi(\frac{\varepsilon_{i0}}{\sigma_0})\frac{1}{\sigma_0}\phi(\varepsilon_{i1} - \rho\varepsilon_{i1})[1 - \Phi(-\alpha_1 - X_i\beta - Z_{i1}\gamma - \rho\varepsilon_{i0})]d\varepsilon_{i1}d\varepsilon_{i0} & \text{if } T_i = 3 \\ \int_{-\infty}^{-\alpha_0 - X_i\beta - Z_{i0}\gamma} \int_{-\infty}^{-\alpha_1 - X_i\beta - Z_{1i}\gamma} \phi(\frac{\varepsilon_{i0}}{\sigma_0})\frac{1}{\sigma_0}\phi(\varepsilon_{i1} - \rho\varepsilon_{i1})\Phi(-\alpha_1 - X_i\beta - Z_{i1}\gamma - \rho\varepsilon_{i0})d\varepsilon_{i1}d\varepsilon_{i0} & \text{if } T_i = 4 \end{cases}
\end{aligned}
$$

The resulting estimated log-likelihood is:

```
## [1] -138618.8
```

A histogram of the estimated likelihoods by duration are below:

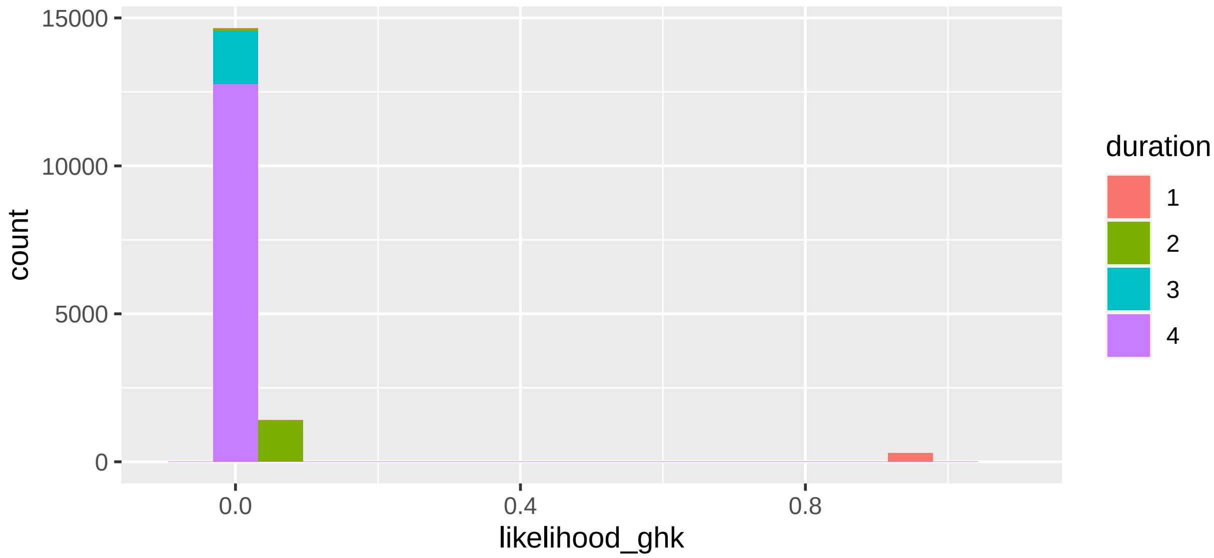


Part 2 - GHK Method

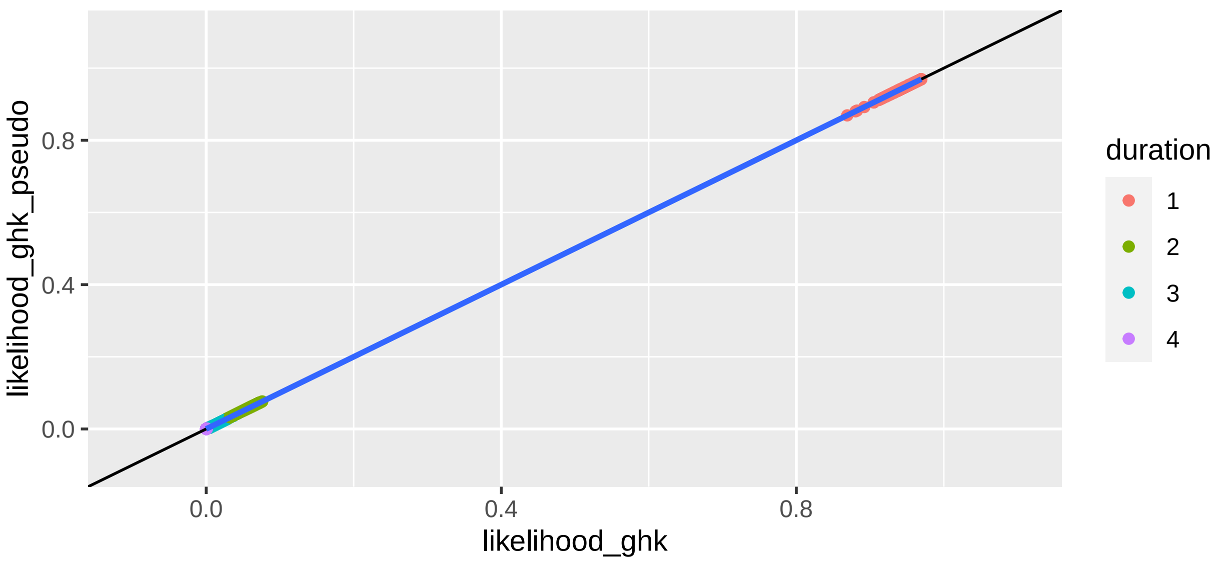
Using the GHK method, the estimated log-likelihood is:

```
## [1] -141525.5
```

The histogram of the estimated likelihoods by duration are below:

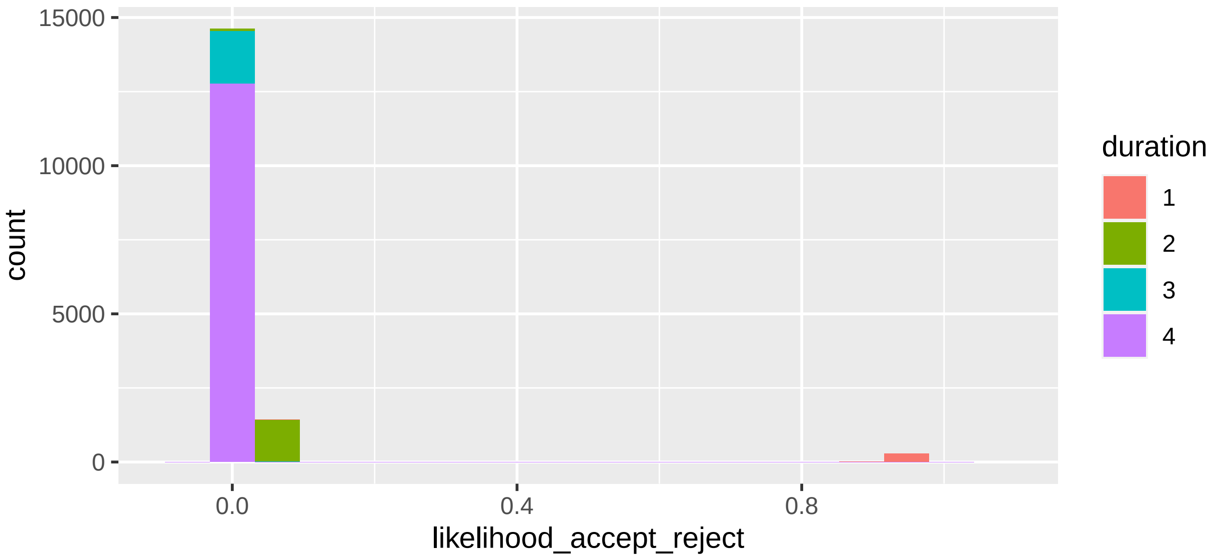


In the above histogram, I use Halton sequences to generate the simulations. In comparison, I also used Julia's built-in pseudo random number generation in the GHK method. The estimates are different, but there's effectively no difference. The black line is a 45 degree line and the blue line is a least squares regression line.

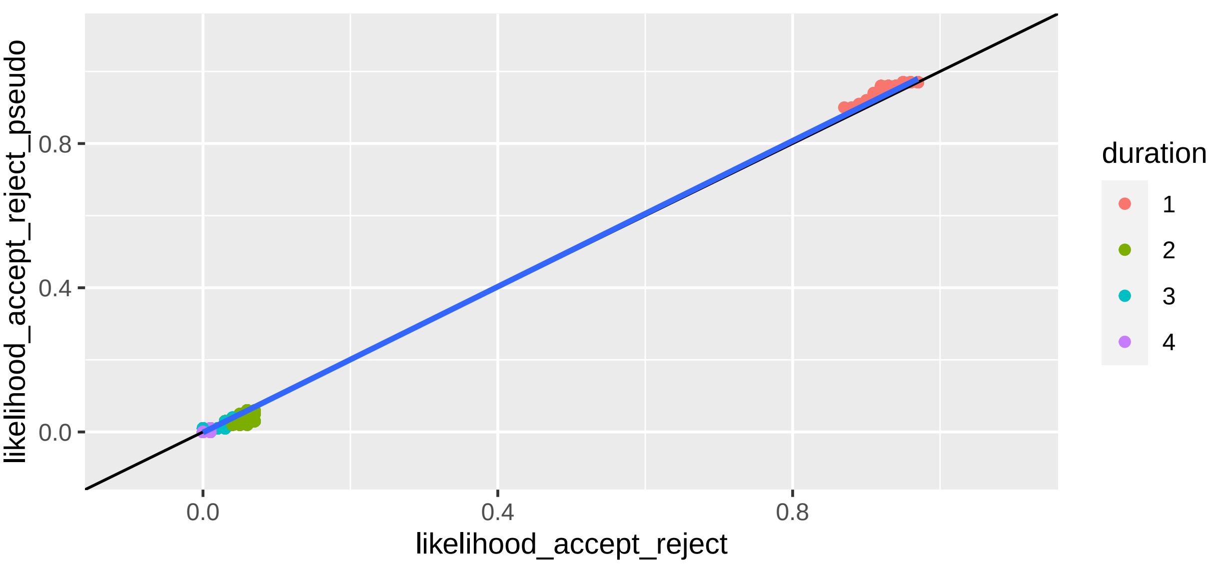


Part 3 - Accept-Reject Method

Using the accept-reject method, some of the estimated likelihoods are zero, so the estimated log-likelihood is negative infinity. The histogram of the likelihoods are below:

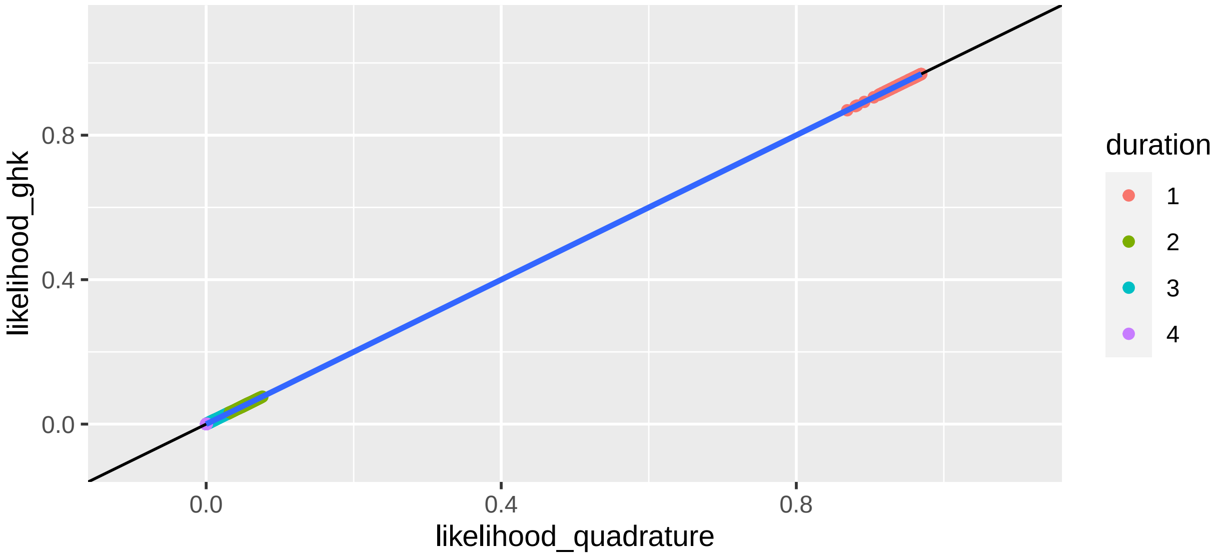


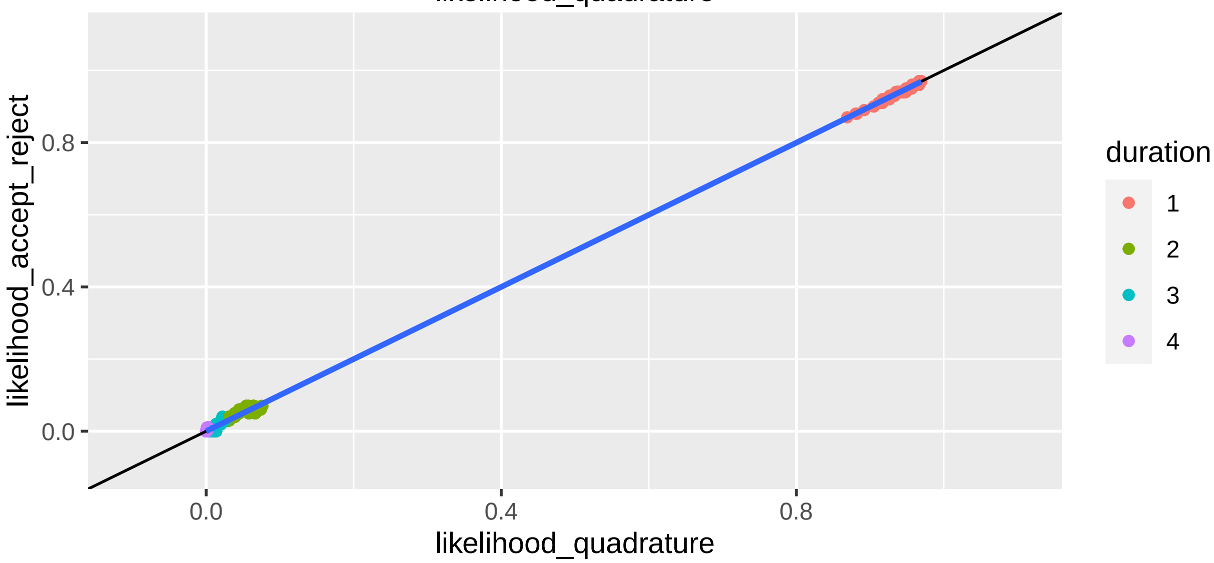
In the above histogram, I use Halton sequences to generate the simulations. In comparison, I also used Julia's built-in pseudo random number generation in the accept-reject method. Unlike using Halton sequences for the GHK method, there is a noticeable difference between using Halton sequences and pseudo-random numbers. More likely observations have higher likelihoods when using pseudo random numbers; less likely observations have lower likelihoods when using pseudo random numbers. This is consistent with Halton sequences having better coverage than pseudo random numbers.

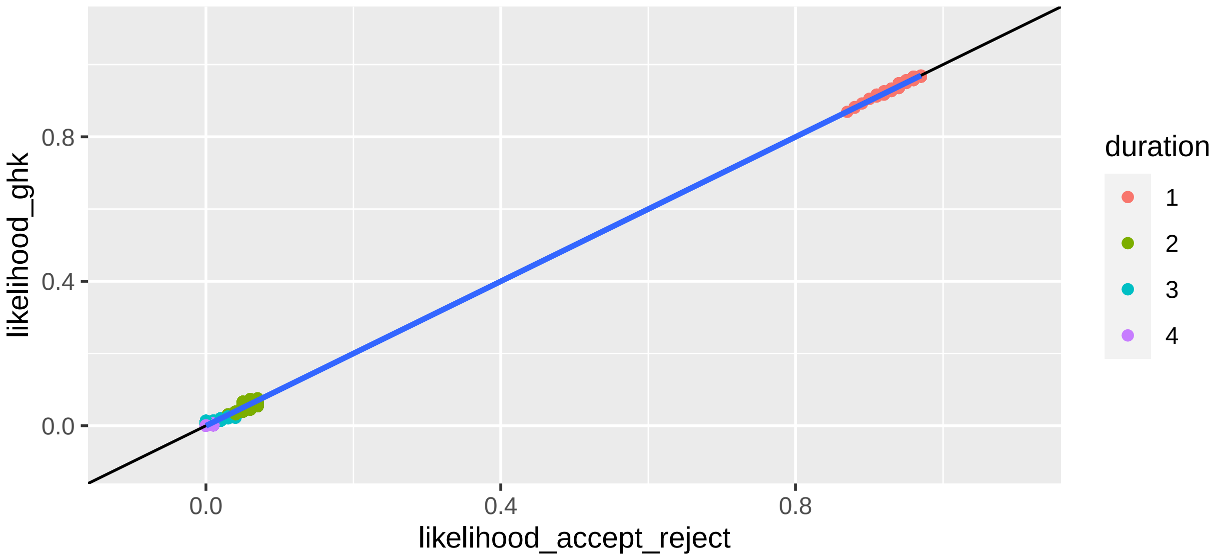


Part 4 - Comparison

Below are three scatterplots that pairwise compare these methods. Generally, the quadrature and GHK methods produce very similar estimated likelihoods while the accept-reject method produces noisier estimates that







likelihood_ghk
0.8
0.4
0.0
0.0
0.4
0.8
likelihood_accept_reject
duration
1
2
3
4

Part 5 - Optimization