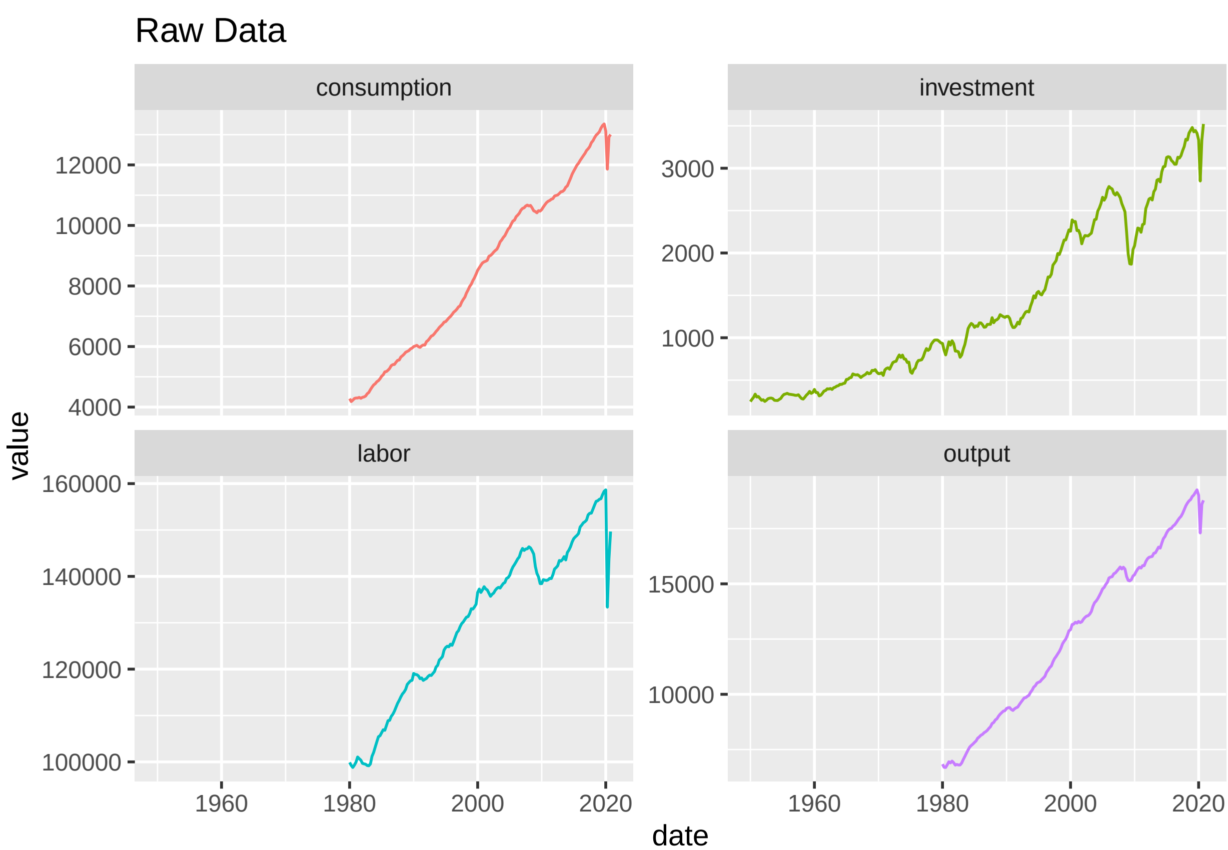
# ECON 714A - Problem Set 3


Alex von Hafften*


2/15/2020

This problem asks you to update the CKM (2007) wedge accounting using more recent data.You are encouraged to use Matlab for the computations. Consider a standard RBC model with the CRRA preferences $E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$, $U(C, L) = \frac{C^{1-\sigma}-1}{1-\sigma} - \frac{L^{1+\phi}}{1+\phi}$, a Cobb-Douglas production function $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, a standard capital law of motion $K_{t+1} = (1-\delta)K_t + I_t$, and four wedges $\tau_t = \{a_t, g_t, \tau_{Lt}, \tau_{It}\}$. Each wedge $\tau_{it}$ follows an AR(1) process $\tau_{it} = \rho_i \tau_{it-1} + \varepsilon_{it}$ with innovations $\varepsilon_{it}$ potentially correlated across $i$. One period corresponds to a quarter.

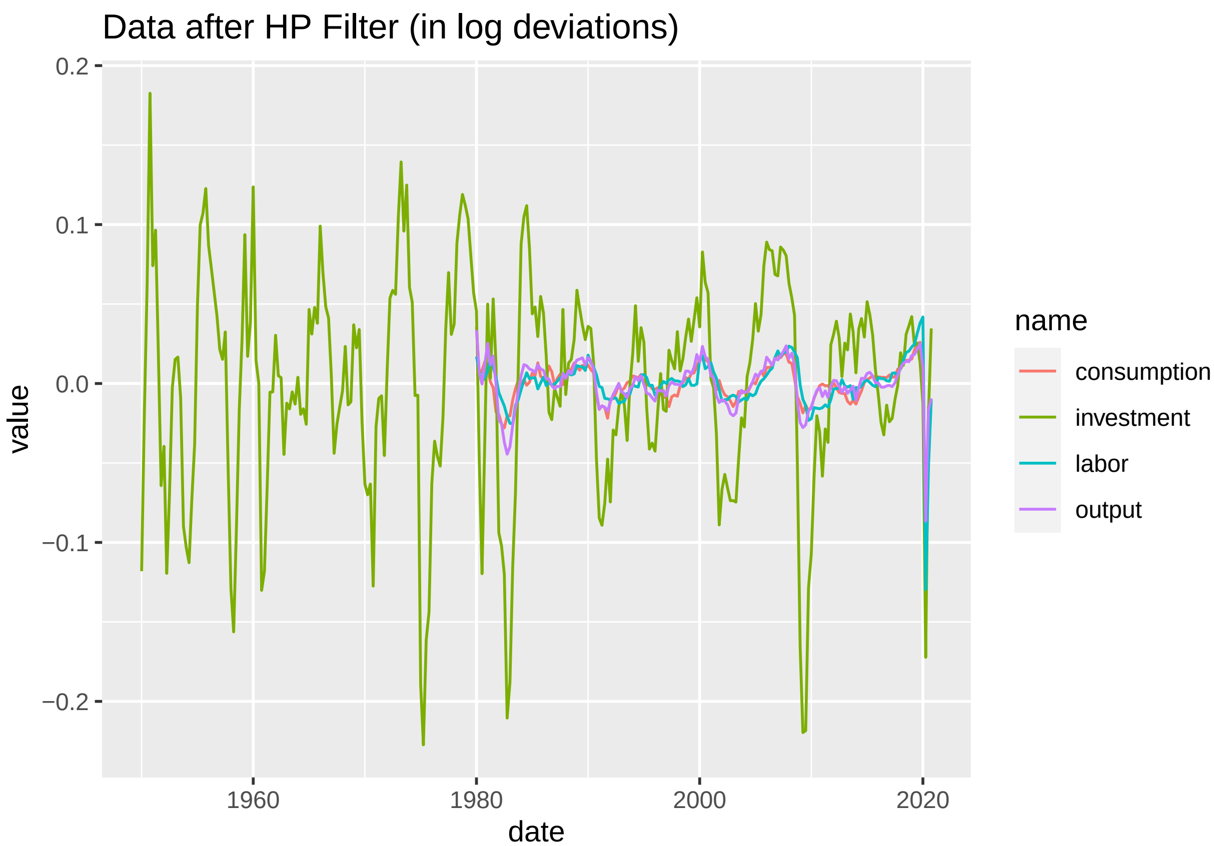
1. Download quarterly data for real seasonally adjusted consumption, employment, and output in the U.S. from 1980–2020 from FRED database. The series for capital are not readily available, but can be constructed using the "perpetual inventory method". To this end, download the series for (real seasonally adjusted) investment from 1950-2020.





*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

2. Convert all variables into logs and de-trend using the Hodrick-Prescott filter.

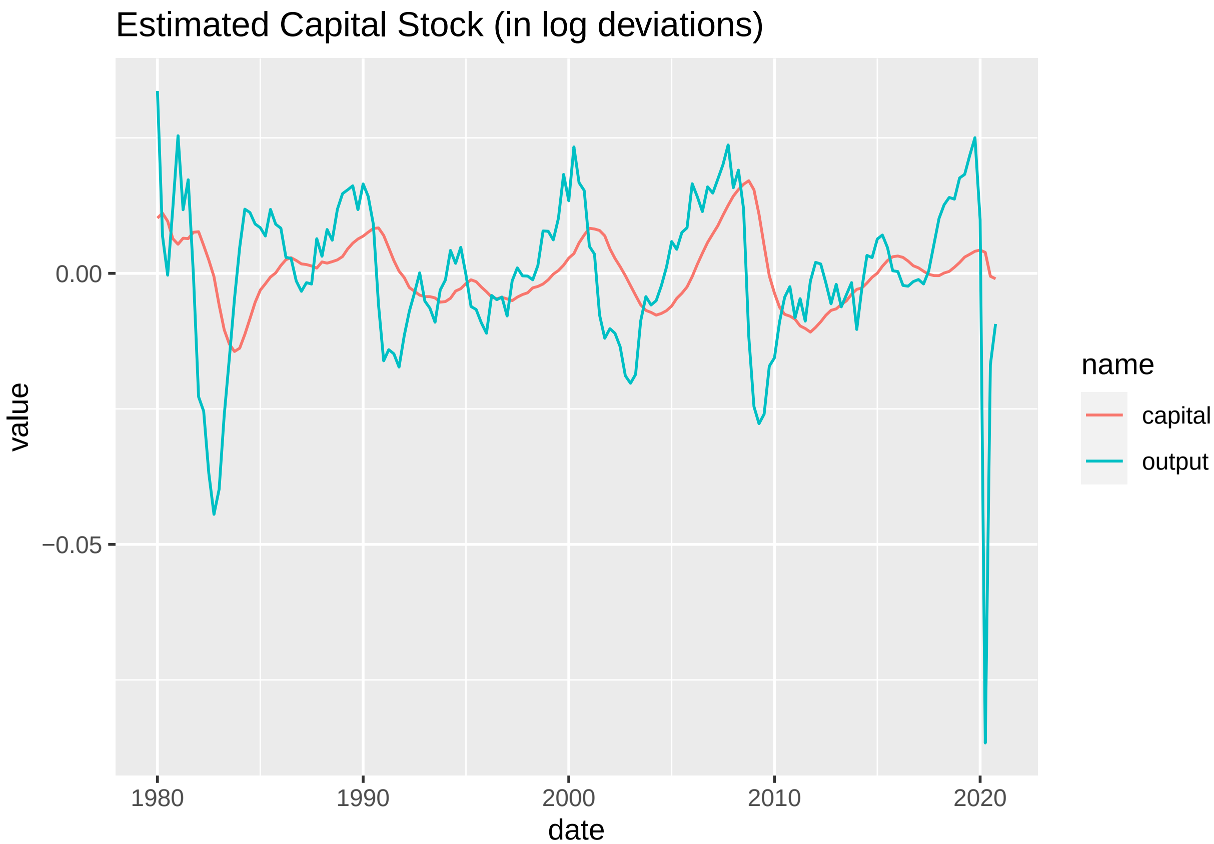



3. Assume that capital was at the steady-state level in 1950 and the rate of depreciation is $\delta = 0.025$ and use the linearized capital law of motion and the series for investment to estimate the capital stock (in log deviations) in 1980-2020. Justify this approach.

Log-linearizing the capital law of motion,

$$
\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
\implies (1+k_{t+1})\bar{K} &= (1-\delta)(1+k_t)\bar{K} + \bar{I}(1+i_t) \\
\implies \bar{K} + k_{t+1}\bar{K} &= (1-\delta)\bar{K} + (1-\delta)\bar{K}k_t + \bar{I} + \bar{I}i_t \\
\implies k_{t+1} &= (1-\delta)k_t + \frac{\bar{I}}{\bar{K}}i_t \\
\implies k_{t+1} &= (1-\delta)k_t + \delta i_t
\end{aligned}
$$

If we assume that we're in the steady state in 1950, then $k_t = 0$. Thus, we can interate forward using $i_t$ from the data over the next thirty years until 1980.




4. Linearize the equilibrium conditions. Assuming $\beta = 0.99$, $\alpha = 1/3$, $\sigma = 1$, $\phi = 1$ and the steady-state share of government spendings in GDP equal 1/3, estimate $a_t$, $g_t$ and $\tau_{Lt}$ for 1980-2020. Run the OLS regression for each of these wedges to compute their persistence parameters $\rho_i$.

The equilibrium conditions from the lecture 3 notes are:

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
A_t F(K_t, L_t) &= C_t + I_t + G_t \\
-\frac{U_{L_t}}{U_{C_t}} &= (1 - \tau_{L_t}) A_t F_{L_t} \\
U_{C_t}(1 + \tau_{I_t}) &= \beta E_t U_{C_{t+1}} (A_{t+1} F_{K_{t+1}} + (1 - \delta)(1 + \tau_{I_{t+1}})) \\
K_{t+1} &= (1 - \delta) K_t + I_t
\end{aligned}
$$

The functional forms from the prompt:

$$
\begin{aligned}
U(C_t, L_t) &= \ln(C_t) - \frac{L_t^2}{2} \\
\implies U_{C_t} &= \frac{1}{C_t} \\
\implies U_{L_t} &= -L_t \\
F(K_t, L_t) &= K_t^{\alpha} L_t^{1-\alpha} \\
\implies F_{L_t} &= (1 - \alpha) K_t^{\alpha} L_t^{-\alpha} \\
\implies F_{K_t} &= \alpha K_t^{\alpha - 1} L_t^{1-\alpha}
\end{aligned}
$$

Thus, the equilibrium conditions become:

$$\begin{aligned}
A_t K_t^\alpha L_t^{1-\alpha} &= C_t + I_t + G_t \\
C_t L_t &= (1-\tau_{L_t}) A_t (1-\alpha) K_t^\alpha L_t^{-\alpha} \\
\frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t \frac{1}{C_{t+1}} (A_{t+1}\alpha K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}}))
\end{aligned}$$

In the steady state, $K_t = K_{t+1} = \bar{K}$, $C_t = C_{t+1} = \bar{C}$, $L_t = L_{t+1} = \bar{L}$, $I_t = I_{t+1} = \delta\bar{K}$, $G_t = G_{t+1} = \bar{G} = \bar{Y}/3$, $A_t = A_{t+1} = \bar{A} = 1$, and $\bar{\tau_L} = \bar{\tau_I} = 0$.

$$\begin{aligned}
(2/3)\bar{K}^\alpha \bar{L}^{1-\alpha} &= \bar{C} + \delta\bar{K} \\
\bar{C} &= (1-\alpha)\bar{K}^\alpha \bar{L}^{-1-\alpha} \\
1 &= \beta(\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha} + (1-\delta))
\end{aligned}$$

We can solve for the labor steady state:

$$\begin{aligned}
(2/3)\bar{K}^\alpha \bar{L}^{1-\alpha} - \delta\bar{K} &= (1-\alpha)\bar{K}^\alpha \bar{L}^{-1-\alpha} \\
\implies \delta &= (2/3)\bar{K}^{\alpha-1}\bar{L}^{1-\alpha} - (1-\alpha)\bar{K}^{\alpha-1}\bar{L}^{-1-\alpha} \\
\implies \bar{K} &= (\frac{\delta}{(2/3)\bar{L}^{1-\alpha} - (1-\alpha)\bar{L}^{-1-\alpha}})^{1/(\alpha-1)} \\
\implies 1 &= \beta(\alpha\frac{\delta}{(2/3)L^{1-\alpha} - (1-\alpha)\bar{L}^{-1-\alpha}}\bar{L}^{1-\alpha} + (1-\delta) \\
\implies 1/\beta - 1 + \delta &= \frac{\alpha\delta}{(2/3) - (1-\alpha)\bar{L}^{-2}} \\
\implies \bar{L} &= \sqrt{\frac{(1-\alpha)}{(2/3) - \frac{\alpha\delta}{(1/\beta - 1 + \delta)}}}
\end{aligned}$$

The steady state values of $\bar{Y}, \bar{K}, \bar{C}$, and $\bar{G}$ are implied by the equations above:

| variable | value |
|---|---|
| L_bar | 1.246 |
| K_bar | 36.470 |
| I_bar | 0.912 |
| Y_bar | 3.840 |
| C_bar | 1.649 |
| G_bar | 1.280 |

We now log-linearize around the steady state. The log-linearized Euler equation is:

$$
\begin{aligned}
\frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t \frac{1}{C_{t+1}}(A_{t+1}\alpha K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}})) \\
X_{t+1} &:= A_{t+1}\alpha K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}}) \\
\bar{X} &= \bar{A}\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha} + 1 - \delta \\
&= \frac{1}{\beta} \\
\implies x_{t+1} &= \frac{\bar{A}\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}}{\bar{X}}(a_{t+1} + (\alpha-1)k_{t+1} + (1-\alpha)l_{t+1}) + \frac{(1-\delta)}{\bar{X}}\hat{\tau}_{I_{t+1}} \\
&= \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + \beta(1-\delta)\hat{\tau}_{I_{t+1}} \\
\implies \frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t \frac{1}{C_{t+1}} X_{t+1} \\
\implies \hat{\tau}_{I_t} - c_t &= E_t[x_{t+1}] - E_t[c_{t+1}] \\
\implies \hat{\tau}_{I_t} - c_t &= E_t[\beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + \beta(1-\delta)\hat{\tau}_{I_{t+1}}] - E_t[c_{t+1}] \\
\implies E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta E_t[\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + (1-\delta)\hat{\tau}_{I_{t+1}}]
\end{aligned}
$$

The log-linearized law of motion of capital is:

$$
\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
\implies k_{t+1} &= \frac{(1-\delta)\bar{K}}{(1-\delta)\bar{K} + \bar{I}}k_t + \frac{\bar{I}}{(1-\delta)\bar{K} + \bar{I}}i_t \\
&= \frac{(1-\delta)\bar{K}}{\bar{K}}k_t + \frac{\bar{I}}{\bar{K}}i_t \\
&= (1-\delta)k_t + \delta i_t
\end{aligned}
$$

The log-linearized consumption-labor equation is:

$$
\begin{aligned}
C_t L_t^{1+\alpha} &= (1-\tau_{L_t})A_t(1-\alpha)K_t^{\alpha} \\
\implies c_t + (1+\alpha)l_t &= -\hat{\tau}_{L_t} + a_t + \alpha k_t
\end{aligned}
$$

The log-linearized output equations are:

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
\implies y_t &= a_t + \alpha k_t + (1-\alpha)l_t \\
Y_t &= C_t + I_t + G_t \\
\implies \bar{Y}(1+y_t) &= \bar{C}(1+c_t) + \bar{I}(1+i_t) + \bar{G}(1+g_t) \\
\implies \bar{Y}y_t &= \bar{C}c_t + \bar{I}i_t + \bar{G}g_t
\end{aligned}
$$

The efficiency wedge can be derived from the first log-linearized output equation:

$$a_t = y_t - \alpha k_t - (1-\alpha) l_t$$

The government consumption wedge can be derived from the second log-linearized output equation:

$$g_t = \frac{\bar{Y}}{\bar{G}} y_t - \frac{\bar{C}}{\bar{G}} c_t - \frac{\bar{I}}{\bar{G}} i_t$$

The labor wedge can be derived from the log-linearized consumption-labor equation:

$$\hat{\tau_{L_t}} = a_t + \alpha k_t - c_t - (1+\alpha) l_t$$

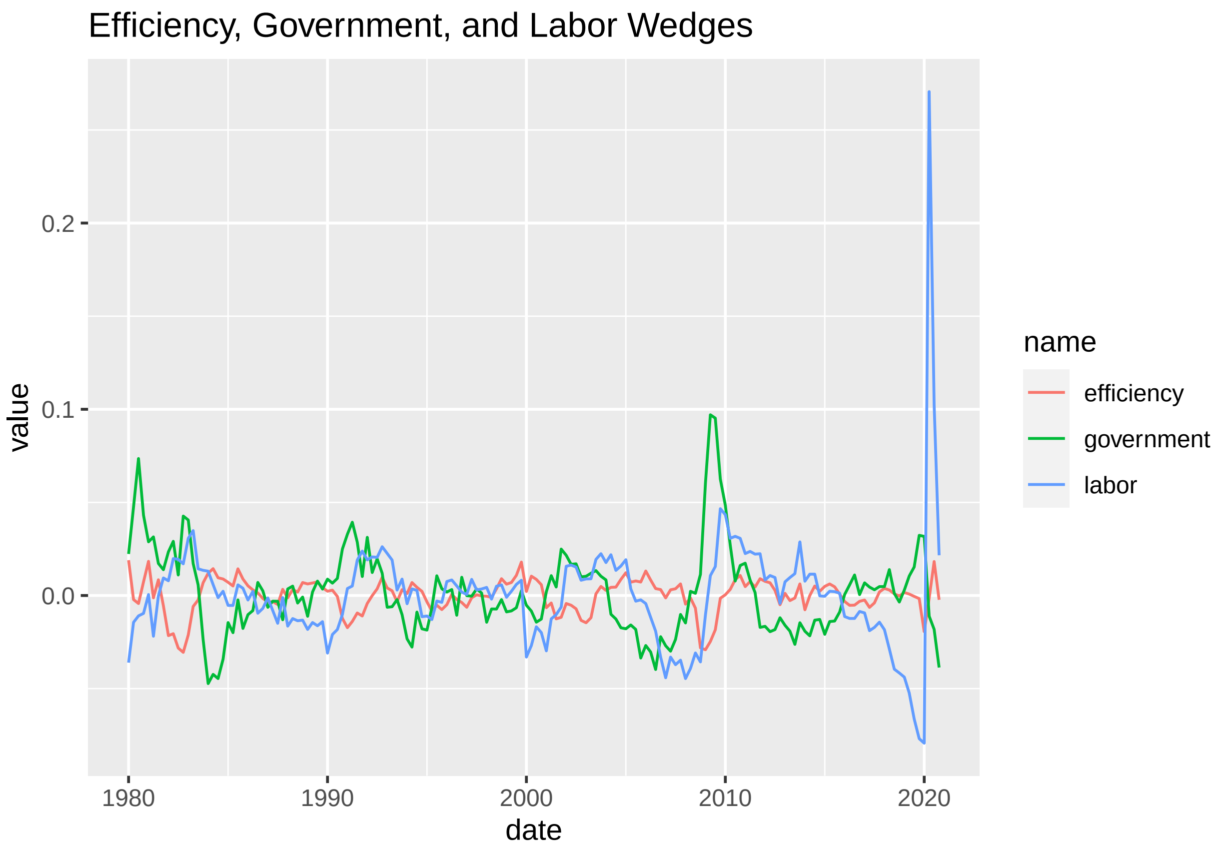


Table 2: Efficiency, Government, and Labor Wedge Persistance (OLS)

| | *Dependent variable:* | | |
|---|---|---|---|
| | efficiency | government | labor |
| | (1) | (2) | (3) |
| lag(efficiency) | 0.711*** | | |
| | (0.054) | | |
| lag(government) | | 0.856*** | |
| | | (0.042) | |
| lag(labor) | | | 0.451*** |
| | | | (0.070) |
| Observations | 163 | 163 | 163 |
| Adjusted $R^2$ | 0.517 | 0.722 | 0.200 |
| *Note:* | *p<0.1; **p<0.05; ***p<0.01 | | |

5. Write down a code that implements the Blanchard-Kahn method to solve the model. Use the values of parameters, including $\rho_a$, $\rho_g$, and $\rho_{\tau_L}$, obtained above, and assume $\rho_{\tau_I} = 0$ for now.

The log-linearized consumption-labor equation implies

$$l_t = \frac{\alpha}{1+\alpha} k_t + \frac{-1}{1+\alpha} c_t + \frac{1}{1+\alpha} a_t + \frac{-1}{1+\alpha} \hat{\tau}_{L_t}$$

The log-linearized output equations implies

$$\begin{aligned}
i_t &= (1-\alpha)\frac{\bar{Y}}{\bar{I}} l_t + \alpha \frac{\bar{Y}}{\bar{I}} k_t + \frac{-\bar{C}}{\bar{I}} c_t + \frac{\bar{Y}}{\bar{I}} a_t + \frac{-\bar{I}}{\bar{Y}} g_t \\
&= (1-\alpha)\frac{\bar{Y}}{\bar{I}} \left[ \frac{\alpha}{1+\alpha} k_t + \frac{-1}{1+\alpha} c_t + \frac{1}{1+\alpha} a_t + \frac{-1}{1+\alpha} \hat{\tau}_{L_t} \right] + \alpha \frac{\bar{Y}}{\bar{I}} k_t + \frac{-\bar{C}}{\bar{I}} c_t + \frac{\bar{Y}}{\bar{I}} a_t + \frac{-\bar{I}}{\bar{Y}} g_t \\
&= \frac{2\bar{Y}\alpha}{\bar{I}(1+\alpha)} k_t + \frac{-\bar{Y}(1-\alpha) - \bar{C}(1+\alpha)}{\bar{I}(1+\alpha)} c_t + \frac{2\bar{Y}}{(1+\alpha)\bar{I}} a_t + \frac{-\bar{Y}(1-\alpha)}{\bar{I}(1+\alpha)} \hat{\tau}_{L_t} + \frac{-\bar{I}}{\bar{Y}} g_t
\end{aligned}$$

The law of motion of capital implies

$$\begin{aligned}
k_{t+1} &= (1-\delta) k_t + \delta i_t \\
&= (1-\delta) k_t + \delta \left[ \frac{2\bar{Y}\alpha}{\bar{I}(1+\alpha)} k_t + \frac{-\bar{Y}(1-\alpha) - \bar{C}(1+\alpha)}{\bar{I}(1+\alpha)} c_t + \frac{2\bar{Y}}{(1+\alpha)\bar{I}} a_t + \frac{-\bar{Y}(1-\alpha)}{\bar{I}(1+\alpha)} \hat{\tau}_{L_t} + \frac{-\bar{I}}{\bar{Y}} g_t \right] \\
&= \left[ 1 - \delta + \frac{2\alpha\delta\bar{Y}}{(1+\alpha)\bar{I}} \right] k_t + \frac{-\delta\bar{Y}(1-\alpha) - \bar{C}(1+\alpha)}{\bar{I}(1+\alpha)} c_t + \frac{2\delta\bar{Y}}{(1+\alpha)\bar{I}} a_t + \frac{-\delta\bar{Y}(1-\alpha)}{\bar{I}(1+\alpha)} \hat{\tau}_{L_t} + \frac{-\delta\bar{I}}{\bar{Y}} g_t
\end{aligned}$$

Based on the AR(1) process for $a_t$, $g_t$, $\hat{\tau}_{L_t}$, and $\hat{\tau}_{I_t}$:

$$\begin{aligned}
E_t[a_{t+1}] &= \rho_a a_t \\
E_t[g_{t+1}] &= \rho_g g_t \\
E_t[\hat{\tau}_{L_{t+1}}] &= \rho_{\tau_L} \hat{\tau}_{L_t} \\
E_t[\hat{\tau}_{I_{t+1}}] &= \rho_{\tau_I} \hat{\tau}_{I_t} = 0
\end{aligned}$$

The log-linearized Euler equation implies:

$$\begin{aligned}
E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta E_t\Bigg[\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\Bigg(a_{t+1} + (1-\alpha)\Bigg[\frac{\alpha}{1+\alpha}k_{t+1} + \frac{-1}{1+\alpha}c_{t+1} \\
&\quad + \frac{1}{1+\alpha}a_{t+1} + \frac{-1}{1+\alpha}\hat{\tau}_{L_{t+1}}\Bigg] - (1-\alpha)k_{t+1}\Bigg) + (1-\delta)\hat{\tau}_{I_{t+1}}\Bigg] \\
\implies E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta E_t\Bigg[\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\Bigg(\frac{\alpha-1}{1+\alpha}k_{t+1} + \frac{\alpha-1}{1+\alpha}c_{t+1} \\
&\quad + \frac{2}{1+\alpha}a_{t+1} + \frac{\alpha-1}{1+\alpha}\hat{\tau}_{L_{t+1}}\Bigg)\Bigg] + (1-\delta)E_t[\hat{\tau}_{I_{t+1}}] \\
\implies E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\Bigg(\frac{\alpha-1}{1+\alpha}E_t[k_{t+1}] + \frac{\alpha-1}{1+\alpha}E_t[c_{t+1}] \\
&\quad + \frac{2}{1+\alpha}E_t[a_{t+1}] + \frac{\alpha-1}{1+\alpha}E_t[\hat{\tau}_{L_{t+1}}]\Bigg) \\
\implies E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\Bigg(\frac{\alpha-1}{1+\alpha}E_t[k_{t+1}] + \frac{\alpha-1}{1+\alpha}E_t[c_{t+1}] \\
&\quad + \frac{2}{1+\alpha}\rho_a a_t + \frac{\alpha-1}{1+\alpha}\rho_{\tau_L}\hat{\tau}_{L_t}\Bigg) \\
\implies \Bigg[1 - \beta\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha}\Bigg]E_t[c_{t+1}] &- \beta\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha}E_t[k_{t+1}] \\
&= c_t + \beta\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{2}{1+\alpha}\rho_a a_t + \beta\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha}\rho_{\tau_L}\hat{\tau}_{L_t} - \hat{\tau}_{I_t}
\end{aligned}$$

Thus, we can represent the model with a matrix:

$$
\begin{aligned}
x_t &:= \begin{pmatrix} k_t \\ c_t \end{pmatrix} \\
z_t &:= \begin{pmatrix} a_t \\ g_t \\ \hat{\tau}_{L_t} \\ \hat{\tau}_{I_t} \end{pmatrix} \\
\tilde{C}E[x_{t+1}] &= \tilde{A}x_t + \tilde{B}z_t \\
\tilde{C} &:= \begin{pmatrix} 1 & 0 \\ -\beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha} & 1-\beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha} \end{pmatrix} \\
\tilde{A} &:= \begin{pmatrix} 1-\delta+\frac{2\alpha\delta\bar{Y}}{(1+\alpha)\bar{I}} & \frac{-\delta\bar{Y}(1-\alpha)-\bar{C}(1+\alpha)}{\bar{I}(1+\alpha)} \\ 0 & 1 \end{pmatrix} \\
\tilde{B} &:= \begin{pmatrix} \frac{2\delta\bar{Y}}{(1+\alpha)\bar{I}} & \frac{-\delta\bar{I}}{\bar{Y}} & \frac{-\delta\bar{Y}(1-\alpha)}{\bar{I}(1+\alpha)} & 0 \\ \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{2}{1+\alpha}\rho_a & 0 & \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha}\rho_{\tau_L} & -1 \end{pmatrix} \\
A &:= \tilde{C}^{-1}\tilde{A} \\
B &:= \tilde{C}^{-1}\tilde{B}
\end{aligned}
$$

# Appendix

```
# PCECC96 Real Personal Consumption Expenditures  Billions of Chained 2012 Dollars,
#                                                 Seasonally Adjusted Annual Rate
# GDPC1   Real Gross Domestic Product             Billions of Chained 2012 Dollars
#                                                 Seasonally Adjusted Annual Rate
# CE16OV  Employment Level                        Thousands of Persons
#                                                 Seasonally Adjusted
# GPDIC1  Real Gross Private Domestic Investment  Billions of Chained 2012 Dollars
#                                                 Seasonally Adjusted Annual Rate
```

Log-linearizing the investment and labor wage:

Thus, the equilibrium conditions become:

$$\begin{aligned}
A_t K_t^{\alpha} L_t^{1-\alpha} &= C_t + I_t + G_t \\
C_t L_t &= (1-\tau_{L_t}) A_t (1-\alpha) K_t^{\alpha} L_t^{-\alpha} \\
\frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t \frac{1}{C_{t+1}} (A_{t+1} \alpha K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}}))
\end{aligned}$$

In the steady state, $K_t = K_{t+1} = \bar{K}$, $C_t = C_{t+1} = \bar{C}$, $L_t = L_{t+1} = \bar{L}$, $I_t = I_{t+1} = \delta\bar{K}$, $G_t = G_{t+1} = \bar{G} = \bar{Y}/3$, and $A_t = A_{t+1} = \bar{A} = 1$.

$$\begin{aligned}
(2/3)\bar{K}^{\alpha}\bar{L}^{1-\alpha} &= \bar{C} + \delta\bar{K} \\
\bar{C} &= (1-\bar{\tau_L})(1-\alpha)\bar{K}^{\alpha}\bar{L}^{-1-\alpha} \\
1 + \bar{\tau_I} &= \beta(\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha} + (1-\delta)(1+\bar{\tau_I}))
\end{aligned}$$

The labor wedge is:

$$\begin{aligned}
C_t L_t^{1+\alpha} &= (1-\tau_{L_t}) A_t (1-\alpha) K_t^{\alpha} \\
\implies c_t + (1+\alpha) l_t &= \hat{\tau_{L_t}} \frac{-\bar{\tau}_L}{1-\bar{\tau}_L} + a_t + \alpha k_t \\
\implies \hat{\tau_{L_t}} &= \frac{1-\bar{\tau}_L}{\bar{\tau}_L} a_t + \alpha \frac{1-\bar{\tau}_L}{\bar{\tau}_L} k_t - \frac{1-\bar{\tau}_L}{\bar{\tau}_L} c_t - (1+\alpha) \frac{1-\bar{\tau}_L}{\bar{\tau}_L} l_t
\end{aligned}$$

The log-linearized Euler is:

$$
\begin{aligned}
\frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t \frac{1}{C_{t+1}}(A_{t+1}\alpha K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}})) \\
X_{t+1} &:= A_{t+1}\alpha K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}}) \\
\bar{X} &= \bar{A}\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha} + (1-\delta)(1+\bar{\tau}_I) \\
&= \frac{1+\bar{\tau}_I}{\beta} \\
\implies x_{t+1} &= \frac{\bar{A}\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}}{\bar{X}}(a_{t+1} + (\alpha-1)k_{t+1} + (1-\alpha)l_{t+1}) + \frac{(1-\delta)(1+\bar{\tau}_I)}{\bar{X}}(\frac{\bar{\tau}_I}{1+\bar{\tau}_I})\hat{\tau}_{I_{t+1}} \\
&= \beta\frac{\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}}{1+\bar{\tau}_I}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + \beta\frac{(1-\delta)\bar{\tau}_I\hat{\tau}_{I_{t+1}}}{1+\bar{\tau}_I} \\
\implies \frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t\frac{1}{C_{t+1}}X_{t+1} \\
\implies \frac{\bar{\tau}_I}{1+\bar{\tau}_I}\hat{\tau}_{I_t} - c_t &= E_t[x_{t+1}] - E_t[c_{t+1}] \\
\implies \frac{\bar{\tau}_I}{1+\bar{\tau}_I}\hat{\tau}_{I_t} - c_t &= E_t[\beta\frac{\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}}{1+\bar{\tau}_I}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + \beta\frac{(1-\delta)\bar{\tau}_I\hat{\tau}_{I_{t+1}}}{1+\bar{\tau}_I}] - E_t[c_{t+1}] \\
\implies E_t[c_{t+1}] - c_t + \frac{\bar{\tau}_I}{1+\bar{\tau}_I}\hat{\tau}_{I_t} &= \beta E_t[\frac{\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}}{1+\bar{\tau}_I}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + \frac{(1-\delta)\bar{\tau}_I}{1+\bar{\tau}_I}\hat{\tau}_{I_{t+1}}]
\end{aligned}
$$

$$
E_t[c_{t+1}] = c_t - \hat{\tau}_{I_t} + \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}E_t[a_{t+1}] + \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(1-\alpha)E_t[l_{t+1}] - \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(1-\alpha)E_t[k_{t+1}] + \beta(1-\delta)E_t[\hat{\tau}_{I_{t+1}}] \implies
$$