ECON 709 - PS 1

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9/14/2020

(1) For two events $A, B \in S$, prove that $A \cup B = (A \cap B) \cup ((A \cap B^C) \cup (B \cap A^C))$.

Proof: Applying the partition rule and the properties of set operators,

$$\begin{aligned} A \cup B &= ((A \cap B) \cup (A \cap B^C)) \cup B \\ &= ((A \cap B) \cup (A \cap B^C)) \cup ((B \cap A) \cup (B \cap A^C)) \\ &= ((A \cap B) \cup (B \cap A)) \cup ((A \cap B^C) \cup (B \cap A^C)) \\ &= (A \cap B) \cup ((A \cap B^C) \cup (B \cap A^C)) \end{aligned}$$

□

(2) Prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof: Applying the partition rule and axioms of the probability measure,

$$\begin{aligned} P(A \cup B) &= P(((A \cap B) \cup (A \cap B^C)) \cup ((B \cap A) \cup (B \cap A^C))) \\ &= P((A \cap B) \cup (A \cap B^C) \cup (B \cap A^C)) \\ &= P(A \cap B) + P(A \cap B^C) + P(B \cap A^C) \\ &= P(A \cap B) + P(A \cap B^C) + P(B \cap A^C) + P(A \cap B) - P(A \cap B) \\ &= P((A \cap B) \cup (A \cap B^C)) + P((B \cap A^C) \cup (A \cap B)) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

□

(3) Suppose that the unconditional probability of a disease is 0.0025. A screening test for this disease has a detection rate of 0.9, and has a false positive rate of 0.01. Given that the screening test returns positive, what is the conditional probability of having the disease?

Proof: Define A the event that you have the the disease, so $P(A) = 0.0025$ and $P(A^C) = 0.9975$. Define B as the event you test positive, so $P(B|A) = 0.9$ and $P(B|A^C) = 0.01$. We want to know the probability you have the disease conditional on a positive test result, or $P(A|B)$. Thus, using Bayes Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^C)P(A^C)} = \frac{(0.9)(0.0025)}{(0.9)(0.0025) + (0.01)(0.9975)} \approx 0.184$$

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□

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

(4) Suppose that a pair of events A and B are mutually exclusive, i.e., $A \cap B = \emptyset$, and that $P(A) > 0$ and $P(B) > 0$. Prove that A and B are not independent.

Proof: Assume for the sake of a contradiction that A and B are mutually exclusive events that independently occur with nonzero probabilities. Since they are independent, we know $P(A)P(B) = P(A \cap B)$. Since $P(B)$ is nonzero, $P(A) = P(A \cap B)/P(B)$. Since A and B are mutually exclusive, $P(A) = P(\emptyset)/P(B) = 0/P(B) = 0$. $\Rightarrow\Leftarrow$. A and B cannot be independent. $\square$

(5) (Conditional Independence) Sometimes, we may also use the concept of conditional independence. The definition is as follows: let A, B, C be three events with positive probabilities. Then A and B are independent given C if $P(A \cap B|C) = P(A|C)P(B|C)$. Consider the experiment of tossing two dice. Let $A = \{\text{First die is 6}\}$, $B = \{\text{Second die is 6}\}$, and $C = \{\text{Both dice are the same}\}$.

(a) Show that A and B are independent (unconditionally), but A and B are dependent given C.

Proof: Define (x, y) where $x = \{1, 2, 3, 4, 5, 6\}$ equals the number rolled on the first die and $y = \{1, 2, 3, 4, 5, 6\}$ equals the number rolled on the second die. Thus,

$$\begin{aligned}
A &= \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \\
B &= \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6)\} \\
C &= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\} \\
A \cap B = A \cap C = B \cap C = A \cap B \cap C &= \{(6,6)\}
\end{aligned}$$

If the dice are fair, each (x, y) has probability of $1/36$, so

$$\begin{aligned}
P(A) = P(B) = P(C) &= 6/36 = 1/6 \\
P(A \cap B) = P(A \cap C) = P(B \cap C) = P(A \cap B \cap C) &= 1/36
\end{aligned}$$

Since $P(A \cap B) = 1/36 = (1/6)(1/6) = P(A)P(B)$, A and B are independent.

$$\begin{aligned}
P(A|C) = P(A \cap C)/P(C) &= (1/36)/(1/6) = 1/6 \\
P(B|C) = P(B \cap C)/P(C) &= (1/36)/(1/6) = 1/6 \\
P(A|C)P(B|C) &= (1/6)(1/6) = 1/36 \\
P(A \cap B|C) = P(A \cap B \cap C)/P(C) &= (1/36)/(1/6) = 1/6
\end{aligned}$$

Since $P(A \cap B|C) \neq P(A|C)P(B|C)$, A and B are dependent given C. $\square$

(b) Consider the following experiment: let there be two urns, one with 9 black balls and 1 white balls and the other with 1 black ball and 9 white balls. First randomly (with equal probability) select one urn. Then take two draws with replacement from the selected urn. Let A and B be drawing a black ball in the first and the second draw, respectively, and let C be the event urn 1 is selected. Show that A and B are not independent, but are conditionally independent given C.

Proof: First, notice that since we are drawing with replacement, the ball drawn first does not affect the probability of which color of the ball drawn second, so $P(A|C) = P(B|C) = 9/10$ and $P(A \cap B|C) = P(A|C)P(B|C)$. Thus, conditional on C, A and B are independent. Now, let's find $P(A)$, $P(B)$, and $P(A \cap B)$ using the partition rule and the definition of conditional probability:

$$\begin{aligned}
P(A) &= P((A \cap C) \cup (A \cap C^C)) \\
&= P(A \cap C) + P(A \cap C^C) \\
&= P(A|C)P(C) + P(A|C^C)P(C^C) \\
&= (9/10)(1/2) + (1/10)(1/2) \\
&= 9/20 + 1/20 \\
&= 1/2
\end{aligned}$$

$$\begin{aligned}
P(B) &= P((B \cap C) \cup (B \cap C^C)) \\
&= P(B \cap C) + P(B \cap C^C) \\
&= P(B|C)P(C) + P(B|C^C)P(C^C) \\
&= (9/10)(1/2) + (1/10)(1/2) \\
&= 9/20 + 1/20 \\
&= 1/2
\end{aligned}$$

$$\begin{aligned}
P(A \cap B) &= P(((A \cap B) \cap C) \cup ((A \cap B) \cap C^C)) \\
&= P((A \cap B) \cap C) + P((A \cap B) \cap C^C) \\
&= P(A \cap B|C)P(C) + P(A \cap B|C^C)P(C^C) \\
&= P(A|C)P(B|C)P(C) + P(A|C^C)P(B|C^C)P(C^C) \\
&= (9/10)(9/10)(1/2) + (1/10)(1/10)(1/2) \\
&= 82/200
\end{aligned}$$

Since $P(A)P(B) \neq P(A \cap B)$, A and B are not independent. $\square$

(6) A CDF F_X is stochastically greater than a CDF F_Y if $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Prove that if $X \sim F_X$ and $Y \sim F_Y$, then

$$P(X > t) \geq P(Y > t) \text{ for every } t,$$

and

$$P(X > t) > P(Y > t) \text{ for some } t,$$

that is, X tends to be bigger than Y.

Proof: If $X \sim F_X$ and $Y \sim F_Y$, then, for every t,

$$\begin{aligned} F_X(t) &= P(X \leq t) = 1 - P(X > t) \\ F_Y(t) &= P(Y \leq t) = 1 - P(Y > t) \end{aligned}$$

Furthermore, for every t, we know that

$$\begin{aligned} F_X(t) &\leq F_Y(t) \\ 1 - P(X > t) &\leq 1 - P(Y > t) \\ P(X > t) &\geq P(Y > t). \end{aligned}$$

In addition, we know for some t,

$$\begin{aligned} F_X(t) &< F_Y(t) \\ 1 - P(X > t) &< 1 - P(Y > t) \\ P(X > t) &> P(Y > t). \end{aligned}$$

□

(7) Show that the function $F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - \exp(-x), & x \geq 0 \end{cases}$ is a CDF, and find $f_X(x)$ and $F_X^{-1}(y)$.

Proof: CDFs have three unique properties: (1) $\lim_{x\to\infty} F_X(X) = 1, \lim_{x\to-\infty} F_X(X) = 0$, (2) $F_X(x)$ is non-decreasing, (3), $F_X(x)$ is right-continuous, that is $\lim_{x\to x_0^+}$.

For (1), since $\lim_{x\to\infty} \exp(-x) = 0$, $\lim_{x\to\infty}(1 - \exp(-x)) = 1$ and $\lim_{x\to-\infty}(0) = 0$.

For (2), we show that $x_1 \leq x_2 \implies F_X(x_1) \leq F_X(x_2)$. If $x_1 < 0$, then $F_X(x_1) = 0$. Since $F_X(x_2) \geq 0$, then $F_X(x_1) = 0 \leq F_X(x_2)$. If $x_1 \geq 0$, then $F_X(x_1) = 1 - \exp(-x_1)$ and $F_X(x_2) = 1 - \exp(-x_2)$. $x_1 \leq x_2 \implies 1 - \exp(-x_1) \leq 1 - \exp(-x_2) \implies F_X(x_1) \leq F_X(x_2)$. Thus, F_X is nondecreasing.

For (3), $F_X(x) = 0$ is continuous for $x < 0$ and $x > 0$ because both a constant function and $1 - \exp(-x)$ is continuous. Finally, at $x = 0$, $\lim_{x\to0^+} F_X(x) = \lim_{x\to0^+}(1 - \exp(-x)) = 1 - \exp(-0) = 0$. □

To find the PDF, take the derivative of the CDF:

$$\begin{aligned}
f_X(x) &= \frac{d}{dx} F_X(x) \\
&= \begin{cases} \frac{d}{dx}(0), & x < 0 \\ \frac{d}{dx}(1 - \exp(-x)), & x \geq 0 \end{cases} \\
&= \begin{cases} (0), & x < 0 \\ 0 - \exp(-x)(-1), & x \geq 0 \end{cases} \\
&= \begin{cases} 0, & x < 0 \\ \exp(-x), & x \geq 0 \end{cases}
\end{aligned}$$

Notice that $F_X^{-1}(y)$ is defined over $y \in [0, 1)$ because of the properties of CDFs (i.e., $G(x) \in [0, 1] \forall x \in \mathbb{R}$ for any CDF G) and $F_X(x)$ asymptotically converges to 1 as $x \to \infty$. So, for $x \geq 0$,

$$\begin{aligned}
y &= 1 - \exp(-x) \\
-\ln(1 - y) &= x
\end{aligned}$$

Thus,

$$F_X^{-1}(y) = -\ln(1 - y), y \in [0, 1)$$

Therefore, $F_X^{-1}(F_X(x)) = -\ln(1 - (1 - \exp(-x))) = x$ for $x \geq 0$.