ECON 714A - Problem Set 5

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A representative household maximizes lifetime expected utility

$$E_0 \sum_{t=0}^{\infty} \beta^t (\log C_t - L_t)$$

subject to the budget constraint

$$P_t C_t + B_t = W_t L_t + \Pi_t + (1 + i_{t-1}) B_{t-1} + T_t$$

where consumption bundle C_t is a standard CES aggregator of individual varieties with the elasticity of substitution θ. Firms are monopolistic competitors and use a linear technology $Y_{it} = A_t L_{it}$ to produce a continuum of unique varieties $i \in [0, 1]$. Each firm has to hire additional $\frac{\varphi}{2}(\frac{Pit - Pit-1}{Pit-1})^2$ units of labor to adjust the price from P_{it-1} to P_{it} in period t.

1. Consider a flexible-price version of the model with $\varphi = 0$. Describe the deterministic steady state of the economy and the linearized dynamics around it.

The household problem is to choose consumption of each product, supply labor, and purchase bonds:

$$\max_{\{(C_{it}, L_t, B_t)\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t (\log C_t - L_t)$$

$$\text{s.t. } P_t C_t + B_t = W_t L_t + \Pi_t + (1 + i_{t-1}) B_{t-1} + T_t$$

$$\text{and } C_t = \left(\int C_{it}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$$

The resulting optimality conditions are the same as in the RBC and Dixit-Stiglitz models.

Demand for products:

$$C_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\theta} C_t$$

Labor supply condition:

$$C_t = \frac{W_t}{P_t} \implies L_t = 1$$

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The Euler equation:

$$\beta E_t \left(\frac{C_{t+1}}{C_t}\right)^{-1} \frac{P_t}{P_{t+1}}(1+i_t) = 1$$

With $\varphi = 0$, the problem of each firm to maximize profits becomes static:

$$\begin{gathered}\max P_{it}C_{it} - W_t L_{it} \\ \text{s.t. } C_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\theta} C_t \\ \text{and } C_{it} = A_t L_{it}\end{gathered}$$

From the Dixit-Stiglitz model, the optimal price is:

$$P_{it} = \frac{\theta}{\theta - 1}\frac{W_t}{A_t}$$

$$P_t = \left(\int P_{it}^{1-\theta} di\right)^{\frac{1}{1-\theta}}$$

In the steady state, the labor supply condition implies:

$$\bar{C} = \bar{W}/\bar{P}$$

From the price aggregation condition:

$$\bar{P} = \left(\int \bar{P}_i^{1-\theta} di\right)^{\frac{1}{1-\theta}}$$

From the optimal markup condition:

$$\bar{P} = \frac{\theta}{\theta - 1}\frac{\bar{W}}{\bar{A}}$$

From the Euler equation (and symmetry of $\bar{P}_i = \bar{P}$):

$$\beta E_t \left(\frac{\bar{C}}{\bar{C}}\right)^{-1} \frac{\bar{P}}{\bar{P}}(1+\bar{i}) = 1 \implies \bar{i} = 1/\beta - 1$$

From the output equation:

$$\bar{C} = \bar{A}\bar{L}$$

Thus,

$$\bar{L} = \frac{\bar{C}}{\bar{A}} = \frac{\bar{W}}{\bar{P}\bar{A}} = \frac{\bar{W}}{\bar{A}}\frac{\theta - 1}{\theta}\frac{\bar{A}}{\bar{W}} = \frac{\theta - 1}{\theta}$$

Now let us consider the linearized dynamics around the steady state. Inelastic labor supply means that $l_t = 0$. The labor supply condition implies:

$$c_t = w_t - p_t$$

The Euler equation implies:

$$E_t[c_t - c_{t+1} + p_t - p_{t+1} + i_t] = 0$$

The optimal price implies:

$$p_{it} = w_t - a_t$$

Finally the price aggregation condition implies:

$$p_t = p_{it}$$

The linearized labor supply condition, optimal price condition, and price aggregation condition imply:

$$c_t = w_t - (w_t - a_t) \implies c_t = a_t$$

2. Derive the NKPC following these steps:

(a) write the FOC of an individual firm,

From the Euler equation, we can define the stochastic discount factor: $\Theta_{t,t+j} = \beta^j \frac{C_t}{C_{t+j}} \frac{P_t}{P_{t+j}}$. The firm's problem with sticky prices is to maximize discounted profits:

$$\max_{\{P_{it}\}_{t=0}^{\infty}} E_t \sum_{j=0}^{\infty} \Theta_{t,t+j} \left[P_{i,t+j} C_{i,t+j} - \frac{W_{t+j}}{A_{t+j}} C_{i,t+j} - W_{t+j} \frac{\varphi}{2} \left(\frac{P_{i,t+j} - P_{i,t+j-1}}{P_{i,t+j-1}} \right)^2 \right]$$

$$\text{s.t. } C_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\theta} C_t$$

$$\implies \max_{\{P_{it}\}_{t=0}^{\infty}} E_t \sum_{j=0}^{\infty} \Theta_{t,t+j} \left[P_{i,t+j}^{1-\theta} P_{t+j}^{\theta} C_{t+j} - \frac{W_{t+j}}{A_{t+j}} P_{i,t+j}^{-\theta} P_{t+j}^{\theta} C_{t+j} - W_{t+j} \frac{\varphi}{2} \left(\frac{P_{i,t+j} - P_{i,t+j-1}}{P_{i,t+j-1}} \right)^2 \right]$$

$$\begin{aligned} \implies \max_{\{P_{it}\}_{t=0}^{\infty}} E_t & \left[P_{i,t}^{1-\theta} P_t^{\theta} C_t - \frac{W_t}{A_t} P_{i,t}^{-\theta} P_t^{\theta} C_t - W_t \frac{\varphi}{2} \left(\frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \right)^2 \right] \\ & + \Theta_{t,t+1} \left[P_{i,t+1}^{1-\theta} P_{t+1}^{\theta} C_{t+1} - \frac{W_{t+1}}{A_{t+1}} P_{i,t+1}^{-\theta} P_{t+1}^{\theta} C_{t+1} - W_{t+1} \frac{\varphi}{2} \left(\frac{P_{i,t+1} - P_{i,t}}{P_{i,t}} \right)^2 \right] \\ & + \sum_{j=2}^{\infty} \Theta_{t,t+j} \left[P_{i,t+j}^{1-\theta} P_{t+j}^{\theta} C_{t+j} - \frac{W_{t+j}}{A_{t+j}} P_{i,t+j}^{-\theta} P_{t+j}^{\theta} C_{t+j} - W_{t+j} \frac{\varphi}{2} \left(\frac{P_{i,t+j} - P_{i,t+j-1}}{P_{i,t+j-1}} \right)^2 \right] \end{aligned}$$

FOC $[P_{i,t}]$:

$$E_t\left[(1-\theta)P_{i,t}^{-\theta}P_t^{\theta}C_t-(-\theta)\frac{W_t}{A_t}P_{i,t}^{-\theta-1}P_t^{\theta}C_t-\frac{W_t\varphi}{P_{i,t-1}}\left(\frac{P_{i,t}-P_{i,t-1}}{P_{i,t-1}}\right)+\Theta_{t,t+1}\frac{W_{t+1}\varphi P_{i,t+1}}{P_{i,t}^2}\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right)\right]=0$$

$$\implies (1-\theta)C_{i,t}+\theta\frac{W_t}{A_t}P_{i,t}^{-1}C_{i,t}-\frac{W_t\varphi}{P_{i,t-1}}\left(\frac{P_{i,t}-P_{i,t-1}}{P_{i,t-1}}\right)=-E_t\left[\beta\frac{C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\frac{W_{t+1}\varphi P_{i,t+1}}{P_{i,t}^2}\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right)\right]$$

$$\implies (1-\theta)C_{i,t}P_{i,t}+\theta\frac{W_t}{A_t}C_{i,t}-\frac{W_t\varphi P_{i,t}}{P_{i,t-1}}\left(\frac{P_{i,t}-P_{i,t-1}}{P_{i,t-1}}\right)=-E_t\left[\beta\frac{C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\frac{W_{t+1}\varphi P_{i,t+1}}{P_{i,t}}\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right)\right]$$

(b) impose symmetry across producers and define inflation rate as $\pi_t=\frac{P_t}{P_{t-1}}-1$,

Symmetry across producers implies that $P_{i,t}=P_t$ and $C_{t,t}=C_t$:

$$(1-\theta)C_tP_t+\theta\frac{W_t}{A_t}C_t-\frac{W_t\varphi P_t}{P_{t-1}}\left(\frac{P_t-P_{t-1}}{P_{t-1}}\right)=-E_t\left[\beta\frac{C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\frac{W_{t+1}\varphi P_{t+1}}{P_t}\left(\frac{P_{t+1}-P_t}{P_t}\right)\right]$$

$$(1-\theta)C_tP_t+\theta\frac{W_t}{A_t}C_t-\frac{W_t\varphi P_t}{P_{t-1}}\left(\frac{P_t-P_{t-1}}{P_{t-1}}\right)=-E_t\left[\beta\frac{C_t}{C_{t+1}}W_{t+1}\varphi\left(\frac{P_{t+1}-P_t}{P_t}\right)\right]$$

Define inflation rate as $\pi_t=\frac{P_t}{P_{t-1}}-1=\frac{P_t-P_{t-1}}{P_{t-1}}$:

$$(1-\theta)P_t+\theta\frac{W_t}{A_t}=\varphi\left(\frac{W_t}{C_t}\pi_t(\pi_t+1)-\beta E_t\left[\frac{W_{t+1}}{C_{t+1}}\pi_{t+1}\right]\right)$$

(c) take the first-order approximation,

Define

$$\begin{aligned}U_t &:= (1-\theta)P_t+\theta\frac{W_t}{A_t}\\ V_t &:= \varphi\left(\frac{W_t}{C_t}\pi_t(\pi_t+1)-\beta E_t\left[\frac{W_{t+1}}{C_{t+1}}\pi_{t+1}\right]\right)\end{aligned}$$

Notice that $\bar{\pi}=0 \implies \bar{V}=0 \implies \bar{U}=0$:

$$\begin{aligned}u_t &= (1-\theta)\bar{P}(1+p_t)+\theta\frac{\bar{W}}{\bar{A}}(1+w_t-a_t)\\ &= (1-\theta)\bar{P}p_t+\theta\frac{\bar{W}}{\bar{A}}(w_t-a_t)+(1-\theta)\bar{P}+\theta\frac{\bar{W}}{\bar{A}}\\ &= (1-\theta)\bar{P}p_t+\theta\frac{\bar{W}}{\bar{A}}(w_t-a_t)+\bar{U}\\ &= (1-\theta)\bar{P}p_t+\theta\frac{\bar{W}}{\bar{A}}(w_t-a_t)\end{aligned}$$

Turning to V_t, define

$$M_t := \frac{W_t}{C_t}\pi_t$$
$$N_t := \frac{W_t}{C_t}\pi_t(\pi_t + 1)$$

Notice that $\bar{M} = \bar{N} = 0$:

$$m_t = \frac{\bar{W}}{\bar{C}}(1 + w_t - c_t)\pi_t = \frac{\bar{W}}{\bar{C}}[\pi_t + \pi_t(w_t - c_t)] \approx \frac{\bar{W}}{\bar{C}}\pi_t$$

Notice that the first order approximation of $\pi_t(\pi_t + 1)$ is π_t, so $n_t = m_t$. Thus, $u_t = v_t$ implies:

$$(1-\theta)\bar{P}p_t + \theta\frac{\bar{W}}{\bar{A}}(w_t - a_t) = \varphi\frac{\bar{W}}{\bar{C}}\Big(\pi_t - \beta E_t[\pi_{t+1}]\Big)$$

(d) write the NKPC in terms of inflation rate and output gap,

The log-linearized labor supply condition is $c_t = w_t - p_t$ and market clearing for consumption is $c_t = y_t$.

$$(1-\theta)\bar{P}p_t + \theta\frac{\bar{W}}{\bar{A}}(y_t + p_t - a_t) = \varphi\frac{\bar{W}}{\bar{C}}\Big(\pi_t - \beta E_t[\pi_{t+1}]\Big)$$

$$\implies \left[(1-\theta)\bar{P} + \theta\frac{\bar{W}}{\bar{A}}\right]p_t + \theta\frac{\bar{W}}{\bar{A}}(y_t - a_t) = \varphi\frac{\bar{W}}{\bar{C}}\Big(\pi_t - \beta E_t[\pi_{t+1}]\Big)$$
$$\implies \theta\frac{\bar{W}}{\bar{A}}(y_t - a_t) = \varphi\frac{\bar{W}}{\bar{C}}\Big(\pi_t - \beta E_t[\pi_{t+1}]\Big)$$

because $(1-\theta)\bar{P} + \theta\frac{\bar{W}}{\bar{A}} = 0$. Thus, inflation can be written as a function of the output gap ($x_t = y_t - a_t$) and expected inflation:

$$\begin{aligned}
\pi_t &= \frac{\theta}{\varphi}\frac{\bar{C}}{\bar{A}}x_t + \beta E_t[\pi_{t+1}] \\
&= \frac{\theta}{\varphi}\frac{\bar{A}\bar{L}}{\bar{A}}x_t + \beta E_t[\pi_{t+1}] \\
&= \frac{\theta}{\varphi}\frac{\theta - 1}{\theta}x_t + \beta E_t[\pi_{t+1}] \\
&= \frac{\theta - 1}{\varphi}x_t + \beta E_t[\pi_{t+1}]
\end{aligned}$$

Defining $\kappa := \frac{\theta-1}{\varphi}$, the NKPC is:

$$\pi_t = \kappa x_t + \beta E_t[\pi_{t+1}]$$

(e) compare the NKPC to the one under the Calvo pricing.

From lecture the NKPC with Calvo pricing ($\sigma = 1$ and $\varphi = 0$) is:

$$\pi_t = \frac{(1-\lambda)(1-\beta\lambda)}{\lambda} x_t + \beta E_t[\pi_{t+1}]$$

The coefficient on expected inflation matches, but the coefficient on the output gap differs due to the different mechanism that add stickiness to prices. With Calvo pricing, λ is the fraction of firms that cannot change prices. The lower λ, the more inflation is determined by the output gap. When λ gets close to 1, the coefficient on output gap goes to zero and inflation is determined by future inflation. With labor price adjustment, φ is the coefficient on the additional units of labor that are needed to change the price. When φ is low, inflation is more influenced by the output gap. When φ increases, the coefficient on the output gap decreases and the more inflation is determined by expected inflation.

3. What is the source of inflation costs in this model? Is it different from the one in the Calvo model?

In the Calvo model, the inflation costs are due to labor misallocation. When some firms cannot change their prices, their prices do not reflect the costs of production and consumption and production is not optimal. Thus, labor is misallocated between firms. In this model, the inflation costs are due to firms needing to hire additional labor to change their prices.

4. Let the monetary policy be described by the Taylor rule $i_t = \phi x_t + u_t$. What restrictions on the coefficient ϕ_x ensure uniqueness of the equilibrium?

Linearize the Euler equation:

$$E_t[c_{t+1}] - c_t = i_t - E_t[\pi_{t+1}]$$

Define the "natural rate" as the real rate that prevails under flexible prices: $r_t^n = E_t[a_{t+1}] - a_t$. Express the Euler equation in terms of the output gap to derive the NKIS curve:

$$\begin{aligned} E_t[c_{t+1}] - c_t - (E_t[a_{t+1}] - a_t) &= i_t - E_t[\pi_{t+1}] - r_t^n \\ \implies E_t[c_{t+1} - a_{t+1}] - (c_t - a_t) &= i_t - E_t[\pi_{t+1}] - r_t^n \\ \implies E_t[x_{t+1}] - x_t &= i_t - E_t[\pi_{t+1}] - r_t^n \end{aligned}$$

Substituting the Taylor rule and the NKPC into the NKIS:

$$E_t[x_{t+1}] - x_t = (\phi x_t + u_t) - (\frac{1}{\beta}\pi_t - \frac{\kappa}{\beta}x_t) - r_t^n$$

$$\implies E_t[x_{t+1}] = (1 + \phi + \frac{\kappa}{\beta})x_t - \frac{1}{\beta}\pi_t + u_t - r_t^n$$

We can writing the NKPC and the NKIS in matrix form:

$$E_t \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & \frac{-\kappa}{\beta} \\ \frac{-1}{\beta} & 1 + \phi + \frac{\kappa}{\beta} \end{pmatrix} \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (u_t - r_t^n)$$

With two control variables, both eigenvalues of the law of motion matrix must be larger than 1:

$$
\begin{aligned}
f(\lambda) &= (\frac{1}{\beta} - \lambda)(1 + \phi + \frac{\kappa}{\beta} - \lambda) - \frac{-\kappa}{\beta}\frac{-1}{\beta} \\
= \lambda^2 - (1 + \phi + \frac{\kappa}{\beta} + \frac{1}{\beta})\lambda &+ (\frac{1}{\beta})(1 + \phi + \frac{\kappa}{\beta}) - \frac{\kappa}{\beta^2} \\
&= \lambda^2 - (1 + \phi + \frac{\kappa}{\beta} + \frac{1}{\beta})\lambda + \frac{1 + \phi}{\beta}
\end{aligned}
$$

For both eigenvalues to be greater than one, it is necessary and sufficient that $f(0) > f(1) > 0$:

$$f(0) = \frac{1 + \phi}{\beta}$$

$$f(1) = 1 - (1 + \phi + \frac{\kappa}{\beta} + \frac{1}{\beta}) + \frac{1 + \phi}{\beta} = (\frac{1}{\beta} - 1)\phi - \frac{\kappa}{\beta}$$

These inequalized imply restrictions on ϕ:

$$f(0) > f(1) \implies \phi > \frac{1 + \kappa}{-\beta}$$

$$f(1) > 0 \implies \phi > \frac{\kappa}{1 - \beta}$$