Problem 1 Each of the following set closed, open, and/or compact.

a. $X = \{ x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 2,\ 0 \leq x_2 < \infty \}$ in $(\mathbb{R}^2, d_E)$

Consider $X^c = \{ x \in \mathbb{R}^2 \mid -\infty < x_1 < 0 \text{ or } 2 < x_1 < \infty,\ -\infty < x_2 < 0 \}$.

X^c is open because $\forall x \in X^c\ \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset X^c$. Thus X is closed and not open.

X is not compact because it is not bounded.

[Contraposition of Heine-Borel]

b. $Y = \{1, 2\}$ in $(\mathbb{N}, d_E)$

Y is open because if $\varepsilon = 1/2$ and $x = 1 \in Y$ $B_{1/2}(1) = \{1\} \subset Y$. Similarly, if $x = 2 \in Y$ then $B_{1/2}(2) = \{2\} \subset Y$.

$Y^c = \mathbb{N} \setminus Y = \{3, 4, \ldots\}$

Y^c is open because, if $\varepsilon = 1/2$ and $\forall x \in Y^c$, $B_{1/2}(x) = \{x\} \subset Y^c$. Thus Y is closed.

Y is compact: Let $\mathcal{U}$ be an open cover for Y. Construct a finite subcover from $\mathcal{U}$ as the union of U_{λ_1} and U_{λ_2}, where $U_{\lambda_1} \supset \{1\}$ and $U_{\lambda_2} \supset \{2\}$.

Problem 1

c. $Z = \{(0,1) \cup \{2\}\}$ in $([0,1] \cup \{2\}, d_E)$

Notice $Z^c = \{0, 1\}$ in $([0,1] \cup \{2\}, d_E)$

Z is open because (1) $\forall x \in (0,1)$ $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset [0,1] \subset Z$. (2) for $x = 2$ $\varepsilon = 1/2$ then $B_{1/2}(2) = \{2\} \subseteq \{2\} \subset Z$.

Z^c is not open because (1) for $x = 0 \in Z^c$ $\forall \varepsilon > 0$ $B_\varepsilon(0) = \{x \mid 0 \le x < \varepsilon\} \not\subset Z^c$. Thus Z is not closed.

Since Z is not closed, Z can not be compact. (Contrapositive of if X is compact $\Rightarrow$ X is closed).

Problem 2

Proof To show that d is a metric function, we show that (1) $d(x,y) \geq 0$ $\forall x,y \in M$ w/ $d(x,y)=0$ only for $x=y$, (2) $d(x,y)=d(y,x)$, and (3) $d(x,y) \leq d(x,z)+d(z,y)$ for $\forall$ $x,y,z \in M$.

For (1), $d(x,y) \geq 0$ $\forall x,y \in M$ because two word can — at the least — have 0 different letters. If x and y are the same word, then all of their letters are common so $d(x,y)=0$. If $d(x,y)=0$, then ~~they~~ x and y have no letters that are different, so they must be the same word.

For (2), $d(x,y)=d(y,x)$ because as we compare each letter from each word the order of whether we compare letters in x first or y first doesn't matter. ~~[illegible]~~

For (3), notice that the triangle inequality trivially hold if $z=x$ or $z=y$ ($f(x,y) \leq f(x,x)+f(x,y)=f(x,y)$). Thus, ~~z~~ Consider z where $z \neq x$ and $z \neq y$. Let x_i be the letter in the ith place in x w/ $1 \leq i \leq n$. Thus, $d(x,y)=\sum_{i=1}^{n} \mathbb{1}(x_i \neq y_i)$. Consider $x_i = y_i$, thus ~~[illegible]~~ it doesn't ~~contribute~~ to $d(x,y)$. If $z_i = x_i = y_i$, then it doesn't contribute to $d(x,z)+d(y,z)$. If $z_i \neq x_i = y_i$, then 1 is add to $d(x,z)+d(y,z)$ thus preserves the greater than or equal inequality. If $x_i \neq y_i$, ~~[illegible]~~ this letter position contributes 1 to $d(x,y)$. If $z_i \neq x_i$ and $z_i \neq y_i$, this letter position contributes 2 to $d(x,z)+d(z,y)$. If $z_i = x_i$ or $z_i = y_i$, this letter position contributes 1 to $d(x,z)+d(z,y)$ matching ~~the~~

Problem 2 con't

the contribution to $d(x,y)$. Thus $d(x,y) \leq d(x,z) + d(z,y)$.

Thus, d is a metric function.

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Problem 3 $\quad F = \{f : \mathbb{Z} \to \mathbb{R}\}$

(a) Show that F is a vector ~~state~~ space.

Proof: For $\forall x \in \mathbb{Z}$ and $\forall f \in F$, $f(x) \in \mathbb{R}$. Thus F being a vector space flows from the properties of $\mathbb{R}$. Let $a, b, c \in F$, $\alpha, \beta \in \mathbb{R}$, $\bar{0} \in F$ s.t. $\bar{0}(x) = 0$ $\forall x \in \mathbb{Z}$, and $(-a) \in F$ s.t. $(-a)(x) = -a(x)$ $\forall x \in \mathbb{Z}$.

- $\forall x \in \mathbb{Z}$, $a(x), b(x), c(x) \in \mathbb{R}$. Thus, $(a(x) + b(x)) + c(x) = a(x) + (b(x) + c(x))$
- $\forall x \in \mathbb{Z}$, $a(x), b(x) \in \mathbb{R}$, so $a(x) + b(x) = b(x) + a(x)$.
- $\forall x \in \mathbb{Z}$, $a(x) \in \mathbb{R}$ and $\bar{0}(x) = 0$, so $a(x) + \bar{0}(x) = a(x) + 0 = a(x)$
  $\bar{0}(x) + a(x) = 0 + a(x) = a(x)$.
- $\forall x \in \mathbb{Z}$, $a(x), (-a)(x) \in \mathbb{R}$, so $a(x) + (-a)(x) = a(x) - a(x)$
  $= 0 = \bar{0}(x)$
- $\forall x \in \mathbb{Z}$, $a(x), b(x) \in \mathbb{R}$, so $\alpha(a(x) + b(x)) = \cancel{\alpha a(x) + \alpha b(x)}$
  $= \alpha a(x) + \alpha b(x)$
- $\forall x \in \mathbb{Z}$, $a(x) \in \mathbb{R}$, so $(\alpha + \beta) a(x) = \alpha a(x) + \beta a(x)$.
- $\forall x \in \mathbb{Z}$, $a(x) \in \mathbb{R}$, so $(\alpha \cdot \beta) a(x) = \alpha(\beta a(x))$
- $\forall x \in \mathbb{Z}$, $a(x) \in \mathbb{R}$, so $1 \cdot a(x) = a(x)$.

Thus, F is a vector space.

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(b) $T: F \to F$ defined by $[T(f)](x) = \frac{1}{2}(f(x) + f(-x))$, $x \in \mathbb{Z}$ is linear.

Proof: Let $a, b \in F$ and $\alpha, \beta \in \mathbb{R}$. $\exists c \in F$, s.t $c(x) = \alpha a(x) + \beta b(x)$ $\forall x \in \mathbb{Z}$. Apply T to $c(x)$:

$$
\begin{aligned}
[T(c)](x) &= \frac{1}{2}(c(x) + c(-x)) \\
&= \frac{1}{2}\Big[(\alpha a(x) + \beta b(x)) + (\alpha a(-x) + \beta b(-x))\Big] \\
&= \alpha \frac{1}{2}(a(x) + a(-x)) + \beta \frac{1}{2}(b(x) + b(-x)) \\
&= \alpha [T(a)](x) + \beta [T(b)](x).
\end{aligned}
$$

Thus, T is linear. $\blacksquare$

Problem 3

c. Calculate Ker T and Im T

$\operatorname{Ker} T = \{ f \in F \mid T(f)(x) = 0 \quad \forall x \in \mathbb{Z} \}$

For $T(f)(x) = 0 \implies \frac{1}{2}(f(x) + f(-x)) = 0$

$\implies f(x) + f(-x) = 0$

$\implies f(x) = -f(-x)$

$\implies -f(x) = f(-x)$

$\implies$ f is an odd function.

$$\boxed{\begin{aligned} \operatorname{Ker} T &= \{ f \in F \mid f(x) = -f(-x) \quad \forall x \in \mathbb{Z} \} \\ &= \{ f \in F \mid f \text{ is an odd function} \} \end{aligned}}$$

$\operatorname{Im} T = \{ T(f)(x) \mid f \in F \}$

An arbitrary element of F is

$$f(x) = \begin{cases} \vdots & \vdots \\ a & x = -2 \\ b & x = -1 \\ c & x = 0 \\ d & x = 1 \\ e & x = 2 \\ \vdots & \vdots \end{cases} \qquad a, b, c, d, e \in \mathbb{R}$$

An arbitrary element of $T(f)$ $\forall f \in F$ is

$$T(f)(x) = \begin{cases} \vdots & \vdots \\ \frac{1}{2}(a+e) & x = -2 \\ \frac{1}{2}(b+d) & x = -1 \\ c & x = 0 \\ \frac{1}{2}(d+b) & x = 1 \\ \frac{1}{2}(e+a) & x = 2 \\ \vdots & \vdots \end{cases}$$

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$$\Rightarrow T(f)(x) = \begin{cases} c & x = 0 \\ \tfrac{1}{2}(b+d) & x \in \{-1,1\} \\ \tfrac{1}{2}(a+e) & x \in \{-2,2\} \\ \vdots & \vdots \end{cases}$$

Since $a, b, c, d, e \in \mathbb{R}$, $\tfrac{1}{2}(b+d), \tfrac{1}{2}(a+e) \in \mathbb{R}$.

Thus,

$$\Rightarrow T(f)(x) = \begin{cases} p & x = 0 \\ q & x \in \{-1,1\} \\ r & x \in \{-2,2\} \\ \vdots & \vdots \end{cases} \quad p, q, r \in \mathbb{R}$$

$$\Rightarrow \boxed{\operatorname{Im} T = \{ f \in F \mid f(x) = f(-x) \quad x \in \mathbb{Z} \}}$$

Problem 4

Proof: I show this by induction.

For the base step, assume $K=4$. We need to show that all info can become common knowledge with no more than $2(4)-4=4$ calls. ~~Let~~ Let us denote the 4 people, A, B, C, and D. First, A calls B and now A and B know each others' info. Second, C calls D and now C and D know each others info. Third, A calls C and now A and C know all the info. Fourth, B calls D and now B and D know all the info.

For the induction step, assume that ~~there~~ there are $K \geq 4$ people and that all private info can become common know with no more than $2K-4$ calls. Now, we add one more person. We now need to show that all $K+1$ people can know all information after ~~at~~ no more than $2(K+1)-4 = (2K-4)+2$. Notice that we have two more calls than we did for just K people. ~~First, the new person call one of existing people the other people Let So, the person knows both their info at the info of one of the other people First, let the (K+1)th person call the 1st person~~ For sake of clarity, denote each person as $1, \dots, K, K+1$. With the $(K+1)$th person, the newest addition. First the $(K+1)$th person calls person numbered 1. Then, omitting ~~the~~ person $K+1$, the K people call each as per the induction hypothesis. ~~Then~~ So, now K people know all information. Finally, person $(K+1)$ calls the first person back and gets all information. Thus, all $(K+1)$ people know ~~all private know~~ everything. ∎