ECON 703 - PS 2

Alex von Hafften*

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(1) Consider the set $A = \{\frac{1}{n}\}_{n \in \mathbb{N}} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. Does there exist $S \subset \mathbb{R}$, such that the set of S's limit points equals A?

No.

Proof: For the sake of a contradiction, assume that $\exists S \subset \mathbb{R}$ such that the set of S's limit points, called B, equals $A = \{\frac{1}{n}\}_{n \in \mathbb{N}}$. For $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$ for all $n \geq N$. Notice that there must exist an $s \in S$ such that $s \in B_{\frac{1}{4n}}(\frac{1}{2n}) = (\frac{1}{2n} - \frac{1}{4n}, \frac{1}{2n} + \frac{1}{4n})$. Furthermore, $|s - 0| < \frac{1}{2n} + \frac{1}{4n} < \frac{1}{n} < \varepsilon$, so 0 must be a limit point of S and therefore $0 \in B$. However, $A \cap \{0\} = \emptyset$, so $A \neq B$. $\Rightarrow\Leftarrow$. No such $S \subset \mathbb{R}$ exists. $\square$

(2) Prove that $f(x) = \cos x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof: Let $\varepsilon = 1$. Define $x = \sqrt{(2k+1)\pi}$ and $y = \sqrt{2k\pi}$ where $k \in \mathbb{N}$. Notice that $f(x) = \cos x^2 = -1$ and $f(y) = \cos y^2 = 1$, thus $|f(x) - f(y)| = |(-1) - 1| = 2 > \varepsilon$. Now consider

$$\begin{aligned}
|x - y| &= |\sqrt{(2k+1)\pi} - \sqrt{2k\pi}| \\
&= \frac{|(2k+1)\pi - 2k\pi|}{|\sqrt{(2k+1)\pi} + \sqrt{2k\pi}|} \\
&= \frac{\pi}{|\sqrt{(2k+1)\pi} + \sqrt{2k\pi}|}
\end{aligned}$$

For any $\delta > 0$, choose $K = \frac{(\pi/\delta)^2}{2\pi}$, so for all $k > K$.

$$\frac{\pi}{\sqrt{(2k+1)\pi} + \sqrt{2k\pi}} < \frac{\pi}{\sqrt{2k\pi}} = \frac{\pi}{\sqrt{2\frac{(\pi/\delta)^2}{2\pi}\pi}} = \delta$$

Thus, $f(x) = \cos x^2$ is not uniformly continuous on $\mathbb{R}$. $\square$

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(3) Show that if the function $f : \mathbb{R} \to \mathbb{R}_{++}$ is continuous on an interval $[a, b]$, where $\mathbb{R}_{++} = \{x \in \mathbb{R} | x > 0\}$, then the reciprocal of this function $(\frac{1}{f})$ is bounded on this same interval.

Proof: By the extreme value theorem, we know that $f(x)$ attains its maximum and minimum on $x \in [a, b]$. Define $M_1 := \frac{1}{\sup f(x)}$ and $M_2 := \frac{1}{\inf f(x)}$ for $x \in [a, b]$. Notice that $\inf f(x) > 0$ because f maps to $\mathbb{R}_{++}$. Thus, $\forall x \in [a, b]$, $0 < \frac{1}{M_2} \leq f(x) \leq \frac{1}{M_1} \implies M_1 \leq \frac{1}{f(x)} \leq M_2$. Thus, $\frac{1}{f(x)}$ is bounded on $[a, b]$. $\square$

(4) ***Bisection method*****. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, $a < b, a, b \in \mathbb{R}$. Assume that $f(a) < 0 < f(b)$. We want to show that $\exists c \in (a, b)$ such that $f(c) = 0$. To do this, construct the following sequences:**

(I): Set $l_1 = a$, $u_1 = b$.

(II): For each n, let $m_n = (l_n + u_n)/2$.

- if $f(m_n) > 0$, then set $l_{n+1} = l_n, u_{n+1} = m_n$;

- if $f(m_n) < 0$, then set $l_{n+1} = m_n, u_{n+1} = u_n$;

- if $f(m_n) = 0$, then stop and set $l_k = u_k = m_n$ for all $k \geq n$.

Using what you have learned about the limits of real sequences, prove

(a) Sequences $\{l_n\}$ and $\{u_n\}$ both converge.

Proof: To prove that $\{l_n\}$ and $\{u_n\}$ both converge, we show that $l_n \leq u_n$, $l_n \leq l_{n+1}$, and $u_n \geq u_{n+1}$ $\forall n \in \mathbb{N}$ by induction. If these three conditions hold, both sequences converge by the monotone convergence theorem because $\{l_n\}$ is an increasing sequence bounded from above by b and $\{u_n\}$ is a decreasing sequence bounded from below by a.

For the base step, $l_1 = a < b = u_1$ by assumption.

For the induction step, assume $l_n \leq u_n$. Furthermore,

$$m_n - l_n = (l_n + u_n)/2 - l_n = (u_n - l_n)/2 \geq 0 \implies m_n \leq l_n$$
$$u_n - m_n = u_n - (l_n + u_n)/2 = (u_n - l_n)/2 \geq 0 \implies u_n \geq m_n$$

Thus, $l_n \leq m_n \leq u_n$. If $f(m_n) > 0$, $l_{n+1} = l_n \leq m_n = u_{n+1} \leq u_n$. If $f(m_n) < 0$, $l_n \leq m_n = l_{n+1} \leq u_n = u_{n+1}$. If $f(m_n) = 0$, we stop and the sequences ends. Clearly, $\min\{l_n\} = l_1 = a$ and $\max\{u_n\} = u_1 = b$. Hence, $\{l_n\}$ is an increasing sequence bounded from above by b and $\{u_n\}$ is a decreasing sequence bounded from below by a, so both sequences converge. $\square$

(b) Both sequences converge to the same limit, i.e. $\lim_{n\to\infty} l_n = \lim_{n\to\infty} u_n$**. (Hint: Show that** $\{u_n - l_n\} \to 0$**).**

Proof: First, I show that $u_n - l_n \leq \frac{b-a}{2^{n-1}}$ by induction. Notice that,

$$
\begin{aligned}
f(m_n) > 0 &\implies u_{n+1} - l_{n+1} = m_n - l_n = (l_n + u_n)/2 - l_n = (u_n - l_n)/2 \\
f(m_n) < 0 &\implies u_{n+1} - l_{n+1} = u_n - m_n = u_n - (l_n + u_n)/2 = (u_n - l_n)/2 \\
f(m_n) = 0 &\implies u_{n+1} - l_{n+1} = m_n - m_n = 0
\end{aligned}
$$

For the base step, $u_1 - l_1 = b - a = \frac{b-a}{2^{1-1}}$. For the induction step, if $f(m_n) = 0$, then $l_k = u_k \implies u_k - l_k = 0$ for all $k > n$. If $f(m_n) \neq 0$, assume $u_n - l_n = \frac{b-a}{2^{n-1}}$:

$$
\begin{aligned}
u_{n+1} - l_{n+1} &= \frac{b-a}{2^n} \\
&= \frac{b-a}{2^{n-1}}\frac{1}{2} \\
&= (u_n - l_n)/2
\end{aligned}
$$

Thus, $u_n - l_n \leq \frac{b-a}{2^{n-1}}$. For $\varepsilon > 0$, choose $N = \log_2(\frac{b-a}{\varepsilon}) + 1$. For $n > N$,

$$
|\frac{b-a}{2^{n-1}} - 0| < |\frac{b-a}{2^{\log_2(\frac{b-a}{\varepsilon})+1-1}} - 0| = \varepsilon
$$

Thus, $\{u_n - l_n\} \to 0$ by the squeeze theorem. Since from (a), we know $u_n \to u$ and $l_n \to l$, this result implies that $u = l$. $\square$

(c) Define the common limit of two sequences c **and show that** $f(c) = 0$**. (Hint: Use the continuity of** f **and the fact that taking limits preserves weak inequalities).**

Proof: Define c as the limit of u_n and l_n. Because f is continuous, $f(u_n) \to f(c)$ and $f(l_n) \to f(c)$.

Now, I will show that $f(l_n) \leq 0 \leq f(u_n)$ for all $n \in \mathbb{N}$ using induction. For the base step, $f(l_1) = f(a) < 0 < f(b) = f(u_1)$. For the induction step, assume that $f(l_n) \leq 0 \leq f(u_n)$. If $f(m_n) > 0$, then $f(l_n) = f(l_{n+1}) \leq 0 \leq f(m_n) = f(u_{n+1})$. If $f(m_n) < 0$, then $f(l_{n+1}) = f(m_n) \leq 0 \leq f(u_n) = f(u_{n+1})$. Thus, $f(l_n) \leq 0 \leq f(u_n)$. Since taking limits preserve weak inequalities: $f(l) \leq 0 \leq f(u) \implies f(c) \leq 0 \leq f(c) \implies f(c) = 0$. $\square$

(5) Prove that at any time there are two antipodal points (diametrically opposite) on Earth that share the same temperature. (Hint: Use the Intermediate Value Theorem).

Proof: I will show that two antipodal points on any great circle, C, of the Earth share the same temperature. A great circle of a sphere is the intersection of the sphere and a plane that passes through the center point of the sphere. (e.g., the Equator). Notice that if two antipodal points on any great circle of the Earth share the same temperature then any two arbitrary antipodal points on Earth must share the same temperature. In addition, I assume that temperature is continuous when moving across the global and can be represented by a continuous function.

Let x_0 be the point that C intersects the prime meridian. If C follows the prime meridian, let x_0 be Null Island (i.e. the point that the prime meridian intersects with the Equator). Construct a function $f(\theta) : [0, 2\pi] \to \mathbb{R}$ that equals the temperature at a point on C such that the angle between the point and x_0 is θ radians. Assume that f is a continuous function.

Let $g(\theta) = f(\theta) - f(\theta + \pi)$. Notice that $g(\theta)$ is the difference between the temperatures at antipodal points on C. As the difference of two continuous functions, $g(\theta)$ is also continuous.

If $g(0) = 0$, $f(0) - f(0 + \pi) = 0 \implies f(0) = f(\pi)$. Thus, the point x_0 and its antipodal point (i.e., the point at a π radian angle from x_0) share the same temperature.

If $g(0) > 0$, then

$$\begin{aligned} f(0) - f(0 + \pi) > 0 &\implies 0 > f(\pi) - f(0) \\ &\implies 0 > f(\pi) - f(2\pi) \\ &\implies 0 > g(\pi) \end{aligned}$$

By the intermediate value theorem, there exists $\tilde{\theta} \in [0, \pi]$ such that $g(\tilde{\theta}) = 0 \implies f(\tilde{\theta}) = f(\tilde{\theta} + \pi)$. The point at a $\tilde{\theta}$ radian angle from x_0 and its antipodal point (i.e., the point at a $\tilde{\theta} + \pi$ radian angle from x_0 along C) share the same temperature.

If $g(0) < 0$, then

$$\begin{aligned} f(0) - f(0 + \pi) < 0 &\implies 0 < f(\pi) - f(0) \\ &\implies 0 < f(\pi) - f(2\pi) \\ &\implies 0 < g(\pi) \end{aligned}$$

By the intermediate value theorem, there exists $\hat{\theta} \in [0, \pi]$ such that $g(\hat{\theta}) = 0 \implies f(\hat{\theta}) = f(\hat{\theta} + \pi)$. The point at a $\hat{\theta}$ radian angle from x_0 and its antipodal point (i.e., the point at a $\hat{\theta} + \pi$ radian angle from x_0 along C) share the same temperature. $\square$