ECON 710A - Problem Set 6

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1. Consider a random effects model with an intercept only: $Y_{it} = \mu_0 + \alpha_i + \varepsilon_{it}$ $(i = 1, ..., n, t = 1, ..., T_i)$ with $\varepsilon_i = (\varepsilon_{i1}, ..., \varepsilon_{iT_i})'$ independent across i, $E[\alpha_i] = E[\varepsilon_i] = 0$, and

$$Var\begin{pmatrix}\alpha_i \\ \varepsilon_i\end{pmatrix} = \begin{bmatrix}\sigma_\alpha^2 & 0 \\ 0 & \sigma^2 I_{T_i}\end{bmatrix} \text{ where } \sigma^2 > 0$$

(i) Argue that the OLS estimator of μ_0 (the sample average), can be represented as $\hat{\mu}_{OLS} = \frac{\sum_{i=1}^n 1_i' Y_i}{\sum_{i=1}^n 1_i' 1_i}$ where $1_i = (1, ..., 1)' \in \mathbb{R}^{T_i}$ and $Y_i = (Y_{i1}, ..., Y_{iT_i})'$.

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(ii) For a non-random instrument $Z_i \in \mathbb{R}^{T_i}$, consider the IV estimator $\hat{\mu}_{IV} = \frac{\sum_{i=1}^n Z_i' Y_i}{\sum_{i=1}^n Z_i' 1_i}$. Show that $Var(\hat{\mu}_{IV}) = \frac{\sum_{i=1}^n Z_i' \Omega_i Z_i}{(\sum_{i=1}^n Z_i' 1_i)^2}$ for some $\Omega_i = \Omega_i(\sigma_\alpha^2, \sigma^2, T_i)$ and find Ω_i.

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(iii) Show that $Var(\hat{\mu}_{IV}) \geq (\sum_{i=1}^n 1_i' \Omega_i^{-1} 1_i)^{-1}$ and find an instrument $\tilde{Z}_i$ (possibly depending on Ω_i) such that $Var(\frac{\sum_{i=1}^n \tilde{Z}_i' y_i}{\sum_{i=1}^n \tilde{Z}_i' 1_i}) = (\sum_{i=1}^n 1_i' \Omega_i^{-1} 1_i)^{-1}$.

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(iv) The IV estimator that uses $\tilde{Z}_i$ is often referred to as generalized least squares (GLS) and it is (weakly) more efficient than the OLS estimator. Is GLS strictly more efficient than OLS if $T_i = T$ for all i?

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(v) Show that $\hat{\sigma}_i^2 = \frac{1}{T_i - 1} \sum_{t=1}^{T_i} (Y_{it} - \bar{Y}_i)^2$ where $\bar{Y}_i = \frac{1}{T_i} \sum_{i=1}^{T_i} Y_{it}$ has expectation equal to σ^2 and argue (informally) that $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \hat{\sigma}_i{}^2$ is consistent for σ^2 as $n \to \infty$.

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(vi) Show that $\hat{\sigma}_{\alpha,i}(\mu) = \frac{1}{T_i} \sum_{t=1}^{T_i} (Y_{it} - \mu)^2 - \hat{\sigma}_i^2$ has expectation equal to σ_α^2 when $\mu = \mu_0$ and argue (informally) that $\hat{\sigma}_\alpha^2 = \frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{\alpha,i}^2(\hat{\mu}_{OLS})$ is consistent for $\hat{\sigma}_\alpha^2$ as $n \to \infty$.

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(vii) If we let $\hat{\Omega}_i = \Omega_i(\hat{\sigma}_\alpha^2, \hat{\sigma}^2, T_i)$ and use this to construct a feasible version of $\tilde{Z}_i$, then we obtain the feasible GLS estimator. In the panel data context, this estimator is also referred to as the random effects estimator. The previous arguments imply that $\sqrt{n}\hat{\mu}_{FGLS}$ and $\sqrt{n}\hat{\mu}_{GLS}$ has the same asymptotic variances as $n \to \infty$. Propose a variance estimator $\hat{V}$ for which you expect (based on the previous questions) that $\hat{V}^{-1/2}(\hat{\mu}_{FGLS} - \mu_0) \to_d N(0, 1)$.

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*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

2. Consider a fixed effects regression model $Y_{it} = X_{it}\beta_0 + \alpha_i + \varepsilon_{it}$ $(i = 1, ..., n, t = 1, ..., T)$ where data is independent across i, X_{it} is independent of X_{is} for $s \neq t$ with $E[X_{it}] = 0$ and $E[X_{it}^2] = \sigma_x^2$ for all t, and strict exogeneity fails since $E[X_{is}\varepsilon_{it}] = \delta\sigma_x^2 1\{s = t+1\}$. Such failure can hapen if the regressor is a response to a previous shock, e.g., $X_{it} = \delta\varepsilon_{i,t-1} + u_{it}$ for $t \geq 1$ and ε_{it} independent of u_{it}.

(i) Derive asymptotic biases of the fixed effects estimator and the first differences estimator as $n \to \infty$. [1]

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(ii) Is there a value of T so that the two asymptotic biases are identical?

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[1] You may use without proof that $\hat{\beta}_{FE} \to_p \beta_0 + \frac{\sum_{t=1}^{T}(X_{i,t}-\bar{X})\varepsilon_{i,t}}{\sum_{t=1}^{T}(X_{i,t}-\bar{X})^2}$ and $\hat{\beta}_{FD} \to_p \beta_0 + \frac{\sum_{t=1}^{T}(X_{i,t}-X_{i,t-1})(\varepsilon_{it}-\varepsilon_{it-1})}{\sum_{t=1}^{T}(X_{i,t}-X_{i,t-1})^2}$

3. Consider panel data $\{\{(Y_{it}, X_{it})'\}_{t=1}^{T}\}_{i=1}^{n}$ generated by the model $Y_{it} = X_{it}\beta_0 + \delta_t + \alpha_i + \varepsilon_{it}$, where $\{(Y_{it}, X_{it})'\}_{t=1}^{T}$ is i.i.d. across i, $T = 4$, and $(X_{i1}, ..., X_{i4})' = (0, 0, 1, 1)' \cdot 1\{\alpha_i > .6\}$. The error terms has an autoregressive structure $\varepsilon_{it} = \phi\varepsilon_{i,t-1} + u_{it}$ for $t \geq 1$, and α_i, ε_{i0}, and $u_{i1}, ..., u_{i4}$ are i.i.d. $N(0, 1)$. The parameters take the values $\beta_0 = \delta_2 = \delta_3 = \delta_4 = 1$ while δ_1 is normalized to zero and omitted from the model. We will vary the autoregressive parameter in $\{0, 0.8\}$ and consider sample sizes $n \in \{40, 70, 100\}$.

(i) In a statistical software of your choice, generate data according to (1), estimate $(\beta_0, \delta_2, \delta_3, \delta_4)$ using the FE estimator, and calculate both a heteroskedasticity robust variance estimate and a cluster robust variance estimate where the clustering is at the individual level. Use these variance estimators to construct two different 95% confidence intervals for β_0. Additionally, calculate the OLS estimator of the regression coefficients in the (misspecified) common intercepts model $Y_{it} = \alpha + X_{it}\beta_0 + \delta_t + \varepsilon_{it}$.

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(ii) Across 10000 simulated repetitions of the above, report the mean of the two point estimators for β_0 (OLS and FE) and the coverage rate for the two confidence intervals that rely on different variance estimators.

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(iii) Discuss the results and relate them to the theory presented in lecture.

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