

ECON 712 - PS 2

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Problem 1: Two-dimensional non-linear system

Consider the Ramsey model of consumption c_t and capital k_t :

$$k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t \quad (1)$$

$$\beta u'(c_{t+1}) = \frac{u'(c_t)}{1 - \delta + f'(k_{t+1})} \quad (2)$$

parametrized by: $f(k) = zk^\alpha$, $z = 1$, $\alpha = 0.3$, $\delta = 0.1$, $\beta = 0.97$, $u(c) = \log(c)$.

1. Solve for steady state (\bar{k}, \bar{c}) .

The functional forms provided imply:

$$\begin{aligned} f(k) = zk^\alpha &\implies f'(k) = z\alpha k^{\alpha-1} \\ u(c) = \log(c) &\implies u'(c) = c^{-1} \end{aligned}$$

Setting $\bar{c} := c_t = c_{t+1}$, $\bar{k} := k_t = k_{t+1}$:

$$\begin{aligned} (2) &\implies c_{t+1} = \beta c_t (1 - \delta + z\alpha k_{t+1}^{\alpha-1}) \\ &\implies 1 = \beta (1 - \delta + z\alpha \bar{k}^{\alpha-1}) \\ &\implies \bar{k} = \left(\frac{\beta^{-1} - 1 + \delta}{z\alpha} \right)^{\frac{1}{\alpha-1}} \\ &\implies \bar{k} \approx 3.2690 \end{aligned}$$

$$\begin{aligned} (1) &\implies \bar{c} = z\bar{k}^\alpha + (1 - \delta)\bar{k} - \bar{k} \\ &\implies \bar{c} = z \left(\frac{\beta^{-1} - 1 + \delta}{z\alpha} \right)^{\frac{\alpha}{\alpha-1}} + \delta \left(\frac{\beta^{-1} - 1 + \delta}{z\alpha} \right)^{\frac{1}{\alpha-1}} \\ &\implies \bar{c} \approx 1.0998 \end{aligned}$$

The steady state is $(\bar{k}, \bar{c}) = (3.2690, 1.0998)$.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

2. Linearize the system around its steady state.

(a) Rewrite equations (1) and (2) as

$$\begin{aligned}
 k_{t+1} &= g(k_t, c_t) \\
 (1) \implies k_{t+1} &= zk_t^\alpha + (1 - \delta)k_t - c_t \\
 c_{t+1} &= h(k_t, c_t) \\
 (2) \implies c_{t+1} &= \beta c_t (1 - \delta + z\alpha k_{t+1}^{\alpha-1}) \\
 \implies c_{t+1} &= \beta c_t (1 - \delta + z\alpha(zk_t^\alpha + (1 - \delta)k_t - c_t)^{\alpha-1})
 \end{aligned}$$

(b) Analytically calculate Jacobian $J = \begin{pmatrix} dk_{t+1}/dk_t & dk_{t+1}/dc_t \\ dc_{t+1}/dk_t & dc_{t+1}/dc_t \end{pmatrix}$ (use provided functional forms, but don't plug in parameters yet).

$$\begin{aligned}
 dk_{t+1}/dk_t &= z\alpha k_t^{\alpha-1} + 1 - \delta \\
 dk_{t+1}/dc_t &= -1 \\
 dc_{t+1}/dk_t &= z\alpha\beta c_t(\alpha - 1)(zk_t^\alpha + (1 - \delta)k_t - c_t)^{\alpha-2}(z\alpha k_t^{\alpha-1} + 1 - \delta) \\
 dc_{t+1}/dc_t &= (1 - \delta)\beta + z\alpha\beta[(zk_t^\alpha + (1 - \delta)k_t - c_t)^{\alpha-1} - c_t(\alpha - 1)(zk_t^\alpha + (1 - \delta)k_t - c_t)^{\alpha-2}]
 \end{aligned}$$

(c) Using Taylor expansion (first-order approximation here), systems can be written in terms of deviations from steady state $\tilde{k}_t = k_t - \bar{k}$ and $\tilde{c}_t = c_t - \bar{c}$:

$$\begin{pmatrix} \tilde{k}_{t+1} \\ \tilde{c}_{t+1} \end{pmatrix} = J \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix}$$

3. Compute numerically eigenvalues and eigenvectors of the Jacobian at the steady state. Verify that the system has a saddle path. What is the slope of the saddle path at the steady state?

At $(\bar{k}, \bar{c}) = (3.269, 1.100)$ and the above parameters (from Matlab).

$$J = \begin{pmatrix} 1.0309 & -1 \\ -0.0308 & 1.0299 \end{pmatrix}$$

From Matlab, the eigenvectors and eigenvalues for J are:

$$\begin{aligned}
 \Lambda &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1.2060 & 0 \\ 0 & 0.8548 \end{pmatrix} \\
 E &= \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 0.9850 & 0.9848 \\ -0.1725 & 0.1734 \end{pmatrix} \\
 \begin{pmatrix} k_t \\ c_t \end{pmatrix} &= \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} x_1 \lambda_1^t \\ x_2 \lambda_2^t \end{pmatrix}
 \end{aligned}$$

The system has a saddle path because the absolute value of one eigenvalue is greater than one and the absolute value of the other eigenvalue is less than one. The saddle path is (k_t, c_t) where $x_1 = 0$.

$$\begin{pmatrix} k_t \\ c_t \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \lambda_2^t \end{pmatrix} = \begin{pmatrix} e_{12} x_2 \lambda_2^t \\ e_{22} x_2 \lambda_2^t \end{pmatrix}$$

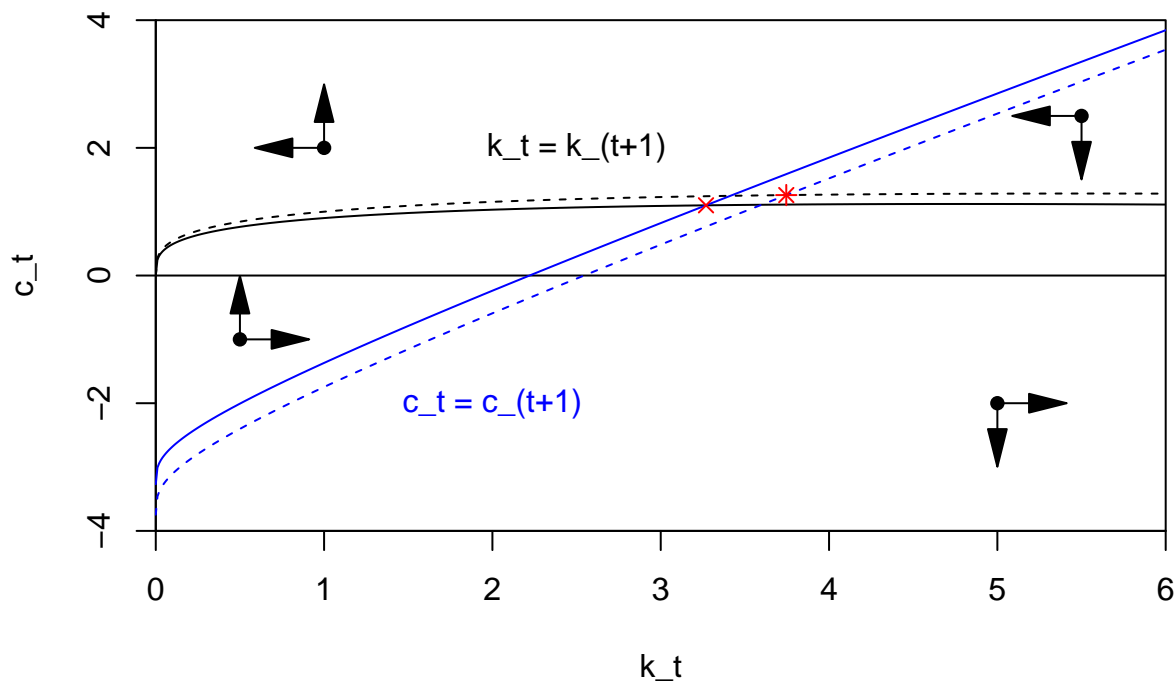
So, the slope of the saddle path at the steady state is $\frac{e_{22} x_2 \lambda_2^t}{e_{12} x_2 \lambda_2^t} = \frac{e_{22}}{e_{12}} = \frac{0.1734}{0.9848} = 0.1761$.

4. On a phase diagram in (k_t, c_t) show how the system evolves after an unexpected permanent positive productivity shock at $t_0, z' > z$. (You don't need to plot lines precisely - do this by hand, but pay attention to vector field (arrows), relative position of old and new steady states, directions of saddle paths and system trajectory after the shock.)

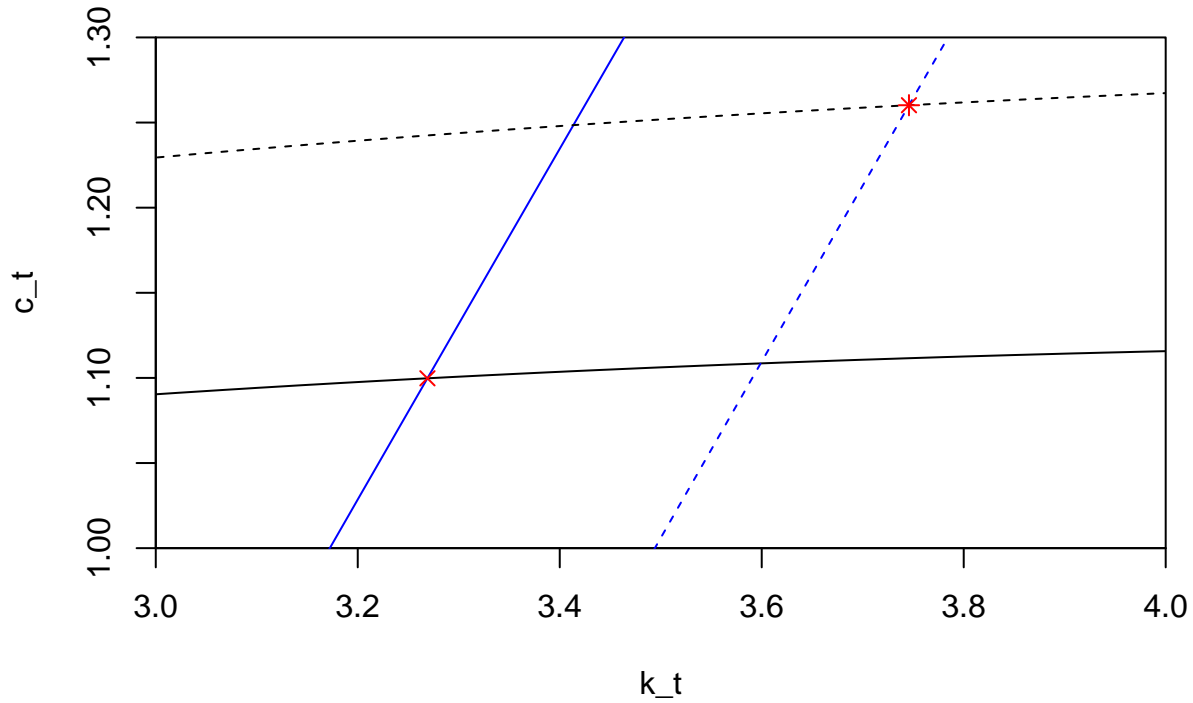
$$\begin{aligned}
 \Delta k_{t+1} &= 0 \\
 \implies k_{t+1} - k_t &= 0 \\
 \implies (f(k_t) + (1 - \delta)k_t - c_t) - k_t &= 0 \\
 \implies c_t &= f(k_t) - \delta k_t
 \end{aligned}$$

$$\begin{aligned}
 \Delta c_{t+1} &= 0 \\
 \implies c_{t+1} &= c_t \\
 \implies \beta u'(c_{t+1}) &= \beta u'(c_t) \\
 \implies \frac{u'(c_t)}{1 - \delta + f'(k_{t+1})} &= \beta u'(c_t) \\
 \implies f'(k_{t+1}) &= \beta^{-1} - 1 + \delta \\
 \implies k_{t+1} &= \bar{k} \\
 \implies c_t &= f(k_t) + (1 - \delta)k_t - \bar{k}
 \end{aligned}$$

Phase Diagram with Vector Field Arrows



Phase Diagram with Saddle Paths



5. (continuing from 4) Compute numerically and plot trajectories of k_t and c_t for $t = 1, 2, \dots, 20$ if the productivity shock occurs at $t_0 = 5$ and $z = z + 0.1$. For this question, we will be looking at the linearized version of the nonlinear system around the new steady state.

- (a) Compute the new steady state (\bar{k}', \bar{c}') and Jacobian matrix at that point.

From Matlab:

$$(\bar{k}', \bar{c}') \approx (3.7458, 1.2424)$$

The new Jacobian equals

$$J = \begin{pmatrix} 1.0440 & -1 \\ -0.0319 & 1.0392 \end{pmatrix}$$

- (b) Diagonalize the system using eigenvectors and rewrite it in terms of \hat{k}_t and \hat{c}_t .

From Matlab:

The new matrix of eigenvalues is:

$$\Lambda \approx \begin{pmatrix} 1.2203 & 0 \\ 0 & 0.8629 \end{pmatrix}$$

The new matrix of eigenvectors is:

$$E \approx \begin{pmatrix} 0.9848 & 0.9840 \\ -0.1736 & 0.1783 \end{pmatrix}$$

Define

$$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = E^{-1} \begin{pmatrix} k_t \\ c_t \end{pmatrix} \implies \begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = \Lambda \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}$$

- (c) Write down non-explosive solution for (\hat{k}_t, \hat{c}_t) , rewrite in terms of original variables (k_t, c_t) .
- (d) Pin down a particular saddle path trajectory using a boundary condition $k_{t_0} = \bar{k}$ (capital can't jump from the old steady state at the time of the shock, so pick suitable c_{t_0}).
- (e) Use the particular solution to compute and graph k_t and c_t after the shock.
- 6. For this question, we explore the nonlinear nature of the system and numerically solve the actual transition path using the “shooting method”.
 - (a) In the previous question, you solve c_{t_0} under the linear system. Put (k_{t_0}, c_{t_0}) into the nonlinear system (1) and (2). Compute and graph how the system evolves. Does it converge to a steady state?
 - (b) Use “shooting method” to find the actual c_{t_0} needed. The method is to try different values of c_{t_0} such that after long enough time, the system will converge to the new steady state.

Problem 2: Setting up a model

- For the problems below, state the Social Planner Problem (SPP), the Consumer Problem (CP), and define the Competitive Equilibrium (CE). (Don't solve).
- 1. Consider an overlapping generations economy of 2-period-lived agents. There is a constant measure of N agents in each generation. New young agents enter the economy at each date $t \geq 1$. Half of the young agents are endowed with w_1 when young and 0 when old. The other half are endowed with 0 when young and w_2 when old. There is no savings technology. Agents order their consumption steady by $U(c_t^t, c_{t+1}^t) = \ln c_t^t + \ln c_{t+1}^t$. There is a measure N of initial old agents. Half of them are endowed with w_2 and the other half endowed with 0. Each old agent order their consumption by c_1^0 . Each old agent is endowed with M units of fiat currency. No other generation is endowed with fiat currency, and the stock of fiat currency is fixed over time.
- 2. Consider an overlapping generations economy with 3-period-lived agents. Denote these periods as young, mid, and old. At each date $t \geq 1$, N_t new young agents enter the economy, each endowed with w_1 units of the consumption good when young, w_2 units when mid, and w_3 units when old. The consumption good is non-storable. The population is described by $N_{t+1} = n * N_t$, where $n > 0$. Consumption preference is described by $\ln c_t^t + \ln c_{t+1}^t + \ln c_{t+2}^t$. At time $t = 1$, there is a measure N_{-1} of old agents, each endowed with w_3 units of the consumption good, and a measure of N_0 mid agents, each endowed with w_2 units of the consumption good at $t = 1$ and w_3 units at $t = 2$. Additionally, each initial old agent is endowed with 1 unit of a fiat currency.
- (Cake eating problem) Consider a single infinitely lived agent with preference over their consumption stream $\mathbf{c} = \{c_t\}$, given by $U(\mathbf{c}) = \sum_{t=1}^{\infty} \beta^t u(c_t)$, where $\beta < 1$ and $u(\cdot)$ is increasing and concave. Consumption cannot be negative in any period. The agent is endowed with k_1 units of the consumption good in period $t = 1$. There is a perfect storage technology, such that the consumption good is effectively infinity durable. State the agent's problem (don't solve).