ECON 712B - Problem Set 3

Alex von Hafften*

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1. An endowment economy consists of two types of consumers indexed by $i = 1, 2$. There is one nonstorable consumption good. Consumer 1 has initial endowment stream $e_0^1 = 1$ while consumer 2 has initial endowment $e_0^2 = 0$. At some random date $s > 0$ the endowments will switch so $e_t^1 = 0$ and $e_t^2 = 1$ for all $t \geq s$. Suppose the switch happens with probability δ each period until it is realized. Both consumers have preferences ordered by $E_0 \sum_{t=0}^{\infty} \beta^t \log c_t^i$. For this problem you can either consider an Arrow-Debreu market structure or a recursive equilibrium with a sequential market structure. For the Arrow-Debreu market, consider a single market at date zero with claims to consumption in any future date and any state. For the recursive structure, consider a market each period in claims to the two "trees," and use backward induction to find the values after and then before the switch of the endowments.

(a) Define a competitive equilibrium for this economy for your choice of market structure, being explicit about the objects which make up the equilibrium and the conditions they must satisfy.

I approach this problem using an Arrow-Debreu market structure. There is a Markov state $s_t \in \{0, 1\}$ that governs the realization of endowment uncertainty. The Markov state starts at $s_0 = 0$ (i.e., $P(s_0 = 0) = 1$) and the transition kernel is

$$
\begin{aligned}
P(s_{t+1} = 0 | s_t = 0) &= 1 - \delta \\
P(s_{t+1} = 1 | s_t = 0) &= \delta \\
P(s_{t+1} = 0 | s_t = 1) &= 0 \\
P(s_{t+1} = 1 | s_t = 1) &= 1
\end{aligned}
$$

Thus, the associated Markov transition matrix is:

$$
Q = \begin{bmatrix} 1 - \delta & \delta \\ 0 & 1 \end{bmatrix}
$$

Based on this Markov state, the endowments $e_t^1 = e^1(s_t)$ and $e_t^2 = e^2(s_t)$ are defined as

$$
\begin{aligned}
e^1(s_t) &= 1 - s_t \\
e^2(s_t) &= s_t
\end{aligned}
$$

Define $s^t = (s_0, s_1, s_2, ..., s_t)$ as the history of shock up to and including period t. Define preferences for consumer i over state-contigent consumption plans $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ as:

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

$$U^i(c^i) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \log(c_t^i(s^t))\pi_t(s^t)$$

Consumers trade dated, state-contingent claims to consumption at date 0 at price $q_t^0(s^t)$. Thus, budget constraint of consumer i is:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) e^i(s_t)$$

Thus, the problem of consumer i is:

$$\begin{aligned}
\max_{\{c_t^i(s^t)\}_{i=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \log(c_t^i(s^t))\pi_t(s^t) \\
\text{s.t.} & \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) e^i(s_t)
\end{aligned}$$

A competitive equilibrium is a price system $\{q_t^0(s^t)\}_{t=0}^{\infty}$ and an allocation $\{c^1, c^2\}$ such that consumers optimize (i.e., c^i solves the problem of consumer i) and markets clear (i.e., $e^1(s_t) + e^2(s_t) = c_t^1(s^t) + c_t^2(s^t)$ for all s^t).

(b) Compute a competitive equilibrium, finding the consumption allocation and the prices (as of date zero) of claims to each consumer's endowment process.

Attach Lagrange multipler μ_i on the budget constraint for consumer i:

$$\max_{\{c_t^i(s^t)\}_{i=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \log(c_t^i(s^t))\pi_t(s^t) + \mu_i \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)(e^i(s_t) - c_t^i(s^t))$$

Thus, the FOC with respect to $c_t^i(s^t)$ is:

$$\beta^t \frac{\pi_t(s^t)}{c_t^i(s^t)} = \mu_i q_t^0(s^t)$$

The ratio of the FOC for consumer 1 and consumer 2 imply that their consumption is constant (because they each consume a constant fraction of the total endowment, which is constant):

$$\frac{\beta^t \frac{P(s^t|s_0=0)}{c_t^1(s^t)}}{\beta^t \frac{P(s^t|s_0=0)}{c_t^2(s^t)}} = \frac{\mu_1 q_t^0(s^t)}{\mu_2 q_t^0(s^t)} \implies c_t^1(s^t) = (\frac{\mu_1}{\mu_2})^{-1} c_t^2(s^t)$$

Thus, define $\bar{c}^1$ as consumer 1's consumption and $\bar{c}^2$ consumer 2's consumption. Substituting the FOC into the budget constraint:

$$\sum_{t=0}^{\infty}\sum_{s^t}(\beta^t \frac{\pi_t(s^t)}{c_t^i(s^t)\mu_i})(e^i(s_t) - c_t^i(s^t)) = 0$$

$$\sum_{t=0}^{\infty}\sum_{s^t}(\beta^t \frac{\pi_t(s^t)}{\bar{c}^i(1/\bar{c}^i)})(e^i(s_t) - \bar{c}^i) = 0$$

$$\sum_{t=0}^{\infty}\sum_{s^t}(\beta^t \pi_t(s^t))(e^i(s_t) - \bar{c}^i) = 0$$

$$\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t(s^t)e^i(s_t) - \sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t(s^t)\bar{c}^i = 0$$

$$\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t(s^t)e^i(s_t) = \sum_{t=0}^{\infty}\beta^t\bar{c}^i$$

$$\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t(s^t)e^i(s_t) = \frac{1}{1-\beta}\bar{c}^i$$

$$\bar{c}^i = (1-\beta)\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t(s^t)e^i(s_t)$$

As of period t, the expected endowment of consumer 1 is:

$$\begin{aligned}
\sum_{s^t}\pi_t(s^t)e^1(s_t) &= (1)(1) + (1-\delta)(1) + (1-\delta)(1-\delta)(1) + ... + (1-\delta)^t(1) \\
&= \sum_{l=0}^{t}(1-\delta)^l \\
&= \frac{1-(1-\delta)^{t+1}}{1-(1-\delta)} \\
&= \frac{1-(1-\delta)^{t+1}}{\delta}
\end{aligned}$$

Thus, consumer 1's consumption is:

$$\begin{aligned}
\bar{c}^1 &= (1-\beta)\sum_{t=0}^{\infty}\beta^t\frac{1-(1-\delta)^{t+1}}{\delta} \\
&= \frac{1-\beta}{1-\beta+\beta\delta}
\end{aligned}$$

So the consumer 2's consumption is:

$$\bar{c}^2 = 1 - \bar{c}^1 = \frac{\beta\delta}{1-\beta+\beta\delta}$$

To find prices, notice that $\pi(s^0) = 1$. Normalizing $q_0^0(s^0) = 1$, the FOC implies that

$$\mu_i = \frac{1}{c_0^i(s^0)} = \frac{1}{\bar{c}^i}$$

We can substitute this equation into the FOC for period t:

$$\begin{aligned}
q_t^0(s^t) &= \beta^t \frac{\pi_t(s^t)}{c_t^i(s^t)\mu_i} \\
&= \beta^t \frac{\pi_t(s^t) c_0^i(s^0)}{c_t^i(s^t)} \\
&= \beta^t \frac{\pi_t(s^t)\bar{c}^i}{\bar{c}^i} \\
&= \beta^t \pi_t(s^t)
\end{aligned}$$

For the prices of the claims to consumer 1's endowments, we need to consider histories that do not include the endowment process switch. For these histories, $\pi(s^t) = \delta^t$. For the prices of the claims to consumer 2's endowments, we need to consider histories that include the endowment process switch. For these histories, notice that $\pi(s^t) = \pi(s^s) = \delta^{s-1}(1-\delta)$ when s^t is a history with $s_l = 0$ for $l = 1, ..., s-1$ and $s_l = 1$ for $l = s, ..., t$. Therefore,

$$q_t^0(s^t) = \begin{cases} \beta^t \delta^t & \text{if } s_l = 0 \; \forall l = \{1, ..., t\} \\ \beta^t \delta^{s-1}(1-\delta), & \text{if } s_l = 0 \; \forall l = \{1, ..., s-1\} \text{ and } s_l = 0 \; \forall l = \{s, ..., t\} \end{cases}$$

(c) What is price of a claim to the aggregate endowment? What is the risk-free interest rate?

A claim bought in period 0 to the aggregate endowment in period t, q_t^0, does not depend on the history of shocks s^t because the aggregate endowment does not change based on the shocks. Thus, the FOC from part (B) implies that $q_t^0 = \beta^t$ (i.e., set $\pi_t = 1$).

This claim to the aggregate endowment in period t can be thought of a t-period risk-free bond that pays off one unit of consumption good in period t and costs β^t. With no arbitrage, this bond is equivalent to rolling over t 1-period risk-free bonds that pay off one unit of consumption and cost β. Thus, the return on these one-period risk-free bonds (i.e., the risk-free interest rate) is $R = \frac{1}{\beta}$.

(d) Now suppose that $\lambda \in (0, 1)$ is the Pareto weight on consumer 1 with $1 - \lambda$ the weight on consumer 2. Formulate the social planner's problem and characterize the solution of the Pareto optimal allocation for a given λ.

The social planner's problem is:

$$\begin{aligned}
\max_{\{(c_t^1, c_t^2)\}_{t=1}^{\infty}} \; & \lambda \sum_{t=1}^{\infty} \beta^t \log c_t^1 + (1-\lambda) \sum_{t=1}^{\infty} \beta^t \log c_t^2 \\
\text{s.t. } & c_t^1 + c_t^2 = 1
\end{aligned}$$

The problem can be rewritten as:

$$\max_{\{c_t^1\}_{t=1}^{\infty}} \lambda \sum_{t=1}^{\infty} \beta^t \log c_t^1 + (1-\lambda) \sum_{t=1}^{\infty} \beta^t \log(1 - c_t^1)$$

FOC (c_t^1):

$$\frac{\lambda \beta^t}{c_t^1} = \frac{(1-\lambda)\beta^t}{1 - c_t^1}$$

$$\implies c_t^1 = \lambda \implies c_t^2 = 1 - \lambda$$

(e) Relate the competitive equilibrium that you found earlier with the Pareto optima. Is the competitive equilibrium optimal? (Can you find a particular λ so that they are equivalent?) For a given λ, can you find a way to decentralize the Pareto optimum as a competitive equilibrium?

In part (c), we found that the solution to social planner's problem is for each consumer to consumer their Pareto weight. In part (b), we found that consumer consume a constant amount. Thus, we can set the Pareto weight to the fraction consumed in the competitive equilibrium: $\lambda = \frac{1-\beta}{1-\beta+\beta\delta}$.

Yes, based on a λ, we can alter the endowment functions to achieve the Pareto optimum. In part (b), we found that the fraction of the aggregate endowment consumed by each consumer depended on their expected total endowment. If you adjust the endowment amount you can achieve the Pareto optimum as a competitive equilibrium.

2. Suppose that a representative agent has preferences: $E\sum_{t=0}^{\infty}\beta^t u(c_t)$ over the single nonstorable consumption good ("fruit"), where $u(c) = c^{1-\gamma}/(1-\gamma)$ for $\gamma > 0$. Her endowment of the good is governed by a Markov process with transition function $F(x', x)$.

(a) Define a recursive competitive equilibrium with a market in claims to the endowment process ("trees").

Let a_t be the representative agent's claim to "trees" in period t, p_t be the price of trees in period t, and s_t be the stochastic output from each "trees" in period t. The representative agent problem is:

$$\max_{\{c_t, a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + p_t a_{t+1} \leq (p_t + s_t) a_t$$

where $s_t \sim F(ds', s)$. We posit that $p_t = p(s_t)$. The Bellman equation is

$$\begin{aligned} V(a, s) = \max_{\{c, a'\}} &\{u(c) + \beta \int V(a', s') F(ds', s)\} \\ \text{s.t. } c + p(s)a' &\leq (p(s) + s)a \\ c &\geq 0 \\ 0 \leq a' &\leq 1 \end{aligned}$$

A recursive competitive equilibrium is continuous pricing function $p(s)$ and a continuous and bounded function $V^*(a, s)$ such that (i) $V^*(a, s)$ solves the Bellman equation and (ii) $\forall s$, $V^*(1, s)$ is attained by $c \geq s$ and $s' = 1$. Rewriting the state variables $\{a, s\}$ as wealth $(p(s) + s)a$ and substituting $c = c(s) = (p(s) + s)a - p(s)a'$, the Bellman equation becomes:

$$V((p(s) + s)a) = \max_{a'} \{u((p(s) + s)a - p(s)a') + \beta \int V((p(s') + s')a') F(ds', s)\}$$

FOC:

$$u'(c(s))p(s) = \beta \int V'((p(s') + s')a')(p(s') + s') F(ds', s)$$

Envelope condition:

$$V'((p(s) + s)a) = u'(c(s))$$

FOC and envelope condition imply an Euler equation:

$$u'(c(s)) = \beta \int u'(c(s')) \frac{p(s') + s'}{p(s)} F(ds', s)$$

Defining $R_{t+1} = \frac{p_{t+1} + s_{t+1}}{p_t}$, the Euler equation becomes:

$$u'(c_t) = \beta E_t[u'(c_{t+1}) R_{t+1}]$$

At equilibrium, $c = s$ and $a' = a = 1$. Thus the pricing function is determined by:

$$p(s) = \beta \int \frac{u'(s')}{u'(s)}(p(s') + s')F(ds', s)$$

With CRRA utility, the pricing function becomes:

$$p(s) = \beta \int \left(\frac{s}{s'}\right)^{\gamma} (p(s') + s')F(ds', s)$$

(b) When the consumer has logarithmic utility, $u(c) = \log c$, what is the equilibrium price/dividend ratio of a claim to the entire consumption stream? How does it depend on the distribution of consumption growth?

With log utility, the pricing function becomes:

$$p(s) = \beta \int \frac{s}{s'}(p(s') + s')F(ds', s)$$

Thus, the price/dividend ratio is

$$\frac{p(s)}{s} = \frac{\beta \int \frac{s}{s'}(p(s') + s')F(ds', s)}{s} = \beta \int \frac{s}{s'} \frac{p(s') + s'}{s} F(ds', s) = \beta \int \frac{s}{s'} R' F(ds', s)$$

If consumption is growing, the price/dividend is lower than if consumption is shrinking,

$$c' > c \implies s' > s \implies \downarrow \frac{p(s)}{s}$$

$$c' < c \implies s' < s \implies \uparrow \frac{p(s)}{s}$$

(c) Suppose there is news at time t that future consumption will be higher. How will prices respond to this news? How does this depend on the consumer's preferences (which are CRRA, but not necessarily log)? Interpret your results.

The pricing function is:

$$p(s) = E\left[\left(\frac{c}{c'}\right)^{\gamma} (p(s') + s')\right]$$

If c' increases, prices drop. A higher γ results in a larger drop in price. For CRRA utility, $1/\gamma$ is the the intertemporal elasticity of substitution (IES). A higher IES is associated with a stronger preference for consumption smoothing across periods. If γ is lower then $1/\gamma$ is higher, so households have a stronger preference for consumption smoothing. Thus, the news that future consumption will be higher has less of an effect of today's prices. If γ is higher, then $1/\gamma$ is lower, so households will change their consumption pattern more in response to this news.

(d) Consider an option which is bought or sold in period t and entitles the current owner to exercise the right to buy one "tree" in period $t+1$ at the fixed price $\bar{p}$ specified at date t. (The buyer may choose not to exercise this option.) Find a formula for the price of this option in terms of the parameters describing preferences and endowments.

The pricing function implies a pricing kernal:

$$q(s, s') = \beta \frac{u'(s')}{u'(s)} f(s', s)$$

where f is the PDF that corresponds to F, or $F(s', s) = \int_{-\infty}^{s'} f(u, s) du$.

The call option is in-the-money if the strike price $\bar{p}$ is lower than the future market price $p(s')$. So, the payoff function for the call option is

$$g(s') = \begin{cases} p(s') - \bar{p}, & p(s') > \bar{p} \\ 0, & p(s') \leq \bar{p} \end{cases}$$

Thus, the price of the call option is

$$\begin{aligned} p^c(s) &= \int_{-\infty}^{\infty} q(s, s') g(s') ds' \\ &= \int_{-\infty}^{\infty} \beta \frac{u'(s')}{u'(s)} (p(s') - \bar{p}) 1\{p(s') > \bar{p}\} f(s', s) ds' \\ &= \int_{-\infty}^{\infty} \beta \frac{u'(s')}{u'(s)} (p(s') - \bar{p}) 1\{s' > p^{-1}(\bar{p})\} f(s', s) ds' \\ &= \int_{p^{-1}(\bar{p})}^{\infty} \beta \frac{u'(s')}{u'(s)} (p(s') - \bar{p}) f(s', s) ds' \end{aligned}$$

The last equality requires $p(\cdot)$ to strictly monotonically increasing in s and thus invertible. With CRRA utility, the call option price formula becomes:

$$p^c(s) = \int_{p^{-1}(\bar{p})}^{\infty} \beta \left(\frac{s}{s'} \right)^{\gamma} (p(s') - \bar{p}) f(s', s) ds'$$

3. Suppose consumers have preferences given by: $E\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$ with $\beta \in (0,1)$ and $\gamma > 1$. The consumer's budget constraint is: $c_t + a_{t+1} = wl_t + (1+r)a_t$ where l_t is the labor endowment, a_t is the amount of assets, $c_t \geq 0$ is consumption, $w > 0$ is the (constant) wage, and $r > 0$ is the (constant) real interest rate. The consumer also faces the borrowing constraint $a_t \geq 0$. Suppose that the labor endowment follows a finite-state Markov chain with transition matrix Q.

(a) Write down the consumer's Bellman equation, and state the optimality conditions.

$$
\begin{aligned}
V(a,l) &= \max_{(c,a')\in\Gamma(a,l)} \left\{ u(c) + \beta \int V(a',l')Q(l,dl') \right\} \\
\implies V(a,l) &= \max \left\{ u(wl + (1+r)a - a') + \beta \int V(a',l')Q(l,dl') \right\}
\end{aligned}
$$

FOC:

$$
\begin{aligned}
0 &= (-1)u(wl + (1+r)a - a') + \beta \int V_a(a',l')Q(l,dl') \\
\implies u'(c) &= \beta \int V_a(a',l')Q(l,dl')
\end{aligned}
$$

Envelope condition:

$$
\begin{aligned}
V_a(a,l) &= u'(wl + (1+r)a - a')(1+r) \\
\implies V_a(a,l) &= (1+r)u'(c)
\end{aligned}
$$

FOC and envelope condition implies a stochastic Euler equation:

$$
\begin{aligned}
u'(c) &= \beta(1+r) \int u'(c',l')Q(l,dl') \\
\implies u'(c_t) &= \beta(1+r)E_t u'(c_{t+1})
\end{aligned}
$$

Plugging in the functional form for the utility function:

$$
c_t^{-\gamma} = \beta(1+r)E_t c_{t+1}^{-\gamma}
$$

(b) The rest of the question will numerically solve for the policy functions and the stationary distribution. Suppose that l_t takes on the following values: $l_t \in \{0.7, 1.1\}$, with the transition matrix given by: $Q = \begin{bmatrix} 0.85 & 0.15 \\ 0.05 & 0.95 \end{bmatrix}$, where $Q(i,j) = P(l_{t+1} = l_j | l_t = l_i)$. Find the stationary distribution and the unconditional mean of the labor endowment.

The stationary distribution of a Markov Chain is π such that $\pi Q = \pi$. Thus we find the eigenvalues and eigenvectors of Q. The characteristic polynomial of Q is

$$
(0.85-\lambda)(0.95-\lambda) - (0.15)(0.05) = \lambda^2 - 1.8\lambda + 0.8 = (\lambda - 1)(\lambda - 0.8)
$$

So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.8$. For λ_1,

$$
\begin{aligned}
0.85x + 0.15y &= x \\
0.95x + 0.05y &= y
\end{aligned}
$$

These equations imply that $(0.5, 0.5)'$ is a corresponding eigenvector. For λ_1,

$$
\begin{aligned}
0.85x + 0.15y &= 0.8x \\
0.95x + 0.05y &= 0.8y
\end{aligned}
$$

These equations imply that $(0.25, 0.75)'$ is the corresponding eigenvector. Double-checking, we see that $(0.25, 0.75)'$ is the stationary distribution:

$$
\begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} \begin{bmatrix} 0.85 & 0.15 \\ 0.05 & 0.95 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}
$$

The unconditional mean of labor endowment is $(0.25)(0.7) + (0.75)(1.1) = 1$.

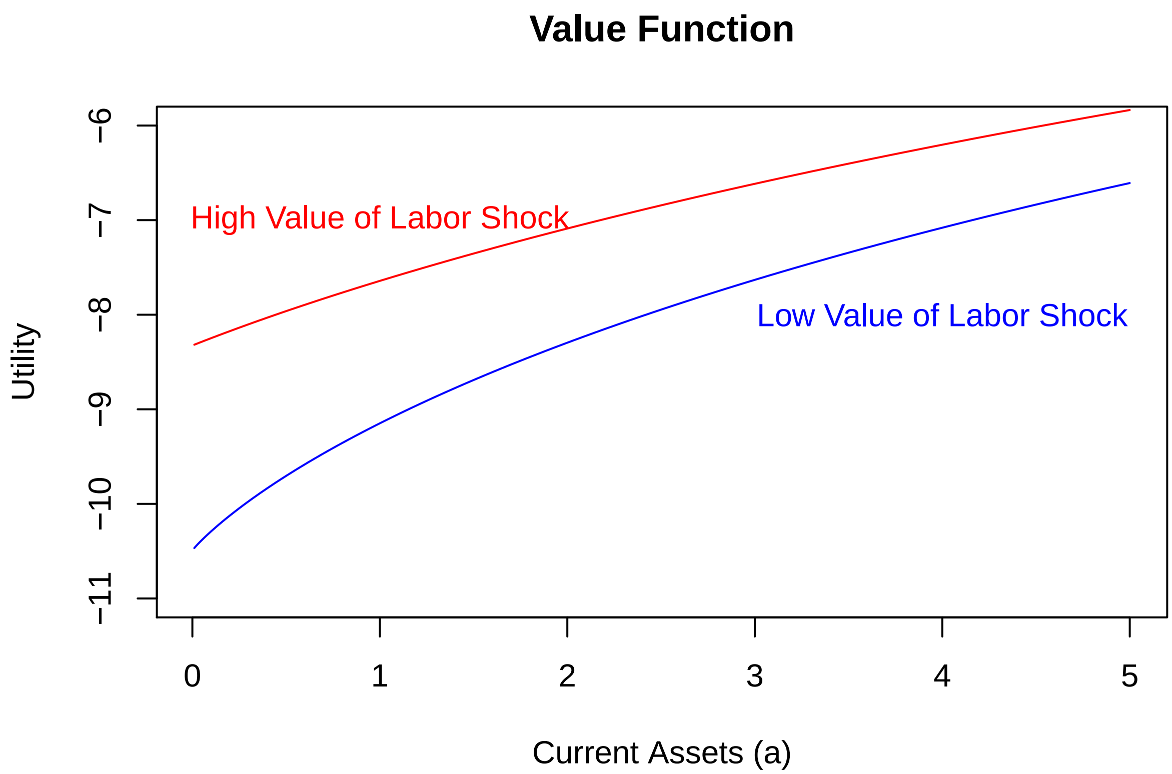
```r
Q <- matrix(c(0.85, 0.05, 0.15, 0.95), nrow = 2)

# Numerical estimation
x <- c(0.5, 0.5)
for (i in 1:100) x <- x %*% Q
x
```

```
##      [,1] [,2]
## [1,] 0.25 0.75
```

(c) Take the following as parameters: $\beta = 0.95, \gamma = 3, r = 0.03, w = 1.1$. Discretize the asset choice into a finite grid. Note that this directly restricts the assets to lie in a compact set. Numerically solve for the value functions by iterating on the Bellman equation.

i. Plot the value functions for the high and low values of the labor shocks. Do they appear to have the properties that we have assumed?

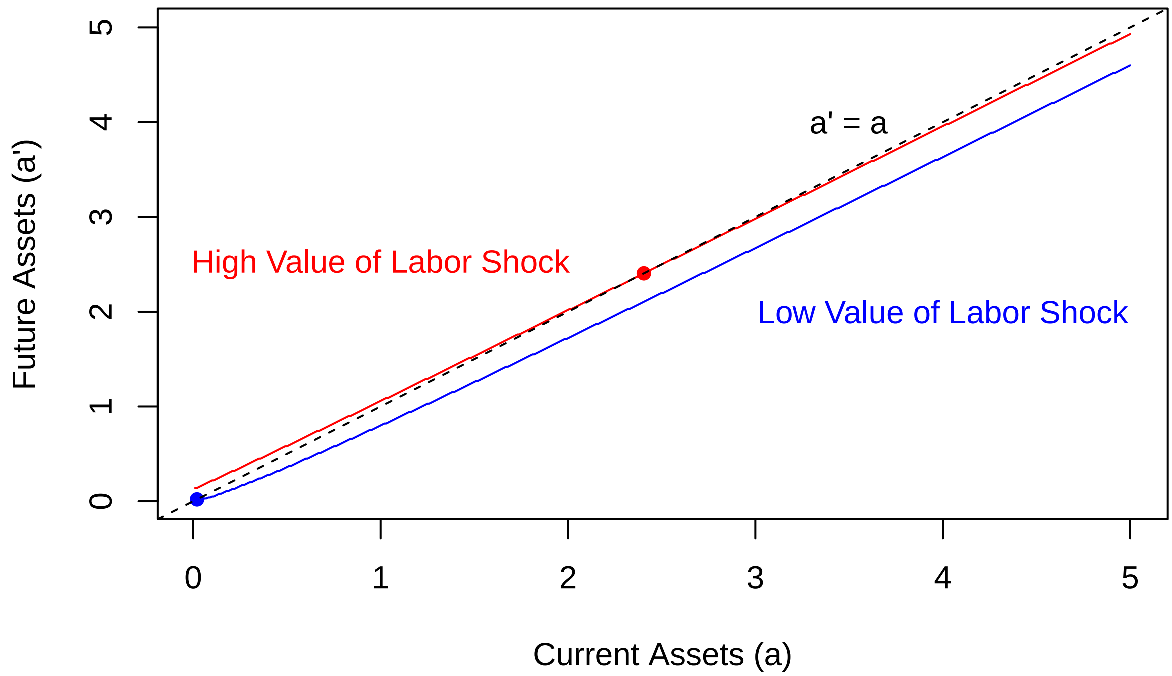
Yes, the value function appears continuous, increasing, and differentiable. In addition, the value function for the high labor shocks is larger than the value function for low labor shocks. The difference decreases as the level of current assets increases.



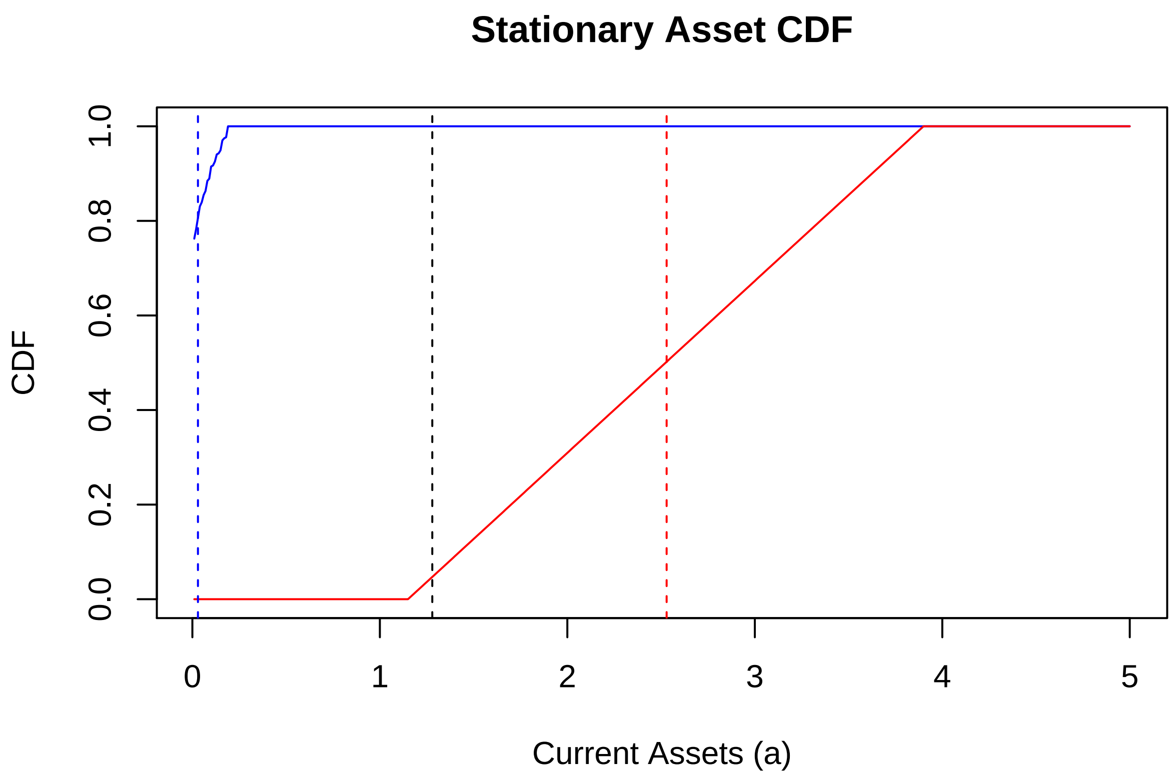
ii. Plot the asset holdings decision rules $a'(a, l_i)$ for the two labor shocks. Is there a value $\bar{a}$ such that $a'(a, l_2) < a$ for all $a \geq \bar{a}$? What do the features of the decision rules suggest about the long-run behavior of assets a_t?

As the chart shows at $\bar{a} \approx 2.2575$, $a'(a, l_2) < a$ for all $a \geq \bar{a}$.

The decision rules suggest that when a high labor shock is realized, agents save more than their current asset holdings and when a low labor shock is realized, agents draw down their current asset holdings. In the long run, the transition matrix implies where along the range from 0 to $\bar{a}$, agents converge to.



(d) To find the stationary distribution over assets and labor, first construct a (large) transition matrix over (a, l). Then starting from an arbitrary initial distribution, iterate until convergence on the mapping from the current distribution into the distribution in the next period. What is the mean of the stationary asset distribution? Plot the (marginal) asset distribution.



Appendix

```
# parameters
beta <- .95
gamma <- 3
r <- 0.03
w <- 1.1
l_low <- 0.7
l_high <- 1.1

# assets grid
min_a <- 0.01
max_a <- 5
inc_a <- min_a
a <- seq(min_a, max_a, by = inc_a)
n <- length(a)

# initialize value grid
v_low <- rep(0, times=n)
v_high <- rep(0, times=n)

# initialize decision rule grid
decision_low <- rep(0, times=n)
decision_high <- rep(0, times=n)

# create consumption matrices
# (columns are different values of a; rows are values of a')
ones <- rep(1, times=n)
c_matrix_low <- w * l_low + (1 + r)*ones %*% t(a)  - a %*% t(ones)
c_matrix_high <- w * l_high + (1 + r)*ones %*% t(a)  - a %*% t(ones)

# create flow utility matrices and set negative consumption levels to low value
utility_matrix_low <- (c_matrix_low^(1-gamma))/(1-gamma)
utility_matrix_low[c_matrix_low < 0|!is.finite(utility_matrix_low)] <- -1000

utility_matrix_high <- (c_matrix_high^(1-gamma))/(1-gamma)
utility_matrix_high[c_matrix_high < 0|!is.finite(utility_matrix_high)] <- -1000

# solve bellman equations:
test <- 10
iter <- 1
max_iter <- 10000

while (test != 0 & iter < max_iter) {
  # create value matrix for different a and a' values with transition probabilities
  # to next period shock
  value_matrix_low <- utility_matrix_low + beta * Q[1, 1] * v_low %*% t(ones) +
    beta * Q[1, 2] * v_high %*% t(ones)
  value_matrix_high <- utility_matrix_high + beta * Q[2, 1] * v_low %*% t(ones) +
    beta * Q[2, 2] * v_high %*% t(ones)

  # find max utility
  tv_low <- apply(value_matrix_low, 2, max)
```

```r
  tv_high <- apply(value_matrix_high, 2, max)

  # find column with max utility
  tdecision_low <- apply(value_matrix_low, 2, which.max)
  tdecision_high <- apply(value_matrix_high, 2, which.max)

  # update test variable so loop ends if no changes from previous iteration
  test <- max(tdecision_low - decision_low) + max(tdecision_high - decision_high)

  # update value grid
  v_low <- tv_low
  v_high <- tv_high

  # update decision rule
  decision_low <- tdecision_low
  decision_high <- tdecision_high

  iter <- iter + 1
}

# save policy function based on decision rule
policy_function_low <- tibble(a = a, a_prime = min_a + decision_low*inc_a) %>%
  mutate(c = w * l_low + (1 + r)*a  - a_prime)

policy_function_high <- tibble(a = a, a_prime = min_a + decision_high*inc_a) %>%
  mutate(c = w * l_high + (1 + r)*a  - a_prime)
```

```r
tol <- .05
diff <- 1000

p_low0 <- matrix(rep(1/n, times = 2*n), ncol = 2)
p_high0 <- matrix(rep(1/n, times = 2*n), ncol = 2)

p_low <- p_low0
p_high <- p_high0

iter <- 1

while (diff > tol & iter < max_iter ) {

  p_low <- 0 * p_low0
  p_high <- 0 * p_high0

  for (i in 1:n) {
    if (p_high0[i] > .Machine$double.xmin) {
      p_high[decision_high[i], 1] <- p_high[decision_high[i]] + p_high0[i, 1] * Q[2, 1]
      p_high[decision_high[i], 2] <- p_high[decision_high[i]] + p_high0[i, 2] * Q[2, 2]
    }

    if (p_low0[i] > .Machine$double.xmin) {
      p_low[decision_low[i], 1] <- p_low[decision_low[i]] + p_low0[i, 1] * Q[1, 1]
      p_low[decision_low[i], 2] <- p_low[decision_low[i]] + p_low0[i, 2] * Q[1, 2]
    }
```

```
  }

  diff <- sum(abs(p_low - p_low0) + abs(p_high - p_high0))

  p_low0 <- p_low
  p_high0 <- p_high

  iter <- iter + 1
}

p_low_marginal <- apply(p_low, 1, sum) / sum(p_low)
p_high_marginal <- apply(p_high, 1, sum) / sum(p_high)

mean_assets_low <- sum(a * p_low_marginal)
mean_assets_high <- sum(a * p_high_marginal)
mean_assets <- (mean_assets_low + mean_assets_high)/2
```