Incomplete Markets and Aggregate Demand*

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I study the relationship between aggregate consumption and interest rates when markets are incomplete. I first provide a generalized Euler relation involving the real interest rate, current and future aggregate consumption under extreme illiquidity (no borrowing and no outside assets). This provides a tractable way of incorporating incomplete markets into macroeconomic models. When household income risk is acyclical I show that this relation coincides with that of a representative agent, although time-varying discount factors may potentially act as aggregate demand shocks. The same representation extends to the case with positive liquidity as long as liquidity relative to income is acyclical. A corollary of these 'as if' results is that forward guidance policies are as powerful as in representative agent models. Away from the 'as if' benchmark, I show that aggregate consumption becomes more sensitive to interest rates, especially future ones, when idiosyncratic income risk is countercyclical or when liquidity is procyclical. Finally, I also apply my analysis to a Real Business Cycle model, providing an exact analytical aggregation result that complements existing numerical findings.

1 Introduction

Macroeconomic models are minimalist at heart. For example, the basic New Keynesian model can be reduced to two fundamental components,

1. the Intertemporal Euler equation or "Demand block";
2. the Phillips curve or "Supply block";

the picture may be completed by describing monetary policy, providing a third equation. A similar description can be applied to models without nominal rigidities, such as the Real Business Cycle model, which also features the Euler equation at center stage and adopts the flexible price limit as its supply block.

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Microfoundations are available for both blocks, but at best these should be viewed as extremely simplified approximations of a deeper reality. In particular, underlying the standard Euler equation is the assumption of either complete markets or the adoption of a representative agent. Similarly, underlying the New Keynesian Phillips curve is the simplifying assumption that opportunities to change prices arrive at random, i.e. Calvo pricing. These assumptions can be easily rejected at face value, but, at the same time, it is hard to deny their usefulness as tractable starting points.

This paper is concerned with the 'demand block' and asks: What are the effects of market incompleteness on aggregate demand? How should incomplete markets be incorporated into macroeconomic models and what can we expect from doing so? How is it relevant for business cycles and policy analyses? Given the well-documented importance of idiosyncratic uncertainty and the limits to insurance at the household level, these questions hardly require motivation. Less formally, or introspectively, nobody living in the real world can possibly feel well represented by the representative agent lurking in our models! Indeed, anyone outside our models would quickly point out that idiosyncratic uncertainty looms large relative to the business cycle, that the risk of unemployment is the main concern with recessions, that current or future borrowing constraints limit the smoothing of consumption, etcetera. Intuitively, a more realistic model could potentially make a difference and this is the motivation for a large and important literature on incomplete markets. Despite recent progress, the literature still finds itself far from offering a comprehensive and conclusive answer to the questions posed above.

To address these questions, I study a Bewley-Huggett-Aiyagari incomplete-market economy: households save and borrow, but lack insurance for the idiosyncratic uncertainty they face. A key feature of my formulation is to postulate that each household earns income that depends on an idiosyncratic shock (the microeconomic component) as well as on aggregate spending (the macroeconomic component). This allows me to impose the general-equilibrium condition that aggregate consumption equal aggregate income. With this aggregate demand setup in place, the main goal is to describe the paths for aggregate consumption and interest rates that are consistent with one another. In the complete market or representative agent case this boils down to the standard intertemporal Euler equation, i.e. $U'(C_t) = \beta_t R_t U'(C_{t+1})$ where the discount factor β_t is often taken to be constant.

Crucially, this exercise goes beyond aggregating under partial equilibrium, that is, for fixed household income processes and arbitrary interest rates. Indeed, for a given interest rate path household income must be endogenously determined using the requirement that aggregate income equal aggregate consumption, so that the asset market is cleared.

It is also important to note that my setup captures the aggregate demand block for a wide class of models, with or without nominal rigidities, avoiding the need to fully specify the supply block. I will illustrate how this structure serves as a component for the New Keynesian and the Real Business Cycle models.

A necessary condition for incomplete markets to matter at the aggregate level is for it to make a difference at the household level. The capacity of the market to self insure agents depends on the amount of liquidity—the value of available assets and the amount of borrowing permitted. I first explore the case of extreme illiquidity, with no borrowing and no outside assets. This natural benchmark captures the greatest deviation from complete markets; it is also relatively tractable thanks to the fact that, in equilibrium, no intertemporal trade is possible.[1] In this context I provide a simple, yet general and insightful, result, deriving an equilibrium relation involving the real interest rate, current and future aggregate consumption, i.e. $g_t(R_t, C_t, C_{t+1}) = 0$. According to this generalized Euler relation, current consumption may be more or less sensitive to the current interest rate relative to the intertemporal elasticity of substitution given by preferences; likewise, consumption may respond more or less than one-for-one with changes in future consumption—upsetting the standard consumption smoothing property. In short, the generalized Euler equation may depart from the standard Euler equation. Indeed, it offers a convenient platform to study departures and I return to this below.

I first show that for a benchmark case, with constant relative risk aversion utility functions and household income that depends proportionally on aggregate income, the general Euler relation reduces to a standard representative-agent Euler equation, i.e. $U'(C_t) = \beta_t R_t U'(C_{t+1})$. This provides a useful *'as if'* result: as far as the response of aggregate consumption to interest rate changes is concerned, the economy can be summarized by an Euler equation relation *as if* markets were complete or populated by a representative agent. The assumption that household income varies proportionally with aggregate income is equivalent to assuming that the distribution of relative income is unaffected by aggregate income, so that household income risk is acyclical.

I also consider cases with a positive supply of liquidity, when an outside asset is available or when households can borrow from one another. In general, this situation does not lend itself as easily to aggregation, since the allocation no longer coincides with financial autarky and may depend in complex ways on the path for interest rates. Despite these difficulties, I show that when utility is logarithmic and borrowing constraints are

[1]However, despite the lack of borrowing and saving, consumption is still endogenously determined since aggregate and household income are endogenous—the general-equilibrium feedback discussed above.

proportional to aggregate income a similar 'as if' result applies.

The 'as if' result provides an important benchmark, as well as a useful launchpad to investigate other cases, which I will turn to further below. Before doing so it is worth interpreting the result with care, to understand what it does and does not say.

The 'as if' result *does not* say that incomplete markets are irrelevant for aggregates. Idiosyncratic uncertainty, lack of insurance and borrowing constraints all have an impact on the *level* of aggregate consumption: given interest rates, uncertainty and scarce liquidity depress consumption. What the 'as if' result *does* say is that the *sensitivity* or elasticity of aggregate demand to interest rates and future consumption is unaffected by the incompleteness of markets: aggregate consumption reacts to changes in the path of interest rates just as in a representative agent model.

The level effects show up in the subjective discount factors, β_t, of the 'as if' representative agent formulation. Indeed, in the background household consumption may involve rich dynamics, as in the deleveraging episodes in Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2011). While as far as aggregates are concerned, the effects are captured by time varying discounting. This provides a foundation for working a representative-agent formulation augmented with aggregate demand shocks.[2]

The 'as if' result also has some important practical implications. An immediate one is for "forward guidance", defined as the commitment by a central bank to keep future interest rates low. These policies have been advocated as a way to stimulate current aggregate demand during a "liquidity trap"—when the zero lower bound on interest rates is binding. Whenever the 'as if' result is applicable, forward guidance works exactly as it does in the representative agent setting, just as powerfully. The same is true of other related policies that work by committing future policy in other ways, such as price level or nominal GDP targeting. A related implication is for the potency of "news shocks", since these also work by affecting expectations of future consumption and affecting current demand through the consumption-smoothing channel. Under the 'as if' result, these shocks are just as effective as in representative agent models.

Although the 'as if' result does not imply that market incompleteness is irrelevant, it may seem surprising at first. How can the response of aggregate consumption to interest rates not be affected by market incompleteness? After all, some households may find themselves up against, or near, a borrowing constraint and are likely to be insensitive to interest rates. This seems especially relevant for future interest rate changes, since

[2] Interestingly, the literature on "liquidity traps" has often taken the shortcut of assuming a representative agent with temporarily high discount factors, to push the economy towards zero interest rates. My results provide a microfoundation for this approach.

borrowing constraints may not bind today but will bind at some point in the future, frustrating consumption smoothing and, thus, breaking the transmission chain running from future interest rates to present consumption.

My result does not rely on denying any of these considerations. Quite the contrary, it embraces them since it applies when borrowing constraints are binding or may become binding in the future. The problem lies with the notion that these factors necessarily mute the response of consumption to interest rates, since this argument is based on partial equilibrium logic only. General equilibrium effects turn out to be important, especially the feedback loop from consumption to income. When interest rates fall, households that are not liquidity constrained substitute intertemporally to increase spending. They would do so even if their income were to remain unchanged—a partial equilibrium response. However, in equilibrium, their income and that of others increases as a result of this increase in their spending. Liquidity constrained households, in contrast, cannot substitute intertemporally, but they are especially sensitive to the increase in income. As it turns out, both constrained and unconstrained households respond proportionally in equilibrium, but they do so for different reasons. This general equilibrium feedback between consumption and spending ensures that the 'as if' result obtains despite partial equilibrium forces to the contrary. It is worth noting that this is the very same feedback that amplifies the effects of fiscal policy with liquidity constrained consumers (Gali et al., 2007; Farhi and Werning, 2012). A greater multiplier arises when the high propensity to spend by constrained consumers is coupled with the feedback between their spending and their income—the "Keynesian cross". Thus, not only is the mechanism underlying my results not counterintuitive as it may first appear, but it should be recognized as a staple in the macroeconomics toolbox.

I also explore departures from the 'as if' result. With zero liquidity the 'as if' result holds when household income varies proportionally with aggregate income, which amounts to assuming household income risk is acyclical. Instead, when uncertainty is countercyclical, i.e. rising in recessions, I show that aggregate demand becomes more sensitive to interest rates, current and future (the reverse is true when uncertainty is procyclical). Countercyclical risk is widely used in the asset pricing literature to explain the equity premium puzzle, at least since Mankiw (1986), and has received empirical support (Storesletten et al., 2004; Guvenen et al., 2014).

To see the effects from countercyclical risk clearly, I consider a simple setting with an extensive margin for employment. Aggregate income continues to affects income along the intensive margin, but it now also affects the probability of full employment. I show that this makes aggregate demand more sensitive to current interest rates, and even more sensitive to future interest rates. Intuitively, when interest rates are lowered, aggregate income rises. This increases the probability of full employment, lowering income risk. In earlier periods, this diminishes the desire for precautionary savings, stimulating consumption further. This result relies on the fact that the probability of full employment varies with aggregate income. The greater the adjustment along this extensive margin, the greater the sensitivity of aggregate demand. Indeed, in the limit where all the adjustment is along this extensive margin, without an intensive margin, aggregate demand becomes infinitely elastic to interest rates.

With positive liquidity, I isolate an important role for the cyclicality of liquidity relative to income. Working with a three period economy with acyclical income risk, I show that the 'as if' result holds as long as liquidity is acyclical, when borrowing limits and asset values expand proportionally with aggregate income. Instead, aggregate demand becomes more sensitive to interest rates whenever liquidity is procyclical, when borrowing limits and asset values expand more than proportionally with aggregate income. Conversely, aggregate demand becomes less sensitive when liquidity is countercyclical.

To understand this result, consider the case where a lower interest rate increases aggregate income, but increases the value of assets by more. This extra liquidity allows household to buffer idiosyncratic income shocks more effectively, lowering consumption risk. Once again, in earlier periods, this diminishes the desire for precautionary savings, stimulating consumption further. Interestingly, this result does not rely on binding liquidity constraints. However, I also show that aggregate demand becomes even more sensitive if some households are constrained.

The cyclicality of liquidity is partly endogenous, since asset prices must be determined in equilibrium. With log utility the ratio of asset prices relative to aggregate income turns out to be constant, explaining why the 'as if' result holds in this case. In contrast, with more curvature (when the intertemporal elasticity of substitution is less than one) equilibrium asset prices fluctuate more than income, so that liquidity is procyclical. Borrowing constraints can also be modeled endogenously. One common specification assumes that borrowing is limited by the present value of pledgeable income. When interest rates fall, borrowing limits expand more than aggregate income when, once again, the intertemporal elasticity of substitution is less than one. Both results suggest that the case with procyclical liquidity is plausible in the model.

Taking stock, one contribution of this paper is to develop a mapping from assumptions to results, with the hope that it may be used to explore the territory and provide a better understanding of the different possibilities, as well as the economic mechanisms at work. Table 1 provides a schematic representation of the mapping suggested by my results,

Assumptions on			Response of aggregate
Income Risk	**Liquidity**		**consumption to interest rates**
countercyclical	procyclical	$\rightarrow$	higher sensitivity
acyclical	acyclical	$\rightarrow$	'As if' representative agent
procyclical	countercyclical	$\rightarrow$	lower sensitivity

Table 1: Summary schematic mapping assumptions to results.

highlighting the central role for the cyclicality of idiosyncratic income risk and liquidity. The 'as if' result holds when both income risk and liquidity are acyclical. Aggregate demand becomes more sensitive to interest rates, especially future interest rates, when income risk is countercyclical and liquidity is procyclical; the reverse is true if instead income risk is procyclical and liquidity is countercyclical.

Although immediate applicable in contexts with nominal rigidities, where monetary policy affects real interest rates, my methods and results transcend such settings. Indeed, my demand block framework sidesteps modeling the supply side and avoids making assumptions regarding nominal rigidities. To highlight this point, I close the paper studying a Real Business Cycle model featuring capital, labor and productivity shocks. This model is also of interest because the supply of liquidity, given by capital, is then endogenous.

I show that the 'as if' result holds for the Brock-Mirman specification, with log utility and full depreciation. The saving rate is then constant, which implies that the value of assets is proportional to output. In other words, liquidity is acyclical and the 'as if' result follows just as before. Admittedly, full depreciation is not plausible, so the value of this exact analytical aggregation result is conceptual, rather than practical. However, my result complements the numerical results in Krusell and Smith (1998), which showed that approximate aggregation holds when households are able to smooth their consumption effectively. Since my result holds regardless of how well households are able to smoothing their consumption, highlighting a new rationale for aggregation.

Related Literature. This paper belongs to a vast literature in macroeconomics exploring the implications of relaxing the representative agent assumption and adopting an incomplete markets model. A seminal contribution was Krusell and Smith (1998) in the context of a Real Business Cycle model. Useful overviews can be found in Heathcote et al. (2009) and Guvenen (2011). More recent work includes Guerrieri and Lorenzoni (2011), Oh and Reis (2012), Kaplan and Violante (2011), Ravn and Sterk (2012), Eggertsson and Krugman (2012), McKay and Reis (2013), Sterk and Tenreyro (2013), Sheedy (2014), Auclert (2015) and McKay, Nakamura and Steinsson (2015). These contributions focus on various different aspects (e.g. the effects of deleveraging episodes and the zero lower bound, the role of illiquid assets in high marginal propensities to consume, the redistributive effects from inflation or interest rate exposures, etc.) and have uncovered important implications of incomplete markets in different contexts.

A distinguishing feature of my approach is its focus on the aggregate demand side, without specifying a particular supply side. This allows me to study the effects of incomplete markets on aggregate demand under relatively general conditions, and to isolate the key roles played by the cyclicality of risk and liquidity. This is accomplished by deriving a general Euler relation expressed in terms of aggregates, providing conditions under which the 'as if' result holds, as well as exploring departures from this result.

Although not the focus of the present paper, my results have implications for forward guidance that may be contrasted with McKay, Nakamura and Steinsson (2015). That paper sets up a New Keynesian model and shows that forward guidance is less powerful with incomplete markets. My results clarify that it is not market incompleteness per se that affects the power of forward guidance. Instead, it is the interaction of incomplete markets with auxiliary assumptions, such as the cyclicality of risk and liquidity. Two modeling assumptions in their benchmark New Keynesian model tend to make income risk procyclical and relative liquidity countercyclical—placing them in the bottom row of Table 1. First, monopolistic profits are distributed evenly across workers. Thus, they are disproportionately important for low earners. Since profits relative to income are countercyclical, this tends to make income risk procyclical.[3,4] Second, in the model the only asset available to households is a short-term real bond issued by the government and fiscal policy is assumed to keep government debt constant. Thus, liquidity relative to income is countercyclical.[5]

My results are also consistent with the findings in Ravn and Sterk (2012), which ar-

[3]In the benchmark model, as in many basic New Keynesian models, wages are flexible, prices are rigid, desired markups are constant, real marginal costs are constant and there are no fixed costs. This leads to countercyclical markups and, thus, profits relative to income (and potentially even profits) inherit this countercyclicality, as wages rise more than prices. Models with increasing marginal costs, or fixed costs, or other features such as wage rigidity and labor hoarding may have different predictions. Another important component of profits or capital income is the returns to capital, unrelated to markups, and the cyclicality of its return is likely to depend on a number of other assumptions.

[4]Del Negro et al. (2015) show that the effect of forward guidance policies is mitigated in an overlapping generation model, without uninsurable idiosyncratic risk. The source for this result can also be traced back to the countercyclicality of monopolistic profits relative to income. Indeed, this is a corollary of my results because my framework allows for heterogeneity and taste shifters, so it can easily accommodate overlapping generations.

[5]An alternative model in their appendix considers a case with zero liquidity and two idiosyncratic states. In the unemployment state household income is fixed; all earnings and profits accrue in the employment state. Since the probability of the low state is assumed constant, household income risk increases with aggregate income. In contrast, the extensive margin model I develop in 3.4 and Appendix A assumes that the probability of the low state decreases with aggregate output.

gues that incomplete markets and labor markets imperfections lead to deeper recessions when interest rates are not adjusted, e.g. at the zero lower bound. The main mechanism in their model is that household income risk is countercyclical, with the probability of unemployment rising in a recession. This places their model in the top row of Table 1. I develop a related specification with employment risk in Section 3.4.

Countercyclical income risk is also what drives the feedback loop between aggregate demand, idiosyncratic risk and precautionary savings in Chamley (2013) and Beaudry et al. (2014).

An important literature in finance, motivated by the equity premium puzzle, studies the implications of incomplete markets for asset prices. A partial list of papers includes Heaton and Lucas (1996), Constantinides and Duffie (1996), Krusell and Smith (1997), Alvarez and Jermann (2001), Krueger and Lustig (2010), Krusell et al. (2011). Indeed, Constantinides and Duffie (1996), Alvarez and Jermann (2000; 2001) and Krusell et al. (2011) consider environments where the equilibrium coincides with autarky.[6] My zero-liquidity analysis similarly exploits the tractability offered by an autarkic equilibrium, although there appears to be no parallel to the generalized Euler equation for aggregates that I obtain.[7] A common message to emerge from the asset pricing literature is the important role played by the cyclicality of idiosyncratic income risk in shaping the equity premium. My paper also highlights the importance of the cyclicality income risk, although it isolates a distinct role, for the sensitivity of aggregate demand to interest rates. My analysis also uncovers an important role played by the cyclicality of liquidity, which, to the best of my knowledge, has no parallel in the asset pricing literature.

Some other important differences between the asset pricing literature are worth mentioning. First, the asset pricing literature focuses on endowment economies, taking an exogenous path for aggregate income and consumption and solving for asset prices; in contrast, the present paper takes the path of interest rates as given, and solves for the endogenous path of income and consumption. Second, the asset pricing literature focuses

[6] Constantinides and Duffie (1996) reverse engineer an income process to ensure that individual consumption (relative to aggregate consumption) follows a geometric random walk, implying a no-trade equilibrium. The variance of the idiosyncratic shocks then provides an additional factor determining the stochastic discount factor. Krusell, Mukoyama and Smith Jr. (2011) study simple environments with zero liquidity, exploring quantitative implications for asset prices for calibrated income processes. Alvarez and Jermann (2000; 2001) consider endogenous insurance limited by lack of commitment; however, autarky is always an equilibrium in their model, and autarky may be the only equilibrium in some cases.

[7] Heaton and Lucas (1996), Alvarez and Jermann (2001) and Krueger and Lustig (2010) consider environments where the equilibrium does not coincide with autarky. Heaton and Lucas (1996) provides an early quantitative study of the role of incomplete markets. Krueger and Lustig (2010) focuses on environments where the equity premium coincides with that of a representative agent. Alvarez and Jermann (2001) allow for insurance limited by the lack of commitment.

on the equity premium—the difference in the average returns to risky stocks and riskless bonds. The present paper focuses instead on interest rates, abstracting from stock returns and aggregate shocks. Finally, and most importantly, the asset pricing literature studies the dynamic properties of a single equilibrium along its stochastic path. In contrast, the present paper is interested in the effect of incomplete markets on the level of aggregate consumption and its sensitivity to different interest rate paths. This requires a comparative static analysis across equilibria with different interest rates, leading to different methods and results.

2 A General Incomplete Market Setting

This section introduces the incomplete market model where households make consumption and savings decisions. The setup is fairly general and will be specialized in different directions later. The focus will be on households, with firms and the government largely relegated to the background. To complete the model, this "demand block" could be combined with a "supply block", together with specifications for monetary and fiscal policy. For my purposes, it is best to avoid committing to particular ways of doing this and focus on the "demand block" only.

The main goal is to understand how market incompleteness affects macroeconomic aggregates. Of particular interest is the relationship between the paths for household spending and interest rates. Ideally, one aims to obtain something like the standard representative Euler equation for the incomplete-market model. This turns out to be possible in some interesting cases.

2.1 Economic Environment

The framework I develop is based on standard Bewley-Huggett-Aiyagari incomplete market models. The time horizon is infinite, with discrete periods $t = 0, 1, \ldots$ Each period, there is a single final good. The economy is populated by a unit measure of infinitely-lived households that are subject to idiosyncratic shocks to their income and spending needs. Insurance against these shocks is absent and credit may be limited.

To simplify, I abstract from aggregate uncertainty, focusing exclusively on idiosyncratic uncertainty. Aggregate shocks could be added at some expense, but they do not seem essential to understanding how aggregate demand is affected by uninsurable idiosyncratic uncertainty.

Household heterogeneity. There is a finite set of household types indexed by $i \in I$. Household type i represents a fractions $\mu^i > 0$ of the population. Types may differ with respect to their preferences, including discounting. They may also have different labor earnings process and different degrees of access to credit, i.e. borrowing constraints.

Preferences. Household of type $i \in I$ has preferences over consumption given by the utility function

$$\sum_{t=0}^{\infty} \beta_{i,t} \mathbb{E}_0[u_t^i(c(s^t), s_t)] \tag{1}$$

where c_t denotes consumption of the single final good, $s_t \in S^i$ denotes an idiosyncratic state of nature that follows a stochastic process, discussed further below, and s^t is the history of s_t. Shocks to utility are included for both generality and realism. They may capture important lifetime events, such as health shocks or family size changes, that affect the relative desirability of current spending.

Although typical Bewley-Huggett-Aiyagari models are populated by a fixed cohort of infinitely-lived agents, the dependence of the utility function on t can be combined with household heterogeneity to capture overlapping generation frameworks: households within a generation born in period t may be associated with a particular household type i and value consumption only from period t onwards while alive.

Because the focus is on consumption and savings choices, I first directly postulate a labor income process, without deriving it from labor supply choices. This also keeps us closer to the incomplete markets literature, which studies consumption and savings taking income processes as given. With this in mind, there is no need to describe preferences over leisure or labor at this stage.

Budget Constraints. Households of type i face the budget constraints

$$c(s^t) + q_t \cdot a(s^t) + b(s^t) \leq y_t^i(s_t) + (q_t + d_t)a(s^{t-1}) + R_{t-1} \cdot b(s^{t-1}) \tag{2}$$

for all $t = 0, 1, \ldots$ and histories $s^t \in S^{t+1}$; here $a(s^t)$ and $b(s^t)$ denote savings in the outside asset and riskless one-period bonds, respectively; q_t is the price of the outside asset and d_t is its dividend; R_t denotes the interest rate on riskless bonds, in real terms; y_t^i is income from labor.

Labor income depends on the household state s_t and aggregate income according to

$$y_t^i(s_t) = \gamma_t^i(s_t, Y_t), \tag{3}$$

for some function γ_t^i. The nature of the relationship between household and aggregate income encapsulated by γ_t^i, will turn out to be crucial.

Borrowing Constraints. Households of type $i \in I$ are also subject to borrowing constraints

$$b(s^t) + q_t \cdot a(s^t) \geq -B_t^i(s_t, Y_t), \tag{4}$$

which limits how negative their wealth is allowed to become. Here B^i is a nonnegative borrowing limit, determined as a function of the current household state and aggregate income.[8]

Idiosyncratic Uncertainty. Uncertainty is purely idiosyncratic: the realization of states are independent across agents and for each type $i \in I$ the probability of a certain set of histories s^t equals the fraction of agents in the cross section experiencing this history. For each household type $i \in I$, the exogenous state $\{s_t\}$ follows a stochastic process. An important case is when s_t follows a Markov process, although this assumption is not required for most of my analysis. For each household type $i \in I$, the stochastic process for household states $\{s_t\}$ is independent of the path for aggregate income $\{Y_t\}$. This assumption is essentially a normalization, since we have not placed restrictions on the functions γ_t^i and B_t^i.

Initial Conditions. At $t = 0$ the economy inherits, for each household type $i \in I$, a joint distribution over initial states, asset and bonds $\Lambda_0^i(s_0, a_0, b_0)$. The stochastic process then induces a joint distribution $\Lambda_t^i(s^t, a_0, b_0)$ of histories s^t and initial (a_0, b_0).

Outside Asset. The outside asset is in fixed unit supply[9]and provides a dividend stream that in each period is a function of current aggregate income,

$$d_t = D_t(Y_t). \tag{5}$$

The case with zero assets is captured by setting dividends to zero: $D_t(Y_t) = 0$ for all t.[10]

[8]The constraint on borrowing is specified in terms of total wealth; alternatively, one can impose separate constraints for assets and bonds, $b(s^t) \geq -B^i(s_t, Y_t)$ and $a(s^t) \geq 0$, and the results would be similar.

[9]The assumption that the outside asset is in fixed supply is relaxed in Section 6, which extends the analysis to a model with capital and investment.

[10]When $D = 0$ there always exists an equilibrium where the asset price is zero. In some cases there may be other equilibria, akin to monetary equilibria where fiat money has value, I shall not consider these equilibria.

To ensure that Y_t can be interpreted as aggregate income from labor and capital, we require that the functions γ_t and D_t satisfy the identity

$$\sum_{i \in I} \mu^i \int \gamma_t^i(s_t, Y_t)\, d\Lambda_t^i + D_t(Y_t) = Y_t, \tag{6}$$

for all Y_t. Note that given $\{\gamma_t^i\}$ the dividend function $D_t(Y_t)$ can be backed out from this identity.

2.2 Equilibrium

I now introduce a natural equilibrium concept for this framework and provide a simple characterization, reducing the conditions to a few equations.

Equilibrium Definition. An equilibrium specifies interest rates and consumption decisions that are required to be optimal as well as consistent with aggregate income. Formally, given initial conditions R_{-1} and Λ_0^i, an equilibrium is a path for aggregates

$$\{C_t, Y_t, A_t, B_t, R_t, q_t\},$$

and household choices, conditional on initial conditions,

$$\{c^i(s^t; a_0, b_0), a^i(s^t; a_0, b_0), b^i(s^t; a_0, b_0)\},$$

satisfying the following:

1. *household optimization:* taking as given the path for aggregate income and interest rates $\{Y_t, R_t\}$, household choices maximize utility (1) subject to (2), (3) (4) and (5);

2. *market clearing:* for all $t = 0, 1, \ldots$ the good, asset and bond markets clear,

$$C_t = Y_t,$$
$$A_t = 1,$$
$$B_t = 0;$$

3. *aggregation:* the aggregate quantities are consistent with household quantities,

$$C_t = \sum_{i \in I} \mu^i \int c^i(s^t; a_0, b_0)\, d\Lambda^i_t(s^t, a_0, b_0),$$
$$A_t = \sum_{i \in I} \mu^i \int a^i(s^t, a_0, b_0)\, d\Lambda^i_t(s^t, a_0, b_0),$$
$$B_t = \sum_{i \in I} \mu^i \int b^i(s^t, a_0, b_0)\, d\Lambda^i_t(s^t, a_0, b_0).$$

Given a path for interest rates, one may seek a path for aggregate consumption that forms part of an equilibrium. It is important to stress that this goes beyond the pure aggregation of household consumption choices, for given income processes. Indeed, the equilibrium notion used here also incorporates the general equilibrium feedback from consumption to income.[11]

Implementability Conditions. The equilibrium requirements can be reduced to a small set of conditions as follows. First, the two riskless assets must satisfy a no-arbitrage condition equating returns,[12]

$$\frac{q_{t+1} + d_{t+1}}{q_t} = R_t.$$

This no-arbitrage condition is implied by imposing that the asset price equal the present value of its dividends,

$$q_t = \sum_{s=0}^{\infty} \frac{1}{R_t R_{t+1} \cdots R_{t+s}} D_{t+s}(Y_{t+s}). \tag{7}$$

Household optimality requires budget constraints to hold with equality. Defining total wealth $\hat{a}^i(s^t; a_0, b_0) \equiv q_t \cdot a^i(s^t) + b^i(s^t)$, this requires for $t \geq 1$

$$c^i(s^t; a_0, b_0) + \hat{a}^i(s^t; a_0, b_0) = \gamma^i_t(s_t, Y_t) + R_{t-1} \cdot \hat{a}^i(s^{t-1}; a_0, b_0), \tag{8a}$$

Similarly, the budget constraint at $t = 0$ requires

$$c^i(s_0; a_0, b_0) + \hat{a}^i(s_0; a_0, b_0) = \gamma^i_0(s_0, Y_0) + (q_0 + D_0(Y_0))a^i_0 + R_{-1} \cdot b^i_0, \tag{8b}$$

[11]This perspective can be contrasted some well-known aggregation exercises. For example, Huggett (1993) and Aiyagari (1994) aggregate consumption and savings for given interest rates, but taking the income process as given. They then employ this aggregate relationship graphically to determine an equilibrium that clears the market.

[12]Strictly speaking, when borrowing is completely ruled out, so that $B^i = 0$, an equilibrium requires only that $R^a_t \geq R_t$. However, in such cases, the equilibrium with $R^a_t > R_t$ is not robust to the introduction of vanishingly small amounts of borrowing.

Wealth must satisfy the borrowing constraints

$$\hat{a}^i(s^t; a_0, b_0) \geq -B_t^i(s_t, Y_t). \tag{9}$$

Household optimization reduces to the Euler condition,

$$u_{c,t}^i(c^i(s^t; a_0, b_0), s_t) \geq \frac{\beta_{t+1}^i}{\beta_t^i} R_t \mathbb{E}_t[u_{c,t+1}^i(c^i(s^{t+1}; a_0, b_0), s_{t+1})], \tag{10}$$

with the complementary slackness requirement that this condition hold with equality in period t whenever the borrowing constraint (9) in period t holds with strict inequality. Finally, we impose the market clearing condition $C_t = Y_t$ and the aggregation condition for C_t. All the other market clearing conditions are then implied.

To summarize, an equilibrium can be reduced to aggregates

$$\{C_t, R_t\}$$

and household consumption and wealth $\{c^i(s^t; a_0, b_0), \hat{a}^i(s^t; a_0, b_0)\}$ satisfying aggregation $C_t = \sum_{i \in I} \mu^i \int c^i(s^t; a_0, b_0) d\Lambda^i(s^t, a_0, b_0)$, the budget constraints (8), borrowing constraints (9) and Euler condition (10) with complementary slackness. In these conditions we obtain q_0 by (7) and $Y_t = C_t$.

When these conditions hold, one can find the remaining equilibrium objects as follows. The asset price in all periods is given by (7). Asset and bond holdings, however, are indeterminate: any portfolio split satisfying $\hat{a}^i(s^{t-1}; a_0, b_0) = q_t \cdot a^i(s^t) + b^i(s^t)$ constitutes an equilibrium.

2.3 Pitfalls of Aggregation under Partial Equilibrium

Before stating my aggregation results, it is worth briefly reviewing common pitfalls of aggregation under partial equilibrium. This discussion will help underscore how my results rely on general equilibrium considerations.

The Euler condition is nonlinear, but absent uncertainty, absent borrowing constraints and assuming homogeneous discounting and constant relative risk aversion utility functions $u(c) = c^{1-\sigma}/(1-\sigma)$ it can be transformed into a linear relationship and aggregated. To see this, note that the household Euler equation implies $c^i(s^t) = (\beta R_t)^{-\frac{1}{\sigma}} c^i(s^{t+1})$ and thus, aggregating,

$$C(s^t) = (\beta R_t)^{-\frac{1}{\sigma}} C(s^{t+1}) \tag{11}$$

Unfortunately, this aggregation is delicate and easily upset by idiosyncratic uncertainty and borrowing constraints, among other things.

In the presence of idiosyncratic uncertainty and borrowing constraints, we start from $c^i(s^t) \leq (\beta R_t)^{-\frac{1}{\sigma}} \cdot \left(\mathbb{E}_t[c^i(s^{t+1})^{-\sigma}]\right)^{-\frac{1}{\sigma}}$, so that aggregating and using Jensen's inequality gives

$$C(s^t) < (\beta R_t)^{-\frac{1}{\sigma}} C(s^{t+1}).$$

as long as some households are constrained or face uncertainty. The magnitude of the departure from (11) varies with the amount of uncertainty and the extent to which borrowing constraints bind. This makes it difficult to expect an equation such as (11) to hold, even for a modified discount factor. The problem is that the departures from (11) depend on precautionary effects and binding borrowing constraints, which are not invariant to the distribution of wealth and the path of interest rates.

To overcome these difficulties I take a different route: rather than simply working with Euler equations, I also incorporate the fact that aggregate income must equal aggregate consumption. I find tractable aggregate relations only after imposing this additional equilibrium condition. In sum, my results do not rely on cleverly overcoming standard microeconomic aggregation problems. Indeed, the aggregate relation I obtain does not always take the form of (11). Instead, I solve for aggregate consumption by exploiting its general equilibrium relation with aggregate income.[13]

3 Zero Liquidity

For market incompleteness to matter at the aggregate macroeconomic level, it must first make a difference at the microeconomic level and generate significant departures from complete markets for household consumption. Without complete markets, perfect insurance is generally impossible. However, as has been widely noted, due to the incentive to smooth consumption and the precautionary motives to accumulate assets, general equilibrium outcomes under incomplete markets may approximate the outcome under complete markets.

The extent to which this is true depends on the capacity offered by the market for self insurance. This in turn depends crucially on the amount of liquidity—the value of available assets and the amount of borrowing permitted. When liquidity is plentiful the

[13]By the same token, my results are designed for comparative static exercises, e.g. changing the interest rate path. They may not resolve the empirical problem faced by microeconomic tests (e.g. Attanasio and Weber, 1993).

outcome is closer to complete markets; when liquidity is scarce the outcome is closer to financial autarky.

Along this spectrum, the simplest results obtain in the extreme case of complete illiquidity: no outside asset and no borrowing. This situation is best thought of as a limiting case of extreme scarcity in liquidity, with small asset values and very limited borrowing. While extreme, this case is of special interest since it lies on the opposite side of the spectrum from a situation with plentiful liquidity which may mimic complete markets. Thus, it is the natural place to start to look for effects of incomplete markets on aggregate demand.

3.1 A Generalized Euler Relation

To study situations with zero liquidity assume that

$$D_t(Y_t) = 0,$$
$$B_t^i(s_t, Y_t) = 0,$$

and the initial conditions $\hat{a}_0^0 = b_0^i = 0$ for all agents. Savers can only save by lending to borrowers and borrowing is ruled out. Thus, no intertemporal trade is possible in equilibrium and the allocation is one of financial autarky

$$c^i(s^t) = y_t^i(s_t) = \gamma_t^i(s_t, C_t),$$

where I have substituted the equilibrium condition $C_t = Y_t$. The equilibrium interest rate path that sustains the equilibrium must, in each period, ensure that no household has an incentive to save, which requires

$$R_t \leq R_t^* \equiv \min_{i \in I, s^t} \left(\frac{\beta_{t+1}^i}{\beta_t^i} \mathbb{E} \left[\frac{u_{c,t+1}(\gamma_{t+1}^i(s_{t+1}, C_{t+1}), s_{t+1})}{u_{c,t}(\gamma_t^i(s_t, C_t), s_t)} \mid s^t \right] \right)^{-1}. \tag{12}$$

Any interest rate below R_t^* ensure that all households find the corner solution, with $b_t^i = 0$, optimal; in this sense, equilibrium interest rates are not uniquely determined. However, interest rates strictly below R_t^* are not robust to the introduction of small amounts of liquidity. Positive but vanishing levels of liquidity require $R_t = R_t^*$, since the Euler equation must hold with equality for some agents when $D_t(Y) > 0$ and $B_t^i(s, Y) > 0$. The equilibrium is unique subject to this refinement, which I, henceforth adopt. The next proposition summarizes this characterization.

Proposition 1. *A path for aggregate consumption and interest rates $\{C_t, R_t\}$ is part of an equilibrium with vanishing liquidity if and only if*

$$g_t(R_t, C_t, C_{t+1}) = 0, \tag{13}$$

where the function g_t is given by

$$g_t(R, C, C') \equiv \log R + \log \left(\max_{i \in I, s^t} \frac{\beta^i_{t+1}}{\beta^i_t} \mathbb{E} \left[\frac{u^i_{c,t+1}(\gamma^i_{t+1}(s_{t+1}, C'), s_{t+1})}{u^i_{c,t}(\gamma^i_t(s_t, C), s_t)} \mid s^t \right] \right).$$

This proposition provides a simple and condensed way of exploring the aggregate consequences of incomplete markets. It all boils down to a single relation (13), a generalized Euler equation, which involves the same variables as the standard Euler equation: the current interest rate R_t, present consumption C_t and future consumption C_{t+1}. The standard Euler equation, is a special case with

$$g_t(R, C, C') = \log R + \log \beta + \log U'(C') - \log U'(C),$$

for some constant discount factor β and utility function $U(c)$. While the generalized Euler condition (13) does not necessarily take this functional form, it is nevertheless a simple condition involving only aggregates, much like the standard Euler condition. This makes the aggregate equilibrium conditions tractable, despite market incompleteness. Indeed, relation (13) can be handled, alongside other equilibrium conditions, as easily as the standard Euler equation in any positive or normative analysis.[14]

In general the function g_t varies over time. This is not surprising, since no stationarity assumptions have been placed on primitives (utility functions, discounting, the stochastic process, the income function γ^i_t, etc.). This may help capture interesting situations, such as temporary episodes of heightened idiosyncratic uncertainty.[15]

[14] Computing an equilibrium does not require carrying a large endogenous state variable and confronting the curse of dimensionality, as in Krusell and Smith (1998). The simplifying assumption of zero liquidity ensures that wealth is zero for all agents, at all times, so that there is no wealth distribution to keep track of.

[15] Even if primitives are stationary, the function g_t may vary simply because the cross-section of states s_t is not at an invariant steady-state distribution. This time dependence, however, vanishes in the long run if the Markov process is ergodic, since the cross section of states then converges to its invariant distribution in the long run.

3.2 'As If' Representative Agent Result

I now apply the general characterization obtained above to an important benchmark specification with constant elasticity utility functions and multiplicative taste shocks

$$u_t^i(c,s) = \theta_t^i(s) \cdot U(c) \qquad \text{with} \qquad U^i(c) = \frac{c^{1-\sigma}}{1-\sigma}, \tag{14}$$

for $\sigma > 0$; household income is proportional to aggregate income,

$$\gamma_t^i(s,Y) = \tilde{\gamma}_t^i(s)Y, \tag{15}$$

for some function $\tilde{\gamma}_t^i$.

The next result shows that aggregate consumption and interest rates are related by a standard Euler equation, just as in complete-market or representative-agent economies.

Proposition 2. *Suppose utilities satisfy (14) and household income satisfy (15). Then a sequence $\{C_t, R_t\}$ is part of a equilibrium with vanishing liquidity if and only if*

$$U'(C_t) = \beta_t R_t U'(C_{t+1}),$$

with the discount factors given by

$$\beta_t = \max_{i \in I, s^t} \frac{\beta_{t+1}^i}{\beta_t^i} \cdot \mathbb{E}\left[\frac{\theta_{t+1}^i(s_{t+1})}{\theta_t^i(s_t)} U'\left(\frac{\tilde{\gamma}_{t+1}^i(s_{t+1})}{\tilde{\gamma}_t^i(s_t)}\right) \mid s^t\right]. \tag{16}$$

According to this proposition, aggregate demand is determined *as if* the economy were populated by a single representative agent with discount factor β_t. By implication, changes in the path for interest rates have the same effect on the aggregate consumption path as they do in the representative-agent benchmark. In this sense, the response of consumption to interest rates, is not affected by incomplete markets.[16]

It is worth noting that this result applies in overlapping-generation economies. As discussed earlier, different generations can be modeled as different household types, which value consumption over different time intervals, i.e. $\theta_t = 1$ while alive.

[16]If household are heterogeneous with $U^i(C) = C^{1-\sigma^i}/(1-\sigma^i)$ then this result extends, and one obtains

$$U_t(C_t) = \beta_t U_{t+1}(C_{t+1})$$

where $\beta_t = \max_{i \in I, s^t} \beta_t^i \cdot \mathbb{E}[U^{i\prime}(\frac{\tilde{\gamma}_{t+1}^i(s_{t+1})\theta_{t+1}^i(s_{t+1})}{\tilde{\gamma}_t^i(s_t)\theta_t^i(s_t)}) \mid s^t]$ and the utility function U_t in period t equals the utility function of the household $i \in I$ that attains the maximum in the definition of β_t.

Are Incomplete Markets Irrelevant? No, not at all. In fact, Proposition 2 not only implies that market incompleteness is *not* irrelevant, it identifies very clearly the influence. Due to market incompleteness the discount factor β_t is a function of idiosyncratic uncertainty, as shown in (16). For instance, due to the convexity of the marginal utility function $U'(c)$, in periods with greater uncertainty or downward tail risk (for the growth rate of household income) we can expect the discount factor to be higher; given interest rate and future consumption, R_t and C_{t+1}, this implies lower current consumption, C_t. In contrast, with complete markets the aggregate Euler equation holds with a discount factor that does not depend on idiosyncratic uncertainty (just as in (11)) implying that idiosyncratic uncertainty has no effect on current consumption.

This discussion underscores that incomplete markets matters, affecting the *level* of demand, even if according to Proposition it does not affect the responsiveness of demand to current and future interest rates.

3.3 Departures from 'As If' Result: Cyclical Income Risk

It is useful to recast Proposition 2 as providing conditions for

$$g_t(R_t, C_t, C_{t+1}) = \log R_t + \log \beta_t + \sigma \log C_t - \sigma \log C_{t+1},$$

so that g_t is exactly log linear, with equal coefficients in absolute value on $\log C_t$ and $\log C_{t+1}$ given by σ, the reciprocal of the intertemporal elasticity of substitution. This implies the exact relationship for the log changes,

$$d \log R_t + \sigma d \log C_t - \sigma d \log C_{t+1} = 0.$$

Below I investigate departures from this representation by characterizing the analog first-order expansion

$$d \log R_t + \alpha_{C,t} d \log C_t - \alpha_{C',t} d \log C_{t+1} = 0.$$

I will study this expansion as an exact relation characterizing the derivatives of g_t, rather than an approximation, i.e. the coefficients $\alpha_{C',t}$ depend on where they are evaluated.

Define the elasticity of γ_t^i by

$$\varepsilon_t^i(s, Y) \equiv \frac{\frac{\partial}{\partial Y}\gamma_t^i(s, Y)}{\gamma_t^i(s, Y)} Y.$$

This elasticity measures the responsiveness of household income to aggregate income.

Then next result shows that the coefficients depend can be expressed in terms of this elasticity.

Proposition 3. *Suppose utilities satisfy (14) without taste shocks, i.e.* $\theta_t^i(s) = 1$. *Then*

$$\alpha_{C,t} = Cg_{C,t}(R, C, C') = \sigma \varepsilon_t^i,$$

$$\alpha_{C',t} = -C' g_{C',t}(R, C, C') = \sigma \frac{\mathbb{E}_t[u_{c,t+1}^i \varepsilon_{t+1}^i]}{\mathbb{E}_t[u_{c,t+1}^i]}.$$

for the household type $i \in I$ *and history* s^t *that attains the maximum in (16).*

Note that

$$\left.\frac{d \log C_t}{d \log C_{t+1}}\right|_{d \log R_t = 0} = \frac{\alpha_{C',t}}{\alpha_{C,t}} = \mathbb{E}_t \left[\frac{u_{c,t+1}^i}{\mathbb{E}_t[u_{c,t+1}^i]} \cdot \frac{\varepsilon_{t+1}^i}{\varepsilon_t^i} \right]$$
$$= \mathbb{E}_t \left[\frac{\varepsilon_{t+1}^i}{\varepsilon_t^i} \right] + Cov_t \left[\frac{\lambda_{t+1}^i}{\mathbb{E}\left[\lambda_{t+1}^i \mid s_t\right]}, \frac{\varepsilon_{t+1}^i}{\varepsilon_t^i} \right].$$

Under the conditions for Proposition 2 one has $\varepsilon_t^i = 1$, so that the first term equals 1 and the covariance is zero. Then current consumption C_t varies proportionally with future consumption C_{t+1}.

Away from this neutrality case, what are the plausible possibilities? Consider a simple, but telling, example. Assume γ_t^i is independent of the period t and s_t is a Markov chain, with finitely many values for s_t, that is mean reverting. Suppose further that γ^i is an increasing function of s_t, so that a higher state s_t is associated with a higher income level. Suppose that the household $i \in I$ and history s^t pinning down the interest rate, the one attaining the maximum in (16), has the highest value of s_t. By mean reversion, this agent expects s_{t+1} to fall below its current level, $s_{t+1} < s_t$.

Now suppose ε_{t+1}^i is a decreasing function, so that income at the bottom is more sensitive to aggregate income. Then the first term is greater than one and the covariance term is positive. Current consumption is then more sensitive to future consumption. This implies that changes in future interest rates have *stronger* effects on current consumption than changes in current rates, so that forward guidance is more powerful than in the standard Euler equation case.

Income Growth Rate Perspective. The characterization above relied on the elasticity of income γ_t^i and adopted a perspective in terms of *levels* of income. Perhaps a better perspective is that of *growth* in income. Growth rates are more directly relevant since, absent taste shocks and with constant elasticity utility functions, it is the growth rate of income that enters households' Euler equations. Define the growth rate

$$\Gamma_t^i(s,s',C,C') \equiv \frac{\gamma_{t+1}^i(s',C')}{\gamma_t^i(s,C)}.$$

Then we obtain

$$\alpha_{C,t} = \sigma \frac{\mathbb{E}\left[U'(\Gamma_t^i)\frac{\Gamma_{C,t}^i}{\Gamma_t^i}C \mid s_t^i\right]}{\mathbb{E}[U'(\Gamma_t^i) \mid s_t^i]} \qquad \text{and} \qquad \alpha_{C',t} = -\sigma \frac{\mathbb{E}\left[U'(\Gamma_t^i)\frac{\Gamma_{C,t'}^i}{\Gamma_t^i}C' \mid s_t^i\right]}{\mathbb{E}[U'(\Gamma_t^i) \mid s_t^i]},$$

for the household i with history s^t that attains the maximum in (16).

By implication

$$\alpha_{C',t} - \alpha_{C,t} = -\sigma\mathbb{E}\left[\frac{U'(\Gamma_t^i)}{\mathbb{E}[U'(\Gamma_t^i) \mid s_t^i]}\left(\frac{\Gamma_{C',t}^i}{\Gamma_t^i}C' + \frac{\Gamma_{C,t}^i}{\Gamma_t^i}C\right) \mid s_t^i\right].$$

Thus, the sum of the two elasticities $\frac{\Gamma_{C'}}{\Gamma}C' + \frac{\Gamma_C}{\Gamma}C$ is crucial. This represents the effect that increasing aggregate consumption today and tomorrow proportionally has on the growth rate of consumption for the household pricing the bond. If higher aggregate income diminishes mean reversion, uncertainty and downside risk in consumption then $\frac{\Gamma_{C'}}{\Gamma}C' + \frac{\Gamma_C}{\Gamma}C > 0$ and we conclude that

$$\left.\frac{d\log C_t}{d\log C_{t+1}}\right|_{d\log R_t=0} = \frac{\alpha_{C',t}}{\alpha_{C,t}} > 1,$$

So that consumption reacts more than one-to-one with future consumption.

3.4 Countercyclical Employment Risk

To illustrate the previous results, consider the following simple specification motivated by employment risk. The example features both an intensive and extensive margin—full employment versus underemployment. Importantly, the probability of full employment rises with aggregate demand.

There is a single household type. There are no taste shocks and utility is $U(c) = c^{1-\sigma}/(1-\sigma)$. Each period households may be fully employed or underemployed. Those fully employed earn $\bar{y}Y^\psi$; the rest earn $\underline{y}Y^\psi < \bar{y}Y^\psi$ with $\psi \in [0,1]$.[17] The parameter ψ

[17]Underemployed households earn a positive amount, to avoid zero consumption. This can be motivated

determines the intensive margin adjustment; when $\psi = 1$ all the adjustment is along the intensive margin; when $\psi = 0$ all the adjustment is along the extensive margin.

Let $\lambda(Y)$ denote the fraction of underemployed households, which must satisfy the income identity

$$Y \equiv (1 - \lambda(Y))\bar{y}Y^{\psi} + \lambda(Y)\underline{y}Y^{\psi}.$$

We limit attention to values of Y that imply $\lambda \in (0,1)$. If $\psi = 1$ then λ is constant; as long as $\psi < 1$ then $\lambda(Y)$ is a strictly decreasing function of Y. We assume that $\lambda(Y)$ also represents the probability each household faces of being unemployed.[18]

The bond is priced by the fully employed and their Euler equation can be written as

$$U'(\bar{y}Y_t^{\psi}) = \beta R\left((1 - \lambda(Y_{t+1}))U'(\bar{y}Y_{t+1}^{\psi}) + \lambda(Y_{t+1})U'(\underline{y}Y_{t+1}^{\psi})\right).$$

This reflects the fact that their current consumption and income is $\bar{y}Y_t^{\psi}$ while their consumption next period is either $\bar{y}Y_{t+1}^{\psi}$, with probability $1 - \lambda(Y_{t+1})$, or $\underline{y}Y_{t+1}^{\psi}$, with probability $\lambda(Y_{t+1})$. Using the fact that $U'(c) = c^{-\sigma}$ this equilibrium condition can be represented as follows.

Proposition 4. *In the intensive-extensive margin example economy with varying underemployment we have*

$$\hat{U}'(C_t) = \hat{\beta}(C_{t+1})R\hat{U}'(C_{t+1}),$$

where $\hat{U}(c) = c^{1-\sigma\psi}/(1-\sigma\psi)$ and the discount rate function $\hat{\beta}$ is decreasing and given by

$$\hat{\beta}(C) \equiv \beta\left(1 - \lambda(C) + \lambda(C)U'(\underline{y}/\bar{y})\right).$$

In particular, $g_t(R_t, C_t, C_{t+1})$ is such that $\alpha_{C,t} = \sigma\alpha$ and $\alpha_{C',t} > \alpha_{C,t}$.

This proposition shows that the general Euler relation has two important differences from the standard Euler equation, $U'(C_t) = \beta R U'(C_{t+1})$. First, the utility function $\hat{U}$ has less curvature than U, with elasticity of substitution $\frac{1}{\sigma\psi}$ instead of $\frac{1}{\sigma}$. By implication, consumption is *more* elastic to the interest rate under incomplete markets, compared to complete markets or a representative agent. This elasticity becomes larger when ψ is smaller, so that most of the adjustment takes place along the extensive margin, rather

by supposing each household has many individual members and at least one of these is always able to work, or by extending the model to include an unemployment insurance benefit.

With zero liquidity an unemployment insurance system requires a balanced budget: taxing employed workers and rebating the proceeds to unemployed workers.

[18] To fit this into our notation, assume $s \in [0,1]$ is uniformly distributed and i.i.d. over time and across agents and that $\gamma_t^i(Y_t)$ is a step function with an upward discontinuity at $s = \lambda(Y)$.

than the intensive margin; indeed, the elasticity becomes infinite as $\psi \to 0$. This result is a direct implication of Proposition 3. Second, the discount factor $\hat{\beta}(C_{t+1})$ is lower and strictly decreasing, rather than constant. This implies that current aggregate consumption C_t reacts more than proportionally to changes in future aggregate consumption C_{t+1}, for a given interest rate R_t. By implication, future interest rates have greater effect than current interest rates on current consumption.

This latter result regarding the discount factor is driven by the fact that when $\psi < 1$ lower spending leads to higher employment risk. As a result, households have higher precautionary savings motives. This depresses current consumption further. The assumption that labor income risk is countercyclical is standard in the asset pricing literature seeking to explain high historical equity premia (Constantinides and Duffie, 1996; Alvarez and Jermann, 2001).[19] The assumption of countercyclical idiosyncratic risk has also received empirical support (Storesletten et al., 2004; Guvenen et al., 2014).[20]

A similar feedback loop between the aggregate spending and idiosyncratic risk is at the heart of the amplification mechanism in Ravn and Sterk (2012) (see also Chamley, 2013; Beaudry et al., 2014). They build a full macroeconomic model with search unemployment, mismatch shocks, incomplete markets, nominal rigidities, taking into account the zero lower bound . They calibrate their model to the US economy and show that it is capable of inducing deep recessions.[21]

The present formulation assumed for simplicity that the intensive margin effect is symmetric in the two employment states, so that household income had elasticity ψ with respect to aggregate income. Appendix A treats a case of extreme asymmetry: the high state is unchanged, but income is fixed in the low state. The appendix shows that the

[19] Although similar in spirit, the asset pricing literature often postulates the relationship as operating in *growth rates*, i.e. higher aggregate consumption growth is correlated with higher volatility in the growth of individual consumption. In contrast, here it is the *level* of output that determines individual uncertainty, consistent with macroeconomic models (e.g. Ravn and Sterk 2012, Chamley 2013 and Beaudry et al. 2014). When output is low, employment risk is high, regardless of whether output was low in the previous period or not.

This difference is responsible for the difference between the Euler representation obtained in Proposition 4, showing less curvature in the utility function and an endogenous discount factor, with that obtained by Constantinides and Duffie (1996), with greater curvature in the utility function and a constant discount factor.

[20] Earlier literature specified and found support for income processes with a countercyclical variance for the shocks e.g. Storesletten et al. (2004). Recent work with administrative data has instead found support for a different specification. In particular, Guvenen et al. (2014) find that it is the left-skewness of shocks that is strongly cyclical. For our purposes downside risk is likely to be what matters most and their evidence supports strong countercyclicality of this risk.

[21] They also exploit the tractability afforded by zero liquidity. However, their quantitative approach still requires solving their full model numerically and they do not express the aggregate Euler relation featuring only aggregate consumption and interest rates as done here.

same results are then guaranteed to hold as long as λ or ψ are not too large, since this guarantees that uncertainty rises when aggregate income falls.

4 Positive Liquidity

Incomplete markets are likely to have more bite when liquidity is scarce, since this makes it harder for households to smooth transitory income fluctuations or cushion permanent labor income shocks using their accumulated wealth. Thus, the extreme case of zero liquidity studied in the previous section isolates an important case.

Nevertheless, there are good reasons to study situations with positive liquidity. First and foremost, for realism's sake, since modern economies offer nontrivial access to credit and assets. In addition, liquidity may not only be positive but fluctuate. Indeed, below I uncover a role for the cyclicality of liquidity in the sensitivity of aggregate consumption to interest rates.

4.1 'As If' Representative Agent Result: Acyclical Liquidity

Obtaining sharp results is more challenging with positive liquidity because the allocation no longer coincides with autarky. Liquidity allows agents to smooth their consumption and the resulting equilibrium allocation is nontrivial, as in the general equilibrium studies of income fluctuations problems (e.g. Huggett, 1993; Aiyagari, 1994). Despite these challenges, I now show that with logarithmic utility aggregate implications can be worked out. Indeed, I obtain a standard representative agent Euler equation condition.

Utility is given (14) with $\sigma = 1$ so that the utility function is logarithmic

$$U(c) = \log(c).$$

As before household labor income satisfies (15), so that it is proportional to aggregate income. It then follows from identity (6) that

$$D_t(Y_t) = \tilde{d}_t \cdot Y_t,$$

for some $\{d_t\}$, so that dividends are also proportional to total income. Finally, we also assume that borrowing constraints are proportional to aggregate income

$$B^i_t(s, Y) = \tilde{B}^i_t(s)Y, \tag{17}$$

for some function $\tilde{B}_t^i(s)$.

When $\tilde{d}_t = 0$ and $\tilde{B}_t^i(s) = 0$ we are back to the case zero liquidity studied in the previous section. We say there is positive liquidity if the asset's dividend is positive or if borrowing is allowed, if $\tilde{d}_t > 0$ or $\tilde{B}_t^i(s) > 0$ for some t and s.[22]

Revaluation Effects. As it turns out, when studying comparative statics of equilibria with given initial conditions, the initial portfolio held by households matters. This is due to the asset revaluation effects that occur when interest rates are changed. Indeed, at $t = 0$ the budget constraint features an expression for initial wealth given by $q_0 a_0^i + R_- b_0^i$. The interest rate R_- is predetermined and fixed, but q_0 is endogenous. Changes in q_0 affect the value of initial wealth, impacting households differentially. This revaluation channel is the focus of Auclert (2015), who gives a detailed analysis of the different possibilities.

In the present paper, these revaluation effects are present, but play out in the background. I will consider two cases depending on the initial asset and bond holdings. In the first case, the revaluation effect is proportional to aggregate income, leading to the standard Euler representation.

4.1.1 Zero Initial Bond Holdings

Recall that, along an equilibrium, bonds and assets are perfect substitutes and households are indifferent to borrowing and saving in one or the other. As a result, the equilibrium is indeterminate and for any equilibrium with $b_t^i \neq 0$ there is another equilibrium with $b_t^i = 0$ with identical interest rates and allocations.

It is convenient to consider first the case where initial bond holdings are zero for all households,

$$b_0^i = 0.$$

Formally, the initial distribution Λ_0^i has full mass over $b_0^i = 0$. Note that this initial condition does *not* rule out or even constrain borrowing. It restricts initial bonds holdings to be zero, but as long as $\tilde{B}_t^i(s) > 0$ future borrowing and saving in bonds, $b_t^i \neq 0$ for $t > 0$, is permitted. Indeed, even initial indebtedness is possible if it takes the form of negative positions in the asset, so that $a_0^i < 0$.

The next result shows the under these conditions the standard Euler equation characterizes equilibrium aggregate responses to changes in interest rates.

[22]Strictly speaking, when $d_t = 0$ and $\tilde{B}_t^i(s) > 0$ for some s, then in some cases the equilibrium coincides with autarky. This is the case if borrowing is only allowed for the agent setting the interest rate.

Proposition 5. *Suppose utilities satisfy (14) with $\sigma = 1$, household income satisfies (15) and borrowing constraints satisfy (17). In addition, suppose initial bond holdings are zero $b_0^i = 0$ for all households. Then $\{C_t, R_t\}$ is part of an equilibrium if and only if*

$$U'(C_t) = \beta_t R_t U'(C_{t+1}) \tag{18}$$

for some sequence of discount factors $\{\beta_t\}$, independent of both $\{R_t\}$ and $\{C_t\}$.

Proposition 2 applied with zero liquidity where the outcome was financial autarky. In contrast, Proposition 5 applies to situations with positive liquidity with nontrivial allocations that depart from financial autarky, with households attempting to smooth their consumption by saving and borrowing. Indeed, the equilibrium may be nonstationary and involve rich dynamics, as in the deleveraging episodes modeled by Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2011) where borrowing constraints suddenly tighten and upset the original steady state, inducing a slow transition.

How is it that all these nontrivial household decisions under uncertainty can be subsumed in as simple an aggregate relation such as (18)? Indeed, the underlying household allocation cannot typically be solved in closed form—it is well known that incomplete market models demand a numerical solution approach. The key innovation behind Proposition 5 is to change the question one seeks to answer, so as to avoid having to solve all equilibria. Instead, I turn to the more manageable problem of characterizing how a given reference equilibrium adjusts to changes in interest rates. Although the household allocations in the reference equilibrium are nontrivial, across equilibria with different interest rates, household consumption scales up and down proportionally with aggregate consumption. Obtaining the sequence of discount factors $\{\beta_t\}$ requires computing the reference equilibrium, one with constant output. In general this may be a demanding task—as demanding as any of the existing equilibrium analysis of incomplete market models, e.g. Guerrieri and Lorenzoni (2011).

To see this more clearly, it is useful to spell out the proof for the result in some detail. The discount factors β_t and the household allocation behind this result are obtained as follows. Consider an economy with constant income $\tilde{Y}_t = 1$ and its associated equilibrium, including the interest rate $\{\tilde{R}_t\}$, household consumption and wealth $\{\tilde{c}(s^t; a_0), \tilde{a}(s^t; a_0)\}$ and asset prices $\tilde{q}_t = \sum_{s=0}^{\infty} (\tilde{R}_t \tilde{R}_{t+1} \cdots \tilde{R}_{t+s})^{-1} \tilde{d}_{t+1+s}$. I call this equilibrium the reference equilibrium. Defining $\beta_t \equiv \frac{1}{\tilde{R}_t}$ the aggregate Euler equation (18) holds for $C_t = Y_t = 1$.

Now, for any other sequence $\{C_t, R_t\}$ satisfying the aggregate Euler equation (18), construct the equilibrium objects as follows. Rescale household consumption and wealth proportionally to aggregate consumption

$$c^i(s^t;a_0) = \tilde{c}^i(s^t;a_0)C_t \qquad \text{and} \qquad \hat{a}^i(s^t;a_0) = \tilde{a}^i(s^t;a_0)C_t$$

and adjust the interest rate and asset prices by

$$R_t = \tilde{R}_t \frac{C_{t+1}}{C_t} \qquad \text{and} \qquad q_t = \tilde{q}_t C_t.$$

With this guess, one can verify all equilibrium conditions. The crucial observation is that the Euler equation, the budget constraints and borrowing constraints are all linear homogeneous in $Y_t = C_t$. To see this, start with the Euler equation for the reference equilibrium

$$U'(\tilde{c}^i(s^t;a_0)) \geq \beta \tilde{R}_t \mathbb{E}[U'(\tilde{c}^i(s^{t+1};a_0))],$$

using the fact $\beta_t = \frac{1}{\tilde{R}_t}$ and that $U'(x) = U'(xy)\frac{1}{U'(y)}$ for any $x, y > 0$ we can write

$$U'(\tilde{c}^i(s^t;a_0)C_t)\frac{1}{U'(C_t)} \geq \beta \frac{1}{\beta_t U'(C_{t+1})} \mathbb{E}[U'(\tilde{c}^i(s^{t+1};a_0)C_{t+1})],$$

then using the fact that $U'(C_t) = \beta_t R_t U'(C_{t+1})$ we obtain

$$U'(c^i(s^t;a_0)C_t) \geq \beta R_t \mathbb{E}[U'(c^i(s^{t+1};a_0))].$$

Start with the budget constraint in period $t = 1, 2, \ldots$ for the reference equilibrium

$$\tilde{c}^i(s^t;a_0) + \tilde{a}^i(s^t;a_0) = \tilde{\gamma}^i_t(s_t) + \tilde{R}_{t-1}\tilde{a}^i(s^{t-1};a_0),$$

multiply by C_t to obtain

$$c^i(s^t;a_0) + \hat{a}^i(s^t;a_0) = \tilde{\gamma}^i_t(s_t)C_t + \tilde{R}_{t-1}\frac{C_t}{C_{t-1}}\hat{a}^i(s^{t-1};a_0),$$

using $U'(c) = \frac{1}{C}$ and $U'(C_t) = \beta_t R_t U'(C_{t+1})$ this is equivalent to

$$c^i(s^t;a_0) + \hat{a}^i(s^t;a_0) = \tilde{\gamma}^i_t(s_t)C_t + R_{t-1}\hat{a}^i(s^{t-1};a_0).$$

Finally, the budget constraint at $t = 0$ for the reference equilibrium is

$$\tilde{c}^i(s_0;a_0) + \tilde{a}^i(s_0;a_0) = \tilde{\gamma}^i_0(s_0) + (\tilde{q}_0 + \tilde{d}_0)\tilde{a}^i_0,$$

multiply by C_0 to obtain

$$c^i(s_0;a_0) + \hat{a}^i(s_0;a_0) = \tilde{\gamma}^i_0(s_0)C_0 + (q_0 + d_0)a^i_0.$$

One also verifies that the borrowing constraints are satisfied, since they are homogenous of degree one. Finally, starting from $\tilde{q}_t = \sum_{s=0}^{\infty}(\tilde{R}_t\tilde{R}_{t+1}\cdots\tilde{R}_{t+s})^{-1}\tilde{d}_{t+1+s}$ and using that $R_t = \tilde{R}_t\frac{C_{t+1}}{C_t}$ one arrives at $\tilde{q}_tC_t = q_t = \sum_{s=0}^{\infty}(R_tR_{t+1}\cdots R_{t+s})^{-1}\tilde{d}_{t+1+s}C_{t+1+s}$. Thus, we have verified all the conditions for an equilibrium.

Does the Level of Liquidity Matter? Yes and no. As Proposition 5 makes clear, the level of liquidity has absolutely no effect on the response of consumption to current and future interest rates. Indeed, the response is identical to that of a representative agent.

However, the level of liquidity is important at the microeconomic level. Greater liquidity allows better consumption smoothing and affects the discount factors β_t in the Euler representation (18). This in turn has an aggregate effect in levels, affecting the consumption path for a given interest rate path. However, the level of liquidity does not affect aggregate responses to changes in the interest rate path.

Discounting and Natural Interest Rates. When primitives are stationary (i.e. s_t is a Markov process) and the economy is initialized with an invariant distribution for (s_0, a_0) then the discount factor is constant $\beta_t = \frac{1}{\tilde{R}}$, where $\tilde{R}$ is the steady state interest rate. As shown by Huggett (1993) and Aiyagari (1994) steady states require $\beta\tilde{R} < 1$, implying that a higher discount factor appears in the representation, compared to the true subjective discount factor.

In general, the discount factors β_t may not be constant, but this is a feature, not a bug. For example, periods of higher idiosyncratic uncertainty are likely to increase β_t temporarily. Likewise, deleveraging episodes—situation where some household start with high initial debt but must lower their debt over time—as modeled, for example, by Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2011), may also increase the discount factor temporarily. A higher discount factor β_t is associated with a lower natural interest rate, in the New Keynesian model. Indeed, the economy may be pushed up against the zero interest rate bound if β_t is high enough, as stressed by the liquidity trap literature.

Overlapping Generations. Just as in the case with zero liquidity, Proposition 5 can be applied to overlapping-generation frameworks. Each generation, born in a particular period, can be modeled as a distinct household type, that then values consumption over different time intervals, i.e. while alive. This can be captured by setting $\theta_t^i = 1$ while alive and $\theta_t^i = 0$ otherwise. This immediately implies that the results in Del Negro et al. (2015), which assume logarithmic utility, are not driven by overlapping generations per se, nor by the relative myopia of shorter horizons, nor the higher discounting that overlapping generation models may entail.[23] Instead, Proposition 5 and the results below suggest that the key determinant is the interaction of overlapping generations with the countercyclical behavior of monopolistic profits relative to income in standard New Keynesian models.

4.1.2 Arbitrary Initial Bond Holdings

I now allow any initial distribution for initial bonds holdings, b_0^i. The main result is a similar characterization to that obtained when initial bonds holdings are zero, except for an adjustment to the discount factors that depends on initial consumption.

Proposition 6. *Suppose utilities satisfy (14), household income satisfies (15) and borrowing constraints satisfy (17). Then $\{C_t, R_t\}$ is part of an equilibrium if and only if*

$$U'(C_t) = \beta_t R_t U'(C_{t+1}) \tag{19}$$

for some a sequence of discount factors $\{\beta_t\}$ that depends on C_0, i.e. $\beta_t = \hat{\beta}_t(C_0)$.

According to this result, a standard Euler equation continues to hold, except that now the relevant discount factors are endogenous. Conveniently, these new effects are entirely summarized by the initial aggregate consumption level, C_0. Thus, the aggregate dynamics for consumption remains tractable and tied to a standard Euler equation.

Why are the discount factors dependent on initial consumption? When output expands in the first period this diminishes the relative value of bonds. In contrast, the asset price and dividends expand proportionally with output. Indeed, a simple variation of the previous argument shows that the discount factors are now precisely $\beta_t = \frac{1}{\tilde{R}_t}$ where $\tilde{R}_t$ is the equilibrium interest rates for a reference economy with constant output and initial bond holdings rescaled to $\tilde{b}_0^i = \frac{1}{C_0} b_0^i$. This works since then the budget constraint at $t = 0$ for the reference equilibrium is

$$\tilde{c}^i(s_0; a_0) + \tilde{a}^i(s_0; a_0) = \tilde{\gamma}_0^i(s_0) + (\tilde{q}_0 + \tilde{d}_0)\tilde{a}_0^i + R_- \tilde{b}_0^i,$$

[23] Indeed, the key feature of overlapping generation models is new generations being born, rather than shorter life spans and older generations passing away.

which then implies

$$c^i(s_0; a_0) + \hat{a}^i(s_0; a_0) = \tilde{\gamma}_0^i(s_0)C_0 + (q_0 + d_0)a_0^i + R_- b_0^i.$$

Implications. How is the dependence of β_t on C_0 likely to play out? What are the implications of this adjustment? I offer a tentative, but informed, speculation.

As I just argued, a higher C_0 diminishes the dispersion in bond holding levels relative to income, scaling it down towards zero. For given interest rates, it is reasonable to expect lower dispersion in initial bond holdings to increase consumption (since consumption functions are concave as a function of wealth), at least in earlier periods. This pushes the interest rates $\tilde{R}_t$ upward to reestablish equilibrium with constant unitary output. This argument suggests that the discount factors $\beta_t = \frac{1}{\tilde{R}_t}$ decrease with C_0. The effects on β_t are likely to die out in later time periods t, since $\tilde{R}_t$ is likely to converge to a common steady state $\tilde{R}$, regardless of initial conditions.

If, as seems likely, discount factors are decreasing in consumption, then interest rate paths that increase consumption, e.g. lower interest rates, are likely to have an additional effect, especially in earlier periods. In other words, non-zero initial bond holdings are likely to amplify the effects of interest rate changes.

4.2 Departures from 'As If' Result: Procyclical Liquidity

Away from the conditions required by Proposition 5 no general aggregation result appears available. This makes studying the effects of interest rate changes on aggregate demand challenging. Fortunately, it is possible to work out some simple cases analytically using a three period economy. The results shed light on what to expect when departing from Proposition 5.

A Three Period Economy. Consider an economy with three period labeled $t = 0, 1, 2$. It is best to think of period $t = 2$ as collapsing the entire future in an infinite horizon setting. Households are of a single type with utility functions

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}.$$

There is no uncertainty at $t = 0$ nor at $t = 2$, so that $y_0 = (1 - d_0)Y_0$ and $y_2 = (1 - d_2)Y_2$. At $t = 1$ households experience an idiosyncratic shock s_1 with c.d.f. F and receive income $y_1 = s_1(1 - d_1)Y_1$. By implication, the asset's dividend equals $D_t(Y_t) = d_t Y_t$. For simplicity set $d_0 = d_1 = 0$. All households are initially endowed with one unit of the asset and cannot borrow in any period, so that $B_t(s) = 0$.[24]

Since households are initially identical at $t = 0$, in equilibrium they all hold on to the asset into period $t = 1$. Households then observe their temporary income shock. Those with a negative enough shock sell some or all of their asset position, while those with a positive enough shock buy too add to their asset position. This helps households smooth consumption between $t = 1$ and $t = 2$, reducing the dispersion of consumption at $t = 1$.

Sensitivity to Interest Rates. I now turn to the sensitivity of aggregate consumption with respect to changes in interest rates. To do so, fix aggregate income in the last period at a constant Y_2 and solve for aggregate consumption in earlier periods C_1 and C_0 as functions of R_0 and R_1.[25] My results are expressed in terms of the elasticities of these functions.

I first consider the case where no household finds itself liquidity constrained. That is, where no household sells off entire asset holdings.

Proposition 7. *Consider the three-period economy. Suppose no household is liquidity constrained. If $\sigma \geq 1$ then*

$$\frac{d \log C_0}{d \log R_1} \leq \frac{d \log C_1}{d \log R_1} = \frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma};$$

with equality if $\sigma = 1$ and strict inequality with $\sigma > 1$; the inequality is reversed if $\sigma < 1$. Finally, $\frac{d \log C_1}{d \log R_0} = 0$.

The effect of the interest rate on current spending at $t = 1$ is standard, exactly as in the representative agent case: a one percent drop in the interest rate R_1 leads to a rise in consumption of $\frac{1}{\sigma}$ percent. The effects on earlier periods, however, depends on the value of σ. When $\sigma > 1$ this interest rate drop makes consumption at $t = 0$ rise by a greater percentage; when $\sigma < 1$ the effect is smaller. In the logarithmic case $\sigma = 1$, which was already covered by Proposition 5, consumption rises by the same percentage in both periods, i.e. the standard consumption-smoothing property.

Underlying these results and their dependence on σ is the behavior of the equilibrium asset price relative to income. The asset price affects the degree of liquidity in the market

[24] Identical results obtain with borrowing instead of outside assets if the borrowing constraints are specified as a present value constraint, a common specification in the literature. In particular, assuming the borrowing limit $B_1 = \frac{1}{R_1} d_2 Y_2$ and that all income is labor income $y_2 = Y_2$, then d_2 can be interpreted as the pledgeable fraction of labor income. All the results below apply to this formulation.

[25] Consumption in all periods scales proportionally with Y_2 given R_0 and R_1. In a monetary economy a motivation for fixing Y_2 is to assume that prices are flexible at $t = 2$ or that monetary policy achieves the flexible price equilibrium then; in this case, output is at the flexible price equilibrium in the long run at $t = 2$ and the analysis focuses on aggregate demand in the short and medium run $t = 0, 1$.

and the ability of households to smooth consumption in the intermediate period. To see this, note that the asset price at $t = 1$ rises one for one with the drop in R_1, while output rises by $\frac{1}{\sigma}$ of the drop in R_1. Thus, when $\sigma > 1$ the asset value rises relative to income. As a result, in this case, consumption at $t = 1$ becomes less sensitive to the current idiosyncratic income shock. Consumption at $t = 0$ then rises for two reasons: because of the higher level of consumption at $t = 1$ as well as the lower consumption uncertainty at $t = 1$. When $\sigma < 1$ this effect is reversed, since consumption at $t = 1$ expands more than the asset value.

Note that σ plays a role only through its indirect implication for the equilibrium asset price relative to income. Indeed, if the asset price were proportional to income regardless of σ, then the 'as if' result obtains for all σ. Similarly, if relative liquidity is procyclical, then consumption at $t = 0$ becomes more sensitive to consumption at $t = 1$ and future interest rates. Proposition 9 below liberates the asset price from the risk-free present value condition to establish these results.

This result underscores the importance of the response of asset prices to interest rate changes. It is worth clarifying that for any value of σ, regardless of whether $\sigma > 1$, $\sigma < 1$ or $\sigma = 1$, there are revaluation effects working through in the asset price. However, whether or not revaluation effects increase or decrease the sensitivity of consumption to interest rate changes depends on σ since this determines the relative strength of the standard substitution channel. What is relevant is whether the asset value rises *relative* to income.[26]

I now consider the case where some households sell off all their assets at $t = 0$. These households are liquidity constrained and their Euler equation holds with strict inequality.

Proposition 8. *Consider the three-period economy and suppose some households are liquidity constrained at $t = 1$. If $\sigma \geq 1$ then*

$$\frac{d \log C_0}{d \log R_1} \leq \frac{d \log C_1}{d \log R_1} \leq \frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma};$$

with equality if $\sigma = 1$ and strict inequality with $\sigma > 1$; the inequality is reversed if $\sigma < 1$. Finally, $\frac{d \log C_1}{d \log R_0} = 0$.

When some households are liquidity constrained, the asset value relative to income determines the degree to which this constraint binds. When $\sigma > 1$, the asset value rises

[26]The revaluation effects I focus on here work through outside assets that are in net positive supply with their own stream of dividends. Thus, these effects are distinct from the redistribution effects between creditor and debtors focused on in Auclert (2015) or Sheedy (2014).

relative to income, and this rise in value has a direct one-for-one impact on the consumption of constrained households, amplifying the effect of R_1 on C_1.

4.3 Ad Hoc Asset Price

The crucial feature determining the sensitivity of aggregate consumption to interest rates is the cyclical behavior of the asset price relative to income. The role played by the preference parameter σ is in this sense indirect. In reality, one expects many factors affecting the behavior of asset prices, such as risk premia, absent here. To explore this point in a simple way, I now liberate the asset pricing condition.

I continue to assume that the Euler equation holds for the riskless bond, but assume that outside asset is not priced the same way, so that we no longer have $q_1 = \frac{1}{R_1} d_2 Y_2$. Indeed, I postulate that

$$q_1 = \bar{q} d_2 Y_1^{\phi}, \tag{20}$$

for some $\bar{q} \geq 0$ and $\phi \in \mathbb{R}$. This can be accommodated by assuming that the discount factor $\hat{\beta}_1(Y_1, Y_2, R_1)$ applied to the asset satisfies

$$q_1 u'(c_1(s)) = \hat{\beta}_1(Y_1, Y_2, R_1) R_1 u'(c_2(s)) d_2 Y_2.$$

There are many functions $\hat{\beta}_1$ that give the desired price (20). In the context of the present model assuming the asset is priced in this way is, no doubt, ad hoc. Extensions of the model, perhaps with uncertainty and risk premia, could perhaps provide a justification, but delivering microfoundations is beyond the scope of the present inquiry. Instead, for present purposes it is simply interesting and useful to liberate oneself from the risk-free pricing condition $q_1 = \frac{1}{R_1} d_2 Y_2$. The point is not to have a full model, but rather a suggestive one that disentangles the effects of σ and the behavior of the asset price relative to income.

For simplicity I limit myself to situations where agents are not liquidity constrained, providing the analog to Proposition 7.

Proposition 9. *Consider the three-period economy and suppose no household is liquidity constrained. Assume the discount factor on the outside asset is such that (20) holds. If $\phi \geq 1$ then*

$$\frac{d \log C_0}{d \log R_1} \leq \frac{d \log C_1}{d \log R_1} = \frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma};$$

with equality if $\phi = 1$ and strict inequality if $\phi > 1$. If $\phi > 1$ the strict inequality is reversed.

When the asset's price relative to income is acylical then the 'as if' result holds for any value of σ. Also, when the asset price is procyclical relative to income, then aggregate consumption becomes more sensitive to future aggregate consumption and interest rates.

5 Labor Markets, Nominal Rigidities and Supply Side

Up to this point I have specified labor income and dividends directly, taking them to be a function of aggregate income. This enabled me to focus on consumption-saving decisions, leaving aside household labor supply choices and firm decisions—the supply side. One benefit of this approach is that it isolates the demand side, without the need to make particular choices for the supply side. Another benefit is that, by placing conditions directly on income and dividend processes, results are more easily stated and interpreted. Nevertheless, it is of interest to spell out a few possibilities for how the demand side fits in with the supply side. To this end, I consider a few variants of the New Keynesian model. Since this is a well-known model, I will omit the details and focus on elements that determine the cyclicality of income and liquidity.

Common Elements: Preferences and Technology. Two specifications for utility will be useful. The first has additively separable between consumption and labor

$$\sum_{t=0}^{\infty} \beta_{i,t} \mathbb{E}_0[u_t^i(c(s^t), s_t) - v_t^i(n(s^t), s_t)], \tag{21a}$$

the second assumes quasilinearity to rule out income effects for labor

$$\sum_{t=0}^{\infty} \beta_{i,t} \mathbb{E}_0[u_t^i(c(s^t) - v_t^i(n(s^t), s_t), s_t)]. \tag{21b}$$

For both cases, assume the disutility function takes the form

$$v_t^i(n, s) = \bar{v}^i(s) \frac{n^{1+\gamma_t}}{1+\gamma_t}$$

with $\gamma_t > 0$. Households are also subject to productivity shocks, so that when working $n(s^t)$ they supply $N(s^t) = a(s_t)n(s^t)$ effective units of labor.

There is a single final good that is produced competitively, combining a continuum of intermediate varieties with capital according to the constant returns production function

$$Y = A \left(\int \lambda(j) Y(j)^{1-\frac{1}{\varepsilon}} dj \right)^{\frac{1-\alpha}{1-\frac{1}{\varepsilon}}} K^{\alpha},$$

for $\varepsilon > 1$ and $\alpha \in [0,1)$, where A is total factor productivity, $\lambda(j)$ shifts the productivity and demand for different varieties; in the expression above, we omitted their dependence, but all the parameters may fluctuate with time and the demand shifters for varieties may be subject to idiosyncratic shocks. Since the final good is produced competitively with a constant returns technology, profits are zero. Capital is in fixed unitary supply and held by households. Note that given the Cobb-Douglas specification the returns to capital equals a fraction α_t of total output.

Each variety i is produced one-for-one from effective units of labor. The production of varieties is done by specialized firms that may earn profits.

Next I describe a few different assumptions on the way good and labor markets can be arranged.

Yeomen Farmers. In the first and simplest variant, intermediate goods are produced by households themselves, just as in the "yeomen farmer" specification in Blanchard and Kiyotaki (1987) and Ball and Romer (1990). Since households own the production of a variety, there is no clear distinction between labor earnings and monopolistic profits. The outside asset is capital and its dividends equal the returns to capital.

Each household produces a particular variety exclusively. Thus, there is no labor market, or alternatively, the labor market coincides with the goods market and intermediate goods can be reinterpreted as specialized labor. The price/wage of each variety is sticky and households supply everything demanded at the going price. For simplicity, consider the extreme case where prices/wages are completely rigid.

Household income is a function of aggregate demand and idiosyncratic demand for their own variety, which may fluctuate according to the household's idiosyncratic state s_t^i. Income in this case does not depend on household labor productivity.

Sticky Wages and Rationing in the Labor Market. Assume now that intermediate varieties are produced competitively by hiring labor in a centralized labor market. Prices are flexible and set to marginal cost, so profits are zero. Again, the outside asset is capital and its dividends equal the returns to capital.

The nominal wage is sticky and labor is rationed. Since the wage is sticky, marginal costs are sticky; thus, prices inherit this stickiness. The exact nature of the nominal rigidity is unimportant for our purposes, but, for simplicity, assume wages are always at a level where labor supply is greater than labor demand, so that labor is demand constrained. Then the aggregate demand for labor is determined by the aggregate demand for goods. When aggregate demand is low, labor demand is low and labor is rationed across households. Various rationing mechanisms are possible. One option is to ration labor hours proportionally across households. In this case, earnings fluctuate with household's current idiosyncratic productivity shock. Another option is to also ration along an extensive employment margin, potentially leading to outcomes similar to those described in Section 4.2. In this case, household income depends on the idiosyncratic productivity shock and also on the household's luck of the draw in the rationing mechanism.

An alternative to a rationing mechanism is to model costly search and matching, as in Hall (2005) and Ravn and Sterk (2012), among others.

Sticky Prices and Flexible Labor Market. The standard New Keynesian model assumes that each intermediate variety is produced by a single firm that acts monopolistically, setting the price and maintaining it for some time due to some nominal rigidity, meanwhile meeting demand at the posted price. These firms are assumed to hire labor in a centralized competitive labor market with a flexibly determined wage.

Monopolistic firms earn profits equal to the profit margin—the difference between price and marginal cost—multiplied by the quantity sold. When demand rises, real wages must rise to increase the supply of labor, squeezing profit margins. This makes monopolistic profits relative to total income countercyclical; by implication, labor income relative to total income becomes procyclical.

In the basic New Keynesian model, depending on the ownership structure assumed for these profits, this can provide a force for procyclical income risk or countercyclical liquidity. For example, McKay et al. (2015) assumes monopolistic profits are evenly distributed across all households, which provides a force for procyclical income risk. Del Negro et al. (2015) assumes instead that the ownership of monopolistic profits are a tradable asset, which provides a force for countercyclical liquidity.

Within the confines of the basic New Keynesian model, one alternative way to distribute monopolistic profits is to assume that they are taxed 100% by the government and the revenue is rebated back to household by way of a labor subsidy. This assumption is not uncommon in the literature and has the virtue of achieving the efficient level of output and labor, since the labor subsidy effectively undoes the monopolistic markup distortion. Since profits are fully taxed, the only outside asset with positive dividend is then capital. Labor earnings are a function of the economy-wide wage and the supply decision of households.

Consider first the case case with zero liquidity, we can then solve for the static labor supply decision for each household as a function of the wage. This then implies an aggregate level of labor and, hence, output. This provides a relation between aggregate income and individual household earnings, providing a construction of the γ_t^i function. One exception occurs when utility over consumption is logarithmic and preferences are additively separable as in (21a), since then labor supply is constant for all wage levels.[27] However, if preferences are quasilinear as in (21b) then logarithmic utility poses no challenge.[28] The same arguments apply in the case of positive liquidity. In particular, Proposition 5 applies in this case.

All these possibilities not withstanding, in basic New Keynesian models, countercyclical profits relative to income are an unintended artifact of simplifying assumptions, such as the absence of fixed costs, labor hoarding, rigid wages and other realistic features. Extensions along these lines are likely to make profits relative to income procyclical and more in line with the facts, as already noted by Blanchard and Kiyotaki (1987, Section IV). Given the important role highlighted by my analysis for the cyclicality of monopolistic profits it may be important to model these other features to get a more realistic behavior for profits.

6 Exact Aggregation in a Real Business Cycle Model

Up to this point, I have considered economies with a fixed or mechanical supply of liquidity.[29] In addition, consumption has been assumed to be the only component of demand, since the model lacked investment. Finally, although the results apply more generally and make no explicit assumptions regarding nominal rigidities, their natural application is the study of monetary policy in the presence of nominal rigidities.

The purpose of this section is to show that the results can be generalized and applied in a very different setting. To this end, I now consider a Real Business Cycle model where capital is accumulated by investment and as the outside asset. This real economy is assumed to operate under flexible prices and perfect competition.

[27] A similar possibility is discussed at length in Broer, Krusell, Hansen and Oberg (2015) without the taxation of profits for a two class economy with workers and capitalists.

[28] In this case, to relate it to our previous notation it is useful to define consumption net of disutility of labor $c_t^i - v_t^i$ and define aggregate income as the sum of earnings in the labor market and "household production", v_t^i.

[29] Even with a fixed supply of assets, liquidity is arguably not exogenous, since dividends are a function on aggregate income, an endogenous variable, and the asset price is determined endogenously.

My main result assumes utility is logarithmic and supposes full depreciation of capital, a well-known case referred to as Brock-Mirman. Under these conditions, aggregate dynamics behave exactly as those obtained in representative agent or complete market version of the model. This occurs despite potentially arbitrarily large departures at the microeconomic household level in the allocation.

Krusell and Smith (1998) studied a similar real business cycle model numerically, but without the restriction to full depreciation. Their main conclusion was approximate aggregation. My result complements theirs, providing conditions for exact aggregation.

6.1 Economic Environment

The economic environment is a standard real business cycle model, augmented to include idiosyncratic uncertainty and incomplete markets. Because most of it is well known, I will keep the description brief.

Preferences. All households have utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t - v(n_t)\right)$$

where

$$v(n) = \bar{v}\frac{n^{1+\gamma}}{1+\gamma}$$

with $\bar{v} > 0$ and $\gamma \geq 0$.

Uncertainty. Agents are subject to idiosyncratic productivity shocks so that if they work n_t^i they supply $z_t^i \cdot n_t^i$ to the market. Denote the history of idiosyncratic shocks by z^t, and the history of aggregate shocks by s^t. Let Λ_t denote the cross sectional distribution of history of shocks for households of type i.

Technology. Output is given by a constant returns to scale Cobb-Douglas production function,

$$Y(s^t) = A(s_t)F(K(s^{t-1}), N(s^t)) = A(s_t)K(s^{t-1})^{\alpha}N(s^t)^{1-\alpha}.$$

The resource constraint is

$$C(s^t) + K_{t+1}(s^t) \leq A(s_t)F(K_t(s^{t-1}), N_t(s^t)) + (1-\delta)K(s^t).$$

We shall focus on the case with $\delta = 1$.

Budget Constraints. Households are subject to the budget constraints

$$c^i(z^t, s^t) + k^i(z^t, s^t) \leq W_t(s^t) n^i(z^t, s^t) + k^i(z^{t-1}, s^{t-1}) R_t(s^t)$$

To simplify, I have assumed no borrowing. The results are robust to the introduction of borrowing, as long as the borrowing constraints are proportional to output.

Equilibrium Conditions. The equilibrium conditions are standard and we relegate the details to the appendix. Households maximize utility subject to the budget constraints. Aggregate variables must be consistent with individual household choices. Firms maximize. Finally, the market for goods (consumption and investment), the market for labor and the market for capital must clear.

6.2 'As if' Representative Agent Result

The next proposition states the main result of this section.

Proposition 10. *Consider the Real Business Cycle model with $\delta = 1$. Then the aggregate dynamics of capital and labor are equivalent to their counterparts in the complete market, or representative agent economy. Namely,*

$$\begin{aligned} C(s^t) &= (1 - \omega_t) \, Y(s^t) \\ K(s^t) &= \omega_t Y(s^t) \\ N(s^t) &= \bar{N}_t, \end{aligned}$$

for some deterministic sequence of saving rates $\{\omega_t\}$ and labor $\{\bar{N}_t\}$. Saving rates ω_t are constant if the initial distribution of wealth is at an invariant steady state.

Individual household consumption and labor are given by

$$\begin{aligned} c^i(z^t, s^t) &= c^i(z^t) C(s^t), \\ n^i(z^t, s^t) &= n^i(z^t) N(s^t), \end{aligned}$$

where $c^i(z^t)$ and $n^i(s^t)$ is computed from an incomplete market equilibrium for a normalized economy with $C(s^t) = 1$ and $N(s^t) = 1$.

7 Conclusion

This paper studied the effect of financial market imperfections on aggregate consumption. Idiosyncratic uncertainty, incomplete markets and borrowing constraints affect the level of aggregate demand. As expected, greater uncertainty or tighter borrowing constraints tend to depress consumption.

However, my results show that the sensitivity of aggregate consumption to interest rates may not be affected by these financial frictions. Indeed, the responses may be identical to those in a frictionless representative-agent model. To determine these elasticities, my analysis isolates two key players: the cyclicality of income risk and the cyclicality of liquidity. When income risk is countercyclical and liquidity is procyclical aggregate consumption becomes more sensitive to current and, especially, future interest rates. More generally, my results suggest that it is not incomplete markets *per se* that determines the sensitivity of demand to interest rates, but, rather, the interaction of incomplete markets with other auxiliary assumptions.

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A Asymmetric Intensive Margin

Assume now that income equals $\bar{y}Y^{\psi}$ with probability $1-\lambda$ and, otherwise, income is fixed at $\underline{y}$. Define $\lambda(Y)$ from the identity

$$Y \equiv (1-\lambda(Y))\bar{y}Y^{\psi} + \lambda(Y)\underline{y},$$

and consider the range of Y is such that $\lambda(Y) \in (0,1)$ and $\bar{y}Y^{\psi} > \underline{y}$. Note that since $\psi < 1$ then $\lambda(Y)$ is decreasing.

The Euler equation becomes

$$U'(\bar{y}Y^{\psi}) = \beta R\left((1-\lambda(Y'))U'(\bar{y}Y'^{\psi}) + \lambda(Y')U'(\underline{y})\right),$$

where $U'(c) = c^{-\sigma}$. Equivalently

$$U'(Y^{\psi}) = \hat{\beta}(Y')RU'(Y'^{\psi})$$

where

$$\hat{\beta}(Y) \equiv \beta\Bigg((1-\lambda(Y) + \lambda(Y)\frac{1}{U'(\frac{\bar{y}}{\underline{y}}Y^{\psi})}\Bigg).$$

Differentiating and rearranging,

$$\hat{\beta}'(Y) = \beta\frac{\lambda(Y)}{YU'(\rho)}\sigma\psi\left(\frac{-1}{\psi\sigma}\frac{Y\lambda'(Y)}{\lambda(Y)}\left(U'(\rho)-1\right)+1\right) \tag{22}$$

where $\rho \equiv \frac{\bar{y}}{\underline{y}} Y^{\psi}$ is relative income in the two states.

Differentiating the identity defining $\lambda(Y)$ one obtains

$$\frac{-1}{\sigma\psi} \frac{Y\lambda'(Y)}{\lambda(Y)} = \frac{1}{\psi} \left((1-\psi) \frac{1-\lambda(Y)}{\lambda(Y)} \rho + 1 \right) \frac{1}{\sigma} \frac{1}{\frac{\bar{y}}{\underline{y}} Y^{\psi} - 1}.$$

Substituting into (22) and rearranging gives,

$$\hat{\beta}'(Y) = \beta \frac{\lambda}{YU'(\rho)} \psi\sigma \left(\frac{1}{\psi} \left((1-\psi) \frac{1-\lambda}{\lambda} \rho + 1 \right) \frac{1}{\sigma} \frac{U'(\rho) - 1}{\rho - 1} + 1 \right)$$

It follows that $\hat{\beta}'(Y) \leq 0$ if and only if

$$1 \leq \frac{1}{\psi} \left((1-\psi) \frac{1-\lambda}{\lambda} \rho + 1 \right) \left(-\frac{1}{\sigma} \frac{U'(\rho) - 1}{\rho - 1} \right). \tag{23}$$

This condition is relaxed for lower ψ and for lower λ. Indeed, it is always satisfied if either ψ or λ close to 0. Thus, for any λ there is an interval of ψ containing zero for which the condition is satisfied. Moreover, this interval expands for lower λ, and converges to the entire support $(0, 1)$ as $\lambda \to 0$.

A simpler sufficient condition is possible. Since $U'(c) = c^{-\sigma}$ is convex then

$$-\sigma\rho^{-\sigma-1} = U''(\rho) \geq \frac{U'(\rho) - 1}{\rho - 1}.$$

Thus, a sufficient condition for (23) is

$$\rho^{\sigma+1} \leq \frac{1}{\psi} \left((1-\psi) \frac{1-\lambda}{\lambda} \rho + 1 \right).$$

B Proof of Proposition 7

To recap, the economy is described by the following primitives

$$\gamma_0(Y_0) = Y_0 \qquad d_0 = 0$$

$$\gamma_1(Y_1) = sY_1 \qquad d_1 = 0$$

$$\gamma_2(Y) = (1 - d_2)Y \qquad d_2 > 0$$

where s is a random variable with c.d.f. F and $\mathbb{E}[s] = 1$. Finally, no borrowing is allowed: $B_t(s) = 0$ for all s and t. Fix Y_2 at a constant.

In equilibrium the asset price is

$$q_1 = \frac{1}{R_1} d_2 Y_2. \tag{24}$$

The present value budget constraint at $t = 1$ is

$$c_1(s) + \frac{1}{R_1}(c_2(s) - (1 - d_2)Y_2) = sY_1 + q_1. \tag{25}$$

Assuming borrowing constraints do not bind, the Euler equation between $t = 1$ and $t = 2$ gives

$$c_1(s) = (\beta R_1)^{-\frac{1}{\sigma}} c_2(s). \tag{26}$$

Substituting (24) and (26) into (25) gives

$$c_1(s) + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma}-1} c_1(s) = sY_1 + \frac{1}{R_1} Y_2.$$

Solving gives

$$c_1(s) = \frac{sY_1 + \frac{1}{R_1} Y_2}{1 + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma}-1}}.$$

Setting $C_1 = Y_1$ and integrating gives the relation

$$C_1 = \int c_1(s) dF(s) = \frac{C_1 + \frac{1}{R_1} Y_2}{1 + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma}-1}},$$

and solving for C_1 gives

$$C_1 = R_1^{-\frac{1}{\sigma}} \beta^{-\frac{1}{\sigma}} Y_2,$$

so that $\frac{d \log C_1}{d \log R_1} = -\frac{1}{\sigma}$.

Returning to individual household consumption, one observes that

$$c_1(s) = \hat{c}(s; R) C_1$$

where

$$\hat{c}(s; R_1) = \omega(R_1) s + 1 - \omega(R_1),$$

$$\omega(R_1) = \frac{\beta^{-\frac{1}{\sigma}} R_1^{1-\frac{1}{\sigma}}}{\beta^{-\frac{1}{\sigma}} R_1^{1-\frac{1}{\sigma}} + 1} \in (0,1).$$

Thus, $c(s)$ is a proportion $\hat{c}(s; R_1)$ of aggregate income C_1; for $\sigma = 1$ this proportion varies with s but is independent of the interest rate R_1. However, $\hat{c}(s; R_1)$ varies with R_1 when $\sigma \neq 1$. When $\sigma > 1$ we have ω increasing and so an increase in R_1 increases $\hat{c}$ for $s > 1$ and decreases $\hat{c}$ for $s < 1$, i.e. if $R_1' > R_1$ then $\hat{c}(\cdot; R_1')$ single crosses $\hat{c}(\cdot; R_1)$ from below at $s = 1$; this implies a mean-preserving spread in the distribution for $\hat{c}$. For $\sigma < 1$, we have ω decreasing and so the reverse is true: a decrease in R_1 implies a mean preserving spread in $\hat{c}$.

Turning to $t = 0$ we have

$$C_0 = (\beta R_0)^{-\frac{1}{\sigma}} \cdot \left(\int \hat{c}(s; R_1)^{-\sigma} dF(s) \right)^{-\frac{1}{\sigma}} C_1,$$

so that $\frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma}$. Since the marginal utility function $\hat{c}^{-\sigma}$ is convex for all $\sigma > 0$ then the expression

$$\left(\int \hat{c}(s; R_1)^{-\sigma} dF(s) \right)^{-\frac{1}{\sigma}}$$

is decreasing in R_1 when $\sigma > 1$. This follows by Jensen's inequality since R_1 implies a mean preserving spread in $\hat{c}$. This implies that C_0 changes more than proportionally with C_1, due to changes in R_1; this establishes $\frac{d \log C_0}{d \log R_1} < -\frac{1}{\sigma}$ when $\sigma > 1$. The reverse inequality holds when $\sigma < 1$, since then a decrease in R_1 implies a mean-preserving spreads in $\hat{c}$.

C Proof of Proposition 8

The argument is similar to proof of Proposition 7, so I only sketch a proof of the arguments that are different.

We now impose the liquidity constraint $a_2(s) \geq 0$, which is equivalent to

$$c_1(s; Y_1) \leq sY_1 + q_1,$$

and consider the case where this constraint binds for some households, but not all households (in equilibrium some agents must hold the asset). It follows that

$$c_1(s) = \min \left\{ \frac{sY_1 + \frac{1}{R_1} Y_2}{1 + \beta^{\frac{1}{\sigma}} R_1^{\frac{1}{\sigma} - 1}}, sY_1 + \frac{1}{R_1} d_2 Y_2 \right\} \tag{27}$$

or equivalently

$$c_1(s;Y_1,\hat{s}) = \begin{cases} sY_1 + \frac{1}{R_1}d_2Y_2 & s \leq \hat{s} \\ \frac{sY_1+\frac{1}{R_1}Y_2}{1+\beta^{\frac{1}{\sigma}}R_1^{\frac{1}{\sigma}-1}} & s > \hat{s} \end{cases}$$

where the cutoff $\hat{s}$ is defined by the equality

$$\frac{\hat{s}Y_1 + \frac{1}{R}Y_2}{1+\beta^{\frac{1}{\sigma}}{R_1}^{\frac{1}{\sigma}-1}} = \hat{s}Y_1 + \frac{1}{R_1}d_2Y_2,$$

This provides a strictly decreasing relation between Y_1 and $\hat{s}$, given R_1.

Aggregating,

$$\begin{aligned} C_1 &= \min_{\tilde{s}} \int c_1(s;C_1;\tilde{s})dF(s) \\ &= \int_{\hat{s}} \left(\frac{sC_1 + \frac{1}{R_1}Y_2}{1+\beta^{\frac{1}{\sigma}}{R_1}^{\frac{1}{\sigma}-1}} \right) dF(s) + \int^{\hat{s}} \left(sC_1 + \frac{1}{R_1}d_2Y_2 \right) dF(s) \\ &= \frac{C_1 \int_{\hat{s}} sdF(s) + \frac{1}{R_1}Y_2\left(1-F(\hat{s})\right)}{1+\beta^{\frac{1}{\sigma}}{R_1}^{\frac{1}{\sigma}-1}} + C_1 \int^{\hat{s}} sdF(s) + \frac{1}{R_1}d_2Y_2F(\hat{s}) \end{aligned}$$

The first equality follows from (27). It implies, by an Envelope condition argument, that we can compute the equilibrium derivative of C_1 and R_1 without considering the change in $\hat{s}$. Rearranging, for given $\hat{s}$, we obtain

$$C_1 = h(R_1,\hat{s})\beta^{-\frac{1}{\sigma}}{R_1}^{-\frac{1}{\sigma}}Y_2 = m(R_1,\hat{s})\beta^{-\frac{1}{\sigma}}{R_1}^{-1}Y_2$$

where

$$\begin{aligned} h(R_1,\hat{s}) &\equiv \frac{1-F(\hat{s})+d_2F(\hat{s})+\beta^{\frac{1}{\sigma}}{R_1}^{\frac{1}{\sigma}-1}d_2F(\hat{s})}{\int_{\hat{s}} s\,dF(s)} \\ m(R_1,\hat{s}) &\equiv \frac{{R_1}^{1-\frac{1}{\sigma}}\left(1-F(\hat{s})+d_2F(\hat{s})\right)+\beta^{\frac{1}{\sigma}}d_2F(\hat{s})}{\int_{\hat{s}} s\,dF(s)} \end{aligned}$$

When $F(\hat{s}) > 0$ then h is decreasing if $\sigma > 1$ and decreasing if $\sigma < 1$. This in turn implies that when $\sigma > 1$ then $\frac{d\log C_1}{d\log R_1} < -\frac{1}{\sigma}$ and the inequality is reversed when $\sigma < 1$.

When $\sigma > 1$ then m is increasing; if $\sigma < 1$ then m is decreasing. This implies that we can write

$$Y_2 = M(R_1)R_1C_1$$

for some function M that is decreasing if $\sigma > 1$ and increasing if $\sigma < 1$.

To see that a change in R_1 changes C_0 in the same direction but in a greater proportion than the change resulting in C_1, note that in equilibrium

$$c_1(s) = \hat{c}(s, R_1)C_1,$$

where

$$\hat{c}(s, R_1) \equiv \min\left\{\frac{s}{1+\beta^{\frac{1}{\sigma}}R_1^{\frac{1}{\sigma}-1}} + \frac{M(R_1)}{1+\beta^{\frac{1}{\sigma}}R_1^{\frac{1}{\sigma}-1}}, s + M(R_1)d_2\right\}.$$

Write $\hat{c}$ in the form

$$\hat{c}(s, R_1) = \min\left\{\delta(R_1)s + \phi_2(R_1), s + \phi_2(R_1)\right\}.$$

When $\sigma > 1$ then the slope $\delta(R_1)$ is increasing in R_1 and the intercept $\phi_2(R_1)$ is decreasing. An increase in R_1 shifts $\hat{c}$ creating a single crossing at some interior point from below, i.e. if $R_1' > R_1$ then there exists a $S \in (0,1)$ such that $\hat{c}(s, R_1') \leq \hat{c}(s, R_1)$ for $s \leq S$ and $\hat{c}(s, R_1') \geq \hat{c}(s, R_1)$ for $s \geq S$. We also have by construction $\int \hat{c}(s, R_1)dF(s) = 1$ for all R_1. Thus, if $\sigma > 1$ we have a mean preserving spread in $\hat{c}$ when R_1 increases. The reverse is true when $\sigma < 1$. The result then follows as before.

D Proof of Proposition

The Euler equation between $t = 1$ and $t = 2$ gives

$$c_1(s) = (\beta R_1)^{-\frac{1}{\sigma}} c_2(s), \tag{28}$$

and aggregating gives

$$C_1 = (\beta R_1)^{-\frac{1}{\sigma}} C_2$$

so that $\frac{d\log C_1}{d\log R_1} = -\frac{1}{\sigma}$. The budget constraints are

$$c_1(s) + q_1 a_2(s) = sY_1 + q_1, \tag{29}$$

$$c_2(s) = (1 - d_2)Y_2 + d_2 a_2(s)Y_2 \tag{30}$$

Solving gives

$$c_1(s) = \hat{c}(s; C_1)C_1$$

where

$$\hat{c}(s; C_1) = \omega(C_1)s + 1 - \omega(C_1),$$
$$\omega(C_1) = \frac{1}{1 + \bar{q}C_1^{\phi-1}} \in (0,1).$$

Thus, $c(s)$ is a proportion $\hat{c}(s; C_1)$ of aggregate income C_1; for $\phi = 1$ this proportion varies with s but is independent of the interest rate C_1. However, $\hat{c}(s; C_1)$ varies with C_1 when $\phi \neq 1$. When $\phi < 1$ we have ω increasing and so an increase in C_1 increases $\hat{c}$ for $s > 1$ and decreases $\hat{c}$ for $s < 1$, i.e. if $C_1' > C_1$ then $\hat{c}(\cdot; C_1')$ single crosses $\hat{c}(\cdot; C_1)$ from below at $s = 1$; this implies a mean-preserving spread in the distribution for $\hat{c}$. For $\phi > 1$, we have ω decreasing and so the reverse is true: a decrease in C_1 implies a mean preserving spread in $\hat{c}$.

Turning to $t = 0$ we have

$$C_0 = (\beta R_0)^{-\frac{1}{\sigma}} \cdot \left(\int \hat{c}(s; C_1)^{-\sigma} dF(s) \right)^{-\frac{1}{\sigma}} C_1,$$

so that $\frac{d \log C_0}{d \log R_0} = -\frac{1}{\sigma}$. Since the marginal utility function $\hat{c}^{-\sigma}$ is convex for all $\sigma > 0$ then the expression

$$\left(\int \hat{c}(s; C_1)^{-\sigma} dF(s) \right)^{-\frac{1}{\sigma}}$$

is increasing in C_1 when $\phi > 1$. This follows by Jensen's inequality since lower C_1 implies a mean preserving spread in $\hat{c}$. This implies that C_0 changes more than proportionally with C_1, due to changes in R_1; this establishes $\frac{d \log C_0}{d \log R_1} < -\frac{1}{\sigma}$ when $\phi > 1$. The reverse inequality holds when $\phi < 1$, since then an increase in C_1 implies a mean-preserving spreads in $\hat{c}$.

E Proof of Proposition 10

The required equilibrium conditions are

$$c^i(z^t, s^t) + k^i(z^t, s^t) = W_t(s^t)n^i(z^t, s^t) + k^i(z^{t-1}, s^{t-1})R_t(s^t)$$
$$\frac{v'(n^i(z^t, s^t))}{u'(c^i(z^t, s^t))} = z_t W_t$$
$$u'(c^i(z^t, s^t)) \geq \beta \mathbb{E}_t \left[R_{t+1} u'(c^i(z^t, s^t)) \mid z^t, s^t \right]$$

with equality whenever $k^i(z^t, s^t) > 0$. The aggregate conditions are

$$C(s^t) + K_{t+1}(s^t) = A(s_t)F(K_t(s^{t-1}), N_t(s^t)) + (1-\delta)K(s^t)$$

$$\begin{aligned} R(s^t) &= A(s_t)F_k(K(s^{t-1}), N(s^t)) + 1 - \delta \\ W(s^t) &= A(s_t)F_N(K(s^{t-1}), N(s^t)) \end{aligned}$$

with aggregates consistent with household choices

$$\begin{aligned} C(s^t) &= \sum \mu^i \int c^i(z^t, s^t) d\Lambda(z^t), \\ N(s^t) &= \sum \mu^i \int n^i(z^t, s^t) d\Lambda(z^t), \\ K(s^t) &= \sum \mu^i \int k^i(z^t, s^t) d\Lambda(z^t). \end{aligned}$$

Guess and verify that the equilibrium satisfies

$$c^i(z^t, s^t) = c^i(z^t)C(s^t)$$

$$n^i(z^t, s^t) = n^i(z^t)N(s^t)$$

For each period t and aggregate history s^t such a decomposition is without loss of generality; define $c^i(z^t)$ and $n^i(z^t)$ to be the household decomposition of the allocation associated with some aggregate history s^t for each period. I now verify that this decomposition works for all other histories.

Substituting we obtain

$$c^i(z^t)C(s^t) + k^i(z^t)K(s^t) = W(s^t)N(s^t)n^i(z^t) + k^i(z^{t-1})K(s^{t-1})R(s^t)$$

$$\frac{c^i(z^t)}{\bar{v}z_t} v'\left(n^i(z^t)\right) v'(N(s^t))N(s^t)C(z^t) = W(s^t)N(s^t)$$

$$u'(C(s^t)) \geq \left(\frac{\beta \mathbb{E}\left[u'(c^i(z^{t+1})) \mid z^t\right]}{u'(c^i(z^{t+1}))} \right) \cdot \mathbb{E}[R(s^{t+1})u'(C(s^{t+1})) \mid s^t]$$

By definition these equations hold for one history s^t; to check whether these conditions hold for other histories, rewrite them as

$$c^i(z^t)\frac{C(s^t)}{Y(s^t)} + k^i(z^t)\frac{K(s^t)}{Y(s^t)} = \frac{W(s^t)N(s^t)}{Y(s^t)}n^i(z^t) + \frac{K(s^{t-1})R(s^t)}{Y(s^t)} \cdot k^i(z^{t-1})$$

$$\hat{v}v'(N(s^t))N(s^t)\frac{C(s^t)}{Y(s^t)} = \frac{W(s^t)N(s^t)}{Y(s^t)}$$
$$u'(C(s^t)) = \hat{\beta}_t \cdot \mathbb{E}[R(s^{t+1})u'(C(s^t)) \mid s^t]$$

where
$$\hat{v}_t \equiv \int \frac{c^i(z^t)}{\bar{v}z_t} v'\left(n^i(z^t)\right)$$
$$\hat{\beta}_t \equiv \max_{z^t} \frac{\beta \mathbb{E}\left[u'(c^i(z^t)) \mid z^t\right]}{u'(c^i(z^t))}$$

The first equation will hold as long as the terms

$$\frac{C(s^t)}{Y(s^t)} \qquad \frac{K(s^t)}{Y(s^t)} \qquad \frac{W(s^t)N(s^t)}{Y(s^t)} \qquad \frac{K(s^{t-1})R(s^t)}{Y(s^t)}$$

are independent of history s^t; the last two ratios are guaranteed to be constant by the Cobb-Douglas assumption. The second ratio is implied by the first. Thus, we require

$$C(s^t) = (1 - \omega_t)\, Y(s^t)$$

for some saving rate ω_t that does not depend on the history s^t. Turning to the second equation, we determine $N(s^t)$,

$$\hat{v}v'(N(s^t))N(s^t) = \frac{1-\alpha}{1-\omega_t}$$

Finally

$$\frac{1}{C(s^t)} = \hat{\beta}_t \mathbb{E}\left[\frac{\alpha Y(s^{t+1})}{k(s^t)} \frac{1}{C(s^{t+1})} \mid s^t\right]$$
$$\frac{1}{(1-\omega_t)Y(s^t)} = \hat{\beta}_t \frac{1}{k(s^t)} \frac{\alpha}{1-\omega_t}$$
$$\frac{\omega_t}{1-\omega_t} = \hat{\beta}_t \frac{\alpha}{1-\omega_t}$$

Note that at with a steady state invariant distribution we have $\hat{\beta}_t$ constant, so we obtain $\omega_t = \alpha\hat{\beta}$ as in the standard Brock-Mirman solution.