ECON 703 - PS 3

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(1) Let (X, d) be a nonempty complete metric space. Suppose an operator $T : X \to X$ satisfies $d(T(x), T(y)) < d(x, y)$ for all $x \neq y, x, y \in X$. Prove or disprove that T has a fixed point. Compare with the Contraction Mapping Theorem.

I disprove that such a T has a fixed point using a counter example.

Proof: Consider the metric space $([1, \infty), d_E)$ and the operator $T(x) = x + \frac{1}{x}$. T is an operator because if $x \geq 1$, then $x + \frac{1}{x} > x \geq 1$.

Consider $x \neq y, x, y \in [1, \infty)$. If $x > y \implies 0 > \frac{1}{x} - \frac{1}{y}$, then[1]

$$\begin{aligned}
d(x, y) &= |x - y| \\
&= x - y \\
&> x - y + (\frac{1}{x} - \frac{1}{y}) \\
&= (x + \frac{1}{x}) - (y + \frac{1}{y}) \\
&= |(x + \frac{1}{x}) - (y + \frac{1}{y})| \\
&= |T(x) - T(y)| \\
&= d(T(x), T(y))
\end{aligned}$$

If $x < y$, switch x for y and y for x in the logic above. For the sake of a contradiction, assume that T has a fixed point, $x^* \in [1, \infty)$ where $T(x^*) = x^*$. However, $T(x^*) = x^* + \frac{1}{x^*} > x^* \Rightarrow\Leftarrow$. Therefore, T does not have a fixed point. $\square$

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[1] I show that $(x + \frac{1}{x}) - (y + \frac{1}{y}) > 0$ for $x > y \geq 1$. Rewrite $x = y + \varepsilon$ where $\varepsilon > 0$:

$$-(\frac{1}{x} - \frac{1}{y}) = \frac{1}{y} - \frac{1}{y + \varepsilon} = \frac{\varepsilon}{y(y + \varepsilon)} < \varepsilon = (y + \varepsilon) - y = x - y$$

Because $y(y + \varepsilon) > 1$.

(2) Does there exist a countable set, which is compact?

Yes, there exists a countable compact set.

Proof: Consider $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and

$$f(x) = \begin{cases} 0, & n = 1 \\ \frac{1}{n-1}, & n > 1 \end{cases}$$

Since f is a one-to-one mapping between $\mathbb{N}$ and A, A is countable.

Let $\mathcal{U}$ be an arbitrary open cover of A. $\exists U_{\lambda^*} \in \mathcal{U}$ such that $U_{\lambda^*} \supset B_\varepsilon(0)$ for some $\varepsilon > 0$. Let $\{a_n\} = 1/n$ for $n \in \mathbb{N}$. Since $a_n \to 0$, $\exists N$ such that, for all $n \geq N$, $a_n \in B_\varepsilon(0) \implies a_n \in U_{\lambda^*}$. Thus, we can construct a finite subcover of A as the union of U_{λ^*} and a finite number of $U_{\lambda_i} \in \mathcal{U}$, which each contain a $a_n \notin U_{\lambda^*}$ where $n = 1, ..., N$. $\square$

(3) Prove that the function $f(x) = \cos^2(x)e^{5-x-x^2}$ has a maximum on $\mathbb{R}$.

Proof: Using the extreme value theorem, I show that f attains a maximum on $\mathbb{R}$ by (i) finding a compact set $C \subset \mathbb{R}$ where $\exists x \in C$ such that $f(x) > f(y) \forall y \in C^c$ and (ii) showing that f is continuous on $\mathbb{R}$. Define $g(x) = \cos^2(x)$, $h_1(x) = e^x$, $h_2(x) = 5 - x - x^2$, so that $f(x) = g(x)h_1(h_2(x))$.

For (i), define $C = [(-\sqrt{21} - 1)/2, (\sqrt{21} - 1)/2]$. As a closed interval of $\mathbb{R}$, C is compact by the Heine-Borel theorem. For $0 \in C$, $f(0) = \cos^2(0)e^{5-0-0^2} = e^5 > 1$. Since $0 \leq \cos^2(x) \leq 1$, $f(x) \leq h_1(h_2(x))$ for all $x \in \mathbb{R}$. Furthermore, if $h_2(x) < 0 \implies h_1(h_2(x)) < 1$. Because $h_2(0) = 5 - 0 - 0^2 = 5$ and the quadratic roots of h_2 are $x = (\sqrt{21} - 1)/2$ and $x = (-\sqrt{21} - 1)/2$, $f(x) \leq h_1(h_2(x)) < 1$ for all $x \in C^c$. Thus, the maximum of $f(x)$ on $\mathbb{R}$ must occur on C.

For (ii), since g, h_1, and h_2 are continuous functions on $\mathbb{R}$, f is a continuous function on $\mathbb{R}$.[2]

Thus, by the extreme value theorem, f attains a maximum on C and therefore on $\mathbb{R}$. $\square$

(4) Suppose you have two maps of Wisconsin: one large and one small. You put the large one on top of the small one, so that the small one is completely covered by the large one. Prove that it is possible to pierce the stack of those two maps in a way that the needle will go through exactly the same (geographical) points on both maps.

Proof: Draw Cartesian planes on both maps such that the Wisconsin Capital Building in Madison is at the origin. For the larger map, scale the x coordinate such that moving one unit right corresponds to the point on the map that is one inch east of the Capital Building on the map and scale the y coordinate such that moving one unit up corresponds to the point on the map that is one inch north of the Capital Building on the map. Scale the Cartesian plane on the smaller map similarly.

Define β as the the ratio of the miles between geographical points per inch on the larger map to the miles between geographical points per inch on the smaller map. Notice that $\beta < 1$. Define $A \subset \mathbb{R} \times \mathbb{R}$ as the closed set of points on the larger map that are on or within Wisconsin's borders. Notice that the metric space (A, d_E) is complete because A is closed. Define operator $T : A \to \mathbb{R} \times \mathbb{R}$ such that, for $x \in A$, (x_1, x_2) on the larger map and $(T(x_1), T(x_2))$ on the smaller map represent the same geographical point. Notice that, for all $x, y \in A, x \neq y$, $d_E(x, y) = \beta d_E(T(x), T(y))$. Thus, T is a contraction with modulus $\beta < 1$. Thus, T has a fixed point where $x^* = T(x^*)$, where you can pierce the stack of maps at the same geographical point. $\square$

[2] Proof that the products of two continuous functions is continuous.

(5) Consider the set $X = \{-1, 0, 1\}$ and the space of all functions on X, $F_X = \{f : X \to \mathbb{R}\}$.

(a) Show that F_X is a vector space.

Proof: To show that X is a vector space, I satisfy the eight properties of vector space with X:

- Let $a, b, c \in F_X$. $\forall x \in X$, $(a(x) + b(x)) + c(x) = a(x) + (b(x) + c(x))$.
- Let $a, b \in F_X$. $\forall x \in X$, $a(x) + b(x) = b(x) + a(x)$.
- Let $a, \bar{0} \in F_X$ where $\bar{0}(x) = 0$ $\forall x \in X$, so $a(x) + \bar{0}(x) = \bar{0}(x) + a(x) = a(x)$.
- Let $a, (-a) \in F_X$ where $(-a)(x) = -a(x)$ $\forall x \in X$, so $a(x) + (-a)(x) = a(x) - a(x) = 0 = \bar{0}(x)$.
- Let $a, b \in F_X$ and $\alpha \in \mathbb{R}$. $\forall x \in X$, $\alpha(a(x) + b(x)) = \alpha a(x) + \alpha b(x)$.
- Let $a \in F_X$ and $\alpha, \beta \in \mathbb{R}$. $\forall x \in X$, $(\alpha + \beta)a(x) = \alpha a(x) + \beta a(x)$.
- Let $a \in F_X$ and $\alpha, \beta \in \mathbb{R}$. $\forall x \in X$, $(\alpha * \beta)a(x) = \alpha(\beta * a(x))$.
- Let $a \in F_X$. $\forall x \in X$, $1 * a(x) = a(x)$. $\square$

(b) Show that the operator $T : F_X \to F_X$ defined by $T(f)(x) = f(x^2), x \in \{-1, 0, 1\}$ is linear.

An arbitrary $f \in F_X$ is

$$f(x) = \begin{cases} a & x = -1 \\ b & x = 0 \\ c & x = 1 \end{cases}$$

where $a, b, c \in \mathbb{R}$. Applying T to f,

$$T(f)(x) = f(x^2) = \begin{cases} a & x = \{-1, 1\} \\ b & x = 0. \end{cases}$$

Proof: Let $a, b \in F_X$ and $\alpha, \beta \in \mathbb{R}$. $\exists c \in F_X$, such that $c(x) = \alpha a(x) + \beta b(x)$ $\forall x \in X$. Apply T to $c(x)$:

$$\begin{aligned} T(c)(x) &= c(x^2) \\ &= \alpha a(x^2) + \beta b(x^2) \\ &= \alpha T(a)(x) + \beta T(b)(x) \end{aligned}$$

Thus, T is linear. $\square$

(c) Calculate ker T, Im T, and rank T.

ker $T = \{f \in F_X | T(f) = \bar{0}\} = \{f \in F_X | f(0) = f(1) = 0\}$.

Im $T = \{T(f) | f \in F_X\} = \{f(x^2) | f \in F_X\} = \{(a, b) | a, b \in \mathbb{R}\} = \mathbb{R}^2$.

rank T = dim (Im T) = dim ($\mathbb{R}^2$) = 2.

(6) Consider the following system of linear equations

$$x_1 + x_2 + 2x_3 + x_4 = 0 \tag{1}$$
$$3x_1 - x_2 + x_3 - x_4 = 0 \tag{2}$$
$$5x_1 - 3x_2 - 3x_4 = 0 \tag{3}$$

Let X be the set of $\{x_1, x_2, x_3, x_4\}$ which satisfy the system of equations.

(a) Show that X is a vector space.

Proof: First I find a basis for X:

$$(3) \implies 5x_1 = 3x_2 + 3x_4 \tag{4}$$

$$\begin{aligned}
(3)(4) \implies \frac{3}{5}(x_2 + x_4) + x_2 + 2x_3 + x_4 &= 0 \\
2x_3 = -\frac{8}{5}(x_2 + x_4) \\
5x_3 = -4(x_2 + x_4)
\end{aligned}$$

Letting $x_2 = a$ and $x_3 = b$, all elements of X can be represented by

$$\begin{aligned}
y &= \begin{bmatrix} 3(a+b) \\ 5a \\ -4(a+b) \\ 5b \end{bmatrix} \\
&= a \begin{bmatrix} 3 \\ 5 \\ -4 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ -4 \\ 5 \end{bmatrix} \\
&= a\vec{u} + b\vec{v}
\end{aligned}$$

Thus, $\{\vec{u}, \vec{v}\}$ forms a basis for X because they span X and they are linearly independent.

For $x, y, z \in X$, $\exists o, p, q, r, s, t \in \mathbb{R}$ such that $x = o\vec{u} + p\vec{v}$, $y = q\vec{u} + r\vec{v}$, and $y = s\vec{u} + t\vec{v}$. Vector spaces have eight properties:

- $(x + y) + z = (o\vec{u} + p\vec{v} + q\vec{u} + r\vec{v}) + s\vec{u} + t\vec{v} = o\vec{u} + p\vec{v} + (q\vec{u} + r\vec{v} + s\vec{u} + t\vec{v}) = x + (y + z)$.
- $x + y = o\vec{u} + p\vec{v} + q\vec{u} + r\vec{v} = q\vec{u} + r\vec{v} + o\vec{u} + p\vec{v} = y + x$.
- Let $\bar{0} \in X$ be $0 * \vec{u} + 0 * \vec{v}$. $x + \bar{0} = o\vec{u} + p\vec{v} + 0 * \vec{u} + 0 * \vec{v} = (o + 0)\vec{u} + (p+)\vec{v} = o\vec{u} + p\vec{v} = x$.
- Let $(-x) = (-o)\vec{u} + (-p)\vec{v}$. $x + (-x) = o\vec{u} + p\vec{v} + (-o)\vec{u} + (-p)\vec{v} = (o - o)\vec{u} + (p - p)\vec{v} = 0 * \vec{u} + 0 * \vec{v} = \bar{0}$.
- Let $\alpha \in \mathbb{R}$. $\alpha(x + y) = \alpha(o\vec{u} + p\vec{v} + q\vec{u} + r\vec{v}) = \alpha(o\vec{u} + p\vec{v}) + \alpha(q\vec{u} + r\vec{v}) = \alpha x + \alpha y$.
- Let $\alpha, \beta \in \mathbb{R}$. $(\alpha + \beta)x = (\alpha + \beta)(o\vec{u} + p\vec{v}) = \alpha(o\vec{u} + p\vec{v}) + \beta(o\vec{u} + p\vec{v}) = \alpha x + \beta y$.
- Let $\alpha, \beta \in \mathbb{R}$. $(\alpha * \beta)x = (\alpha * \beta)(o\vec{u} + p\vec{v}) = \alpha(\beta(o\vec{u} + p\vec{v})) = \alpha(\beta x)$.
- $1 * x = 1 * (o\vec{u} + p\vec{v}) = 1 * o\vec{u} + 1 * p\vec{v} = o\vec{u} + p\vec{v} = x$.

Thus, X is a vector space. □

(b) Calculate dim X.

From (a), we found that $\{(3, 5, -4, 0), (3, 0, -4, 5)\}$ form a basis for X. Since the dimension of X is the cardinality of a basis for X, dim $X = 2$.