ECON 714B - Problem Set 7

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Exercise 8.1 - Existence of representative consumer

Suppose households 1 and 2 have one-period utility functions $u(c_1)$ and $w(c_2)$, respectively, where u and w are both increasing, strictly concave, twice differentiable functions of a scalar consumption rate. Consider the Pareto problem:

$$v_\theta(c) = \max_{\{c_1, c_2\}} [\theta u(c_1) + (1-\theta) w(c_2)]$$

subject to the constraint $c_1 + c_2 = c$. Show that the solution of this problem has the form of a concave utility function $v_\theta(c)$, which depends on the Pareto weight θ. Show that $v'_\theta(c) = \theta u'(c_1) = (1-\theta)w'(c_2)$.The function $v_\theta(c)$ is the utility function of the representative consumer. Such a representative consumer always lurks within a complete markets competitive equilibrium even with heterogeneous preferences. At a competitive equilibrium, the marginal utilities of the representative agent and each and every agent are proportional.

First, notice that v_θ is increasing, strictly concave, and twice differentiable because $\theta, 1-\theta \geq 0$ and u and w are both increasing, strictly concave, twice differentiable functions.

$$v_\theta(c) = \max_{\{c_1, c_2\}} [\theta u(c_1) + (1-\theta) w(c_2)] \text{ s.t. } c = c_1 + c_2$$

$$\implies v_\theta(c) = \max_{c_1} [\theta u(c_1) + (1-\theta) w(c - c_1)]$$

Since u and w are continuously differentiable functions, we can find the envelope condition:

$$\implies v'_\theta(c) = (1-\theta) w'(c - c_1) = (1-\theta) w'(c_2)$$

FOC $[c_1]$:

$$\theta u'(c_1) = (1-\theta) w'(c - c_1) \implies \theta u'(c_1) = (1-\theta) w'(c_2)$$

Therefore,

$$v'_\theta(c) = \theta u'(c_1) = (1-\theta) w'(c_2)$$

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Exercise 8.3

An economy consists of two infinitely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer i consumes c_{it} at time t. Consumer i ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_{it})$$

where $\beta \in (0, 1)$ and $u(c)$ is increasing, strictly concave, and twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_{1t} = 1, 0, 0, 1, 0, 0, 1, ...$ Consumer 2 is endowed with a stream of the consumption good $y_{2t} = 0, 1, 1, 0, 1, 1, 0, ...$ Assume that there are complete markets with time 0 trading.

a. Define a competitive equilibrium.

First, notice that there is no stochastic states (i.e., endowment process is deterministic). The agent i's problem is:

$$\max_{\{c_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{it}) \text{ s.t. } \sum_{t=0}^{\infty} Q_t c_{it} \leq \sum_{t=0}^{\infty} Q_t y_{it}$$

A competitive equilibrium is a feasible allocation $\{\{c_{it}\}_{t=0}^{\infty}\}_{i=1,2}$ and a price system $\{Q_t\}_{t=0}^{\infty}$ such that given the price system, the allocation maximized the agent's utility subject to their budget constraint.

b. Compute a competitive equilibrium.

Let μ_i be the multipler on the BC for agent i:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_{it}) - \mu_i \left[\sum_{t=0}^{\infty} Q_t y_{it} - \sum_{t=0}^{\infty} Q_t c_{it} \right]$$

FOC $[c_{it}]$:

$$\beta^t u'(c_{it}) = \mu_i Q_t$$

Thus, the ratio of the FOC for 1 and the FOC for 2 is:

$$\frac{u'(c_{1t})}{u'(c_{2t})} = \frac{\mu_1}{\mu_2} \implies c_{1t} = u'^{-1}\left(\frac{\mu_1}{\mu_2} u'(c_{2t}) \right)$$

Plugging into the resource constraint:

$$c_{1t} + c_{2t} = 1 \implies u'^{-1}\left(\frac{\mu_1}{\mu_2} u'(c_{2t}) \right) + c_{2t} = 1$$

This implies that c_{2t} a function of the aggregate endowment for all t. Since the aggregate endowment is constant, $c_{2t} = c_{2,t+1} = c_2$. Thus, c_{1t} is also constant: $c_{1t} = c - c_2 \equiv c_1$. Let the numeraire be date 0 consumption $Q_0 = 1$:

$$\frac{\beta^t u'(c_{1,t})}{\beta^0 u'(c_{1,0})} = \frac{\mu_1 Q_t}{\mu_1} \implies \frac{\beta^t u'(c_1)}{u'(c_1)} = Q_t \implies Q_t = \beta^t$$

The budget constraint implies

$$\sum_{t=0}^{\infty} \beta^t c_{1t} = \sum_{t=0}^{\infty} \beta^t y_{1t} \implies c_1 \frac{1}{1-\beta} = \sum_{t=0}^{\infty} \beta^{3t} \implies c_1 = \frac{1-\beta}{1-\beta^3}$$

Furthermore,

$$c_2 = 1 - \frac{1-\beta}{1-\beta^3} = \frac{\beta - \beta^3}{1-\beta^3}$$

c. Suppose that one of the consumers markets a derivative asset that promises to pay .05 units of consumption each period. What would the price of that asset be?

The price of the derivative asset in period 0 with promises $\{d_t\}_{t=0}^{\infty} = \{0.05\}_{t=0}^{\infty}$ is:

$$P_0^0 = \sum_{t=0}^{\infty} Q_t d_t = \sum_{t=0}^{\infty} \beta^t 0.05 = \frac{0.05}{1-\beta}$$

Exercise 8.4

Consider a pure endowment economy with a single representative consumer; $\{c_t, d_t\}_{t=0}^{\infty}$ are the consumption and endowment processes, respectively. Feasible allocations satisfy $c_t \leq d_t$. The endowment process is described by $d_{t+1} = \lambda_{t+1} d_t$. The growth rate λ_{t+1} is described by a two-state Markov process with transition probabilities $P_{ij} = Prob(\lambda_{t+1} = \bar{\lambda}_j | \lambda_t = \bar{\lambda}_i)$. Assume that

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}, \bar{\lambda} = \begin{bmatrix} .97 \\ 1.03 \end{bmatrix}$$

In addition, $\lambda_0 = .97$ and $d_0 = 1$ are both known at date 0. The consumer has preferences over consumption ordered by

$$E_0 = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

where E_0 is the mathematical expectation operator, conditioned on information known at time 0, $\gamma = 2, \beta = .95$.

Part I

At time 0, after d_0 and λ_0 are known, there are complete markets in date- and history-contingent claims. The market prices are denominated in units of time 0 consumption goods.

a. Define a competitive equilibrium, being careful to specify all the objects composing an equilibrium.

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b. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, ..., \lambda_5) = (.97, .97, 1.03, .97, 1.03)$. Please give a numerical answer.

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c. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, ..., \lambda_5) = (1.03, 1.03, 1.03, 1.03, .97)$.

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d. Give a formula for the price at time 0 of a claim on the entire endowment sequence.

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e. Give a formula for the price at time 0 of a claim on consumption in period 5, contingent on the growth rate λ_5 being .97 (regardless of the intervening growth rates).

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Part II

Now assume a different market structure. Assume that at each date $t \geq 0$ there is a complete set of one-period forward Arrow securities.

f. Define a (recursive) competitive equilibrium with Arrow securities, being careful to define all of the objects that compose such an equilibrium.

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g. For the representative consumer in this economy, for each state compute the “natural debt limits” that constrain state-contingent borrowing.

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h. Compute a competitive equilibrium with Arrow securities. In particular, compute both the pricing kernel and the allocation.

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i. An entrepreneur enters this economy and proposes to issue a new security each period, namely, a risk-free two-period bond. Such a bond issued in period t promises to pay one unit of consumption at time $t + 1$ for sure. Find the price of this new security in period t, contingent on λ_t.

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