ECON 709 - PS 3

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1. A random point (X, Y) is distributed uniformly on the square with vertices $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$. That is, the joint PDF is $f(x, y) = 1/4$ on the square and $f(x, y) = 0$ outside the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$

(b) $|X + Y| < 2$

2. Let the joint PDF of X and Y be given by $f(x, y) = g(x)h(y)\ \forall x, y \in \mathbb{R}$ for some functions $g(x)$ and $h(y)$. Let a denote $\int_{-\infty}^{\infty} g(x)dx$ and b denote $\int_{-\infty}^{\infty} h(x)dx$

(a) What conditions a and b should satisfy in order for $f(x, y)$ to be a bivariate PDF?

For $f(x, y)$ to be a PDF, it should integrate to one:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= 1 \\
\implies \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)dxdy &= 1 \\
\implies \int_{-\infty}^{\infty} g(x)dx \int_{-\infty}^{\infty} h(y)dy &= 1 \\
\implies ab &= 1 \\
\implies a &= b^{-1}
\end{aligned}$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

(b) Find the marginal PDF of X and Y.

The marginal PDF of X:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\infty}^{\infty} g(x)h(y)dy \\ &= g(x)\int_{-\infty}^{\infty} h(y)dy \\ &= b \cdot g(x) \end{aligned}$$

The marginal PDF of Y:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_{-\infty}^{\infty} g(x)h(y)dx \\ &= h(y)\int_{-\infty}^{\infty} g(x)dx \\ &= a \cdot h(y) \end{aligned}$$

(c) Show that X and Y are independent.

Proof: X and Y are independent if the product of their marginal distributions is their joint distribution:

$$\begin{aligned} f_X(x) \cdot f_Y(y) &= b \cdot g(x) \cdot a \cdot h(y) \\ &= b \cdot g(x) \cdot b^{-1} \cdot h(y) \\ &= g(x) \cdot h(y) \\ &= f(x,y) \end{aligned}$$

□

3. Let the joint PDF of X and Y be given by

$$f(x,y) = \begin{cases} cxy & \text{if } x, y \in [0,1], x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c such that $f(x,y)$ is a joint PDF.

$$\begin{aligned} \int_0^1 \int_0^{1-x} f(x,y)dydx &= 1 \\ \implies \int_0^1 \int_0^{1-x} cxydydx &= 1 \\ \implies c\int_0^1 \left[\frac{xy^2}{2}\right]_{y=0}^{1-x} dx &= 1 \\ \implies \frac{c}{2}\int_0^1 x(1-x)^2 dx &= 1 \\ \implies \frac{c}{2}\int_0^1 x - 2x^2 + x^3 dx &= 1 \\ \implies \frac{c}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{x=0}^{1} &= 1 \\ \implies \frac{c}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] &= 1 \\ \implies \frac{c}{2}\left(\frac{1}{12}\right) &= 1 \\ \implies c &= 24 \end{aligned}$$

(b) Find the marginal distributions of X and Y.

$$\begin{aligned} f_X(x) &= \int_0^{1-x} f(x,y)dy \\ &= \int_0^{1-x} 24xydy \\ &= \left[12xy^2\right]_{y=0}^{1-x} \\ &= \begin{cases} 12x(1-x)^2, x \in [0,1] \\ 0, \text{ otherwise.} \end{cases} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_0^{1-y} f(x,y)dx \\ &= \int_0^{1-y} 24xydx \\ &= \left[12x^2y\right]_{x=0}^{1-y} \\ &= \begin{cases} 12(1-y)^2y, y \in [0,1] \\ 0, \text{ otherwise.} \end{cases} \end{aligned}$$

(c) Are X and Y independent? Compare your answer to Problem 2 and discuss.

X and Y independent if the product of the marginal distributions equals their joint distribution at all points in the support. If $x = y = 0.9$, $f(0.9, 0.9) = 0$ because $(0.9, 0.9)$ is not in the support, $x+y = 0.9+0.9 = 1.8 > 1$. But each marginal distribution is define over $[0, 1]$, so the product of the marginals is positive at $(0.9, 0.9)$: $f_X(0.9)f_Y(0.9) = [12(0.9)(1-(0.9))^2][12(1-0.9)^2(0.9)] = 0.0117$.

In (2), the support for the joint distribution is $\mathbb{R}^2$, whereas the support for the joint distribution depends on the realization of the random variable.

4. Show that any random variable is uncorrelated with a constant.

Proof: Let $a \in \mathbb{R}$ and X be a random variable with distribution F_X. Define random variable Y as the degenerate random variable that equals a. Thus, the distribution Y is

$$F_Y(y) = \begin{cases} 0, & y < a \\ 1, & y \geq a \end{cases}$$

To show X is uncorrelated with a constant, I show that X and Y are independent and then, by a theorem in the Lecture 3 Notes, we know that X and Y are uncorrelated.

To find the joint distribution of X and Y, consider two cases: $y < a$ and $y \geq a$. For $y < a$,

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x \text{ and } Y \leq a) \\ &= 0 \end{aligned}$$

For $y \geq a$:

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x) \\ &= F_X(x) \end{aligned}$$

Thus, the joint distribution is

$$F(x, y) = \begin{cases} 0, & y < a \\ F_X(x), & y \geq a \end{cases}$$

The joint distribution equals the product of the marginals:

$$F(x, y) = \begin{cases} 0 * F_X(x), & y < a \\ 1 * F_X(x), & y \geq a \end{cases} = \begin{cases} F_Y(y) * F_X(x), & y < a \\ F_Y(y) * F_X(x), & y \geq a \end{cases}.$$

□

5. Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for the correlation of XY and Y in terms of these means and variances.

6. Prove the following: For any random vector $(X_1, X_2, ..., X_n)$,

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \le i < j \le n} Cov(X_i, X_j).$$

Proof by induction.

7. Suppose that X and Y are joint normal, i.e. they have the joint PDF:

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp(-(2(1-\rho^2))^{-1}(x^2/\sigma_X - 2xy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))$$

(a) Derive the marginal distributions of X and Y, and observe that both normal distributions.

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(b) Derive the conditional distribution of Y given $X = x$. Observe that it is also a normal distribution.

(c) Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X and Z are independent.

8. Consider a function $g : \mathbb{R} \to \mathbb{R}$. Recall that the inverse image of a set A, denoted $g^{-1}(A)$ is $g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$. Let there be functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$. Let X and Y be two random variables that are independent. Suppose that g_1 and g_2 are both Borel-measurable, which means that $g_1^{-1}(A)$ and $g_2^{-1}(A)$ are both in the Borel σ-field whenever A is in the Borel σ-field. Show that the two random variables $Z := g_1(X)$ and $W := g_2(Y)$ are independent. (Hint: use the 1st or the 2nd definition of independence.)