

ECON 710A - Problem Set 3

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1. Let $(Y, X', Z')'$ be a random vector such that $Y = X'\beta + U$, $E[U|Z] = 0$ where $E[ZX']$ is invertible and $E[Y^4 + \|X\|^4 + \|Z\|^4] < \infty$. Also, let $\{(Y_i, X'_i, Z'_i)'\}_{i=1}^n$ be a random sample from the distribution of $(Y, X', Z')'$. We showed in lecture 4 that $\sqrt{n}(\hat{\beta}^{IV} - \beta) \rightarrow_d N(0, \Omega)$, $\Omega = E[ZX']^{-1} E[ZZ'U^2] E[XZ']^{-1}$. Now suppose that $X = (X_1, X_2)'$, $Z = (Z_1, X_2)'$, and $E[U^2|Z] = \sigma_U^2$ and let Ω_{11} be the upper left entry in Ω .

- (i) Show that $E[ZX']$ is invertible iff $E[ZZ']$ is invertible and $\pi_1 \neq 0$ where $(\pi_1, \pi_2)' = E[ZZ']^{-1} E[ZX_1]$.

Observe that

$$E[ZX'] = E\left[\begin{pmatrix} Z_1 \\ X_2 \end{pmatrix} \begin{pmatrix} X_1 & X_2' \end{pmatrix}\right] = \begin{pmatrix} E[Z_1 X_1] & E[Z_1 X_2'] \\ E[X_2 X_1] & E[X_2 X_2'] \end{pmatrix}$$

$$E[ZZ'] = E\left[\begin{pmatrix} Z_1 \\ X_2 \end{pmatrix} \begin{pmatrix} Z_1 & X_2' \end{pmatrix}\right] = \begin{pmatrix} E[Z_1^2] & E[Z_1 X_2'] \\ E[X_2 Z_1] & E[X_2 X_2'] \end{pmatrix}$$

First, if either $E[ZZ']$ or $E[ZX']$ are invertible then, $E[X_2 X_2']$ is invertible. Assume for sake of a contradiction that $E[X_2 X_2']$ is not invertible. Then there exists some nonzero t such that $E[X_2 X_2']t = 0 \implies t'E[X_2 X_2']t = t'0 = 0 \implies E[X_2^2 t] = 0 \implies E[Z_1 X_2']t = 0$. This implies that $E[ZX']$ and $E[ZZ']$ are not invertible, which is a contradiction:

$$E[ZX'](0, t')' = \begin{pmatrix} E[Z_1 X_1] & E[Z_1 X_2'] \\ E[X_2 X_1] & E[X_2 X_2'] \end{pmatrix} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} E[Z_1 X_1](0) + E[Z_1 X_2']t' \\ E[X_2 X_1](0) + E[X_2 X_2']t' \end{pmatrix} = 0$$

$$E[ZZ'](0, t')' = \begin{pmatrix} E[Z_1^2] & E[Z_1 X_2'] \\ E[X_2 Z_1] & E[X_2 X_2'] \end{pmatrix} \begin{pmatrix} 0 \\ t' \end{pmatrix} = \begin{pmatrix} E[Z_1^2](0) + E[Z_1 X_2']t \\ E[X_2 Z_1](0) + E[X_2 X_2']t \end{pmatrix} = 0$$

By the block inversion formula, if $E[ZX']$ is invertible iff

$$E[Z_1 X_1] - E[Z_1 X_2'] E[X_2 X_2']^{-1} E[X_2 X_1] \neq 0 \iff E[\tilde{Z}_1 X_1] \neq 0$$

where $\tilde{Z}_1 := Z_1 - X_2' E[X_2 X_2']^{-1} E[X_2 Z_1]$. Denote $\pi_1 := \frac{E[\tilde{Z}_1 X_1]}{E[\tilde{Z}_1^2]}$,

$$E[\tilde{Z}_1 X_1] \neq 0 \iff \pi_1 E[\tilde{Z}_1^2] \neq 0 \iff \pi_1 \neq 0 \text{ and } E[\tilde{Z}_1^2] = E[Z_1^2] - E[Z_1 X_2'] E[X_2 X_2']^{-1} E[X_2 Z_1] \neq 0$$

By the block inversion formula, $E[Z_1^2] - E[Z_1 X_2'] E[X_2 X_2']^{-1} E[X_2 Z_1] \neq 0 \iff E[ZZ']$ is invertible. \square

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(ii) Show that $\Omega_{11} = \frac{\sigma_U^2}{E[\tilde{Z}_1^2]\pi_1^2}$ where $\tilde{Z}_1 = Z_1 - X_2'E[X_2X_2']^{-1}E[X_2Z_1]$.

Using the block inversion formula, define A and B :

$$\begin{aligned}
A &= E[ZX']^{-1} \\
&= \begin{pmatrix} E[Z_1X_1] & E[Z_1X_2'] \\ E[X_2X_1] & E[X_2X_2'] \end{pmatrix}^{-1} \\
B &= E[XZ']^{-1} \\
A_{11} &= B_{11} \\
&= (E[Z_1X_1] - E[X_2X_1]E[X_2X_2']^{-1}E[X_2X_1])^{-1} \\
&= E[\tilde{Z}_1X_1]^{-1} \\
A_{12} &= B_{21} \\
&= -E[Z_1X_1]^{-1}E[Z_1X_2']E[\tilde{Z}_1X_1]^{-1} \\
A_{21} &= B_{12} \\
&= -E[\tilde{Z}_1X_1]^{-1}E[X_2X_1]E[Z_1X_1]^{-1} \\
A_{22} &= B_{22} \\
&= E[X_2X_2']^{-1} + E[X_2X_2']^{-1}E[X_2X_1]E[\tilde{Z}_1X_1]^{-1}E[Z_1X_2']E[X_2X_2']^{-1}
\end{aligned}$$

Homoskedastic variance-covariance matrix:

$$\begin{aligned}
\Omega &= \sigma_U^2 E[ZX']^{-1} E[ZZ'] E[XZ']^{-1} \\
&= \sigma_U^2 AE[ZZ']B \\
&= \sigma_U^2 \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} E[Z_1^2] & E[Z_1X_2'] \\ E[X_2Z_1] & E[X_2X_2'] \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\
&= \sigma_U^2 \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11}E[Z_1^2] + B_{21}E[Z_1X_2'] & B_{12}E[Z_1^2] + B_{22}E[Z_1X_2'] \\ B_{11}E[X_2Z_1] + B_{21}E[X_2X_2'] & B_{12}E[X_2Z_1] + B_{22}E[X_2X_2'] \end{pmatrix} \\
&= \sigma_U^2 \begin{pmatrix} A_{11}B_{11}E[Z_1^2] + A_{11}B_{21}E[Z_1X_2'] + A_{12}B_{11}E[X_2Z_1] + A_{12}B_{21}E[X_2X_2'] & A_{11}B_{12}E[Z_1^2] + A_{11}B_{22}E[Z_1X_2'] + A_{12}B_{12} \\ A_{21}B_{11}E[Z_1^2] + A_{21}B_{21}E[Z_1X_2'] + A_{22}B_{11}E[X_2Z_1] + A_{22}B_{21}E[X_2X_2'] & A_{21}B_{12}E[Z_1^2] + A_{21}B_{22}E[Z_1X_2'] + A_{22}B_{12} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\Omega_{11} &= \sigma_U^2 [A_{11}B_{11}E[Z_1^2] + A_{11}B_{21}E[Z_1X_2'] + A_{12}B_{11}E[X_2Z_1] + A_{12}B_{21}E[X_2X_2']] \\
&= \sigma_U^2 [E[\tilde{Z}_1X_1]^{-1}E[\tilde{Z}_1X_1]^{-1}E[Z_1^2] \\
&\quad + E[\tilde{Z}_1X_1]^{-1}(-E[Z_1X_1]^{-1}E[Z_1X_2']E[\tilde{Z}_1X_1]^{-1})E[Z_1X_2'] \\
&\quad + (-E[Z_1X_1]^{-1}E[Z_1X_2']E[\tilde{Z}_1X_1]^{-1})E[\tilde{Z}_1X_1]^{-1}E[X_2Z_1] \\
&\quad + (-E[Z_1X_1]^{-1}E[Z_1X_2']E[\tilde{Z}_1X_1]^{-1})(-E[Z_1X_1]^{-1}E[Z_1X_2']E[\tilde{Z}_1X_1]^{-1})E[X_2X_2']] \\
&= \frac{\sigma_U^2}{E[\tilde{Z}_1^2]} \frac{E[\tilde{Z}_1^2]^2}{E[\tilde{Z}_1X_1]^2} \\
&= \frac{\sigma_U^2 E[\tilde{Z}_1^2]}{E[\tilde{Z}_1X_1]^2} \\
&= \frac{\sigma_U^2}{E[\tilde{Z}_1^2]\pi_1^2}
\end{aligned}$$

- (iii) Explain, using “regression language”, what $(\pi_1, \pi_2)'$ and \tilde{Z}_1 is.
- (iv) Show that $\Omega_{11} \geq \frac{\sigma_U^2}{E[Z_*^2]}$ where $Z_* = E[X_1|Z] - X_2'E[X_2X_2']^{-1}E[X_2E[X_1|Z]]$ and provide restrictions on $E[X_1|Z]$ such that $\Omega_{11} = \frac{\sigma_U^2}{E[Z_*^2]}$.
- ...
- (v) Suppose that $X_2 = 1$. Write Ω_{11}/σ_U^2 as a function of variances and covariances involving Z_1 and X_1 .
- ...

2. Let $(Y, X, Z)'$ be a random vector such that $Y = X\beta_1 + U$, $E[U|Z] = 0$ where $E[Y_4 + X_4 + Z_4] < \infty$. Also, let $\{(Y_i, X_i, Z_i)'\}_{i=1}^n$ be a random sample from the distribution of $(Y, X', Z)'$. Let h be a function of Z such that $E[h(Z)^4] < \infty$.

(i) Provide conditions such that $E[h(Z)(Y - X\beta_1)] = 0$ iff $\beta = \beta_1$.

...

(ii) Derive a method of moments estimator of β_1 , say $\hat{\beta}_1^h$, using the IV moment in (i).

...

(iii) Under the conditions provided in (i), show that $\sqrt{n}(\hat{\beta}_1^h - \beta_1) \rightarrow_d N(0, \Omega^h)$ for some asymptotic variance $\Omega^h \geq 0$.

...

(iv) Show that $\Omega^h \geq E[\frac{E[X|Z]^2}{E[U^2|Z]}]^{-1}$ and find a function h , such that Ω^h achieves this lower bound.

...

3. Consider the data from Angrist and Krueger (1991) provided on the course website and the following linear model for $\log(wage)$ as a function of education and additional control variables: $\log(wage) = \beta_0 + educ \cdot \beta_1 + \sum_{t=31}^{39} 1\{yob = t\}\beta_t + \sum_{s=1}^{50} 1\{sob = s\}\gamma_s + U$, where yob is year of birth and sob is state of birth. As instruments for $educ$ consider three instruments: $1\{qob = 2\}, 1\{qob = 3\}, 1\{qob = 4\}$ where qob is quarter of birth. Using matrix algebra and your preferred statistical software, write code that loads the data and computes the 2SLS estimate of β_1 and the heteroskedasticity robust standard error stemming from the variance estimator formula (12.40) on page 354 of Bruce Hansen's textbook. The solution to this exercise should include code and the two numbers produced by it.

```
data <- read_csv("AK91.csv", col_types = "ddddd")

n <- nrow(data)

# prep variables
y <- data$lwage
x_1 <- data$educ
controls <- cbind(rep(1, n),
                  to.dummy(data$yob, prefix = "yob")[,2:10],
                  to.dummy(data$sob, prefix = "sob")[,2:51])
instruments <- to.dummy(data$qob, prefix = "qob")[,2:4]

z <- cbind(instruments, controls)
x <- cbind(x_1, controls)

k <- ncol(x)
l <- ncol(z)

# Estimating 2sls beta using 12.29 in Hansen
pi_hat <- solve(t(z) %*% z) %*% t(z) %*% x_1
z_2 <- cbind(z %*% pi_hat, controls)
beta_2sls <- solve(t(z_2) %*% x) %*% t(z_2) %*% y

# Estimating heteroskedastic robust standard errors using 12.40 in Hansen
e_hat <- as.numeric(y - x %*% beta_2sls)

omega <- crossprod(z_2 * e_hat)

varcov <- solve(t(z_2) %*% x) %*% omega %*% solve(t(x) %*% z_2)

print(beta_2sls[1])

## [1] 0.1083936
print(sqrt(varcov[1,1]))

## [1] 0.01954637
```