Firm Default with Long-Term Debt

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Motivation

- Firm bankruptcy is important for firm dynamics through debt prices
- Corbae and D'Erasmo (2021) extend structural corporate finance models (i.e., Gomes (2001), and Hennessy and Whited (2007)) by adding Ch. 7 liquidation and Ch. 11 reorganization bankruptcy
- All debt in CD (2021) is short-term in the sense that it matures before new debt can be issued
- Long-term debt allows debt dilution; issuing new debt reduces value of existing debt obligations
- But long-term debt increases computational complexity (i.e., adds state variable)
- Arellano (2008) sovereign default model is related to Corbae and D'Erasmo (2021) More
- Bornstein (2020) recasts Arellano (2008) in continuous time, uses methods of Achdou et al (2022), and then adds long-term debt a la Chatterjee and Eyigungor (2012)
- I want to apply Bornstein (2020) approach to Corbae and D'Erasmo (2021)

Continuous Time Methods and Incomplete Markets

- Growing use of continuous time methods to incomplete-markets models
  - Financial frictions (Brunnermeier and Sannikov 2014)
  - Dynamics of inequality (Gabaix et al 2016)
- Achdou et al (2022) develop efficient and portable algorithm to solve Aiyagari-Bewley-Huggett style models in continuous time
- Recent applications of Achdou et al (2022) methods
  - Monetary policy with heterogeneous agents (Kaplan et al 2016)
  - Mortgage refinancing (Laibson et al 2020)
  - Durable consumption (McKay and Wieland 2021)
  - Precautionary savings with housing (Guerrieri et al 2020)
  - Marital wage premia (Pilossoph and Wee 2021)
- Bornstein (2020) is only strategic equilibrium default model to use Achdou et al (2022) methods

Achdou et al (2022) Summary Details

- Recast Aiyagari-Bewley-Huggett in continuous time
- Key advantages over discrete time:
  1. Optimal for assets to evolve continuously
  2. Simultaneous change in asset and change in earning state is measure 0 event
- From (1), agent in interior knows she will reoptimize before hitting BC. So given guess of HJB, BC "disappears" for agents in interior and can "brute force" asset decision at BC
- From (1) and (2), in an instant in the future, agent at (a, y) can be in at most four states
  - One asset grid point higher and same earning state: $(a + \Delta a, y)$
  - Same level of assets and same earning state: (a, y)
  - One asset grid point lower and same earning state: $(a - \Delta a, y)$
  - Same level of assets and different earning state: $(a, \tilde{y})$

  So, transition matrix is sparse and near block diagonal $\implies$ easy to invert
- Anecdotal experience:
  - My discrete time Huggett (from ECON 899) took about 40 seconds
  - My version of Achdou et al (2022) took about 0.1 second
  - Achdou et al (2022) is "less robust"

Bornstein (2022) with Short-Term Debt

- As in Arellano (2008), a sovereign faces fluctuations in domestic output and can borrow or save a constant world interest rate to smooth domestic consumption
- At any time, the sovereign can default on debt obligations and be excluded from world financial markets for a stochastic period of time and face output losses
- Sovereign problem is close to Achdou et al (2022) but no BC and r depends on state variables

$$
\begin{aligned}
&E_0 \int_0^\infty e^{-\rho t} u(c_t) dt \\
\text{s.t. } \dot{a}_t &= y_t - c_t + r(y_t, a_t) a_t && \text{out of financial autarky} \\
0 &= y_t - \phi(y_t) - c_t && \text{in financial autarky}
\end{aligned}
$$

where y_t follows compound Poisson process with arrival rate and distribution $F(y', y)$ and $\phi(\cdot)$ is output loss in autarky

- Jumps are critical for endowment process for sovereigns (or productivity process for firms)
  - Endowment process is diffusion $\implies$ probability of default on default frontier is one $\implies$ interest rate on default frontier is infinity $\implies$ sovereign never defaults in equilibrium

Bornstein (2022) with Short-Term Debt (con't)

- HJB of sovereign in autarky

$$\rho w(y) = u(y - \phi(y)) + \lambda_y \int_0^\infty (w(y') - w(y)) f(y', y) dy' + \lambda_D [v(0, y) - w(y)]$$

where sovereign gets out of autarky with Poisson intensity λ_D

- HJB of sovereign out of autarky

$$\rho v(a, y) = \max_c \left\{ \rho w(y), u(c) + v_a(a, y)[y - c + r(a, y)a] + \lambda_y \int_0^\infty (v(a, y') - v(a, y)) f(y', y) dy' \right\}$$

- Zero profit condition for outside investor

$$r(a, y) - \lambda_y \int_0^\infty D(a, y') f(y', y) dy' = r^f$$

where $D(a, y')$ indicates whether sovereign defaults

- Follow combination of Achdou et al (2022) and Hennessy and Whited (2007) algorithms to solve

Why is long-term debt pricing easier in continuous time?

- Bornstein (2020) sets up problem with long-term debt a la Chatterjee and Eyigungor (2012)
- Long-term bonds mature with Poisson intensity λ_b and pays flow payment z before maturing
- Flow budget constraint

$$c_t + q(a_t, y_t)\lambda_b a_t + q(a_t, y_t)\dot{a}_t = y_t + z a_t + \lambda_b a_t$$
$$\implies s_t \equiv \dot{a}_t = \frac{y_t + (z + \lambda_b)a_t - c_t}{q(y_t, a_t)} - \lambda_b a_t$$

- Zero profit condition

$$q(a, y) = \frac{1}{r^f}\left[z + \lambda_b(1 - q(a, y)) + \lambda_y \int_0^\infty [q(a, y') - q(a, y)]dF(y'|y) + s(a, y)q_a(a, y)\right]$$

- In discrete time, we have to solve a fixed point problem
- In continuous time, we can use finite differences and sparse matrix inversion

Conclusion

- Today, I discussed recent methods for continuous time incomplete-markets models
- I discussed how these methods are applied in equilibrium sovereign default models
- Long-term debt is easier to work with in these models
- My plan going forward is to try to implement this approach in a model looking at firm dynamics/bankruptcy/liquidation/reorganization problem

Equilibrium Strategic Default Models

Topic	Discrete Time	Continuous Time
Consumer Default	Chatterjee et al (2007)	—
Sovereign Default	Arellano (2008) Chatterjee and Eyigungor (2012)	Bornstein (2020)
Firm Default	Corbae and D'Erasmo (2021)	—

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Achdou et al (2022) Details

- HH maximize lifetime expected utility subject to exogenous BC

$$
\begin{aligned}
E_0 \int_0^\infty e^{-\rho t} u(c_t) dt \\
\text{s.t. } \dot{a}_t = y_t - c_t + r a_t \\
a_t \geq \underline{a}
\end{aligned}
$$

where y_t follows Poisson process with two states $y_L < y_H$ with arrival rate λ

- Stationary Hamilton-Jacob-Bellman equation

$$
\rho V(a, y) = \max_c \{u(c) + \dot{a} V_a(a, y)\} + \lambda [V(a, \tilde{y}) - V(a, y)]
$$

- FOC: $u_c(c) = V_a(a, y) \implies c(a, y) = (u_c)^{-1}(V_a(a, y))$

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Achdou et al (2022) (con't) Details

- Discretize $\implies$ $V_{i,j}$ is HJB and $\partial V_{i,j}$ is partial derivative of HJB wrt a at grid point (a_i, y_j)
- Given $V_{i,j}$, approximate derivative using finite forward and backward differences

$$\partial V_{i,j}^F \equiv \frac{V_{i,j} - V_{i+1,j}}{\Delta a}, \quad \partial V_{i,j}^B \equiv \frac{V_{i,j} - V_{i-1,j}}{\Delta a}$$

- FOCs implies asset drift for each finite difference

$$\dot{a}_{i,j}^F = y_i + ra_i - (u_c)^{-1}(\partial V_{i,j}^F), \quad \dot{a}_{i,j}^B = y_i + ra_i - (u_c)^{-1}(\partial V_{i,j}^B)$$

- V, u are concave $\implies$ $\partial V_{i,j}^F < \partial V_{i,j}^B$ $\implies$ $\dot{a}_{i,j}^F < \dot{a}_{i,j}^B$
- Use "upwinding" rule to chose which finite difference:

$$\partial V_{i,j} \equiv \begin{cases} \partial V_{i,j}^F, & \text{if } \dot{a}_{i,j}^B > \dot{a}_{i,j}^F > 0 \\ \partial V_{i,j}^B, & \text{if } 0 > \dot{a}_{i,j}^B > \dot{a}_{i,j}^F \\ \partial V_{i,j}^0, & \text{if } \dot{a}_{i,j}^B \geq 0 \geq \dot{a}_{i,j}^F \end{cases}$$

where $\dot{a}_{i,j}^0 \equiv 0 \implies c_{i,j}^0 = y_j + ra_i \implies \partial V_{i,j}^0 \equiv u_c(y_j + ra_i)$

- Brute force at BC $\implies$ $\partial V_{1,j}^B = u_c(y_j + ra_1)$

Achdou et al (2022) (con't) Details

- How to update HJB guess? Intuition is to update nonstationary HJB until it converges:

$$\frac{\partial V(t,a,y)}{\partial t} + \rho V(t,a,y) = \max_c \{u(c) + \dot{a} V_a(t,a,y)\} + \lambda[V(t,a,\tilde{y}) - V(t,a,y)]$$

- Natural first guess $\implies$ the "explicit method"

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n} = u(c_{i,j}^{n}) + \dot{a}_{i,j}^{n} \partial V_{i,j}^{n} + \lambda[V_{i,-j}^{n} - V_{i,j}^{n}]$$

However, explicit method is "badly behaved"

- "Implicit method" works better

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} + \rho V_{i,j}^{n+1} = u(c_{i,j}^{n}) + \dot{a}_{i,j}^{n} \partial V_{i,j}^{n+1} + \lambda[V_{i,-j}^{n+1} - V_{i,j}^{n+1}]$$

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Achdou et al (2022) (con't) Details

- Implicit method results in closed form expression for updating HJB:

$$\frac{V^{n+1} - V^n}{\Delta} + \rho V^{n+1} = u^n + \mathbf{A}^n V^{n+1}$$

$$\text{where } V^n = [V^n_{1,1}, ..., V^n_{N,1}, V^n_{1,2}, ..., V^n_{N,2}]^T$$

$$u^n = [u(c^n_{1,1}), ..., u(c^n_{N,1}), u(c^n_{1,2}), ..., u(c^n_{N,2})]^T$$

$$\mathbf{A}^n = \begin{bmatrix} y_{1,1} & z_{1,1} & 0 & \dots & 0 & 0 & \lambda & 0 & \dots & 0 \\ x_{2,1} & y_{2,1} & z_{2,1} & \dots & 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & x_{N_a,1} & y_{N_a,1} & 0 & 0 & \dots & \lambda \\ \lambda & 0 & 0 & \dots & 0 & 0 & y_{1,2} & z_{1,2} & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 & 0 & x_{2,2} & y_{2,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & \lambda & 0 & 0 & \dots & y_{N_a,2} \end{bmatrix}$$

- $\mathbf{A}^n$ is large but it is sparse with at most 4 nonzero entries per row
- Solving for the stationary distribution essentially boils down to inverting A^n