Alex von Hafften
Dec. 16
UW ID# 907 934 4306

ECON 712

① a. The social planner maximizes household utility subject to resource feasibility and the firm problem.
Since the firm production is CRS, there's no profit. Thus, the planner problem is

$$\max_{\{C_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad K_{t+1} = (1-\delta)K_t + Y_t - C_t$$
$$Y_t = F(A_t K_t, N_t)$$
$$N_t = 1$$

Define $f(A_t K_t) = F(A_t K_t, 1)$

So the social planner's problem becomes

$$\max_{\{K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{\left[f(A_t K_t) + (1-\delta)K_t - K_{t+1}\right]^{1-\gamma}}{1-\gamma}$$

$$\cancel{FOC[K_t]: \beta^t[f'(A_t K_t)A_t + (1-\delta)] - \beta^{t-1} = 0}$$

$$FOC[K_t]\; 0 = \beta^t \left[f(A_t K_t) + (1-\delta)K_t - K_{t+1}\right]^{-\gamma}$$
$$\cdot \left[f'(A_t K_t)A_t + (1-\delta)\right]$$
$$+ \beta^{t-1}\left[f(A_t K_{t-1}) + (1-\delta)K_{t-1} - K_t\right]^{-\gamma}$$
$$\cdot [-1]$$

(1) a. con't We get Euler equation:

$$\Rightarrow \beta^t [C_t]^{-\gamma} \cdot (f'(A_t K_t) A_t + 1 - \delta) = \beta^{t-1} [C_{t-1}]^{-\gamma}$$

$$\Rightarrow \beta \, C_t^{-\gamma} \, (f'(A_t K_t) A_t + 1 - \delta) = C_{t-1}^{-\gamma}$$

Iterate forward one period.

$$\Rightarrow \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \left[f'(A_{t+1} K_{t+1}) A_{t+1} + 1 - \delta \right] = 1 \quad (*)$$

The law of motion of capital

$$K_{t+1} = (1-\delta) K_t + Y_t - C_t$$
$$K_{t+1} = (1-\delta) K_t + f(A_t K_t) - C_t \quad (**)$$

$(*)$ and $(**)$ are two equation for the two unknowns : C_t and K_{t+1}.

① (b) Normalize by the capital-augmenting technology: A_t

Define $c_t = \frac{C_t}{A_t}$ and $k_t = \frac{K_t}{A_t}$

Since the production function satisfies usual assumptions: [i.e. CRS].

$$y_t = \frac{Y_t}{A_t} = \frac{F(A_t K_t, 1)}{A_t} = \frac{f(A_t K_t)}{A_t} = f(k_t).$$

So the LOM for capital in transformed variables becomes:

$$\frac{K_{t+1}}{A_t} = (1-\delta)\frac{K_t}{A_t} + \frac{Y_t}{A_t} - \frac{C_t}{A_t}$$

$$\Rightarrow \frac{K_{t+1}}{A_t}\frac{A_{t+1}}{A_{t+1}} = (1-\delta)k_t + y_t - c_t$$

$$\Rightarrow (1-g)k_{t+1} = (1-\delta)k_t + y_t - c_t$$

On BGP: $k_{t+1} = k_t = \bar{k}$, and $c_t = \bar{c}$,

$$(1-g)\bar{k} = (1-\delta)\bar{k} + f(\bar{k}) - \bar{c}$$

$$\Rightarrow \bar{c} = (1-\delta)\bar{k} + f(\bar{k}) - (1-g)\bar{k}$$

$$\Rightarrow \bar{c} = (g-\delta)\bar{k} + f(\bar{k}).$$

① (b) The Euler equation becomes:

$$\beta \left[\frac{c_{t+1}/A_{t+1}}{c_t/A_t} \left(\frac{A_{t+1}}{A_t} \right) \right]^{-\gamma}$$

$$\left[f'(A_{t+1}(A_{t+1} k_{t+1}))A_{t+1} + 1 - \delta \right] = 1$$

$$\Rightarrow \beta \left[\frac{c_{t+1}}{c_t} (1+g) \right]^{-\gamma} \cdot \left[f'(k_{t+1}) A_{t+1}^3 + 1 - \delta \right] = 1$$

...

1(c) ~~increases~~

If $g \uparrow$ to g', capital has become more productive. Thus, in the ~~[illegible]~~ long-run, the level of capital and consumption are both higher.

On impact, consumption increases to the saddle path associated with the new balanced growth path and capital slowly increase to its long-run rate of growth.

② a. Conjecture that $p_t = p(B, s)$.

Thus, the Bellman equation is

$$V(a, B, s) = \max_{c, a'} \left\{ u(c, B) + \beta E\left[V(a', B', s')\right] \right\}$$

$$\text{s.t.} \quad c + p(B,s) a' = (p(B,s) + s) a$$

$$\Rightarrow V(a, B, s) = \max_{a'} \left\{ u\left[(p(B,s) + s) a - p(B,s) a', B\right] + \beta \iint V(a', B', s') F(B, s, dB') Q(s, ds') \right\}$$

The individual state variables are
- a, the holding of trees coming into the period
- B, the realization of the preference shock.
- s, the realization of the stochastic dividend.

FOC $[a']$:

$$0 = u'\left[(p(B,s) + s) a - p(B,s) a', B\right] (-p(B,s)) + \beta \iint V'(a', B', s') F(B, s, dB') Q(s, ds')$$

$$\Rightarrow p(B,s) u'(c, B) = \beta E\left[V'(a', B', s')\right]$$

Envelope Condition

$$V'(a, B, s) = u'\left[(p(B,s) + s) a - p(B,s) a', B\right] \cdot (p(B,s) + s)$$

$$\Rightarrow V'(a, B, s) = u'\left[c, B\right] \cdot (p(B,s) + s)$$

(2)(a) ~~[illegible]~~ Con't

Iterate Env condition forward one period:

$$V'(a', B', s') = u'(c', B') \cdot [P(B', s') + s']$$

FOC + Env => Euler equation

$$\Rightarrow u'(c, B)\, p(B, s) = \beta E\left[u'(c', B') \cdot [P(B', s') + s']\right]$$

$$\Rightarrow 1 = \beta \iint \frac{u'(c', B')}{u'(c, B)} \frac{P(B', s') + s'}{P(B, s)} F(B, s, dB')\, Q(s, ds')$$

In equilibrium, $a = a' = 1$ and $c(s) = s$:

$$(*) \Rightarrow 1 = \beta \iint \frac{u'(s', B')}{u'(s, B)} \frac{P(B', s') + s'}{P(B, s)} F(B, s, dB')\, Q(s, ds')$$

The assumptions that allow me to do this are

- u is bounded and continuous w/ $0 < \beta < 1$.
-
-
-
-

As in lecture, we can argue that the relevant state variable is realized wealth $(p(B,s) + s)a$. Thus the Bellman equation for this formulation is

$$V((p(B,s) + s)a, B) = \max_{a'} \Big\{ u\big((p(B,s) + s)a - p(B,s)a', B\big) + \beta \iint V\big((p(B', s') + s')a'\big)\, F(B, s, dB')\, Q(s, ds') \Big\}$$

(2) (a) con't

FOC (a'):

$$0 = -u(c, B)\, P(B,s) + \beta\, E\left[V'((P(B',s')+s')a') \right.$$
$$\left. (P(B',s')+s') \right]$$

$$\Rightarrow u(c,B)\, P(B,s) = \beta E\left[V'((P(B',s')+s')\, a') \right.$$
$$\left. (P(B',s')+s') \right]$$

ENV: $V'((P(B,s)+s)a, B) = u'(c, B)$

Euler EQ:

$$u(c,B)\, P(B,s) = \beta E\left[u'(c',B')(P(B',s')+s') \right]$$

Thus, it results in the same Euler equation.

(2) (b) A recursive equilibrium is a continuous function $p(B, s)$ and a continuous, bounded function $v(a, B, s)$ such that

1. $v(a, B, s)$ solves the Bellman Eq in part (a).

2. $\forall s$, $v(1, B, s)$ is attained by $c = s$, $a' = 1$.

(2) (C) The Euler equation implies that the equilibrium price of a tree is:

$$P(B, s) = \beta \iint \frac{u'(s', B')}{u'(s, B)} \left(P(B', s') + s'\right) F(B, s, dB') \cdot Q(s, ds')$$

It differs from the standard case because ~~the~~ the price depends on the realization of the preference shock.

(2) (d) We can price any contingent claim $g(s', B')$ one-period ahead as:

$$p^g(s,B) = \mathbb{E}\left[\beta \frac{u'(s',B')}{u'(s,B)} g(s',B') \mid s,B\right]$$

Specializing to the functional form of the utility function:

$$p^g(s,B) = E\left[\beta \frac{B'}{B} \cdot \frac{s}{s'} g(s',B') \mid s, B\right]$$

For asset 1, $g_1(s', B')$ is positively correlated w/ B' and $g_2(s,B)$ is positively correlated w/ s'.

Clearly asset 1 will have a higher price than asset 2 because asset 1 pays off more when the agent cares more about consuming more and pays off less when agents care about consuming less. ~~Asset 2 on the other hand pays off more when~~

Asset 2 is independent of the preference shocks and payoff more in "good" states of the world when s_t is high.

(3) (a) True ~~If the feasible set $\Gamma(x)$ is nonempty $\forall x \in X$, the choice set $\forall x_0 \in X$ and $x \in \Gamma(x_0)$~~

Let X be the choiceset, $\Gamma(x)$ be the feasible set correspondence, Π be the feasible plan correspondence, and $F: A \to \mathbb{R}$ be objective function w/ $A = \text{graph}\,\Gamma$.

If ① $\Gamma(x)$ is nonempty $\forall x \in X$,
② $\forall x_0 \in X$ and $x \in \Pi(x_0)$

$$\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) \text{ exists.}$$ Then the solution of a dynamic optimization problem

can be found either w/ a sequence of recursive formulation

(b) True, if there are complete markets. In complete markets, there is an asset for each state of the world that can be price independently of assets that payoff is other states of the world. In the recursive equilibrium, that means that there must be an asset for each of the state in the next period. In the Arrow Debreu equilibrium, that means that there must be an asset for each history of states of the world. In other words, there need to be a full array of state-contigent claims.

(3)(c) Generally false, in incomplete markets there are fewer assets than the state of world. In incomplete markets agents bear some of the risk of fluctuations in their income.

This statement would be true if the markets are so incomplete that there is no consumption smoothing. If there are no assets, or trading, then agent can only consume their income, so then they would bear all the risk of fluctuations of their income.

(d) ~~Generally~~ False, in the setup of Kehoe-Robinson general equilibrium model, agents could differ in their preferences, discount rates, and endowments. We saw that differences in endowments did not lead to differences in allocations across agents. However, differences in preferences and specific their discount rate did lead to heterogeneity in the allocations across agents. For example,

~~Generally however, if agents are identical~~

in the LR, the most patient agent in the economy consumes the entire endowment.