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Analytical Approximations in Models of Hysteresis

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Decisions made under ongoing uncertainty and costly reversibility entail a range of the state variable where inaction is optimal, which in turn produces hysteresis—permanent effects of temporary shifts The range is usually defined by non-linear equations that need numerical solutions. In this paper a technique of analytical approximations is developed and applied to two models—menu costs and investment. The resulting explicit solutions help clarify why hysteresis is important even for small irreversibility. In the menu cost model, hysteresis is two orders of magnitude larger than under the Akerlof-Yellen or Mankiw assumptions

1. INTRODUCTION

When decisions that are costly to reverse are made under conditions of ongoing uncertainty, the optimum policy usually entails a range of inertia. The action should be taken only when the value of the underlying state variable becomes especially favourable, and reversed only when it becomes especially unfavourable. There is an intermediate range over which inaction is optimal. This in turn gives rise to hysteresis. Starting from a value in the range of inertia, if the state variable crosses the action trigger point and then returns to its original level, the action will be taken but not reversed. Thus a temporary change will leave a permanent effect.

Numerical solutions for models of irreversible investment show that the range of inaction is remarkably large even when the cost of reversal is small; see Brennan and Schwartz (1985), Constantinides (1986) and Dixit (1989*a*, *b*) for examples of this. Unfortunately, the equations that determine the optimum policy are highly non-linear. They do not have analytical solutions. Therefore we lack a clearer and more general understanding of why hysteresis is numerically so important. In this paper I derive analytical approximations for the solutions of more general problems of this kind when the costs of change are small.

The idea that small costs of change induce large amounts of inertia has been popularized in the model of Akerlof and Yellen (1985), Mankiw (1985) and others. In these models, second-order small costs of making price changes yield price-stickiness in response to first-order changes in the money supply. This is a simple application of the envelope theorem. But these models are static, and their application to a dynamic reality must assume either permanent shocks or static expectations. When there is ongoing uncertainty, which is rationally expected by the decision-maker, the option value of the status quo enlarges the zone where inaction is optimal. Adding these features to the Mankiw-Akerlof-Yellen (MAY) model, I show that *fourth*-order small costs of change generate first-order inertia.

I also consider the model of entry and exit of a firm, or equivalently, the activation and abandonment of an investment project, as the flow profits from its operation fluctuate.

Once again I find that small sunk costs have large effects in the form of inertia of the optimal decision.

2. THE QUADRATIC MENU COST MODEL

Consider a dynamic system with a state variable x. In the MAY model, this can be the logarithm of the price of one firm relative to the general price level. The ideal value of x is zero. Any departure from 0 entails a flow cost of penalty of $f(x) = kx^2$. This can be thought of as the leading term in a smooth convex adjustment cost; the first-order term vanishes as a condition of optimality at 0. It is possible to change the value of x by applying an instantly effective control at a lump-sum cost g. Suppose x is initially at 0, and a shock moves it to s. If this is a once-and-for-all shock, or if the decision-maker has static expectations, then the discounted present value of the cost of leaving things alone is ks^2/ρ, where ρ is the rate of interest. The cost of restoring x to 0 is g. Therefore it is optimal to leave things alone when s is sufficiently small. The critical value where it is only just optimal to act is given by

$$s = \sqrt{\rho g/k}. \tag{1}$$

If g is of the second-order of smallness (ε^2 say), then the range $(-s, s)$ where inaction is optimal is of the first-order of smallness (ε). It is in this sense that small menu costs have a relatively large effect.

The MAY model is static; in each period the cost of the deviation is compared to the cost of restoration, so the comparison is between kx^2 and g. The result is similar. But in reality, such periods follow one another. Then the decision-maker should recognize that the successive shocks to x constitute a stochastic process, and find the optimal plan of adjustment. That is the contribution of this section.

To conform with the symmetry assumed for the flow cost of deviation, I shall assume that x follows a symmetric random walk, or a driftless Brownian motion in continuous time. If, in the course of its stochastic evolution, x reaches $\pm s$, it is no longer optimal to incur the cost g to restore it to 0. One can wait for a small amount of time to see if x moves back toward zero of its own accord. This "option value" of the status quo means that the zone of inertia is wider. It is interesting to note that going from static expectations to rational ones enlarges the importance of menu costs. My purpose is to obtain a quantitative estimate of this enlargement.

Write the stochastic process as

$$dx = \sigma dz, \tag{2}$$

where z is the standard Wiener process. We want to find

$$V(x) = \min E\left\{ \int_0^{\infty} kx_t^2 e^{-\rho t} dt + \textstyle\sum_i g e^{-\rho t_i} \,\middle|\, x_0 = x \right\}, \tag{3}$$

where t_i denotes the instants where x_t is shifted by exercising the control.

This is a problem of "impulse control"; see Harrison, Sellke and Taylor (1983), or the simplified exposition in Dixit (1989c). The solution consists of three numbers $x_1 < x_2 < x_3$, such that in the interval (x_1, x_3) no action is taken, but if x is initially outside the interval, or if it hits either endpoint during its course of evolution, it is immediately moved to the central value x_2. Here, by symmetry we have $x_2 = 0$ and $x_1 = -h$, $x_3 = h$, where h is to be determined.

In the interval $(-h, h)$, we can express the right-hand side of the integral in (3) as the sum of the flow cost in a small interval dt and the expectation of the Bellman value function at the end of this interval:

$$V(x) = kx^2 dt + E[V(x+dx)e^{-\rho dt}].$$

Expanding the right-hand side using Itô's Lemma, we get the differential equation

$$\tfrac{1}{2}\sigma^2 V''(x) - \rho V(x) + kx^2 = 0. \tag{4}$$

Writing

$$\alpha = \sqrt{2\rho/\sigma^2}, \tag{5}$$

and noting symmetry, the general solution becomes

$$V(x) = A(e^{\alpha x} + e^{-\alpha x}) + kx^2/\rho + k\sigma^2/\rho^2, \tag{6}$$

where A is a constant to be determined. Harrison (1985, p. 50) shows that the last two terms on the right-hand side constitute the discounted present value of the flow cost if we let x_t drift for ever and never exercise the control. Then the terms involving A must be the cost reduction that can be achieved by optimal control; therefore A should be negative.

The conditions to determine A and h are as follows. (1) The reduction in the value of V by moving from h to 0 should equal g, the cost of this move. That is, $V(h) - V(0) = g$. This is the Value Matching Condition.

$$A(e^{\alpha h} + e^{-\alpha h}) + kh^2/\rho + k\sigma^2/\rho^2 - 2A - k\sigma^2/\rho^2 = g,$$

or

$$A(e^{\alpha h} + e^{-\alpha h} - 2) = -(kh^2/\rho - g). \tag{7}$$

(2) The first-order condition for the optimum choice of h, or the Smooth Pasting Condition. This says that $V'(h) = 0$, or

$$A\alpha(e^{\alpha h} - e^{-\alpha h}) + 2kh/\rho = 0. \tag{8}$$

Note that, with $\alpha > 0$, the bracketed expression on the left-hand side of (8) is positive. Therefore $A < 0$. Also, in (7),

$$e^{\alpha h} + e^{-\alpha h} - 2 = 2[\cosh(\alpha h) - 1] > 0,$$

so $kh^2/\rho > g$, or comparing (1), $h > s$. This confirms the intuition stated earlier: the range of inertia is wider under ongoing uncertainty and rational expectations than it is with one-off shocks or static expectations.

To derive the analytical approximation to h when menu costs are small, we divide (7) by (8) and further by h to write

$$\frac{e^{\alpha h} + e^{-\alpha h} - 2}{\alpha h(e^{\alpha h} - e^{-\alpha h})} = \frac{1}{2}\left\{1 - \frac{\rho g}{kh^2}\right\}. \tag{9}$$

Expand the left-hand side of this, assuming that αh is sufficiently small; I shall postpone the discussion of the validity of this assumption to a later point. We have

$$\frac{2(1 + \frac{1}{2}\alpha^2 h^2 + \frac{1}{24}\alpha^4 h^4 + \cdots) - 2}{2\alpha h(\alpha h + \frac{1}{6}\alpha^3 h^3 + \cdots)},$$

or

$$\tfrac{1}{2}(1+\tfrac{1}{12}\alpha^2h^2+\cdots)(1-\tfrac{1}{6}\alpha^2h^2+\cdots)=\tfrac{1}{2}(1-\tfrac{1}{12}\alpha^2h^2+\cdots).$$

Using this on the left-hand side in (9) and comparing leading terms, we have

$$h^4=\frac{12\rho g}{\alpha^2 k}=\frac{6\sigma^2 g}{k}, \tag{10}$$

or

$$h=\left(\frac{6\sigma^2 g}{k}\right)^{1/4}. \tag{11}$$

The zone of inertia is fully two orders of magnitude larger than in the MAY model—menu costs that are small of the fourth-order ($g\sim\varepsilon^4$) generate a first-order range where inaction is optimal.

It would be useful to have some numerical feel for the magnitudes involved. This is hard to do in the MAY model, because small costs of price changes are hard to quantify. But a different interpretation readily yields some back-of-the-envelope calculations. Consider the problem of the optimum management of exchange rates. Suppose x is the log of the exchange rate. Take $\sigma=0{\cdot}1$; this is roughly the right magnitude for the variation of real exchange rates over a year in recent times. Let the real interest rate be $\rho=0{\cdot}05$. In the days when exchange rates were fixed, and shocks were more like one-time events, a 10% deviation of the rate was thought to be very large. Therefore suppose that, under the alternative assumption of one-time shocks or static expectations, it would have been optimal to act when the exchange rate deviated by that much from its ideal level, that is, $s=0{\cdot}1$. This pins down the magnitudes that are otherwise hard to find, namely g and k, as $g/k=s^2/\rho=\frac{1}{5}$ using (1). Then (11) gives $h=0{\cdot}331$, so the zone of inaction is more than three times as wide for the case of rational expectations under a managed float than it would be with static expectations.

This also gives us the opportunity to check the validity of the approximation in practice. An exact calculation using (9) gives $h=0{\cdot}340$, so the approximation is good to within 3%.

We can also find the algebraic conditions for the approximation to be valid. Using (11), we have

$$\alpha h=(24\, g\rho^2/(k\sigma^2))^{1/4}. \tag{12}$$

We want this to be small. In the numerical example, we have $\alpha h=1{\cdot}075$, which is hardly small, but the approximation proves quite accurate. The reason is that the higher-order terms in the expansion of the left-hand side of (9) get small very rapidly. For example, in the numerator, the next term is $(\alpha h)^6/720$, and in the denominator the next term is $(\alpha h)^5/120$. Therefore the approximation that omits such terms is quite robust.

It is also useful to check that the expression in (12) is a pure number, to be sure that nothing in this procedure depends on any arbitrary choice of units. Denote the units of length, time and cost by [L], [T] and [\$] respectively. Then the units of g are [\$], those of ρ^2 are $[\mathrm{T}]^{-2}$, those of k are $[\$][\mathrm{L}]^{-2}[\mathrm{T}]^{-1}$, and those of σ^2 are $[\mathrm{L}]^2[\mathrm{T}]^{-1}$. It is easy to check that the result is dimensionless.

3. MORE GENERAL MENU COST MODELS

Next suppose the flow cost of deviation from the ideal $x=0$ is $f(x)=k|x|$. This is not differentiable at 0. The envelope theorem does not apply, and in the MAY model with one-time shocks or static-expectations, first-order small menu costs are needed to generate first-order inertia. Using the notation introduced in the previous section, the maximum deviation that goes uncorrected is given by $s = \rho g/k$.

But when x follows the driftless Brownian motion $dx = \sigma dz$, and optimal adjustments are made with rational expectations about the stochastic process, a MAY-type result still obtains: *third*-order small menu costs have first-order inertia effects.

To see this, proceed as in the previous section. Define $V(x)$ to be the minimum expected discounted present value of the costs, including the costs of control, when the process starts at x. In the region where no control is applied, we have the differential equation

$$\tfrac{1}{2}\sigma^2 V''(x) - \rho V(x) + k|x| = 0. \tag{13}$$

This must be solved separately for positive and negative regions of x. The general solution is easily seen to be

$$V(x) = \begin{cases} A_- e^{-\alpha x} + B_- e^{\alpha x} - kx/\rho & \text{when } x < 0 \\ A_+ e^{-\alpha x} + B_+ e^{\alpha x} + kx/\rho & \text{when } x > 0, \end{cases}$$

where α is defined in (5) above, and A_-, B_-, A_+ and B_+ are constants to be determined. By symmetry, we set $A_+ = B_- = a$, and $A_- = B_+ = b$, to write

$$V(x) = \begin{cases} ae^{\alpha x} + be^{-\alpha x} - kx/\rho & \text{when } x < 0 \\ ae^{-\alpha x} + be^{\alpha x} + kx/\rho & \text{when } x > 0. \end{cases} \tag{14}$$

When f is continuous, V is differentiable; this follows from Theorem 4.4.9 of Karatzas and Shreve (1988). Therefore the solutions for the two sides must be smoothly pasted together at zero: $V'(0+) = V'(0-)$. This gives

$$-\alpha a + \alpha b + k/\rho = \alpha a - \alpha b - k/\rho.$$

Then each side must be zero, so

$$b - a + k/(\alpha\rho) = 0. \tag{15}$$

The other two equations to determine the constants a, b and the optimal choice of h are once again the Value Matching and Smooth Pasting conditions at $\pm h$. We find

$$ae^{-\alpha h} + be^{\alpha h} + kh/\rho - g = a + b, \tag{16}$$

and

$$-ae^{-\alpha h} + be^{\alpha h} + k/(\alpha\rho) = 0. \tag{17}$$

Solve (15) and (17) for a and b, and substitute the result in (16). This gives

$$\frac{2(e^{\alpha h} - 1)(1 - e^{-\alpha h})}{e^{\alpha h} - e^{-\alpha h}} = \frac{kh/\rho - g}{k/(\alpha\rho)},$$

or

$$\frac{2(e^{\alpha h} + e^{-\alpha h} - 2)}{e^{\alpha h} - e^{-\alpha h}} = \alpha(h - \rho g/k). \tag{18}$$

Expanding the left-hand side as in the previous section, we get

$$\alpha h(1 - \tfrac{1}{12}\alpha^2 h^2 + \cdots).$$

Comparing this with the right-hand side of (18), we have

$$h^3 = 12\rho g/(k\alpha^2) = 6\sigma^2 g/k, \tag{19}$$

or

$$h=\left(\frac{6\sigma^2 g}{k}\right)^{1/3}. \tag{20}$$

As in the previous section, we can find the condition for the validity of the approximation, and check that the expression is a pure number. We have

$$\alpha h=(12\sqrt{2}\rho^{3/2}g/(\sigma k))^{1/3}. \tag{21}$$

Now the dimension of k is [\$]/[L][T]; the other dimensions are as before. I shall leave the details to the reader. Incidentally, the expression in the parentheses on the right-hand side of (20) happens to be the same as the corresponding expression in (11), but that is a coincidence.

The two cases examined so far suggest that if the flow costs of the deviation of x from 0 are given by $f(x)=k|x|^n$, then h should be of the order of g raised to the power $1/(n+2)$. I have verified this for $n=4$, and find

$$h=\left(\frac{15}{2}\frac{\sigma^2 g}{k}\right)^{1/6}. \tag{22}$$

But the calculations get rapidly more tedious, and I shall leave the general case as a conjecture.

4. THE ENTRY-EXIT MODEL

Brennan and Schwartz (1985) and Dixit (1989b) consider models of investment and disinvestment. The project is of a fixed size, and the revenue R follows a geometric Brownian motion

$$dR/R=\mu dt+\sigma dz. \tag{23}$$

The flow cost of operation is C. The lump-sum cost of making the investment is k, and that of abandonment is l.

The value function $V_1(R)$ of an active project satisfies the differential equation

$$\tfrac{1}{2}\sigma^2 R^2 V_1''(R)+\mu R V_1'(R)-\rho V_1(R)=C-R, \tag{24}$$

while the corresponding function $V_0(R)$ for an idle project satisfies

$$\tfrac{1}{2}\sigma^2 R^2 V_0''(R)+\mu R V_0'(R)-\rho V_0(R)=0. \tag{25}$$

Note the slight difference from equations like (4) and (13) of the previous section, corresponding to the difference between the "absolute" Brownian motion (2) for x and the "geometric" Brownian motion (23) for R.

The general solutions of (24) and (25) are

$$V_1(R)=A_1 R^{-\alpha}+B_1 R^{\beta}+R/(\rho-\mu)-C/\rho, \tag{26}$$

and

$$V_0(R)=A_0 R^{-\alpha}+B_0 R^{\beta}. \tag{27}$$

The powers $-\alpha$ and β are the roots of the associated quadratic

$$\tfrac{1}{2}\sigma^2\xi(\xi-1)+\mu\xi-\rho=0.$$

Therefore

$$\beta=\frac{\sigma^2-2\mu+((\sigma^2-2\mu)^2+8\rho\sigma^2)^{1/2}}{2\sigma^2}>1, \tag{28}$$

and

$$-\alpha = \frac{\sigma^2 - 2\mu - ((\sigma^2 - 2\mu)^2 + 8\rho\sigma^2)^{1/2}}{2\sigma^2} < 0. \tag{29}$$

The constants A_i, B_i in (26) and (27) remain to be determined. In this, it helps to interpret the value functions. An idle project has an "option" value because if R rises sufficiently high, the project is optimally activated. As R goes to zero, the expected date for the exercise of this activation option recedes into the far future, so the option becomes worthless. For this, we need $A_0 = 0$. Similarly, the value of an active project is the expected present value of the profits, which consists of the last two terms in (26), plus the "option" value of terminating the project if R falls sufficiently low. If the current R is very high, the option is nearly worthless. Therefore B_1 should be zero.

Now we are left with A_1 and B_0, for notational simplicity I shall replace them by A and B respectively, and write

$$V_1(R) = AR^{-\alpha} + R/(\rho - \mu) - C/\rho, \tag{26}$$

and

$$V_0(R) = BR^{\beta}. \tag{27}$$

The optimum policy is characterized by two numbers R_H and R_L; invest when R rises to R_H, disinvest when it falls to R_L, and continue the status quo over the range (R_L, R_H). The optimum policy parameters R_H, R_L are determined jointly with the constants A, B by the value matching conditions

$$V_1(R_H) = V_0(R_H) + k, \qquad V_0(R_L) = V_1(R_L) + l,$$

and the smooth pasting conditions

$$V_1'(R_H) = V_0'(R_H), \qquad V_0'(R_L) = V_1'(R_L).$$

Written out using (26) and (27), we have

$$BR_H^{\beta} - AR_H^{-\alpha} - R_H/(\rho - \mu) + C/\rho = -k, \tag{30}$$

$$BR_L^{\beta} - AR_L^{-\alpha} - R_L/(\rho - \mu) + C/\rho = l, \tag{31}$$

$$\beta BR_H^{\beta} + \alpha AR_H^{-\alpha} - R_H/(\rho - \mu) = 0, \tag{32}$$

$$\beta BR_L^{\beta} + \alpha AR_L^{-\alpha} - R_L/(\rho - \mu) = 0. \tag{33}$$

Numerical solutions of these over a wide range of parameter values were examined in Dixit (1989*b*). Here I shall obtain analytical approximations in two distinct cases of parameter values. The first corresponds to the case of small sunk costs studied in the previous section: k and l are small, but μ, ρ and σ are arbitrary, except the convergence condition $\rho > \mu$ and the requirement $\sigma > 0$ for a non-trivially stochastic problem.

The algebraic details are similar to those of the previous two sections but considerably more tedious, therefore I shall relegate them to the Appendix. Define a measure of the range of inaction,

$$z = \tfrac{1}{2} \ln (R_H/R_L). \tag{34}$$

The approximate expression for z is

$$z = \left(\frac{3}{4} \frac{\sigma^2 (k + l)}{C} \right)^{1/3}. \tag{35}$$

Small costs of change again give rise to a disproportionately large optimal range of inertia. For example, if we set $l=0$ and consider the behaviour of z as k changes, $\partial z/\partial k=\infty$ at $k=0$. The numerical calculations in Dixit (1989*b*, Figure 2) show this vividly.[1]

In fact the result that z varies as the third power of the sunk costs conforms to the general conjecture made for the menu cost model. The flow profit of the entry-exit model is linear in the state variable R, so the conjecture does suggest the power 3 in the approximation.

The numerical calculations also enable us to check the accuracy of the approximation. This is shown in Table 1. The parameters are set at $C=1$ (a choice of units), $l=0$, $\sigma=0{\cdot}1$ and $\rho=0{\cdot}025$. The entry cost k is allowed to vary over a wide range. The analytical approximation of (35) is compared with the exact numerical solutions obtained as in Dixit (1989*b*). The highest value of k considered is $k=20$, when the annual flow equivalent of the entry cost is $\rho k=0{\cdot}5$, fully half of the variable cost C of operation of the project. Even for such a high entry cost, the approximation is good to within 10% of the exact numerical calculation.

TABLE 1

Accuracy of the approximation

k	Approximate z from (35)	Numerical solution for z
0·1	0·091	0·091
0·4	0·144	0·146
1·0	0·196	0·199
4·0	0·310	0·325
20·0	0·531	0·587

One other interesting feature should be pointed out. When ρ is zero, the entry cost k and the exit cost l have a symmetric effect on the decisions to invest and disinvest. At the time of investment, the cost of disinvestment matters only because the Brownian motion will reach R_L in the future, but with no discounting this matters just as much as the investment cost currently being incurred. What is surprising is that the same symmetry applies to the leading terms in the approximation when the costs k and l are themselves small, irrespective of the discount rate. Again the symmetry was seen in the numerical calculations in Dixit (1989*b*, Figure 2); here we have a theoretical confirmation of the phenomenon.

The other approximation is for small ρ. Here, to ensure convergence, we must set $\mu=0$, but k, l and σ are arbitrary. Now we find

$$\frac{(e^{z}-e^{-z})^2-4z^2}{e^{z}-e^{-z}}=\frac{\sigma^2(k+l)}{2C}. \tag{36}$$

This looks complicated, but has a close parallel (35). If z happens to be sufficiently small to allow us to retain only the leading term on the left-hand side, this term is $\frac{2}{3}z^3$, and (36) turns into (35).

The next natural case to consider is the one where σ is small. But this has so far proved intractable. The difficulty is that while one expects x to be small, the magnitudes

1. At the other extreme, if one of k or l is infinite, the problem permits an explicit analytical solution, see Bertola (1989), Pindyck (1988) and Dixit (1989*b*).

of the roots α and β is also affected by σ. As σ becomes small, β goes to the limit ρ/μ, while α behaves like $2\mu/\sigma^2$, tending to ∞. Then it is not clear whether the power αz in (A.1) will be large or small. I have not been able to find a valid approximation for this case.

5. CONCLUDING REMARKS

In this paper I have developed a method of analytical approximations for problems of optimal decision-making under conditions of ongoing uncertainty and costly reversibility, and obtained expressions for the size of the range of optimal inertia. This led to a stronger result on the magnitude of sluggishness in adjustment caused by menu costs.

APPENDIX

The appendix contains technical details of the derivation of the approximation formula for the entry-exit model.

Solve the smooth pasting conditions (32) and (33) of the text for the constants A and B, and substitute the result in the value matching conditions (30) and (31) to obtain two equations in the control parameters R_H and R_L. Next we add and subtract the equations to get two others that are better suited for the approximation. Finally, define

$$M = (R_H R_L)^{1/2}, \qquad z = \tfrac{1}{2}\ln(R_H/R_L),$$

or

$$R_H = Me^{z}, \qquad R_L = Me^{-z}.$$

The subtracted equation becomes

$$\frac{(\rho-\mu)(k+l)}{M} = (e^{z} - e^{-z}) - \frac{\alpha(e^{\beta z} - e^{-\beta z})(e^{(\alpha+1)z} - e^{-(\alpha+1)z}) + \beta(e^{\alpha z} - e^{-\alpha z})(e^{(\beta-1)z} - e^{-(\beta-1)z})}{\alpha\beta(e^{(\alpha+\beta)z} - e^{-(\alpha+\beta)z})}. \qquad (A.1)$$

The approximations require appropriate expansions of the right-hand side.

First consider the case when k and l are small, we proceed on the assumption that z is sufficiently small that the exponentials can be expanded in a Taylor series and the leading terms retained. To facilitate this, we define the numerator and the denominator functions

$$\begin{aligned} N(z) &= \alpha(e^{\beta z} - e^{-\beta z})(e^{(\alpha+1)z} - e^{-(\alpha+1)z}) + \beta(e^{\alpha z} - e^{-\alpha z})(e^{(\beta-1)z} - e^{-(\beta-1)z}) \\ &= 4\alpha \sinh(\beta z)\sinh((\alpha+1)z) + 4\beta \sinh(\alpha z)\sinh((\beta-1)z), \end{aligned}$$

and

$$\begin{aligned} D(z) &= \alpha\beta(e^{(\alpha+\beta)z} - e^{-(\alpha+\beta)z}) \\ &= 2\alpha\beta \sinh((\alpha+\beta)z) \end{aligned}$$

Now we must use the rules for differentiation of hyperbolic sines and cosines, and the "binomial" rule for higher-order derivatives of a product,[2]

$$\frac{d^n(f(x)g(x))}{dx^n} = \sum_{i=0}^{n} \binom{n}{i} \frac{d^i f(x)}{dx^i}\,\frac{d^{n-i} g(x)}{dx^{n-i}}.$$

This gives

$$N(0) = 0, \qquad N'(0) = 0, \qquad N''(0) = 4\alpha\beta(\alpha+\beta), \qquad N'''(0) = 0,$$

and

$$N''''(0) = 8\alpha\beta(\alpha+\beta)(\alpha^2 + \beta^2 - 2\beta + 2\alpha + 3).$$

2. See Hardy (1952) for details of such arcana.

Similarly

$$D(0)=0,\ D'(0)=\alpha\beta(\alpha+\beta),\ D''(0)=0,\ D'''(0)=\alpha\beta(\alpha+\beta)^3.$$

Then the ratio $R(z)=N(z)/D(z)$ becomes

$$\begin{aligned}
R(z)&=\frac{\frac{1}{2}N''(0)z^2+\frac{1}{24}N''''(0)z^4+\cdots}{D'(0)z+\frac{1}{6}D'''(0)z^3+\cdots}\\
&=\tfrac{1}{2}z\frac{N''(0)+\frac{1}{12}N''''(0)z^2+\cdots}{D'(0)+\frac{1}{6}D'''(0)z^2+\cdots}\\
&=\tfrac{1}{2}z\left\{\frac{N''(0)}{D'(0)}+\frac{\frac{1}{12}N''''(0)D'(0)-\frac{1}{6}N''(0)D'''(0)}{D'(0)^2}z^2+\cdots\right\}\\
&=2z+(1-\tfrac{2}{3}(\beta-\alpha)-\tfrac{2}{3}\beta\alpha)z^3+\cdots.
\end{aligned}$$

Then (A.1) becomes

$$\begin{aligned}
\frac{(\rho-\mu)(k+l)}{C}&=2z+\tfrac{1}{3}z^3-2z-(1-\tfrac{2}{3}(\beta-\alpha)-\tfrac{2}{3}\beta\alpha)z^3+\cdots\\
&=\tfrac{2}{3}[\beta\alpha+(\beta-\alpha-1)]z^3\\
&=\tfrac{2}{3}[2\rho/\sigma^2-2\mu/\sigma^2]z^3\\
&=\frac{4}{3}\frac{(\rho-\mu)}{\sigma^2}z^3.
\end{aligned}$$

This immediately gives equation (35) of the text.

Note that for geometric Brownian motion, σ has dimension $1/[\mathrm{T}]$. The dimensions of the lump-sum costs k and l are [\$], while that of the flow cost C is [\$]/[T]. Then (35) shows at once that z is a pure number.

Next consider the case where ρ is small and $\mu=0$. Then (28) and (29) become

$$\alpha=2\rho/\sigma^2,\qquad \beta=1+\alpha, \tag{A.2}$$

ignoring higher-order terms in ρ. Thus α is small, but β is not and z need not be. Therefore the right-hand side of (A.1) should be expanded in powers of α. Write the numerator of the complicated ratio on the right-hand side of (A.1) as $N(\alpha)$ and the denominator as $D(\alpha)$. Using (A.2), we find

$$N(\alpha)=\alpha(e^{2z+2\alpha z}+e^{-2z-2\alpha z}-2)+(1+\alpha)(e^{2\alpha z}+e^{-2\alpha z}-2).$$

It is easy to verify that

$$N(0)=0,\qquad N'(0)=(e^{z}-e^{-z})^2,\qquad N''(0)=4z(e^{2z}-e^{-2z})+8z^2.$$

Also

$$D(\alpha)=\alpha(1+\alpha)(e^{z+2\alpha z}-e^{-z-2\alpha z}),$$

and

$$D(0)=0,\qquad D'(0)=e^{z}-e^{-z},\qquad D''(0)=2(e^{z}-e^{-z})+4z(e^{z}+e^{-z}).$$

The ratio $R(\alpha)$ can then be written as

$$\begin{aligned}
R(\alpha)&=\frac{N'(0)\alpha+\frac{1}{2}N''(0)\alpha^2+\cdots}{D'(0)\alpha+\frac{1}{2}D''(0)\alpha^2+\cdots}\\
&=\frac{N'(0)+\frac{1}{2}N''(0)\alpha+\cdots}{D'(0)+\frac{1}{2}D''(0)\alpha+\cdots}\\
&=\frac{N'(0)}{D'(0)}+\frac{1}{2}\frac{D'(0)N''(0)-N'(0)D''(0)}{D'(0)^2}\alpha+\cdots\\
&=(e^{z}-e^{-z})-\frac{(e^{z}-e^{-z})^2-4z^2}{e^{z}-e^{-z}}\alpha+\cdots
\end{aligned}$$

Substituting in (A.1), we have

$$\frac{\rho(k+l)}{M}=\frac{(e^{z}-e^{-z})^2-4z^2}{e^{z}-e^{-z}}\alpha.$$

The leading term in the equation obtained by adding the two value matching conditions can be simplified similarly, and the leading terms give $M=C$. Using this and $\alpha=2\rho/\sigma^2$, we get equation (36) of the text.[3]

3. For related problems, see Miller and Orr (1966), Mossin (1968), Constantinides and Richard (1978), and Smith (1989).

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