FIN 970: Final Exam

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1 Problem 1a

1. Using SDF approach:

   Conjecture $P_t^n = \exp(A_n + B_n' X_t)$. Proof by induction.

   For $n = 0$,

$$
\begin{aligned}
P_t^1 &= E_t[M_{t+1} \cdot 1] \\
\implies \exp(A_1 + B_1' X_t) &= E_t\left[\exp\left(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1}\right)\right] \\
\implies E_t\left[-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1}\right] &= -\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t' \lambda_t \\
Var_t\left[-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1}\right] &= \lambda_t' \lambda_t \\
\implies \exp(A_1 + B_1' X_t) &= \exp\left(-\delta_0 - \delta_1 X_t\right) \\
\implies &\begin{cases} A_1 = -\delta_0 \\ B_1 = -\delta_1' \end{cases}
\end{aligned}
$$

   For some n, the Euler equation holds:

$$
\begin{aligned}
P_t^n &= E_t[M_{t+1} P_{t+1}^{n-1}] \\
\exp(A_n + B_n' X_t) &= E_t\left[\exp\left(-r_t - \frac{1}{2}\lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1}\right) \exp(A_{n-1} + B_{n-1}' X_{t+1})\right] \\
&= E_t\left[\exp\left(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1} + A_{n-1} + B_{n-1}'(\mu + \Phi X_t + \Sigma \varepsilon_{t+1})\right)\right] \\
&= E_t\left[\exp\left(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t' \lambda_t + A_{n-1} + B_{n-1}' \mu + B_{n-1}' \Phi X_t + [B_{n-1}' \Sigma - \lambda_t'] \varepsilon_{t+1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&E_t\Bigg[-\delta_0-\delta_1 X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t+[B_{n-1}'\Sigma-\lambda_t']\varepsilon_{t+1}\Bigg]\\
&=-\delta_0-\delta_1 X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t\\
&Var_t\Bigg[-\delta_0-\delta_1 X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t+[B_{n-1}'\Sigma-\lambda_t']\varepsilon_{t+1}\Bigg]\\
&=[B_{n-1}'\Sigma-\lambda_t'][B_{n-1}'\Sigma-\lambda_t']'\\
&=B_{n-1}'\Sigma\Sigma'B_{n-1}+\lambda_t'\lambda_t-2B_{n-1}'\Sigma\lambda_t
\end{aligned}
$$

$$
\begin{aligned}
\exp(A_n+B_n'X_t)&=\exp\Bigg(-\delta_0-\delta_1 X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t\\
&\quad+\frac{1}{2}B_{n-1}'\Sigma\Sigma'B_{n-1}+\frac{1}{2}\lambda_t'\lambda_t-B_n'\Sigma(\lambda_0+\lambda_1 X_t)\Bigg)\\
&=\exp\Bigg(-\delta_0+A_{n-1}+B_{n-1}'\mu-B_{n-1}'\Sigma\lambda_0+\frac{1}{2}B_{n-1}'\Sigma\Sigma'B_{n-1}+(-\delta_1+B_{n-1}'\Phi-B_{n-1}'\Sigma\lambda_1)X_t)\Bigg)\\
&\implies\begin{cases}A_n=-\delta_0+A_{n-1}+B_{n-1}'(\mu-\Sigma\lambda_0)+\frac{1}{2}B_{n-1}'\Sigma\Sigma'B_{n-1}\\ B_n=-\delta_1+(\Phi-\Sigma\lambda_1)'B_{n-1}\end{cases}
\end{aligned}
$$

2. Conjecture $P_t^n=\exp(C_n+D_n'X_t)$. Proof by induction.
   For $n=0$,

$$
\begin{aligned}
P_t^1&=e^{-r_t}E_t^Q[1]\\
\exp(C_1+D_1'X_t)&=\exp(-\delta_0-\delta_1 X_t)\\
&\implies\begin{cases}C_1=-\delta_0\\ D_1=-\delta_1'\end{cases}
\end{aligned}
$$

   For some n, the Euler equation holds:

$$
\begin{aligned}
P_t^n&=e^{-r_t}E_t^Q[P_{t+1}^{n-1}]\\
\exp(C_n+D_n'X_t)&=e^{-r_t}E_t^Q[\exp(C_{n-1}+D_{n-1}'X_{t+1})]\\
&=e^{-r_t}E_t^Q[\exp(C_{n-1}+D_{n-1}'(\mu^Q+\Phi^Q X_t+\Sigma\varepsilon_{t+1}^Q))]\\
&=e^{-r_t}E_t^Q[\exp(C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t+D_{n-1}'\Sigma\varepsilon_{t+1}^Q)]
\end{aligned}
$$

$$
\begin{aligned}
E_t^Q[C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t+D_{n-1}'\Sigma\varepsilon_{t+1}^Q]&=E_t[C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t]\\
Var_t^Q[C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t+D_{n-1}'\Sigma\varepsilon_{t+1}^Q]&=D_{n-1}'\Sigma\Sigma'D_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\exp(C_n+D_n'X_t)&=\exp\Bigg(-\delta_0-\delta_1 X_t+C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t+\frac{1}{2}D_{n-1}'\Sigma\Sigma'D_{n-1}\Bigg)\\
&\begin{cases}C_n=-\delta_0+C_{n-1}+D_{n-1}'\mu^Q+\frac{1}{2}D_{n-1}'\Sigma\Sigma'D_{n-1}\\ D_n=-\delta_1'+\Phi^{Q'}D_{n-1}\end{cases}
\end{aligned}
$$

3. Show that this is a one-to-one mapping between the risk-neutral parameters (μ^Q, Φ^Q) and the market prices of risk (λ_0, λ_1).

   Clearly, parts (1) and (2) are equivalent iff

$$\mu^Q = \mu - \Sigma\lambda_0 \iff \lambda_0 = \Sigma^{-1}(\mu - \mu^Q)$$
$$\Phi^Q = \Phi - \Sigma\lambda_1 \iff \lambda_1 = \Sigma^{-1}(\Phi - \Phi^Q)$$

   with $A_n = C_n$ and $B_n = D_n$. Thus, we can go back and forth from SDF to risk-neutral densities to price bonds of any maturity.