ECON 703 - PS 7

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(1) Let $X \subset \mathbb{R}^n$ be a convex set, and $\lambda_1, ..., \lambda_k \geq 0$ with $\sum_{i=1}^k \lambda_i = 1$. Prove that if $x_1, ..., x_k \in X$, then $\sum_{i=1}^k \lambda_i x_i \in X$.

Proof (by induction): For the base step, choose $\lambda_1^2, \lambda_2^2 \geq 0$ such that $\lambda_1^2 + \lambda_2^2 = 1$.[1] For any $x_1, x_2 \in X \subset \mathbb{R}^n$, $\lambda_1^2 x_1 + \lambda_2^2 x_2 \in X$ because X is convex. For some k, assume that $\sum_{i=1}^k \lambda_i^k x_i \in X$ for $x_1, ..., x_k \in X$ with $\lambda_1^k, ..., \lambda_k^k \geq 0$ and $\sum_{i=1}^k \lambda_i^k = 1$. Consider $k+1$. Choose $\lambda_1^{k+1}, ..., \lambda_{k+1}^{k+1} \geq 0$ such that $\sum_{i=1}^{k+1} \lambda_i^{k+1} = 1$:

$$\sum_{i=1}^{k+1} \lambda_i^{k+1} x_i = \sum_{i=1}^{k} \lambda_i^{k+1} x_i + \lambda_{k+1}^{k+1} x_{k+1} = \left(\sum_{i=1}^{k} \lambda_i^{k+1} \right) \sum_{i=1}^{k} \left(\frac{\lambda_i^{k+1}}{\sum_{i=1}^k \lambda_i^{k+1}} x_i \right) + \lambda_{k+1}^{k+1} x_{k+1}$$

By the induction hypothesis, $y := \sum_{i=1}^k \left(\frac{\lambda_i^{k+1}}{\sum_{i=1}^k \lambda_i^{k+1}} x_i \right) \in X$ because $\sum_{i=1}^k \frac{\lambda_i^{k+1}}{\sum_{i=1}^k \lambda_i^{k+1}} = 1$. Thus,

$$\sum_{i=1}^{k+1} \lambda_i^{k+1} x_i = \left(\sum_{i=1}^{k} \lambda_i^{k+1} \right) y + \lambda_{k+1}^{k+1} x_{k+1}$$

By the definition of convexity, $\sum_{i=1}^{k+1} \lambda_i^{k+1} x_i \in X$ because $\sum_{i=1}^k \lambda_i^{k+1} + \lambda_{k+1}^{k+1} = 1$. $\square$

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[1] A note on notation; λ_i^j denotes the coefficient on x_i when the convex combination is composed of j elements. For example, λ_i^2 pertains to the base step, λ_i^k pertains to the induction hypothesis, and λ_i^{k+1} pertains to the induction step.

(2) The sum $\sum_{i=1}^{k} \lambda_i x_i$ defined in Problem (1) is called a convex combination. The convex hull of a set S, denoted by $\operatorname{co}(S)$, is the intersection of all convex sets which contain S. Prove that the set of all convex combinations of the elements of S is exactly $\operatorname{co}(S)$.

(3) For any set $X \subset \mathbb{R}^n$, let its closure be $\mathrm{cl}X = X \cup \{\text{all limit points of } X\}$. Show that the closure of a convex set is convex.

(4) The function $f : X \to \mathbb{R}$, where X is a convex set in $\mathbb{R}^n$, is concave if $\forall \lambda \in [0,1], x', x'' \in X f((1-\lambda)x' + \lambda x'') \geq (1-\lambda)f(x') + \lambda f(x'')$. Given a function $f : X \to \mathbb{R}$, its hypograph is the set of points (y, x) lying on or below the graph of the function: $\text{hyp} f = \{(y,x) | x \in X, y \leq f(x)\}$. Show that the function f is concave if and only if its hypograph is a convex set.

(5) Let X and Y be disjoint, closed, and convex sets in $\mathbb{R}^n$, one of which is compact. Show that there exists a hyperplane $H(p, \alpha)$ that strictly separates X and Y.

(6) Call a vector $\pi \in \mathbb{R}^n$ a probability vector if $\sum_{i=1}^n \pi_i = 1$ and $\pi_i \geq 0$ for all $i = 1, ..., n$. Interpretation is that there are n states of the world and π_i is the probability that state i occurs. Suppose that Alice and Bob each have a set of probability distributions (Π_A and Π_B) which are nonempty, convex, and compact. They propose bids on each state of the world. A vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$, where x_i denotes the net transfer Alice recieves from Bob in state i, is called a trade (Thus, $-x$ is the net transfer Bob receives in each state of the world.) A trade is agreeable if $\inf_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i x_i > 0$ and $\inf_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i(-x_i) > 0$. The above means that both Alice and Bob expect to strictly gain from the trade. Prove that there exists an agreeable trade iff there is no common prior (i.e., $\Pi_A \cap \Pi_B = \emptyset$).