ECON 711 - PS 7

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A Risky Investment

You have wealth $w > 0$ and preferences over lotteries represented by a von Neumann-Morgenstern expected utility function with Bernoulli utility u which is strictly increasing, twice differentiable, and weakly concave. Your friend wants you to invest in his startup; you can choose any amount $a \leq w$ to invest, and your investment will either triple in value (with probability p) or become worthless (with probability $1 - p$). Your expected utility if you invest a is therefore

$$U(a) = pu(w - a + 3a) + (1 - p)u(w - a) = pu(w + 2a) + (1 - p)u(w - a)$$

(a) Show that if u is linear, then you invest all your wealth if $p > \frac{1}{3}$ and nothing if $p < \frac{1}{3}$.

If u is linear and strictly increasing, u can be represented as $u(x) = mx + b$ for some $m \in \mathbb{R}_{++}, b \in \mathbb{R}$:

$$\begin{aligned}
U(a) &= pu(w + 2a) + (1 - p)u(w - a) \\
&= p(m(w + 2a) + b) + (1 - p)(m(w - a) + b) \\
&= pwm + 2pam + pb + wm - pwm - am + pam + b - pb \\
&= (3p - 1)ma + mw + b
\end{aligned}$$

If $p > \frac{1}{3} \implies 3p - 1 > 0$, so the coefficent on a in utility function is positive. Thus, to maximize U, you want to invest as much as possible, which is all your wealth. If $p < \frac{1}{3} \implies 3p - 1 < 0$, so the coefficent on a in utility function is negative. Thus, to maximize U, you want to invest as little as possible, which is nothing.

From here on, assume $p > \frac{1}{3}$, so the expected value of the investment is positive; and assume that you are strictly risk-averse ($u'' < 0$).

(b) Show that it's optimal to invest a strictly positive amount.[1]

$$U'(a) = pu'(w + 2a)(2) + (1 - p)u'(w - a)(-1) = 2pu'(w + 2a) - (1 - p)u'(w - a)$$

$$U'(0) = 2pu'(w + 2(0)) - (1 - p)u'(w - (0)) = 2pu'(w) - (1 - p)u'(w) = (3p - 1)u'(w)$$

$U'(0) > 0$ because $3p - 1 > 0$ and $u'(w) > 0$.

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[1]You can do this by showing that $U'(0) > 0$ - the marginal expected utility of increasing a is positive when $a = 0$.

(c) Show that $U(a)$ is strictly concave in a, so that except at a corner solution, the first-order condition is necessary and sufficient to find a^*.

$U(a)$ is strictly concave in a iff $U(ta + (1-t)b) < tU(a) + (1-t)U(b)$ for $a, b \in [0, w]$ and $t \in [0, 1]$. Because $u'' < 0$,

$$\begin{aligned} U(ta + (1-t)b) &= pu(w + 2(ta + (1-t)b)) + (1-p)u(w - (ta + (1-t)b)) \\ &= pu(t(w + 2a) + (1-t)(w + 2b)) + (1-p)u(t(w-a) + (1-t)(w-b)) \\ &< p(tu(w + 2a) + (1-t)u(w + 2b)) + (1-p)(tu(w-a) + (1-t)u(w-b)) \\ &= t(pu(w + 2a) + (1-p)u(w-a)) + (1-t)(pu(w + 2b) + (1-p)u(w-b)) \\ &= tU(a) + (1-t)U(b) \end{aligned}$$

(d) Show that if $u'(0)$ is infinite, it's not optimal to invest all your wealth; and that if $u'(0)$ is finite, then there's a cutoff $\bar{p}$ such that it's optimal to invest all of your wealth if $p \geq \bar{p}$.

From (c), we know that the first-order condition is necessary and sufficient to find a^*. The derivative of the utility function at $a = w$ is

$$U'(w) = 2pu'(w + 2(w)) - (1-p)u'(w - (w)) = 2pu'(3w) - (1-p)u'(0)$$

Thus, if $u'(0)$ is infinite, $U'(w) = -\infty$, so the first order condition cannot hold at w.

If $u'(0)$ is finite, the first order condition is:

$$0 = 2\bar{p}u'(3w) - (1-\bar{p})u'(0) \implies \bar{p} = \frac{u'(0)}{2u'(3w) + u'(0)}$$

Thus, if $p \geq \bar{p}$ investing all of your wealth is optimal.

From here on, assume that either $u'(0)$ is infinite or $p \in (\frac{1}{3}, \bar{p})$, so the optimal level of investment a^* is strictly positive but below w.

(e) Show that if $u(x) = 1 - e^{-cx}$ (the Constant Absolute Risk Aversion or CARA utility function), your optimal investment a^* does not depend on w.

(f) For general u, show that if your Coefficient of Absolute Risk Aversion $A(x) = -\frac{u''(x)}{u'(x)}$ is decreasing, you invest more as w increases.

Now reframe the question as deciding what fraction t of your wealth to invest; writing $a = tw$,

$$U(t) = pu(w(1 + 2t)) + (1-p)u(w(1-t))$$

(g) Show that if $u(x) = \frac{1}{1-\rho}x^{1-\rho}$, with $\rho \leq 1$ and $\rho \neq 0$ (the Constant Relative Risk Aversion or CRRA utility function), you invest the same fraction of your wealth regardless of w.

(h) For general u, show that if your Coefficient of Relative Risk Aversion $R(x) = -\frac{xu''(x)}{xu'(x)}$ is increasing, you invest a smaller fraction of your wealth as w increases.