ECON 709 - PS 5

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1. For the following sequences, show $a_n \to 0$ as $n \to \infty$:

(a) $a_n = 1/n$

Fix $\varepsilon > 0$. Choose $\bar{n} > \frac{1}{\varepsilon}$. For all $n \geq \bar{n}$,

$$|1/n - 0| = |1/n| = \varepsilon.$$

Thus, $a_n = 1/n \to 0$ as $n \to \infty$.

(b) $a_n = \frac{1}{n}\sin(\frac{n\pi}{2})$

Fix $\varepsilon > 0$. Notice that $|\sin(x)| \leq 1\ \forall x$. Choose $\bar{n} > \frac{1}{\varepsilon}$. For all $n \geq \bar{n}$,

$$\left|\frac{1}{n}\sin\left(\frac{n\pi}{2}\right) - 0\right| = \left|\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right| \leq |1|\left|\frac{1}{n}\right| = \left|\frac{1}{n}\right| \leq \varepsilon$$

Thus, $a_n = \frac{1}{n}\sin(\frac{n\pi}{2}) \to 0$ as $n \to \infty$.

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2. Consider a random variable X^n with the probability function

$$X_n = \begin{cases} -n, & \text{with probability } 1/n \\ 0, & \text{with probability } 1-2/n \\ n, & \text{with probability } 1/n \end{cases}$$

(a) Does $X_n \to_p 0$ as $n \to \infty$?

Fix $\varepsilon > 0$. Choose $\bar{n} > \varepsilon$. For $n \geq \bar{n}$,

$$P(|X_n| \geq \varepsilon) \leq P(|X_n| \geq n) = P(X_n = -n) + P(X_n = n) = 1/n + 1/n = 2/n$$

Since $1/n \to 0$, $2/n \to 0$. Thus, $X_n \to_p 0$ as $n \to \infty$.

(b) Calculate $E(X_n)$.

$$E(X_n) = \sum_{x \in \text{Supp}(X)} \pi(x)x = (1/n) * (-n) + (1-2/n)(0) + (1/n)(n) = -1 + 1 = 0.$$

(c) Calculate $Var(X_n)$.

$$Var(X_n) = E(X_n^2) - E(X_n)^2 = E(X_n^2) = \sum_{x \in \text{Supp}(X)} \pi(x)x^2 = (1/n)*(-n)^2 + (1-2/n)(0)^2 + (1/n)(n)^2 = n+n = 2n.$$

(d) Now suppose the distribution is

$$X_n = \begin{cases} 0, & \text{with probability } 1-1/n \\ n, & \text{with probability } 1/n \end{cases}$$

Calculate $E(X_n)$.

$$E(X_n) = \sum_{x \in \text{Supp}(X)} \pi(x)x = (1-1/n)(0) + (1/n)(n) = 0 + 1 = 1$$

(e) Conclude that $X_n \to_p 0$ is not sufficient for $E(X_n) \to 0$.

Fix $\varepsilon > 0$. Choose $\bar{n} > \varepsilon$. For $n > \bar{n}$

$$P(|X_n| \geq \varepsilon) \leq P(|X_n| \geq n) = P(X_n = n) = 1/n$$

Since $1/n \to 0$, $X_n \to_p 0$ as $n \to \infty$. Thus, $X_n \to_p 0$ is not sufficient for $E(X_n) \to 0$.

3. A weighted sample mean takes the form $\bar{Y}^* = \frac{1}{n}\sum_{i=1}^n w_i Y_i$ for some non-negative constants w_i satisfying $\frac{1}{n}\sum_{i=1}^n w_i = 1$. Assume that $Y_i : i = 1, ..., n$ are i.i.d.

(a) Show that $\bar{Y}^*$ is unbiased for $\mu = E(Y_i)$.

$$E(\bar{Y}^*) = E\left(\frac{1}{n}\sum_{i=1}^n w_i Y_i\right) = \frac{1}{n}\sum_{i=1}^n w_i E(Y_i) = \frac{1}{n}\sum_{i=1}^n w_i \mu = (1)\mu = \mu$$

(b) Calculate $Var(\bar{Y}^*)$.

$$Var(\bar{Y}^*) = Var\left(\frac{1}{n}\sum_{i=1}^n w_i Y_i\right) = \frac{1}{n^2}\sum_{i=1}^n w_i^2 Var(Y_i)$$

(c) Show that a sufficient condition for $\bar{Y}^* \to_p \mu$ is that $\frac{1}{n^2}\sum_{i=1}^n w_i^2 \to 0$. (Hint: use the Markov's or Chebyshev's Inequality).

Fix $\varepsilon > 0$. Because $\frac{1}{n^2}\sum_{i=1}^n w_i^2 \to 0$, there exists $\bar{n}$ such that for $n \geq \bar{n}$,

$$\left|\frac{1}{n^2}\sum_{i=1}^n w_i^2\right| \leq \varepsilon$$

From (a) we know that $E(\bar{Y}^*) = \mu$ and from (b) we know that $Var(\bar{Y}^*) = \frac{1}{n^2}\sum_{i=1}^n w_i^2 Var(Y_i)$, so by Chebychev's Inequality,

$$P(|\bar{Y}^* - \mu| \geq \lambda) \leq \frac{Var(\bar{Y}^*)}{\lambda^2} = \frac{\frac{1}{n^2}\sum_{i=1}^n w_i^2 Var(Y_i)}{\lambda^2} \leq \frac{\varepsilon Var(Y_i)}{\lambda^2} = \frac{\varepsilon Var(Y_i)}{\left(\sqrt{Var(Y_i)}\right)^2} = \varepsilon$$

where $\lambda = \sqrt{Var(Y_i)}$. Thus $\bar{Y}^* \to_p \mu$.

(d) Show that the sufficient condition for the condition in part (c) is $\max_{i \leq n} w_i/n \to 0$.

Fix $\varepsilon > 0$. Let $\delta = \sqrt{\frac{\varepsilon}{n}}$. Because $\max_{i \leq n} w_i/n \to 0$, there exists a $\bar{n}$ such that for $n \geq \bar{n}$,

$$\begin{aligned}
\left|\max_{i \leq n} w_i/n\right| \leq \delta &\implies |w_i/n| \leq \delta \ \forall i \in \{1, ..., n\} \\
&\implies (w_i/n)^2 \leq \delta^2 \ \forall i \in \{1, ..., n\} \\
&\implies \sum_{i=1}^n \frac{w_i^2}{n^2} \leq n\delta^2 \\
&\implies \sum_{i=1}^n \frac{w_i^2}{n^2} \leq n\left(\sqrt{\frac{\varepsilon}{n}}\right)^2 \\
&\implies \sum_{i=1}^n \frac{w_i^2}{n^2} \leq \varepsilon
\end{aligned}$$

Thus, $\frac{1}{n^2}\sum_{i=1}^n w_i^2 \to 0$.

4. Take a random sample $\{X_1, ..., X_n\}$. Which statistic converges in probability by the weak law of large numbers and continuous mapping theorem, assuming the moment exists?

(a) $\frac{1}{n}\sum_{i=1}^{n} X_i^2$

Transform $\{X_1, ..., X_n\}$ to $\{Y_1, ..., Y_n\}$ such that $Y_i = X_i^2$. Thus, $\{Y_1, ..., Y_n\}$ is an i.i.d. sequence with $E(|Y_i|) = E(X_i^2) = \mu_2 < \infty$. By the weak law of large numbers $\bar{Y}_N \to_p \mu_2$ as $n \to \infty$. Thus, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to_p \mu_2$ as $n \to \infty$.

(b) $\frac{1}{n}\sum_{i=1}^{n} X_i^3$

Transform $\{X_1, ..., X_n\}$ to $\{Y_1, ..., Y_n\}$ such that $Y_i = X_i^3$. Thus, $\{Y_1, ..., Y_n\}$ is an i.i.d. sequence with $E(|Y_i|) = E(X_i^3) = \mu_3 < \infty$. By the weak law of large numbers $\bar{Y}_N \to_p \mu_3$ as $n \to \infty$. Thus, $\frac{1}{n}\sum_{i=1}^{n} X_i^3 \to_p \mu_3$ as $n \to \infty$.

(c) $\max_{i\leq n} X_i$

This statistic does not converge in probability by the weak law of large numbers and continuous mapping theorem. Instead we could apply the Fisher-Tippett-Gnedenko theorem, which can characterize the asymptotic distribution of extreme order statistics.

(d) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2$

From (a), we know that $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to_p \mu_2$. An immediate result of the weak law of large numbers is $\frac{1}{n}\sum_{i=1}^{n} X_i \to_p \mu$. By the continuous mapping theorem, $(\frac{1}{n}\sum_{i=1}^{n} X_i)^2 \to_p \mu^2$. Thus, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2 \to_p \mu_2 - \mu^2$.

(e) $\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} X_i}$ assuming $\mu = E(X_i) > 0$.

From (a), we know that $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to_p \mu_2$. An immediate result of the weak law of large numbers is $\frac{1}{n}\sum_{i=1}^{n} X_i \to_p \mu$. Thus, by the Continuous Mapping Theorem, $\frac{n^{-1}\sum_{i=1}^{n} X_i^2}{n^{-1}\sum_{i=1}^{n} X_i} = \frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} X_i} \to_p \frac{\mu_2}{\mu}$.

(f) $1(\frac{1}{n}\sum_{i=1}^{n} X_i > 0)$ where

$$1(a) = \begin{cases} 1 & \text{if } a \text{ is true} \\ 0 & \text{if } a \text{ is not true} \end{cases}$$

is called the indicator function of event a.

Notice that $1(\frac{1}{n}\sum_{i=1}^{n} X_i > 0) \sim Bernoulli(P(\frac{1}{n}\sum_{i=1}^{n} X_i > 0))$. By the weak law of large numbers, $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n \to_p \mu$. So, if $\mu > 0$, $1(\frac{1}{n}\sum_{i=1}^{n} X_i > 0) \to_p 1$. if $\mu \leq 0$, $1(\frac{1}{n}\sum_{i=1}^{n} X_i > 0) \to_p 0$.

5. Take a random sample $\{X_1, ..., X_n\}$ where the support X_i is a subset of $(0, \infty)$. Consider the sample geometric mean $\hat{\mu} = (\Pi_{i=1}^n X_i)^{1/n}$ and population geometric mean $\mu = \exp(E(\log(X)))$. Assuming that μ is finite, show that $\hat{\mu} \to_p \mu$ as $n \to \infty$.

Assuming that μ is finite,

$$\log(\hat{\mu}) = \log((\Pi_{i=1}^n X_i)^{1/n}) = \frac{1}{n}\log(\Pi_{i=1}^n X_i) = \frac{1}{n}\sum_{i=1}^n \log(X_i)$$

By the weak law of large numbers, $\log(\hat{\mu}) = \frac{1}{n}\sum_{i=1}^n \log(X_i) \to_p E(\log(X))$. By the continuous mapping theorem with $g(x) = \exp(x)$, we know that $\hat{\mu} \to_p \exp(E(\log(X))) = \mu$.

6. Let $\mu_k = E(X^k)$ for some integer $k \geq 1$.

(a) Write down the natural moment estimator $\hat{\mu}_k$ of μ_k.

For i.i.d. sample $X_i : i = 1, ..., n$, the "plug-in" estimator is the sample moment:

$$\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^n X_i^k$$

(b) Find the asymptotic distribution of $\sqrt{n}(\hat{\mu}_k - \mu_k)$ as $n \to \infty$, assuming that $E(X^{2k}) < \infty$.

Notice that the $E(X_i^k) = \mu_k$ and $Var(X_i^k) = E(X_i^{2k}) - (\mu_k)^2 = \mu_{2k} - \mu_k^2$. Thus, by the central limit theorem, $\sqrt{n}(\hat{\mu}_k - \mu_k) \to_d N(0, \mu_{2k} - \mu_k^2)$.

7. Let $m_k = (E(X_k))^{1/k}$ for some integer $k \geq 1$.

(a) Write down the natural moment estimator $\hat{m}_k$ of m_k.

For i.i.d. sample $X_i : i = 1, ..., n$, the "plug-in" estimator is:

$$\hat{m}_k = \left(\frac{1}{n}\sum_{i=1}^n X_i^k\right)^{1/k}$$

(b) Find the asymptotic distribution of $\sqrt{n}(\hat{m}_k - m_k)$ as $n \to \infty$, assuming that $E(X^{2k}) < \infty$.

From 6(b), we know that $\sqrt{n}(\hat{\mu}_k - \mu_k) \to_d N(0, \mu_{2k} - \mu_k^2)$. Define $g(x) = x^{1/k}$ for some $k \in \mathbb{N}$. Notice that g is continuous for all values of k. Furthermore, $g(\hat{\mu}_k) = \hat{m}_k$, $g(\mu_k) = m_k$, and $g'(\mu_k) = (1/k)\mu_k^{(1-k)/k}$. Thus, by the Delta Method,

$$\begin{aligned}
\sqrt{n}(g(\hat{\mu}_k) - g(\mu_k)) &\to_d N(0, (g'(\mu_k))^2(\mu_{2k} - \mu_k^2)) \\
\sqrt{n}(\hat{m}_k - m_k) &\to_d N(0, ((1/k)\mu_k^{(1-k)/k})^2(\mu_{2k} - \mu_k^2)) \\
\sqrt{n}(\hat{m}_k - m_k) &\to_d N\left(0, \frac{\mu_k^{2/k}(\mu_{2k} - \mu_k^2)}{\mu_k^2 k^2}\right)
\end{aligned}$$

8. Suppose $\sqrt{n}(\hat{\mu} - \mu) \to_d N(0, v^2)$ and set $\beta = \mu^2$ and $\hat{\beta} = \hat{\mu}^2$.

(a) Use the Delta Method to obtain an asymptotic distribution for $\sqrt{n}(\hat{\beta} - \beta)$.

Define $g(x) = x^2$. Notice that g is continuous, $\beta = g(\mu)$, and $\hat{\beta} = g(\hat{\mu})$. Thus, by the Delta Method,

$$
\begin{aligned}
&\sqrt{n}(\hat{\beta} - \beta) \to_d N(0, (2\mu)^2 v^2) \\
\implies &\sqrt{n}(\hat{\beta} - \beta) \to_d N(0, 4\beta v^2)
\end{aligned}
$$

(b) Now suppose $\mu = 0$. Describe what happens to the asymptotic distribution from the previous part.

If $\mu = 0$, then $\beta = (0)^2 = 0$ and the variance of the asymptotic distribution is zero, so it become a degenerate normal distribution at zero:

$$\sqrt{n}(\hat{\beta} - \beta) = 0 \implies \hat{\beta} = \beta = 0$$

(c) Improve on the previous answer. Under the assumption $\mu = 0$, find the asymptotic distribution of $n\hat{\beta}$.

$$
\begin{aligned}
\sqrt{n}\hat{\mu} \to_d N(0, v^2) &\implies \frac{\sqrt{n}\hat{\mu}}{v} \to_d N(0, 1) \\
&\implies \frac{n\hat{\mu}^2}{v^2} \to_d \chi^2_1 \\
&\implies n\hat{\beta} \to_d \Gamma(1/2, 2v^2)
\end{aligned}
$$

$n\hat{\beta}$ is distributed gamma with shape parameter, $\alpha = 1/2$, and scale parameter, $\beta = 2v^2$.

(d) Comment on the differences between the answers in parts (a) and (c).

The main difference between (a) and (c) is the term, n^α. In (a), $\alpha = 1/2$ so n^α increases slower than the decrease in the variance of $\hat{\beta}$. Thus, as $n \to \infty$, the asymptotic distribution of $\sqrt{n}(\hat{\beta} - \beta)$ is a degenerate distribution with unit mass at zero. In (c), $\alpha = 1$ so n^α increases at the same rate as the variance of $\hat{\beta}$ decreases. Thus, as $n \to \infty$, the asymptotic distribution of $n\hat{\beta}$ is non-degenerate.