

# FIN 920: Continuous-Time Diffusion Models Notes\*

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December 18, 2021

## 1 Part I

### (Discrete) Random Walks

- Random walk:  $z_t = z_{t-1} + e_t = z_0 + \sum_{s=1}^t e_s$  (often  $z_0 = 0$ ) with  $E[e_t] = 0, \forall t$  and  $e_t \perp e_s, t \neq s$ .
- Random walk with drift:  $z_t = \mu + z_{t-1} + e_t$ .
- Geometric random walk with drift:  $\ln(z_t) = \mu + \ln(z_{t-1}) + e_t$  or  $z_t = z_{t-1} \exp(\mu + e_t)$ .
- Normally distributed increments  $e_t \sim N(0, \sigma^2)$ .

### Standard Brownian Motion

- A Brownian motion is a process  $\{z_t\}_{t \geq 0}$  such that
  - $P(z_0 = 0) = 1$
  - $z_t - z_s \sim N(0, t - s), t > s \geq 0$
  - $\lim_{e \rightarrow 0} z_{t-e} = z_t, t \geq 0$
  - $z_t - z_s \perp z_u - z_v, t > s > u > v \geq 0$
- Brownian motion is Markov:  $E[f(z_t) | \{z_v\}_{v=0}^s] = E[f(z_t) | z_s] =: E_s[f(z_t)]$  for  $t \geq s$ .
- Paths are nowhere differentiable:  $\lim_{t \rightarrow s} \frac{z_t - z_s}{t - s}$  is not defined.
- Paths have unbounded total variation:  $\sum_{v=1}^N |z_{tv/N} - z_{t(v-1)/N}| \rightarrow \infty$  as  $N \rightarrow \infty$ .
- Paths have bounded quadratic variation:  $\sum_{v=1}^N (z_{tv/N} - z_{t(v-1)/N})^2 \rightarrow t$  as  $N \rightarrow \infty$ .
- Conventional expressions:
  - $z_t - z_0 = \sum_{v=1}^N z_{tv/N} - z_{t(v-1)/N} \rightarrow \int_{v=0}^t dz_v$  as  $N \rightarrow \infty$  where  $dz_t \sim N(0, dt)$ .
  - Rules for the product of  $dz$  and  $dt$ :

$$\begin{bmatrix} dz & dz & dt \\ dz & dt & 0 \\ dt & 0 & 0 \end{bmatrix}$$

- For example,  $\sum_{v=1}^N (z_{Tv/N} - z_{T(v-1)/N})(T/N) \rightarrow \int_{t=0}^T dz_t dt = 0$  when  $N \rightarrow \infty$ .

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\*These notes are based on four sets of notes on Continuous-Time Diffusion Models by David Brown. I've done my best to flesh arguments out and add detail.

## Formal Construction of Brownian Motion

- Probability Space  $(\Omega, \mathcal{F}, P)$  with set of states  $\Omega = \{\omega\}$ , tribe  $\mathcal{F}$ , probability measure  $P : \mathcal{F} \rightarrow \mathbb{R}$ .
- A Brownian motion is a measurable function  $z(\omega, t) : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ , such that  $\forall \omega \in \Omega$ ,
  - $z(\omega, 0) = 0$  almost surely,
  - $z(\omega, t) - z(\omega, s) \sim N(0, t - s)$  for  $t > s$ ,
  - $z(\omega, t) - z(\omega, s) \perp z(\omega, u) - z(\omega, v), t > s > u > v \geq 0$
  - $\lim_{t \rightarrow s} z(\omega, t) = z(\omega, s)$
- The standard filtration  $\{\mathcal{F}_t\}_{t \in [0, \infty)}$  is defined by the paths of the process together with the null sets of  $\mathcal{F}$ .

## Scalar Diffusion Processes

- A diffusion (or Ito process) is an adapted process  $x_t$  with continuous paths,

$$x_t = x_0 + \int_{v=0}^t \mu(x_v, v) dv + \int_{v=0}^t \sigma(x_v, v) dz_v$$

$$\iff dx_t = \mu(x_t, t) dt + \sigma(x_t, t) dz_t$$

where  $\mu(x_v, v)$  is a drift coefficient,  $\sigma(x_v, v)$  is a diffusion coefficient, and  $z_t$  is a Brownian motion.

- The Ito integral is defined as

$$\int_{v=0}^t \sigma(x_v, v) dz_v := \lim_{N \rightarrow \infty} \sum_{v=1}^N \sigma(x_{(v-1)t/N}, (v-1)t/N) (z_{tv/N} - z_{t(v-1)/N})$$

- $E_t(dx_t) = E_t(\mu(x_t, t)dt + \sigma(x_t, t)dz_t) = \mu(x_t, t)dt + \sigma(x_t, t)E_t(dz_t) = \mu(x_t, t)dt = \mu_t dt$
- $E_t((dx_t)^2) = E_t(\mu(x_t, t)^2(dt)^2 + 2\mu(x_t, t)\sigma(x_t, t)dtdz_t + \sigma(x_t, t)^2(dz_t)^2) = E_t(\sigma(x_t, t)^2(dz_t)^2) = \sigma_t^2 dt$

## Examples of Scalar Diffusion Processes

- Brownian motion with drift:
  - $Y_t = Y_0 + \mu t + \sigma z_t$
  - $dY_t = \mu dt + \sigma dz_t$
  - $Y_t - Y_s \sim N(\mu(t - s), \sigma^2(t - s))$  for  $t > s$ .
- Geometric Brownian Motion:
  - $dS_t = \mu S_t dt + \sigma S_t dz_t$ , with constants  $\mu, \sigma$ .
  - For example, stock price in Black and Scholes (JPE 1973).
- Ornstein-Uhlenbeck process (mean-reverting):
  - $dr_t = \kappa(\theta - r_t)dt + \sigma dz_t$  with constants  $\kappa, \theta, \sigma > 0$ .
  - Risk-free rate in Vasicek (JFE, 1977)
- Square root process (mean-reverting):
  - $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz_t$ .
  - Risk-free rate in Cox, Ingersoll and Ross (ECTA, 1985)

## Vector Diffusion Processes

- A vector of Brownian motions  $\mathbf{z}_t$  is independent iff  $z_{it} - z_{is} \perp z_{ju} - z_{jv}$  for all  $i \neq j$  and all intervals  $[t, s]$  and  $[u, v]$ .
- A diffusion (or Ito process) is an adapted random vector process  $\mathbf{x}_t$  with continuous paths,

$$\mathbf{x}_t = \mathbf{x}_0 + \int_{v=0}^t \boldsymbol{\mu}(\mathbf{x}_v, v) dv + \int_{v=0}^t \boldsymbol{\sigma}(\mathbf{x}_v, v) d\mathbf{z}_v$$

$$\iff d\mathbf{x}_t = \boldsymbol{\mu}(\mathbf{x}_t, t) dt + \boldsymbol{\sigma}(\mathbf{x}_t, t) d\mathbf{z}_t$$

where  $\boldsymbol{\mu}(\mathbf{x}_t, t)$  is a vector of drift coefficients,  $\boldsymbol{\sigma}(\mathbf{x}_t, t)$  is a diffusion coefficient, and  $\mathbf{z}_t$  is a vector of independent Brownian motions.

- The Ito integral is defined as

$$\int_{v=0}^t \boldsymbol{\sigma}(\mathbf{x}_v, v) d\mathbf{z}_v := \lim_{N \rightarrow \infty} \sum_{v=1}^N \boldsymbol{\sigma}(\mathbf{x}_{(v-1)t/N}, (v-1)t/N) (\mathbf{z}_{vt/N} - \mathbf{z}_{(v-1)t/N})$$

$$\begin{aligned} E_t(d\mathbf{x}_t) &= E_t(\boldsymbol{\mu}(\mathbf{x}_t, t) dt + \boldsymbol{\sigma}(\mathbf{x}_t, t) d\mathbf{z}_t) = \boldsymbol{\mu}(\mathbf{x}_t, t) dt + \boldsymbol{\sigma}(\mathbf{x}_t, t) E_t(d\mathbf{z}_t) = \boldsymbol{\mu}(\mathbf{x}_t, t) dt \\ E_t(d\mathbf{x}_t d\mathbf{x}_t^T) &= E_t((\boldsymbol{\mu}(\mathbf{x}_t, t) dt + \boldsymbol{\sigma}(\mathbf{x}_t, t) d\mathbf{z}_t)(\boldsymbol{\mu}(\mathbf{x}_t, t) dt + \boldsymbol{\sigma}(\mathbf{x}_t, t) d\mathbf{z}_t)^T) \\ &= E_t((dt)^2 \boldsymbol{\mu}(\mathbf{x}_t, t) (\boldsymbol{\mu}(\mathbf{x}_t, t))^T + 2\boldsymbol{\mu}(\mathbf{x}_t, t) (dt d\mathbf{z}_t^T) \boldsymbol{\sigma}(\mathbf{x}_t, t)^T + \boldsymbol{\sigma}(\mathbf{x}_t, t) (d\mathbf{z}_t d\mathbf{z}_t^T) \boldsymbol{\sigma}(\mathbf{x}_t, t)^T) \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t) E_t(d\mathbf{z}_t d\mathbf{z}_t^T) \boldsymbol{\sigma}(\mathbf{x}_t, t)^T \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t) (dt \times I) \boldsymbol{\sigma}(\mathbf{x}_t, t)^T \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t) \boldsymbol{\sigma}(\mathbf{x}_t, t)^T dt \end{aligned}$$

## Examples of Vector Diffusion Processes

- Two Brownian motions with drift and correlation  $\rho \in [-1, 1]$ .

$$d \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} dt + \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}$$

- Multiperiod consumption-savings-portfolio choice in Merton (various 1970s)

$$\begin{aligned} dW_t &= W_t(\boldsymbol{\alpha}_t \cdot (\boldsymbol{\mu}(\mathbf{x}_t, t) - r(\mathbf{x}_t, t)\mathbb{1}) + r(\mathbf{x}_t, t)) dt + W_t \boldsymbol{\alpha}_t^T \boldsymbol{\sigma}(\mathbf{x}_t, t) d\mathbf{z}_t - c_t dt + y_t dt \\ d\mathbf{x}_t &= \boldsymbol{\mu}_x(\mathbf{x}_t, t) dt + \boldsymbol{\sigma}_x(\mathbf{x}_t, t) d\mathbf{z}_t \end{aligned}$$

where  $W_t \geq 0$  and  $W_0$  and  $\mathbf{x}_0$  is given.

- Constant returns-to-scale production and productivity in Ai (JF 2010).

$$\begin{aligned} dK_t &= x_t K_t dt - C_t dt + \sigma_c K_t dz_t^c \\ dx_t &= \kappa(\mu - x_t) dt + \sigma_x dz_t^x \\ dz_t^c dz_t^x &= \rho dt \end{aligned}$$

## Convenient Facts

- For an adapted process  $\gamma_t$  (vector), we can express some functions of the expectation of the adapted process in terms of a change in time instead of a change in the Brownian motion value.
- For example, expectation of quadratic:

$$\begin{aligned}
E_t \left( \left( \int_t^T \gamma_s d\mathbf{z}_s \right)^2 \right) &= E_t \lim_{N \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^N K_i K_j \\
&= E_t \lim_{N \rightarrow \infty} 2 \sum_{i=1}^N \sum_{i < j}^N K_j E_{(T-t)(i-1)/N} K_i + \sum_{j=1}^N E_{(T-t)(j-1)/N} K_j^2 \\
&= E_t \lim_{N \rightarrow \infty} \frac{T-t}{N} \sum_{j=1}^N \gamma_{(T-t)(j-1)/N} \cdot \gamma_{(T-t)(j-1)/N} \\
&= \int_t^T E_t(\gamma_s \cdot \gamma_s) ds
\end{aligned} \tag{1}$$

where  $K_j = \gamma_{T(j-1)/N} \cdot (\mathbf{z}_{(T-t)j/N} - \mathbf{z}_{(T-t)(j-1)/N})$ .

- For example, expectation of exponential:

$$E_t \left( \exp \left( \int_t^T \gamma_s d\mathbf{z}_s \right) \right) = E_t \left( \exp \left( \frac{1}{2} \int_t^T (\gamma_s \cdot \gamma_s) ds \right) \right) \tag{2}$$

- Consider the square-root process,

$$\begin{aligned}
dr_t &= \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz_t \\
\implies e^{\kappa t}dr_t &= e^{\kappa t}\kappa\theta dt - e^{\kappa t}\kappa r_t dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\
\implies e^{\kappa t}dr_t + e^{\kappa t}\kappa r_t dt &= e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\
\implies d(e^{\kappa t}r_t) &= e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\
\implies \int_{-\infty}^t d(e^{\kappa s}r_s) &= \int_{-\infty}^t e^{\kappa s}\kappa\theta ds + \int_{-\infty}^t e^{\kappa s}\sigma\sqrt{r_s}dz_s \\
\implies e^{\kappa t}r_t &= e^{\kappa t}\theta + \sigma \int_{-\infty}^t e^{\kappa s}\sqrt{r_s}dz_s \\
\implies r_t &= \theta + \sigma \int_{-\infty}^t e^{\kappa(s-t)}\sqrt{r_s}dz_s
\end{aligned}$$

- Using (1), we can find the unconditional variance (based on the unconditional expectation):

$$\begin{aligned}
\Rightarrow E[r_t] &= \theta + E\left[\sigma \int_{-\infty}^t e^{\kappa(s-t)} \sqrt{r_s} dz_s\right] \\
&= \theta \\
\Rightarrow \text{Var}(r_t) &= E[r_t^2] - E[r_t]^2 \\
&= E\left[\left(\theta + \sigma \int_{-\infty}^t e^{\kappa(s-t)} \sqrt{r_s} dz_s\right)^2\right] - \theta^2 \\
&= E\left[2\theta\sigma \int_{-\infty}^t e^{\kappa(s-t)} \sqrt{r_s} dz_s\right] + E\left[\left(\sigma \int_{-\infty}^t e^{\kappa(s-t)} \sqrt{r_s} dz_s\right)^2\right] \\
&= \sigma^2 \int_{-\infty}^t e^{2\kappa(s-t)} E[r_s] dz_s \\
&= \sigma^2 \theta e^{-2\kappa t} \int_{-\infty}^t e^{2\kappa s} dz_s \\
&= \sigma^2 \theta e^{-2\kappa t} \left[\frac{1}{2\kappa} e^{2\kappa s}\right]_{-\infty}^t \\
&= \frac{\sigma^2 \theta}{2\kappa}
\end{aligned}$$

## Black and Scholes Structure

- Stock with price  $S_t$ :  $dS_t = \mu S_t dt + \sigma S_t dz_t$ ,  $\mu > 0, \sigma > 0$ .
- Risk-free bond:  $dB_t = B_t r dt$ ,  $\mu > r > 0$ .
- Option with strike price  $k$ : At the exercise date  $T$ , the payoff is  $C(S_T, T) = \max\{0, S_T - K\}$
- Assumptions:
  - No dividend payments on stock.
  - Infinite depth in the stock and bond markets.
  - Constant drift and volatility in the stock return.
  - Constant rate of interest.
  - Frictionless markets (i.e. no transaction costs).
  - European call option (i.e. can only exercise at maturity date  $T$ ).
- Goal is to find equation for  $C(S_t, t)$ ,  $t < T$ .

## Future Values

- To get  $d \ln B_t$  use Ito's lemma [where  $\mu(B_t, t) = B_t r$ ,  $\sigma = 0$  and  $f(x) = \ln x \implies f_x(x) = \frac{1}{x}$ ,  $f_{xx}(x) = -\frac{1}{x^2}$ ,  $f_t(x) = 0$ ]:

$$\begin{aligned}
 d \ln B_t &= \frac{1}{B_t}(0)dz_t + \frac{1}{B_t}B_t r dt + \frac{1}{2} \frac{-1}{x^2}(0)^2 dt + 0 dt \\
 &= r dt \\
 \implies \int_0^t d \ln B_s &= \int_0^t r ds \\
 \implies \ln B_t - \ln B_0 &= r(t - 0) \\
 \implies B_t &= B_0 \exp(rt)
 \end{aligned}$$

- To get  $d \ln S_t$  use Ito's lemma [where  $\mu(S_t, t) = \mu S_t$ ,  $\sigma(S_t, t) = \sigma S_t$ , and  $f(x) = \ln x \implies f_x(x) = \frac{1}{x}$ ,  $f_{xx}(x) = -\frac{1}{x^2}$ ,  $f_t(x) = 0$ ]:

$$\begin{aligned}
 d \ln S_t &= \frac{1}{S_t} \mu S_t dt + \frac{1}{S_t} \sigma S_t dz_t + \frac{1}{2} \frac{-1}{S_t^2} (\sigma S_t)^2 dt + (0) dt \\
 &= \mu dt + \sigma dz_t - \frac{1}{2} \sigma^2 dt \\
 \implies \int_0^t d \ln S_s &= \mu \int_0^t ds + \sigma \int_0^t dz_s - \frac{1}{2} \sigma^2 \int_0^t dt \\
 \implies \ln S_t - \ln S_0 &= \mu t + \sigma z_t - \frac{1}{2} \sigma^2 t \\
 \implies S_t &= S_0 \exp(\mu t + \sigma z_t - \frac{1}{2} \sigma^2 t)
 \end{aligned}$$

where  $z_0 \equiv 0$ .

$$\begin{aligned}
 E[\ln S_t | \ln S_0] &= E[\ln S_0 + \mu t + \sigma z_t - \frac{1}{2} \sigma^2 t | \ln S_0] \\
 &= \ln S_0 + \mu t + \sigma E[z_t | \ln S_0] - \frac{1}{2} \sigma^2 t \\
 &= \ln S_0 + \mu t - \frac{1}{2} \sigma^2 t
 \end{aligned}$$

Using (2),

$$\begin{aligned}
 E[S_t | S_0] &= E[S_0 \exp(\mu t + \sigma z_t - \frac{1}{2} \sigma^2 t)] \\
 &= S_0 \exp(\mu t - \frac{1}{2} \sigma^2 t) E \left[ \exp \left( \int_{v=0}^t \sigma dz_v \right) \right] \\
 &= S_0 \exp(\mu t - \frac{1}{2} \sigma^2 t) E \left[ \exp \left( \frac{1}{2} \int_{v=0}^t \sigma^2 dv \right) \right] \\
 &= S_0 \exp(\mu t - \frac{1}{2} \sigma^2 t) \exp \left( \frac{1}{2} \sigma^2 t \right) \\
 &= S_0 \exp(\mu t)
 \end{aligned}$$

## Ito's Lemma (Scalar)

- Let  $f(x, t)$  be twice differentiable in  $x$  and once in  $t$ . Let  $x$  be a (scalar) diffusion with  $dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dz_t$ , then

$$f(x_t, t) - f(x_0, 0) = \int_{s=0}^t f_x(x_s, s)dx_s + \frac{1}{2} \int_{s=0}^t f_{xx}(x_s, s)\sigma(x_s, s)^2 ds + \int_{s=0}^t f_t(x_s, s)ds$$

$$df = f_x \mu dt + f_x \sigma dz_t + \frac{1}{2} f_{xx} \sigma^2 dt + f_t dt$$

where  $f_x = \frac{\partial f(x, s)}{\partial x}$  ( $f_{xx}$  and  $f_t$  similar).

- Examples

- Consider  $d(z_t^2)$ . Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t, t) &= 0, \sigma(x_t, t) = 1 \quad \forall x_t, t \\ \implies dx_t &= 0 * dt + 1 * dz_t = dz_t \\ f(x, t) &= x^2 \\ \implies f_x(x, t) &= 2x, f_{xx} = 2, f_t = 0 \end{aligned}$$

$$\implies d(z_t^2) = df = (2z_t)(1)dz_t + (2z_t)(0)dt + \frac{1}{2}(2)(1)^2 dt + (0)dt = 2z_t dz_t + dt$$

- Consider  $d \exp(z_t)$ . Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t, t) &= 0, \sigma(x_t, t) = 1 \quad \forall x_t, t \\ \implies dx_t &= 0 * dt + 1 * dz_t = dz_t \\ f(x, t) &= \exp(x) \\ \implies f_x(x, t) &= \exp(x), f_{xx} = \exp(x), f_t = 0 \end{aligned}$$

$$\begin{aligned} \implies d \exp(z_t) &= df = \exp(z_t)(0)dt + \exp(z_t)(1)dz_t + \frac{1}{2} \exp(z_t)(1)^2 dt + (0)dt \\ &= \exp(z_t)dz_t + \frac{1}{2} \exp(z_t)dt \end{aligned}$$

- Consider  $dx_t = \mu dt + \sigma dz_t$  and  $d \exp(x_t)$ . Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t, t) &= \mu, \sigma(x_t, t) = \sigma \quad \forall x_t, t \\ \implies dx_t &= \mu dt + \sigma dz_t \\ f(x, t) &= \exp(x) \\ \implies f_x(x, t) &= \exp(x), f_{xx} = \exp(x), f_t = 0 \end{aligned}$$

$$\begin{aligned} \implies d \exp(z_t) &= df = \exp(x_t)\mu dt + \exp(z_t)\sigma dz_t + \frac{1}{2} \exp(z_t)\sigma^2 dt + (0)dt \\ &= \exp(x_t)\mu dt + \exp(z_t)\sigma dz_t + \frac{1}{2} \exp(z_t)\sigma^2 dt \end{aligned}$$

## No Instantaneous Arbitrage

- Bond increment is  $dB_t = B_t r dt$
- Stock price increment is  $dS_t = \mu S_t dt + \sigma S_t dz_t$
- Option price increment is [by Ito's Lemma where  $\mu(S_t, t) = S_t \mu$ ,  $\sigma(S_t, t) = \sigma S_t$ ,  $f = C$ ,  $f_x = C_s$ , etc.]

$$\begin{aligned} dC(S_t, t) &= C_s S_t \mu dt + C_s \sigma S_t dz_t + \frac{1}{2} C_{ss} (\sigma S_t)^2 dt + C_t dt \\ &= C_s (S_t \mu dt + \sigma S_t dz_t) + \frac{1}{2} C_{ss} (\sigma S_t)^2 dt + C_t dt \\ &= C_s dS_t + \frac{1}{2} C_{ss} (\sigma S_t)^2 dt + C_t dt \end{aligned}$$

where  $C_t = \frac{\partial C(S_t, t)}{\partial t}$  and similar for  $C_s$  and  $C_{ss}$ .

- Portfolio (value) increment:  $dP_t = -dC(S_t, t) + C_s dS_t + (C - C_s S_t) r dt$

This portfolio is where you sell one option (inflow of  $C(S_t, t)$ ), buy  $C_s$  shares of stock at price  $S_t$  (outflow of  $C_s S_t$ ), and invest  $C(S_t, t) - C_s S_t$  dollars in the bond (outflow of  $C(S_t, t) - C_s S_t$ ).

Portfolio cost is zero [i.e.  $C(S_t, t) - C_s S_t - (C(S_t, t) - C_s S_t) = 0$ ] and it is risk-free:

No arbitrage  $\implies dP_t = 0 \implies 0 = -dC(S_t, t) + C_s dS_t + (C(S_t, t) - C_s S_t) r dt$

Substituting in option increment:

$$\begin{aligned} \implies 0 &= -[C_s dS_t + \frac{1}{2} C_{ss} \sigma^2 S_t^2 dt + C_t dt] + C_s dS_t + (C(S_t, t) - C_s S_t) r dt \\ &= -\frac{1}{2} C_{ss} \sigma^2 S_t^2 dt - C_t dt + (C(S_t, t) - C_s S_t) r dt \\ &= -\frac{1}{2} C_{ss} \sigma^2 S_t^2 - C_t + (C(S_t, t) - C_s S_t) r \end{aligned}$$

## Black-Scholes Call Option Price

- The price of a European call option for  $0 \leq t \leq T, 0 \leq S_t$  satisfies:

$$\begin{aligned} 0 &= \frac{1}{2} C_{ss} \sigma^2 S_t^2 + C_t - (C(S_t, t) - C_s S_t) r && \text{[differential equation]} \\ C(S_T, T) &= \max[S_T - K, 0] && \text{[boundary condition]} \\ C(0, t) &= 0, && \forall 0 \leq t < T \end{aligned}$$

- A solution is:

$$\begin{aligned} C(S_t, t) &= S_t \Phi(d_1(S_t)) - K \exp(-r(T-t)) \Phi(d_2(S_t)) \\ d_1(S) &:= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \\ d_2(S) &:= d_1(S) - \sigma \sqrt{T-t} \\ \Phi(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-\frac{v^2}{2}) dv && \text{[standard normal cdf]} \end{aligned}$$



## Ito's Lemma (Vector)

- Let scalar function  $f(\mathbf{x}, t)$  be twice differentiable in vector  $\mathbf{x}$  and once in  $t$ .
- Let  $\mathbf{x}_t$  be a vector diffusion with increment:

$$d\mathbf{x}_t = \boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t$$

where  $\mathbf{z}_t$  is a Brownian Motion vector. Then<sup>1</sup>

$$\begin{aligned} f(\mathbf{x}_t, t) - f(\mathbf{x}_0, 0) &= \int_{s=0}^t f_{\mathbf{x}}(\mathbf{x}_s, s)d\mathbf{x}_s + \frac{1}{2} \int_{s=0}^t \text{tr}[f_{\mathbf{xx}}(\mathbf{x}_s, s)\boldsymbol{\sigma}(\mathbf{x}_s, s)\boldsymbol{\sigma}(\mathbf{x}_s, s)^T]ds + \int_{s=0}^t f_t(\mathbf{x}_s, s)ds \\ \iff df &= f_{\mathbf{x}}^T \boldsymbol{\mu}_x dt + f_{\mathbf{x}}^T \boldsymbol{\sigma}_x d\mathbf{z}_t + \frac{1}{2} \text{tr}[f_{\mathbf{xx}} \boldsymbol{\sigma} \boldsymbol{\sigma}^T]dt + f_t dt \end{aligned}$$

## Ito's Lemma Examples

$$d \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} dt + \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}$$

- Consider  $d(x^2)$ . We know that  $dx = \mu_x dt + \sigma_x dz_1$ . Mapping to Ito's Lemma notation,  $f(x) = x^2, f_x = 2x, f_{xx} = 2, f_t = 0$ :

$$d(x^2) = 2x\mu_x dt + 2x\sigma_x dz_1 + (0)dt + \frac{1}{2}(2)\sigma_x^2 dt = 2x\mu_x dt + 2x\sigma_x dz_1 + \sigma_x^2 dt$$

- Consider  $d(xy)$ . We know that  $dy = \mu_y dt + \sigma_y \rho dz_1 + \sigma_y \sqrt{1-\rho^2} dz_2$

$$\begin{aligned} d(xy) &= xdy + ydx + dxdy \\ &= x[\mu_y dt + \sigma_y \rho dz_1 + \sigma_y \sqrt{1-\rho^2} dz_2] + y[\mu_x dt + \sigma_x dz_1] \\ &\quad + [\mu_x dt + \sigma_x dz_1][\mu_y dt + \sigma_y \rho dz_1 + \sigma_y \sqrt{1-\rho^2} dz_2] \\ &= x\mu_y dt + x\sigma_y \rho dz_1 + x\sigma_y \sqrt{1-\rho^2} dz_2 + y\mu_x dt + y\sigma_x dz_1 \\ &\quad + \mu_x dt \mu_y dt + \mu_x dt \sigma_y \rho dz_1 + \mu_x dt \sigma_y \sqrt{1-\rho^2} dz_2 \\ &\quad + \sigma_x dz_1 \mu_y dt + \sigma_x dz_1 \sigma_y \rho dz_1 + \sigma_x dz_1 \sigma_y \sqrt{1-\rho^2} dz_2 \\ &= x\mu_y dt + x\sigma_y \rho dz_1 + x\sigma_y \sqrt{1-\rho^2} dz_2 \\ &\quad + y\mu_x dt + y\sigma_x dz_1 + \sigma_x \sigma_y \rho dt + \sigma_x \sigma_y \sqrt{1-\rho^2} dt \\ &= (x\mu_y + y\mu_x + \sigma_x \sigma_y \rho + \sigma_x \sigma_y \sqrt{1-\rho^2})dt + (x\sigma_y \rho + y\sigma_x)dz_1 + x\sigma_y \sqrt{1-\rho^2} dz_2 \end{aligned}$$

## Application of the Martingale Property

- Suppose  $X$  is Brownian motion,  $dX = dz$ .
- We know  $X_T | X_t \sim N(X_t, T-t)$ :

$$h(X_t, t) := \Pr(X_T \leq A | X_t) = \Phi\left(\frac{A - X_t}{\sqrt{T-t}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{A-X_t}{\sqrt{T-t}}} \exp\left(-\frac{u^2}{2}\right) du$$

---

<sup>1</sup> $\text{tr}[\mathbf{B}]$  denotes the trace of matrix  $\mathbf{B}$  i.e. the elements of the diagonal.

- By Ito's lemma,  $dh = h_x dX + \frac{1}{2}h_{xx}dt + h_t dt$

Notice that  $E[dh] = h_x E[dX] + \frac{1}{2}h_{xx}dt + h_t dt = \frac{1}{2}h_{xx}dt + h_t dt$

- Also, probabilities are martingales:

$$\frac{1}{v}E_t(h(X_{t+v}, t+v) - h(X_t, t)) = \frac{1}{v}E_t(E_{t+v}(\mathbb{1}_{X_T \leq A}) - E_t(\mathbb{1}_{X_T \leq A})) = 0$$

Taking  $v$  small, so  $dt = v$ :

$$\implies 0 = E_t(dh)/dt = \frac{1}{2}h_{xx} + h_t, \text{ subject to } h(X_T, T) = \begin{cases} 1 & \text{if } X_T \leq A \\ 0 & \text{otherwise} \end{cases}$$

- Show that  $0 = \frac{1}{2}h_{xx} + h_t$ :

$$\begin{aligned} \Phi'(x) &= \phi(x) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \\ \phi'(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)(-x) \\ &= -x * \phi(x) \\ h_t &= \frac{\partial h(X_t, t)}{\partial t} \\ &= \phi\left(\frac{A - X_t}{\sqrt{T - t}}\right) \frac{(A - X_t)\frac{1}{2}(T - t)^{-1/2} - (0)\sqrt{T - t}}{T - t} \\ &= \frac{1}{2}\phi\left(\frac{A - X_t}{\sqrt{T - t}}\right) \frac{A - X_t}{(T - t)^{3/2}} \\ h_x &= \frac{\partial h(X_t, t)}{\partial X_t} \\ &= \phi\left(\frac{A - X_t}{\sqrt{T - t}}\right) \frac{-1}{\sqrt{T - t}} \\ h_{xx} &= \frac{\partial^2 h(X_t, t)}{\partial^2 X_t} \\ &= \phi'\left(\frac{A - X_t}{\sqrt{T - t}}\right) \frac{1}{T - t} \\ &= \phi\left(\frac{A - X_t}{\sqrt{T - t}}\right) \frac{X_t - A}{(T - t)^{3/2}} \end{aligned}$$

Thus,  $\frac{1}{2}h_{xx} + h_t = 0$ .

## Feynman-Kac I

- Suppose the risk-free rate  $r$  is constant (and technical conditions hold on functions  $\mu$  and  $\sigma^2$ ).
- The function  $f(x, t)$  is a solution to the boundary-value problem:

$$\begin{aligned} 0 &= \frac{1}{2}f_{xx}(x, t)\sigma^2(x, t) + f_x(x, t)\mu(x, t) + f_t(x, t) - rf(x, t), 0 \leq t < T \\ f(x, T) &= F(x) \end{aligned}$$

iff  $f$  is given by

$$\begin{aligned} f(x, t) &= \exp(-r(T-t))E[F(X_T)|X_t = x] \\ \text{where } dX_s &= \mu(X_s, s)ds + \sigma(X_s, s)dz_s; X_t = x \end{aligned}$$

- Intuition: Use the factor of integration  $\exp(-rt)$ .

$$\begin{aligned} f(X_T, T)\exp(-rT) - f(X_t, t)\exp(-rt) &= \int_{s=t}^T d(f(X_s, s)\exp(-rs)) \\ &= \int_{s=t}^T \exp(-rs)f_x(X_s, s)\sigma(X_s, s)dz_s \end{aligned}$$

## Black-Scholes and Feynman-Kac

- The price of a European call option for  $0 \leq t \leq T, 0 \leq S_t$  satisfies:

$$\begin{aligned} 0 &= \frac{1}{2}C_{ss}\sigma^2S_t^2 + C_t - (C(S_t, t) - C_sS_t)r \\ C(S_T, T) &= \max[S_T - K, 0] \\ C(0, t) &= 0 \end{aligned}$$

- A solution is:

$$\begin{aligned} C(s, t) &= \exp(-r(T-t))E[\max\{S_T^* - K, 0\}|S_t^* = s] \\ \text{where } \frac{dS_t^*}{S_t^*} &= rdt + \sigma dz_t \end{aligned}$$

## Feynman-Kac II

- Under technical conditions on functions  $g$ ,  $r$ ,  $\mu$ , and  $\sigma^2$ , the function  $f(x, t)$  is a solution to the boundary-value problem:

$$\begin{aligned} 0 &= \frac{1}{2}f_{xx}(x, t)\sigma^2(x, t) + f_x(x, t)\mu(x, t) + f_t(x, t) + g(x, t) - r(x, t)f(x, t), 0 \leq t < T \\ f(x, T) &= F(x) \end{aligned}$$

iff  $f$  is given by

$$f(x, t) = E \left[ \int_{v=t}^T \exp \left( \int_{v=t}^T -r(X_v, v) dv \right) g(X_s, s) ds + \exp \left( \int_{s=t}^T -r(X_s, s) ds \right) F(X_T) \middle| X_t = x \right]$$

where  $dX_s = \mu(X_s, s)ds + \sigma(X_s, s)dz_s$ ;  $X_t = x$

## 2 Part II

### Diffusion Processes

- In this section, any vector stochastic process  $S$  is an Ito process  $dS_t = \mu(\omega, t)dt + \sigma(\omega, t)dz_t$  or (shorthand)  $dS_t = \mu_t dt + \sigma_t dz_t$  (the subscripts remind us that the coefficient are functions of state and time).
- We casually speak of  $\mu_t$  as the conditional expected growth rate (or drift) of  $S_t$  per unit of time, while the conditional covariance matrix of the increment  $dS_t$  is  $\sigma_t \sigma_t^T$ .

### Continuous-Time Budget Constraint

- In continuous time, wealth is equal before and after a trade:

$$W_t = \underbrace{\theta_t \cdot S_t}_{\text{At time } t, \text{ after trade}} = \underbrace{\theta_{t-\Delta} \cdot S_t + \theta_{t-\Delta} \cdot D_t \Delta - c_t \Delta + y_t \Delta}_{\text{At time } t, \text{ before trade}}$$

where  $\theta_t$  is the vector of positions,  $S_t$  is the price vector,  $D_t$  is the dividend vector,  $c_t$  is the consumption rate, and  $y_t$  is the labor income rate.

$$\begin{aligned} \Rightarrow \underbrace{\theta_{t-\Delta} \cdot D_t \Delta}_{\text{dividend income}} - \underbrace{c_t \Delta}_{\text{consumption}} + \underbrace{y_t \Delta}_{\text{labor income}} &= \underbrace{(\theta_t - \theta_{t-\Delta}) \cdot S_t}_{\text{asset purchases}} \\ &= (\theta_t - \theta_{t-\Delta}) \cdot S_{t-\Delta} + (\theta_t - \theta_{t-\Delta}) \cdot (S_t - S_{t-\Delta}) \end{aligned} \quad (3)$$

- Change in wealth is:

$$\begin{aligned} W_t - W_{t-\Delta} &= \theta_t \cdot S_t - \theta_{t-\Delta} \cdot S_{t-\Delta} \\ &= \theta_{t-\Delta} \cdot (S_t - S_{t-\Delta}) + (\theta_t - \theta_{t-\Delta}) \cdot S_{t-\Delta} + (\theta_t - \theta_{t-\Delta}) \cdot (S_t - S_{t-\Delta}) \end{aligned} \quad (4)$$

- Combining (3) and (4):

$$\underbrace{W_t - W_{t-\Delta}}_{\text{Change in wealth}} = \underbrace{\theta_{t-\Delta} \cdot (S_t - S_{t-\Delta}) + \theta_{t-\Delta} \cdot D_t \Delta}_{\text{Portfolio gains or losses}} - \underbrace{c_t \Delta}_{\text{consumption}} + \underbrace{y_t \Delta}_{\text{labor income}}$$

- Let  $\Delta \rightarrow 0$ :

$$\begin{aligned} dW_t &= \theta_t \cdot dS_t + \theta_t \cdot D_t dt - c_t dt + y_t dt \\ &= W_t (\alpha_t \cdot ((dS_t + D_t dt) / S_t - r_t \mathbb{1} dt) + r_t dt) - c_t dt + y_t dt \end{aligned}$$

where  $\alpha_t$  is the portfolio weight in each risky security,  $r_t$  is the instantaneous risk-free rate, and “./” denotes element-by-element division.

## Normally Distributed Returns

- We write  $(dS_t + D_t dt)/S_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{z}_t$  when prices are diffusion process, where expected rate of return is  $\boldsymbol{\mu}_t$  is the sum of the expected capital gain and dividend yield.
- The instantaneous rate of return on a portfolio is distributed normal, with expected return and variance of return:

$$\begin{aligned}\mu_p &= \alpha_t \cdot (\boldsymbol{\mu}_t - r_t \mathbb{1}) + r_t \\ \sigma_p^2 &= \alpha_t^T \mathbf{V}_t \alpha_t = \alpha_t^T \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t^T \alpha_t\end{aligned}$$

- Furthermore, if an investor has state-independent utility, an optimal portfolio is on the instantaneous minimum variance frontier.
- Given an instantaneous risk-free rate, the MVF is defined by the risk-free asset and the tangency portfolio of risky assets with the highest Sharpe ratio.
- The portfolio of (only) risky assets with the maximum Sharpe ratio is

$$\boldsymbol{\alpha}_t = \frac{\mathbf{V}_t^{-1}(\boldsymbol{\mu}_t - r_t \mathbb{1})}{\mathbb{1}^T \mathbf{V}_t^{-1}(\boldsymbol{\mu}_t - r_t \mathbb{1})}$$

## Self-Financing Portfolio

- The process for wealth is  $dW_t = \theta_t \cdot (dS_t + D_t dt) - c_t dt + y_t dt$ .
- A **self-financing portfolio** has no inflows or outflows. Here this means that  $c_t = y_t$  and all dividends are reinvested, so  $dW_t = d(\theta_t \cdot S_t) = \theta_t \cdot (dS_t + D_t dt)$ .
- Given processes  $S$  and  $D$ , the **gains process** is defined as  $G_t := S_t + \int_{s=0}^t D_s ds$ .
- Then, a self-financing strategy  $\theta$  satisfies  $dW_t = \theta_t \cdot dG_t$  and  $W_t = W_0 + \int_{s=0}^t \theta_s \cdot dS_s + \int_{s=0}^t \theta_s \cdot D_s ds$ .

## Stochastic Discount Factor

- A **continuous-time stochastic discount factor** is a positive process  $\pi > 0$ , such that:

$$S_t = \frac{1}{\pi_t} E_t \int_{s=t}^{\infty} D_s \pi_s ds$$

- Define a **discounted gains process**  $G_t^\pi = \pi_t S_t + \int_{s=0}^t \pi_s D_s ds$ . Then, for  $T > t$ ,

$$G_t^\pi = \pi_t S_t + \int_{s=0}^t \pi_s D_s ds = E_t \left( \int_{s=0}^T D_s \pi_s ds + \pi_T \frac{1}{\pi_T} \int_{s=T}^{\infty} D_s \pi_s ds \right) = E_t(G_T^\pi)$$

Thus, the discounted gains process is a martingale.

- For a stock that first pays dividends at date  $T > t$ ,  $\pi_t S_t = E_t(\pi_T S_T)$ .
- For a discount bond that pays \$1 at date  $T > t$ ,  $P(t, T) = \exp(-YTM(t, T)(T - t)) = E_t(\frac{\pi_T}{\pi_t})$ .
- For a strategy of rolling over an instantaneously risk-free asset,  $1 = E_t(\exp(\int_t^T r_s ds) \frac{\pi_T}{\pi_t})$

## Martingale Representation Theorem

- Theorem: If  $M$  is a (local) martingale, then there exists an adapted process  $\theta$  such that

$$M_t = M_0 + \int_0^t \theta_s dz_s, \text{ with } Pr\left(\int_0^t \theta_s \cdot \theta_s ds < \infty\right) = 1.$$

- The discounted process is an adapted process and a martingale. Thus,  $dG_t^\pi = \sigma_G(\omega, t) d\mathbf{z}_t$ , for some adapted  $\sigma_G(\omega, t)$ .

## Risk Premiums

- Assume the stochastic discount factor  $\pi_t$  is a diffusion. Then,

$$S_t \pi_t = E_t\left(\int_{s=t}^{\infty} D_s \pi_s ds\right) = E_t\left(\int_{s=t}^{t+\Delta} D_s \pi_s ds + E_t S_{t+\Delta} \pi_{t+\Delta}\right)$$

Using element-by-element division

$$0 = \frac{1}{\Delta} E_t\left(\int_{s=t}^{t+\Delta} (D_s \pi_s / S_t \pi_t) ds + \frac{1}{\Delta} (E_t(S_{t+\Delta} \pi_{t+\Delta}) - S_t \pi_t) / S_t \pi_t\right)$$

- Taking  $\Delta \rightarrow 0$ :

$$\begin{aligned} 0 &= D_t dt / S_t + E_t[d(S_t \pi_t) / S_t \pi_t] \\ &= D_t dt / S_t + E_t[dS_t / S_t] + E_t\left[\frac{d\pi_t}{\pi_t} \mathbb{1}\right] + E_t\left[\frac{d\pi_t}{\pi_t} dS_t / S_t\right] \\ \implies \underbrace{E_t[dS_t / S_t + D_t dt / S_t]}_{\text{Expected Rate of Return}} &= -E_t\left[\frac{d\pi_t}{\pi_t} \mathbb{1}\right] - E_t\left[\frac{d\pi_t}{\pi_t} dS_t / S_t\right] \end{aligned}$$

- For the risk-free security:

$$\frac{d\pi_t}{\pi_t} r_t dt = 0 \implies r_t dt = -E_t\left[\frac{d\pi_t}{\pi_t}\right] \quad (5)$$

- For the risky securities,

$$\begin{aligned} \underbrace{E_t[dS_t / S_t + D_t dt / S_t] - r_t \mathbb{1} dt}_{\text{Risk Premium}} &= -E_t\left[\frac{d\pi_t}{\pi_t} \mathbb{1}\right] - E_t\left[\frac{d\pi_t}{\pi_t} dS_t / S_t\right] - E_t\left[\frac{d\pi_t}{\pi_t} \mathbb{1}\right] \\ &= \underbrace{-E_t\left[\frac{d\pi_t}{\pi_t} dS_t / S_t\right]}_{\text{Covariance of price and SDF changes}} \end{aligned} \quad (6)$$

## Marginal Utility is a SDF

- Consider the lifetime expected utility  $E_t[\int_{s=t}^{\infty} u(c_s, s)ds]$ , where wealth follows the process  $dW_t = \theta_t \cdot dS_t + \theta \cdot D_t dt - c_t dt + y_t dt, \forall t; W_t \geq 0$ ; and initial wealth  $W_0$  is given.
- Pick an asset with price  $S_t$  and consider 2 perturbations:

1. Increase the consumption rate by  $\varepsilon S_t/\Delta$  for a brief interval of time  $[t, t + \Delta]$ , raising expected utility:

$$E_t \left( \int_{s=t}^{t+\Delta} [u(c_s + \varepsilon S_t/\Delta, s) - u(c_s, s)] ds \right) \approx E_t \left( \int_{s=t}^{t+\Delta} u'(c_s, s) (\varepsilon S_t/\Delta) ds \right) \rightarrow \varepsilon S_t u'(c_t, t)$$

2. Increase and hold forever  $\varepsilon$  more shares of an asset, raising expected utility

$$E_t \left( \int_{s=t}^{\infty} [u(c_s + \varepsilon D_s, s) - u(c_s, s)] ds \right) \approx \varepsilon E_t \left( \int_{s=t}^{\infty} u'(c_s, s) D_s ds \right)$$

- Each perturbation costs  $\varepsilon S_t$ , so optimality requires that

$$S_t u'(c_t, t) = E_t \left( \int_{s=t}^{\infty} u'(c_s, s) D_s ds \right)$$

- Consequently, we can use marginal utility as an SDF in (5) and (6):

$$E_t \left( \frac{du'(c_s, s)}{u'(c_s, s)} \right) = -r_t dt$$

$$\underbrace{\frac{D_t}{S_t} dt + E_t \frac{dS_t}{S_t} - r_t dt}_{\text{Risk Premium}} = - \underbrace{E_t \left( \frac{dS_t du'(c_s, s)}{S_t u'(c_s, s)} \right)}_{\text{Covariance}}$$

## Consumption-Based Asset Pricing

- Let the process for per capita consumption be  $\frac{dc_t}{c_t} = \mu_c dt + \sigma_c d\mathbf{z}_t$ .  
[Notice that consumption is a scalar, so  $\sigma_c$  is a row vector, which is different from almost all notion used here. Generally,  $\sigma$  is  $N$  by  $Q$  where  $N$  is length of the diffusion process vector and  $Q$  is the number of Brownian motions. Here  $N = 1$  and  $Q$  is arbitrary.]
- Let the process for the vector of risky asset prices be  $dS_t./S_t + D_t dt./S_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{z}_t$ .  
Thus, the risk premium of these risky assets is  $E_t[dS_t./S_t + D_t dt./S_t] - r_t \mathbb{1} = \boldsymbol{\mu}_t dt - r_t \mathbb{1} dt$
- Let time  $t$  utility be  $\exp(-\beta t)u(c_t)$ .
- Use Ito's Lemma to find  $d(\exp(-\beta t)u'(c_t))$ :  

$$- \mu(c_t, t) = \mu_c c_t \text{ and } \sigma(c_t, t) = \sigma_c c_t$$

$$-f(c_t, t) = \exp(-\beta t)u'(c_t) \implies f_c = \exp(-\beta t)u''(c_t), f_{cc} = \exp(-\beta t)u'''(c_t), f_t = -\beta \exp(-\beta t)u'(c_t)$$

$$\begin{aligned} d(\exp(-\beta t)u'(c_t)) &= f_c \boldsymbol{\sigma}(c_t, t) d\mathbf{z}_t + \left[ f_c \mu(c_t, t) + \frac{1}{2} f_{cc} \boldsymbol{\sigma}(c_t, t) \boldsymbol{\sigma}(c_t, t)^T + f_t \right] dt \\ &= \exp(-\beta t)u''(c_t) c_t \boldsymbol{\sigma}_c d\mathbf{z}_t + \left[ \exp(-\beta t)u''(c_t) \mu_c c_t \right. \\ &\quad \left. + \frac{1}{2} \exp(-\beta t)u'''(c_t) \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T c_t^2 - \beta \exp(-\beta t)u'(c_t) \right] dt \\ &= \exp(-\beta t) \left[ u''(c_t) c_t \boldsymbol{\sigma}_c d\mathbf{z}_t + \left[ u''(c_t) \mu_c c_t + \frac{1}{2} u'''(c_t) \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T c_t^2 - \beta u'(c_t) \right] dt \right] \end{aligned} \quad (7)$$

- Using marginal utility of a representative investor as the SDF in (6):

$$\begin{aligned} \boldsymbol{\mu}_t dt - r_t \mathbb{1} dt &= -E_t \left[ (dS_t./S_t) \frac{d(\exp(-\beta t)u'(c_t))}{\exp(-\beta t)u'(c_t)} \right] \\ \implies \boldsymbol{\mu}_t - r_t \mathbb{1} &= \frac{-1}{dt} E_t \left[ (dS_t./S_t) \frac{d(\exp(-\beta t)u'(c_t))}{\exp(-\beta t)u'(c_t)} \right] \\ &= -E_t \left[ (dS_t./S_t) \left( R_c \boldsymbol{\sigma}_c \frac{d\mathbf{z}_t}{dt} + R_c \mu_c + \frac{1}{2} \frac{u'''(c_t)}{u'(c_t)} \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T c_t^2 - \beta \right) \right] \\ &= \dots \\ &= R_c \boldsymbol{\sigma}_t \boldsymbol{\sigma}_c^T \\ &= R_c \mathbf{V} \mathbf{V}^{-1} \boldsymbol{\sigma}_t \boldsymbol{\sigma}_c^T \\ &= R_c \mathbf{V} \hat{\boldsymbol{\alpha}}_t \end{aligned}$$

where  $R_c = \frac{-c_t u''(c_t)}{u'(c_t)}$  is relative risk aversion,  $\mathbf{V} = \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t^T$ , and  $\hat{\boldsymbol{\alpha}}_t = \mathbf{V}^{-1} \boldsymbol{\sigma}_t \boldsymbol{\sigma}_c^T$  is a portfolio with maximum correlation with consumption.

- Thus,

$$\begin{aligned} \boldsymbol{\mu}_t - r_t \mathbb{1} &= R_c \mathbf{V} \hat{\boldsymbol{\alpha}}_t \\ \implies \underbrace{\hat{\boldsymbol{\alpha}}_t^T \boldsymbol{\mu}_t}_{:=\hat{\mu}} - r_t \underbrace{\hat{\boldsymbol{\alpha}}_t^T \mathbb{1}}_{=1} &= R_c \hat{\boldsymbol{\alpha}}_t^T \mathbf{V} \hat{\boldsymbol{\alpha}}_t \\ \implies R_c &= \frac{\hat{\mu} - r_t}{\hat{\boldsymbol{\alpha}}_t^T \mathbf{V} \hat{\boldsymbol{\alpha}}_t} \\ \implies \boldsymbol{\mu}_t - r_t \mathbb{1} &= \underbrace{\frac{\mathbf{V} \hat{\boldsymbol{\alpha}}_t}{\hat{\boldsymbol{\alpha}}_t^T \mathbf{V} \hat{\boldsymbol{\alpha}}_t}}_{:=\hat{\beta}} (\hat{\mu}_t - r_t) \end{aligned}$$



## The Instantaneous Risk-Free Rate

- Let per capita consumption follow process:  $\frac{dc_t}{c_t} = \mu_c dt + \sigma_c dz_t$ .
- Use the marginal utility of consumption as the discount factor and using (7),

$$\begin{aligned} r_t &= \frac{-1}{dt} E_t \left[ \frac{d(\exp(-\beta t) u'(c_t))}{\exp(-\beta t) u'(c_t)} \right] \\ &= \beta - \frac{u''(c_t)}{u'(c_t)} \mu_c c_t - \frac{1}{2} \frac{u'''(c_t)}{u'(c_t)} \sigma_c \sigma_c^T c_t^2 \end{aligned}$$

- Example using CRRA utility

$$\begin{aligned} u(c_t) &= \frac{c_t^{1-B}}{1-B} \\ u'(c_t) &= c_t^{-B} \\ u''(c_t) &= -B c_t^{-B-1} \\ u'''(c_t) &= B(B-1) c_t^{-B-2} \\ r_t &= \beta - \frac{-B c_t^{-B-1}}{c_t^{-B}} \mu_c c_t - \frac{1}{2} \frac{B(B-1) c_t^{-B-2}}{c_t^{-B}} \sigma_c \sigma_c^T c_t^2 \\ &= \beta + B \mu_c - \frac{1}{2} B(B-1) \sigma_c \sigma_c^T \\ &= \beta + B \left( \mu_c + \frac{1}{2} \sigma_c \sigma_c^T \right) - \frac{1}{2} B^2 \sigma_c \sigma_c^T \end{aligned}$$

## The Risk-Free Rate is a State Variable

- Suppose a representative investor with time-additive logarithmic utility lives in an economy with one productive asset.
- Assume marginal utility is  $u'(c_t, t) = \exp(-\beta t) c_t^{-1} = \exp(-\beta t - \ln c_t)$ .
- Assume wealth in units of consumption follows  $dW_t = W_t x_t dt + W_t \sigma \sqrt{x_t} dz_{1t} - c_t dt$ .
- Assume the productivity rate  $x$  follows  $dx_t = \kappa(\theta - x_t) dt + \sigma_x \sqrt{x_t} dz_{2t}$ .
- In the equilibrium of CIR, consumption is  $c_t = \beta W_t$  and it follows the subordinated process:

$$d \ln c_t = -\beta dt + (1 - \frac{1}{2} \sigma^2) x_t dt + \sigma \sqrt{x_t} dz_{1t}$$

As a consequence,

$$r_t = \frac{-1}{dt} E_t \left[ \frac{d(\exp(-\beta t) u'(c_t))}{\exp(-\beta t) u'(c_t)} \right] = (1 - \sigma^2) x_t$$

and

$$dr_t = \kappa(\theta_r - r_t) dt + \sigma_r \sqrt{r_t} dz_{2t}$$

where  $\theta_r = (1 - \sigma^2)\theta$  and  $\sigma_r \sqrt{1 - \sigma^2} = \sigma_x$

### 3 Part III

#### Hamilton-Jacobi-Bellman Equation

- Consider the problem:

$$J(W_t, \mathbf{X}_t, t) = \max_{\{\alpha_s, c_t\}} E_t \left[ \int_{s=t}^{\infty} u(c_s, s) ds \right]$$

$$dW_t = W_t(\alpha_t^T(\mu(\mathbf{X}_t, t) - r(\mathbf{X}_t, t)\mathbb{1}) + r(\mathbf{X}_t, t))dt + W_t\alpha_t^T\sigma(\mathbf{X}_t, t)d\mathbf{z}_t - c_tdt + y_tdt, \forall t;$$

$$d\mathbf{X}_t = \mu(\mathbf{X}_t, t)dt + \sigma_x(\mathbf{X}_t, t)d\mathbf{z}_t$$

where  $W_t \geq 0$  and  $W_0$  and  $\mathbf{X}_0$  is given.

- Suppose the optimal policy for consumption and investment is followed for all times  $s \geq t + \Delta$ .

$$J(W_t, \mathbf{X}_t, t) = \max_{\{\alpha_s, c_t\}_{t \leq s < t+\Delta}} E_t \left[ \int_{s=t}^{t+\Delta} u(c_s, s) ds + J(W_{t+\Delta}, \mathbf{X}_{t+\Delta}, t + \Delta) \right]$$

$$\implies 0 = \max_{\{\alpha_s, c_s\}_{t \leq s < t+\Delta}} \frac{1}{\Delta} \left[ E_t[J(W_{t+\Delta}, \mathbf{X}_{t+\Delta}, t + \Delta)] - J(W_t, \mathbf{X}_t, t) + E_t \left[ \int_{s=t}^{t+\Delta} u(c_s, s) ds \right] \right]$$

- Taking the limit as  $\Delta \rightarrow 0$ , optimal policies satisfy:

$$0 = \max_{\{\alpha_t, c_t\}} E_t[dJ(W_t, \mathbf{X}_t, t)]/dt + u(c_t, t)$$

#### $K + 2$ Fund Separation

- For simplicity, drop the explicit dependence of  $\mu, \mu_x, r, \sigma, \sigma_x$  on  $(\mathbf{X}_t, t)$ .
- Applying Ito's lemma, the HJB is:

$$0 = \max_{\{\alpha, c\}} J_{\mathbf{x}}^T \mu_x + J_W(W_t(\alpha^T(\mu - r\mathbb{1}) + r) - c + y_t) + \frac{1}{2} tr \left( \begin{bmatrix} J_{\mathbf{x}\mathbf{x}} & J_{\mathbf{x}W}^T \\ J_{\mathbf{x}W} & J_{WW} \end{bmatrix} \begin{bmatrix} \sigma_x \sigma_x^T & W_t \alpha^T \sigma \sigma_x^T \\ \sigma_x \sigma^T \alpha W_t & \alpha^T \sigma \sigma^T \alpha W_t^2 \end{bmatrix} \right) + J_t + u(c_t, t)$$

The FOCs are:

$$0 = u_c - J_w \quad [c]$$

$$\implies c_t^* = u_c^{-1}(J_w(W_t, \mathbf{X}_t, t), t)$$

$$0 = (\mu - r\mathbb{1})J_W + \sigma\sigma_x^T J_{\mathbf{x}W} + \sigma\sigma^T \alpha_t^* J_{WW} W_t \quad [\alpha]$$

$$\implies \alpha_t^* = T \underbrace{(\sigma\sigma^T)^{-1}(\mu - r\mathbb{1})}_{\text{Maximum Sharpe Ratio portfolio}} + \underbrace{(\sigma\sigma^T)^{-1}\sigma\sigma_x^T}_{\text{Portfolios with maximum correlations with } \mathbf{X}_t} \mathbf{T}_{\mathbf{x}}$$

$$\text{where } T \equiv \frac{-J_W}{J_{WW}W_t}$$

$$\mathbf{T}_{\mathbf{x}} \equiv \frac{-J_{\mathbf{x}W}}{J_{WW}W_t}$$

## Maximum Correlation Portfolios

- Consider the portfolio with instantaneous return

$$\frac{dP_t}{P_t} = (\boldsymbol{\alpha}_t^T (\boldsymbol{\mu}(\mathbf{X}_t, t) - r(\mathbf{X}_t, t)\mathbb{1}) + r(\mathbf{X}_t, t))dt + \boldsymbol{\alpha}_t^T \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{z}_t$$

the state variables follow

$$d\mathbf{X}_t = \boldsymbol{\mu}_x(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}_x(\mathbf{X}_t, t)d\mathbf{z}_t$$

where  $P_0$  and  $X_0$  are given.

- The correlation of the portfolio return with state variable  $j$ :

$$Cor\left(\frac{dP_t}{P_t}, dX_{jt}\right) = E\left(\frac{\frac{dP_t}{P_t} dX_{jt}}{\sqrt{\left(\frac{dP_t}{P_t}\right)^2} \sqrt{(dX_{jt})^2}}\right) = \frac{\boldsymbol{\alpha}_t^T \boldsymbol{\sigma} \boldsymbol{\sigma}_{jx}^T}{\sqrt{\boldsymbol{\alpha}_t^T \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t^T \boldsymbol{\alpha}_t} \sqrt{\boldsymbol{\sigma}_{jx} \boldsymbol{\sigma}_{jx}^T}}$$

- The portfolio with maximum correlation is

$$\arg \max_{\boldsymbol{\alpha}} Cor\left(\frac{dP_t}{P_t}, dX_{jt}\right) \propto (\boldsymbol{\sigma} \boldsymbol{\sigma}^T)^{-1} \boldsymbol{\sigma} \boldsymbol{\sigma}_{jx}^T$$

## The Value Function

- For function  $J(w, \mathbf{x}, t)$ , policies  $c$  and  $\boldsymbol{\alpha}$ , and the processes for  $W_t$  and  $\mathbf{X}_t$ , the operator  $D^{(c, \boldsymbol{\alpha})}$  is convenient shorthand, where:

$$D^{(c, \boldsymbol{\alpha})} J(w, \mathbf{x}, t) \equiv J_{\mathbf{x}}^T \boldsymbol{\mu}_x + J_w (w(\boldsymbol{\alpha}^T (\boldsymbol{\mu} - r\mathbb{1}) + r) - c + y_t) + \frac{1}{2} tr \left( \begin{bmatrix} J_{\mathbf{x}\mathbf{x}} & J_{\mathbf{x}w}^T \\ J_{\mathbf{x}w} & J_{ww} \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_x \boldsymbol{\sigma}_x^T & w \boldsymbol{\alpha}^T \boldsymbol{\sigma} \boldsymbol{\sigma}_x^T \\ \boldsymbol{\sigma}_x \boldsymbol{\sigma}^T \boldsymbol{\alpha} w & \boldsymbol{\alpha}^T \boldsymbol{\sigma} \boldsymbol{\sigma}^T \boldsymbol{\alpha} w^2 \end{bmatrix} \right) + J_t$$

Thus, the HJB is:

$$0 = \max_{\{\boldsymbol{\alpha}, c\}} D^{(c, \boldsymbol{\alpha})} J(w, \mathbf{x}, t) + u(c, t)$$

Given the optimal policies  $c_t^*$  and  $\boldsymbol{\alpha}_t^*$ , the value function solves:

$$0 = D^{(c_t^*, \boldsymbol{\alpha}_t^*)} J(W_t, \mathbf{X}, t) + u(c_t^*, t)$$

## Example - CRRA with no labor income, constant risk-free rate, and no risky assets

- Consider the problem:

$$J(W_t, t) \equiv \max_{\{c_s\}_{t \leq s}} E_t \left[ \int_{s=t}^{\infty} e^{-\rho s} \frac{c_s^{1-B}}{1-B} ds \right]$$

$$dW_t = W_t r dt - c_t dt, \forall t$$

where  $W_t \geq 0$  and  $W_0$  is given.

- HJB:  $0 = \max_{c_t} J_W(W_t r - c_t) + J_t + e^{-\rho t} \frac{c_t^{1-B}}{1-B}$
- FOC:  $e^{-\rho t} c_t^{-B} = J_W$
- Guess:  $J(W_t, t) = e^{-\rho t} H^B \frac{W_t^{1-B}}{1-B}$ , where  $H = W/C$  is the wealth-consumption ratio.
- Check:

$$\begin{aligned}
J_W &= e^{-\rho t} H^B W_t^{-B} \\
\implies c_t &= H^{-1} W_t \text{ and } J_t = -\rho e^{-\rho t} H^B \frac{W_t^{1-B}}{1-B} \\
\implies 0 &= e^{-\rho t} H^B W_t^{-B} (W_t r - H^{-1} W_t) - \rho e^{-\rho t} H^B \frac{W_t^{1-B}}{1-B} + e^{-\rho t} \frac{H^{-1+B} W_t^{1-B}}{1-B} \\
\implies H^{-1} &= \frac{1}{B} \rho - \frac{1-B}{B} r
\end{aligned}$$

### Example II - CRRA with geometric labor income, constant risk-free rate, and no risky assets

- Consider the problem:

$$\begin{aligned}
J(W_t, y_t) &\equiv \max_{\{c_s\}_{t \leq s}} E_t \left[ \int_{s=t}^{\infty} e^{-\rho s} \frac{c_s^{1-B}}{1-B} ds \right] \\
dW_t &= W_t r dt - c_t dt + y_t dt, \forall t \\
dy_t &= g y_t dt
\end{aligned}$$

where  $W_t \geq 0$  and  $W_0$  and  $y_0$  are given.

- Notice that  $y_t = y_0 e^{gt}$ .
- HJB:  $0 = \max_{c_t} J_W(W_t r - c_t + y_t) + J_t + e^{-\rho t} \frac{c_t^{1-B}}{1-B}$
- FOC:  $e^{-\rho t} c_t^{-B} = J_W$
- Guess:  $J(W_t, t) = e^{-\rho t} H^B \frac{(W_t + Y_t)^{1-B}}{1-B}$ , where  $H = W/C$  and  $Y_t = \frac{y_t}{r-g}$
- Check:

$$\begin{aligned}
J_W &= e^{-\rho t} H^B (W_t + Y_t)^{-B} \\
\implies c_t &= H^{-1} (W_t + Y_t) \\
\text{and } J_t &= -\rho e^{-\rho t} H^B \frac{(W_t + Y_t)^{1-B}}{1-B} + \rho e^{-\rho t} H^B (W_t + Y_t)^{-B} g Y_t \\
\implies 0 &= e^{-\rho t} H^B (W_t + Y_t)^{-B} (W_t r - H^{-1} (W_t + Y_t) + Y_t (r - g)) - \rho e^{-\rho t} H^B \frac{(W_t + Y_t)^{1-B}}{1-B} \\
&\quad + e^{-\rho t} H^B (W_t + Y_t)^{-B} g Y_t + e^{-\rho t} \frac{H^{-1+B} (W_t + Y_t)^{1-B}}{1-B} \\
\implies H^{-1} &= \frac{1}{B} \rho - \frac{1-B}{B} r
\end{aligned}$$

### Example III - Log utility with one risky asset and mean-reverting state variable

- Consider the problem:

$$J(W_t, x_t, t) \equiv \max_{\{c_s, \alpha_s\}_{t \leq s}} E_t \left[ \int_{s=t}^{\infty} e^{-\rho s} \log(c_s) ds \right]$$

$$dW_t = W_t(\alpha_t(x_t - r(x_t)) + r(x_t))dt + W_t \alpha_t \sigma \sqrt{x_t} dz_{1t} - c_t dt, \forall t$$

$$dx_t = \kappa(\theta - x_t) + \sigma_x \sqrt{x_t} dz_{2t}$$

where  $W_t \geq 0$ ,  $W_0$  and  $x_0$  is given, and  $z_1$  independent of  $z_2$ .

- HJB:

$$0 = \max_{c_t, \alpha_t} J_W(W_t(\alpha_t(x_t - r_t) + r_t) - c_t) + J_t + e^{-\rho t} \log(c_t) + \frac{1}{2} J_{WW} W_t^2 \alpha_t^2 \sigma^2 x_t + J_x \kappa(\theta - x_t) + \frac{1}{2} J_{xx} \sigma_x^2 x_t$$

- Guess:  $J(W_t, x_t, t) = e^{-\rho t} \frac{\log(W_t)}{\rho} + f(x_t)$
- FOC

$$0 = -e^{-\rho t} \frac{1}{\rho W_t} + e^{-\rho t} \frac{1}{c_t} \quad [c]$$

$$\implies c_t = \rho W_t$$

$$0 = -e^{-\rho t} \frac{1}{\rho W_t} W_t(x_t - r_t) + \frac{1}{2} e^{-\rho t} \frac{1}{\rho W_t^2} 2\alpha_t \sigma^2 x_t \quad [\alpha_t]$$

$$\implies \alpha_t = \frac{x_t - r_t}{\sigma^2 x_t}$$

- Thus, consumption is a constant wealth-consumption ratio because of log utility.
- HBJ:

$$0 = \frac{1}{\rho} \left[ \frac{(x_t - r_t)^2}{x_t^2 \sigma^2} + r_t - \rho \right] + \log(\rho) - \frac{1}{2} \frac{1}{\rho} \frac{(x_t - r_t)^2}{x_t^2 \sigma^2} + f_x \kappa(\theta - x_t) + \frac{1}{2} f_{xx} \sigma_x^2 x_t$$

- One possible solution is  $\alpha_t = 1 \implies r_t = x_t(1 - \sigma^2)$

### Example IV - CRRA on terminal wealth with single risky asset and constant risk-free rate

- Consider the problem:

$$J(W_t, t) \equiv \max_{\{\alpha_s\}_{t \leq s \leq T}} E_t \left[ \frac{W_T^{1-B}}{1-B} \right]$$

$$dW_t = W_t(\alpha_t(\mu - r) + r)dt + W_t \alpha_t \sigma dz_t, \forall t$$

where  $0 < B \neq 1$ ,  $W_t \geq 0$ , and  $W_0$  is given.

- HJB:  $0 = \max_{\alpha_t} J_W W_t (\alpha_t (\mu - r) + r) + J_t + \frac{1}{2} J_{WW} W_t^2 \alpha_t^2 \sigma^2$
- Guess:  $J(W_t, t) = e^{-k(T-t)} \frac{W_t^{1-B}}{1-B}$
- FOC:

$$0 = e^{-k(T-t)} W_t^{1-B} (\mu - r) - \frac{1}{2} B e^{-k(T-t)} W_t^{1-B} 2 \alpha_t \sigma^2$$

$$\implies \alpha_t = \frac{\mu - r}{B \sigma^2}$$

- HJB:

$$0 = e^{-k(T-t)} W_t^{1-B} \left( \frac{(\mu - r)^2}{B \sigma^2} + r \right) + k e^{-k(T-t)} \frac{W_t^{1-B}}{1-B} - \frac{1}{2} B e^{-k(T-t)} W_t^{1-B} \left( \frac{\mu - r}{B \sigma^2} \right)^2 \sigma^2$$

$$k = -(1-B) \left( \frac{1}{2} \frac{(\mu - r)^2}{B \sigma^2} + r \right)$$

## Intertemporal CAPM (ICAPM)

- By  $K + 2$  fund separation, the optimal portfolio for individual  $i$  is:

$$\alpha_t^i W_t^i = \underbrace{(\sigma \sigma^T)^{-1} (\mu_t - r_t \mathbb{1})}_{\text{Maximum Sharpe ratio portfolio}} T^i W_t^i + \underbrace{(\sigma \sigma^T)^{-1} \sigma \sigma_x^T}_{\text{Maximum correlation portfolios}} \mathbf{T}_x^i W_t^i$$

where  $T \equiv \frac{-J_W}{J_{WW} W_t}$  and  $\mathbf{T}_x \equiv \frac{-J_{xW}}{J_{WW} W_t}$ .

- The market portfolio  $\alpha_t^m$  is:

$$\alpha_t^m \equiv \sum_i \alpha_t^i W_t^i / W_t^m = (\sigma \sigma^T)^{-1} (\mu_t - r_t \mathbb{1}) T^m + \alpha_{MCP,t} \mathbf{T}_x^m$$

where  $T^m \equiv \sum_i T^i W_t^i / W_t^m$

$$\mathbf{T}_x^m \equiv \sum_i \mathbf{T}_x^i W_t^i / W_t^m$$

$$\alpha_{MCP,t} \equiv (\sigma \sigma^T)^{-1} \sigma \sigma_x^T$$

- Expected returns are:

$$\mu_t - r_t \mathbb{1} = \sigma \sigma^T \begin{bmatrix} \alpha_t^m & \alpha_{MCP,t} \end{bmatrix} \begin{bmatrix} 1/T^m \\ -\mathbf{T}_x^m / T^m \end{bmatrix}$$

$$= \text{Cov}(\text{Var})^{-1} \begin{bmatrix} \mu_{mt} - r_t \\ \mu_{MCP,t} - r_t \mathbb{1} \end{bmatrix}$$

$$\text{Cov} \equiv \sigma \sigma^T \begin{bmatrix} \alpha_t^m & \alpha_{MCP,t} \end{bmatrix}$$

$$\text{Var} \equiv \begin{bmatrix} \alpha_t^m & \alpha_{MCP,t} \end{bmatrix}^T \sigma \sigma^T \begin{bmatrix} \alpha_t^m & \alpha_{MCP,t} \end{bmatrix}$$

## Stochastic Differential Utility

- Consider the problem:

$$J(W_t, \mathbf{X}_t) \equiv \max_{\{\alpha_s, c_t\}_{t \leq s}} E_t \left[ \int_{s=t}^{\infty} f(c_s, J(W_s, \mathbf{X}_s)) ds \right]$$

$$dW_t = W_t(\alpha_t^T(\mu(\mathbf{X}_t, t) - r(\mathbf{X}_t, t)\mathbb{1}) + r(\mathbf{X}_t, t))dt + W_t\alpha_t^T\sigma(\mathbf{X}_t, t)d\mathbf{z}_t - c_t dt + y_t dt, \forall t;$$

$$d\mathbf{X}_t = \mu(\mathbf{X}_t, t)dt + \sigma_x(\mathbf{X}_t, t)d\mathbf{z}_t$$

where  $W_t \geq 0$ ,  $W_0$  and  $\mathbf{X}_0$  is given, and where the felicity function is defined

$$f(c, J) = \phi\theta \left( \frac{c^{1-1/\psi}}{((1-\gamma)J)^{1/\theta}} - 1 \right) J$$

with  $\theta = \frac{1-\gamma}{1-1/\psi}$ , rate of time preference if  $\phi$ , risk aversion is  $\gamma$ , and elasticity of intertemporal substitution is  $\psi$ .

- The HJB equation now is:

$$0 = \max_{\alpha_t, c_t} E_t dJ(W_t, \mathbf{X}_t) \frac{1}{dt} + f(c_t, J(W_t, \mathbf{X}_t)).$$

- Standard result:

$$J(W_t, \mathbf{X}_t) = \frac{W_t^{1-\gamma}}{1-\gamma} (\phi H_t^{1/\psi})^\theta$$

where  $H_t = H(\mathbf{X}_t) = \frac{W_t}{c_t}$

## Special Case: Time Additive

- Set  $\gamma = 1/\psi$  and  $\theta = 1$ , implying  $f(c, J) = \phi(\frac{c^{1-\gamma}}{1-\gamma} - J)$ .
- Under an optimal policy,

$$\begin{aligned} 0 &= E_t \left[ dJ_t + \phi \left( \frac{c_t^{1-\gamma}}{1-\gamma} - J_t \right) dt \right] \\ &= E_t \left[ d(e^{-\phi t} J_t) \right] + \phi e^{-\phi t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \\ &= E_t \left[ \int_{s=t}^{\infty} d(e^{-\phi s} J_s) \right] + \phi E_t \int_{s=t}^{\infty} e^{-\phi s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \end{aligned}$$

[second step is introduce factor of integration, third step is integrate and use iterated expectations.]

- Assume transversality conditions holds.

$$\implies J(W_t, \mathbf{X}_t) = \phi E_t \left[ \int_{s=t}^{\infty} e^{\phi(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right]$$

## Example V

Setting  $\psi \rightarrow 1$  with SDU:

$$J(W_t, y_t) = \max_{\{\alpha_s, c_s\}_{t \leq s}} E_t \left[ \int_{s=t}^{\infty} f(c_s, J(W_s, y_s)) ds \right]$$

$$\text{Felicity: } f(c, J) = \phi(1 - \gamma) \left( \log(c) - \frac{1}{1 - \gamma} \log((1 - \gamma)J) \right) J$$

$$\text{Wealth: } dW_t = W_t(\alpha_t(\mu - r) + r)dt + W_t\alpha_t\sqrt{\frac{1}{y_t}}dz_{1t} - c_tdt, \forall t$$

$$\text{Precision: } dy_t = \kappa(\bar{y} - y_t)dt + \sigma\sqrt{y_t}dz_{2t}$$

where  $W_t \geq 0$ ,  $W_0$  given, and  $dz_{1t}dz_{2t} = \rho dt$ .

- HJB:

$$0 = \max_{\alpha_t, c_t} J_W(W(\alpha_t(\mu - r) + r) - c_t) + h(c_t, J) + \frac{1}{2}J_{WW}W_t^2\alpha_t^2\frac{1}{y_t} + J_y\kappa(\bar{y} - y_t) + \frac{1}{2}J_{yy}\sigma^2y_t + J_{Wy}W_t\alpha_t\rho\sigma.$$

- Guess:  $J(W, y) = \frac{W^{1-\gamma}}{1-\gamma}h(y)$
- Solution:

$$c_t = \phi W_t$$

$$\alpha_t = \frac{1}{\gamma}(\mu - r)y_t + (1 - 1/\gamma)(-\rho)\sigma\frac{A}{1 - \gamma}y_t$$

$$h(y) = \exp(Ay + B)$$

where  $A$  and  $B$  solve a system of quadratic equations.

## Challenge

- In Ai (JF, 2010), an investor chooses consumption and savings when capital (wealth) follows CRS process.
- The productivity rate of capital  $x_t$  is unobservable and the investor continuously updates a posterior distribution that has mean  $m_t$ .
- The investor solves

$$J(K, m_t) = \max_{\{c_s\}_{t \leq s}} E_t \left[ \int_{s=t}^{\infty} f(c_s, J(K_s, m_s)) ds \right]$$

$$dK_t = K_t m_t dt - c_t dt + K_t \sigma dz_t$$

$$dm_t = \kappa(\mu - m_t)dt + \sigma_m dz_t^m$$

where  $K_t \geq 0$ ,  $K_0$  and  $m_0$  given,  $dz_t dz_t^m = \rho dt$ ,  $\sigma > 0, \kappa > 0, \mu > 0, \sigma_m > 0, \rho > 0$  constants.

- To do:



1. Write out the HJB equation.
2. Show that the standard result for  $J_t$  holds, where  $H$  is the wealth consumption ratio.
3. Show that the HJB (given the optimal consumption policy) is a partial differential equation in  $H$  as a function of  $m$ . (The HJB has no known general solution.)
4. Assume a special case holds:  $m_t = \mu$ , a constant. Solve for  $H$  as a constant, and show that it satisfies the Constant Dividend Growth Model of Gordon.

## 4 Part IV

### Setup

- We assume:
  1. A probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ .
  2. A  $d$ -dimensional Brownian motion  $\mathbf{z} : \Omega \times [0, \infty) \rightarrow \mathbb{R}^d$ .
  3. An investor's information (filtration  $\mathcal{F}_t$ ) is defined by the history of  $\mathbf{z}$ .
- All adapted processes are Ito process with increments written:

$$dS_t = \mu(\omega, t)dt + \sigma(\omega, t)d\mathbf{z}_t$$

- If  $S_t > 0$  assumed for all  $t$ , an alternative expression is  $dS_t./S_t = \mu(\omega, t)dt + \sigma(\omega, t)d\mathbf{z}_t$  with  $\mu(\omega, t)$  and  $\sigma(\omega, t)$  appropriately redefined.
- Any scalar martingale satisfies  $dM_t = \gamma(\omega, t)d\mathbf{z}_t$  and any positive martingale satisfies  $dM_t = M_t\gamma(\omega, t) \cdot d\mathbf{z}_t$  for some adapted  $\gamma(\omega, t)$ .

### Deflated Gains and SDF

- A deflator is a process  $Y$  such that  $P(Y_t > 0) = 1$  for all  $t$ . A deflator changes the numeraire.
- For example if  $Y_t = \exp(-\int_0^t r_s ds)$ , which is the inverse of gains rolling over at the risk-free rate  $r_s$ ,  $S^Y := YS$  is the (vector) of units of an account with value  $\exp(\int_0^t r_s ds)$  required to buy one unit of each asset at the prices in  $S_t$ .
- A deflated gains process if  $G_t^Y := S_t Y_t + \int_{s=0}^t D_s Y_s ds$ .
- Another example: A stochastic discount factor is a deflator  $\pi$  such that  $G_t^\pi$  is a martingale:  $E_t(G_s^\pi) = G_t^\pi, t < s$ .
- If  $dS_t./S_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma} d\mathbf{z}_t$  and if  $G_t^Y$  is a martingale with  $Y_t = \exp(-\int_0^t r_s ds)$ , then

$$0 = E_t dG_t^Y = S_t Y_t (-r_t \mathbb{1} dt + E_t[dS_t./S_t + D_t./S_t dt]) \implies r_t \mathbb{1} = \boldsymbol{\mu}_t + D_t./S_t$$

- Let a pure discount bond have payoff of 1 at time  $T > 0$  and price  $P_t$ . If the deflated gains are a martingale, and if the risk-free rate is Markov, say  $dr_t = \mu(r_t)dt + \sigma(r_t) \cdot d\mathbf{z}_t$ ,

$$\begin{aligned}
P_t \exp \left( - \int_0^t r_s ds \right) &= E_t \left( \exp \left( - \int_0^t r_s ds \right) \right) \\
P_t &= E_t \left( \exp \left( - \int_t^T r_s ds \right) \right) \\
&:= p(r_t, t) \\
p(r_t, t) r_t &= p_r \mu(r_t) + \frac{1}{2} p_{rr} \sigma(r_t) \cdot \sigma(r_t) + p_t
\end{aligned}$$

## Arbitrage

- A self-financing portfolio  $\theta_t, 0 \leq t \leq T$  (trading strategy) is an arbitrage if for any  $T > 0$  either:
  1.  $W_0 \leq 0, W_t \geq 0$  with probability one, and  $W_t > 0$  with positive probability, or
  2.  $W_0 < 0$  and  $W_t \geq 0$  with probability one.
- A trading strategy is self-financing if  $\theta_t \cdot S_t = \theta_0 \cdot S_0 + \int_{v=0}^t \theta_v \cdot dS_v + \int_{v=0}^t \theta_v \cdot D_v dv$ , which is alternatively written  $d(\theta_t \cdot S_t) = \theta_t \cdot dS_t + \theta_t \cdot D_t dt$ .
- Numeraire Invariance Theorem: A trading strategy  $\theta$  is self-financing with respect to the gains process  $G$  iff it is self-financing with respect to the deflated process  $G^Y$  for any positive deflator.
- Corollary. A trading strategy is an arbitrage with respect to  $G$  iff it is an arbitrage with respect to  $G^Y$ .

## No Arbitrage

- Consider two spaces of trading strategies,  $\mathcal{H}^2(S^\pi)$  and  $\Theta(S^\pi)$ , where

$$\begin{aligned}
\mathcal{H}^2(S^\pi) &:= \begin{cases} \theta \text{ adapted;} \\ P(\int_0^\tau |\theta_t \cdot \mu_t \pi_t| dt < \infty) = 1; \\ E[(\int_0^\tau \theta_t \cdot \mu_t \pi_t dt)^2] < \infty; \\ P(\int_0^\tau (\theta_t \cdot \mu_t \pi_t)^2 dt < \infty) = 1; \\ E[(\int_0^\tau (\theta_t \cdot \mu_t \pi_t)^2 dt) < \infty, \\ \text{for } \tau > 0 \end{cases} \\
\Theta(S^\pi) &:= \begin{cases} \theta \text{ adapted;} \\ P(\theta_t \cdot S_t \pi_t \geq k) = 1, t > 0, \text{ for constant } k \end{cases}
\end{aligned}$$

- Proposition: Assume dividends are zero. If a sdf  $\pi$  exists such that  $E_t(S_s \pi_s) = S_t \pi_t, s > t$ , there is no arbitrage in either  $\mathcal{H}^2(S^\pi)$  or  $\Theta(S^\pi)$ .

## Doubling Strategy

- For a stock with driftless price  $S$  with  $dS_t = S_t dz_t$ , and bond with price  $\beta_t = 1$  (and zero risk-free rate), consider the trading strategy  $\theta_t(a_t, b_t)$  where

$$\begin{aligned}
a_t &= \begin{cases} \frac{1}{S_t \sqrt{T-t}}, & t \leq \tau \\ 0, & \tau < t \end{cases} \\
b_t &= -a_t S_t + \int_{v=0}^t a_v S_v \\
\tau &= \inf\{t : \int_{v=0}^t \frac{1}{\sqrt{T-v}} dz_v = \alpha > 0\}
\end{aligned}$$

- For this strategy,  $W_0 = b_0 + a_0 S_0 = 0$ , while  $W_\tau = b_\tau + a_\tau S_\tau = \int_{v=0}^\tau a_v dS_v = \alpha$ .
- Also,  $P(0 < \tau < T) = 1$ .

## Equivalent Martingale Measure

- We say the deflated price and dividend processes admit an equivalent martingale measure if there is a measure  $Q$  equivalent to  $P$  such that  $E_t^Q(G_T^Y) = G_t^Y$  for all dates  $T > t$ .
- Theorem: Assume dividends are zero. If the deflated gains process admits an equivalent martingale measures, there is no arbitrage in either  $\mathcal{H}^2(S^\pi)$  or  $\Theta(S^\pi)$ .

## SDF and Measure Q

- Assume zero dividends. The value  $\xi_t = \exp(\int_0^t r_s ds) \frac{\pi_t}{\pi_0}$  is the density of an equivalent martingale measure.
- We have  $E_t(\xi_T) = \xi_t$  and

$$S_t = E_t\left[\frac{\xi_T}{\xi_t} \exp(-\int_t^T r_s ds) S_T\right] = E_t^Q(\exp(-\int_t^T r_s ds) S_T)$$

- Alternatively, if  $\xi_t$  is the density for an equivalent martingale measure, meaning  $E_t(\xi_T) = \xi_t$  and  $S_t = E_t^Q(\exp(-\int_t^T r_s ds) S_T)$ ,  $\pi_t = \exp(-\int_0^t r_s ds) \xi_t$  is a stochastic discount factor and  $\pi_t S_t = E_t(\pi_T S_T)$ .

## Girsanov's Theorem

- Because  $\xi_t$  is adapted, positive almost surely, and is a martingale,  $d\xi_t = \gamma_t \cdot d\mathbf{z}_t = -\xi_t \eta_t \cdot d\mathbf{z}_t$ , where  $\eta_t := -\frac{\gamma_t}{\xi_t}$  for some (vector) adapted process  $\gamma_t$ , and  $\xi_t = \exp(-\int_0^t \eta_s \cdot d\mathbf{z}_s - \frac{1}{2} \int_0^t \eta_s \cdot \eta_s ds)$ .
- Also,  $\frac{d\pi_t}{\pi_t} = -r_t dt - \eta_t \cdot d\mathbf{z}_t$  where  $\pi_t = \exp(-\int_0^t r_s ds) \xi_t$ .
- Girsanov's Theorem: Under measure  $Q$ ,  $\mathbf{z}_t^Q = \mathbf{z}_t + \int_0^t \eta_s ds$  is standard Brownian motion.

## Market Prices of Risk

- Under  $P$ :  $dS_t/S_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{z}_t$ , and

$$\frac{1}{\exp(-\int_0^t r_s ds) S_t} d(\exp(-\int_0^t r_s ds) S_t) = (\boldsymbol{\mu}_t - r_t \mathbb{1}) dt + \boldsymbol{\sigma}_t (d\mathbf{z}_t + \eta_s^T dt - \eta_s^T dt) = (\boldsymbol{\mu}_t - r_t - \boldsymbol{\sigma}_t \eta_s^T) dt + \boldsymbol{\sigma}_t d\mathbf{z}_t^Q$$

- Under  $Q$ :  $dS_t/S_t = \boldsymbol{\mu}_t^Q dt + \boldsymbol{\sigma}_t d\mathbf{z}_t^Q$ , where  $\boldsymbol{\mu}_t^Q = \boldsymbol{\mu}_t - \boldsymbol{\sigma}_t \eta_s^T = r_t \mathbb{1}$  and

$$\frac{1}{\exp(-\int_0^t r_s ds) S_t} d(\exp(-\int_0^t r_s ds) S_t) = \boldsymbol{\sigma}_t d\mathbf{z}_t^Q$$

## The Risk-Free Rate is a State Variable

- Suppose a representative investor with time-additive logarithmic utility lives in an economy with one productive asset.

1. Marginal utility is  $u'(c_t, t) = \exp(-\beta t)c_t^{-1} = \exp(-\beta t - \ln c_t)$ .
2. Wealth in units of consumptions follows  $dW_t = W_t x_t dt + W_t \sigma_w \sqrt{x_t} dz_{1t} - c_t dt$ .
3. The productivity rate  $x$  follows  $dx_t = \kappa(\theta_x - x_t)dt + \sigma_x \sqrt{x_t}(\rho dz_{1t} + \sqrt{1 - \rho^2} dz_{2t})$ .

- In the equilibrium of CIR, consumption is  $c_t = \beta W_t$  and it follows the subordinated process:

$$d \ln c_t = -\beta dt + (1 - \frac{1}{2}\sigma_w^2)x_t dt + \sigma_w \sqrt{x_t} dz_{1t}$$

- As a consequence,

$$\begin{aligned} \frac{d\pi_t}{\pi_t} &= \beta dt - d \ln c_t + \frac{1}{2}(d \ln c_t)^2 \\ &= -(1 - \sigma_w^2)x_t dt - \sigma_w \sqrt{x_t} dz_{1t} \\ &:= -\delta x_t dt - \eta_t dz_{1t} \\ r_t &= -E_t \frac{d \exp(-\beta t - \ln c_t)}{\exp(-\beta t - \ln c_t) dt} \\ &= \delta x_t, \\ dr_t &= \kappa(\theta - r_t)dt + \sigma \sqrt{r_t}(\rho dz_{1t} + \sqrt{1 - \rho^2} dz_{2t}) \\ &= \kappa(\theta - r_t)dt - \sigma \sqrt{r_t} \rho \eta_t dt + \sigma \sqrt{r_t}(\rho(dz_{1t} + \eta_t dt) + \sqrt{1 - \rho^2} dz_{2t}) \\ &= \kappa(\theta - r_t)dt - \lambda r_t dt + \sigma \sqrt{r_t} dz^Q \end{aligned}$$

where  $\theta = \delta \theta_x$  and  $\sigma = \sqrt{\delta} \sigma_x$ .

## Cox, Ingersoll, and Ross

- Consider a short rate that follows the process:

$$dr_t = \kappa(\theta - r_t)dt + \sigma \sqrt{r_t} dz,$$

where  $z$  is scalar Brownian motion under the statistical measure  $P$ .

- Suppose also that under the risk-neutral measure  $Q$ , the short rate follows

$$dr_t = \kappa(\theta - r_t)dt + \sigma \sqrt{r_t} dz^Q - \lambda r_t dt,$$

where  $\lambda$  is a constant and  $z^Q$  is scalar Brownian motion. Then the price of a discount bond maturing at date  $T$  is:

$$\begin{aligned} P(t, T) &= \exp(-YTM(t, T)(T - t)) = E_t^Q(\exp(-\int_t^T r_s ds)) = A(t, T) \exp(-B(t, T)r_t) \\ \text{where } A(t, T) &\equiv \left[ \frac{2\gamma \exp([\kappa + \lambda + \gamma](T - t)/2)}{(\gamma + \kappa + \lambda)(\exp(\gamma(T - t)) - 1) + 2\gamma} \right]^{2\kappa\theta/\sigma^2} \\ B(t, T) &\equiv \frac{2(\exp(\gamma(T - t)) - 1)}{(\gamma + \kappa + \lambda)(\exp(\gamma(T - t)) - 1) + 2\gamma} \\ \gamma &\equiv ((\kappa + \lambda)^2 + 2\sigma^2)^{1/2} \end{aligned}$$

## Solution

- Three alternative methods of solution:
  1. Calculate the expectation
  2. Discretize the Ito process for  $r_t$ , and use Monte Carlo to approximate the expectation
  3. Solve a p.d.e.
- Suppose the bond price is a function  $P(t, T) = p(r_t, t)$  that satisfies the conditions required for Ito's lemma.

Then  $p(r_t, t) = E_t^Q(\exp(-\int_t^T r_s ds))$ , with  $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz^Q - \lambda r_t dt$ .

By Feynman-Kac,

$$0 = p_r(\kappa\theta - (\kappa + \lambda)r) + p_{rr}\frac{1}{2}\sigma^2 r + p_t - pr$$

with  $p(r_T, T) = 1$  for all  $r_T$ .

- Discretize:  $r_{i,j}\Delta = \kappa(\theta - r_{i,(j-1)\Delta})\Delta - \lambda r_{i,(j-1)\Delta}\Delta + \sigma\sqrt{r_{i,(j-1)\Delta}}\sqrt{\Delta}z_{i,j}$ ,  $r_{i,0} = r_t$ , where  $j = 1, \dots, n$  with  $\Delta = (T - t)/n$ ;  $i = 1, \dots, m$ , and  $z_{i,j} \sim N(0, 1)$  for all  $i, j$ .
- Approximate:  $E_t^Q(\exp(-\int_t^T r_s ds)) \approx \frac{1}{m} \sum_{i=1}^m \exp(-\frac{1}{2} \sum_{j=1}^n (r_{i,j}\Delta + r_{i,(j-1)\Delta})\Delta)$