ECON 709 - PS 3

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1. A random point (X, Y) is distributed uniformly on the square with vertices $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$. That is, the joint PDF is $f(x, y) = 1/4$ on the square and $f(x, y) = 0$ outside the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$

$X^2 + Y^2 < 1 \implies -\sqrt{1 - X^2} < Y < \sqrt{1 - X^2}$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4} dy dx = \int_{-1}^{1} \left[\frac{1}{4} y \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} dx$$

Define $x = \sin\theta \implies dx = \cos\theta d\theta$.

$$\begin{aligned}
\frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} dx &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sqrt{1 - (\sin\theta)^2} \cos\theta d\theta \\
&= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta \\
&= \frac{1}{4} \int_{-\pi/2}^{\pi/2} 1 + \cos(2\theta) d\theta \\
&= \frac{1}{4} \left[\theta + \frac{\sin(2\theta)}{2} \right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{4} \left[(\pi/2) + \frac{0}{2} - (-\pi/2) - \frac{0}{2} \right] \\
&= \frac{\pi}{4}
\end{aligned}$$

(b) $|X + Y| < 2$

$|X + Y| < 2 \implies -2 < X + Y < 2 \implies -2 - X < Y < 2 - X$. Since X ranges from -1 to 1, $-2 - X < Y < 2 - X \implies -1 < Y < 1$

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} dy dx = \frac{1}{4} \int_{-1}^{1} [y]_{-1}^{1} dx = \frac{1}{2} \int_{-1}^{1} dx = \frac{1}{2} [x]_{-1}^{1} = 1$$

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2. Let the joint PDF of X and Y be given by $f(x,y) = g(x)h(y)\ \forall x, y \in \mathbb{R}$ for some functions $g(x)$ and $h(y)$. Let a denote $\int_{-\infty}^{\infty} g(x)dx$ and b denote $\int_{-\infty}^{\infty} h(x)dx$

(a) What conditions a and b should satisfy in order for $f(x,y)$ to be a bivariate PDF?

For $f(x,y)$ to be a PDF, it should integrate to one:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= 1 \\
\implies \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)dxdy &= 1 \\
\implies \int_{-\infty}^{\infty} g(x)dx \int_{-\infty}^{\infty} h(y)dy &= 1 \\
\implies ab &= 1 \\
\implies a &= b^{-1}
\end{aligned}$$

(b) Find the marginal PDF of X and Y.

The marginal PDF of X:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} g(x)h(y)dy = g(x)\int_{-\infty}^{\infty} h(y)dy = b \cdot g(x)$$

The marginal PDF of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} g(x)h(y)dx = h(y)\int_{-\infty}^{\infty} g(x)dx = a \cdot h(y)$$

(c) Show that X and Y are independent.

Proof: X and Y are independent if the product of their marginal distributions is their joint distribution:

$$\begin{aligned}
f_X(x) \cdot f_Y(y) &= b \cdot g(x) \cdot a \cdot h(y) \\
&= b \cdot g(x) \cdot b^{-1} \cdot h(y) \\
&= g(x) \cdot h(y) \\
&= f(x,y)
\end{aligned}$$

□

3. Let the joint PDF of X and Y be given by

$$f(x,y) = \begin{cases} cxy & \text{if } x,y \in [0,1], x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c such that $f(x,y)$ is a joint PDF.

$$\begin{aligned} \int_0^1 \int_0^{1-x} f(x,y)dydx &= 1 \\ \implies \int_0^1 \int_0^{1-x} cxydydx &= 1 \\ \implies c\int_0^1 \left[\frac{xy^2}{2}\right]_{y=0}^{1-x} dx &= 1 \\ \implies \frac{c}{2}\int_0^1 x(1-x)^2 dx &= 1 \\ \implies \frac{c}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{x=0}^{1} &= 1 \\ \implies \frac{c}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] &= 1 \\ \implies c &= 24 \end{aligned}$$

(b) Find the marginal distributions of X and Y.

$$f_X(x) = \int_0^{1-x} f(x,y)dy = \int_0^{1-x} 24xydy = \left[12xy^2\right]_{y=0}^{1-x} = \begin{cases} 12x(1-x)^2, x \in [0,1] \\ 0, \text{ otherwise.} \end{cases}$$

$$f_Y(y) = \int_0^{1-y} f(x,y)dx = \int_0^{1-y} 24xydx = \left[12x^2y\right]_{x=0}^{1-y} = \begin{cases} 12(1-y)^2y, y \in [0,1] \\ 0, \text{ otherwise.} \end{cases}$$

(c) Are X and Y independent? Compare your answer to Problem 2 and discuss.

X and Y independent if the product of the marginal distributions equals their joint distribution at all points in the support. If $x = y = 0.9$, $f(0.9, 0.9) = 0$ because $(0.9, 0.9)$ is not in the support, $x+y = 0.9+0.9 = 1.8 > 1$. But each marginal distribution is define over $[0,1]$, so the product of the marginals is positive at $(0.9, 0.9)$: $f_X(0.9)f_Y(0.9) = [12(0.9)(1-(0.9))^2][12(1-0.9)^2(0.9)] = 0.0117$.

In (2), the support for the joint distribution is $\mathbb{R}^2$, whereas the support for the joint distribution depends on the realization of the random variable.

4. Show that any random variable is uncorrelated with a constant.

Proof: Let $a \in \mathbb{R}$ and X be a random variable with distribution F_X. Define random variable Y as the degenerate random variable that equals a. Thus, the distribution Y is

$$F_Y(y) = \begin{cases} 0, & y < a \\ 1, & y \geq a \end{cases}$$

To show X is uncorrelated with a constant, I show that X and Y are independent and then, by a theorem in the Lecture 3 Notes, we know that X and Y are uncorrelated.

To find the joint distribution of X and Y, consider two cases: $y < a$ and $y \geq a$. For $y < a$,

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x \text{ and } Y \leq a) \\ &= 0 \end{aligned}$$

For $y \geq a$:

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x) \\ &= F_X(x) \end{aligned}$$

Thus, the joint distribution is

$$F(x, y) = \begin{cases} 0, & y < a \\ F_X(x), & y \geq a \end{cases}$$

The joint distribution equals the product of the marginals:

$$F(x, y) = \begin{cases} 0 * F_X(x), & y < a \\ 1 * F_X(x), & y \geq a \end{cases} = \begin{cases} F_Y(y) * F_X(x), & y < a \\ F_Y(y) * F_X(x), & y \geq a \end{cases}.$$

□

5. Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for the correlation of XY and Y in terms of these means and variances.

Note that $Var(X) = E(X^2) - E(X)^2 \implies E(X^2) = Var(X) + E(X)^2$.

$$\begin{aligned}
Corr(XY, Y) &= \frac{Cov(XY, Y)}{\sqrt{Var(XY)Var(Y)}} \\
&= \frac{E(XY^2) - E(XY)E(Y)}{\sigma_Y \sqrt{E((XY)^2) - E(XY)^2}} \\
&= \frac{E(X)E(Y^2) - E(X)E(Y)E(Y)}{\sigma_Y \sqrt{E(X^2)E(Y^2) - (E(X)E(Y))^2}} \\
&= \frac{\mu_X (Var(Y) + E(Y)^2) - \mu_X \mu_Y^2}{\sigma_Y \sqrt{(Var(X) + E(X)^2)(Var(Y) + E(Y)^2) - (\mu_X \mu_Y)^2}} \\
&= \frac{\mu_X \sigma_Y^2 + \mu_X \mu_Y^2 - \mu_X \mu_Y^2}{\sigma_Y \sqrt{\sigma_Y^2 \sigma_X^2 + \mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2 + \mu_Y^2 \mu_X^2 - \mu_X^2 \mu_Y^2}} \\
&= \frac{\mu_X \sigma_Y}{\sqrt{\sigma_Y^2 \sigma_X^2 + \mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2}}
\end{aligned}$$

6. Prove the following: For any random vector $(X_1, X_2, ..., X_n)$,

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \le i < j \le n} Cov(X_i, X_j).$$

Proof (by induction): For $n = 2$, $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$ from lecture 3 notes. Assume that the formula holds for some n, then

$$\begin{aligned}
&Var\left(\sum_{i=1}^{n+1} X_i\right)\\
&= Var\left(\sum_{i=1}^{n} X_i + X_{n+1}\right)\\
&= Var\left(\sum_{i=1}^{n} X_i\right) + Var(X_{n+1}) + 2Cov\left(\sum_{i=1}^{n} X_i, X_{n+1}\right)\\
&= \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \le i < j \le n} Cov(X_i, X_j) + Var(X_{n+1}) + 2\left[E\left(X_{n+1}\sum_{i=1}^{n} X_i\right) + E\left(X_{n+1}\right)E\left(\sum_{i=1}^{n} X_i\right)\right]\\
&= \sum_{i=1}^{n+1} Var(X_i) + 2 \sum_{1 \le i < j \le n} Cov(X_i, X_j) + 2\left[\sum_{i=1}^{n} E\left(X_{n+1}X_i\right) + E\left(X_{n+1}\right)\sum_{i=1}^{n} E\left(X_i\right)\right]\\
&= \sum_{i=1}^{n+1} Var(X_i) + 2 \sum_{1 \le i < j \le n} Cov(X_i, X_j) + 2\left[\sum_{i=1}^{n} E\left(X_{n+1}X_i\right) + E\left(X_{n+1}\right)E\left(X_i\right)\right]\\
&= \sum_{i=1}^{n+1} Var(X_i) + 2 \sum_{1 \le i < j \le n} Cov(X_i, X_j) + 2\sum_{i=1}^{n} Cov\left(X_{n+1}X_i\right)\\
&= \sum_{i=1}^{n+1} Var(X_i) + 2 \sum_{1 \le i < j \le n+1} Cov(X_i, X_j)
\end{aligned}$$

□

7. Suppose that X and Y are joint normal, i.e. they have the joint PDF:

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp(-(2(1-\rho^2))^{-1}(x^2/\sigma_X^2 - 2\rho xy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))$$

(a) Derive the marginal distributions of X and Y, and observe that both normal distributions.

The marginal distribution of X is

$$\begin{aligned}
&f_X(x) \\
&= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2} + \frac{\rho^2x^2}{\sigma_X^2} - \frac{\rho^2x^2}{\sigma_X^2}\right)\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{\rho^2x^2}{\sigma_X^2}\right) - \frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{\rho^2x^2}{\sigma_X^2}\right)\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy(\sigma_Y/\sigma_X)}{\sigma_Y^2} + \frac{\rho^2x^2(\sigma_Y^2/\sigma_X^2)}{\sigma_Y^2}\right) - \frac{(1-\rho^2)}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}\right)\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{(y-\rho x(\sigma_Y/\sigma_X))^2}{2(1-\rho^2)\sigma_Y^2} - \frac{x^2}{2\sigma_X^2}\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{(y-\rho x(\sigma_Y/\sigma_X))^2}{2(1-\rho^2)\sigma_Y^2}\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)
\end{aligned}$$

Thus, $X \sim Normal(0, \sigma_X^2)$. By swapping x and y in the algebra above, we see that $Y \sim Normal(0, \sigma_Y^2)$.

(b) Derive the conditional distribution of Y given $X = x$. Observe that it is also a normal distribution.

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right)\left[\frac{1}{\sigma_X\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)\right]^{-1} \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right) + \frac{x^2}{2\sigma_X^2}\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2} - \frac{x^2(1-\rho^2)}{\sigma_X^2}\right)\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{x^2\rho^2}{\sigma_X^2}\right)\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y}{\sigma_Y} - \frac{x\rho}{\sigma_X}\right)^2\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(y - \frac{\sigma_Y}{\sigma_X}\rho x)^2}{(1-\rho^2)\sigma_Y^2}\right)
\end{aligned}
$$

Observe that $Y|X \sim Normal(\frac{\sigma_Y}{\sigma_X}\rho x, \sigma_Y(1-\rho^2))$.

(c) Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X and Z are independent.

Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(x, y) = (x, (y/\sigma_Y) - (\rho x)/\sigma_X)$. Notice that g is one-to-one, so it is invertible. Define $h = g^{-1}$ such that $h(x, z) = (x, \sigma_Y(z+\rho x/\sigma_X))$. The determinant of the Jacobian of the tranformation is

$$
\begin{aligned}
|J| &= \begin{vmatrix} \frac{\partial h_1(x,z)}{\partial x} & \frac{\partial h_1(x,z)}{\partial z} \\ \frac{\partial h_2(x,z)}{\partial x} & \frac{\partial h_2(x,z)}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ \frac{\rho\sigma_Y}{\sigma_X} & \sigma_Y \end{vmatrix} \\
&= \sigma_Y
\end{aligned}
$$

Thus, from lecture 3 notes, we know that the joint distribution

$$
\begin{aligned}
&f_{X,Z}(x, z) \\
&= f_{X,Y}(h(x, z))|J| \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_X^2} - \frac{2\rho x(\sigma_Y(z+\rho x/\sigma_X))}{\sigma_X\sigma_Y} + \frac{(\sigma_Y(z+\rho x/\sigma_X))^2}{\sigma_Y^2} \right)\right)\sigma_Y \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_X^2} - \frac{2\rho x(z+\rho x/\sigma_X)}{\sigma_X} + (z+\rho x/\sigma_X)^2 \right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xz}{\sigma_X} - \frac{2\rho^2 x^2}{\sigma_X^2} + z^2 + \frac{2z\rho x}{\sigma_X} + \frac{\rho^2 x^2}{\sigma_X^2} \right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_X^2} - \frac{\rho^2 x^2}{\sigma_X^2} + z^2 \right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{(1-\rho^2)x^2}{\sigma_X^2} + z^2 \right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \left(\frac{x^2}{\sigma_X^2} + \frac{z^2}{(1-\rho^2)} \right)\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2} \left(\frac{x^2}{\sigma_X^2} \right)\right) \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \left(\frac{z^2}{(1-\rho^2)} \right)\right) \\
&= f_X(x) \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \left(\frac{z^2}{(1-\rho^2)} \right)\right)
\end{aligned}
$$

Thus, X and Z are independent with $Z \sim Normal(0, 1-\rho^2)$.

8. Consider a function $g : \mathbb{R} \to \mathbb{R}$. Recall that the inverse image of a set A, denoted $g^{-1}(A)$ is $g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$. Let there be functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$. Let X and Y be two random variables that are independent. Suppose that g_1 and g_2 are both Borel-measurable, which means that $g_1^{-1}(A)$ and $g_2^{-1}(A)$ are both in the Borel σ-field whenever A is in the Borel σ-field. Show that the two random variables $Z := g_1(X)$ and $W := g_2(Y)$ are independent. (Hint: use the 1st or the 2nd definition of independence.)

Proof: Let A, B be events in the Borel σ-field. Then,

$$P(Z \in A, W \in B) = P(g_1(X) \in A, g_2(Y) \in B) = P(X \in g_1^{-1}(A), Y \in g_2^{-1}(B))$$

Since g_1, g_2 are Borel measurable, $g_1^{-1}(A), g_2^{-1}(B)$ are events in the Borel σ-field. Thus, by the independence of X and Y:

$$P(X \in g_1^{-1}(A), Y \in g_2^{-1}(B)) = P(X \in g_1^{-1}(A))P(Y \in g_2^{-1}(B)) = P(g_1(X) \in A)P(g_2(Y) \in B) = P(Z \in A)P(W \in B)$$

Thus, Z and W are independent. $\square$