ECON 709 - PS 3

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1. A random point (X, Y) is distributed uniformly on the square with vertices $(1,1), (1,-1), (-1,1)$, and $(-1,-1)$. That is, the joint PDF is $f(x,y) = 1/4$ on the square and $f(x,y) = 0$ outside the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$

(b) $|X + Y| < 2$

2. Let the joint PDF of X and Y be given by $f(x,y) = g(x)h(y)\ \forall x, y \in \mathbb{R}$ for some functions $g(x)$ and $h(y)$. Let a denote $\int_{-\infty}^{\infty} g(x)dx$ and b denote $\int_{-\infty}^{\infty} h(x)dx$

(a) What conditions a and b should satisfy in order for $f(x,y)$ to be a bivariate PDF?

(b) Find the marginal PDF of X and Y.

(c) Show that X and Y are independent.

3. Let the joint PDF of X and Y be given by

$$f(x,y) = \begin{cases} cxy \text{ if } x, y \in [0,1], x + y \leq 1 \\ 0 \text{ otherwise} \end{cases}$$

(a) Find the value of c such that $f(x,y)$ is a joint PDF.

(b) Find the marginal distributions of X and Y.

(c) Are X and Y independent? Compare your answer to Problem 2 and discuss.

4. Show that any random variable is uncorrelated with a constant.

5. Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for the correlation of XY and Y in terms of these means and variances.

6. Prove the following: For any random vector $(X_1, X_2, ..., X_n)$,

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \leq i < j \leq n} Cov(X_i, Y_i).$$

7. Suppose that X and Y are joint normal, i.e. they have the joint PDF:

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp(-(2(1-\rho^2))^{-1}(x^2/\sigma_X - 2xy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))$$

(a) Derive the marginal distributions of X and Y, and observe that both normal distributions.

(b) Derive the conditional distribution of Y given $X = x$. Observe that it is also a nromal distribution.

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(c) Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X and Z are independent.

8. Consider a function $g : \mathbb{R} \to \mathbb{R}$. Recall that the inverse image of a set A, denoted $g^{-1}(A)$ is $g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$. Let there be functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$. Let X and Y be two random variables that are independent. Suppose that g_1 and g_2 are both Borel-measurable, which means that $g_1^{-1}(A)$ and $g_2^{-1}(A)$ are both in the Borel σ-field whenever A is in the Borel σ-field. Show that the two random variables $Z := g_1(X)$ and $W := g_2(Y)$ are independent. (Hint: use the 1st or the 2nd definition of independence.)