FIN 970: Final Exam

Alex von Hafften

April 25, 2022

1 Problem 1a

1. Using SDF approach:

   Conjecture $P_t^n = \exp(A_n + B_n' X_t)$. Proof by induction.

   For $n = 0$,

$$
\begin{aligned}
P_t^1 &= E_t[M_{t+1} \cdot 1] \\
\implies \exp(A_1 + B_1' X_t) &= E_t\Bigg[\exp\Bigg(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\Bigg)\Bigg] \\
\implies E_t\Bigg[-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\Bigg] &= -\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t \\
Var_t\Bigg[-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\Bigg] &= \lambda_t'\lambda_t \\
\implies \exp(A_1 + B_1' X_t) &= \exp\Bigg(-\delta_0 - \delta_1 X_t\Bigg) \\
\implies &\begin{cases} A_1 = -\delta_0 \\ B_1 = -\delta_1' \end{cases}
\end{aligned}
$$

   For some n, the Euler equation holds:

$$
\begin{aligned}
P_t^n &= E_t[M_{t+1} P_{t+1}^{n-1}] \\
\exp(A_n + B_n' X_t) &= E_t\Bigg[\exp\Bigg(-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\Bigg)\exp(A_{n-1} + B_{n-1}' X_{t+1})\Bigg] \\
&= E_t\Bigg[\exp\Bigg(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1} + A_{n-1} + B_{n-1}'(\mu + \Phi X_t + \Sigma\varepsilon_{t+1})\Bigg)\Bigg] \\
&= E_t\Bigg[\exp\Bigg(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t + A_{n-1} + B_{n-1}'\mu + B_{n-1}'\Phi X_t + [B_{n-1}'\Sigma - \lambda_t']\varepsilon_{t+1}\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&E_t\Bigg[-\delta_0-\delta_1X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t+[B_{n-1}'\Sigma-\lambda_t']\varepsilon_{t+1}\Bigg]\\
&=-\delta_0-\delta_1X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t\\
&Var_t\Bigg[-\delta_0-\delta_1X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t+[B_{n-1}'\Sigma-\lambda_t']\varepsilon_{t+1}\Bigg]\\
&=[B_{n-1}'\Sigma-\lambda_t'][B_{n-1}'\Sigma-\lambda_t']'\\
&=B_{n-1}'\Sigma\Sigma'B_{n-1}+\lambda_t'\lambda_t-2B_{n-1}'\Sigma\lambda_t
\end{aligned}
$$

$$
\begin{aligned}
\exp(A_n+B_n'X_t)=\exp\Bigg(&-\delta_0-\delta_1X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t\\
&+\frac{1}{2}B_{n-1}'\Sigma\Sigma'B_{n-1}+\frac{1}{2}\lambda_t'\lambda_t-B_n'\Sigma(\lambda_0+\lambda_1X_t)\Bigg)\\
&=\exp\Bigg(-\delta_0+A_{n-1}+B_{n-1}'\mu-B_{n-1}'\Sigma\lambda_0+\frac{1}{2}B_{n-1}'\Sigma\Sigma'B_{n-1}+(-\delta_1+B_{n-1}'\Phi-B_{n-1}'\Sigma\lambda_1)X_t)\Bigg)\\
&\Longrightarrow\begin{cases}A_n=-\delta_0+A_{n-1}+B_{n-1}'(\mu-\Sigma\lambda_0)+\frac{1}{2}B_{n-1}'\Sigma\Sigma'B_{n-1}\\ B_n=-\delta_1+(\Phi-\Sigma\lambda_1)'B_{n-1}\end{cases}
\end{aligned}
$$