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Problem 1

(1) The HH problem is

$$\max_{\{c_t, k_t, n_{1t}, n_{2t}, B_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_{1t}-n_{2t})$$

$$\text{s.t.} \quad c_t + k_{t+1} + B_t \leq (1-\tau_{1t}^n) w_{1t} n_{1t} + (1-\tau_{2t}^n) w_{2t} n_{2t} + (1-\tau_t^k) r_t k_t + k_t + R_t^B B_{t-1} \quad (1)$$

The firm problem is

$$\max_{\{n_{1t}, n_{2t}, k_t\}} F(k_t, n_{1t}, n_{2t}) - r_t k_t - w_1 n_{1t} - w_2 n_{2t} \quad \forall t \ (2)$$

The Government budget constraint is

$$g_t + R_t^B B_{t-1} = \tau_t^k r_t k_t + \tau_{1t}^n w_{1t} n_{1t} + \tau_{2t}^n w_{2t} n_{2t} + B_t \quad \forall t \ (3)$$

Problem 1 con't

(1) con't

A CE is an allocation $\{(C_t, K_t, n_{1t}, n_{2t})\}_{t=0}^{\infty}$ and a price System $\{(r_t, w_{1t}, w_{2t}, R_t^B)\}_{t=0}^{\infty}$ and a Government policy $\{(\tau_{1t}^n, \tau_{2t}^n, \tau_t^K, B_t)\}_{t=0}^{\infty}$ such

(1) HHs solve their problem (1)

(2) Firms solve their problem (2)

(3) Government budget constraint is satisfied (3)

(4) Markets clear

From the firm problem and market clearing, we get

$$r_t = F_1(K_t, n_{1t}, n_{2t})$$

$$w_{1t} = F_2(K_t, n_{1t}, n_{2t})$$

$$w_{2t} = F_3(K_t, n_{1t}, n_{2t})$$

if firms are perfectly competitive. Also if production is CRS, $\pi_t = 0$.

Problem 1

(2) The lagrangian for HH problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big[u(c_t, 1-n_{1t}-n_{2t})$$

$$+ \lambda_t \Big[(1-\tau^n_{1t}) w_{1t} n_{1t} + (1-\tau^n_{2t}) w_{2t} n_{2t}$$

$$+ (1-\tau^k_t) r_t K_t + K_t + R^B_t B_{t-1}$$

$$- c_t - K_{t+1} - B_t \Big]\Big]$$

FOC:

$[c_t]$: $u_1(c_t, 1-n_{1t}-n_{2t}) = \lambda_t$

$[n_{1t}]$: $u_2(c_t, 1-n_{1t}-n_{2t}) = \lambda_t (1-\tau^n_{1t}) w_{1t}$

$[n_{2t}]$: $u_2(c_t, 1-n_{1t}-n_{2t}) = \lambda_t (1-\tau^n_{2t}) w_{2t}$

$[K_t]$: $\beta \lambda_t [(1-\tau^k_t) r_t + 1] = \lambda_{t-1}$

$[B_t]$: $\lambda_t = \beta \lambda_{t+1} R^B_{t+1}$

Problem 1 con't

(2) con't

The HH BC multiplied by λ_t and summed across t

$$\sum_{t=0}^{\infty} \lambda_t \left[C_t + K_{t+1} + B_t \right] = \sum_{t=}^{\infty} \lambda_t \Big[(1-\tau_{1t}^n) w_{1t} n_{1t}$$
$$+ (1-\tau_{2t}^n) w_{2t} n_{2t}$$
$$+ (1-\tau_t^k) r_t K_t$$
$$+ K_t$$
$$+ R_t^B B_{t-1} \Big]$$

Substituting in HH FOCs, we get the IC constraint

$$\sum_{t=0}^{\infty} \beta^t \Big[U_1(C_t, 1-n_{1t}-n_{2t}) C_t$$
$$- U_2(C_t, 1-n_{1t}-n_{2t}) (1-n_{1t}-n_{2t}) \Big] \qquad (4)$$
$$= U_1(C_t, 1-n_{1t}, n_{2t}) \Big[[(1-\tau_0^k) r_0 + 1] K_{-1}$$
$$+ R_0^B B_{-1} \Big]$$

The RC constraint is

$$C_t + K_{t+1} + g_t \leq F(K_t, n_{1t}, n_{2t}) + K_t \qquad (5)$$

Problem 1

(2) cont

The Ramsay Problem is

$$\max \sum \beta^t u(c_t, 1 - n_{1t} - n_{2t})$$

st IC holds (4)

RC holds (5)

Define $w(c_t, l_t, \mu) = u(c_t, l_t)$

$$+ \mu \left[u_1(c_t, l_t) c_t + u_2(c_t, l_t) l_t \right]$$

$\Rightarrow$ The Ramsay problem is

$$\max \sum \beta^t w(c_t, 1 - n_{1t} - n_{2t}, \mu)$$

st RC holds (5)

Assume that τ_0^k is bounded.

Problem 1 con't

(3) FOCs of RP:

$$FOC\ [n_{1t}]:\ -w_2(c_t, 1-n_{1t}-n_{2t}) = \gamma_t F_2(K_t, n_{1t}, n_{2t})$$

$$FOC\ [n_{2t}]:\ w_2(c_t, 1-n_{1t}-n_{2t}) = \gamma_t F_3(K_t, n_{1t}, n_{2t})$$

Where γ_t is the multiplier on the resource constraint.

These FOCs suggest:

$$F_2(K_t, n_{1t}, n_{2t}) = F_3(K_t, n_{1t}, n_{2t})$$

In words, the optimal τ^n_{1t} and τ^n_{2t} should equalize the marginal product of the two labor goods.

The problem is very similar to the problem we discussed in class, so the takeaways about no capital taxes in SS hold.

$\boxed{\text{Problem 1}}$ cont

(4) The RP is

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_{1t}, l_{2t})$$

s.t.

$$c_t + K_{t+1} + g_t \leq F(K_t, 1-l_{1t}, 1-l_{2t}) + K_t \quad \text{(RC)}$$

$$\sum \beta^t \big[u_1(c_t, l_{1t}, l_{2t})\, c_t + u_2(c_t, l_{1t}, l_{2t})\, l_{1t} + u_3(c_t, l_{1t}, l_{2t})\, l_3 \big] = \quad \text{(IC)}$$

$$u_1(c_0, l_{10}, l_{20}) \left[R_0^b B_{-1} + (1+\tau_0) K_{-1} \right]$$

$$\frac{u_2(c_t, l_{1t}, l_{2t})}{F_2(K_t, l_{1t}, l_{2t})} = \frac{u_3(c_t, l_{1t}, l_{2t})}{F_3(K_t, l_{1t}, l_{2t})}$$

(constraint based on $\tau_{1t}^n = \tau_{2t}^n$)

FOC of HH problem w.r.t. l_{1t}

$$\tau_{1t}^n = \tau_{2t}^n = \tau_t^n \Rightarrow \frac{u_2(c_t, l_{1t}, l_{2t})}{F_2(K_t, l_{1t}, l_{2t})} = \lambda_t (1-\tau_t^n)$$

Question 2

1) The problem of agent θ is

$$\max_{c,y} \; u(c) - v\left(\frac{y}{\theta}\right)$$

s.t. $c \leq y$ ← Holds w/ equality

$$\Rightarrow \quad u(y) - v\left(\frac{y}{\theta}\right)$$

$$\text{FOC}[y]: \; u'(y) = \frac{1}{\theta} v'\left(\frac{y}{\theta}\right)$$

$$\Rightarrow \begin{cases} u'(y_H) = \dfrac{1}{\theta_H} v'\left(\dfrac{y_H}{\theta_H}\right) \\ u'(y_L) = \dfrac{1}{\theta_L} v'\left(\dfrac{y_L}{\theta_L}\right) \end{cases}$$

$$\theta_H > \theta_L \Rightarrow \frac{1}{\theta_H} v'\left(\frac{y_H}{\theta_H}\right) < \frac{1}{\theta_L} v'\left(\frac{y_L}{\theta_L}\right)$$

$$\Rightarrow \quad u'(y_H) < u'(y_L)$$

$$\Rightarrow \quad y_H > y_L$$

$$\Rightarrow \quad c_H > c_L$$

The high type produces and consumes more. Moreover, there is no trade because any trade would make the high type worse off, so they would not accept any trades.

(2) The problem facing an agent is [w/ π probability of being high type]

$$\max_{\{C_H, C_L, y_H, y_L\}} \pi\left[u(C_H) - v\left(\frac{y_H}{\theta_H}\right)\right] + (1-\pi)\left[u(C_L) - v\left(\frac{y_L}{\theta_L}\right)\right]$$

s.t.
$$C_H + A = y_H$$
$$C_L - A = y_L$$

Where A is the transfer between high and low productivity agents. Multiply BC_H by π and BC_L by $1-\pi$ and add λ_H and λ_L as Lagrangian multipliers.

FOC
$$[C_H]: \quad \pi u'(C_H) = \lambda_H \pi$$
$$[C_L]: \quad (1-\pi) u'(C_L) = \lambda_L (1-\pi)$$
$$[A]: \quad \lambda_H = \lambda_L$$
$$[y_H]: \quad \pi v\left(\frac{y_H}{\theta_H}\right)\frac{1}{\theta_H} = \lambda_H \pi$$
$$[y_L]: \quad (1-\pi) v\left(\frac{y_L}{\theta_L}\right)\frac{1}{\theta_L} = \lambda_L (1-\pi)$$

$$\Rightarrow u'(C_H) = u'(C_L) \Rightarrow C_H = C_L$$

$$\Rightarrow v'\left(\frac{y_H}{\theta_H}\right)\frac{1}{\theta_H} = v'\left(\frac{y_L}{\theta_L}\right)\frac{1}{\theta_L}$$

$$\Rightarrow \frac{v'\left(\frac{y_H}{\theta_H}\right)}{v'\left(\frac{y_L}{\theta_L}\right)} = \frac{\theta_H}{\theta_L} < 1$$

$$\Rightarrow v'\left(\frac{y_H}{\theta_H}\right) < v'\left(\frac{y_L}{\theta_L}\right)$$

$$\Rightarrow y_H > y_L$$

(2) con't

- Consumption will be equal across θ_H & θ_L.
- High type will work more.

Welfare

- Autarky (found in (1)) is possible, so ex ante welfare must be higher in this outcome.
- Ex post, low type is better off.
- Ex post, high type is worse off.

(3) (a) The contracting problem for the planner is

$$\max \ \pi\left[u(c_H) - v\left(\frac{y_H}{\theta_H}\right)\right] + (1-\pi)\left[u(c_L) - v\left(\frac{y_L}{\theta_L}\right)\right]$$

$$\text{s.t. } \pi c_H + (1-\pi) c_L \leq \pi y_H + (1-\pi) y_L \qquad (RC)$$

$$u(c_H) - v\left(\frac{y_H}{\theta_H}\right) \geq u(c_L) - v\left(\frac{y_L}{\theta_H}\right) \qquad (IC_H)$$

$$u(c_L) - v\left(\frac{y_L}{\theta_L}\right) \geq u(c_H) - v\left(\frac{y_H}{\theta_L}\right) \qquad (IC_L)$$

(b) The IC constraint for the high type is binding.

Suppose not and IC_H is slack. Then an addition amount of consumption good could be transferred from the high type to the low type w/o violating IC_H. Since $c_H < c_L$, this transfer increases aggregate utility. Thus, the solution is not an optimum. $\Rightarrow\Leftarrow$ IC_H holds w/ equality

③ (c) the Relaxed Problem is

$$\max \quad \pi\left[u(c_H) - v\left(\frac{y_H}{\theta_H}\right)\right] + (1-\pi)\left[u(c_L) - v\left(\frac{y_L}{\theta_L}\right)\right]$$

$$\text{st} \quad \pi c_H + (1-\pi)c_L = \pi y_H + (1-\pi)y_L \qquad (RC)$$

$$u(c_H) - v\left(\frac{y_H}{\theta_H}\right) = u(c_L) - v\left(\frac{y_L}{\theta_H}\right) \qquad (IC_H)$$

$$\text{FOC } [c_H]: \ \pi u'(c_H) = \mu\pi + \lambda u'(c_H)$$

$$[c_L]: \ (1-\pi)u'(c_L) = \mu(1-\pi) - \lambda u'(c_L)$$

$$[y_H]: \ \pi v'\left(\frac{y_H}{\theta_H}\right)\frac{1}{\theta_H} = \mu\pi + \lambda v'\left(\frac{y_H}{\theta_H}\right)\frac{1}{\theta_H}$$

$$[y_L]: \ (1-\pi)v'\left(\frac{y_L}{\theta_L}\right)\frac{1}{\theta_L} = \mu(1-\pi) - \lambda\frac{1}{\theta_H}v'\left(\frac{y_L}{\theta_H}\right)$$

FOC $[c_H]$ + FOC $[y_H]$ $\Rightarrow$ $u'(c_H) = \frac{1}{\theta_H}v'\left(\frac{y_H}{\theta_H}\right)$ [No distortion at the top]

FOC $[c_L]$ + FOC $[y_L]$ $\Rightarrow$ $(1-\pi)u'(c_L) + \lambda u'(c_L) = (1-\pi)v'\left(\frac{y_L}{\theta_L}\right)\frac{1}{\theta_L} + \lambda\frac{1}{\theta_H}v'\left(\frac{y_L}{\theta_H}\right)$

$$u'(c_L) = \frac{1}{\theta_L}v'\left(\frac{y_L}{\theta_L}\right) - \lambda\left[u'(c_L) + \frac{1}{\theta_H}v'\left(\frac{y_L}{\theta_H}\right)\right]$$

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