ECON 709 - PS 3

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9/27/2020

1. A random point (X, Y) is distributed uniformly on the square with vertices $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$. That is, the joint PDF is $f(x, y) = 1/4$ on the square and $f(x, y) = 0$ outside the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$

$X^2 + Y^2 < 1 \implies -\sqrt{1 - X^2} < Y < \sqrt{1 - X^2}$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4} dy dx = \int_{-1}^{1} \left[\frac{1}{4} y \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} dx$$

Define $x = \sin \theta \implies dx = \cos \theta d\theta$.

$$\begin{aligned}
\frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} dx &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sqrt{1 - (\sin \theta)^2} \cos \theta d\theta \\
&= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta \\
&= \frac{1}{4} \int_{-\pi/2}^{\pi/2} 1 + \cos(2\theta) d\theta \\
&= \frac{1}{4} \left[\theta + \frac{\sin(2\theta)}{2} \right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{4} \left[(\pi/2) + \frac{0}{2} - (-\pi/2) - \frac{0}{2} \right] \\
&= \frac{\pi}{4}
\end{aligned}$$

(b) $|X + Y| < 2$

$|X + Y| < 2 \implies -2 < X + Y < 2 \implies -2 - X < Y < 2 - X$. Since X ranges from -1 to 1, $-2 - X < Y < 2 - X \implies -1 < Y < 1$

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} dy dx = \frac{1}{4} \int_{-1}^{1} [y]_{-1}^{1} dx = \frac{1}{2} \int_{-1}^{1} dx = \frac{1}{2} [x]_{-1}^{1} = 1$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

2. Let the joint PDF of X and Y be given by $f(x,y) = g(x)h(y)\ \forall x,y \in \mathbb{R}$ for some functions $g(x)$ and $h(y)$. Let a denote $\int_{-\infty}^{\infty} g(x)dx$ and b denote $\int_{-\infty}^{\infty} h(x)dx$

(a) What conditions a and b should satisfy in order for $f(x,y)$ to be a bivariate PDF?

For $f(x,y)$ to be a PDF, it should integrate to one:

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= 1 \\ \implies \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)dxdy &= 1 \\ \implies \int_{-\infty}^{\infty} g(x)dx \int_{-\infty}^{\infty} h(y)dy &= 1 \\ \implies ab &= 1 \\ \implies a &= b^{-1} \end{aligned}$$

(b) Find the marginal PDF of X and Y.

The marginal PDF of X:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} g(x)h(y)dy = g(x)\int_{-\infty}^{\infty} h(y)dy = b \cdot g(x)$$

The marginal PDF of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} g(x)h(y)dx = h(y)\int_{-\infty}^{\infty} g(x)dx = a \cdot h(y)$$

(c) Show that X and Y are independent.

Proof: X and Y are independent if the product of their marginal distributions is their joint distribution:

$$\begin{aligned} f_X(x) \cdot f_Y(y) &= b \cdot g(x) \cdot a \cdot h(y) \\ &= b \cdot g(x) \cdot b^{-1} \cdot h(y) \\ &= g(x) \cdot h(y) \\ &= f(x,y) \end{aligned}$$

□

3. Let the joint PDF of X and Y be given by

$$f(x,y) = \begin{cases} cxy & \text{if } x, y \in [0,1], x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c such that $f(x,y)$ is a joint PDF.

$$\begin{aligned} \int_0^1 \int_0^{1-x} f(x,y) dy dx &= 1 \\ \implies \int_0^1 \int_0^{1-x} cxy dy dx &= 1 \\ \implies c \int_0^1 \left[\frac{xy^2}{2}\right]_{y=0}^{1-x} dx &= 1 \\ \implies \frac{c}{2} \int_0^1 x(1-x)^2 dx &= 1 \\ \implies \frac{c}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{x=0}^1 &= 1 \\ \implies \frac{c}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] &= 1 \\ \implies c &= 24 \end{aligned}$$

(b) Find the marginal distributions of X and Y.

$$f_X(x) = \int_0^{1-x} f(x,y) dy = \int_0^{1-x} 24xy dy = \left[12xy^2\right]_{y=0}^{1-x} = \begin{cases} 12x(1-x)^2, x \in [0,1] \\ 0, \text{ otherwise.} \end{cases}$$

$$f_Y(y) = \int_0^{1-y} f(x,y) dx = \int_0^{1-y} 24xy dx = \left[12x^2 y\right]_{x=0}^{1-y} = \begin{cases} 12(1-y)^2 y, y \in [0,1] \\ 0, \text{ otherwise.} \end{cases}$$

(c) Are X and Y independent? Compare your answer to Problem 2 and discuss.

X and Y independent if the product of the marginal distributions equals their joint distribution at all points in the support. If $x = y = 0.9$, $f(0.9, 0.9) = 0$ because $(0.9, 0.9)$ is not in the support, $x + y = 0.9 + 0.9 = 1.8 > 1$. But each marginal distribution is define over $[0,1]$, so the product of the marginals is positive at $(0.9, 0.9)$: $f_X(0.9) f_Y(0.9) = [12(0.9)(1-(0.9))^2][12(1-0.9)^2(0.9)] = 0.0117$.

In (2), the support for the joint distribution is $\mathbb{R}^2$, whereas the support for the joint distribution depends on the realization of the random variable.

4. Show that any random variable is uncorrelated with a constant.

Proof: Let $a \in \mathbb{R}$ and X be a random variable with distribution F_X. Define random variable Y as the degenerate random variable that equals a. Thus, the distribution Y is

$$F_Y(y) = \begin{cases} 0, & y < a \\ 1, & y \geq a \end{cases}$$

To show X is uncorrelated with a constant, I show that X and Y are independent and then, by a theorem in the Lecture 3 Notes, we know that X and Y are uncorrelated.

To find the joint distribution of X and Y, consider two cases: $y < a$ and $y \geq a$. For $y < a$,

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x \text{ and } Y \leq a) \\ &= 0 \end{aligned}$$

For $y \geq a$:

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x) \\ &= F_X(x) \end{aligned}$$

Thus, the joint distribution is

$$F(x, y) = \begin{cases} 0, & y < a \\ F_X(x), & y \geq a \end{cases}$$

The joint distribution equals the product of the marginals:

$$F(x, y) = \begin{cases} 0 * F_X(x), & y < a \\ 1 * F_X(x), & y \geq a \end{cases} = \begin{cases} F_Y(y) * F_X(x), & y < a \\ F_Y(y) * F_X(x), & y \geq a \end{cases}.$$

□

5. Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for the correlation of XY and Y in terms of these means and variances.

Note that $Var(X) = E(X^2) - E(X)^2 \implies E(X^2) = Var(X) + E(X)^2$.

$$
\begin{aligned}
Corr(XY, Y) &= \frac{Cov(XY, Y)}{\sqrt{Var(XY)Var(Y)}} \\
&= \frac{E(XY^2) - E(XY)E(Y)}{\sigma_Y \sqrt{E((XY)^2) - E(XY)^2}} \\
&= \frac{E(X)E(Y^2) - E(X)E(Y)E(Y)}{\sigma_Y \sqrt{E(X^2)E(Y^2) - (E(X)E(Y))^2}} \\
&= \frac{\mu_X(Var(Y) + E(Y)^2) - \mu_X \mu_Y^2}{\sigma_Y \sqrt{(Var(X) + E(X)^2)(Var(Y) + E(Y)^2) - (\mu_X \mu_Y)^2}} \\
&= \frac{\mu_X \sigma_Y^2 + \mu_X \mu_Y^2 - \mu_X \mu_Y^2}{\sigma_Y \sqrt{\sigma_Y^2 \sigma_X^2 + \mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2 + \mu_Y^2 \mu_X^2 - \mu_X^2 \mu_Y^2}} \\
&= \frac{\mu_X \sigma_Y}{\sqrt{\sigma_Y^2 \sigma_X^2 + \mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2}}
\end{aligned}
$$

6. Prove the following: For any random vector $(X_1, X_2, ..., X_n)$,

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \leq i < j \leq n} Cov(X_i, X_j).$$

Proof by induction.

7. Suppose that X and Y are joint normal, i.e. they have the joint PDF:

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp(-(2(1-\rho^2))^{-1}(x^2/\sigma_X - 2xy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))$$

(a) Derive the marginal distributions of X and Y, and observe that both normal distributions.

Gaussian integrals

(b) Derive the conditional distribution of Y given $X = x$. Observe that it is also a normal distribution.

(c) Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X and Z are independent.

Equation 23 in lecture 3.

8. Consider a function $g : \mathbb{R} \to \mathbb{R}$. Recall that the inverse image of a set A, denoted $g^{-1}(A)$ is $g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$. Let there be functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$. Let X and Y be two random variables that are independent. Suppose that g_1 and g_2 are both Borel-measurable, which means that $g_1^{-1}(A)$ and $g_2^{-1}(A)$ are both in the Borel σ-field whenever A is in the Borel σ-field. Show that the two random variables $Z := g_1(X)$ and $W := g_2(Y)$ are independent. (Hint: use the 1st or the 2nd definition of independence.)