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ECON 713B Midterm

Problem 1

(a) $$u_i(b_i, b_j, v_i) = \begin{cases} v_i - b_i - f & b_i > b_j \\ \frac{1}{2} v_i - b_i - f & b_i = b_j > 0 \\ -b_i - f & b_j > b_i > 0 \\ 0 & b_i = 0 \end{cases}$$

(b) Bidder i's action is a bid.
The set of strategies is a bidding function that maps from valuations (types) to bids (actions). $b_i : [0,1] \to [0, \infty)$
A BNE for this game is the strategy profile $b^*(\cdot) = (b_1^*(\cdot), b_2^*(\cdot))$ if

$$E[u_i(b_i^*(v_i), b_j^*(v_j), v_i) \mid v_i]$$

$$\geq E[a_i, b_j^*(v_j), v_i) \mid v_i]$$

$$\forall i, \forall a_i \in [0, \infty) \text{ and } \forall v_i \in [0,1].$$

The expectations are taken over the realization of bidder j's valuations condition on bidder i's valuation.

Problem 1

(c) Assume symmetric BNE. $\Rightarrow b_1(\cdot) = b_2(\cdot) = b(\cdot)$

Assume bidder 1 w/ $v_1 \in [0,1)$ bids a positive amount.

$$\Rightarrow b(v_1) = b_1 > 0.$$

Assume that the symmetric BNE is based on a bidding function that is weakly increasing. For $v_2 > v_1$,

$$\Rightarrow b(v_2) = b_2 \geq b_1 > 0$$

because b is weakly increasing.

Problem 1

(d) The probability of $b_i = b_j$ and $b_i = 0$ is zero. Thus the expected utility of i is

$$\cancel{E[u_i] = (v_i - b_i - f)\, P(b_i > b_j) + (-b}$$

$$E[u_i] = v_i\, P(b_i > b_j) - b_i - f$$

Assume j bids $b_j = \alpha + \beta v_j^2$

$$\Rightarrow P(b_i > \alpha + \beta v_j^2) = P\left(\sqrt{\frac{b_i - \alpha}{\beta}} > v_j\right)$$

$$= \sqrt{\frac{b_i - \alpha}{\beta}}$$

$$\Rightarrow E[u_i] = v_i \sqrt{\frac{b_i - \alpha}{\beta}} - b_i - f$$

FOC: $\frac{1}{2} v_i \left(\frac{b_i - \alpha}{\beta}\right)^{-1/2} \frac{1}{\beta} - 1 = 0$

$$\Rightarrow \quad v_i \left(\frac{b_i - \alpha}{\beta}\right)^{-1/2} = 2\beta$$

$$\Rightarrow \quad \left(\frac{b_i - \alpha}{\beta}\right)^{-1/2} = \frac{2\beta}{v_i}$$

(d) cont

$$\Rightarrow \quad \left(\frac{b_i - \alpha}{\beta}\right)^{1/2} = \frac{v_i}{2\beta}$$

$$\Rightarrow \quad \frac{b_i - \alpha}{\beta} = \left(\frac{v_i}{2\beta}\right)^2$$

$$\Rightarrow \quad b_i = \beta\left(\frac{v_i}{2\beta}\right)^2 + \alpha$$

$$\Rightarrow \quad b_i = \frac{1}{4\beta} v_i^2 + \alpha$$

Matching coefficients

$$\frac{1}{4\beta} = \beta$$

$$1 = 4\beta^2$$

$$1/4 = \beta^2$$

$$1/2 = \beta$$

$$\Rightarrow \quad b(v_i) = \frac{1}{2} v_i^2 + \alpha$$

If ~~xxx~~ $v_i = 0$, bidder i bids zero

$$\Rightarrow b(0) = \frac{1}{2}(0)^2 + \alpha \Rightarrow \alpha = 0$$

$$\Rightarrow b(v_i) = \frac{1}{2} v_i^2$$

Problem 2

(a) $u_i(b_i, b_j, v_i) = \begin{cases} v_i & b_i > b_j \\ \frac{1}{2} v_i - \frac{1}{2} b_i & b_i = b_j \\ -b_i & b_i < b_j \end{cases}$

(b) The probability of $b_i = b_j$ is zero if the bidding function is continuous.

$$E[u_i] = v_i \, P(b_i > b_j) - b_i \, P(b_i < b_j)$$

Assume $b_j = b(v_j)$ where b is strictly increasing and continuously differentiable. $\Rightarrow$ invertible.

$$\Rightarrow E[u_i] = v_i \, P(b_i > b(v_j)) - b_i \, P(b_i < b(v_j))$$

$$P(b_i > b(v_j)) = P(b^{-1}(b_i) > v_j)$$

$$= F(b^{-1}(b_i))$$

$$= 2 b^{-1}(b_i)$$

$$P(b_i < b(v_j)) = 1 - P(b_i > b(v_j))$$

$$= 1 - 2b^{-1}(b_i)$$

$\Rightarrow E[u_i] = v_i \, 2b^{-1}(b_i) - b_i \, (1-2b^{-1}(b_i))$

FOC: $0 = \frac{2v_i}{b'(b^{-1}(b_i))} - (1-2b^{-1}(b_i)) - b_i\left(-\frac{2}{b'(b^{-1}(b_i))}\right)$

Apply symmetry $b_i = b(v_i)$. $\Rightarrow b^{-1}(b_i) = v_i$.

$\Rightarrow 0 = \frac{2v_i}{b'(v_i)} - (1-2v_i) + \frac{2b(v_i)}{b'(v_i)}$

$\Rightarrow 0 = 2v_i - b'(v_i)(1-2v_i) + 2b(v_i)$

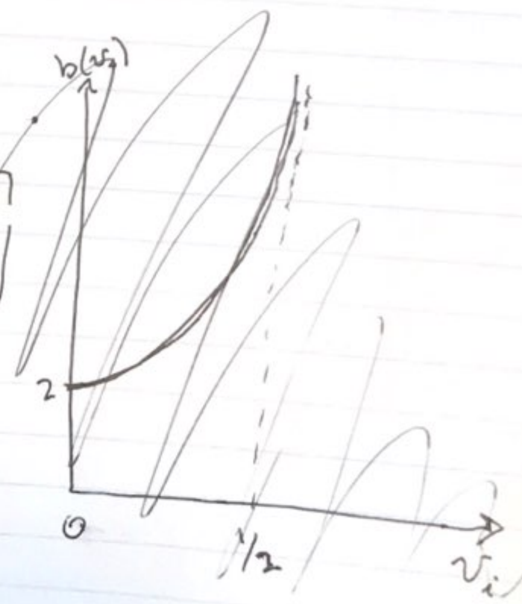
$\Rightarrow -2v_i = 2b(v_i) + b'(v_i)(2v_i-1)$

$\Rightarrow -2v_i = \left(b(v_i)(2v_i-1)\right)'$

$\Rightarrow \int -2v_i \, dv_i = \int \left(b(v_i)(2v_i-1)\right)' \, dv_i$

$\Rightarrow -v_i^2 + c = b(v_i)(2v_i-1)$

$\cancel{-\frac{-1}{1-2v_i}}$



$$\Rightarrow b(v_i) = \frac{v_i^2 + c}{1 - 2v_i}$$

$$b(0) = 0$$

$$\Rightarrow b(0) = \frac{0^2 + c}{1 - 2(0)}$$

$$0 = c$$

$$\Rightarrow b(v_i) = \frac{v_i^2}{1 - 2v_i}$$

(c) In FPA, the symmetric bidding function is:

$$b^{FPA}(v) = \frac{1}{F(v)^{I-1}} \int_0^v x\,(I-1)\,f(x)\,F(x)^{I-2}\,dx$$

[From discussion section 2 notes]

$F(v) = 2v \qquad I = 2$

$f(v) = 2$

$$\Rightarrow b^{FPA}(v) = \frac{1}{2v}\int_0^v x\,(1)(2)(2x)^0\,dx$$

$$= \frac{1}{2v}\int_0^v 2x\,dx$$

$$= \frac{1}{2v}\left[x^2\right]_{x=0}^{v}$$

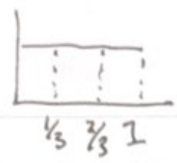
$$= \frac{1}{2v}\left(v^2\right)$$

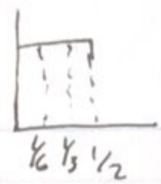
$$= \frac{v}{2}$$

~~E. Expected revenue from FPA~~

E

$u \sim [0, 1]$ $\qquad\qquad$ $u \sim [0, 1/2]$





(c) con't

Expected revenue from this auction is

$E[b_i \mid b_i < b_j]$

For FPA:

$$E\left[\frac{v_i}{2} \;\middle|\; \frac{v_i}{2} < \frac{v_j}{2}\right] = \frac{1}{2} E[v_i \mid v_i < v_j]$$

$$= \frac{1}{2}\left(1/6\right)$$

$$= \frac{1}{12}$$

[because the first order statistic from $u \sim [0, 1/2]$ when $n=2$ equals $1/6$.]

For zero auction, the expected revenue is:

$$E\left[\frac{v_i^2}{1-2v_i} \;\middle|\; \frac{v_i^2}{1-2v_i} < \frac{v_j^2}{1-2v_j}\right]$$

$$= E\left[\frac{v_i^2}{1-2v_i} \;\middle|\; v_i < v_j\right]$$

$$= \frac{E[v_i^2 \mid v_i < v_j]}{1 - 2E[v_i \mid v_i < v_j]}$$

$$= \frac{\frac{1}{36}}{1-2(1/6)} = \frac{\frac{1}{36}}{1-1/3} = \frac{\frac{1}{36}}{2/3} = \frac{1}{24}$$

Yes, the seller will earn more w/ a FPA