ECON 709 - PS 3

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1. A random point (X, Y) is distributed uniformly on the square with vertices $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$. That is, the joint PDF is $f(x, y) = 1/4$ on the square and $f(x, y) = 0$ outside the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$

$X^2 + Y^2 < 1 \implies -\sqrt{1 - X^2} < Y < \sqrt{1 - X^2}$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4} dy dx = \int_{-1}^{1} \left[\frac{1}{4} y \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} dx$$

Define $x = \sin\theta \implies dx = \cos\theta d\theta$.

$$\begin{aligned}
\frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} dx &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sqrt{1 - (\sin\theta)^2} \cos\theta d\theta \\
&= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta \\
&= \frac{1}{4} \int_{-\pi/2}^{\pi/2} 1 + \cos(2\theta) d\theta \\
&= \frac{1}{4} \left[\theta + \frac{\sin(2\theta)}{2} \right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{4} \left[(\pi/2) + \frac{0}{2} - (-\pi/2) - \frac{0}{2} \right] \\
&= \frac{\pi}{4}
\end{aligned}$$

(b) $|X + Y| < 2$

$|X + Y| < 2 \implies -2 < X + Y < 2 \implies -2 - X < Y < 2 - X$. Since X ranges from -1 to 1, $-2 - X < Y < 2 - X \implies -1 < Y < 1$

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} dy dx = \frac{1}{4} \int_{-1}^{1} [y]_{-1}^{1} dx = \frac{1}{2} \int_{-1}^{1} dx = \frac{1}{2} [x]_{-1}^{1} = 1$$

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2. Let the joint PDF of X and Y be given by $f(x,y) = g(x)h(y)\ \forall x, y \in \mathbb{R}$ for some functions $g(x)$ and $h(y)$. Let a denote $\int_{-\infty}^{\infty} g(x)dx$ and b denote $\int_{-\infty}^{\infty} h(x)dx$

(a) What conditions a and b should satisfy in order for $f(x,y)$ to be a bivariate PDF?

For $f(x,y)$ to be a PDF, it should integrate to one:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= 1 \\
\implies \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)dxdy &= 1 \\
\implies \int_{-\infty}^{\infty} g(x)dx \int_{-\infty}^{\infty} h(y)dy &= 1 \\
\implies ab &= 1 \\
\implies a &= b^{-1}
\end{aligned}$$

(b) Find the marginal PDF of X and Y.

The marginal PDF of X:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} g(x)h(y)dy = g(x)\int_{-\infty}^{\infty} h(y)dy = b \cdot g(x)$$

The marginal PDF of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} g(x)h(y)dx = h(y)\int_{-\infty}^{\infty} g(x)dx = a \cdot h(y)$$

(c) Show that X and Y are independent.

Proof: X and Y are independent if the product of their marginal distributions is their joint distribution:

$$\begin{aligned}
f_X(x) \cdot f_Y(y) &= b \cdot g(x) \cdot a \cdot h(y) \\
&= b \cdot g(x) \cdot b^{-1} \cdot h(y) \\
&= g(x) \cdot h(y) \\
&= f(x,y)
\end{aligned}$$

□

3. Let the joint PDF of X and Y be given by

$$f(x,y) = \begin{cases} cxy & \text{if } x,y \in [0,1], x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c such that $f(x,y)$ is a joint PDF.

$$\begin{aligned} \int_0^1 \int_0^{1-x} f(x,y)dydx &= 1 \\ \implies \int_0^1 \int_0^{1-x} cxydydx &= 1 \\ \implies c\int_0^1 \left[\frac{xy^2}{2}\right]_{y=0}^{1-x} dx &= 1 \\ \implies \frac{c}{2}\int_0^1 x(1-x)^2 dx &= 1 \\ \implies \frac{c}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{x=0}^{1} &= 1 \\ \implies \frac{c}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] &= 1 \\ \implies c &= 24 \end{aligned}$$

(b) Find the marginal distributions of X and Y.

$$f_X(x) = \int_0^{1-x} f(x,y)dy = \int_0^{1-x} 24xydy = \left[12xy^2\right]_{y=0}^{1-x} = \begin{cases} 12x(1-x)^2, x \in [0,1] \\ 0, \text{ otherwise.} \end{cases}$$

$$f_Y(y) = \int_0^{1-y} f(x,y)dx = \int_0^{1-y} 24xydx = \left[12x^2y\right]_{x=0}^{1-y} = \begin{cases} 12(1-y)^2y, y \in [0,1] \\ 0, \text{ otherwise.} \end{cases}$$

(c) Are X and Y independent? Compare your answer to Problem 2 and discuss.

X and Y independent if the product of the marginal distributions equals their joint distribution at all points in the support. If $x = y = 0.9$, $f(0.9, 0.9) = 0$ because $(0.9, 0.9)$ is not in the support, $x+y = 0.9+0.9 = 1.8 > 1$. But each marginal distribution is define over $[0,1]$, so the product of the marginals is positive at $(0.9, 0.9)$: $f_X(0.9)f_Y(0.9) = [12(0.9)(1-(0.9))^2][12(1-0.9)^2(0.9)] = 0.0117$.

In (2), the support for the joint distribution is $\mathbb{R}^2$, so the support for each random variable does not depend on the realization of the other random variable. In contrast in (3), the support for each random variable depends on the realization of the other random variable. Thus in (3), the random variables cannot be independent.

4. Show that any random variable is uncorrelated with a constant.

Proof: Let $a \in \mathbb{R}$ and X be a random variable with distribution F_X. Define random variable Y as the degenerate random variable that equals a. Thus, the distribution Y is

$$F_Y(y) = \begin{cases} 0, & y < a \\ 1, & y \geq a \end{cases}$$

To show X is uncorrelated with a constant, I show that X and Y are independent and then, by a theorem in the Lecture 3 Notes, we know that X and Y are uncorrelated.

To find the joint distribution of X and Y, consider two cases: $y < a$ and $y \geq a$. For $y < a$,

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x \text{ and } Y \leq a) \\ &= 0 \end{aligned}$$

For $y \geq a$:

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x) \\ &= F_X(x) \end{aligned}$$

Thus, the joint distribution is

$$F(x, y) = \begin{cases} 0, & y < a \\ F_X(x), & y \geq a \end{cases}$$

The joint distribution equals the product of the marginals:

$$F(x, y) = \begin{cases} 0 * F_X(x), & y < a \\ 1 * F_X(x), & y \geq a \end{cases} = \begin{cases} F_Y(y) * F_X(x), & y < a \\ F_Y(y) * F_X(x), & y \geq a \end{cases}.$$

□

5. Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for the correlation of XY and Y in terms of these means and variances.

Note that $Var(X) = E(X^2) - E(X)^2 \implies E(X^2) = Var(X) + E(X)^2$.

$$\begin{aligned}
Corr(XY, Y) &= \frac{Cov(XY, Y)}{\sqrt{Var(XY)Var(Y)}} \\
&= \frac{E(XY^2) - E(XY)E(Y)}{\sigma_Y \sqrt{E((XY)^2) - E(XY)^2}} \\
&= \frac{E(X)E(Y^2) - E(X)E(Y)E(Y)}{\sigma_Y \sqrt{E(X^2)E(Y^2) - (E(X)E(Y))^2}} \\
&= \frac{\mu_X(Var(Y) + E(Y)^2) - \mu_X \mu_Y^2}{\sigma_Y \sqrt{(Var(X) + E(X)^2)(Var(Y) + E(Y)^2) - (\mu_X \mu_Y)^2}} \\
&= \frac{\mu_X \sigma_Y^2 + \mu_X \mu_Y^2 - \mu_X \mu_Y^2}{\sigma_Y \sqrt{\sigma_Y^2 \sigma_X^2 + \mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2 + \mu_Y^2 \mu_X^2 - \mu_X^2 \mu_Y^2}} \\
&= \frac{\mu_X \sigma_Y}{\sqrt{\sigma_Y^2 \sigma_X^2 + \mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2}}
\end{aligned}$$

6. Prove the following: For any random vector $(X_1, X_2, ..., X_n)$,

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1\leq i<j\leq n} Cov(X_i, X_j).$$

Proof (by induction): For $n = 2$, $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$ from lecture 3 notes. Assume that the formula holds for some n, then

$$\begin{aligned}
&Var\left(\sum_{i=1}^{n+1} X_i\right) \\
&= Var\left(\sum_{i=1}^{n} X_i + X_{n+1}\right) \\
&= Var\left(\sum_{i=1}^{n} X_i\right) + Var(X_{n+1}) + 2Cov\left(\sum_{i=1}^{n} X_i, X_{n+1}\right) \\
&= \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1\leq i<j\leq n} Cov(X_i, X_j) + Var(X_{n+1}) + 2\left[E\left(X_{n+1}\sum_{i=1}^{n} X_i\right) + E\left(X_{n+1}\right)E\left(\sum_{i=1}^{n} X_i\right)\right] \\
&= \sum_{i=1}^{n+1} Var(X_i) + 2 \sum_{1\leq i<j\leq n} Cov(X_i, X_j) + 2\left[\sum_{i=1}^{n} E\left(X_{n+1}X_i\right) + E\left(X_{n+1}\right)\sum_{i=1}^{n} E\left(X_i\right)\right] \\
&= \sum_{i=1}^{n+1} Var(X_i) + 2 \sum_{1\leq i<j\leq n} Cov(X_i, X_j) + 2\sum_{i=1}^{n}\left[E\left(X_{n+1}X_i\right) + E\left(X_{n+1}\right)E\left(X_i\right)\right] \\
&= \sum_{i=1}^{n+1} Var(X_i) + 2 \sum_{1\leq i<j\leq n} Cov(X_i, X_j) + 2\sum_{i=1}^{n} Cov\left(X_{n+1}X_i\right) \\
&= \sum_{i=1}^{n+1} Var(X_i) + 2 \sum_{1\leq i<j\leq n+1} Cov(X_i, X_j)
\end{aligned}$$

$\square$

7. Suppose that X and Y are joint normal, i.e. they have the joint PDF:

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp(-(2(1-\rho^2))^{-1}(x^2/\sigma_X^2 - 2\rho xy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))$$

(a) Derive the marginal distributions of X and Y, and observe that both normal distributions.

The marginal distribution of X is

$$\begin{aligned}
&f_X(x) \\
&= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2} + \frac{\rho^2x^2}{\sigma_X^2} - \frac{\rho^2x^2}{\sigma_X^2}\right)\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{\rho^2x^2}{\sigma_X^2}\right) - \frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{\rho^2x^2}{\sigma_X^2}\right)\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy(\sigma_Y/\sigma_X)}{\sigma_Y^2} + \frac{\rho^2x^2(\sigma_Y^2/\sigma_X^2)}{\sigma_Y^2}\right) - \frac{(1-\rho^2)}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}\right)\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{(y-\rho x(\sigma_Y/\sigma_X))^2}{2(1-\rho^2)\sigma_Y^2} - \frac{x^2}{2\sigma_X^2}\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)\int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}}\exp\left(-\frac{(y-\rho x(\sigma_Y/\sigma_X))^2}{2(1-\rho^2)\sigma_Y^2}\right)dy \\
&= \frac{1}{\sigma_X\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)
\end{aligned}$$

Thus, $X \sim Normal(0,\sigma_X^2)$. By swapping x and y in the algebra above, we see that $Y \sim Normal(0,\sigma_Y^2)$.

(b) Derive the conditional distribution of Y given $X = x$. Observe that it is also a normal distribution.

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right)\left[\frac{1}{\sigma_X\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)\right]^{-1} \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right) + \frac{x^2}{2\sigma_X^2}\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2} - \frac{x^2(1-\rho^2)}{\sigma_X^2}\right)\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{x^2\rho^2}{\sigma_X^2}\right)\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y}{\sigma_Y} - \frac{x\rho}{\sigma_X}\right)^2\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(y-\frac{\sigma_Y}{\sigma_X}\rho x)^2}{(1-\rho^2)\sigma_Y^2}\right)
\end{aligned}
$$

Observe that $Y|X \sim Normal(\frac{\sigma_Y}{\sigma_X}\rho x, \sigma_Y^2(1-\rho^2))$.

(c) Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X and Z are independent.

Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(x, y) = (x, (y/\sigma_Y) - (\rho x)/\sigma_X)$. Notice that g is one-to-one, so it is invertible. Define $h = g^{-1}$ such that $h(x, z) = (x, \sigma_Y(z+\rho x/\sigma_X))$. The determinant of the Jacobian of the transformation is

$$
\begin{aligned}
|J| &= \begin{vmatrix} \frac{\partial h_1(x,z)}{\partial x} & \frac{\partial h_1(x,z)}{\partial z} \\ \frac{\partial h_2(x,z)}{\partial x} & \frac{\partial h_2(x,z)}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ \frac{\rho\sigma_Y}{\sigma_X} & \sigma_Y \end{vmatrix} \\
&= \sigma_Y
\end{aligned}
$$

Thus, from lecture 3 notes, we know that the joint distribution

$$
\begin{aligned}
& f_{X,Z}(x, z) \\
&= f_{X,Y}(h(x, z))|J| \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho x(\sigma_Y(z+\rho x/\sigma_X))}{\sigma_X\sigma_Y} + \frac{(\sigma_Y(z+\rho x/\sigma_X))^2}{\sigma_Y^2} \right)\right)\sigma_Y \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho x(z+\rho x/\sigma_X)}{\sigma_X} + (z+\rho x/\sigma_X)^2 \right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xz}{\sigma_X} - \frac{2\rho^2x^2}{\sigma_X^2} + z^2 + \frac{2z\rho x}{\sigma_X} + \frac{\rho^2x^2}{\sigma_X^2} \right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{\rho^2x^2}{\sigma_X^2} + z^2 \right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(1-\rho^2)x^2}{\sigma_X^2} + z^2 \right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2} + \frac{z^2}{(1-\rho^2)} \right)\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2} \right)\right) \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{z^2}{(1-\rho^2)} \right)\right) \\
&= f_X(x) \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{z^2}{(1-\rho^2)} \right)\right)
\end{aligned}
$$

Thus, X and Z are independent with $Z \sim Normal(0, 1-\rho^2)$.

8. Consider a function $g : \mathbb{R} \to \mathbb{R}$. Recall that the inverse image of a set A, denoted $g^{-1}(A)$ is $g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$. Let there be functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$. Let X and Y be two random variables that are independent. Suppose that g_1 and g_2 are both Borel-measurable, which means that $g_1^{-1}(A)$ and $g_2^{-1}(A)$ are both in the Borel σ-field whenever A is in the Borel σ-field. Show that the two random variables $Z := g_1(X)$ and $W := g_2(Y)$ are independent. (Hint: use the 1st or the 2nd definition of independence.)

Proof: Let A, B be events in the Borel σ-field. Then,

$$P(Z \in A, W \in B) = P(g_1(X) \in A, g_2(Y) \in B) = P(X \in g_1^{-1}(A), Y \in g_2^{-1}(B))$$

Since g_1, g_2 are Borel measurable, $g_1^{-1}(A), g_2^{-1}(B)$ are events in the Borel σ-field. Thus, by the independence of X and Y:

$$P(X \in g_1^{-1}(A), Y \in g_2^{-1}(B)) = P(X \in g_1^{-1}(A))P(Y \in g_2^{-1}(B)) = P(g_1(X) \in A)P(g_2(Y) \in B) = P(Z \in A)P(W \in B)$$

Thus, Z and W are independent. $\square$