ECON 703 - PS 7

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(1) Let $X \subset \mathbb{R}^n$ be a convex set, and $\lambda_1, ..., \lambda_k \geq 0$ with $\sum_{i=1}^k \lambda_i = 1$. Prove that if $x_1, ..., x_k \in X$, then $\sum_{i=1}^k \lambda_i x_i \in X$.

Proof (by induction): For the base step, choose $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$. For any $x_1, x_2 \in X \subset \mathbb{R}^n$, $\lambda_1 x_1 + \lambda_2 x_2 \in X$ because X is convex. For some k, assume that $\sum_{i=1}^k \lambda_i x_i \in X$ for $x_1, ..., x_k \in X$ with $\lambda_1, ..., \lambda_k \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$. Consider $k+1$. Choose $\lambda_1', ..., \lambda_{k+1}' \geq 0$ such that $\sum_{i=1}^{k+1} \lambda_i' = 1$:

$$\sum_{i=1}^{k+1} \lambda_i' x_i = \sum_{i=1}^{k} \lambda_i' x_i + \lambda_{k+1}' x_{k+1} = \left(\sum_{i=1}^{k} \lambda_i' \right) \sum_{i=1}^{k} \left(\frac{\lambda_i'}{\sum_{i=1}^k \lambda_i'} x_i \right) + \lambda_{k+1}' x_{k+1}$$

By the induction hypothesis, $y := \sum_{i=1}^k \left(\frac{\lambda_i'}{\sum_{i=1}^k \lambda_i'} x_i \right) \in X$ because $\sum_{i=1}^k \frac{\lambda_i'}{\sum_{i=1}^k \lambda_i'} = 1$. Thus,

$$\sum_{i=1}^{k+1} \lambda_i' x_i = \left(\sum_{i=1}^{k} \lambda_i' \right) y + \lambda_{k+1}' x_{k+1}$$

By the definition of convexity, $\sum_{i=1}^{k+1} \lambda_i' x_i \in X$ because $\sum_{i=1}^k \lambda_i' + \lambda_{k+1}' = 1$. $\square$

(2) The sum $\sum_{i=1}^k \lambda_i x_i$ defined in Problem (1) is called a convex combination. The convex hull of a set S, denoted by $\text{co}(S)$, is the intersection of all convex sets which contain S. Prove that the set of all convex combinations of the elements of S is exactly $\text{co}(S)$.

Proof: We show that an arbitrary convex combination of elements of S is in $\text{co}(S)$ and an arbitrary point in $\text{co}(S)$ can be represented by a convex combination of elements of S. First, notice that $S \subset \text{co}(S)$ and $\text{co}(S)$ is convex because it is the intersection of convex sets.

Consider an arbitrary convex combination of elements of S, $\sum_{i=1}^k \lambda_i s_i$ with $s_1, ..., s_k \in S$. Since $s_i \in S$, $s_i \in \text{co}(S)$ for $i \in \{1, ..., k\}$. Since $\text{co}(S)$ is convex, $\sum_{i=1}^k \lambda_i s_i \in \text{co}(S)$.

Consider $x \in \text{co}(S)$. Assume for the sake of a contradiction that x cannot be represented as a convex combination of elements of S. Then there exists a convex set Y such that $S \subset Y$ and $x \notin Y$. This is a contradiction because $\text{co}(S)$ is the intersection of all convex sets which contain S. Thus, x can be represented as a convex combination of elements of S. $\square$

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(3) For any set $X \subset \mathbb{R}^n$, let its closure be $\text{cl}X = X \cup \{\text{all limit points of } X\}$. Show that the closure of a convex set is convex.

Proof: Let X be a convex set. Choose two points $x, y \text{cl}X$. Thus, there exists sequences $\{x_n\}, \{y_n\} \in X$ such that $\{x_n\} \to x$ and $\{y_n\} \to y$. Since X is convex, $\lambda x_n + (1-\lambda)y_n \in X$ for all n with $\lambda \in [0,1]$. Because cl$X$ contains all limit points of X, $\lambda x + (1-\lambda)y = \lim_{n\to\infty}(\lambda x_n + (1-\lambda)y_n) \in \text{cl}X$. □

(4) The function $f : X \to \mathbb{R}$, where X is a convex set in $\mathbb{R}^n$, is concave if $\forall \lambda \in [0,1], x', x'' \in X$, $f((1-\lambda)x' + \lambda x'') \geq (1-\lambda)f(x') + \lambda f(x'')$. Given a function $f : X \to \mathbb{R}$, its hypograph is the set of points (y, x) lying on or below the graph of the function: $\text{hyp}f = \{(y,x) \in \mathbb{R}^{n+1} | x \in X, y \leq f(x)\}$. Show that the function f is concave if and only if its hypograph is a convex set.

Proof: Assume a function $f : X \to \mathbb{R}$ is concave where X is a convex set in $\mathbb{R}^n$. To show that its hypograph is a convex set, we need to show that, for any $\lambda \in [0,1]$ and $(y', x'), (y'', x'') \in \text{hyp}f$, $\lambda(y', x') + (1-\lambda)(y'', x'') = (\lambda y' + (1-\lambda)y'', \lambda x' + (1-\lambda)x'') \in \text{hyp}f$. First, notice that since X is convex, $\lambda x' + (1-\lambda)x'' \in X$. Since f is concave, $f(\lambda x' + (1-\lambda)x'') \geq (1-\lambda)f(x') + \lambda f(x'') \geq (1-\lambda)y' + \lambda y''$. Thus, $\lambda(y', x') + (1-\lambda)(y'', x'') \in \text{hyp}f$.

Assume that the hypograph of a function $f : X \to \mathbb{R}$ is convex. Choose $(x', y'), (x'', y'') \in \text{hyp}f$. To show that X is convex, we need to show that, for any $\lambda \in [0,1]$, $f((1-\lambda)x' + \lambda x'') \geq (1-\lambda)f(x') + \lambda f(x'')$. Since hyp$f$ is convex, we know that $\lambda f(x') + (1-\lambda)f(x'') \leq \lambda y' + (1-\lambda)y'' \leq f((1-\lambda)x' + \lambda x'')$. Thus, f is concave. □

(5) Let X and Y be disjoint, closed, and convex sets in $\mathbb{R}^n$, one of which is compact. Show that there exists a hyperplane $H(p, \alpha)$ that strictly separates X and Y.

Proof: Let X and Y be disjoint, closed, and convex sets in $\mathbb{R}^n$ and X be compact. Let $Z := X - Y = \{z \in \mathbb{R}^n | z = x - y \text{ for some } x \in X, y \in Y\}$. The set Z is convex and $\bar{0} \notin Z$ because $X \cap Y = \emptyset$. By the theorem on slide 6 of the lecture 14 slides, there exists a hyperplane $H(p, \beta)$ that strictly separates Z and $\{\bar{0}\}$. Thus, for all $z \in Z, x \in X, y \in Y$,

$$p \cdot \bar{0} < p \cdot z \implies 0 < p \cdot (x - y) \implies p \cdot y < p \cdot x$$

Pick $x' \in X$. Since $p \cdot y$ is bounded from above by $p \cdot x'$ for all $y \in Y$, define $\beta' := \sup_{y \in Y}\{p \cdot y\} \in \mathbb{R}$. Define $f : X \to \mathbb{R}$ as $f(x) = p \cdot x$. Since f is continuous and X is compact, f attains its minimum on X, by the extreme value theorem. Define $\beta'' := \min_{x \in X}\{f(x)\} = \min_{x \in X}\{p \cdot x\} \in \mathbb{R}$. Since X and Y are disjoint, β' is strictly less than β''. By the denseness of rational numbers, there exists a $\beta^* \in \mathbb{Q}$ such that $\beta' < \beta^* < \beta''$. Therefore, for all $x \in X$ and $y \in Y$:

$$p \cdot y \leq \sup_{y \in Y} p \cdot y = \beta' < \beta^* < \beta'' = \min_{x \in X}\{p \cdot x\} \leq p \cdot x$$

Thus, X and Y are strictly separated by $H(p, \beta^*)$. □

(6) Call a vector $\pi \in \mathbb{R}^n$ a probability vector if $\sum_{i=1}^n \pi_i = 1$ and $\pi_i \geq 0$ for all $i = 1, ..., n$. Interpretation is that there are n states of the world and π_i is the probability that state i occurs. Suppose that Alice and Bob each have a set of probability distributions (Π_A and Π_B) which are nonempty, convex, and compact. They propose bids on each state of the world. A vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$, where x_i denotes the net transfer Alice receives from Bob in state i, is called a trade (Thus, $-x$ is the net transfer Bob receives in each state of the world.) A trade is agreeable if $\inf_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i x_i > 0$ and $\inf_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i(-x_i) > 0$. The above means that both Alice and Bob expect to strictly gain from the trade. Prove that there exists an agreeable trade iff there is no common prior (i.e., $\Pi_A \cap \Pi_B = \emptyset$).

Proof: Suppose that Alice and Bob each have a set of nonempty, convex, and compact probability distributions, Π_A and Π_B.

($\Rightarrow$) Assume x is an agreeable trade, so $\inf_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i x_i > 0$ and $\inf_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i(-x_i) > 0$. Thus, $\pi \cdot x > 0$ and $\pi' \cdot (-x) > 0$ for all $\pi \in \Pi_A$ and $\pi' \in \Pi_B$. Because $\pi' \cdot (-x) > 0 \implies \pi' \cdot x < 0$, we can construct a hyperplane $h(x, \bar{0})$ which strictly separates Π_A and Π_B. Assume for sake of a contradiction that $\pi'' \in \Pi_A \cap \Pi_B$, then $\pi'' \cdot x < 0$ and $\pi'' \cdot x > 0$. $\Rightarrow\Leftarrow$ Therefore, $\Pi_A \cap \Pi_B = \emptyset$.

($\Leftarrow$) Assume there are no common priors (i.e., $\Pi_A \cap \Pi_B = \emptyset$) or, in other words, Π_A and Π_B are disjoint. Notice that Π_A and Π_B are convex, positive subsets of the hyperplane $h(\vec{1}, 1)$ because $\sum_{i=1}^n \pi_i = \vec{1} \cdot \pi = 1$.[1] Thus, we can reduce dimensions and construct $\tilde{\Pi_A} = \{(\pi_1, ..., \pi_{n-1}) | \forall (\pi_1, ..., \pi_n) \in \Pi_A\}$ and $\tilde{\Pi_B} = \{(\pi_1, ..., \pi_{n-1}) | \forall (\pi_1, ..., \pi_n) \in \Pi_B\}$ because conditional on the first $n-1$ elements π_n must bring the sum up to one. For all $\pi \in \Pi_A \cup \Pi_B$:

$$\sum_{i=1}^n \pi_i = 1 \implies \pi_n = 1 - \sum_{i=1}^{n-1} \pi_i$$

Notice that $\tilde{\Pi_A}$ and $\tilde{\Pi_B}$ inherit closedness, convexity, compactness, and disjointness. By problem (5), there exists a hyperplane $h(p, \beta)$ in $n-1$ dimensions that strictly separates $\tilde{\Pi_A}$ and $\tilde{\Pi_B}$. Construct $h(x, \bar{0})$ as the hyperplane in n dimensions containing $\bar{0}$ and $h(p, \beta)$. Thus, $h(x, \bar{0})$ strictly separates Π_A and Π_B. For $\pi \in \Pi_A$ and $\pi' \in \Pi_B$:

$$x \cdot \pi > x \cdot 0 > x \cdot \pi' \implies x \cdot \pi > 0 \text{ and } (-x) \cdot \pi' > 0$$

We can define $f : \Pi_A \to \mathbb{R}$ as $f(\pi) = \pi \cdot x$ and $g : \Pi_B \to \mathbb{R}$ as $g(\pi) = \pi \cdot (-x)$. Since f and g are both continuous and Π_A and Π_B are both compact, f achieves its minimum on Π_A and g achieves its minimum on Π_B by the extreme value theorem:

$$\inf_{\pi \in \Pi_A} \pi \cdot x = \min_{\pi \in \Pi_A} \pi \cdot x = \min_{\pi \in \Pi_A} f(\pi) > 0 \text{ and } \inf_{\pi \in \Pi_B} \pi \cdot (-x) = \min_{\pi \in \Pi_B} \pi \cdot (-x) = \min_{\pi \in \Pi_B} g(\pi) > 0$$

Therefore, x is an agreeable trade. □

[1] $\vec{1}$ denotes an n-dimensional vector of ones.