ECON 712 - PS 6

Alex von Hafften*

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(Non-) Commitment in a black-box example with discrete choice set

Let there be a continuum of identical households in the economy, taking actions $\xi \in X$.[1] Let the economy wide average (the aggregate) of these actions be x. The benevolent government takes action $y \in Y$. The payoff to households is $u(\xi, x, y)$. Let the optimal choice of households, as a function of aggregates, be $f(x, y) := \arg\max_{\xi \in X} u(\xi, x, y)$.

In a competitive equilibrium, household action is consistent with the aggregate, i.e. $x = f(x, y)$. Now for each y, let $x = h(y)$ be such that $h(y) = f(h(y), y)$. That is, $(x = h(y), y)$ is a competitive equilibrium.

Let $X = \{x_H, x_L\}, Y = \{y_H, y_L\}$. For the one-period economy, with $\xi_i = x_i$, the payoffs $u(x_i, x_i, y_j)$ is given by the following table of one-period payoffs:

	x_L	x_H
y_L	12*	25
y_H	0	24*

Here the values $u(\xi_k, x_i, y_j)$ not reported are such that the outcomes with * are competitive equilibria. For example, $u(\xi_k, x_i, y_j) = -1$ for $k \neq i$ and $i = j$, and $u(\xi_k, x_i, y_j) = 30$ for $k \neq i$ and $i \neq j$.[2]

1. Find the Ramsey outcome, that is when the government has commitment/moves first. Find the outcome when the government cannot commit/moves second (in pure strategies). We will refer to this case as the Nash equilibrium in pure strategies (NE).

For **the Ramsey equilibrium**, the government moves first and the household moves second. Thus, employing backward induction to solve a subgame perfect equilibria, we first solve the household problem conditional on each government move. And then based on the household's move, we solve the government's problem.

Conditional on the government playing y_L, the household is faced by these payoffs:

Conditional on y_L	x_L	x_H
ξ_L	12	30
ξ_H	-1	25

Whether the aggregate households chose x_L or x_H, ξ_L is the dominant strategy for an individual household. By consistency, the aggregate households play x_L, so the payoff is 12.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

[1] These problems draw extensively from Ljungqvist and Sargent's Recursive Macroeconomic Theory.

[2] You should convince yourself that this is the case.

Conditional on the government playing y_H, the household is faced by these payoffs:

Conditional on y_H	x_L	x_H
ξ_L	0	-1
ξ_H	30	24

Whether the aggregate households chose x_L or x_H, ξ_H is the dominant strategy for an individual household. By consistency, the aggregate households play x_H, so the payoff is 24.

Now the government decides between playing y_L or y_H and chooses y_H, so the Ramsey equilibrium is (ξ_H, x_H, y_H) with a payoff of 24.

For **the Nash Equilibrium in pure strategies**, the household plays first and the government plays second. Thus, we can solve for the competitive equilibrium using backward induction. First we solve the government problem conditional on the possible household moves and then solve the household problem.

Conditional on the aggregated households playing x_L, the government will play y_L because 12 is larger than 0 (looking at original payoff table). Conditional on the aggregated households playing x_H, the government will play y_L because 25 is larger than 24.

We now know that the government will always play y_L, so the payoffs for an individual household look like the y_L table above. No matter what the aggregate households play, it's best for individual households to play ξ_L. By consistency, aggregate households play x_L. Thus, the NE is (ξ_L, x_L, y_L) with a payoff of 12.

2. Suppose the economy is repeated 5 times. Can the Ramsey outcome be supported in any period?

No, the Ramsey outcome cannot be supported in any period. Define (ξ_i, x_i, y_i) as the moves of the individual households, aggregate households, and governments in ith repetition. If the economy is repeated 5 times, we can think of repeated game as 10 subsequent moves starting with the household, then the government, then the household, then the government, etc. We can solve this game using backward induction.

Consider the fifth period. Notice that this repetition is equivalent to the single shot of NE from #1, so we know that the the government plays y_L and before which individual households play ξ_L and aggregate households play x_L.

Consider the fourth period. The government knows that in the next period the households will play x_L regardless of what it play in the current period, so there's no reason for the government to change what it would play in the one shot NE setup. Thus, the government plays y_L. Similarly, individual households play ξ_L and in aggregate play x_L.

Consider the third, second, and first periods. The logic is the same as in the fourth period. Thus, the Ramsey outcome cannot be supported in any period.

Now consider the expanded version of the previous economy. The payoffs $u(x_i, x_i, y_j)$ is given by the following the following table of one-period payoffs:

	x_{LL}	x_L	x_H
y_{LL}	2*	6	10
y_L	1	12*	25
y_H	-1	0	24*

3. What are the NEs? Suppose the economy is repeated 3 times, with agents discounting future utilities by $\beta = 0.9$. Can the Ramsey outcome be supported in any period?

In **the Ramsey equilibrium**, the government plays first and households play second. We can apply backward induction. Conditional that the government plays y_{LL}, the individual household cannot do better off than playing ξ_{LL} whatever is played by the aggregate household. So it plays ξ_{LL} and, by consistency, the aggregate household plays x_{LL}.

Conditional on y_{LL}	x_{LL}	x_L	x_H
ξ_{LL}	2	30	30
ξ_L	-1	6	30
ξ_H	-1	30	10

Conditional that the government plays y_L, the individual household cannot do better off than playing ξ_L whatever is played by the aggregate household. By consistency, the aggregate household plays x_L.

Conditional on y_L	x_{LL}	x_L	x_H
ξ_{LL}	1	-1	30
ξ_L	30	12	30
ξ_H	30	-1	25

Conditional that the government plays y_H, the individual household cannot do better than playing ξ_H regardless of the aggregate household's move. By consistency, the aggregate household plays x_H.

Conditional on y_H	x_{LL}	x_L	x_H
ξ_{LL}	-1	30	-1
ξ_L	30	0	-1
ξ_H	30	30	24

Now consider the government problem. If it plays y_{LL}, aggregate households play x_{LL} and the payoff is 2. If it plays y_L, aggregate households play x_L and the payoff is 12. If it plays y_H, aggregate households play x_H and the payoff is 24. Thus, the government plays y_H. Thus, the Ramsey equilibrium is (ξ_H, x_H, y_H).

In the **Nash equilibrium in pure strategy**, households play first and the government plays second. If aggregated households play x_{LL}, the government plays y_{LL} because $\max\{2, 1, -1\} = 2$. If aggregated households play x_L, the government plays y_L because $\max\{6, 12, 0\} = 12$. If aggregated households play x_H, the government plays y_L because $\max\{10, 25, 24\} = 25$.

Now let's consider the individual household problem. As we see in the table above, if the government plays y_{LL}, the payoffs of the individual household plays ξ_{LL} and, by consistency, the aggregate household plays x_{LL}. Thus, we have no commitment equilibrium at $(\xi_{LL}, x_{LL}, y_{LL})$. As we see from the other table above, if the government play y_L, the individual household cannot do better off than playing ξ_L. Thus, individual household plays ξ_L, and by consistency, the aggregate household plays x_L. So there is second no commitment equilibrium at (ξ_L, x_L, y_L).

Consider repeating the economy 3 times. The Ramsey equilibrium can be supported in the first two period. Consider the timeline of moves $x_H, y_H, x_H, y_H, x_L, y_L$ and the household adopting a grim trigger strategy that punishes the government with x_{LL} for the rest of the game if it deviates. This threat is credible because individual households do not have an incentive to deviate from playing ξ_{LL}.

Since the final period is the same as the one-shot, both the government choosing y_L and aggregate household choosing x_L as well as the government choosing y_{LL} and aggregate household choosing x_{LL}.

Consider the government's decision in the second period. Conditional that the government did not deviate in the first period, the aggregate household play x_H. If the government plays y_L, then in the final period aggregate households punish the government with x_{LL} and the government plays y_{LL}. The payoff associated with this outcome is $25+\beta 2 = 26.8$. If the government plays y_H, then the aggregate households play x_L in the final period and the government plays y_L. The payoff associated with this outcome is $25 + \beta 12 = 35.8$. Thus, the government does not deviate and plays y_H. The individual households, knowing that the government will not deviate, play ξ_H and thus aggregate households play x_H.

Consider the first period. Say the aggregate households play x_H. If the government deviates and plays y_L, the households punish the government with x_{LL} in the second and third periods. The payoff associated with this outcome is $25 + \beta 2 + \beta^2 2 = 28.42$. If the government does not deviate and plays y_H, the rest of the game proceeds as explained in the previous paragraph, so the payoff is $24 + \beta 24 + \beta^2 12 = 55.32$. Thus, the government does not deviate. The individual households, knowing that the government will not deviate, play ξ_H and thus aggregate households play x_H.

Static taxation

Let there be a unit measure of households with preferences over leisure, (private) consumption, and public goods (l, c, g), defined by the utility $u(l, c, g) = \ln l + \ln(\alpha + c) + \ln(\alpha + g), \alpha \in (0, 0.5)$. Each household is endowed with 1 unit of time, which can be spent on leisure or labor. Production is linear in labor, i.e. the economy resource constraint is $\bar{l} + g + \bar{c} = 1$ where $\bar{l}, \bar{c}$ are aggregate leisure and consumption. To provide the public good, the government can levy a flat proportional tax τ on labor. That is $g = \tau(1 - l)$.

1. Set up and solve the Planner's problem.

The Planner maximizes utility subject to resource feasibility and the government budget constraint with consistency ($\bar{l} = l$ and $\bar{c} = c$):

$$\begin{aligned} &\max_{l,c,g} \ln l + \ln(\alpha + c) + \ln(\alpha + g) \\ &\text{s.t. } \; l + g + c = 1, g = \tau(1 - l) \text{ and } \tau \in [0, 1] \end{aligned}$$

Substituting in $l = 1 - g - c$, we get

$$\begin{aligned} &\max_{c,g} \ln(1 - g - c) + \ln(\alpha + c) + \ln(\alpha + g) \\ &\text{s.t. } \; g = \tau(1 - l) \text{ and } \tau \in [0, 1] \end{aligned}$$

The FOC with respect to c implies:

$$0 = -\frac{1}{1 - g - c} + \frac{1}{\alpha + c} \implies 1 - g - c = \alpha + c \implies \frac{1 - g - \alpha}{2} = c$$

The FOC with respect to g implies:

$$0 = -\frac{1}{1 - g - c} + \frac{1}{\alpha + g} \implies \frac{1 - c - \alpha}{2} = g$$

Substituting to find c:

$$c = \frac{1 - (\frac{1-c-\alpha}{2}) - \alpha}{2} \implies c = \frac{1 - \alpha}{3} > 0$$

Finding g and l:

$$g = \frac{1 - \frac{1-\alpha}{3} - \alpha}{2} \implies g = \frac{1 - \alpha}{3} > 0 \text{ and } l = 1 - \frac{1 - \alpha}{3} - \frac{1 - \alpha}{3} \implies l = \frac{1 + 2\alpha}{3} > 0$$

Verifying these solution implies an appropriate τ:

$$\tau = \frac{g}{1 - l} = \frac{\frac{1-\alpha}{3}}{1 - \frac{1+2\alpha}{3}} = \frac{1}{2} \in [0, 1]$$

Thus, the solution to the Planner's problem is $(l, c, g) = \left(\frac{1+2\alpha}{3}, \frac{1-\alpha}{3}, \frac{1-\alpha}{3}\right)$.

2. Set up and solve for the Ramsey outcome.

In the Ramsey outcome, the government moves first and the household moves second. Specifically the government chooses τ and, since the individual household cannot change $\bar{l}$, g is also fixed. Thus, we first solve the individual household problem conditional on τ and g. The individual household's real income is $(1-l)$ and they pay $(1-l)\tau$ in taxes, so their discretionary real income is $(1-l)(1-\tau)$. Thus, the household problem is:

$$\begin{aligned}&\max_{c,l} \ln l + \ln(\alpha + c) + \ln(\alpha + g)\\ &\text{s.t. } c \leq (1-l)(1-\tau)\end{aligned}$$

Since the household's income is increasing in c, they will consume their entire discretionary income. We can substitute in $c = (1-l)(1-\tau)$:

$$\max_{l} \ln l + \ln(\alpha + (1-l)(1-\tau)) + \ln(\alpha + g)$$

Thus, the FOC with respect to l implies

$$\frac{1}{l} - \frac{1-\tau}{\alpha + (1-l)(1-\tau)} + 0 = 0 \implies \alpha + (1-\tau) - l(1-\tau) = l(1-\tau) \implies l = \frac{\alpha + (1-\tau)}{2(1-\tau)}$$

Thus, c is:

$$c = \left(1 - \left(\frac{\alpha + (1-\tau)}{2(1-\tau)}\right)\right)(1-\tau) = \frac{1-\tau-\alpha}{2}$$

By consistency, $\bar{l} = l = \frac{\alpha+(1-\tau)}{2(1-\tau)}$ and $\bar{c} = c = \frac{(1-\tau)-\alpha}{2}$. This implies that g equals:

$$g = \tau\left(1 - \frac{\alpha + (1-\tau)}{2(1-\tau)}\right) = \frac{\tau(1-\tau) - \tau\alpha}{2(1-\tau)}$$

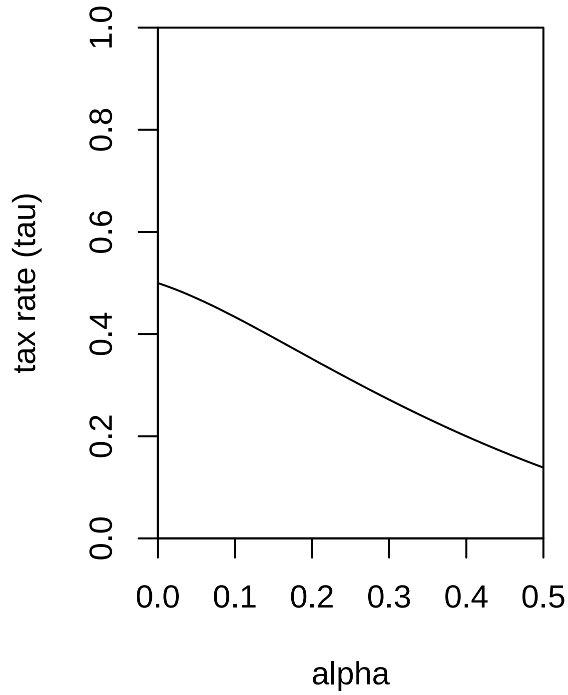
Now let us consider the government problem:

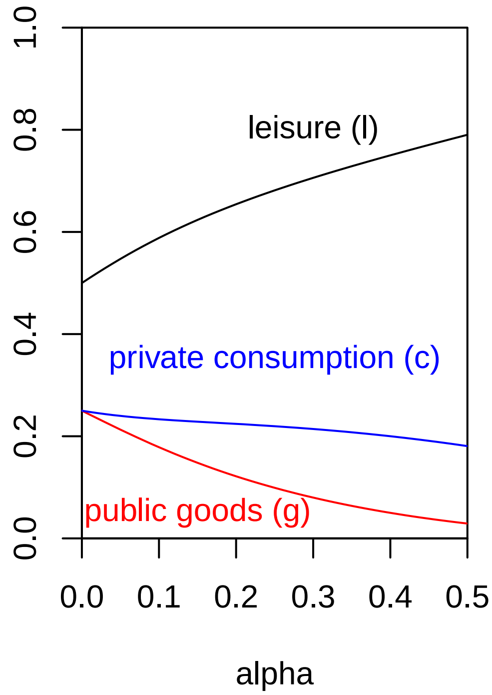
$$\begin{aligned}&\max_{\tau} \ln\left(\frac{\alpha + (1-\tau)}{2(1-\tau)}\right) + \ln\left(\alpha + \frac{(1-\tau)-\alpha}{2}\right) + \ln\left(\alpha + \frac{\tau(1-\tau) - \tau\alpha}{2(1-\tau)}\right)\\
&= \max_{\tau} \ln\left(\frac{\alpha - \tau + 1}{2 - 2\tau}\right) + \ln\left(\frac{1-\tau+\alpha}{2}\right) + \ln\left(\frac{2(1-\tau)\alpha + \tau(1-\tau) - \tau\alpha}{2-2\tau}\right)\\
&= \max_{\tau} \ln\Big(\alpha - \tau + 1\Big) - \ln\Big(2 - 2\tau\Big) + \ln\Big(1 - \tau + \alpha\Big) - \ln\Big(2\Big) + \ln\Big(2(1-\tau)\alpha + \tau(1-\tau) - \tau\alpha\Big) - \ln\Big(2 - 2\tau\Big)\\
&= \max_{\tau} \ln\Big(2(1-\tau)\alpha + \tau(1-\tau) - \tau\alpha\Big) + 2\ln\Big(\alpha - \tau + 1\Big) - 2\ln\Big(2 - 2\tau\Big) - \ln\Big(2\Big)\end{aligned}$$

The FOC with respect to τ implies

$$0 = \frac{1 - 3\alpha - 2\tau}{2(1-\tau)\alpha + \tau(1-\tau) - \tau\alpha} + \frac{4}{2-2\tau} - \frac{2}{\alpha - \tau + 1}$$

Numerically solving the FOC for τ for a grid of α between 0 and 0.5, we get the figure on the left. Plugging τ in for g, c, and l we get the figure on the right.





3. Set up and solve for the NE outcome.

For the NE, the household moves first and the government moves second. Thus, the government problem conditional on $\bar{l}$ and $\bar{c}$ is:

$$\max_{g,\tau} \ln \bar{l} + \ln(\alpha + \bar{c}) + \ln(\alpha + g)$$
$$\text{s.t. } g = (1 - \bar{l})\tau$$

We can substitute in $\bar{c} = 1 - g - \bar{l}$ and $g = (1 - l)\tau$:

$$\max_{\tau} \ln \bar{l} + \ln(\alpha + (1 - (1 - \bar{l})\tau - \bar{l})) + \ln(\alpha + (1 - \bar{l})\tau) \max_{\tau} \ln \bar{l} + \ln(\alpha + 1 - (1 - \bar{l})\tau - \bar{l}) + \ln(\alpha + (1 - \bar{l})\tau)$$

FOC with respect to τ imply

$$0 = \frac{1 - \bar{l}}{\alpha + (1 - \bar{l})\tau} - \frac{1 - \bar{l}}{\alpha + 1 - (1 - \bar{l})\tau - \bar{l}} \implies 2(1 - \bar{l})\tau = 1 - \bar{l} \implies \tau = \frac{1}{2}$$

Now, let us consider the individual household's problem. As mentioned in #2, the individual household's real discretionary income is $(1 - \tau)(1 - l)$ and the individual household cannot change $\bar{l}$. At $\tau = 1/2$, discretionary real income is $(1 - l)/2$. Thus, the household problem is:

$$\max_{c,l} \ln l + \ln(\alpha + c) + \ln(\alpha + g)$$
$$\text{s.t. } c \leq (1 - l)/2$$

Since the household's income is increasing in c, they will consume their entire discretionary income:

$$\max_{l} \ln l + \ln(\alpha + (1 - l)/2) + \ln(\alpha + g)$$

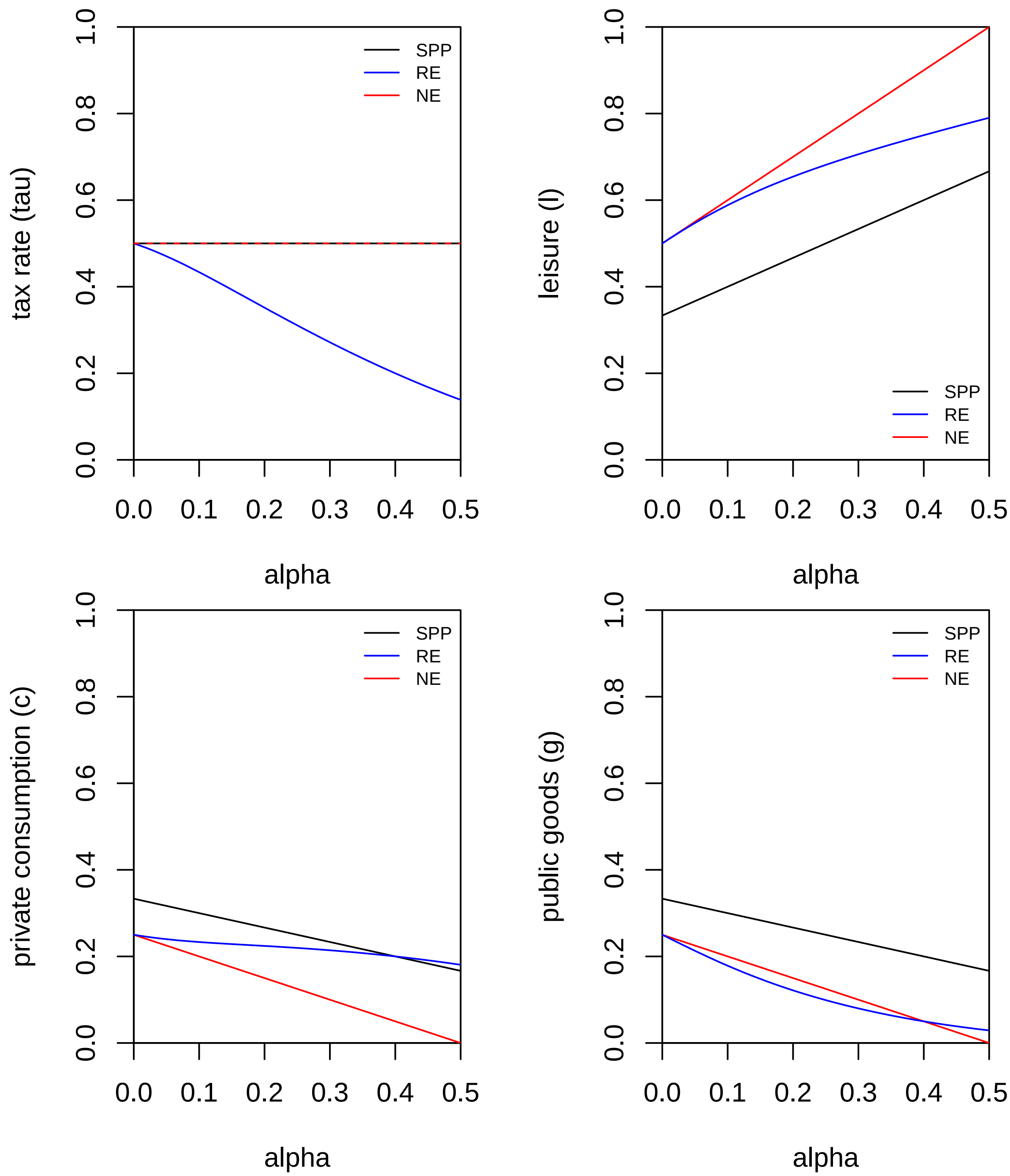
FOC with respect to l imply

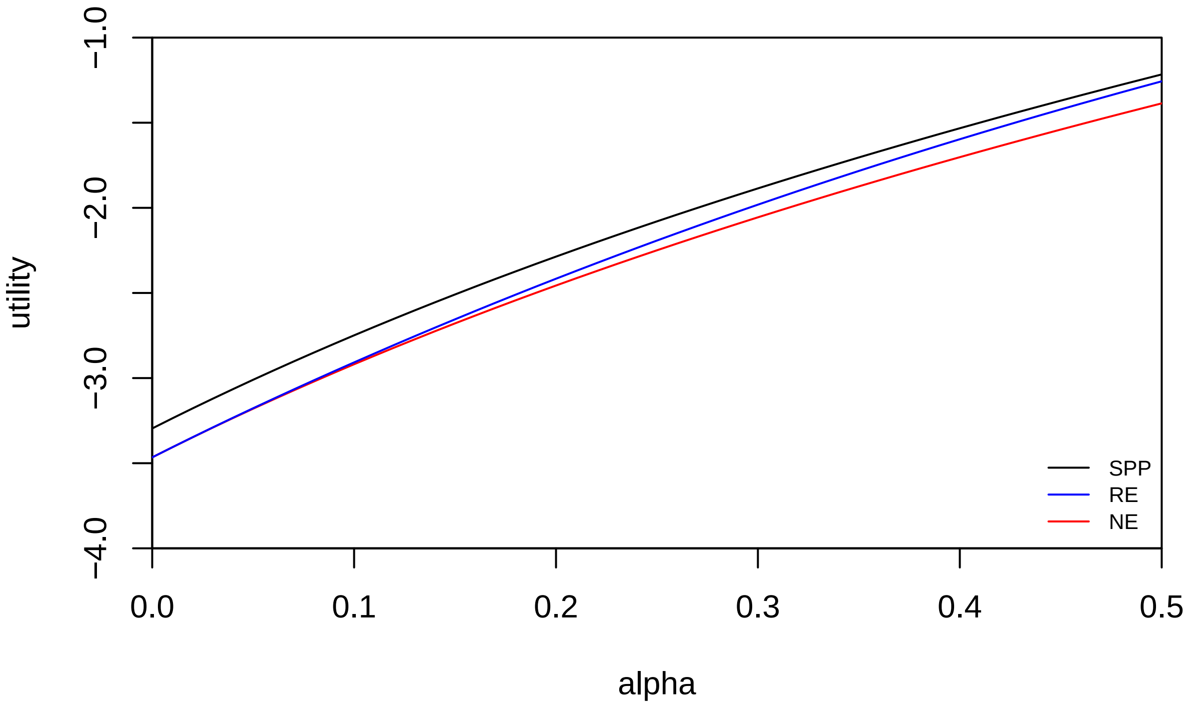
$$\frac{1}{l} + \frac{-1/2}{\alpha + (1 - l)/2} = 0 \implies 2\alpha + 1 - l = l \implies l = \alpha + 1/2$$

So c equals $c = (1 - (\alpha + 1/2))/2 = 1/4 - \alpha/2$. By consistency, $\bar{l} = l = \alpha + 1/2$ and $\bar{c} = c = 1/4 - \alpha/2$. So $g = (1 - \alpha + 1/2)/2 = 1/4 - \alpha/2$. Thus, the NE is $(l, c, g) = \Big(1/2 + \alpha, 1/4 - \alpha/2, 1/4 - \alpha/2\Big)$.

4. Comment on the differences between the above 3 outcomes, and the reason as to why they are different.

Below are plotted the variables of interest across the 3 outcomes. Black lines represent the solution to social planner's problem (SPP), blue lines represent the Ramsey equilibrium (RE), and red lines represent the Nash equilibrium (NE).





In the NE, the government taxes at the SPP level, so households work less to avoid taxes. Thus, the level of private consumption and public goods is lower. In the RE, the government incentives households to work more with a lower tax rate. Thus, household spend more time working. However, with the lower tax rate, there is less public goods (the RE and NE of public goods is roughly the same), so the major change is household consume more in RE compared to their consumption in NE. The household's higher private consumption in the RE means that their utility sits between the utility associated with the NE and the utility associated with the SPP. The difference between the RE and the NE increases as α increases, so the utility associated with the RE increases relative to the utility from the NE as α increases. The utility from the RE stays below the SPP because in the RE there's still an underprovision of public goods.

5. Suppose the economy is repeated for infinite periods, with discount factor $\beta < 1$. For high enough β, can the Ramsey outcome be sustained?

Yes. As seen in #4, the utility from the Ramsey outcome is higher than the utility from the Nash equilibrium at every α. If the household adopts a grim trigger strategy (play Ramsey outcome unless the government deviates and then plays Nash equilibrium for eternity) and if β is high enough, the difference in the discounted utilities would discourage the government from deviating.

If the government does not deviate, the utility is the discounted Ramsey equilibrium into eternity:

$$\sum_{i=0}^{\infty} \beta^i u(l_{RE}, c_{RE}, g_{RE}) = \frac{1}{1-\beta} u(l_{RE}, c_{RE}, g_{RE})$$

If the government deviates, the utility is one period of overtaxing the household's Ramsey labor decision (define $c'_{RE} = (1-\tau_{NE})(1-l_{RE})$ and $g'_{RE} = \tau_{NE}(1-l_{RE})$) and the Nash equilibrium for eternity:

$$u(l_{RE}, c'_{RE}, g'_{RE}) + \sum_{i=1}^{\infty} \beta^i u(l_{NE}, c_{NE}, g_{NE}) = u(l_{RE}, c'_{RE}, g'_{RE}) + \frac{\beta}{1-\beta} u(l_{NE}, c_{NE}, g_{NE})$$

For the government to not deviate, the following inequality needs to hold:

$$\begin{aligned} \frac{1}{1-\beta} u(l_{RE}, c_{RE}, g_{RE}) &> u(l_{RE}, c'_{RE}, g'_{RE}) + \frac{\beta}{1-\beta} u(l_{NE}, c_{NE}, g_{NE}) \\ \implies \beta &> \frac{u(l_{RE}, c'_{RE}, g'_{RE}) - u(l_{RE}, c_{RE}, g_{RE})}{u(l_{RE}, c'_{RE}, g'_{RE}) - u(l_{NE}, c_{NE}, g_{NE})} \end{aligned}$$

The figure shows values for β that can sustain the Ramsey outcome.