# ECON 703 - PS 4

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(1) Let $X, Y$ be two vector spaces such that dim $X = n$, dim $Y = m$. Construct a basis of $L(X, Y)$.

(2) Suppose that $T \in L(X, X)$ and $\lambda$ is $T$'s eigenvalue.

(a) Prove that $\lambda^k$ is an eigenvalue of $T^k$, $k \in \mathbb{N}$.

(b) Prove that if $T$ is invertible, then $\lambda^{-1}$ is an eigenvalue of $T^{-1}$.

(c) Define an operator $S : X \to X$, such that $S(x) = T(x) - \lambda x$ for all $x \in X$. Is $S$ linear? Prove that ker $:= \{x \in X | S(x) = \bar{0}\}$ is a vector space.

(3) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x - y, 2x + 3y)$. Let $W$ be the standard basis of $\mathbb{R}^2$ and let $V$ be another basis of $\mathbb{R}^2, V = \{(1, -4), (-2, 7)\}$ in the coordinates of $W$.

(a) Find $\text{mtx}_W(T)$.

(b) Find $\text{mtx}_V(T)$.

(c) Find $T(1, -2)$ in the basis $V$.

(4) In this exercise you will learn to solve first order linear difference equations in $n$ variables. We want to find an $n$-dimensional process $\{\mathbf{x}_1, \mathbf{x}_2, ...\}$ such that each $\mathbf{x}_i$ is an $n$-dimensional vector and

$$\mathbf{x}_t = A\mathbf{x}_{t-1}, t = 1, 2, ..., \tag{1}$$

where $A \in M_{n\times n}$ and $\mathbf{x}_0 \in \mathbb{R}^n$ are given. Then

$$\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0, \mathbf{x}_t = A^t\mathbf{x}_0 \forall t \in \mathbb{N},$$

where $A^t = A \cdot A \cdot ... \cdot A$ ($t$ times). Thus, we need to calculate $A^t$.
To do this, we diagonalize $A$, $A = PDP^{-1}$, where $D$ is diagonal, $D = diag\{\lambda_1, ..., \lambda_n\}$.
Hence we can rewrite

$$A^t = PDP^{-1}PDP^{-1}...PDP^{-1} = PD^tP^{-1} = Pdiag\{\lambda_1, ..., \lambda_n\}P^{-1},$$

which is now easy to compute. Thus, what you is

Step 1: Calculate $A$'s eigenvalues $\lambda_1, ..., \lambda_n$ and eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$.
Remember that we need to independent eigenvectors (this holds if all eigenvalues are distinct).

Step 2: Set $D = diag\{\lambda_1, ..., \lambda_n\}$ and $P = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ (eigenvectors are columns of $P$).

Step 3: Calculate $P^{-1}$ and $Pdiag\{\lambda_1^t, ..., \lambda_n^t\}P^{-1}$.

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Step 4: Plug $A^t$ from Step 3 to get $\mathbf{x}_t = A^t\mathbf{x}_0$.

Implement the above approach to solve for $\mathbf{x}_t \in \mathbb{R}^2$:

$$\mathbf{x}_t = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \mathbf{x}_{t-1}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Simplify your answer as much as possible.

(5) In this exercise you will learn to to solve $n$th order linear difference equations in one variable. We want to find a sequence of real numbers $\{z_t\}_{t=1}^{\infty}$, which satisfies

$$z_t = a_1 z_{t-1} + a_2 z_{t-2} + ... + a_n z_{t-n}, \tag{2}$$

where $a_1, ..., a_n \in \mathbb{R}$ and $z_0, z_{-1}, ..., z_{-n+1} \in \mathbb{R}$ are given.

(a) Define $\mathbf{x}_t := (z_t, z_{t-1}, ..., z_{t-n+1})'$ and rewrite Eq. (2) in the form of Eq. (1). What is $A$?

(b) Notice that if you find the function form of $z_t = f(t)$, then you do not need to find a similar form for $z_{t-1}, ..., z_{t-n+1}$ (you use the same function $f(\cdot)$ and evaluate it at a different time). Thus, you actually do not need to calculate $Pdiag\{\lambda_1^t, ..., \lambda_n^t\}P^{-1}\mathbf{x}_0$. You only need the first coordinate of that $n$-dimensional vector. The first coordinate takes the form

$$\mathbf{x}_{t1} \equiv z_t = c_1\lambda_1^t + c_2\lambda_2^t + ... + c_n\lambda_n^t, \tag{3}$$

where coefficient $c_1, ..., c_n$ depend on $P$ and $\mathbf{x}_0$.
Given Eq. (3) which holds for any $t$ and initial values $z_0, ..., z_{-n+1}$, which equations must $c_1, ..., c_n$ solve?

(c) Suppose that $n = 3$, $a_1 = 2$, $a_2 = 1$, $a_3 = -2$, and $z_0 = 2$, $z_{-1} = 2$, $z_{-2} = 1$. Find the expression for $a_t$ as a function of $t$.