ECON 899A - Problem Set 7

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1. *Derive the following asymptotic moments associated with $m_3(x)$: mean, variance, first order autocorrelation. Furthermore, compute $\nabla_b g(b_0)$. Which moments are informative for estimating b?*

With $|\rho_0| < 1$, the stochastic process $\{x_t\}$ is stationary, so $E[x_t] = E[x_{t-1}]$ and $Var[x_t] = Var[x_{t-1}]$:[1]

$$E[x_t] = E[\rho_0 x_{t-1} + \varepsilon_t] = \rho_0 E[x_{t-1}] = \rho_0 E[x_t]$$

$$\implies E[x_t] = 0$$

$$Var[x_t] = Var[\rho_0 x_{t-1} + \varepsilon_t] = \rho_0^2 Var[x_{t-1}] + \sigma_0^2 = \rho_0^2 Var[x_t] + \sigma_0^2$$

$$\implies Var[x_t] = \frac{\sigma_0^2}{1 - \rho_0^2}$$

$$Cov[x_t, x_{t-1}] = Cov[\rho_0 x_{t-1} + \varepsilon_t, x_{t-1}] = \rho_0 Cov[x_{t-1}, x_{t-1}] + Cov[\varepsilon_t, x_{t-1}] = \rho_0 Var[x_t] = \frac{\sigma_0^2 \rho_0}{1 - \rho_0^2}$$

*This problem set is for ECON 899A Computational Economics taught by Dean Corbae with assistance from Philip Coyle at UW-Madison. I worked on this problem set with a study group of Michael Nattinger, Sarah Bass, and Xinxin Hu.

[1] Alternatively, we can write x_t in terms of x_0, $\{\varepsilon_i\}_{i=1}^t$, and ρ_0:

$$\begin{aligned}
x_t &= \rho_0 x_{t-1} + \varepsilon_t \\
&= \rho_0(\rho_0 x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \rho_0^2 x_{t-2} + \varepsilon_t + \rho_0 \varepsilon_{t-1} \\
&= \rho_0^2(\rho_0 x_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + \rho_0 \varepsilon_{t-1} \\
&= \rho_0^3 x_{t-3} + \varepsilon_t + \rho_0 \varepsilon_{t-1} + \rho_0^2 \varepsilon_{t-2} \\
&= \rho_0^t x_0 + \sum_{i=1}^t \rho_0^{t-i} \varepsilon_i \\
&= \sum_{i=1}^t \rho_0^{t-i} \varepsilon_i
\end{aligned}$$

$$E[x_t] = E\left[\sum_{i=1}^t \rho_0^{t-i} \varepsilon_i\right] = \sum_{i=1}^t \rho_0^{t-i} E[\varepsilon_i] = 0$$

Thus, the asymptotic moments associated with $m_3(x)$ are:

$$\mu(x) = E[m_3(x)] = \begin{bmatrix} 0 \\ \frac{\sigma_0^2}{1-\rho_0^2} \\ \frac{\sigma_0^2\rho_0}{1-\rho_0^2} \end{bmatrix}$$

To calculate the Jacobian, we compute the derivative of the moment conditions with respect to each parameter:

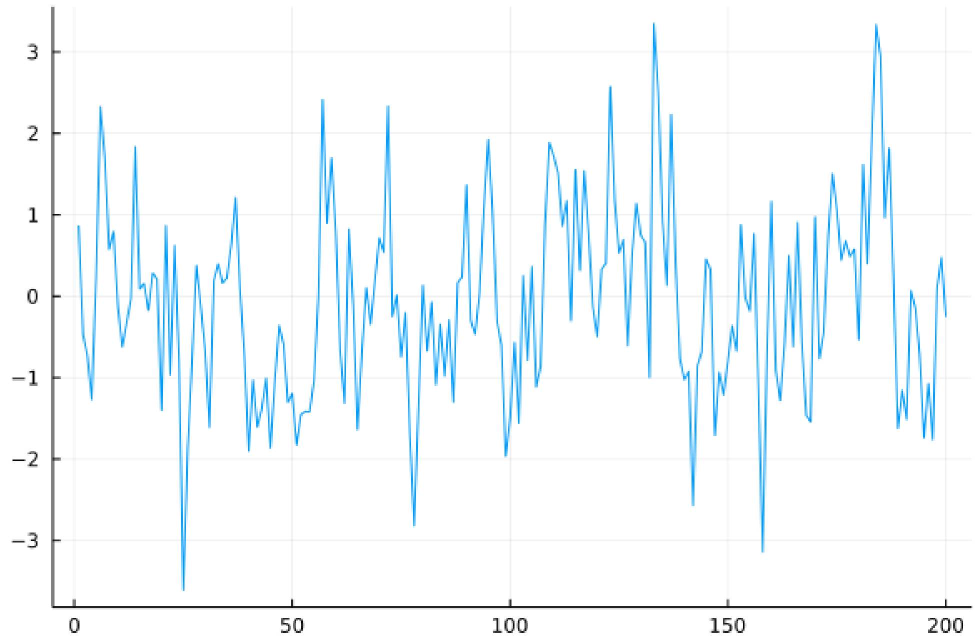
$$\begin{aligned}
\frac{\partial}{\partial\rho}\left(\frac{\sigma^2}{1-\rho^2}\right) &= \frac{\sigma^2(-2\rho) - (1-\rho^2)(0)}{(1-\rho^2)^2} \\
&= \frac{-2\rho\sigma^2}{(1-\rho^2)^2} \\
\frac{\partial}{\partial\sigma}\left(\frac{\sigma^2}{1-\rho^2}\right) &= \frac{2\sigma}{1-\rho^2} \\
\frac{\partial}{\partial\rho}\left(\frac{\sigma^2\rho}{1-\rho^2}\right) &= \frac{\sigma^2\rho(-2\rho) - (1-\rho^2)\sigma^2}{(1-\rho^2)^2} \\
&= \frac{-\sigma^2(1+\rho^2)}{(1-\rho^2)^2} \\
\frac{\partial}{\partial\sigma}\left(\frac{\sigma^2\rho}{1-\rho^2}\right) &= \frac{2\sigma\rho}{1-\rho^2}
\end{aligned}$$

Each cell of the Jacobian is the negative of the derivative of the moment condition:

$$\implies \nabla_b g(b_0) = \begin{bmatrix} 0 & 0 \\ \frac{2\rho_0\sigma_0^2}{(1-\rho_0^2)^2} & \frac{-2\sigma_0}{1-\rho_0^2} \\ \frac{\sigma_0^2(1+\rho_0^2)}{(1-\rho_0^2)^2} & \frac{-2\sigma_0\rho_0}{1-\rho_0^2} \end{bmatrix}$$

Variance and first order autocorrelation are informative for estimating b.

2. *Simulate a series of "true" data of length* $T = 200$ *using (1). We will use this to compute* $M_T(x)$.



$$M_T(x) = \begin{bmatrix} -0.0764 \\ 1.3953 \\ 0.6916 \end{bmatrix}$$

3. *Set* $H = 10$ *and simulate* H *vectors of length* $T = 200$ *random variables* e_t *from* $N(0, 1)$. *We will use this to compute* $M_{TH}(y(b))$. *Store these vectors. You will use the same vector of random variables throughout the entire exercise. Since this exercise requires you to estimate* σ^2, *you want to change the variance of* e_t *during the estimation. You can simply use* σe_t *when the variance is* σ^2.

4-6. *Different moments. Given what you found in part (1), do you think there will be a problem? In general we would not know whether this case would be a problem, so hopefully the standard error of the estimate of b as well as the J test will tell us something.*

(a) *Set* $W = I$ *and graph in three dimensions, the objective function (3) over* $\rho \in [0.35, 0.65]$ *and* $\sigma \in [0.8, 1.2]$. *Obtain an estimate of b by using* $W = I$ *in (4) using fminsearch. Report* $\hat{b}^1_{TH}$.

(b) *Set* $i(T) = 4$. *Obtain an estimate of* W^*. *Using* $\hat{W}^*{}_{TH} = \hat{S}^{-1}_{TH}$ *in (4), obtain an estimate of* $\hat{b}^2_{TH}$. *Report* $\hat{b}^2_{TH}$.

(c) *To obtain standard errors, compute numerically* $\nabla_b g_T(\hat{b}^2_{TH})$ *defined in (6). Report the values of* $\nabla_b g_T(\hat{b}^2_{TH})$. *Next, obtain the* $\ell \times \ell$ *variance-covariance matrix of* $\hat{b}^2_{TH}$ *as in (7). Finally, what are the standard errors defined in (8)? How can we use the information on* $\nabla_b g_T(\hat{b}^2_{TH})$ *to think about local identification?*

(d) *Since we are in the just identified case, the J test should be zero (on a computer this may be not be exact). However, given the identification issues in this particular case where we use mean and variance, the J test may not be zero. Compute the value of the J test:*

$$T\frac{H}{1+H} \times J_{TH}(\hat{b}^2_{TH}) \to \chi^2$$

noting that in this just identified case $n - \ell = 0$ *degrees of freedom recognizing that there really is not distribution.*

- From part (1), I found that in expectation the mean is zero, so the problem is that the mean is not informative for estimating b. Both the variance and the first order covariance were non-zero, so they are likely informative for estimating b. Thus, I think there will be a problem with (4), but not (5) and (6)
- I plot the objective function for both the identity weighting matrix and the estimated optimal weighting matrices for each case on the following pages.
- Table 1 shows the first stage estimates, first stage standard errors, second stage estimates, and second stage standard errors for ρ and σ as well as j-test statistics and p-values.

Table 1: Estimates and Standard Errors

	Mean and Variance	Variance and Covariance	Mean, Variance and Covariance
rho_hat_1	0.707	0.513	0.509
rho_se_1	0.464	0.055	0.055
sigma_hat_1	-0.832	1.015	1.018
sigma_se_1	0.557	0.055	0.054
rho_hat_2	-0.995	0.513	0.519
rho_se_2	8.693	0.185	0.183
sigma_hat_2	0.094	1.015	1.009
sigma_se_2	78.177	0.123	0.124
j_test_stat	0.036	0.000	0.184
j_test_p_value	0.151	0.000	0.332

- Using the mean and variance, the first-stage coefficients estimates are way off. This makes sense, in part 1, I argued that the mean doesn't contain relevant information for estimating the parameters. The first stage standard errors are similarly quite large (about ten times larger than the variance and auto covariance case).
- The second-stage coefficients don't really get much better and the standard errors actually get much larger.
- The j-test statistic is around 0.04, which corresponds to a p-value of around 0.15, so we cannot reject the hypothesis that the model is true.
- Using the mean and variance, the first-stage coefficients estimates are way off. This makes sense, in part 1, I argued that the mean doesn't contain relevant information for estimating the parameters. The first stage standard errors are similarly quite large (about ten times larger than the variance and auto covariance case).
- To obtain the standard errors, I computed the following Jacobians:

Table 2: Jacobian (Mean and Variance): First-Stage

0.258	-0.089
-3.975	3.317

Table 3: Jacobian (Variance and Covariance): First-Stage

-1.902	-2.738
-2.385	-1.359

Table 4: Jacobian (Mean, Variance, and Covariance): First-Stage

-0.109	-0.053
-1.872	-2.730
-2.361	-1.342

Table 5: Jacobian (Mean and Variance): Second-Stage

0.102	-0.016
159.659	-17.752

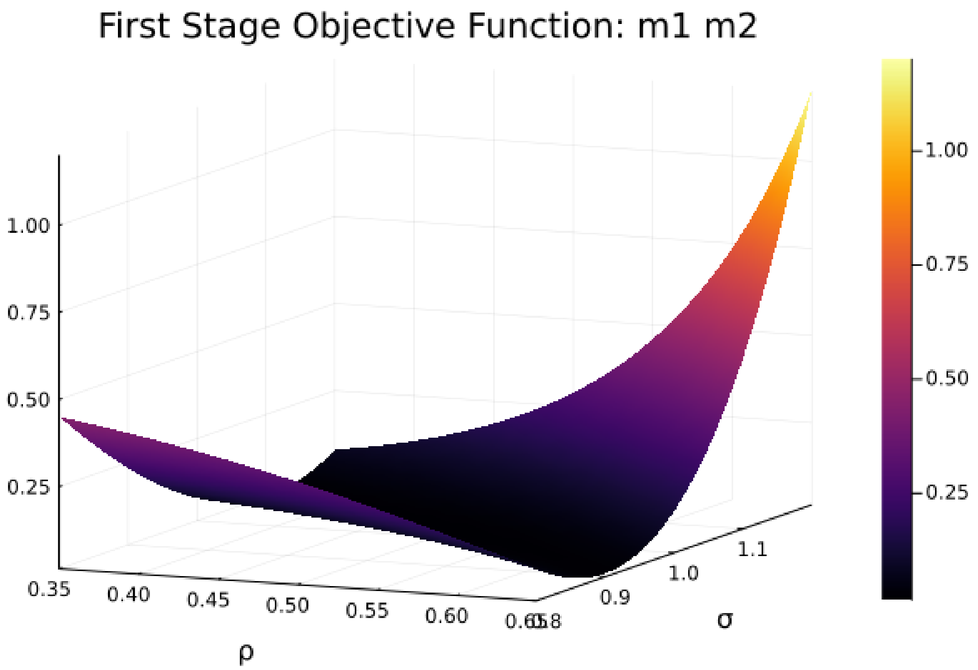
Table 6: Jacobian (Variance and Covariance): Second-Stage

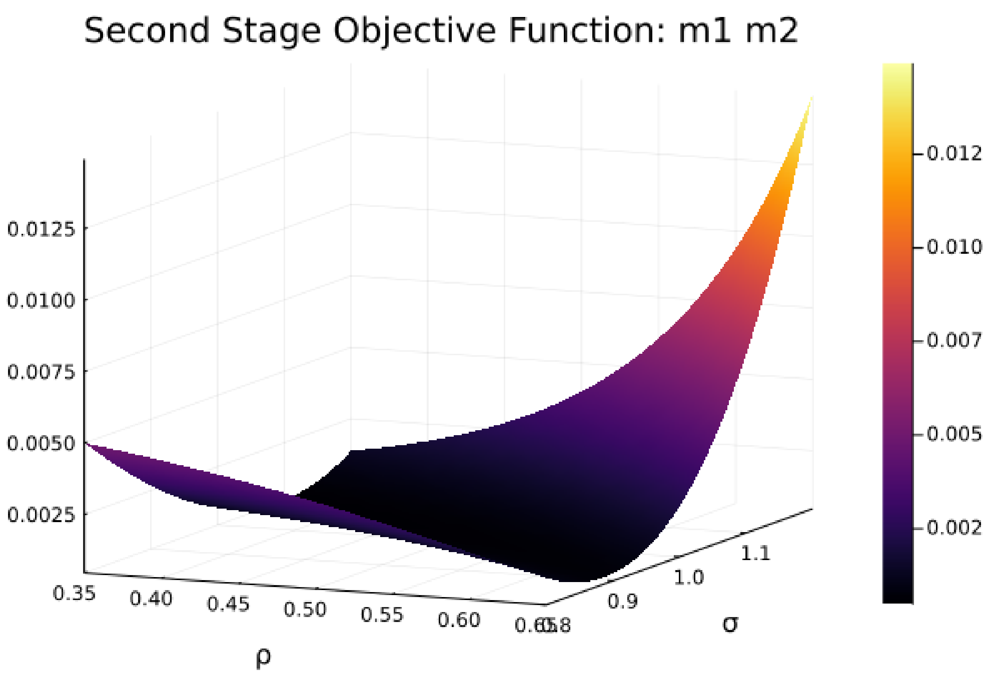
-1.901	-2.737
-2.384	-1.359

Table 7: Jacobian (Mean, Variance, and Covariance): Second-Stage

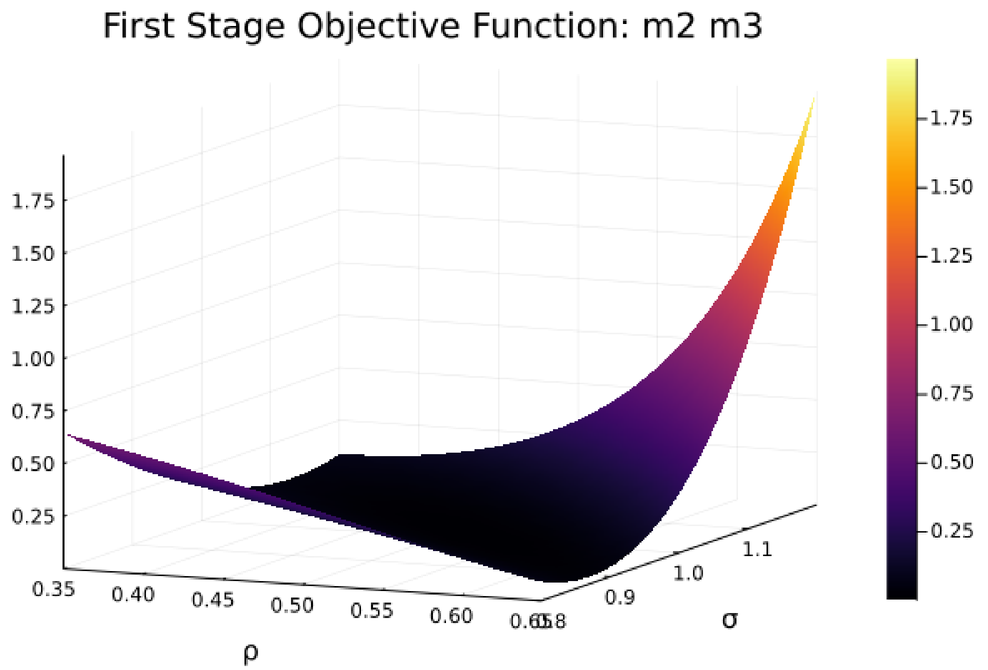
-0.112	-0.054
-1.931	-2.742
-2.405	-1.377

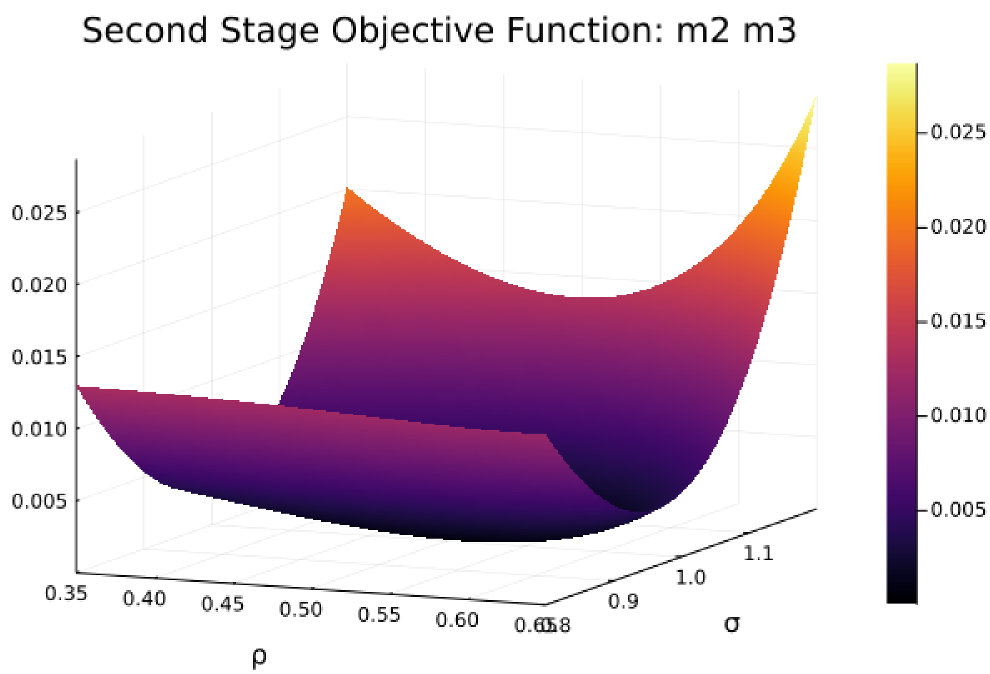
4. The just identified case where m_2 uses mean and variance.



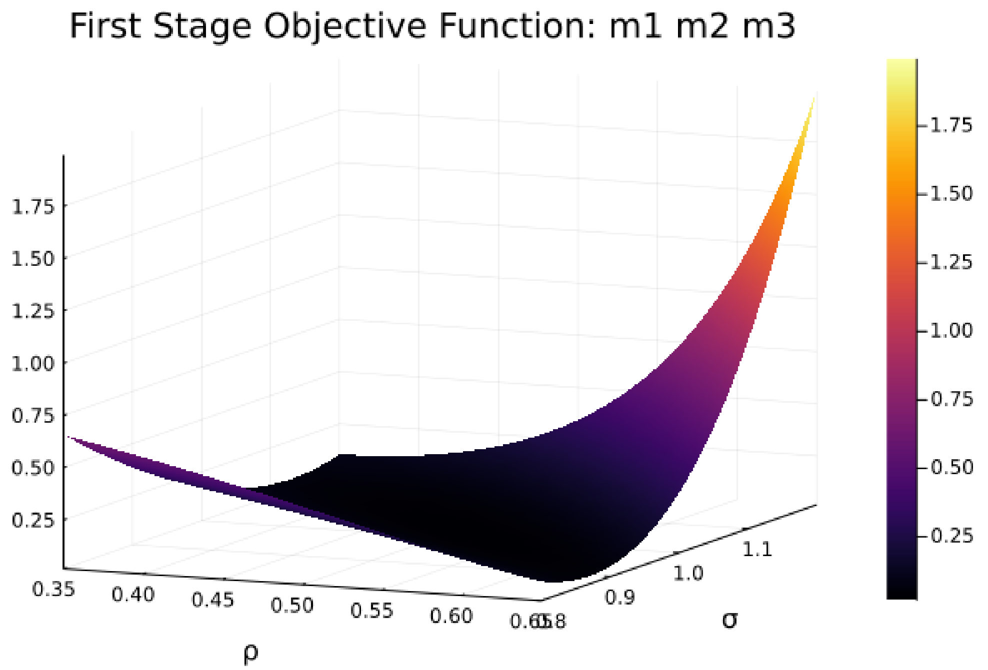


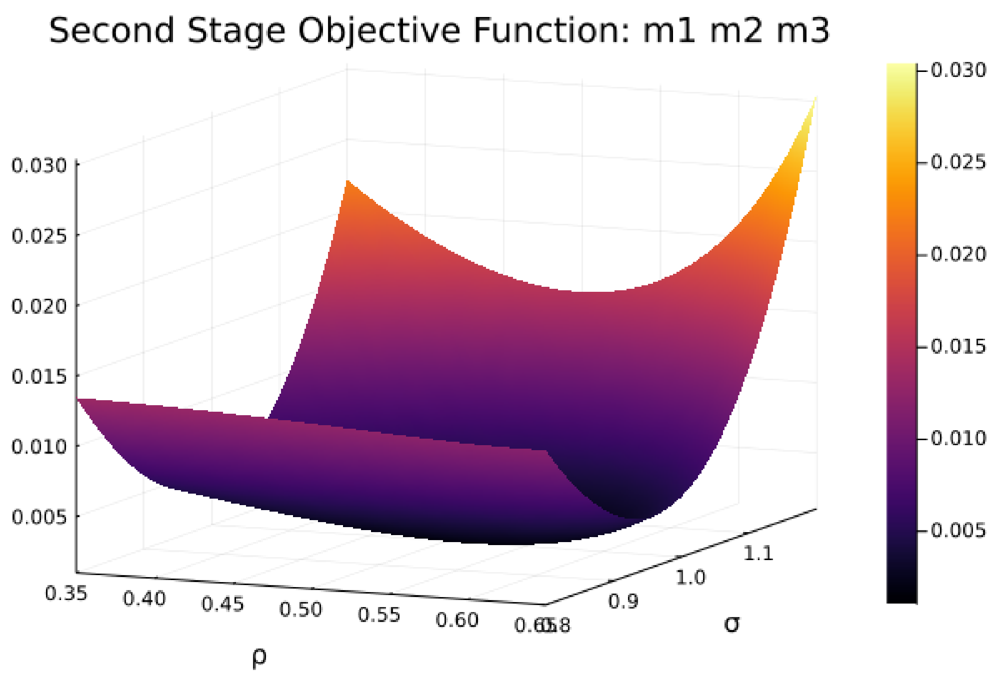
5. The just identified case where m_2 uses the variance and autocorrelation.



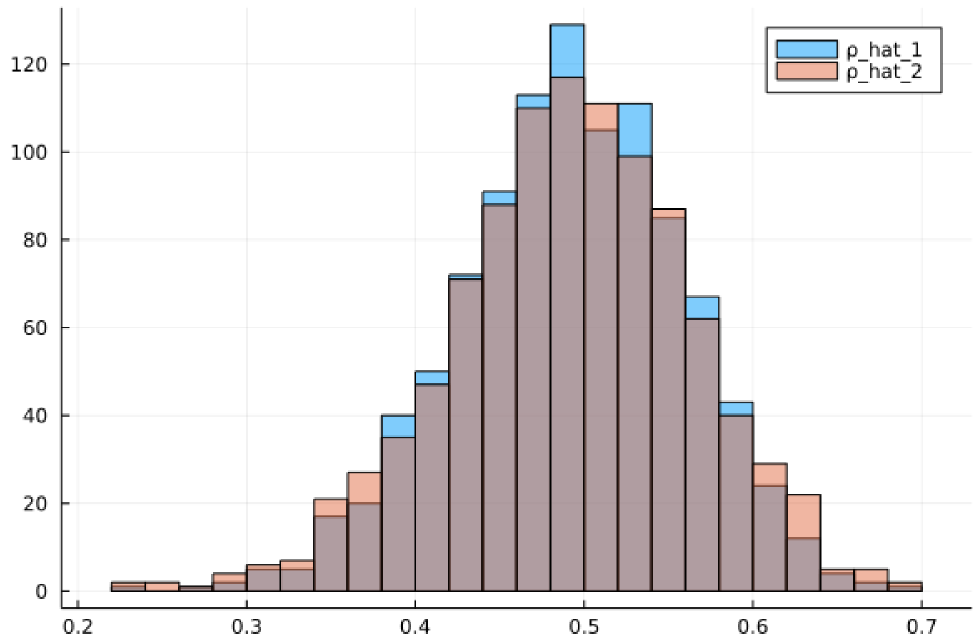


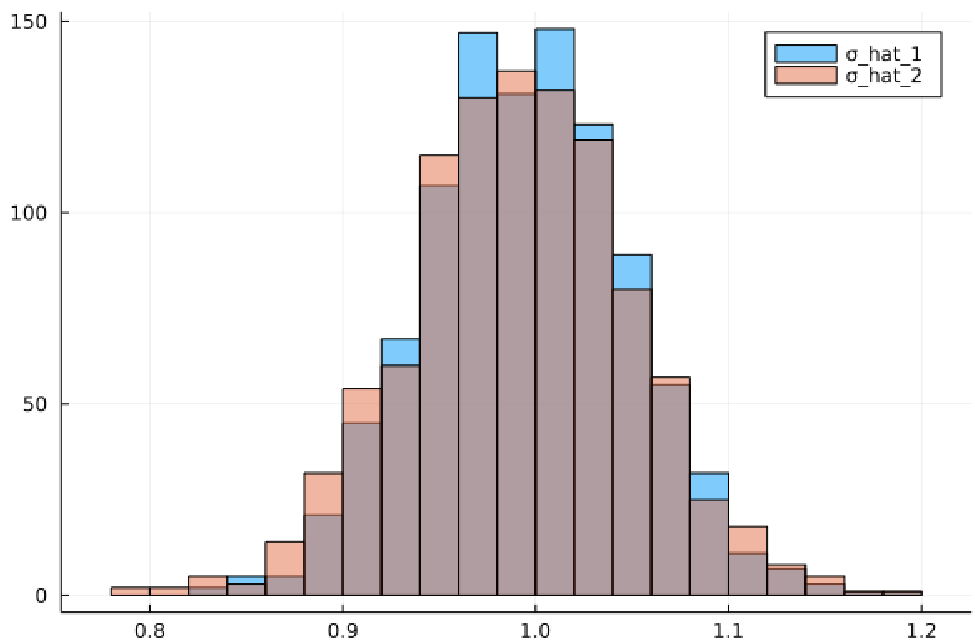
6. The overidentified case where m_3 uses the mean, variance and autocorrelation.





6. (e) Bootstrap the the finite sample distribution of the estimators by repeatedly drawing ε_t and e_t^h from $N(0,1)$ for $t = 1, ..., T$ and $h = 1, ..., H$.





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