ECON 709 - PS 1

Alex von Hafften*

9/14/2020

1. Suppose that $Y = X^3$ and $f_X(x) = 42x^5(1-x), x \in (0,1)$. Find the PDF of Y, and show that the PDF integrates to 1.

Notice that $Y = X^3$ is a monotone transformation, so we can use the following theorem from the lecture notes:

$$
\begin{aligned}
f_Y(y) &= \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|, y \in Y \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} 42(y^{1/3})^5(1-y^{1/3})|(1/3)y^{-2/3}|, y \in (0,1) \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} 14y(1-y^{1/3}), y \in (0,1) \\ 0, \text{ otherwise} \end{cases}
\end{aligned}
$$

where $g^{-1}(y) = y^{1/3}$ and $Y = \{0^3, 1^3\} = \{0, 1\}$.

$f_Y(y)$ integrates to 1:

$$
\begin{aligned}
\int_0^1 14t(1-t^{1/3})dt &= 14\left[y^2/2 - \frac{y^{7/3}}{7/3}\right]_0^1 \\
&= 14\left[\frac{1}{2} - \frac{3}{7}\right] \\
&= 1
\end{aligned}
$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

2. For the following CDF and PDF, show that f_X is the density function of F_X as long as $a \geq 0$. That is, show that for all $x \in [0,1], F_X(x) = \int_0^x f_X(t)dt$.

$$F_X(x) = \begin{cases} 1.2x, x \in [0, 0.5) \\ 0.2 + 0.8x, x \in [0.5, 1] \end{cases}$$

$$f_X(x) = \begin{cases} 1.2, x \in [0, 0.5) \\ a, x = 0.5 \\ 0.8, x \in (0.5, 1] \end{cases}$$

Case 1: $x < 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^x 1.2dt \\ &= 1.2x \\ &= F_X(x) \end{aligned}$$

Case 2: $x = 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} adt \\ &= 1.2(0.5) + 0 \\ &= 0.6 \\ &= 0.2 + 0.8(0.5) \\ &= F_X(0.5) \end{aligned}$$

Case 3: $x > 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} adt + \int_{0.5}^x 0.8dt \\ &= 1.2(0.5) + 0 + 0.8x - 0.8(0.5) \\ &= 0.6 + 0.8x - 0.4 \\ &= 0.2 + 0.8x \\ &= F_X(x) \end{aligned}$$

3.

4.

5.

6.