FIN 920: Continuous-Time Diffusion Models Notes

December 11, 2021

1 Part I

(Discrete) Random Walks

- Random walk: $z_t = z_{t-1} + e_t = z_0 + \sum_{s=1}^{t} e_s$ (often $z_0 = 0$) with $E[e_t] = 0, \forall t$ and $e_t \perp e_s, t \neq s$.
- Random walk with drift: $z_t = \mu + z_{t-1} + e_t$.
- Geometric random walk with drift: $\ln(z_t) = \mu + \ln(z_{t-1}) + e_t$ or $z_t = z_{t-1} \exp(\mu + e_t)$.
- Normally distributed increments $e_t \sim N(0, \sigma^2)$.

Standard Brownian Motion

- A Brownian motion is a process $\{z_t\}_{t \geq 0}$ such that
  - $P(z_0 = 0) = 1$
  - $z_t - z_2 \sim N(0, t - s), t > s \geq 0$
  - $\lim_{e \to 0} z_{t-e} = z_t, t \geq 0$
  - $z_t - z_s \perp z_u - z_v, t > s > u > v \geq 0$
- Brownian motion is Markov: $E[f(z_t)|\{z_v\}_{v=0}^{s}] = E[f(z_t)|z_s] =: E_s[f(z_t)]$ for $t \geq s$.
- Paths are nowhere differentiable: $\lim_{t \to s} \frac{z_t - z_s}{t - s}$ is not defined.
- Paths have unbounded total variation: $\sum_{v=1}^{N} |z_{tv/N} - z_{t(v-1)/N}| \to \infty$ as $N \to \infty$.
- Paths have bounded quadratic variation: $\sum_{v=1}^{N} (z_{tv/N} - z_{t(v-1)/N})^2 \to t$ as $N \to \infty$.
- Conventional expressions:
  - $z_t - z_0 = \sum_{v=1}^{N} z_{tv/N} - z_{t(v-1)/N} \to \int_{v=0}^{t} dz_v$ as $N \to \infty$ where $dz_t \sim N(0, dt)$.
  - Rules for the product of dz and dt:

$$\begin{bmatrix} & dz & dt \\ dz & dt & 0 \\ dt & 0 & 0 \end{bmatrix}$$

  - For example, $\sum_{v=1}^{N} (z_{Tv/N} - z_{T(v-1)/N})(T/N) \to \int_{t=0}^{T} dz_t dt = 0$ when $N \to \infty$.

Formal Construction of Brownian Motion

- Probability Space $(\Omega, \mathcal{F}, P)$ with set of states $\Omega = \{\omega\}$, tribe $\mathcal{F}$, probability measure $P : \mathcal{F} \to \mathbb{R}$.
- A Brownian motion is a measurable function $z(\omega, t) : \Omega \times [0, \infty) \to \mathbb{R}$, such that $\forall \omega \in \Omega$,
  - $z(\omega, 0) = 0$ almost surely,
  - $z(\omega, t) - z(\omega, s) \sim N(0, t - s)$ for $t > s$,
  - $z(\omega, t) - z(\omega, s) \perp z(\omega, u) - z(\omega, v), t > s > u > v \geq 0$
  - $\lim_{t \to s} z(\omega, t) = z(\omega, s)$
- The standard filtration $\{\mathcal{F}_t\}_{t \in [0,\infty)}$ is defined by the paths of the process together with the null sets of $\mathcal{F}$.

Scalar Diffusion Processes

- A diffusion (or Ito process) is an adapted process x_t with continuous paths,

$$
\begin{aligned}
x_t &= x_0 + \int_{v=0}^{t} \mu(x_v, v) dv + \int_{v=0}^{t} \sigma(x_v, v) dz_v \\
\iff dx_t &= \mu(x_t, t) dt + \sigma(x_t, t) dz_t
\end{aligned}
$$

  where $\mu(x_v, v)$ is a drift coefficient, $\sigma(x_v, v)$ is a diffusion coefficient, and z_t is a Brownian motion.
- The Ito integral is defined as

$$
\int_{v=0}^{t} \sigma(x_v, v) dz_v := \lim_{N \to \infty} \sum_{v=1}^{N} \sigma(x_{(v-1)t/N}, (v-1)t/N)(z_{tv/N} - z_{t(v-1)/N})
$$

- $E_t(dx_t) = E_t(\mu(x_t, t)dt + \sigma(x_t, t)dz_t) = \mu(x_t, t)dt + \sigma(x_t, t)E_t(dz_t) = \mu(x_t, t)dt = \mu_t dt$
- $E_t((dx_t)^2) = E_t(\mu(x_t, t)^2(dt)^2 + 2\mu(x_t, t)\sigma(x_t, t)dtdz_t + \sigma(x_t, t)^2(dz_t)^2) = E_t(\sigma(x_t, t)^2(dz_t)^2) = \sigma_t^2 dt$

Examples of Scalar Diffusion Processes

- Brownian motion with drift:
  - $Y_t = Y_0 + \mu t + \sigma z_t$
  - $dY_t = \mu dt + \sigma dz_t$
  - $Y_t - Y_s \sim N(\mu(t - s), \sigma^2(t - s))$ for $t > s$.
- Geometric Brownian Motion:
  - $dS_t = \mu S_t dt + \sigma S_t dz_t$, with constants μ, σ.
  - For example, stock price in Black and Scholes (JPE 1973).
- Ornstein-Uhlenbeck process (mean-reverting):
  - $dr_t = \kappa(\theta - r_t)dt + \sigma dz_t$ with constants $\kappa, \theta, \sigma > 0$.
  - Risk-free rate in Vasicek (JFE, 1977)
- Square root process (mean-reverting):
  - $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t} dz_t$.
  - Risk-free rate in Cox, Ingersoll and Ross (ECTA, 1985)

Vector Diffusion Processes

- A vector of Brownian motions $\mathbf{z}_t$ is independent iff $z_{it} - z_{is} \perp z_{ju} - z_{jv}$ for all $i \neq j$ and all intervals $[t, s]$ and $[u, v]$.
- A diffusion (or Ito process) is an adapted random vector process $\mathbf{x}_t$ with continuous paths,

$$\begin{aligned} \mathbf{x}_t &= \mathbf{x}_0 + \int_{v=0}^{t} \boldsymbol{\mu}(\mathbf{x}_v, v)dv + \int_{v=0}^{t} \boldsymbol{\sigma}(\mathbf{x}_v, v)d\mathbf{z}_v \\ \iff d\mathbf{x}_t &= \boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t \end{aligned}$$

where $\boldsymbol{\mu}(\mathbf{x}_t, t)$ is a vector of drift coefficients, $\boldsymbol{\sigma}(\mathbf{x}_t, t)$ is a diffusion coefficient, and $\mathbf{z}_t$ is a vector of independent Brownian motions.

- The Ito integral is defined as

$$\int_{v=0}^{t} \boldsymbol{\sigma}(\mathbf{x}_v, v)d\mathbf{z}_v := \lim_{N\to\infty} \sum_{v=1}^{N} \boldsymbol{\sigma}(\mathbf{x}_{(v-1)t/N}, (v-1)t/N)(\mathbf{z}_{tv/N} - \mathbf{z}_{t(v-1)/N})$$

$$\begin{aligned} E_t(d\mathbf{x}_t) &= E_t(\boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t) = \boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)E_t(d\mathbf{z}_t) = \boldsymbol{\mu}(\mathbf{x}_t, t)dt \\ E_t(d\mathbf{x}_t d\mathbf{x}^T) &= E_t((\boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t)(\boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t)^T) \\ &= E_t((dt)^2\boldsymbol{\mu}(\mathbf{x}_t, t)(\boldsymbol{\mu}(\mathbf{x}_t, t))^T + 2\boldsymbol{\mu}(\mathbf{x}_t, t)(dt d\mathbf{z}_t^T)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T + \boldsymbol{\sigma}(\mathbf{x}_t, t)(d\mathbf{z}_t d\mathbf{z}_t^T)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T) \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t)E_t(d\mathbf{z}_t d\mathbf{z}_t^T)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t)(dt \times I)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T dt \end{aligned}$$

Examples of Vector Diffusion Processes

- Two Brownian motions with drift and correlation $\rho \in [-1, 1]$.

$$d\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} dt + \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}$$

- Multiperiod consumption-savings-portfolio choice in Merton (various 1970s)

$$\begin{aligned} dW_t &= W_t(\boldsymbol{\alpha}_t \cdot (\boldsymbol{\mu}(\mathbf{X}_t, t) - r(\mathbf{X}_t, t)\mathbf{1}) + r(\mathbf{X}_t, t))dt + W_t\boldsymbol{\alpha}_t^T\boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{z}_t - c_t dt + y_t dt \\ d\mathbf{X}_t &= \boldsymbol{\mu}_x(\mathbf{X}_t, t)dt + \sigma_x(\mathbf{X}_t, t)d\mathbf{z}_t \end{aligned}$$

where $W_t \geq 0$ and W_0 and $\mathbf{X}_0$ is given.

- Constant returns-to-scale production and productivity in Ai (JF 2010).

$$\begin{aligned} dK_t &= x_t K_t dt - C_t dt + \sigma_c K_t dz_t^c \\ dx_t &= \kappa(\mu - x_t)dt + \sigma_x dz_t^x \\ dz_t^c dz_t^x &= \rho dt \end{aligned}$$

Convenient Facts

- For an adapted process γ_t (vector), we can express some functions of the expectation of the adapted process in terms of a change in time instead of a change in the Brownian motion value.
- For example, expectation of quadratic:

$$
\begin{aligned}
E_t\left(\left(\int_t^T \gamma_s d\mathbf{z}_s\right)^2\right) &= E_t \lim_{N\to\infty} \sum_{i=1}^N \sum_{j=1}^N K_i K_j \\
&= E_t \lim_{N\to\infty} 2\sum_{i=1}^N \sum_{i<j}^N K_j E_{(T-t)(i-1)/N} K_i + \sum_{j=1}^N E_{(T-t)(j-1)/N} K_j^2 \\
&= E_t \lim_{N\to\infty} \frac{T-t}{N} \sum_{j=1}^N \gamma_{(T-t)(j-1)/N} \cdot \gamma_{(T-t)(j-1)/N} \\
&= \int_t^T E_t(\gamma_s \cdot \gamma_s) ds \qquad (1)
\end{aligned}
$$

where $K_j = \gamma_{T(j-1)/N} \cdot (\mathbf{z}_{(T-t)j/N} - \mathbf{z}_{(T-t)(j-1)/N})$.

- For example, expectation of exponential:

$$
E_t\left(\exp\left(\int_t^T \gamma_s d\mathbf{z}_s\right)\right) = E_t\left(\exp\left(\frac{1}{2}\int_t^T (\gamma_s \cdot \gamma_s) ds\right)\right) \qquad (2)
$$

- Consider the square-root process,

$$
\begin{aligned}
dr_t &= \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz_t \\
\implies e^{\kappa t} dr_t &= e^{\kappa t}\kappa\theta dt - e^{\kappa t}\kappa r_t dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\
\implies e^{\kappa t} dr_t + e^{\kappa t}\kappa r_t dt &= e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\
\implies d(e^{\kappa t} r_t) &= e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\
\implies \int_{-\infty}^t d(e^{\kappa s} r_s) &= \int_{-\infty}^t e^{\kappa s}\kappa\theta ds + \int_{-\infty}^t e^{\kappa s}\sigma\sqrt{r_s}dz_s \\
\implies e^{\kappa t} r_t &= e^{\kappa t}\theta + \sigma\int_{-\infty}^t e^{\kappa s}\sqrt{r_s}dz_s \\
\implies r_t &= \theta + \sigma\int_{-\infty}^t e^{\kappa(s-t)}\sqrt{r_s}dz_s
\end{aligned}
$$

- Using (1), we can find the unconditional variance (based on the unconditional expectation):

$$
\begin{aligned}
\implies E[r_t] &= \theta + E\left[\sigma \int_{-\infty}^{t} e^{\kappa(s-t)} \sqrt{r_s} dz_s\right] \\
&= \theta \\
\implies Var(r_t) &= E[r_t^2] - E[r_t]^2 \\
&= E\left[\left(\theta + \sigma \int_{-\infty}^{t} e^{\kappa(s-t)} \sqrt{r_s} dz_s\right)^2\right] - \theta^2 \\
&= E\left[2\theta\sigma \int_{-\infty}^{t} e^{\kappa(s-t)} \sqrt{r_s} dz_s\right] + E\left[\left(\sigma \int_{-\infty}^{t} e^{\kappa(s-t)} \sqrt{r_s} dz_s\right)^2\right] \\
&= \sigma^2 \int_{-\infty}^{t} e^{2\kappa(s-t)} E[r_s] dz_s \\
&= \sigma^2 \theta e^{-2\kappa t} \int_{-\infty}^{t} e^{2\kappa s} dz_s \\
&= \sigma^2 \theta e^{-2\kappa t} \left[\frac{1}{2\kappa} e^{2\kappa s}\right]_{-\infty}^{t} \\
&= \frac{\sigma^2 \theta}{2\kappa}
\end{aligned}
$$

Black and Scholes Structure

- Stock with price S_t: $dS_t = \mu S_t dt + \sigma S_t dz_t, \mu > 0, \sigma > 0$.
- Risk-free bond: $dB_t = B_t r dt, \mu > r > 0$.
- Option with strike price k: At the exercise date T, the payoff is $C(S_T, T) = \max\{0, S_t - K\}$
- Assumptions:
  - No dividend payments on stock.
  - Infinite depth in the stock and bond markets.
  - Constant drift and volatility in the stock return.
  - Constant rate of interest.
  - Frictionless markets (i.e. no transaction costs).
  - European call option (i.e. can only exercise at maturity date T).
- Goal is to find equation for $C(S_t, t), t < T$.

Future Values

- To get $d\ln B_t$ use Ito's lemma [where $\mu(B_t,t) = B_t r$, $\sigma = 0$ and $f(x) = \ln x \implies f_x(x) = \frac{1}{x}, f_{xx}(x) = \frac{-1}{x^2}, f_t(x) = 0$]:

$$
\begin{aligned}
d\ln B_t &= \frac{1}{B_t}(0)dz_t + \frac{1}{B_t}B_t r dt + \frac{1}{2}\frac{-1}{x^2}(0)^2 dt + 0dt \\
&= rdt \\
\implies \int_0^t d\ln B_s &= \int_0^t r ds \\
\implies \ln B_t - \ln B_0 &= r(t-0) \\
\implies B_t &= B_0 \exp(rt)
\end{aligned}
$$

- To get $d\ln S_t$ use Ito's lemma [where $\mu(S_t,t) = \mu S_t$, $\sigma(S_t,t) = \sigma S_t$, and $f(x) = \ln x \implies f_x(x) = \frac{1}{x}, f_{xx}(x) = \frac{-1}{x^2}, f_t(x) = 0$]:

$$
\begin{aligned}
d\ln S_t &= \frac{1}{S_t}\mu S_t dt + \frac{1}{S_t}\sigma S_t dz_t + \frac{1}{2}\frac{-1}{S_t^2}(\sigma S_t)^2 dt + (0)dt \\
&= \mu dt + \sigma dz_t - \frac{1}{2}\sigma^2 dt \\
\implies \int_0^t d\ln S_s &= \mu \int_0^t ds + \sigma \int_0^t dz_s - \frac{1}{2}\sigma^2 \int_0^t dt \\
\implies \ln S_t - \ln S_0 &= \mu t + \sigma z_t - \frac{1}{2}\sigma^2 t \\
\implies S_t &= S_0 \exp(\mu t + \sigma z_t - \frac{1}{2}\sigma^2 t)
\end{aligned}
$$

where $z_0 \equiv 0$.

$$
\begin{aligned}
E[\ln S_t | \ln S_0] &= E[\ln S_0 + \mu t + \sigma z_t - \frac{1}{2}\sigma^2 t | \ln S_0] \\
&= \ln S_0 + \mu t + \sigma E[z_t | \ln S_0] - \frac{1}{2}\sigma^2 t \\
&= \ln S_0 + \mu t - \frac{1}{2}\sigma^2 t
\end{aligned}
$$

Using (2),

$$
\begin{aligned}
E[S_t | S_0] &= E[S_0 \exp(\mu t + \sigma z_t - \frac{1}{2}\sigma^2 t)] \\
&= S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t) E\left[\exp\left(\int_{v=0}^t \sigma dz_v\right)\right] \\
&= S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t) E\left[\exp\left(\frac{1}{2}\int_{v=0}^t \sigma^2 dv\right)\right] \\
&= S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t) \exp\left(\frac{1}{2}\sigma^2 t\right) \\
&= S_0 \exp(\mu t)
\end{aligned}
$$

Ito's Lemma (Scalar)

- Let $f(x,t)$ be twice differentiable in x and once in t. Let x be a (scalar) diffusion with $dx_t = \mu(x_t,t)dt + \sigma(x_t,t)dz_t$, then

$$\begin{aligned} f(x_t,t) - f(x_0,0) &= \int_{s=0}^{t} f_x(x_s,s)dx_s + \frac{1}{2}\int_{s=0}^{t} f_{xx}(x_s,s)\sigma(x_s,s)^2 ds + \int_{s=0}^{t} f_t(x_s,s)ds \\ df &= f_x\mu dt + f_x\sigma dz_t + \frac{1}{2} f_{xx}\sigma^2 dt + f_t dt \end{aligned}$$

where $f_x = \frac{\partial f(x,s)}{\partial x}$ (f_{xx} and f_t similar).

- Examples
  - Consider $d(z_t^2)$. Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t,t) &= 0, \sigma(x_t,t) = 1 \; \forall x_t, t \\ \implies dx_t &= 0 * dt + 1 * dz_t = dz_t \\ f(x,t) &= x^2 \\ \implies f_x(x,t) &= 2x, f_{xx} = 2, f_t = 0 \end{aligned}$$

$$\implies d(z_t^2) = df = (2z_t)(1)dz_t + (2z_t)(0)dt + \frac{1}{2}(2)(1)^2 dt + (0)dt = 2z_t dz_t + dt$$

  - Consider $d\exp(z_t)$. Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t,t) &= 0, \sigma(x_t,t) = 1 \; \forall x_t, t \\ \implies dx_t &= 0 * dt + 1 * dz_t = dz_t \\ f(x,t) &= \exp(x) \\ \implies f_x(x,t) &= \exp(x), f_{xx} = \exp(x), f_t = 0 \end{aligned}$$

$$\begin{aligned} \implies d\exp(z_t) = df &= \exp(z_t)(0)dt + \exp(z_t)(1)dz_t + \frac{1}{2}\exp(z_t)(1)^2 dt + (0)dt \\ &= \exp(z_t)dz_t + \frac{1}{2}\exp(z_t)dt \end{aligned}$$

  - Consider $dx_t = \mu dt + \sigma dz_t$ and $d\exp(x_t)$. Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t,t) &= \mu, \sigma(x_t,t) = \sigma \; \forall x_t, t \\ \implies dx_t &= \mu dt + \sigma dz_t \\ f(x,t) &= \exp(x) \\ \implies f_x(x,t) &= \exp(x), f_{xx} = \exp(x), f_t = 0 \end{aligned}$$

$$\begin{aligned} \implies d\exp(z_t) = df &= \exp(x_t)\mu dt + \exp(z_t)\sigma dz_t + \frac{1}{2}\exp(z_t)\sigma^2 dt + (0)dt \\ &= \exp(x_t)\mu dt + \exp(z_t)\sigma dz_t + \frac{1}{2}\exp(z_t)\sigma^2 dt \end{aligned}$$

Second term in Ito's Lemma

No Instantaneous Arbitrage

Back-Scholes Call Option Price

Ito's Lemma (Vector)

Ito's Lemma Examples

Application of the Martingale Property

Feynman-Kac I

Black-Scholes and Feynman-Kac

Feynman-Kac II

2 Part II

- To do

3 Part III

- To do

4 Part IV

- To do