ECON 712B - Lecture Notes

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Lecture 1 - HH Optimization over Finite Time Horizon

- Consumption-savings problem for $T < \infty$.

Preferences

$$\sum_{t=1}^{T} \beta^t u(c_t) \text{ where } 0 < \beta < 1$$

- Assumptions about preferences: u is strictly increasing, strictly concave, and continuously differentiable (C^2).
- We also often assume that utility is bounded ($|u(c)| < K \forall c$) and Inada conditions hold ($\lim_{c \to 0} u'(c) = +\infty$ and $\lim_{c \to +\infty} u'(c) = 0$).
- Think log utility.

Constraints

- R is gross interest rate ($R = 1 + r$ where r is net interest rate).
- $\{y_t\}_{t=0}^T$ is labor income.
- x_0 is initial wealth.
- Intertemporal budget constraint:

$$\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t c_t \le \sum_{t=0}^{T} \left(\frac{1}{R}\right)^t y_t + x_0$$

- s_t is savings.
- Flow budget constraint implies law of motion for wealth:

$$s_t = x_t - c_t + y_t \implies x_{t+1} = Rs_t = R(x_t - c_t + y_t)$$

- Intertemporal budget constraint embeds assumption that HH can borrow any amount at rate r as long as budget constraint is satisfied.

HH Problem

$$\max_{\{c_t\}_{t=0}^T} \sum_{t=1}^{T} \beta^t u(c_t) \text{ s.t. } \sum_{t=0}^{T} \left(\frac{1}{R}\right)^t c_t \le \sum_{t=0}^{T} \left(\frac{1}{R}\right)^t y_t + x_0$$

- Where $x_0 \ge 0$ and $y_t \ge 0 \forall t$.
- Since $u(c_t)$ continuous, $U_T(c) = \sum_{t=1}^T \beta^t u(c_t)$ is continuous on $c = \{c_0, ..., c_t\}$.

- Given $\{y_t\}_{t=0}^T$, the feasible set (below) is a compact subset of $\mathbb{R}^T$ by Heine-Borel.

$$0 \leq \sum_{t=0}^{T} \left(\frac{1}{R}\right)^t c_t \leq \sum_{t=0}^{T} \left(\frac{1}{R}\right)^t y_t + x_0$$

- Existence: Close, bounded subset of $\mathbb{R}_+^T$ are compact $\implies \exists c^*$ that maximizes $U(c)$ subject to budget constraint (Weinstrass).
- Uniqueness: Since the feasible set is convex and $U(c)$ is strictly concave (because $u(c^*)$ is strictly concave), $\exists$ unique c^* that maximizes $U(c)$ subject to budget constraint.
- The Lagrangian is:

$$\mathcal{L} = \sum_{t=1}^{T} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t (y_t - c_t) + x_0 \right]$$

Recursive Formulation

- We use backward induction.
- At T, choose x_T given $\max u(c_T)$ s.t. $c_T \leq x_T + y_T$. $x_{T+1} = 0 \implies c_T = x_T + y_T$. So, $V_T(x) = u(x + y_T)$.
- At $T-1$, choose x_{T_1}. From law of motion, $x_T = Rs_{T-1} = R(x_{T-1} - c_{T-1} + y_{T-1})$.

$$\max_{c_{T-1}} \left\{ u(c_{T-1}) + \beta u(R(x_{T-1} - c_{T-1} + y_{T-1}) + y_T) \right\} \text{ s.t. } 0 \leq c_{T-1} \leq \min\{\bar{c}(x) : c_T = 0\}$$

- For simplicity, $y_t = y > 0\ \forall t$, so $\bar{c}(x) = \frac{R(x+y)+y}{R}$. (HH cannot borrow so much in period $T-1$ that they can't pay it all back in period T).

$$V_{T-1}(x) = \max_c \{u(c) + \beta V_T(R(x - c + y))\} \text{ s.t. } 0 \leq c \leq \bar{c}(x)$$

- $c(x)$ is continuous.
- Policy function $\{c_t\} \implies c_t(x)$.
- Thus, the Bellman equation is

$$V_t(x) = \max_{0 \leq c \leq \bar{c}} \{u(c) + \beta V_{t+1}(R(x - c + y))\}$$

- A Bellman equation is the key result of the theory of dynamic programming.

Theorem of the Maximum

- Lower hemi-continuous (lhc): $\forall y \in \Gamma(x), \{x_n\} \to x, \exists \{y_n\}$ s.t. $y_n \to y$ and $y_n \in \Gamma(x_n)$.
- Upper hemi-continuous (uhc): $\{x_n\} \to x$, all $\{y_n\}$ s.t. $y_n \in \Gamma(x), y \in \Gamma(x)$.

Suppose:

- $x \in X \subseteq \mathbb{R}^L$
- $y \in Y \subseteq \mathbb{R}^M$
- $f : X \times Y \to \mathbb{R}$ is continuous.

- $\Gamma : X \rightrightarrows Y$ compact-valued and continuous (uhc and lhc).

Then:

- $h(x) = \max_{y \in \Gamma(x)} f(x, y)$ is continuous.
- $G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$ is nonempty, compact-valued, and uhc.
- In the HH problem, the theorem of the maximum implies that $V_T(x)$ is continuous.

Principle of Optimality

- Let $\{c_t^*\}_{t=0}^T$ solve sequence problem above. with initial $x_0 \geq 0$. From law of motion $x_{t+1} = R(x_t - c_t + y_t)$, we can derive $\{x_t^*\}$.
- For arbitrary dates $0 \leq a < b \leq T - 1$, let x_a^* and x_{b+1}^* be optimal states at those dates. Then the solution to the subproblem is still $\{c_t^*\}_a^b$:

$$\max_{\{c_t\}_{t=a}^b} \sum_{t=a}^{b} \beta^{t-1} u(c_t) \text{ s.t. } x_{t+1} = R(x_t - c_t + y_t) \text{ and given } x_a^*, x_{b+1}^*$$

Back to Consumer Probem - FOC and Euler Equations

- In period $T - 1$,

$$V_{T-1}(x) = \max_{0 \leq c \leq \bar{c}} \{u(c) + \beta V_T(R(x - c + y))\}$$

- Inada conditions rule out corner solutions $\implies$ c^* is interior.
- Strictly concave, differentiable function and convex choice set $\implies$ FOC determine optimal choice.
- FOC: $u'(c) = \beta R V_T'(R(x - c + y))$.
- Because $V_T(x) = u(x + y) \implies V_T'(x) = u'(x + y)$, FOC $\implies u'(c) = \beta R u'(R(x - c + y) + y)$ (Euler Equation).
- Using policy function $c_{t-1} = c(x)$, $u'(c_{T-1}) = \beta R u'(c_T) \implies$

$$u'(c_t) = \beta R u'(c_{t+1})$$

- Since u is strictly concave ($u'' < 0$) $\implies$ consumption smoothing.
- $\beta R = 1 \implies u'(c_t) = u'(c_{t+1}) \implies c_t = \bar{c}$.
- $\beta R < 1 \implies u'(c_t) < u'(c_{t+1}) \implies c_t > c_{t+1}$.
- $\beta R > 1 \implies u'(c_t) > u'(c_{t+1}) \implies c_t < c_{t+1}$.

Lecture 2 - HH Optimization over Infinity Time Horizon

We'll consider the consumption-saving problem from last class but over an infinite horizon:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } x_{t+1} = R(x_t - c_t + y), c_t \geq 0, x_0 \text{ given.}$$

$c_0, c_1, c_2, \ldots$

Recursive Formulation of Consumption-Savings Problem

$$V(x_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } x_{t+1} = R(x_t - c_t + y)$$

- Since u is bounded and $0 < \beta < 1 \implies |V(x_0)| < \infty$.
- These notes is all about justifying that we can break apart the maximum and apply it to today's value and the rest of time so that the consumer problem becomes:

$$\begin{aligned} V(x_0) &= \max_{\{c_t\}_{t=1}^{\infty}} \left\{ u(c_0) + \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \\ &= \max_{c_0} \left\{ u(c_0) + \beta \max_{\{c_t\}_{t=1}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(c_{s+1}) \right\} \\ &= \max_{c_0} \{ u(c_0) + \beta V(x_1) \} \\ \text{s.t. } x_1 &= R(x_0 + y - c_0) \end{aligned}$$

- Under what conditions, can we find a solution to the sequence problem by using this recursive formulation?

Sequence Problem (SP)

- Consider more general notation following Stokey, Lucas, and Prescott.
- Let F be the objective function that we're maximizing. x_t is savings; it is the state variable. We can reformulate F to be the utility function on consumption from above.

$$V^*(x_0) = \sup_{\{x_{t+1}\}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \text{ s.t. } x_{t+1} \in \Gamma(x_t), x_0 \text{ given.}$$

- $\Gamma(x_t)$ is the feasible correspondence.
- $x_t \in X$
- $A = \text{graph}\Gamma = \{(x, y) \in X \times X : y \in \Gamma(x)\}$
- $F : A \to \mathbb{R}$
- $\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t)\}$ is the set of feasible plans.

Recursive Formulation

- The Functional Equation (FE) (aka the Bellman equation) is $V(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\}$.
- Today's question is when is V^* equal to V. If so, we've made forward progress because solving for a function is easier than solving for an infinite series.
- What do we know about V^* and V?
- To make things easier, assume $|V^*(x_0)| < \infty$ and $|V(x_0)| < \infty$.
- Define $U(x) := \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ and $U(x) := \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ for some $x \in \Pi(x_0)$.
- Thus, we know that the solution to the SP $V^*(x_0) \geq U(x)\ \forall x \in \Pi(x_0)$.
- In addition, we know that for any $\varepsilon > 0$, the solution to the SP $V^*(x_0) \leq U(x) + \varepsilon$ for some $x \in \Pi(x_0)$.
- Similarly, we know that the FE $V(x_0) \geq F(x_0, y) + \beta V(y)\ \forall y \in \Gamma(x_0)$ and $V(x_0) \leq F(x_0, y) + \beta V(y) + \varepsilon$ for some $y \in \Gamma(x_0)$.

Theorem:

- Suppose $\Gamma(x)$ is nonempty $\forall x \in X$.
- Suppose $\forall x_0 \in X$ and $x \in \Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists.
- Then if V solves (FE) and
- $\lim_{n\to\infty} \beta^n V(x_n) = 0\ \forall x \in \Pi(x_0)$, $\forall x_0$ (a tail boundedness restriction)
- Then $V = V^*$ (i.e., V solves the sequence problem).

Proof:

- Suppose $|V| < \infty$ and $|V^*| < \infty$ (not necessarily, but convienent).
- Say we have a solution V of the Bellman equation (FE).
- By definition, we know that $\forall x \in \Pi(x_0)$:

$$\begin{aligned} V(x_0) &\geq F(x_0, x_1) + \beta V(x_1) \\ &\geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 V(x_2) \\ &... \\ &\geq U_n(x) + \beta^{n+1} V(x_{n+1}) \end{aligned}$$

- Take $\lim_{n\to\infty}$:

$$V(x_0) \geq U(x) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})\ \forall x \in \Pi(x_0)$$

- [We're halfway done; we showed that V dominates for all feasible plans, and we now need to show that we can get within ε away with some feasible plan.]
- Having a solution to the FE also implies that $\exists x_1 \in \Gamma(x_0), x_2 \in \Gamma(x_1), ...$ such that

$$V(x_t) \leq F(x_t, x_{t+1}) + V(x_{t+1}) + \delta_{t+1}\ \forall t$$

- Where $\{\delta_t\}$ s.t. $\sum_{t=0}^{\infty} \beta^t \delta_t \leq \varepsilon/2$.

$$\implies V(x_0) \le F(x_0, x_1) + V(x_1) + \delta_1$$
$$\cdots$$
$$\le \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} V(x_{n+1}) + \sum_{t=1}^{n+1} \beta^{t-1} \delta_t$$

- For sufficiently large n, we can bound $\beta^{n+1} V(x_{n+1})$ by $\varepsilon/2$.
- By construction $\sum_{t=1}^{n+1} \beta^{t-1} \delta_t \le \sum_{t=0}^{\infty} \beta^t \delta_t \le \varepsilon/2$.
- Taking the limit as $n \to \infty$:

$$V(x_0) \le \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) + \varepsilon$$

Background on Contractions

- An $T : S \to S$ on (S, μ) metric space is a contraction if $\mu(Tx, Ty) \le \beta \mu(x, y)\ \forall x, y \in S$.

Contraction Mapping Theorem:

- If (S, μ) is complete and T is a contraction, then T has a unique fixed point, $Tv = v$.
- And for any $v_0 \in S$, $\mu(T^n v_0, v) \le \beta^n \mu(v_0, v) \iff \lim_{n \to \infty} T^n v_0 = v$.

Corollary:

- Suppose $S' \subseteq S$, S' closed.
- Then $T(S') \subseteq S' \implies v \in S'$.
- Moreover, if $T(S') \subseteq S'' \subset S'$, then $v \in S''$.
- The last part of this corollary is helpful if we're thinking about strictly increasing functions because we need weak inequalities to get a closed subset.

Blackwell Sufficient Conditions:

- $X \subseteq \mathbb{R}^n$
- $B(X)$ is the set of bounded functions $f : X \to \mathbb{R}$.
- Let $||f|| = \sup_{x \in X} |f(x)| \implies \mu(f, g) = ||f - g||$. Thus, this metric space is complete.
- Suppose $T : B(X) \to B(X)$ satisfies monotonicity and discounting.
  - Monotonicity: $f, g \in B(X), f(x) \le g(x)\ \forall x \in X$, then $Tf(x) \le Tg(x)\ \forall x \in X$.
  - Discounting: $\exists \beta \in (0, 1)$ s.t. $T(f + a)(x) \le Tf(x) + \beta a$ where $a > 0$ and $f \in B(X)$.
- Then T is a contraction with modulus β.

Applying Blackwell to Bellman Equations

- Recall we have only made some weak assumptions (nonempty and tail boundedness). Now, we're added a few more assumptions about the feasible set ($\Gamma 1$) and the objective function ($F1$):
- ($\Gamma 1$): $X \in \mathbb{R}^\ell$ is convex. $\Gamma : X \to X$ nonempty, compact-valued, and continuous (uhc and lhc). [These assumptions are similar to those in the theorem of the maximum.]
- ($F1$): $F : A \to \mathbb{R}$ is bounded and continuous with $0 < \beta < 1$ (recall $A = \text{graph}\Gamma$).
- Define $C(X)$ as the metric space of bounded continuous functions on X with $||f|| = \sup_{x \in X} |f(x)|$. [This metric space is complete.]
- Define the Bellman operator as $(Tf)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\}$ for $f \in C(X)$.
- With T defined this way, the Bellman equation is the fixed point of the Bellman operator: $Tv = v$.

Theorem:

- Under ($\Gamma 1$) and ($F1$).
- The Bellman operator $T : C(X) \to C(X)$ is a contraction, hence T has a unique fixed point (i.e., $v \in C(X)$ s.t. $v = Tv$) and $v_0 \in C(X)$, $||T^n v_0 - v|| \leq \beta^n ||v_0 - v||$.
- Moreover, the optimal policy correspondence $G(x) = \{y \in \Gamma(x) : V(x) = F(x, y) + \beta V(y)\}$ is compact-valued and uhc (conclusions from the Theorem of the Maximum).

Proof:

- The Bellman operator is in fact an operator. $T : C \to C$ by the Theorem of the Maximum.
- $f, g \in C(X), f \leq g$.
- T is monotone:

$$(Tf)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} \leq \max_{y \in \Gamma(x)} \{F(x, y) + \beta g(y)\} = (Tg)(x)$$

- T discounts:

$$T(f(x) + a) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta(f(y) + a)\} = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} + \beta a = Tf(x) + \beta a$$

- By Blackwell, T is a contraction. By CMT, the Bellman equation has a unique solution.