ECON 703 - PS 5

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(1) In this exercise you will prove the following theorem. Suppose X and Y are normed vector spaces and $T \in L(X, Y)$. The inverse function $T^{-1}(\cdot)$ exists and is a continuous linear operator on $T(X)$ if and only if there exists some $m > 0$ such that $m||x|| \leq ||T(x)||$ for all $x \in X$.

(a) Show that if there exists some $m > 0$ such that $m||x|| \leq ||T(x)||$, then T is one-to-one (and therefore invertible on $T(X)$). Hint: Think about the norm of elements which are glued together if T is not one-to-one.

Proof: A theorem on slide 11 of lecture 8 states that $T \in L(X, Y)$ is one-to-one iff $\ker T \equiv \{\bar{0}\}$. Consider $x \in \ker\{T\}$, $m||x|| \leq ||T(x)|| \implies m||x|| \leq 0$. Since $m > 0$, $||x|| = 0$ because norms cannot be negative. By definition of a norm, $||x|| = 0 \iff x = \bar{0}$. Thus, T is one-to-one. $\square$

(b) Use theorem with five equivalent properties (various continuity notions and boundedness) from the lecture notes to show that $T^{-1}(\cdot)$ is continuous on $T(X)$.

Proof: By (a), T is invertible. Thus, for all $x \in X$, $m||x|| \leq ||T(x)|| \implies ||T^{-1}(y)|| \leq m^{-1}||y||$ where $y = T(x) \in T(X)$. Thus, because $m > 0 \implies m^{-1} \in \mathbb{R}$, T^{-1} is bounded on $T(X)$. By a theorem on slide 5 of lecture 11, T^{-1} is continuous on $T(X)$. $\square$

(c) Use the same theorem from the lecture notes to show that if T^{-1} is continuous on $T(X)$, then there exists some $m > 0$ such that $m||x|| \leq ||T(x)||$.

Proof: If T^{-1} is continuous on $T(X)$, then T^{-1} is bounded on $T(X)$. Thus, we can choose β such that $||T^{-1}(y)|| \leq \beta||y||$ $\forall y \in T(X)$. Note that, since norms are nonnegative, we can choose $\beta > 0$, so β^{-1} is positive and finite. Thus, $\beta^{-1}||x|| \leq ||T(x)||$ where $x = T^{-1}(y) \in X$. Define $m = \beta^{-1}$, so $m||x|| \leq ||T(x)||$ for $m > 0$. $\square$

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(2) Consider a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 5y, 8x + 7y)$.

(a) Calculate $||T||$ given the norm $||(x, y)||_1 = |x| + |y|$ in $\mathbb{R}^2$.

By the theorem on slide 5 of lecture 11, since $\dim\mathbb{R}^2 = 2$, T is bounded. So,

$$||T|| = \sup_{||(x,y)||_1=1} \{||T(x, y)||_1\}$$

Since $|x|, |y| \geq 0$, we can assume that $x, y \geq 0$ without loss of generality. Further, we can rewrite $y = 1 - x$, so

$$\begin{aligned} ||T|| &= \sup_{x\in[0,1]} \{|x + 5(1 - x)| + |8x + 7(1 - x)|\} \\ &= \sup_{x\in[0,1]} \{|5 - 4x| + |x + 7|\} \\ &= 5 + 7 \\ &= 12 \end{aligned}$$

(b) Calculate $||T||$ given the norm $||(x, y)||_\infty = \max\{|x|, |y|\}$ in $\mathbb{R}^2$.

By the theorem on slide 5 of lecture 11, since $\dim\mathbb{R}^2 = 2$, T is bounded. So,

$$||T|| = \sup_{||(x,y)||_\infty=1} \{||T(x, y)||_\infty\}$$

Define $X = \{(x, y) \in \mathbb{R}^2 : ||(x, y)||_\infty = 1\} = \{(1, w), (x, 1), (-1, y), (-1, z) : w, x, y, z \in [-1, 1]\}$. Since the linear transformation is increasing in x, y, it is maximized at $(1, 1)$. Thus, $||T|| = \sup\{X\} = \max\{6, 15\} = 15$.

(3) Consider the standard basis in $\mathbb{R}^2$, W, and another orthonormal basis $V = \{(a_1, a_2), (b_1, b_2)\}$ (written in coordinates of W). Prove that Euclidean norm (length) of any vector $(x, y) \in \mathbb{R}^2$ is the same in W and V. (Thus, length of a vector does not depend on a choice of orthonormal basis.) Reminder: Orthonormal basis means that $a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1, a_1b_1 + a_2b_2 = 0$.

Proof: Define $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$. Consider $(x, y)'$ in the standard basis for $\mathbb{R}^2$. There exists $(w, z)' \in \mathbb{R}^2$ such that

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= M \begin{pmatrix} w \\ z \end{pmatrix} \\
&= \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \\
&= \begin{pmatrix} wa_1 + zb_1 \\ wa_2 + zb_2 \end{pmatrix} \\
&= w \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + z \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\end{aligned}$$

Thus, $(w, z)'$ represent $(x, y)'$ in basis $V = \{(a_1, a_2), (b_1, b_2)\}$. Notice that $M'M = I$:

$$\begin{aligned}
\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}' \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} &= \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1^2 + a_2^2 & a_1b_1 + a_2b_2 \\ a_1b_1 + a_2b_2 & b_1^2 + b_2^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= I
\end{aligned}$$

Thus, we can show the Euclidean norms of $(w, z)'$ and $(x, y)'$ are equal:

$$\begin{aligned}
||(x, y)'|| &= \sqrt{(x, y)'(x, y)} \\
&= \sqrt{(M(w, z)')'M(w, z)'} \\
&= \sqrt{(w, z)M'M(w, z)'} \\
&= \sqrt{(w, z)(w, z)'} \\
&= \sqrt{(w, z)'(w, z)} \\
&= ||(w, z)'||
\end{aligned}$$

$\square$

(4) In this exercise you will learn to solve first order linear differential equations in n variables. We want to find an n-dimensional process $y(t)$, such that

$$\frac{d}{dt}y(t) = Ay(t) \tag{1}$$

where $A \in M_{n\times n}$ and $y(0) \in \mathbb{R}^n$ are given. When $n = 1$ we know that solution to Eq. (1) is $y(t) = e^{At}y(0)$. Turns out, it remains the same when $n > 1$, thus, it involves exponent of a matrix, which we have not defined before. To properly define $e^{At}, A \in M_{n\times n}$ we use Taylor expansion and say that

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}A^kt^k.$$

To calculate e^{At} we will use diagonalization. Suppose that $A = Pdiag\{\lambda_1, ..., \lambda_n\}P^{-1}$, so that $A^k = Pdiag\{\lambda_1^k, ..., \lambda_n^k\}P^{-1}$ and

$$\begin{aligned} e^{At} &= P\Big(\sum_{k=0}^{\infty} \frac{1}{k!} diag\{t^k\lambda_1^k, ..., t^k\lambda_n^k\}\Big)P^{-1} \\ &= P\Big(diag\Big\{\sum_{k=0}^{\infty} \frac{1}{k!}t^k\lambda_1^k, ..., \sum_{k=0}^{\infty} \frac{1}{k!}t^k\lambda_n^k\Big\}\Big)P^{-1} \\ &= Pdiag\{e^{t\lambda_1}, ..., e^{t\lambda_n}\}P{-}1 \end{aligned}$$

Thus, solution to Eq. (1) is

$$y(t) = Pdiag\{e^{t\lambda_1}, ..., e^{t\lambda_n}\}P{-}1y(0) \tag{2}$$

Implement the above approach to solve for $y(t) \in \mathbb{R}^2$

$$\frac{d}{dt}y(t) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} y(t), y(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Simplify you answer as much as possible.

To find A's eignevalues, use the characteristic polynomial of A:

$$\begin{aligned}(1-\lambda)(-1-\lambda) - 1 * 3 &= \lambda^2 - 4 \\ &= (\lambda - 2)(\lambda + 2)\end{aligned}$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$. The cooresponding eigenvectors are:

$$\begin{aligned}\begin{pmatrix}-1 & 1 \\ 3 & -3\end{pmatrix}\mathbf{v}_1 &= 0 \\ \mathbf{v}_1 &= \begin{pmatrix}1 \\ 1\end{pmatrix} \\ \begin{pmatrix}3 & 1 \\ 3 & 1\end{pmatrix}\mathbf{v}_2 &= 0 \\ \mathbf{v}_2 &= \begin{pmatrix}-1 \\ 3\end{pmatrix}\end{aligned}$$

We have P and P^{-1}.

$$\begin{aligned}P &= \begin{pmatrix}1 & -1 \\ 1 & 3\end{pmatrix} \\ P^{-1} &= \begin{pmatrix}3/4 & 1/4 \\ -1/4 & 1/4\end{pmatrix}\end{aligned}$$

Substituting into Eq. 2,

$$\begin{aligned}y(t) &= Pdiag\{e^{t\lambda_1}, ..., e^{t\lambda_n}\}P{-}1y(0) \\ &= \begin{pmatrix}1 & -1 \\ 1 & 3\end{pmatrix}\begin{pmatrix}e^{2t} & 0 \\ 0 & e^{-2t}\end{pmatrix}\begin{pmatrix}3/4 & 1/4 \\ -1/4 & 1/4\end{pmatrix}\begin{pmatrix}1 \\ 3\end{pmatrix} \\ &= \begin{pmatrix}1 & -1 \\ 1 & 3\end{pmatrix}\begin{pmatrix}e^{2t} & 0 \\ 0 & e^{-2t}\end{pmatrix}\begin{pmatrix}3/2 \\ 1/2\end{pmatrix} \\ &= \begin{pmatrix}1 & -1 \\ 1 & 3\end{pmatrix}\begin{pmatrix}(3/2)e^{2t} \\ (1/2)e^{-2t}\end{pmatrix} \\ &= \begin{pmatrix}(3/2)e^{2t} - (1/2)e^{-2t} \\ (3/2)e^{2t} + (3/2)e^{-2t}\end{pmatrix}\end{aligned}$$

Here's R code that implements this approach as well.

```
library(matlib)
a <- matrix(c(1, 3, 1, -1), ncol = 2)
print(a)
```

```
##      [,1] [,2]
## [1,]    1    1
## [2,]    3   -1
```

```
ev <- eigen(a)
p <- t(t(ev$vectors))
print(p)
```

```
##           [,1]       [,2]
## [1,] 0.7071068 -0.3162278
## [2,] 0.7071068  0.9486833
```

```
y_0 <- c(1, 3)
for (t in 0:5) {
  print(paste("For t =", t))
  print(p %*% diag(exp(t*ev$values)) %*% inv(p) %*% y_0)
}
```

```
## [1] "For t = 0"
##      [,1]
## [1,]    1
## [2,]    3
## [1] "For t = 1"
##          [,1]
## [1,] 11.01592
## [2,] 11.28659
## [1] "For t = 2"
##          [,1]
## [1,] 81.88807
## [2,] 81.92470
## [1] "For t = 3"
##          [,1]
## [1,] 605.1419
## [2,] 605.1469
## [1] "For t = 4"
##          [,1]
## [1,] 4471.437
## [2,] 4471.437
## [1] "For t = 5"
##         [,1]
## [1,] 33039.7
## [2,] 33039.7
```

(5) Solution to different equation (1) is stable if small perturbation of the initial condition $y(0)$ does not significantly change the solution $y(t)$. Formally, it means that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that if $||y(0) - \tilde{y}(0)|| < \delta$, then $||y(t) - \tilde{y}(t)|| < \varepsilon$, where $\tilde{y}(t)$ is the solution with initial condition $\tilde{y}(0)$. Notice that if one of the eigenvalues λ_i is positive (has positive real part if they are complex), then the solution will have a term $c(y(0))e^{\lambda_i t}$, $\lambda_i > 0$ where $c(\cdot)$ is a constant which depends on the initial condition. Hence, $||y(t) - \tilde{y}(t)|| \geq |c(y(0)) - c(\tilde{y}(0))|e^{\lambda_i t} \to \infty$ as $t \to \infty$. Thus, the solution is not stable. In constrast, if all eigenvalues are negative (have negative real part if they are complex), then for all $i = 1, ..., n, e^{\lambda_i t} \to 0$ as $t \to \infty$, and solutions do not diverge, i.e. are stable. Check whether your solution to Problem 4 is stable.

My solution to Problem 4 is not stable because $\lambda_1 = 2 > 0$.