ECON 711 - PS 5

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Question 1. The Consumer Problem

Solve the Consumer Problem and state the Marshallian demand $x(p, w)$ and indirect utility $v(p, w)$ for the following utility functions:[1]

(a) $u(x) = x_1^\alpha + x_2^\alpha$ for $\alpha < 1$[2]

The consumer problem is $\max_{x_1,x_2}\{x_1^\alpha + x_2^\alpha\}$ subject to $p_1x_1 + p_2x_2 \le w$, $x_1 \ge 0$, and $x_2 \ge 0$. Based on the utility function, it is clear that $x_1^* = x_2^* = 0$ is impossible because the consumer's utility is zero and they would be infinitely better off if they consume ε of either x_1 or x_2. Thus, let us assume $x_1^* > 0$.[3] Notice that u is differentiable and concave in x_i: $\frac{\partial^2 u}{\partial^2 x_i} = \alpha(\alpha - 1)x_i^{\alpha-2} < 0$. So, by Theorem 2 in lecture 10 notes, if (x^*, λ^*, μ^*) satisfies the Kuhn-Tucker conditions, x^* solves the consumer problem. The Legrangian is $\mathcal{L}(x, \lambda, \mu) = (x_1^\alpha + x_2^\alpha) + \lambda(w - p_1x_1 - p_2x_2) + \mu_1x_1 + \mu_2x_2$. The Kuhn-Tucker FOC are

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \implies \alpha(x_i^*)^{\alpha-1} - \lambda^* p_i + \mu_i^* = 0 \implies x_i^* = \left(\frac{\lambda^* p_i - \mu_i^*}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

Since $x_1^* > 0$, the above equation implies that $\lambda^* > 0$. By complementary slackness, $\lambda^* > 0 \implies w = p_1x_1^* + p_2x_2^* \implies x_2^* = \frac{p_1}{p_2}x_1^* > 0$. Thus, again by complementary slackness, $x_1^* > 0$ and $x_2^* > 0 \implies \mu_1^* = \mu_2^* = 0$.

$$\left(\frac{\lambda^* p_1}{\alpha}\right)^{\frac{1}{\alpha-1}} + \frac{p_1}{p_2}\left(\frac{\lambda^* p_1}{\alpha}\right)^{\frac{1}{\alpha-1}} = w \implies \lambda^* = \frac{\alpha}{p_1}\left(\frac{p_2 w}{p_1 + p_2}\right)^{(\alpha-1)}$$

$$x_1^* = \left(\frac{\frac{\alpha}{p_1}\left(\frac{p_2 w}{p_1+p_2}\right)^{(\alpha-1)} p_1}{\alpha}\right)^{\frac{1}{\alpha-1}} = \frac{p_2}{p_1 + p_2}w \implies x_2^* = \frac{p_1}{p_2}\frac{p_2}{p_1 + p_2}w = \frac{p_1}{p_1 + p_2}w$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, Tyler Welch, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

[1] For parts (e) and (f), you may describe the Marshallian demand in words rather than giving mathematical formulas if you prefer, and you can ignore the "knife-edge" cases where two prices or sums of prices are exactly equal, but you should still give formulas for the indirect utility function.

[2] This answer assumes $\alpha > 0$.

[3] This assumption is for expositional clarity. Since the utility function is symmetric, results do not change if we had assumed $x_2^* > 0$.

Thus, the Marshallian demand and the indirect utility are

$$x(p,w) = \left(\frac{p_2}{p_1+p_2}w, \frac{p_1}{p_1+p_2}w\right)$$
$$v(p,w) = \left(\frac{p_2}{p_1+p_2}w\right)^{\alpha} + \left(\frac{p_1}{p_1+p_2}w\right)^{\alpha} = \frac{(p_1^{\alpha}+p_2^{\alpha})w^{\alpha}}{(p_1+p_2)^{\alpha}}$$

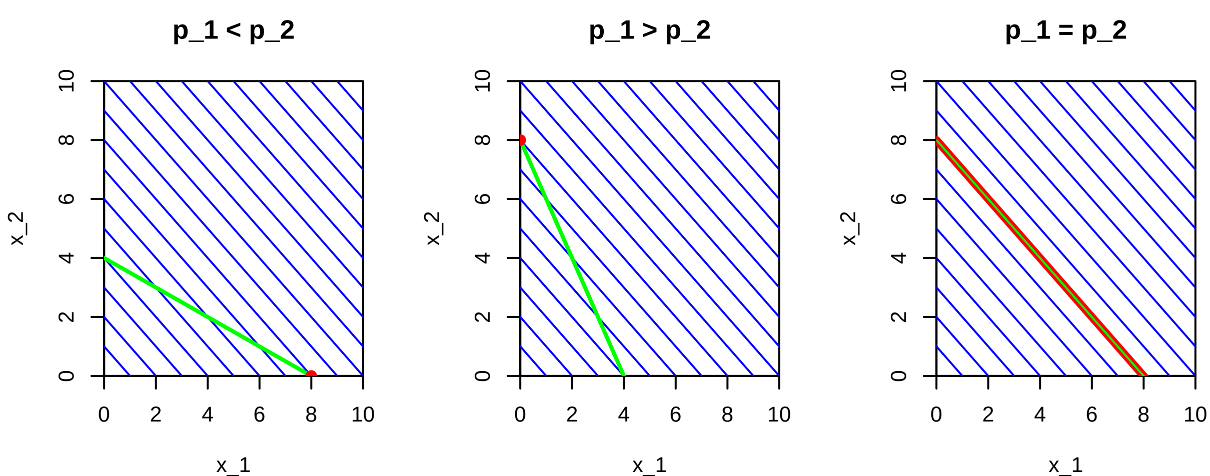
(b) $u(x) = x_1 + x_2$

The consumer problem is $\max_{x_1,x_2}\{x_1+x_2\}$ subject to $p_1x_1 + p_2x_2 \leq w$, $x_1 \geq 0$, and $x_2 \geq 0$. Notice that although u is differentiable, it is not concave in x_i: $\frac{\partial^2 u}{\partial^2 x_i} = 0$. Thus, (x^*, λ^*, μ^*) satisfying the Kuhn-Tucker conditions do not guarantee a solution to the consumer problem. The linear utility function can be represented as straight indifference lines with slopes of -1 (see figure for examples with blue indifference curves, green budget constraints, and red solutions). Since the budget constraint is also a straight line, the consumer chooses corner solutions when $p_1 \neq p_2$. When $p_1 < p_2$, the consumer can afford more of x_1, so she buys $x_1 = \frac{w}{p_1}$ and none of x_2. When $p_1 > p_2$, the consumer can afford more of x_2, so she buys $x_2 = \frac{w}{p_2}$ and none of x_1. When $p_1 = p_2$, there is a continuum of solutions along the overlaid indifference curve and budget constraint.

The Marshallian demand and the indirect utility are

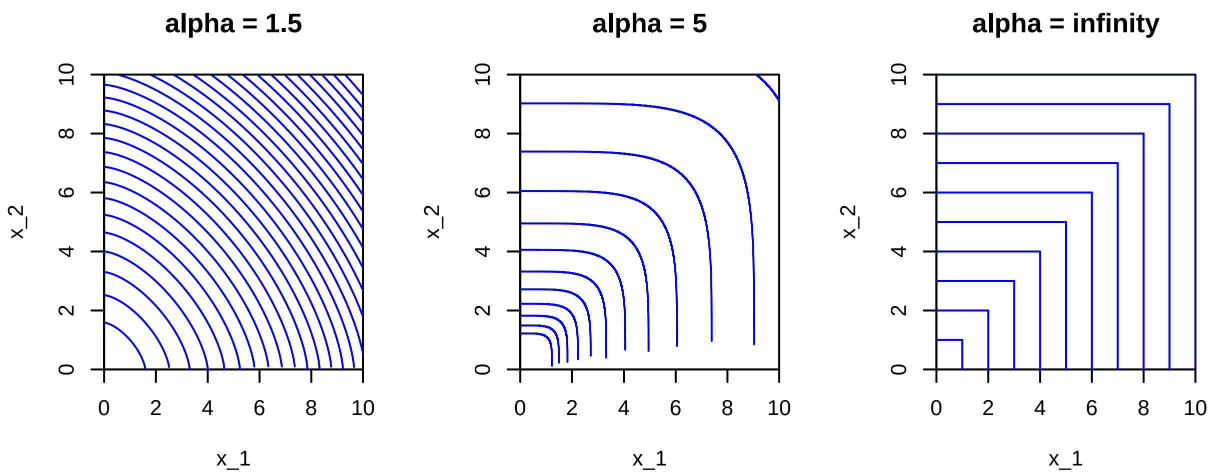
$$x(p,w) = \begin{cases} (w/p_1, 0) & \text{if } p_1 < p_2 \\ (0, w/p_2) & \text{if } p_1 > p_2 \\ \{(tw/p_1, (t-1)w/p_1)\ \forall t \in [0,1]\} & \text{if } p_1 = p_2 \end{cases}$$

$$v(p,w) = \begin{cases} w/p_1 & \text{if } p_1 \leq p_2 \\ w/p_2 & \text{if } p_1 > p_2 \end{cases}$$



(c) $u(x) = x_1^\alpha + x_2^\alpha$ for $\alpha > 1$

The consumer problem is $\max_{x_1,x_2}\{x_1^\alpha + x_2^\alpha\}$ subject to $p_1x_1 + p_2x_2 \leq w$, $x_1 \geq 0$, and $x_2 \geq 0$. Notice that although u is differentiable, it is not concave in x_i: $\frac{\partial^2 u}{\partial^2 x_i} = \alpha(\alpha-1)x_i^{\alpha-2} \geq 0$. Thus, (x^*, λ^*, μ^*) satisfying the Kuhn-Tucker conditions do not guarantee a solution to the consumer problem. As shown in the figure, the indifference curves when $\alpha > 1$ are concave to the origin: $x_i = f(x_j) = (\bar{u} - x_j^\alpha)^{1/\alpha} \implies \frac{\partial^2 f}{\partial^2 x_j} = -(1/\alpha - 1)(\bar{u} - x_j^\alpha)^{1/\alpha-2}(x_j^{\alpha-1})(x_j^{\alpha-1}) - (\bar{u} - (\alpha-1)x_j^\alpha)^{1/\alpha-1}(x_j^{\alpha-2})$.



Since the budget constraint is linear, the consumer chooses corner solutions for all prices. When $p_1 < p_2$, the consumer can afford more of x_1, so she buys $x_1 = \frac{w}{p_1}$ and none of x_2. When $p_1 > p_2$, the consumer can afford more of x_2, so she buys $x_2 = \frac{w}{p_2}$ and none of x_1.

The Marshallian demand and the indirect utility are

$$x(p, w) = \begin{cases} (w/p_1, 0) & \text{if } p_1 < p_2 \\ (0, w/p_2) & \text{if } p_1 > p_2 \\ \{(w/p_1, 0), (0, w/p_2)\} & \text{if } p_1 = p_2 \end{cases}$$

$$v(p, w) = \begin{cases} w/p_1 & \text{if } p_1 \leq p_2 \\ w/p_2 & \text{if } p_1 > p_2 \end{cases}$$

(d) $u(x) = \min\{x_1, x_2\}$ (Leontief utility)

The consumer problem is $\max_{x_1,x_2}\{\min\{x_1, x_2\}\}$ subject to $p_1x_1 + p_2x_2 \leq w$, $x_1 \geq 0$, and $x_2 \geq 0$. Since u is not differentiable, we cannot rely on Kuhn-Tucker conditions to solve the consumer problem. Notice that Leontief utility is associated with the case of perfect complements. Thus, the solution is for the consumer to consume equal amounts of $x_1^* = x_2^*$ and exhaust their wealth. If $x_2^* > x_1^*$, the consumer would increase her utility if she consumed less x_2 and more x_1. Similarly, if $x_1^* > x_2^*$, the consumer should consume less x_1 and more x_2. Therefore, the consumer consumes $\frac{w}{p_1+p_2}$ of each x_i and gets $\frac{w}{p_1+p_2}$ in utility.

The Marshallian demand and the indirect utility are

$$x(p, w) = \left(\frac{w}{p_1 + p_2}, \frac{w}{p_1 + p_2}\right)$$
$$v(p, w) = \frac{w}{p_1 + p_2}$$

(e) $u(x) = \min\{x_1 + x_2, x_3 + x_4\}$

The consumer problem is $\max_{x_1,x_2}\{\min\{x_1 + x_2, x_3 + x_4\}\}$ subject to $p \cdot x \leq w$ and $x_i \geq 0$. Similar to (d), we cannot rely on Kuhn-Tucker conditions to solve the consumer problem, but we know that $x_1^* + x_2^* = x_3^* + x_4^*$ based on the same logic. Now let us just consider the consumer's consumption of x_1 and x_2. Similar to (b), we know that the consumer choose a corner solution and consume all of one good and none of the other. The same logic applies to the consumption of x_3 and x_4. Note that we can ignore knife-edge cases in which $p_1 = p_2$ and/or $p_3 = p_4$.

The Marshallian demand and the indirect utility are

$$x(p, w) = \begin{cases} (w/(p_1 + p_3), 0, w/(p_1 + p_3), 0) & \text{if } p_1 < p_2 \text{ and } p_3 < p_4 \\ (w/(p_1 + p_4), 0, 0, w/(p_1 + p_4)) & \text{if } p_1 < p_2 \text{ and } p_3 > p_4 \\ (0, w/(p_2 + p_3), w/(p_2 + p_3), 0) & \text{if } p_1 > p_2 \text{ and } p_3 < p_4 \\ (0, w/(p_2 + p_4), 0, w/(p_2 + p_4)) & \text{if } p_1 > p_2 \text{ and } p_3 > p_4 \end{cases}$$

$$v(p, w) = \begin{cases} w/(p_1 + p_3) & \text{if } p_1 < p_2 \text{ and } p_3 < p_4 \\ w/(p_1 + p_4) & \text{if } p_1 < p_2 \text{ and } p_3 > p_4 \\ w/(p_2 + p_3) & \text{if } p_1 > p_2 \text{ and } p_3 < p_4 \\ w/(p_2 + p_4) & \text{if } p_1 > p_2 \text{ and } p_3 > p_4 \end{cases}$$

(f) $u(x) = \min\{x_1, x_2\} + \min\{x_3, x_4\}$

The consumer problem is $\max_{x_1,x_2}\{\min\{x_1, x_2\} + \min\{x_3, x_4\}\}$ subject to $p \cdot x \leq w$ and $x_i \geq 0$. Similar to (e), we cannot rely on Kuhn-Tucker conditions to solve the consumer problem because u is not differentiable, but we can combine the logic from (b) and (d). From (d), we know that $x_1 = x_2$ and $x_3 = x_4$. From (b), we know that if $p_1 + p_2 < p_3 + p_4$, the consume will choose the corner solution of consuming all x_1 and x_2; if $p_3 + p_4 < p_1 + p_2$, the consume will choose the corner solution of consuming all x_3 and x_4. The same logic applies to the consumption of x_3 and x_4. Note that we can ignore knife-edge cases in which $p_1 + p_2 = p_3 + p_4$.

The Marshallian demand and the indirect utility are

$$x(p, w) = \begin{cases} (w/(p_1 + p_2), w/(p_1 + p_2), 0, 0) & \text{if } p_1 + p_2 < p_3 + p_4 \\ (0, 0, w/(p_3 + p_4), w/(p_3 + p_4)) & \text{if } p_1 + p_2 > p_3 + p_4 \end{cases}$$

$$v(p, w) = \begin{cases} w/(p_1 + p_2) & \text{if } p_1 + p_2 < p_3 + p_4 \\ w/(p_3 + p_4) & \text{if } p_1 + p_2 > p_3 + p_4 \end{cases}$$

Question 2. CES Utility

Throughout this problem, let $X = \mathbb{R}^k_+$, and let $(a_1, a_2, ..., a_k)$ be a set of strictly positive coefficients which sum to 1. You may assume prices and wealth are strictly positive, and ignore cases where two or more prices are identical.

(a) For each of the following utility functions, solve the consumer problem and state $x(p, w)$:

i. linear utility $u(x) = x_1 + x_2 + ... + x_k$
ii. Cobb-Douglas utility $u(x) = x_1^{a_1} x_2^{a_2} ... x_k^{a_k}$
iii. Leontief utility $u(x) = \min\{\frac{x_1}{a_1}, \frac{x_2}{a_2}, ..., \frac{x_k}{a_k}\}$

(b) Consider the Constant Elasticity of Substitution (CES) utility function $u(x) = \left(\sum_{i=1}^k a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}} \right)^{\frac{s}{s-1}}$ with $s \in (0, 1) \cup (1, +\infty)$. Solve the consumer problem and state $x(p, w)$.[4]

(c) Show that CES utility gives the same demand as linear utility in the limit $s \to +\infty$, as Cobb-Douglas utility in the limit $s \to 1$, and as Leontief utility in the limit $s \to 0$.

(d) The Elasticity of Substitution between goods 1 and 2 is defined as $\xi_{1,2} = -\frac{\partial \log\left(\frac{x_1(p,w)}{x_2(p,w)}\right)}{\partial log\left(\frac{p_1}{p_2}\right)} = -\frac{\partial\left(\frac{x_1(p,w)}{x_2(p,w)}\right)}{\partial\left(\frac{p_1}{p_2}\right)} \frac{\frac{p_1}{p_2}}{\frac{x_1(p,w)}{x_2(p,w)}}$ While this looks complicated, in the case of CES demand, we can write the ratio $\frac{x_1}{x_2}$ as a relatively simple function of the price ratio $\frac{p_1}{p_2}$, and calculate this elasticity without much difficulty. Calculate the elasticity of substitution for CES demand, note its value as $s \to +\infty$, $s \to 1$, and $s \to 0$.

[4] Recall that maximizing a function $(f(x))^{\frac{s}{s-1}}$ is the same as maximizing $f(x)$ when $s > 1$, and the same as minimizing $f(x)$ when $s < 1$.

Question 3. Exchange Economies

We've been considering the problem facing a consumer with wealth w at prices p. An "exchange economy" is different model where instead of money, each consumer is endowed with an initial bundle of goods $e \in \mathbb{R}^k_+$, and can either buy or sell any quantity of the goods at market prices p. The consumer's problem is then $\max_{x \in \mathbb{R}^k_+} u(x)$ subject to $p \cdot x \geq p \cdot e$ (i.e., the consumer's "budget" is the market value of the goods they start with). Assume preferences are locally non-satiated and the consumer's problem has a unique solution $x(p, e)$. We'll say the consumer is a net buyer of good i if $x_i(p, e) > e_i$ and a net seller if $x_i(p, e) < e_i$.

(a) Show that if p_i increases, the consumer cannot switch from being a net seller to a net buyer.

(b) Suppose u is differentiable and concave. Use the Legrangian and the envelope theorem to show that $\frac{\partial v}{\partial p_i}$ is negative if the consumer is a net buyer of good i, and positive if the consumer is a net seller.

(c) Consider the following statement. "If the consumer is a net buyer of good i and its price goes up, the consumer must be worse off." True or false? Explain.