ECON 712 - PS 3

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1. Consider the following overlapping generations problem. In each period $t = 1, 2, 3, ...$ a new generation of 2 period lived households are born. Each generation has a unitary mass. There is a unit measure of initial old who are endowed with $\bar{M} > 0$ units of fiat money. Each generation is endowed with w_1 in youth and w_2 in old age of non-storable consumption goods where $w_1 > w_2$. There is no commitment technology to enforce trades. The utility function of a household of generation $t \geq 1$ is $U(c_t^t, c_{t+1}^t) = \ln(c_t^t) + \ln(c_{t+1}^t)$ where (c_t^t, c_{t+1}^t) is consumption of a household of generation t in youth (i.e., in period t) and old age (i.e., in period $t+1$). The preference of the initial old are given by $U(c_1^0) = \ln(c_1^0)$ where c_1^0 is consumption by a household of the initial old.

(a) State and solve the planner problem.

In any period t, the social planner weights agents alive equally and optimally allocates resources between them given preferences and technologies:

$$\max_{(c_t^t, c_t^{t-1}) \in \mathbb{R}_+^2} \ln(c_t^t) + \ln(c_t^{t-1})$$
$$\text{s.t. } c_t^t + c_t^{t-1} \leq w_1 + w_2$$

Since utility is strictly increasing in consumption, we know that the maximum will occur at $c_t^t + c_t^{t-1} = w_1 + w_2 \implies c_t^{t-1} = w_1 + w_2 - c_t^t$. Thus, we can write the social planner's problem as an unconstrained maximization problem:

$$\max_{c_t^t \in \mathbb{R}_+} \ln(c_t^t) + \ln(w_1 + w_2 - c_t^t)$$

Setting the first order condition to zero:

$$\frac{1}{c_t^t} - \frac{1}{w_1 + w_2 - c_t^t} = 0 \implies c_t^t = \frac{w_1 + w_2}{2}$$

Plugging the solution into the equation for the consumption of old agents:

$$c_t^{t-1} = w_1 + w_2 - \frac{w_1 + w_2}{2} = \frac{w_1 + w_2}{2}$$

Thus, the solution to the social planner's problem is the allocation $\left\{ \left(\frac{w_1+w_2}{2}, \frac{w_1+w_2}{2} \right) \right\}_{\forall t}$.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

(b) State the representative household's problem in period $t \geq 0$. Try to write the budget constraints in real terms.

I break the household problem into the problem facing an initial old agent and the problem facing agents born in period $t > 0$. Let p_t be the number of dollar per unit of consumption good in period t.

The problem facing the initial old agents is:

$$\max_{c_1^0 \in \mathbb{R}_+} \ln(c_1^0)$$
$$\text{s.t. } c_1^0 \leq w_2 + \frac{\bar{M}}{p_1}$$

Define $M_{t+1}^t \in \mathbb{R}_+$ as the holding of fiat currency by the representative household in generation born in t between periods t and $t+1$. The problem facing agents born in period $t > 0$ is:

$$\begin{aligned} \max_{(c_t^t, c_{t+1}^t, M_{t+1}^t) \in \mathbb{R}_+^3} \quad & \ln(c_t^t) + \ln(c_{t+1}^t) \\ \text{s.t. } c_t^t &\leq w_1 - \frac{M_{t+1}^t}{p_t} \\ c_{t+1}^t &\leq w_2 + \frac{M_{t+1}^t}{p_{t+1}} \end{aligned}$$

Defining $m_{t+1}^t = \frac{M_{t+1}^t}{p_t}$ as the real value of the holding of fiat currency, we can rewrite the problem facing agents born in period $t > 0$:

$$\begin{aligned} \max_{(c_t^t, c_{t+1}^t, m_{t+1}^t) \in \mathbb{R}_+^3} \quad & \ln(c_t^t) + \ln(c_{t+1}^t) \\ \text{s.t. } c_t^t &\leq w_1 - m_{t+1}^t \\ c_{t+1}^t &\leq w_2 + \frac{p_t}{p_{t+1}} m_{t+1}^t \end{aligned}$$

(c) Define and solve for an autarkic equilibrium, assuming that it exists.

An autarkic equilibrium is the allocation $\{(c_t^t, c_t^{t-1}\}_{\forall t}$ and prices $\{p_t\}_{\forall t}$ such that agents optimize and markets clear when money has no real value, or $m_{t+1}^t = 0\ \forall t$. The market clearing conditions are:

$$\begin{aligned} c_t^t + c_t^{t-1} &= w_1 + w_2 && \text{(Goods Market)} \\ M_{t+1}^t &= \bar{M} && \text{(Money Market)} \end{aligned}$$

The problem facing the initial old agents collapses to:

$$\begin{aligned} \max_{c_1^0 \in \mathbb{R}_+} \ & \ln(c_1^0) \\ \text{s.t. } & c_1^0 \leq w_2 \end{aligned}$$

Trivially, $c_1^0 = w_2$.

The problem facing agents born in period $t > 0$ collapses to:

$$\begin{aligned} \max_{(c_t^t, c_{t+1}^t) \in \mathbb{R}_+^2} \ & \ln(c_t^t) + \ln(c_{t+1}^t) \\ \text{s.t. } & c_t^t \leq w_1 \\ & c_{t+1}^t \leq w_2 \end{aligned}$$

So, $(c_t^t, c_{t+1}^t) = (w_1, w_2)$. For prices, we know that $m_{t+1}^t = \frac{M_{t+1}^t}{p_t} = 0$ and $M_{t+1}^t = \bar{M} > 0 \implies \frac{1}{p_t} = 0$. Thus, the autarkic equilibrium is the allocation $\{(w_1, w_2)\}_{\forall t}$ and prices $\{\infty\}_{\forall t}$.

(d) Define and solve for a competitive equilibrium assuming valued money but with $w_2 = 0$.

A competitive equilibrium is the allocation $\{(c_t^t, c_t^{t-1}\}_{\forall t}$ and prices $\{p_t\}_{\forall t}$ such that agents optimize and markets clear:

$$\begin{aligned} c_t^t + c_t^{t-1} &= w_1 + w_2 && \text{(Goods Market)} \\ M_{t+1}^t &= \bar{M} && \text{(Money Market)} \end{aligned}$$

We can rewrite the problem of agents born in period $t > 0$ with $w_2 = 0$, since utility is strictly increasing in consumption.

$$\max_{M_{t+1}^t \in \mathbb{R}_+} \ln\left(w_1 - \frac{M_{t+1}^t}{p_t}\right) + \ln\left(\frac{M_{t+1}^t}{p_{t+1}}\right)$$

Setting the first order condition to zero:

$$\left(w_1 - \frac{M_{t+1}^t}{p_t}\right)^{-1} (-p_t)^{-1} + \left(\frac{M_{t+1}^t}{p_{t+1}}\right)^{-1} (p_{t+1})^{-1} = 0 \implies M_{t+1}^t = \frac{p_t w_1}{2}$$

Plugging back in for c_t^t and c_{t+1}^t:

$$\begin{aligned} c_t^t &= w_1 - \frac{(p_t w_1)/2}{p_t} = \frac{w_1}{2} \\ c_{t+1}^t &= \frac{(p_t w_1)/2}{p_{t+1}} = \left(\frac{p_t}{p_{t+1}}\right)\frac{w_1}{2} \end{aligned}$$

Setting $w_2 = 0$ and substituting in the household's optimization conditions, the goods market clearing condition becomes:

$$c_t^t + c_t^{t-1} = w_1 + 0 \implies \frac{w_1}{2} + \left(\frac{p_t}{p_{t+1}}\right)\frac{w_1}{2} = w_1 \implies p_t = p_{t+1}$$

The money market condition implies:

$$\frac{p_t w_1}{2} = \bar{M} \implies p_t = \frac{2\bar{M}}{w_1}$$

Thus, the competitive equilibrium is the allocation $\left\{\left(\frac{w_1}{2}, \frac{w_1}{2}\right)\right\}_{\forall t}$ and prices $\left\{\frac{2\bar{M}}{w_1}\right\}_{\forall t}$

(e) Compare the solutions to the planners problem, the autarky equilibrium and the stationary monetary competitive equilibrium with valued money, all with $w_2 = 0$.

With $w_2 = 0$,

- The solution to the planners problem is the allocation $\left\{\left(\frac{w_1}{2}, \frac{w_1}{2}\right)\right\}_{\forall t}$.
- The autarky equilibrium is the allocation $\{(w_1, 0)\}_{\forall t}$.
- The stationary monetary competitive equilibrium with valued money is the allocation $\left\{\left(\frac{w_1}{2}, \frac{w_1}{2}\right)\right\}_{\forall t}$ and prices $\left\{\frac{2\bar{M}}{w_1}\right\}_{\forall t}$.

We see that the stationary monetary competitive equilibrium with valued money achieves the solution to the planners problem with agents consuming half their endowment in each period. The autarky equilibrium results in agents consuming their entire endowment when young and nothing when old. The autarky equilibrium is clearly worse because all agents have negative infinity utility due to consuming nothing when old.

(f) What happens to consumption, money demand and prices in a competitive equilibrium with valued money if the initial money supply is halved, i.e. $\bar{M}' = \frac{\bar{M}}{2}$. Keep the assumption that $w_2 = 0$.

In (d), we found that the consumption allocation under the competitive equilibrium with valued money does not depend on the money supply, so the consumption allocation would stay the same at $\left\{\left(\frac{w_1}{2}, \frac{w_1}{2}\right)\right\}_{\forall t}$ if the initial money supply is halved. In contrast, prices on the other hand would also halve:

$$\left\{\frac{2\bar{M}'}{w_1}\right\}_{\forall t} = \left\{\frac{2(\bar{M}/2)}{w_1}\right\}_{\forall t} = \left\{\frac{\bar{M}}{w_1}\right\}_{\forall t}$$

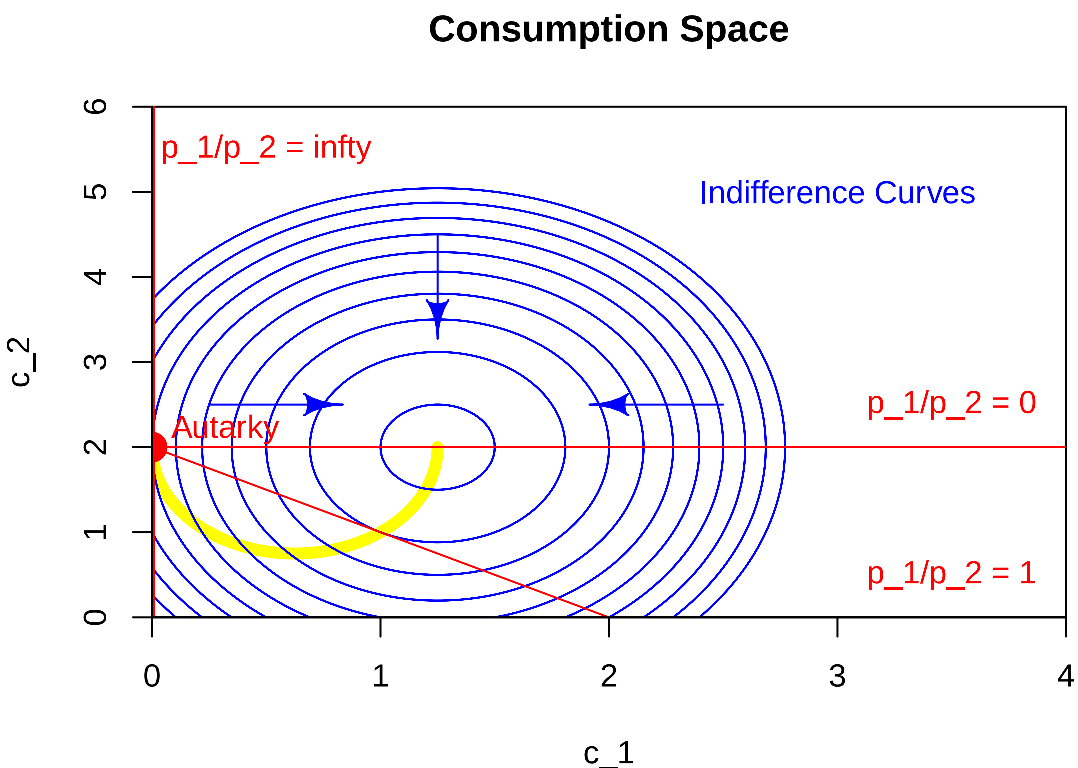
Price dropping by a half would cause money demand to halve as well: $(M^t_{t+1})' = \frac{(p_t'/2)w_1}{2} = \frac{p_t' w_1}{4}$ $\forall t$.

2. Plot the trade offer curves for the following utility functions where the endowment is (w_1, w_2) for goods 1 and 2, respectively.

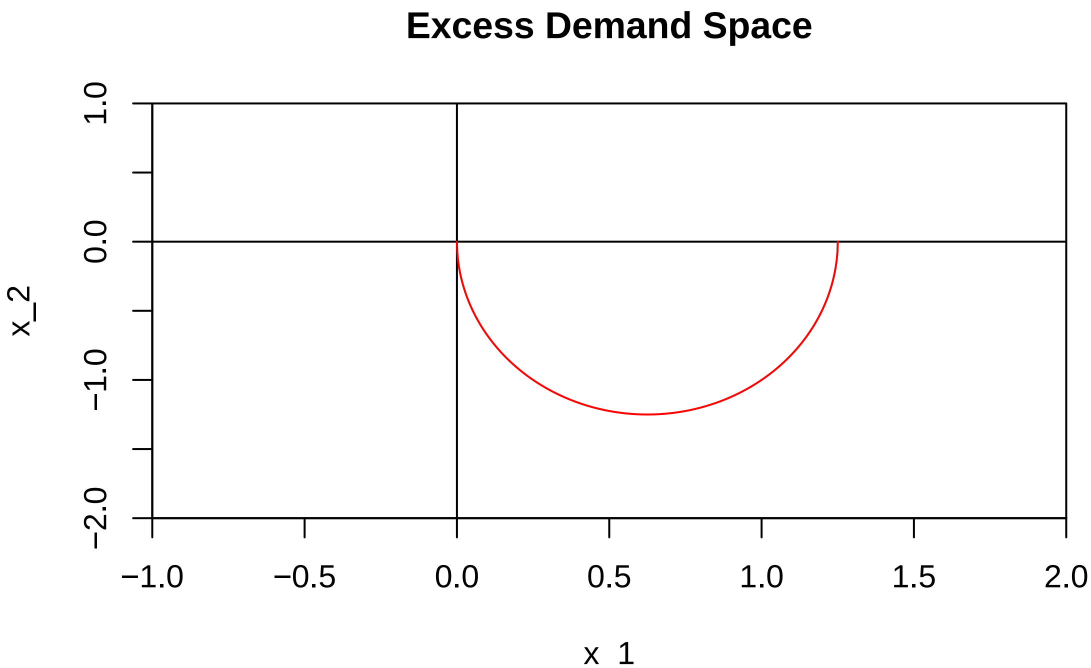
(a) $U = 10c_1 - 4c_1^2 + 4c_2 - c_2^2, (w_1, w_2) = (0, 2)$

In the first figure, we can see that the utility function is an ellipsoid and thus the indifference curves are concentric ellipses centered at $(5/4, 2)$. Higher utility is associated with ellipses closer to $(5/4, 2)$. The budget constraint with a price ratio (p_1/p_2) of infinity is represented as a vertical line at $c_1 = 0$. The highest utility the agent can get is at the autarky point, which is tangent to the indifference curve associated with $\bar{u} = 4$. As the price ratio decreases, the slope of the budget constraint decreases. The agent will choose consumption at the point allow the budget constraint that is tangent to highest utility ellipse. For example, at $p_1/p_2 = 1$, the agent will choose to consume $(1, 1)$ which is tangent to the indifference ellipse associated with $\bar{u} = 9$. At $p_1/p_2 = 0$, the budget constraint is horizontal line at $c_2 = 2$. Thus, the agent will consume $(5/4, 2)$ achieving their maximum utility of $\bar{u} = 10.25$. Thus, the trade offer curve traces the lower half of an ellipse starting at the autarky point through the tangent points to the concentric ellipse and ending at the maximum utility point.

The second figure translates the trade offer curve from consumption space to excess demand space.



Indifference curves (blue), Budget constraints (red), and offer curve (yellow)



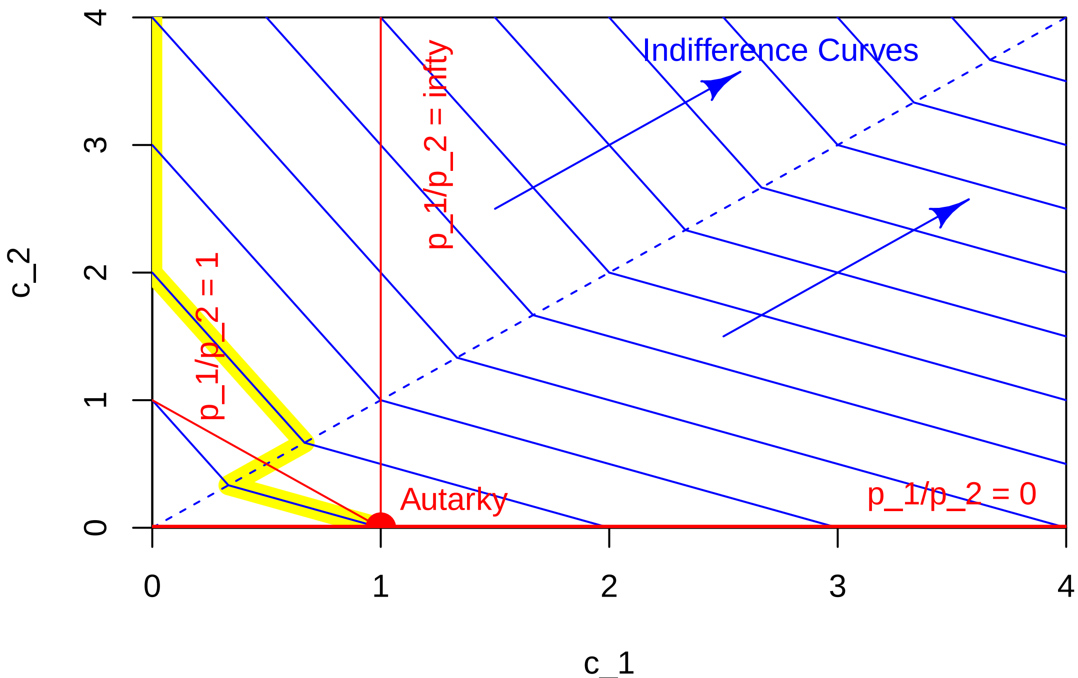
Trade offer curve (red)

(b) $U = \min\{2c_1 + c_2, c_1 + 2c_2\}, (w_1, w_2) = (1, 0)$

In the first figure, we see that the indifference curves are similar to the perfect complements case discussed in section. However, the lines below the identity line have a slope of 1/2 and the lines above the identity line have a slope of 2. At a price ratio (p_1/p_2) of zero, the budget constraint is a horizontal line at $c_2 = 0$. To get the highest indifference curve, the agent will consume infinity units of c_1 and zero units of c_2. Between price ratios of zero and 1/2, the agent will consume at the autarky point of $(1, 0)$. The agent will continue to consume the autarky point until the price ratio is 1/2, at which point the budget constraint will lay on top of the lower half of the indifference curve associated with $\bar{u} = 1/2$. Thus, the agent is indifferent between consuming at all points on the line segment between $(1, 0)$ and $(1/3, 1/3)$. At price ratios between 1/2 and 2, the agent will consume along the identity line. At a price ratio of 2, the budget constraint sits on top of the upper half of the indifference curve associated with $\bar{u} = 1$. At this price ratio, the agent is indifferent between consuming the points along the line segment between $(2/3, 2/3)$ and $(0, 2)$. At price ratio higher than 2, the agent consumes all c_2 at the highest amount possible. At a price ratio of infinity, the agent consumes infinity units of c_2.

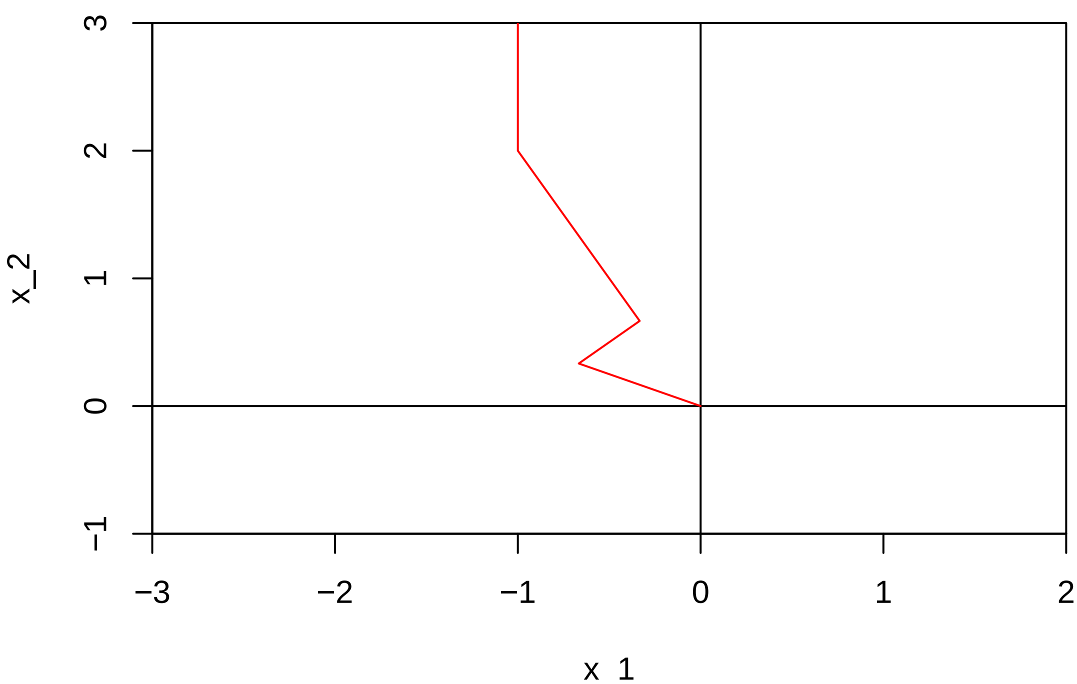
The second figure translates the trade offer curve from consumption space to excess demand space.

Consumption Space



Indifference curves (blue), Budget constraints (red), and offer curve (yellow)

Excess Demand Space

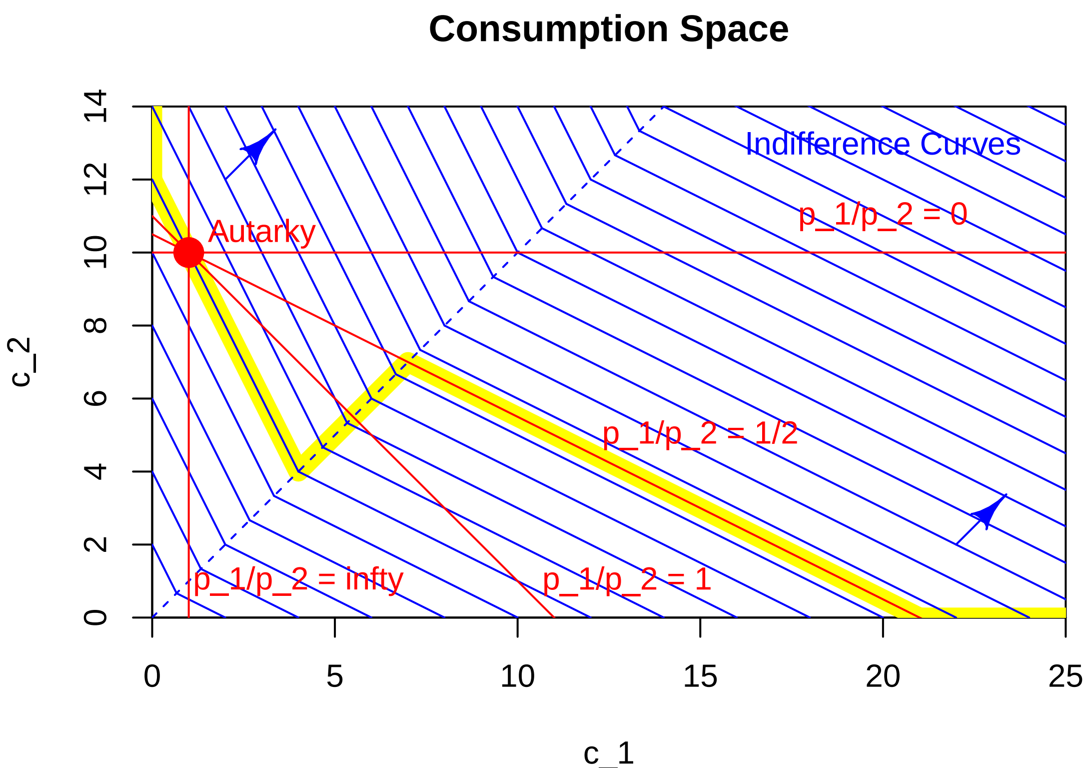


Trade offer curve (red)

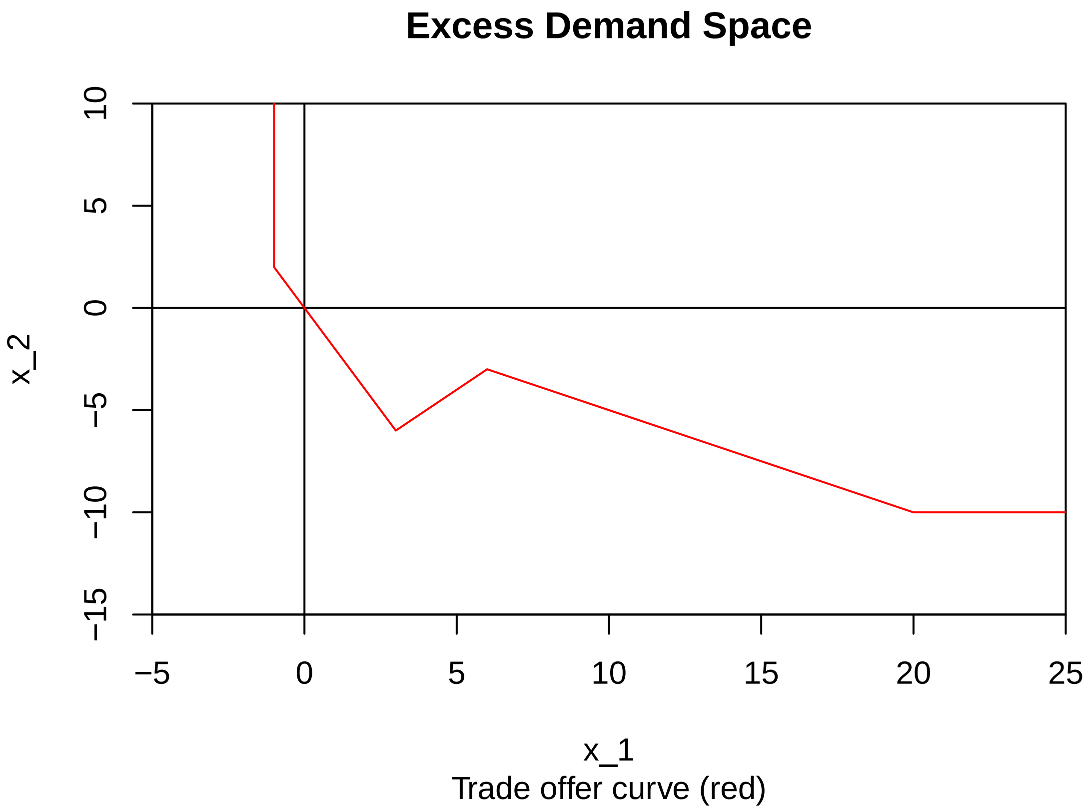
(c) $U = \min\{2c_1 + c_2, c_1 + 2c_2\}, (w_1, w_2) = (1, 10)$

In the first figure, we see that the indifference curves are the same as in (b). At an price ratio (p_1/p_2) of 0, the agent consumes infinity units of c_1 and zero units of c_2. As the price ratio increases, the agent consumes less and less units of c_1. At a price ratio of 1/2, the budget constraint sit on top of the lower half of an indifference curve at $\bar{u} = 10.5$. The agent is indifferent between consuming any point between $(21, 0)$ and $(7, 7)$. For price ratios between 1/2 and 2, the agent consumes along the identity line. At a price ratio of 2, the budget constraint lays on top of the upper half of an indifference curve associated with $\bar{u} = 6$. The agent is indifferent between consuming at points along the line segment between $(4, 4)$ and $(12, 0)$. This segment passes through the autarky point. For higher price ratios, the agent consumes zero units of c_1 and more and more units of c_2. At a price ratio of infinity, the agent would consume infinity units of c_2 and zero units of c_1.

The second figure translates the trade offer curve from consumption space to excess demand space.



Indifference curves (blue), Budget constraints (red), and offer curve (yellow)


Excess Demand Space
x_2
10
5
0
−5
−10
−15
−5
0
5
10
15
20
25
x_1
Trade offer curve (red)