ECON 714B - Problem Set 1

Alex von Hafften*

3/26/2021

Exercise 8.1 - Existence of representative consumer

Suppose households 1 and 2 have one-period utility functions $u(c_1)$ and $w(c_2)$, respectively, where u and w are both increasing, strictly concave, twice differentiable functions of a scalar consumption rate. Consider the Pareto problem:

$$v_\theta(c) = \max_{\{c_1,c_2\}} [\theta u(c_1) + (1-\theta)w(c_2)]$$

subject to the constraint $c_1 + c_2 = c$. Show that the solution of this problem has the form of a concave utility function $v_\theta(c)$, which depends on the Pareto weight θ. Show that $v_\theta'(c) = \theta u'(c_1) = (1-\theta)w'(c_2)$.The function $v_\theta(c)$ is the utility function of the representative consumer. Such a representative consumer always lurks within a complete markets competitive equilibrium even with heterogeneous preferences. At a competitive equilibrium, the marginal utilities of the representative agent and each and every agent are proportional.

First, notice that v_θ is increasing, strictly concave, and twice differentiable because $\theta, 1-\theta \geq 0$ and u and w are both increasing, strictly concave, twice differentiable functions.

$$v_\theta(c) = \max_{\{c_1,c_2\}} [\theta u(c_1) + (1-\theta)w(c_2)] \text{ s.t. } c = c_1 + c_2$$

$$\implies v_\theta(c) = \max_{c_1} [\theta u(c_1) + (1-\theta)w(c - c_1)]$$

Since u and w are continuously differentiable functions, we can find the envelope condition:

$$\implies v_\theta'(c) = (1-\theta)w'(c - c_1) = (1-\theta)w'(c_2)$$

FOC $[c_1]$:

$$\theta u'(c_1) = (1-\theta)w'(c - c_1) \implies \theta u'(c_1) = (1-\theta)w'(c_2)$$

Therefore,

$$v_\theta'(c) = \theta u'(c_1) = (1-\theta)w'(c_2)$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

Exercise 8.3

An economy consists of two infinitely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer i consumes c_{it} at time t. Consumer i ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_{it})$$

where $\beta \in (0,1)$ and $u(c)$ is increasing, strictly concave, and twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_{1t} = 1, 0, 0, 1, 0, 0, 1, ...$ Consumer 2 is endowed with a stream of the consumption good $y_{2t} = 0, 1, 1, 0, 1, 1, 0, ...$ Assume that there are complete markets with time 0 trading.

a. Define a competitive equilibrium.

First, notice that there is no stochastic states (i.e., endowment process is deterministic). The agent i's problem is:

$$\max_{\{c_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{it}) \text{ s.t. } \sum_{t=0}^{\infty} Q_t c_{it} \leq \sum_{t=0}^{\infty} Q_t y_{it}$$

A competitive equilibrium is a feasible allocation $\{\{c_{it}\}_{t=0}^{\infty}\}_{i=1,2}$ and a price system $\{Q_t\}_{t=0}^{\infty}$ such that given the price system, the allocation maximized the agent's utility subject to their budget constraint.

b. Compute a competitive equilibrium.

Let μ_i be the multipler on the BC for agent i:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_{it}) - \mu_i \left[\sum_{t=0}^{\infty} Q_t y_{it} - \sum_{t=0}^{\infty} Q_t c_{it} \right]$$

FOC $[c_{it}]$:

$$\beta^t u'(c_{it}) = \mu_i Q_t$$

Thus, the ratio of the FOC for 1 and the FOC for 2 is:

$$\frac{u'(c_{1t})}{u'(c_{2t})} = \frac{\mu_1}{\mu_2} \implies c_{1t} = u'^{-1} \left(\frac{\mu_1}{\mu_2} u'(c_{2t}) \right)$$

Plugging into the resource constraint:

$$c_{1t} + c_{2t} = 1 \implies u'^{-1} \left(\frac{\mu_1}{\mu_2} u'(c_{2t}) \right) + c_{2t} = 1$$

This implies that c_{2t} a function of the aggregate endowment for all t. Since the aggregate endowment is constant, $c_{2t} = c_{2,t+1} = c_2$. Thus, c_{1t} is also constant: $c_{1t} = c - c_2 \equiv c_1$. Let the numeraire be date 0 consumption $Q_0 = 1$:

$$\frac{\beta^t u'(c_{1,t})}{\beta^0 u'(c_{1,0})} = \frac{\mu_1 Q_t}{\mu_1} \implies \frac{\beta^t u'(c_1)}{u'(c_1)} = Q_t \implies Q_t = \beta^t$$

The budget constraint implies

$$\sum_{t=0}^{\infty} \beta^t c_{1t} = \sum_{t=0}^{\infty} \beta^t y_{1t} \implies c_1 \frac{1}{1-\beta} = \sum_{t=0}^{\infty} \beta^{3t} \implies c_1 = \frac{1-\beta}{1-\beta^3}$$

Furthermore,

$$c_2 = 1 - \frac{1-\beta}{1-\beta^3} = \frac{\beta-\beta^3}{1-\beta^3}$$

c. Suppose that one of the consumers markets a derivative asset that promises to pay .05 units of consumption each period. What would the price of that asset be?

The price of the derivative asset in period 0 with promises $\{d_t\}_{t=0}^{\infty} = \{0.05\}_{t=0}^{\infty}$ is:

$$P_0^0 = \sum_{t=0}^{\infty} Q_t d_t = \sum_{t=0}^{\infty} \beta^t 0.05 = \frac{0.05}{1-\beta}$$

Exercise 8.4

Consider a pure endowment economy with a single representative consumer; $\{c_t, d_t\}_{t=0}^{\infty}$ are the consumption and endowment processes, respectively. Feasible allocations satisfy $c_t \leq d_t$. The endowment process is described by $d_{t+1} = \lambda_{t+1} d_t$. The growth rate λ_{t+1} is described by a two-state Markov process with transition probabilities $P_{ij} = Prob(\lambda_{t+1} = \bar{\lambda}_j | \lambda_t = \bar{\lambda}_i)$. Assume that

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}, \bar{\lambda} = \begin{bmatrix} .97 \\ 1.03 \end{bmatrix}$$

In addition, $\lambda_0 = .97$ and $d_0 = 1$ are both known at date 0. The consumer has preferences over consumption ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

where E_0 is the mathematical expectation operator, conditioned on information known at time 0, $\gamma = 2, \beta = .95$.

Part I

At time 0, after d_0 and λ_0 are known, there are complete markets in date- and history-contingent claims. The market prices are denominated in units of time 0 consumption goods.

a. Define a competitive equilibrium, being careful to specify all the objects composing an equilibrium.

A competitive equilibrium is an allocation $\{c_t(s^t)\}_{t=0}^{\infty}$ and price system $\{Q_t(s^t)\}_{t=0}^{\infty}$ for all histories $s^t = (\lambda_0, ..., \lambda_t)$ such that the allocation is feasible and, given the price system, the allocation maximizes agent's utility subject to their budget constraint.

b. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, ..., \lambda_5) = (0.97, 0.97, 1.03, 0.97, 1.03)$. Please give a numerical answer.

The consumers' problem is

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t|s_0) \beta^t \frac{c_t(s^t)^{1-\gamma}}{1-\gamma} \text{ s.t. } \sum_{t=0}^{\infty} \sum_{s^t} Q_t(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} Q_t(s^t) d_t(s^t)$$

Denote μ as the legrangian multipler on the budget constrant:

$$\mathcal{L} = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t|s_0) \beta^t \frac{(c_t(s^t))^{1-\gamma}}{1-\gamma} + \mu \left[\sum_{t=0}^{\infty} \sum_{s^t} Q_t(s^t) d_t(s^t) - \sum_{t=0}^{\infty} \sum_{s^t} Q_t(s^t) c_t(s^t) \right]$$

FOC $[c_t(s^t)]$:

$$\beta^t \pi_t(s^t|s_0)(c_t(s^t))^{-\gamma} = \mu Q_t(s^t)$$

Feasibility requires $c_t(s^t) = d_t(s^t)$. For $t = 0$, $Q_0(s^0) = \pi_0(s^0|s_0) = 1$ and $c_0(s^0) = d_0 = 1$:

$$\beta^0 \pi_0(s^0|s_0)(c_0(s^0))^{-\gamma} = \mu Q_0(s^0) \implies \mu = 1$$

Thus, for any t, the FOC and feasibility implies:

$$Q_t(s^t) = \beta^t \pi_t(s^t|s_0)(d_t(s^t))^{-\gamma}$$

For $(\lambda_1, \lambda_2, ..., \lambda_5) = (0.97, 0.97, 1.03, 0.97, 1.03)$,

$$\beta^5 = (0.95)^5 = 0.7737809$$

$$\pi_5(s^5 = (0.97, 0.97, 1.03, 0.97, 1.03)|\lambda_0 = 0.97) = 0.8 * 0.8 * 0.2 * 0.1 * 0.2 = 0.00256$$

$$d_5(s^5 = (0.97, 0.97, 1.03, 0.97, 1.03)) = 1 * 0.97 * 0.97 * 1.03 * 0.97 * 1.03 = 0.9682548$$

Thus, $Q_5(s^5 = (0.97, 0.97, 1.03, 0.97, 1.03)) = (0.7737809)(0.00256)(0.9682548)^{-2} = 0.002112899$

c. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, ..., \lambda_5) = (1.03, 1.03, 1.03, 1.03, .97)$.

$$\pi_5(s^5 = (1.03, 1.03, 1.03, 1.03, .97)|\lambda_0 = 0.97) = 0.2 * 0.9 * 0.9 * 0.9 * 0.1 = 0.01458$$

$$d_5(s^5 = (1.03, 1.03, 1.03, 1.03, .97)) = 1 * 1.03 * 1.03 * 1.03 * 1.03 * .97 = 1.091744$$

Thus, $Q_5(s^5 = (1.03, 1.03, 1.03, 1.03, .97)) = (0.7737809)(0.01458)(1.091744)^{-2} = 0.00946529$

d. Give a formula for the price at time 0 of a claim on the entire endowment sequence.

The price at time 0 of an asset that promises $\{d_t(s^t)\}_{t=0}^{\infty}$ is:

$$P_0^0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t} Q_t(s^t) d_t(s^t) = \sum_{t=0}^{\infty} \beta^t \pi_t(s^t|s_0)(d_t(s^t))^{-\gamma} d_t(s^t) = \sum_{t=0}^{\infty} \beta^t \pi_t(s^t|s_0)(d_t(s^t))^{1-\gamma}$$

e. Give a formula for the price at time 0 of a claim on consumption in period 5, contingent on the growth rate λ_5 being 0.97 (regardless of the intervening growth rates).

...

Part II

Now assume a different market structure. Assume that at each date $t \geq 0$ there is a complete set of one-period forward Arrow securities.

f. Define a (recursive) competitive equilibrium with Arrow securities, being careful to define all of the objects that compose such an equilibrium.

...

g. For the representative consumer in this economy, for each state compute the "natural debt limits" that constrain state-contingent borrowing.

...

h. Compute a competitive equilibrium with Arrow securities. In particular, compute both the pricing kernel and the allocation.

...

i. An entrepreneur enters this economy and proposes to issue a new security each period, namely, a risk-free two-period bond. Such a bond issued in period t promises to pay one unit of consumption at time $t+1$ for sure. Find the price of this new security in period t, contingent on λ_t.

...