ECON 703 - PS 5

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9/16/2020

(1) In this exercise you will prove the following theorem. Suppose X and Y are normed vector spaces and $T \in L(X, Y)$. The inverse function $T^{-1}(\cdot)$ exists and is a continuous linear operator on $T(X)$ if and only if there exists some $m > 0$ such that $m||x|| \leq ||T(x)||$ for all $x \in X$.

(a) Show that if there exists some $m > 0$ such that $m||x|| \leq ||T(x)||$, then T is one-to-one (and therefore invertible on $T(X)$). Hint: Think about the norm of elements which are glued together if T is not one-to-one.

Proof: A theorem on slide 11 of lecture 8 states that $T \in L(X, Y)$ is one-to-one iff $\ker T \equiv \{\bar{0}\}$. Consider $x \in \ker\{T\}$, $m||x|| \leq ||T(x)|| \implies m||x|| \leq 0$. Since $m > 0$, $||x|| = 0$ because norms cannot be negative. By definition of a norm, $||x|| = 0 \iff x = \bar{0}$. Thus, T is one-to-one. $\square$

(b) Use theorem with five equivalent properties (various continuity notions and boundedness) from the lecture notes to show that $T^{-1}(\cdot)$ is continuous on $T(X)$.

Proof: By (a), T is invertible. Thus, for all $x \in X$, $m||x|| \leq ||T(x)|| \implies ||T^{-1}(y)|| \leq m^{-1}||y||$ where $y = T(x) \in T(X)$. Thus, because $m > 0 \implies m^{-1} \in \mathbb{R}$, T^{-1} is bounded on $T(X)$. By a theorem on slide 5 of lecture 11, T^{-1} is continuous on $T(X)$. $\square$

(c) Use the same theorem from the lecture notes to show that if T^{-1} is continuous on $T(X)$, then there exists some $m > 0$ such that $m||x|| \leq ||T(x)||$.

Proof: If T^{-1} is continuous on $T(X)$, then T^{-1} is bounded on $T(X)$. Thus, we can choose β such that $||T^{-1}(y)|| \leq \beta||y|| \; \forall y \in T(X)$. Note that, since norms are nonnegative, we can choose $\beta > 0$, so β^{-1} is positive and finite. Thus, $\beta^{-1}||x|| \leq ||T(x)||$ where $x = T^{-1}(y) \in X$. Define $m = \beta^{-1}$, so $m||x|| \leq ||T(x)||$ for $m > 0$. $\square$

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*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

(2) Consider a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 5y, 8x + 7y)$.

(a) Calculate $||T||$ given the norm $||(x, y)||_1 = |x| + |y|$ in $\mathbb{R}^2$.

By the theorem on slide 5 of lecture 11, since $\dim\mathbb{R}^2 = 2$, T is bounded. So,

$$||T|| = \sup_{||(x,y)||_1 = 1} \{||T(x, y)||_1\}$$

Since $|x|, |y| \geq 0$, we can assume that $x, y \geq 0$ without loss of generality. Further, we can rewrite $y = 1 - x$, so

$$\begin{aligned} ||T|| &= \sup_{x \in [0,1]} \{|x + 5(1 - x)|, |8x + 7(1 - x)|\} \\ &= \sup_{x \in [0,1]} \{|5 - 4x|, |x + 7|\} \\ &= \sup_{x \in [0,1]} \{|x + 7|\} \\ &= 1 + 7 \\ &= 8 \end{aligned}$$

(b) Calculate $||T||$ given the norm $||(x, y)||_\infty = \max\{|x|, |y|\}$ in $\mathbb{R}^2$.

By the theorem on slide 5 of lecture 11, since $\dim\mathbb{R}^2 = 2$, T is bounded. So,

$$||T|| = \sup_{||(x,y)||_\infty = 1} \{||T(x, y)||_\infty\}$$

Define $X = \{(x, y) \in \mathbb{R}^2 : ||(x, y)||_\infty = 1\} = \{(1, -1), (1, 0), (1, 1), (0, -1), (0, 1), (-1, -1), (-1, 0), (-1, 1)\}$

$$\begin{aligned} ||T(1, -1)||_\infty &= ||(-4, 1)||_\infty = 4 \\ ||T(1, 0)||_\infty &= ||(1, 8)||_\infty = 8 \\ ||T(1, 1)||_\infty &= ||(6, 15)||_\infty = 15 \\ ||T(0, -1)||_\infty &= ||(-5, -7)||_\infty = 7 \\ ||T(0, 1)||_\infty &= ||(5, 7)||_\infty = 7 \\ ||T(-1, -1)||_\infty &= ||(-6, -15)||_\infty = 15 \\ ||T(-1, 0)||_\infty &= ||(-1, -8)||_\infty = 8 \\ ||T(-1, 1)||_\infty &= ||(4, -1)||_\infty = 4 \end{aligned}$$

Thus, $||T|| = \sup\{4, 7, 8, 15\} = 15$.

(3) Consider the standard basis in $\mathbb{R}^2$, W, and another orthonormal basis $V = \{(a_1, a_2), (b_1, b_2)\}$ (written in coordinates of W). Prove that Euclidean norm (length) of any vector $(x, y) \in \mathbb{R}^2$ is the same in W and V. (Thus, length of a vector does not depend on a choice of orthonormal basis.) Reminder: Orthonormal basis means that $a_1^2 + a_2^2 = b_1^2 + b_2^2 = 1, a_1b_1 + a_2b_2 = 0$.

Proof: We show that $(||(x, y)||_2)^2$ in W equals $(||(x, y)||_2)^2$ in V:

$$\begin{aligned}
\left(\left\|x\begin{pmatrix}a_1\\a_2\end{pmatrix} + y\begin{pmatrix}b_1\\b_2\end{pmatrix}\right\|_2\right)^2 &= \left(\left\|\begin{pmatrix}xa_1 + yb_1\\xa_2 + yb_2\end{pmatrix}\right\|_2\right)^2 \\
&= (xa_1 + yb_1)^2 + (xa_2 + yb_2)^2 \\
&= x^2a_1^2 + y^2b_1^2 + 2a_1b_1xy + x^2a_2^2 + y^2b_2^2 + 2a_2b_2xy \\
&= x^2(a_1^2 + a_2^2) + y^2(b_1^2 + b_2^2) + 2(a_1b_1 + a_2b_2)xy \\
&= x^2(1) + y^2(1) + 2(0)xy \\
&= x^2 + y^2 \\
\left(\left\|x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right\|_2\right)^2 &= \left(\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|_2\right)^2 \\
&= x^2 + y^2
\end{aligned}$$

Thus, the Euclidean norm of any vector in $\mathbb{R}^2$ is the same in W and V. $\square$

(4) In this exercise you will learn to solve first order linear differential equations in n variables. We want to find an n-dimensional process $y(t)$, such that

$$\frac{d}{dt}y(t) = Ay(t) \tag{1}$$

where $A \in M_{n\times n}$ and $y(0) \in \mathbb{R}^n$ are given. When $n = 1$ we know that solution to Eq. (1) is $y(t) = e^{At}y(0)$. Turns out, it remains the same when $n > 1$, thus, it involves exponent of a matrix, which we have not defined before. To properly define $e^{At}, A \in M_{n\times n}$ we use Taylor expansion and say that

$$e^{At} = I + A + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + ... = \sum_{k=0}^{\infty}\frac{1}{k!}A^kt^k.$$

To calculate e^{At} we will use diagonalization. Suppose that $A = Pdiag\{\lambda_1, ..., \lambda_n\}P^{-1}$, so that $A^k = Pdiag\{\lambda_1^k, ..., \lambda_n^k\}P^{-1}$ and

$$\begin{aligned} e^{At} &= P\Big(\sum_{k=0}^{\infty}\frac{1}{k!}diag\{t^k\lambda_1^k, ..., t^k\lambda_n^k\}\Big)P^{-1} \\ &= P\Big(diag\Big\{\sum_{k=0}^{\infty}\frac{1}{k!}t^k\lambda_1^k, ..., \sum_{k=0}^{\infty}\frac{1}{k!}t^k\lambda_n^k\Big\}\Big)P^{-1} \\ &= Pdiag\{e^{t\lambda_1}, ..., e^{t\lambda_n}\}P{-}1 \end{aligned}$$

Thus, solution to Eq. (1) is

$$y(t) = Pdiag\{e^{t\lambda_1}, ..., e^{t\lambda_n}\}P{-}1y(0)$$

Implement the above approach to solve for $y(t) \in \mathbb{R}^2$

$$\frac{d}{dt}y(t) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} y(t), y(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Simplify you answer as much as possible.

(5) Solution to different equation (1) is stable if small perturbation of the initial condition $y(0)$ does not significantly change the solution $y(t)$. Formally, it means that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that if $||y(0) - \tilde{y}(0)|| < \delta$, then $||y(t) - \tilde{y}(t)|| < \varepsilon$, where $\tilde{y}(t)$ is the solution with initial condition $\tilde{y}(0)$. Notice that if one of the eigenvalues λ_i is positive (has positive real part if they are complex), then the solution will have a term $c(y(0))e^{\lambda_i t}$, $\lambda_i > 0$ where $c(\cdot)$ is a constant which depends on the initial condition. Hence, $||y(t) - \tilde{y}(t)|| \geq |c(y(0)) - c(\tilde{y}(0))|e^{\lambda_i t} \to \infty$ as $t \to \infty$. Thus, the solution is not stable. In constrast, if all eigenvalues are negative (have negative real part if they are complex), then for all $i = 1, ..., n, e^{\lambda_i t} \to 0$ as $t \to \infty$, and solutions do not diverge, i.e. are stable. Check whether your solution to Problem 4 is stable.