ECON 710A - Problem Set 5

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1. Suppose that $\{\varepsilon_t\}_{t=0}^T$ are iid random variables with mean zero, variance σ^2 and $E[\varepsilon_t^8] < \infty$. Let $U_t = \varepsilon_t \varepsilon_{t-1}$, $W_t = \varepsilon_t \varepsilon_0$, and $V_t = \varepsilon_t^2 \varepsilon_{t-1}$ where $t = 1, ..., T$.

(i) Show that $\{U_t\}_{t=1}^T$, $\{W_t\}_{t=1}^T$, and $\{V_t\}_{t=1}^T$ are covariance stationary.

For each time series, we check that (1) the second moment is finite, (2) the mean does not depend on t, and (3) the variance does not depend on t.

$\{U_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$
\begin{aligned}
E[U_t^2] &= E[(\varepsilon_t \varepsilon_{t-1})^2] \\
&= E[\varepsilon_t^2 \varepsilon_{t-1}^2] \\
&= E[\varepsilon_t^2] E[\varepsilon_{t-1}^2] \\
&= E[\varepsilon_t^2]^2 \\
&= \sigma^4 \\
&< \infty
\end{aligned}
$$

For (2), $E[U_t] = E[\varepsilon_t \varepsilon_{t-1}] = E[\varepsilon_t]E[\varepsilon_{t-1}] = 0$. For (3),

$$
\begin{aligned}
\gamma(0) &= Cov(U_t, U_t) \\
&= Var(U_t) \\
&= Var(\varepsilon_t \varepsilon_{t-1}) \\
&= Var(\varepsilon_t) Var(\varepsilon_{t-1}) \\
&= \sigma^4
\end{aligned}
$$

$$
\begin{aligned}
\gamma(1) &= Cov(U_t, U_{t+1}) \\
&= E[U_t U_{t+1}] \\
&= E[(\varepsilon_t \varepsilon_{t-1})(\varepsilon_{t+1} \varepsilon_t)] \\
&= E[\varepsilon_t^2] E[\varepsilon_{t-1}] E[\varepsilon_{t+1}] \\
&= 0
\end{aligned}
$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

$$
\begin{aligned}
\gamma(2) &= Cov(U_t, U_{t+2}) \\
&= E[U_t U_{t+2}] \\
&= E[(\varepsilon_t \varepsilon_{t-1})(\varepsilon_{t+2}\varepsilon_{t+1})] \\
&= E[\varepsilon_{t-1}]E[\varepsilon_t]E[\varepsilon_{t+1}]E[\varepsilon_{t+2}] \\
&= 0
\end{aligned}
$$

Thus, $\gamma(k) = \sigma^4$ if $k = 0$ and zero otherwise.

$\{W_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$
\begin{aligned}
E[W_t^2] &= E[(\varepsilon_t \varepsilon_0)^2] \\
&= E[\varepsilon_t^2 \varepsilon_0^2] \\
&= E[\varepsilon_t^2]E[\varepsilon_0^2] \\
&= E[\varepsilon_t^2]^2 \\
&= \sigma^4 \\
&< \infty
\end{aligned}
$$

For (2), $E[W_t] = E[\varepsilon_t \varepsilon_0] = E[\varepsilon_t]E[\varepsilon_0] = 0$. For (3),

$$
\begin{aligned}
\gamma(0) &= Cov(W_t, W_t) \\
&= Var(W_t) \\
&= Var(\varepsilon_t \varepsilon_0) \\
&= Var(\varepsilon_t)Var(\varepsilon_0) \\
&= \sigma^4
\end{aligned}
$$

$$
\begin{aligned}
\gamma(1) &= Cov(W_t, W_{t+1}) \\
&= E[W_t W_{t+1}] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+1}\varepsilon_0)] \\
&= E[\varepsilon_0^2]E[\varepsilon_t]E[\varepsilon_{t+1}] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma(2) &= Cov(W_t, W_{t+2}) \\
&= E[W_t W_{t+2}] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+2}\varepsilon_0)] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+2}\varepsilon_0)] \\
&= E[\varepsilon_0^2]E[\varepsilon_t]E[\varepsilon_{t+2}] \\
&= 0
\end{aligned}
$$

Thus, $\gamma(k) = \sigma^4$ if $k = 0$ and zero otherwise.

$\{V_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$\begin{aligned}
E[V_t^2] &= E[(\varepsilon_t^2 \varepsilon_{t-1})^2] \\
&= E[\varepsilon_t^4 \varepsilon_{t-1}^2] \\
&= E[\varepsilon_t^4] E[\varepsilon_{t-1}^2] \\
&= E[\varepsilon_t^4] \sigma^2 \\
&< \infty
\end{aligned}$$

For (2), $E[V_t] = E[\varepsilon_t^2 \varepsilon_{t-1}] = E[\varepsilon_t^2] E[\varepsilon_{t-1}] = 0$. For (3),

$$\begin{aligned}
\gamma(0) &= Cov(V_t, V_t) \\
&= Var(V_t) \\
&= Var(\varepsilon_t^2 \varepsilon_{t-1}) \\
&= Var(\varepsilon_t^2) Var(\varepsilon_{t-1}) \\
&= E[(\varepsilon_t^2 - E[\varepsilon_t^2])^2] \sigma^2 \\
&= E[(\varepsilon_t^2 - \sigma^2)^2] \sigma^2 \\
&= E[\varepsilon_t^4 - 2\sigma^2 \varepsilon_t^2 + \sigma^4] \sigma^2 \\
&= (E[\varepsilon_t^4] - 2\sigma^2 \sigma^2 + \sigma^4) \sigma^2 \\
&= (E[\varepsilon_t^4] - \sigma^4) \sigma^2 \\
&= \sigma^2 E[\varepsilon_t^4] - \sigma^6
\end{aligned}$$

$$\begin{aligned}
\gamma(1) &= Cov(V_t, V_{t+1}) \\
&= E[V_t V_{t+1}] \\
&= E[(\varepsilon_t^2 \varepsilon_{t-1})(\varepsilon_{t+1}^2 \varepsilon_t)] \\
&= E[\varepsilon_t^3 \varepsilon_{t-1} \varepsilon_{t+1}^2] \\
&= E[\varepsilon_t^3] E[\varepsilon_{t-1}] E[\varepsilon_{t+1}^2] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\gamma(2) &= Cov(V_t, V_{t+2}) \\
&= E[V_t V_{t+2}] \\
&= E[(\varepsilon_t^2 \varepsilon_{t-1})(\varepsilon_{t+2}^2 \varepsilon_{t+1})] \\
&= E[\varepsilon_t^2 \varepsilon_{t-1} \varepsilon_{t+2}^2 \varepsilon_{t+1}] \\
&= E[\varepsilon_t^2] E[\varepsilon_{t-1}] E[\varepsilon_{t+2}^2] E[\varepsilon_{t+1}] \\
&= 0
\end{aligned}$$

Thus, $\gamma(k) = \sigma^2 E[\varepsilon_t^4] - \sigma^6$ if $k = 0$ and zero otherwise.

(ii) Argue that the following three sample means $\bar{U}$, $\bar{W}$, $\bar{V}$ converge in probability to their expectations.

In (i), we found that $E[U_t] = E[W_t] = E[V_t] = 0 \implies E[\bar{U}] = E[\bar{W}] = E[\bar{V}] = 0$. Below I show that $Var(\bar{U}) \to 0$, $Var(\bar{V}) \to 0$, and $Var(\bar{W}) \to 0$, so by Chebyshev's inequality $\bar{U} \to_p E[\bar{U}]$, $\bar{W} \to_p E[\bar{W}]$, and $\bar{V} \to_p E[\bar{V}]$.

$$\begin{aligned}
Var(\bar{U}) &= Var\left(\frac{1}{T}\sum_{t=1}^{T} U_t\right) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} Cov(U_t, U_s) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma(t-s) \\
&= \frac{1}{T^2} T\gamma(0) \\
&= \frac{\gamma(0)}{T} \\
&= \frac{\sigma^2}{T} \\
&\to 0
\end{aligned}$$

As $T \to \infty$. Because V_t and W_t have the same autocovariance function, the variances of $\bar{W}$ and $\bar{V}$ similarly converge to zero.

(iii) Determine whether the following three sample second moments converge in probability to their expectations:

$$\hat{\gamma}_U(0) = \frac{1}{T}\sum_{t=1}^{T} U_t^2, \quad \hat{\gamma}_W(0) = \frac{1}{T}\sum_{t=1}^{T} W_t^2, \quad \hat{\gamma}_V(0) = \frac{1}{T}\sum_{t=1}^{T} V_t^2$$

...

(iv) Determine whether the scaled sample means $\sqrt{T}\bar{U}$, $\sqrt{T}\bar{W}$, and $\sqrt{T}\bar{V}$ are asymptotically normal.

...

2. Consider a time series of length T from the model

$$Y_t = \alpha_0 + t\beta_0 + X_t\delta_0 + Y_{t-1}\rho_1 + U_t$$

where Y_0 and $\{U_t\}_{t=1}^T$ are iid $N(0,1)$, and

$$X_t = X_{t-1} \cdot 0.3 + V_t$$

where X_0 and $\{V_t\}_{t=1}^T$ are iid $N(0,1)$ and independent of Y_0 and $\{U_t\}_{t=1}^T$. We will let $\alpha_0 = \delta_0 = 100$, $\beta_0 = 1$ and consider all combinations of $T \in \{50, 150, 250\}$ and $\rho_1 \in \{0.7, 0.9, 0.95\}$.

(i) In a statistical software of your choice, generate data from (1), estimate the coefficients by OLS, and calculate heteroscedasticity robust two-sided 95% confidence intervals for α_0, δ_0, and ρ_1.

```r
tees <- c(50, 150, 250)
rhos <- c(0.7, 0.9, 0.95)
alpha <- 100
delta <- 100
beta <- 1

results <- NULL

for (t in tees) {
  for (rho in rhos) {
    x_t <- rnorm(1)
    y_t <- rnorm(1)
    v_t <- rnorm(t)
    u_t <- rnorm(t)

    for (i in 1:t) x_t[i+1] <- 0.3 * x_t[i] + v_t[i]
    for (i in 1:t) y_t[i+1] <- alpha + i * beta + x_t[i+1] * delta + y_t[i] * rho + u_t[i]

    x <- cbind(rep(1, t),
               1:t,
               x_t[2:(t+1)],
               y_t[1:t])
    y <- y_t[2:(t+1)]

    ols <- solve(t(x) %*% x) %*% (t(x) %*% y)

    e_hat <- as.numeric(y - x %*% ols)
    omega <- crossprod(x * e_hat)
    varcov <- solve(t(x) %*% x) %*% omega %*% solve(t(x) %*% x)
    se_robust <- sqrt(diag(varcov))

    results <- tibble(t = t,
           rho = rho,
           name = c("alpha", "beta", "delta", "rho"),
           ols = as.numeric(ols),
           se = se_robust) %>%
      bind_rows(results)
  }
}
```

```
results %>%
  mutate(upper_bound = ols + se * 1.96,
         lower_bound = ols - se * 1.96) %>%
  kable(digits = 3)
```

t	rho	name	ols	se	upper_bound	lower_bound
250	0.95	alpha	100.037	0.167	100.364	99.711
250	0.95	beta	0.994	0.003	1.000	0.988
250	0.95	delta	100.072	0.056	100.182	99.961
250	0.95	rho	0.950	0.000	0.950	0.950
250	0.90	alpha	100.199	0.199	100.589	99.810
250	0.90	beta	1.004	0.002	1.009	1.000
250	0.90	delta	100.002	0.056	100.111	99.893
250	0.90	rho	0.900	0.000	0.900	0.899
250	0.70	alpha	100.040	0.183	100.399	99.682
250	0.70	beta	1.000	0.001	1.003	0.997
250	0.70	delta	100.114	0.067	100.246	99.983
250	0.70	rho	0.700	0.000	0.701	0.699
150	0.95	alpha	99.810	0.169	100.142	99.478
150	0.95	beta	0.988	0.007	1.002	0.973
150	0.95	delta	100.117	0.071	100.256	99.979
150	0.95	rho	0.950	0.000	0.951	0.950
150	0.90	alpha	100.509	0.400	101.292	99.726
150	0.90	beta	1.005	0.004	1.013	0.997
150	0.90	delta	100.021	0.104	100.226	99.817
150	0.90	rho	0.899	0.000	0.900	0.899
150	0.70	alpha	99.761	0.209	100.170	99.352
150	0.70	beta	0.999	0.003	1.005	0.993
150	0.70	delta	99.801	0.065	99.929	99.673
150	0.70	rho	0.700	0.001	0.701	0.699
50	0.95	alpha	100.087	0.368	100.807	99.366
50	0.95	beta	1.002	0.033	1.067	0.937
50	0.95	delta	99.959	0.111	100.177	99.742
50	0.95	rho	0.950	0.000	0.951	0.949
50	0.90	alpha	99.798	0.229	100.247	99.349
50	0.90	beta	1.002	0.024	1.050	0.954
50	0.90	delta	100.105	0.128	100.355	99.855
50	0.90	rho	0.900	0.001	0.901	0.899
50	0.70	alpha	100.178	0.323	100.810	99.545
50	0.70	beta	0.969	0.009	0.985	0.952
50	0.70	delta	100.189	0.124	100.432	99.947
50	0.70	rho	0.701	0.001	0.703	0.700

(ii) Across 10000 simulated repetitions of the above, report the simulated mean of the point estimators for α_0, δ_0, and ρ_1 and the simulated coverage rate of the confidence intervals.

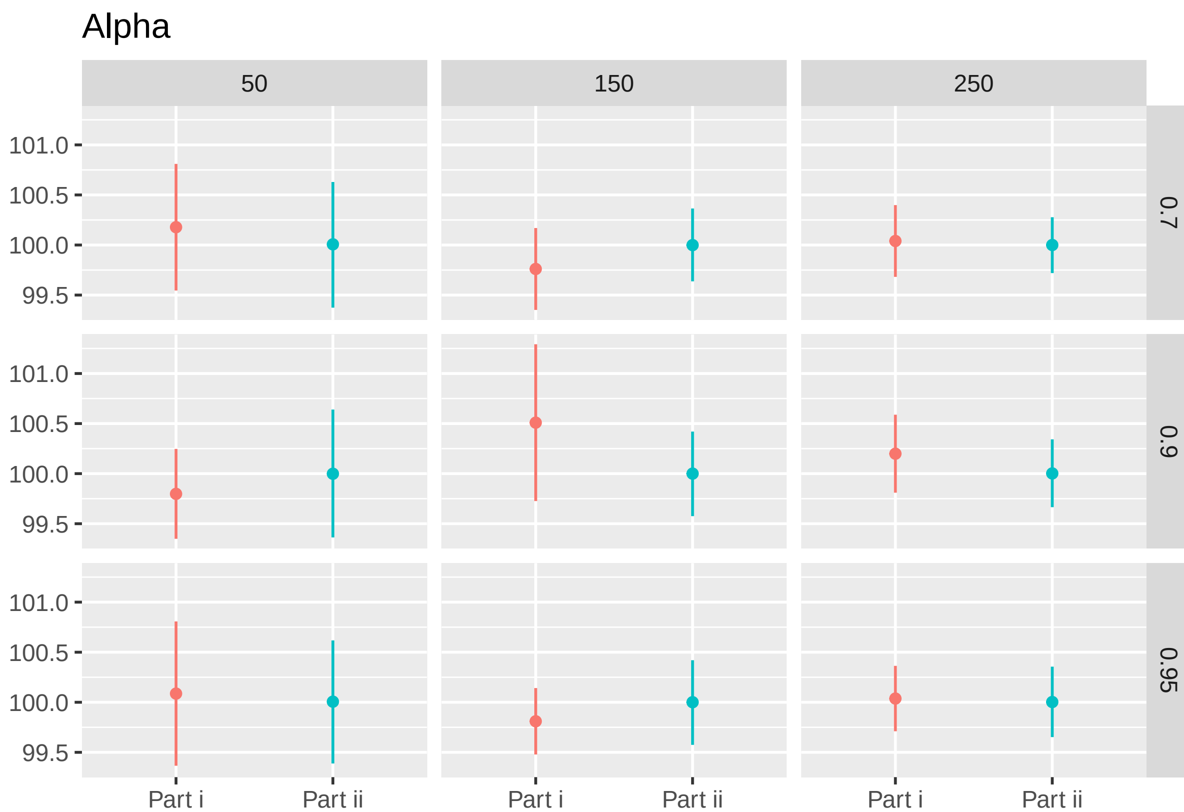
```
ntrials <- 10000
results2 <- NULL

for (t in tees) {
  for (rho in rhos) {
    for (trial in 1:ntrials) {
      print(trial)

      x_t <- rnorm(1)
      y_t <- rnorm(1)
      v_t <- rnorm(t)
      u_t <- rnorm(t)

      for (i in 1:t) x_t[i+1] <- 0.3 * x_t[i] + v_t[i]
      for (i in 1:t) y_t[i+1] <- alpha + i * beta + x_t[i+1] * delta +
        y_t[i] * rho + u_t[i]

      x <- cbind(rep(1, t),
                 1:t,
                 x_t[2:(t+1)],
                 y_t[1:t])
      y <- y_t[2:(t+1)]

      ols <- solve(t(x) %*% x) %*% (t(x) %*% y)

      results2 <- tibble(t = t,
                         rho = rho,
                         trial = trial,
                         name = c("alpha", "beta", "delta", "rho"),
                         ols = as.numeric(ols)) %>%
        bind_rows(results2)
    }
  }
}

save(results2, file = "ps5_vonhafften_temp.RData")
```

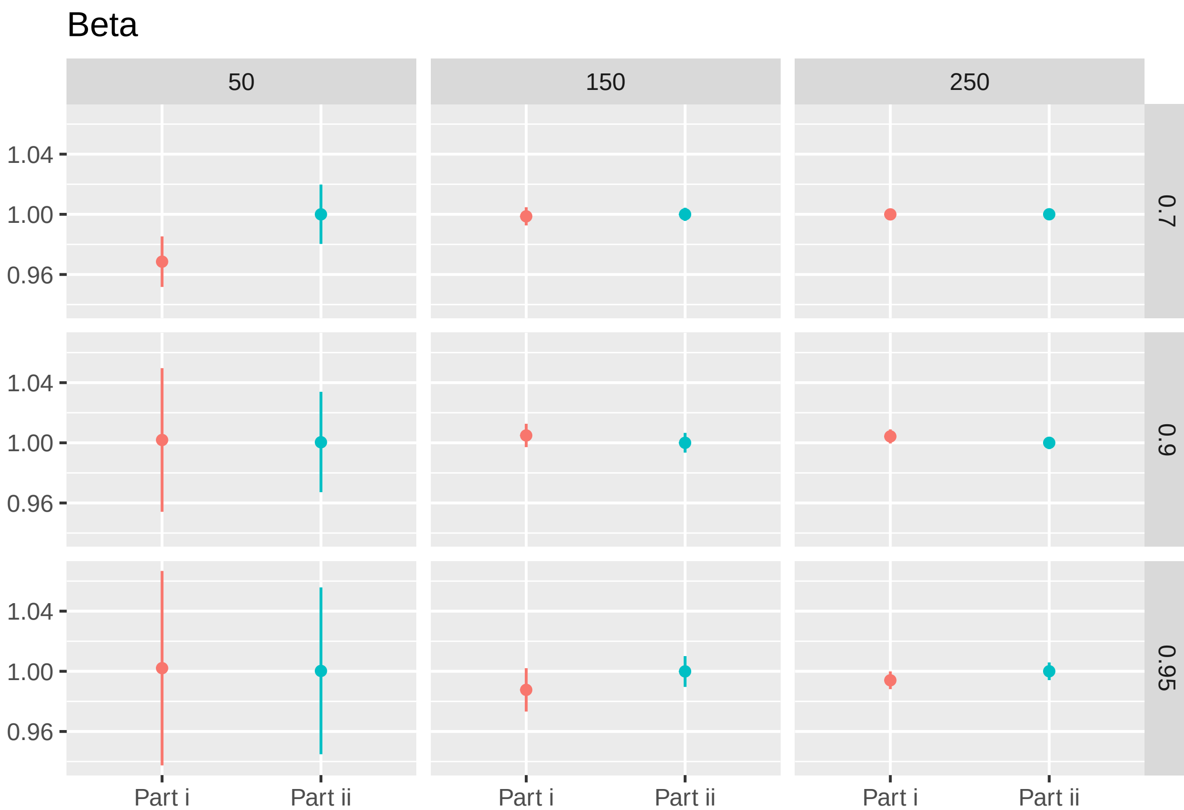
```
load("ps5_vonhafften.RData")

results2 %>%
  group_by(t, rho, name) %>%
  summarise(mean = mean(ols),
            lower_bound = quantile(ols, probs = .05),
            upper_bound = quantile(ols, probs = .95),
            .groups = "keep") %>%
  kable(digits = 3)
```

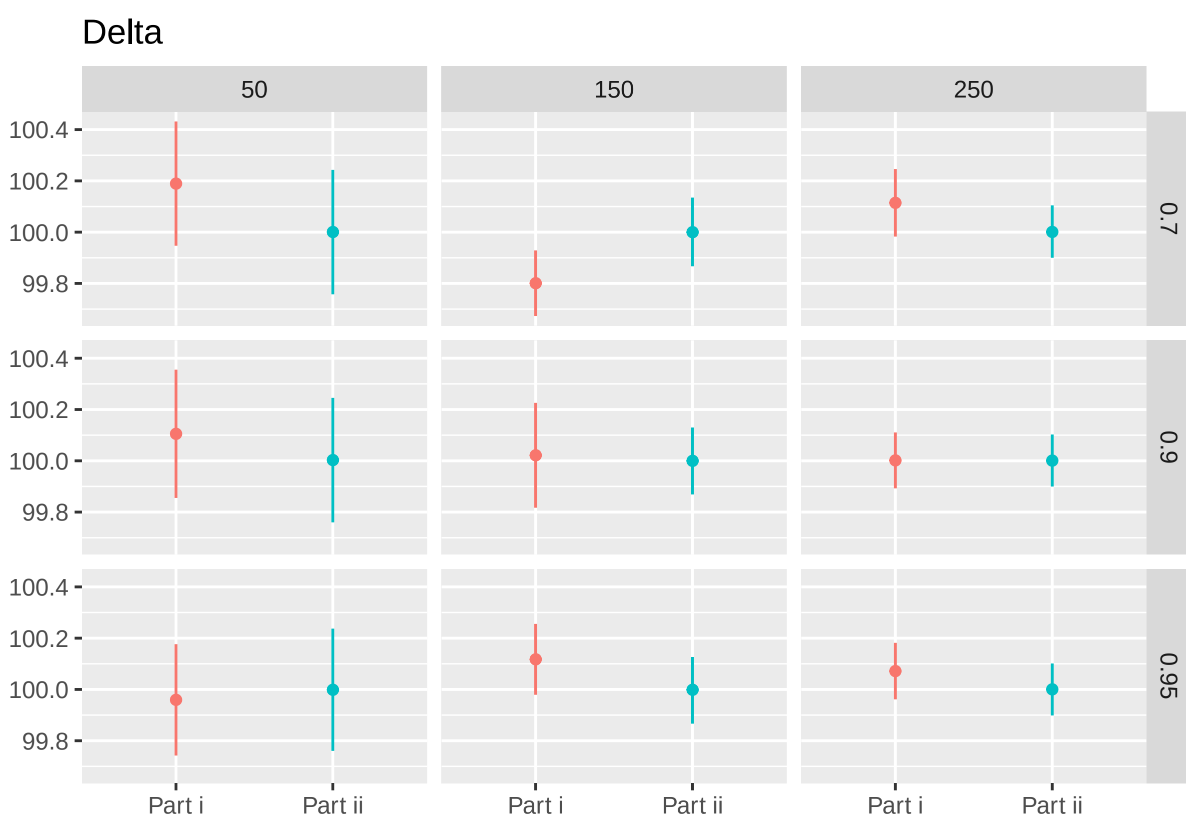
t	rho	name	mean	lower_bound	upper_bound
50	0.70	alpha	100.007	99.374	100.629
50	0.70	beta	1.000	0.980	1.020
50	0.70	delta	100.001	99.758	100.243
50	0.70	rho	0.700	0.698	0.702
50	0.90	alpha	99.999	99.363	100.640
50	0.90	beta	1.000	0.967	1.034
50	0.90	delta	100.003	99.760	100.245
50	0.90	rho	0.900	0.899	0.901
50	0.95	alpha	100.006	99.389	100.618
50	0.95	beta	1.000	0.945	1.056
50	0.95	delta	99.999	99.760	100.237
50	0.95	rho	0.950	0.949	0.951
150	0.70	alpha	99.999	99.637	100.365
150	0.70	beta	1.000	0.996	1.004
150	0.70	delta	100.000	99.867	100.135
150	0.70	rho	0.700	0.699	0.701
150	0.90	alpha	100.000	99.575	100.420
150	0.90	beta	1.000	0.993	1.007
150	0.90	delta	100.000	99.869	100.130
150	0.90	rho	0.900	0.900	0.900
150	0.95	alpha	100.001	99.575	100.420
150	0.95	beta	1.000	0.990	1.010
150	0.95	delta	99.999	99.867	100.127
150	0.95	rho	0.950	0.950	0.950
250	0.70	alpha	100.000	99.719	100.278
250	0.70	beta	1.000	0.997	1.003
250	0.70	delta	100.001	99.900	100.105
250	0.70	rho	0.700	0.699	0.701
250	0.90	alpha	100.002	99.665	100.343
250	0.90	beta	1.000	0.996	1.004
250	0.90	delta	100.001	99.900	100.103
250	0.90	rho	0.900	0.900	0.900
250	0.95	alpha	100.002	99.652	100.356
250	0.95	beta	1.000	0.994	1.006
250	0.95	delta	100.000	99.899	100.101
250	0.95	rho	0.950	0.950	0.950

(iii) How does sample size and the degree of persistence in Y_t affect the results of the simulations.

The three figures below have the point estimate (dots) and confidence intervals (vertical lines) from part i (red) and part ii (blue) where panels differ by sample size (horizontal) and degree of persistence (vertical). The point estimate for part i is the OLS estimate based on a single trial of simulated data and the confidence interval is the heteroskedastic robust standard error. The point estimate for part ii is the mean of OLS estimates over 10,000 trials of simulated data and the confidence interval is the 5th and 95th percentile. Naturally, the point estimates from part ii are closer to the true value than the point estimates from part i. In addition, large sample sizes result in point estimates that are closer to the true value and tighter confidence intervals. For β, we see that higher degrees of persistence dramatically expand confidence intervals particularly for small samples. For δ and α, we that the confidence intervals are similarly sized across degrees of persistence and shrink with larger samples.




Beta
50
150
250
1.04
1.00
0.96
0.7
0.9
0.95
Part i
Part ii


Delta
50
150
250
100.4
100.2
100.0
99.8
0.7
0.9
0.95
Part i
Part ii