Minimum Variance Frontier

Alex von Hafften

10/29/2021

- Objective: Describe the minimum variance frontier (MVF) i.e. the set of minimum variance portfolios. A minimum variance portfolio has the smallest variance for a given expected return.
- Assumption: single period; finite set of basis assets; finite expected returns, variances, and covariances; and nonsingular matrix of second moments.

Setup

- Let $r_b = x_b./S - 1$ be the vector of basis asset returns where x_b is vector of payoffs, S is vector of prices.
- By the law of one price, the return on portfolio α is $r_p = \alpha' r_b$
- Let $\mu \equiv E[r_b]$ be finite.
- Let $V \equiv E[r_b r_b'] - \mu\mu'$ be finite and nonsingular.
- For convenience, define three scalar constants:

$$
\begin{aligned}
A &= \mathbb{1}' V^{-1} \mu = \mu' V^{-1} \mathbb{1} \\
B &= \mu' V^{-1} \mu \\
C &= \mathbb{1}' V^{-1} \mathbb{1}
\end{aligned}
$$

- We first consider two special portfolios on the MVF then derive the MVF.

Global minimum variance portfolio

- The global minimum variance portfolio α_{mvp} satisfies:

$$\min_{\alpha} \frac{1}{2} \alpha' V \alpha + \theta(1 - \alpha' \mathbb{1})$$

- FOC $[\alpha]$:

$$V\alpha = \theta \mathbb{1} \implies \alpha = \theta V^{-1} \mathbb{1}$$

- Plugging back in constraint:

$$\theta \mathbb{1}' V^{-1} \mathbb{1} = 1 \implies \theta = \frac{1}{\mathbb{1}' V^{-1} \mathbb{1}} \implies \alpha_{mvp} = \frac{V^{-1} \mathbb{1}}{\mathbb{1}' V^{-1} \mathbb{1}} = \frac{1}{C} V^{-1} \mathbb{1}$$

- The expected value and variance of r_{mvp}:

$$E[r_{mvp}] = E[x_b'\alpha_{mvp}] = \frac{\mu' V^{-1}\mathbb{1}}{\mathbb{1}' V^{-1}\mathbb{1}} = \frac{A}{C}$$

$$Var[r_{mvp}] = \alpha_{mvp}' Var[x_b']\alpha_{mvp} = \frac{\mathbb{1}' V^{-1} V V^{-1}\mathbb{1}}{\mathbb{1}' V^{-1}\mathbb{1}\mathbb{1}' V^{-1}\mathbb{1}} = \frac{1}{\mathbb{1}' V^{-1}\mathbb{1}} = \frac{1}{C}$$

Tangency Portfolio

- The tangency portfolio α_μ has the maximum expected return per unit of standard deviation.
- Instead of directly finding the tangency portfolio, we can consider the dual problem of the portfolio $\tilde{\alpha}_\mu$ with the minimum variance given an expected return, find the portfolio for unit expected return, and then scale the portfolio so adds up to one:

$$\min_{\alpha} \frac{1}{2}\alpha' V\alpha + \lambda(E[r_p] - \alpha'\mu)$$

- FOC $[\alpha]$

$$V\alpha = \lambda\mu \implies \alpha = \lambda V^{-1}\mu$$

- Plugging into the constraint:

$$E[r_p] = \lambda\mu' V^{-1}\mu \implies \lambda = \frac{E[r_p]}{\mu' V^{-1}\mu} \implies \tilde{\alpha}_\mu(E[r_p]) = \frac{E[r_p]}{\mu' V^{-1}\mu} V^{-1}\mu$$

- At $E[r_p] = 1$,

$$\tilde{\alpha}_\mu(1) = \frac{1}{\mu' V^{-1}\mu} V^{-1}\mu$$

- The sum of the portfolio weight is (necessarily sum to one):

$$\mathbb{1}'\tilde{\alpha}_\mu(1) = \frac{\mathbb{1}' V^{-1}\mu}{\mu' V^{-1}\mu} = A/B$$

- Now, we scale $\tilde{\alpha}_\mu(1)$:

$$\alpha_\mu = \frac{B}{A}\tilde{\alpha}_\mu(1) = \frac{\mu' V^{-1}\mu}{\mathbb{1}' V^{-1}\mu}\frac{1}{\mu' V^{-1}\mu} V^{-1}\mu = \frac{1}{\mathbb{1}' V^{-1}\mu} V^{-1}\mu = \frac{1}{A} V^{-1}\mu$$

- The expected value and variance of r_μ:

$$E[r_\mu] = E[x_b'\alpha_\mu] = \frac{\mu' V^{-1}\mu}{\mathbb{1}' V^{-1}\mu} = \frac{B}{A}$$

$$Var[r_\mu] = \alpha_\mu' Var[x_b']\alpha_\mu = \frac{\mu' V^{-1} V V^{-1}\mu}{\mu' V^{-1}\mathbb{1}\mathbb{1}' V^{-1}\mu} = \frac{\mu' V^{-1}\mu}{\mu' V^{-1}\mathbb{1}\mathbb{1}' V^{-1}\mu} = \frac{B}{A^2}$$

Minimum Variance Frontier

- Portfolios on the minimum variance frontier satisfy

$$\frac{1}{2}\alpha' V\alpha + \theta(1-\alpha'\mathbb{1}) + \lambda(E[r_p] - \alpha'\mu)$$

- FOC $[\alpha]$:

$$V\alpha = \theta\mathbb{1} + \lambda\mu \implies \alpha = \theta V^{-1}\mathbb{1} + \lambda V^{-1}\mu$$

- Plugging into constraints:

$$\begin{aligned}
&\begin{cases} \mathbb{1}'(\theta V^{-1}\mathbb{1} + \lambda V^{-1}\mu) & = 1 \\ \mu'(\theta V^{-1}\mathbb{1} + \lambda V^{-1}\mu) & = E[r_p] \end{cases} \\
\implies &\begin{cases} C\theta + A\lambda & = 1 \\ A\theta + B\lambda & = E[r_p] \end{cases} \\
\implies &\begin{cases} \lambda & = A^{-1} - A^{-1}C\theta \\ \theta & = A^{-1}E[r_p] - A^{-1}B\lambda \end{cases} \\
\implies &\theta = A^{-1}E[r_p] - A^{-1}B[A^{-1} - A^{-1}C\theta] \\
\implies \theta(1 - A^{-2}BC) &= A^{-1}E[r_p] - A^{-2}B \\
\implies \theta &= \frac{A^{-1}E[r_p] - A^{-2}B}{1 - A^{-2}BC} \\
&= \frac{AE[r_p] - B}{A^2 - BC} \\
\implies \lambda &= A^{-1} - A^{-1}C\left[\frac{AE[r_p] - B}{A^2 - BC}\right] \\
&= \frac{A - A^{-1}BC - CE[r_p] + A^{-1}BC}{A^2 - BC} \\
&= \frac{A - CE[r_p]}{A^2 - BC}
\end{aligned}$$

- Notice that λ, θ depend on $E[r_p]$, so we denote with them λ_p, θ_p.

Two fund separation on MVF

- Two fund separation holds on the MVF with α_{mvp} and α_μ:

$$\alpha_p = (\theta_p C)\frac{1}{C}V^{-1}\mathbb{1} + (\lambda_p A)\frac{1}{A}V^{-1}\mu = b_p\alpha_{mvp} + (1 - b_a)\alpha_\mu$$

where

$$b_p = \theta_p C$$

- Notice that

$$1 - b_p = 1 - \theta_p C = \frac{A^2 - BC - ACE[r_p] + BC}{A^2 - BC} = \frac{A^2 - ACE[r_p]}{A^2 - BC} = A\lambda_p$$