ECON 714B - Problem Set 4

Alex von Hafften*

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Problem 1 (50 points)

Suppose that an infinitely lived government has to finance a fixed stream of expenditures, $\{g_t\}_{t\geq 0}$ and can only use consumption taxes for this purpose. Assume that the representative consumer has the utility function:

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + v(\ell_t) \right]$$

where c_t is the consumption in period t and ℓ_t is leisure in period t. Assume that $\sigma > 0$ and v is an increasing function. Also assume that the production function, $F(K, L)$ satisfies all the standard assumptions (i.e., CRS, etc.), that the representative household has an initial endowment of the capital stock, $k_0, \ell_t \leq 1$ and that capital is subject to the usual law of motion, $k_{t+1} = (1-\delta)k_t + x_t$. Set up the Ramsey Problem for this economy, and show that the optimal policy is to set the consumption tax at a constant rate from period one onwards (i.e., show that $\tau_t^{RP} = \tau_{t+1}^{RP}$ for all $t \geq 1$.

[I'm assuming that the HH is endowed with one unit of time with which they can consume $\ell_t \leq 1$ leisure and supply $1 - \ell_t \leq 1$ units of labor.]

The feasibility constraint is:

$$c_t + g_t + k_t = F(k_{t-1}, 1-\ell_t) + (1-\delta)k_{t-1}$$

To find the implementability constraint, we start by defining the HH problem:

$$\max_{c_t, \ell_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + v(\ell_t) \right]$$

$$\text{s.t. } (1+\tau_t)c_t + k_t + b_t = w_t(1-\ell_t) + (1-\delta+r_t)k_{t-1} + R_t^b b_{t-1}$$

Let p_t be the multiplier on the budget constraint, so the legrangian is

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + v(\ell_t) \right] + p_t \Big[w_t(1-\ell_t) + (1-\delta+r_t)k_{t-1} + R_t^b b_{t-1} - (1+\tau_t)c_t - k_t - b_t \Big]$$

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The FOCs are:

$$\beta^t c_t^{-\sigma} = p_t(1+\tau_t) \qquad [c_t] \tag{1}$$
$$\beta^t v'(\ell_t) = p_t w_t \qquad [\ell_t] \tag{2}$$
$$[p_t - p_{t+1}R_{t+1}^b]b_t = 0 \qquad [b_t] \tag{3}$$
$$[p_t - p_{t+1}(1+r_{t+1}-\delta)]k_t = 0 \qquad [k_t] \tag{4}$$

Multiply the HH budget constraint and sum across t:

$$\sum_{t=0}^{\infty} p_t[(1+\tau_t)c_t + k_t + b_t] = \sum_{t=0}^{\infty} p_t[w_t(1-\ell_t) + (1-\delta+r_t)k_{t-1} + R_t^b b_{t-1}]$$

Substituting in (3), we can cancel out bond holdings in every period except for the initial period:

$$\sum_{t=0}^{\infty} p_t[(1+\tau_t)c_t + k_t] = p_0 R_0^b b_{-1} + \sum_{t=0}^{\infty} p_t[w_t(1-\ell_t) + (1-\delta+r_t)k_{t-1}]$$

Substituting in (4), we can cancel out capital holdings in every period except for the initial period:

$$\sum_{t=0}^{\infty} p_t(1+\tau_t)c_t = p_0 R_0^b b_{-1} + p_0(1-\delta+r_0)k_{-1} + \sum_{t=0}^{\infty} p_t w_t(1-\ell_t)$$

Substituting in (5) and (6), we get the implementability constraint:

$$\sum_{t=0}^{\infty} \beta^t c_t^{-\sigma} c_t = p_0[R_0^b b_{-1} + (1-\delta+r_0)k_{-1}] + \sum_{t=0}^{\infty} \beta^t v'(\ell_t)(1-\ell_t)$$

$$\implies \sum_{t=0}^{\infty} \beta^t[c_t^{1-\sigma} - v'(\ell_t)(1-\ell_t)] = \frac{c_0^{-\sigma}}{1+\tau_0}[R_0^b b_{-1} + (1-\delta+r_0)k_{-1}]$$

Thus, the feasibility and implementability constraints are necessary and sufficient conditions for an allocation to be a CE. Thus, the Ramsey problem is

$$\max_{c_t,\ell_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + v(\ell_t)\right]$$

$$\text{s.t.} \ \sum_{t=0}^{\infty} \beta^t[c_t^{1-\sigma} - v'(\ell_t)(1-\ell_t)] = \frac{c_0^{-\sigma}}{1+\tau_0}[R_0^b b_{-1} + (1-\delta+r_0)k_{-1}]$$

$$\text{and } c_t + g_t + k_t = F(k_{t-1}, 1-\ell_t) + (1-\delta)k_{t-1}, \forall t$$

We can rewrite the Ramsay problem as:

$$\max_{c_t,\ell_t}\sum_{t=0}^{\infty}\beta^t\left[\frac{c_t^{1-\sigma}}{1-\sigma}+v(\ell_t)\right]+\lambda\left[\sum_{t=0}^{\infty}\beta^t[c_t^{1-\sigma}-v'(\ell_t)(1-\ell_t)]-\frac{c_0^{-\sigma}}{1+\tau_0}[R_0^b b_{-1}+(1-\delta+r_0)k_{-1}]\right]$$

$$\implies \max_{c_t,\ell_t}\sum_{t=0}^{\infty}\beta^t\left[\frac{c_t^{1-\sigma}}{1-\sigma}+v(\ell_t)+\lambda[c_t^{1-\sigma}-v'(\ell_t)(1-\ell_t)]\right]-\lambda\frac{c_0^{-\sigma}}{1+\tau_0}[R_0^b b_{-1}+(1-\delta+r_0)k_{-1}]$$

$$\implies \max_{c_t,\ell_t}\sum_{t=0}^{\infty}\beta^t w(c_t,\ell_t,\lambda)-\lambda\frac{c_0^{-\sigma}}{1+\tau_0}[R_0^b b_{-1}+(1-\delta+r_0)k_{-1}]$$

$$\text{s.t. } c_t+g_t+k_t=F(k_{t-1},1-\ell_t)+(1-\delta)k_{t-1},\forall t$$

where $w(c_t,\ell_t,\lambda):=\frac{c_t^{1-\sigma}}{1-\sigma}+v(\ell_t)+\lambda[c_t^{1-\sigma}-v'(\ell_t)(1-\ell_t)]$. Assume that τ_0 is bounded. Thus, the Ramsey problem is:

$$\max_{c_t,\ell_t}\sum_{t=0}^{\infty}\beta^t w(c_t,\ell_t,\lambda)$$

$$\text{s.t. } c_t+g_t+k_t=F(k_{t-1},1-\ell_t)+(1-\delta)k_{t-1},\forall t$$

Let γ_t be the multiplier on the feasibility constraint:

$$\sum_{t=0}^{\infty}\beta^t w(c_t,\ell_t,\lambda)+\gamma_t[F(k_{t-1},1-\ell_t)+(1-\delta)k_{t-1}-c_t-g_t-k_t]$$

The FOCs are

$$\begin{aligned}\beta^t w_1(c_t,\ell_t,\lambda)&=\gamma_t && [c_t] && (5)\\ \beta^t w_2(c_t,\ell_t,\lambda)&=\gamma_t F_2(k_{t-1},1-\ell_t) && [\ell_t] && (6)\\ \gamma_{t+t}[F_1(k_t,1-\ell_{t+1})+(1-\delta)]&=\gamma_t && [k_t] && (7)\end{aligned}$$

(5) and (6) imply an intra-temporal FOC:

$$\frac{w_2(c_t,\ell_t,\lambda)}{w_1(c_t,\ell_t,\lambda)}=F_2(k_{t-1},1-\ell_t)$$

(5) and (7) imply an inter-temporal FOC:

$$\frac{w_1(c_t,\ell_t,\lambda)}{w_1(c_{t+1},\ell_{t+1},\lambda)}=\beta[1-\delta+F_1(k_t,1-\ell_{t+1})] \tag{8}$$

Notice that:

$$w_1(c_t, \ell_t, \lambda) = c_t^{-\sigma} + \lambda(1-\sigma)c_t^{-\sigma} = (1+\lambda-\lambda\sigma)c_t^{-\sigma}$$

$$\implies \frac{w_1(c_t, \ell_t, \lambda)}{w_1(c_{t+1}, \ell_{t+1}, \lambda)} = \frac{(1+\lambda-\lambda\sigma)c_{t+1}^{-\sigma}}{(1+\lambda-\lambda\sigma)c_t^{-\sigma}} = \frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}}$$

Thus, (8) becomes:

$$\implies \frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} = \beta[1-\delta+F_1(k_t, 1-\ell_{t+1})] \tag{9}$$

Let us compare (9) with the HH's intertemporal FOC. In a competitive equilibrium, firms optimize so $r_t = F_1(k_{t-1}, 1-\ell_t)$. Combining this with (1) and (4), we get

$$\frac{c_t^{-\sigma}}{c_{t+1}^{-\sigma}} = \beta\frac{1+\tau_t}{1+\tau_{t+1}}[1+F_1(k_{t-1}, 1-\ell_t)-\delta]$$

For the Ramsey intertemporal FOC and the HH intertemporal FOC to both hold:

$$\frac{1+\tau_t}{1+\tau_{t+1}} = 1 \implies \tau_t = \tau_{t+1}$$

Thus, consumption taxes should be constant for all periods $t \geq 1$.

Problem 2 (50 points)

Consider a cash-credit goods economy with preferences given by

$$\log c_{1,t} + \alpha \log c_{2,t} + \gamma \log(1 - n_t)$$

where n_t is the time spent in market activities. The resource constraint is

$$c_{1,t} + c_{2,t} = n_t$$

The cash-in-advance constraint is

$$p_t c_{1,t} \leq M_t$$

The budget constraint for the HH at the beginning of the period is

$$M_t + B_t \leq (M_{t-1} - p_{t-1} c_{1,t-1}) - p_{t-1} c_{2,t-1} + w_{t-1} n_{t-1} + R_{t-1} B_{t-1} - T_t$$

where T_t denotes lump-sum taxes and all the terms are as we discussed in class. The government conducts monetary policy to keep the interest rate fixed at some level R in all periods.

1. Define a competitive equilibrium.

Notice that the production function is $F(n_t) = n_t$. If firms are competitive, then the real wage is the marginal product of labor, so it equals one and the nominal wage is $w_t = p_t$.

A competitive equilibrium is an allocation $x = \{(c_{1,t}, c_{2,t}, n_t)\}_{t=0}^{\infty}$, a price system $q = \{(p_t, R_t)\}_{t=0}^{\infty}$, and a policy $\pi = \{(M_t, B_t, T_t)\}_{t=0}^{\infty}$ such that

(1) Given π and q, x solves the HH problem:

$$\max_{(c_{1,t}, c_{2,t}, n_t)} \sum_{t=0}^{\infty} \beta^t [\log c_{1,t} + \alpha \log c_{2,t} + \gamma \log(1 - n_t)]$$

$$\text{s.t. } M_t + B_t = (M_{t-1} - p_{t-1} c_{1,t-1}) - p_{t-1} c_{2,t-1} + p_{t-1} n_{t-1} + R_{t-1} B_{t-1} - T_t$$

$$\text{and } p_t c_{1,t} = M_t$$

(2) x, q, and π satisfy the government BC:

$$M_t - M_{t-1} + B_t + T_t = R_{t-1} B_{t-1}$$

(3) Markets clear:

$$c_{1,t} + c_{2,t} = n_t$$

From the problem setup, we know that $R_t = R$ for all t.

2. What happens to n_t as R increases. Prove your result.

Assuming that α and $\alpha + \gamma$ are positive, an increase in R leads to be increase in n_t.

Let's solve for a competitive equilibrium. Let λ_t be the multiplier on the HH budget constraint and ξ_t be the multiplier on the cash-in-advance constraint:

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta^t [\log c_{1,t} + \alpha \log c_{2,t} + \gamma \log(1 - n_t)] \\
&\quad + \lambda_t [(M_{t-1} - p_{t-1} c_{1,t-1}) - p_{t-1} c_{2,t-1} + p_{t-1} n_{t-1} + RB_{t-1} - T_t - M_t - B_t] \\
&\quad + \xi_t [M_t - p_t c_{1,t}]
\end{aligned}$$

Thus, the FOCs are:

$$\frac{\beta^t}{c_{1,t}} = \lambda_{t+1} p_t + \xi_t p_t \qquad [c_{1,t}] \tag{10}$$

$$\frac{\beta^t \alpha}{c_{2,t}} = \lambda_{t+1} p_t \qquad [c_{2,t}] \tag{11}$$

$$\frac{\beta^t \gamma}{1 - n_t} = -\lambda_{t+1} p_t \qquad [n_t] \tag{12}$$

$$\lambda_t = \lambda_{t+1} R \qquad [B_t] \tag{13}$$

$$\lambda_t = \lambda_{t+1} + \xi_t \qquad [M_t] \tag{14}$$

(13) and (14) imply

$$\lambda_{t+1} R = \lambda_{t+1} + \xi_t$$

Substituting in (10), we get

$$R \lambda_{t+1} p_t = \frac{\beta^t}{c_{1,t}}$$

Substituting in (12), we get

$$R \frac{\beta^t \alpha}{c_{2,t}} = \frac{\beta^t}{c_{1,t}} \implies c_{1,t} = \frac{c_{2,t}}{R\alpha}$$

Combining (11) and (12), we get

$$\frac{\alpha}{c_{2,t}} = \frac{-\gamma}{1 - n_t} \implies c_{2,t} = \frac{-\alpha(1 - n_t)}{\gamma}$$

Market clearing implies:

$$n_t = \frac{1}{R\alpha} \frac{-\alpha(1 - n_t)}{\gamma} + \frac{-\alpha(1 - n_t)}{\gamma} \implies n_t = \frac{1 + \alpha R}{1 - (\alpha + \gamma) R}$$

Assuming α is positive, an increase in R increases the numerator. Assuming $\alpha + \gamma$ is positive, an increase in R decreases the denominator. Thus, an increase in R increases n_t.