

ECON 713B - Problem Set 1

Alex von Hafften*

4/1/2021

1 All Pay Auction

Consider a symmetric IPV (independent private values) setting with N bidders. Find an equilibrium of the all-pay auction when each bidder's valuation is an iid draw from $F(x) = x^a$ for $a \in (0, \infty)$ and $x \in [0, 1]$.

(a) Define this auction as a Bayesian game.

A Bayesian game is a five-tuple $(I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot))$:

- The set of players is $I = \{1, \dots, N\}$.
- The action set of player $i \in I$ is $S_i = [0, \infty)$.
- The payoff for player $i \in I$ is

$$u_i(b_1, \dots, b_N; v_1, \dots, v_N) = u_i(b_1, \dots, b_N; v_i) \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{2}(v_i - b_i) + \frac{1}{2}(-b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ -b_i & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

- $\Theta = [0, 1] \times \dots \times [0, 1]$.
- $F(x) = x^a$ for $a \in (0, \infty)$

(b) Find equilibrium strategies of all players.

Focus on BNE with symmetric, strictly increasing, and differentiable bids $b(v_i)$. Since the F is continuous and $b(v_i)$ is strictly increasing, the probability of a tie is zero. The expected payoff for bidder i is:

$$\begin{aligned} E[u_i(b_1, \dots, b_N; v_i)] &= (v_i - b_i) \Pr(b_i > b_j, \forall j \neq i) + (-b_i) \Pr(b_i < b_j, \forall j \neq i) \\ &= v_i \Pr(b_i > b_j, \forall j \neq i) - b_i \end{aligned}$$

Suppose bidder $j \neq i$ submit $b(v_j)$:

$$\begin{aligned} \Pr(b_i > b_j, \forall j \neq i) &= \Pr(b(v_i) > b(v_j), \forall j \neq i) \\ &= \Pr(b^{-1}(b(v_i)) > v_j, \forall j \neq i) \\ &\stackrel{iid}{=} F(b^{-1}(b(v_i)))^{N-1} \\ &= ((b^{-1}(b(v_i)))^a)^{N-1} \\ &= (b^{-1}(b(v_i)))^{aN-a} \end{aligned}$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

Thus, the expected payoff of bidder i is:

$$E[u_i(b_1, \dots, b_N; v_i)] = v_i(b^{-1}(b(v_i)))^{aN-a} - b(v_i)$$

FOC $[b(v_i)]$:

$$\begin{aligned} 0 &= (aN - a)v_i(b^{-1}(b(v_i)))^{aN-a-1} \frac{1}{b'(v_i)} - 1 \\ \implies b'(v_i) &= (aN - a)v_i^{aN-a} \\ \implies b(v_i) &= \frac{aN - a}{aN - a + 1} v_i^{aN-a+1} + c_i \end{aligned}$$

For $v_i = 0$, bidder i would bid zero: $b(v_i) = 0 \implies c_i = 0$.

$$\implies b(v_i) = \frac{aN - a}{aN - a + 1} v_i^{aN-a+1}$$

Since $aN - a + 1 > 0$ and $\frac{aN-a}{aN-a+1}$, b is strictly increasing.

(c) Verify that the strategies that you have found do constitute an equilibrium.

We can verify that b is an equilibrium strategy by verifying that $b(v_i)$ is the best response for player i when bidders $j \neq i$ bid $b(v_j)$.

$$\begin{aligned} E[u_i(b_1, \dots, b_N; v_i)] &= v_i \Pr \left(b_i > \frac{aN - a}{aN - a + 1} v_j^{aN-a+1}, \forall j \neq i \right) - b_i \\ &= v_i \Pr \left(\left(\frac{aN - a + 1}{aN - a} b_i \right)^{\frac{1}{aN-a+1}} > v_j, \forall j \neq i \right) - b_i \\ &= v_i \left(\frac{aN - a + 1}{aN - a} b_i \right)^{\frac{aN-a}{aN-a+1}} - b_i \end{aligned}$$

FOC $[b_i]$:

$$\begin{aligned} 0 &= v_i \frac{aN - a}{aN - a + 1} \left(\frac{aN - a + 1}{aN - a} b_i \right)^{\frac{aN-a}{aN-a+1}-1} \frac{aN - a + 1}{aN - a} - 1 \\ \implies \frac{1}{v_i} &= \left(\frac{aN - a + 1}{aN - a} b_i \right)^{\frac{-1}{aN-a+1}} \\ \implies b_i &= b(v_i) = \frac{aN - a}{aN - a + 1} v_i^{aN-a+1} \end{aligned}$$

Thus, all bidders playing $b(\cdot)$ is an equilibrium.

(d) Does the bidding become more competitive when a increases? Explain.

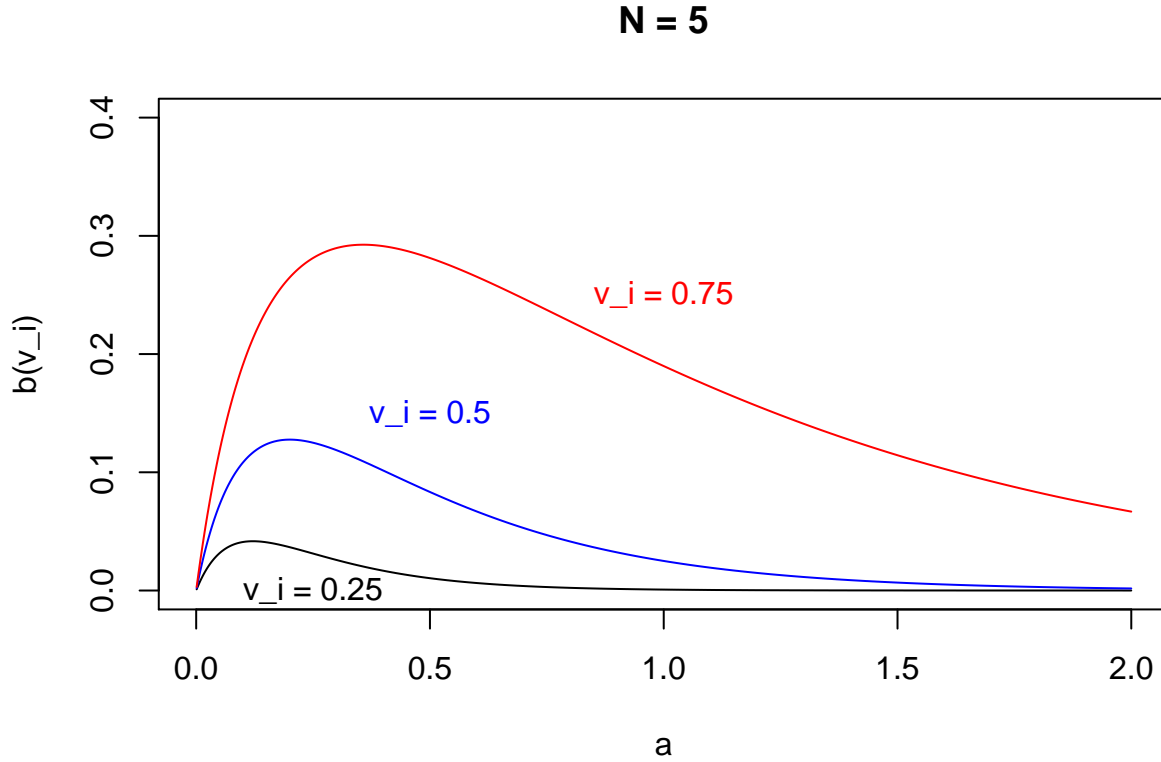
Conditional on v_i and N , a higher a results in more mass closer to 1. Thus, the probability that bidder i wins the auction decreases, so bidder i should decrease her bid. This makes the bidding less competitive. Unconditional on v_i , a higher a results in higher realizations of v_i , so the bids are correspondingly larger.

Consider the derivative of the bidder function with respect to a

$$\begin{aligned}\frac{\partial}{\partial a} b(v_i) &= \frac{\partial}{\partial a} \left(\frac{aN - a}{aN - a + 1} v_i^{aN - a + 1} \right) \\ &= \frac{N - 1}{aN - a + 1} v_i^{aN - a + 1} - \frac{(aN - a)(N - 1)}{(aN - a + 1)^2} v_i^{aN - a + 1} + \frac{(N - 1)(aN - a)}{aN - a + 1} v_i^{aN - a + 1} \log(v_i)\end{aligned}$$

The derivative is not strictly positive or negative, so increasing a may increase competition and may decrease competition. But for a sufficiently large a , the derivative is negative, so an increase in a decreases a bid conditional on v_i .

We can see that a higher a reduces bids for a sufficiently large a in the figure below ($N = 5, a \in (0, 2), v_i \in \{0.25, 0.5, 0.75\}$):



(e) Compute the expected payoff from each bidder before and after she learns her value.

The expected payoff of bidder i conditional on v_i is:

$$\begin{aligned}
 E[u_i(v_i)] &= v_i^{aN-a+1} - b(v_i) \\
 &= v_i^{aN-a+1} - \frac{aN-a}{aN-a+1} v_i^{aN-a+1} \\
 &= \frac{v_i^{aN-a+1}}{aN-a+1}
 \end{aligned}$$

The unconditional expected payoff of bidder i is:

$$\begin{aligned}
 \int_0^1 E[u_i(v_i)] f(v_i) dv_i &= \int_0^1 \frac{v_i^{aN-a+1}}{aN-a+1} a v_i^{a-1} dv_i \\
 &= \int_0^1 \frac{a v_i^{aN}}{aN-a+1} dv_i \\
 &= \left[\frac{a v_i^{aN+1}}{(aN-a+1)(aN+1)} \right]_0^1 \\
 &= \frac{a}{(aN-a+1)(aN+1)}
 \end{aligned}$$

2 Tricky Seller

Two people are interested in one object. Their valuations are drawn independently from $F(x) = x$ and $F(x) = x^2$, respectively, with $x \in [0, 1]$. The seller's value (a cost, perhaps) for the object is known, $c \in [0, 1]$.

- (a) Describe outcome of the First-Price Auction with a reserve price r .

...

- (b) Describe outcome of the Second-Price Auction with a reserve price r .

...

- (c) What auction and what r will the seller choose? Which player wins more often?

...

- (d) Suppose now that $c = 0$ and there is no reserve price. Suppose that a seller can offer discount of α to one of the bidders in the second-price auction. If a bidder is offered a discount $\alpha \in [0; 1]$, then, if she wins, she pays only a fraction α of what she had to pay otherwise. Who should be offered a discount? Compute the optimal discount and expected revenues.

...

3 Third Price Auction

Consider a third-price auction with three players: an auction in which bidder with the highest value wins, but pays only the third highest bid. Assume that valuation of players are iid from the uniform distribution on $[0, 1]$.

- (a) Define the auction as a Bayesian game.

...

- (b) Prove that a bid of $b_i(v_i) = \frac{n-1}{n-2}v_i$ is a symmetric Bayes-Nash equilibrium of the third-price auction.

...

- (c) Show that the expected revenue of a seller in the third-price auction is $R_3 = \frac{n-1}{n+1}$.

...

- (d) What is the symmetric Bayes-Nash equilibrium strategy in a k th price auction? (You need only state how each bidder bids; you need not provide a detailed analysis.)

...