ECON 711 - PS 2

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Question 1. Convex production sets, concave production functions, convex costs

Consider a production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ for a single-output firm.

(a) Prove that if the production set $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$ is convex, the production function f is concave.

Proof: Choose $(q, -z), (q', -z') \in Y$ such that $f(z) = q$ and $f(z') = q'$. The convexity of Y implies that $t(q, -z) + (1-t)(q', -z') \in Y$ for $t \in (0, 1)$. Thus, $f(tz + (1-t)z') \geq tq + (1-t)q'$ by the definition of Y. Our choice of $(q, -z), (q', -z') \implies f(tz + (1-t)z') \geq tf(z) + (1-t)f(z')$. Therefore, f is concave. □

(b) Prove that if f concave, the cost function

$$c(q, w) = \min w \cdot z \text{ subject to } f(z) \geq q$$

is convex in q.

Proof: Fixing $w \in \mathbb{R}^k_+$, choose $q, q' \in \mathbb{R}$. Define $z \in Z^*(q, w)$, $z' \in Z^*(q', w)$, and $\tilde{z} \in Z^*(tq + (1-t)q', w)$ for $t \in (0, 1)$. By the concavity of f,

$$\begin{aligned} \tilde{z} &\leq tz + (1-t)z' \\ \implies w\tilde{z} &\leq w(tz + (1-t)z') \\ \implies w\tilde{z} &\leq twz + (1-t)wz') \\ \implies c(f(\tilde{z}), w) &\leq tc(f(z), w) + (1-t)c(f(z'), w) \\ \implies c(tq + (1-t)q', w) &\leq tc(q, w) + (1-t)c(q', w) \end{aligned}$$

Therefore, c is convex. □

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, Tyler Welch, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

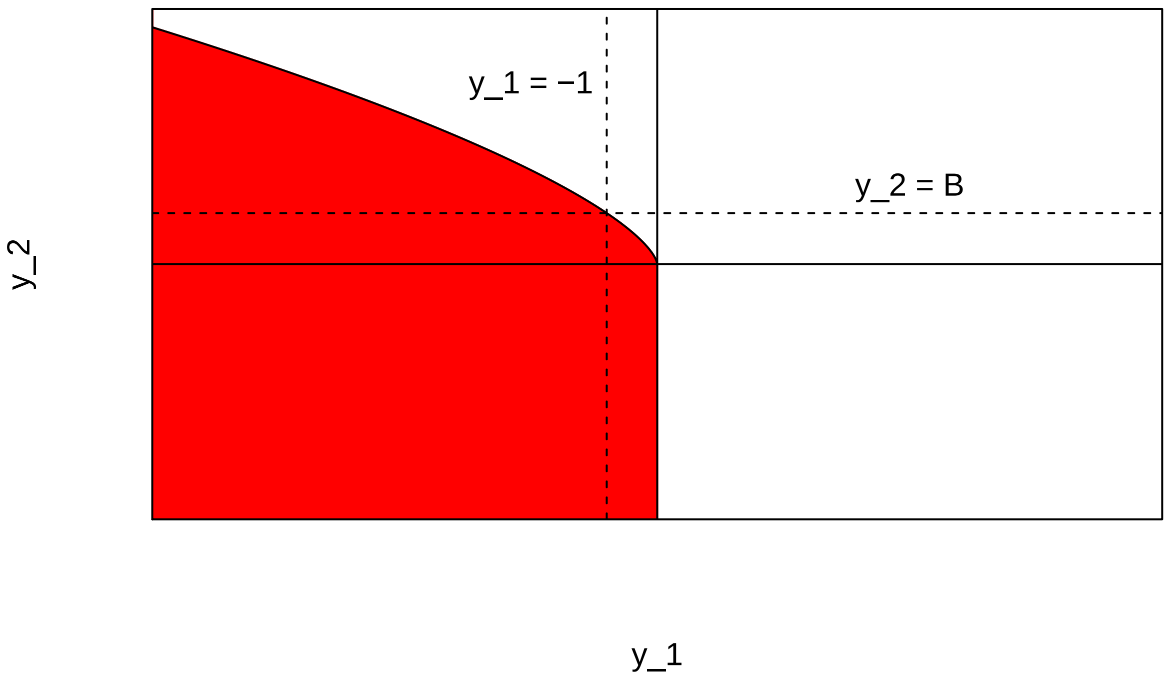
Question 2. Solving for the profit function given technology...

Let $k = 2$, and let the production set be

$$Y = \{(y_1, y_2) : y_1 \leq 0 \text{ and } y_2 \leq B(-y_1)^{\frac{2}{3}}\}$$

where $B > 0$ is a known constant. Assume both prices are strictly positive.

(a) Draw Y, or describe it clearly.



(b) Solve the firm's profit maximization problem to find $\pi(p)$ and $Y^*(p)$.

The firm's profit is

$$\pi(p) = \max_{y_1, y_2 \in Y} \{p_1 y_1 + p_2 y_2\}$$

Define $z = -y_1$. Notice that the firm will produce $y_2 = Bz^{2/3}$ because it is the maximum output given z units of input. Thus, we can rewrite the firm's profit function as

$$\pi(p) = \max_z \{p_1(-z) + p_2 B z^{2/3}\} = \max_z \{p_2 B z^{2/3} - p_1 z\}$$

Setting the first order condition of the profit function to zero:

$$\frac{\partial \pi}{\partial z} = p_2(2/3)Bz^{-1/3} - p_1$$
$$z^* = \left(\frac{2p_2 B}{3p_1}\right)^3$$

Plugging z^* into transformations for y_1, y_2:

$$
\begin{aligned}
y_1^* &= -\left(\frac{2p_2B}{3p_1}\right)^3 \\
y_2^* &= B\left(\left(\frac{2p_2B}{3p_1}\right)^3\right)^{2/3} \\
&= B^3\left(\frac{2p_2}{3p_1}\right)^2
\end{aligned}
$$

Notice that $Y^*(p)$ is single-valued:

$$
Y^*(p) = \left\{(y_1, y_2) : y_1 = -\left(\frac{2p_2B}{3p_1}\right)^3, y_2 = B^3\left(\frac{2p_2}{3p_1}\right)^2\right\} \implies y(p) = \left(-\left(\frac{2p_2B}{3p_1}\right)^3, B^3\left(\frac{2p_2}{3p_1}\right)^2\right)
$$

Thus, the profit function is

$$
\begin{aligned}
\pi(p) &= p_1\left(-\left(\frac{2p_2B}{3p_1}\right)^3\right) + p_2\left(B^3\left(\frac{2p_2}{3p_1}\right)^2\right) \\
&= \frac{3 * 2^2 p_2^3 B^3 - 2^3 p_1^2 p_2^3 B}{3^3 p_1^2} \\
&= \frac{4B^3 p_2^3}{3^3 p_1^2}(3 - 2p_1^2)
\end{aligned}
$$

(c) Since $Y * (p)$ is single-valued, I'll refer to it below as $y(p)$. Verify that $\pi(\cdot)$ is homogeneous of degree 1, and $y(\cdot)$ is homogeneous of degree 0.

(d) Verify that $y_1(p) = \frac{\partial \pi}{\partial p_1}(p)$ and $y_2(p) = \frac{\partial \pi}{\partial p_2}(p)$.

(e) Calculate $D_p y(p)$, and verify it is symmetric, positive semidefinite, and $[D_p y]p = 0$

Question 3 ...and recovering technology from the profit function

Finally, suppose we didn't know a firm's production set Y, but did know its profit function was

$$\pi(p)Ap_1^{-2}p_2^3$$

for all $p_1, p_2 > 0$ and $A > 0$ a known constant.

(a) What conditions must hold for this profit function to be rationalizable? (You don't need to check them.)

(b) Recall that the outer bound was defined