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Econ 709 Final Exam

(1) (a) ~~From the partition regression formula, we know that~~

Define $M_1 = I - X_1(X_1'X_1)^{-1}X_1'$

$M_W = I - W(W'W)^{-1}W'$

From the partition regression formula (Frisch-Waugh) we know that

$$\hat{\gamma}_2 = (W'M_1W)^{-1}(W'M_1y)$$

$$= ((X_1+X_2)'M_1(X_1+X_2))^{-1}((X_1+X_2)'M_1y)$$

[M_1 is ideompotent]

$$= ((M_1(X_1+X_2))'(M_1(X_1+X_2)))^{-1}((M_1(X_1+X_2))'y)$$

$$= ((M_1X_1 + M_1X_2)'(M_1X_1 + M_1X_2))^{-1}(M_1X_1 + M_1X_2)'y$$

$$= ((M_1X_2)'(M_1X_2))^{-1}(M_1X_2)'y$$

[M_1 is ideompotent]

$$= (X_2'M_1X_2)^{-1}(X_2'M_1y)$$

$$= \hat{\beta}_2$$

① (a) con't

From partition regression formula, we know that

$$\hat{\gamma}_1 = (X_1'X_1)^{-1} X_1'(y - W \hat{\gamma}_2)$$

$$= (X_1'X_1)^{-1} X_1'(y - W \hat{\beta}_2)$$

$$= (X_1'X_1)^{-1} X_1'(y - X_1 \hat{\beta}_2 - \hat{X}_2 \hat{\beta}_2)$$

$$= (X_1'X_1)^{-1} X_1'(y - \hat{X}_2 \hat{\beta}_2) - (X_1'X_1)^{-1} X_1' X_1 \hat{\beta}_2$$

$$= \hat{\beta}_1 - \hat{\beta}_2$$

Thus $\hat{\gamma} = (\hat{\beta}_1 - \hat{\beta}_2 \quad \hat{\beta}_2)'$

① (b)

$$\hat{\gamma} = (W'W)^{-1} W'y$$

$$= ((X_1+X_2)'(X_1+X_2))^{-1} (X_1+X_2)'y$$

$$= (X_1'X_1 + X_1'X_2 + X_2'X_1 + X_2'X_2)^{-1} (X_1+X_2)'y$$

$$= (2X_1'X_1)^{-1} (X_1+X_2)'y$$

$$= \frac{1}{2}\left[(X_1'X_1)^{-1} X_1'y + (X_2'X_2)^{-1} X_2'y \right]$$

$$= \frac{1}{2}\left[\hat{\beta}_1 + \hat{\beta}_2 \right]$$

~~Note~~ Note that $X_1' X_2 = 0$ implies that X_1 and X_2 are orthogonal, so $\hat{\beta}_1$ and $\hat{\beta}_2$ are the same in $y = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + \hat{\varepsilon}$ as in $y = X_1\hat{\beta}_1 + \hat{\delta}_1$ and $y = X_2\hat{\beta}_2 + \hat{\delta}_2$.

(1) (c) Let $M_W = I - W(W'W)^{-1}W'$ and
$M_U = I - U(U'U)^{-1}U'$.

$$\cancel{\hat{\alpha}_2 = (U'M_W U)^{-1} U' M_W y}$$

$$\cancel{= ((Z+X_2)' M_W (Z+X_2))^{-1} (Z+X_2)' M_W y}$$

$$\hat{\alpha} = \left[[W\ U]'[W\ U]\right]^{-1} [W\ U]' y$$

$$= \begin{bmatrix} W'W & W'U \\ U'W & U'U \end{bmatrix}^{-1} \begin{bmatrix} W \\ U \end{bmatrix} y$$

$$= \begin{bmatrix} (X_1+X_2)'(X_1+X_2) & (X_1+X_2)'(Z+X_2) \\ (Z+X_2)'(X_1+X_2) & (Z_1+X_2)'(Z_1+X_2) \end{bmatrix}^{-1}$$

$$\begin{bmatrix} (X_1+X_2) y \\ (X_2+Z) y \end{bmatrix}$$

$$\xrightarrow[p]{} \begin{bmatrix} E(X_i'X_1) + E(X_1')E(X_i') + E(X_2^2 \\ \\ \end{bmatrix}$$

1 (d)

(2) (a) ~~Either~~ The coefficient in a regression with just a constant equal $\bar{y}$. The plim of $\bar{y}$ is the population mean of y. The marginal distribution of y is:

$$f_Y(y) = \int_0^1 \frac{6}{13}(3y + 2x^2)\,dx$$

$$= \frac{6}{13}\left[3yx + \frac{2}{3}x^3\right]_0^1$$

$$= \frac{6}{13}\left[3y + \frac{2}{3}\right]$$

$$= \frac{18}{13}y + \frac{4}{13}$$

$$E[y_i] = \int_0^1 y\left(\frac{18}{13}y + \frac{4}{13}\right)dy$$

$$= \int_0^1 \frac{18}{13}y^2 + \frac{4}{13}\,dy$$

$$= \left[\frac{18}{13}\frac{y^3}{3} + \frac{4}{13}y\right]_0^1$$

$$= \frac{18}{13}\frac{1}{3} + \frac{4}{13}$$

$$= \frac{18}{39} + \frac{12}{39}$$

$$= \frac{30}{39}$$

$$\Rightarrow \boxed{\text{plim}(\hat{\alpha}) = \frac{30}{39}} = \frac{10}{13}$$

(2) (b) $R^2 = \frac{\sum_i (\hat{y}_i - \bar{y})^2}{\sum_i (y_i - \bar{y})^2}$

With only an intercept, $\hat{y}_i = \bar{y} \;\forall i$.

$\Rightarrow R^2 = \frac{\sum_i (\bar{y} - \bar{y})^2}{\sum_i (y_i - \bar{y})^2} = 0.$

Thus, the $\operatorname{plim}(R^2) = 0$.

R^2 is a measure of goodness of fit compared to an intercept-only model. The intercept-only model will always ~~???~~ have $R^2 = 0$ no matter how many observation.

(2) (c) $$\hat{\gamma} = \frac{\sum_i y_i x_i}{\sum_i x_i^2}$$

$$= \frac{\frac{1}{n}\sum_i y_i x_i}{\frac{1}{n}\sum_i x_i^2}$$

$$\xrightarrow{p} \frac{E(y_i x_i)}{E(x_i^2)} \quad \text{by WLLN.}$$

$$E(y_i x_i) = \int_0^1 \int_0^1 y x \frac{6}{13}(3y + 2x^2)\, dy\, dx$$

$$= \frac{6}{13}\int_0^1 \int_0^1 3y^2 x + 2x^3 y \, dy\, dx$$

$$= \frac{6}{13}\int_0^1 \left[y^3 x + x^3 y^2 \right]_0^1 dx$$

$$= \frac{6}{13}\int_0^1 [x + x^3]\, dx$$

$$= \frac{6}{13}\left[\frac{x^2}{2} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{6}{13}\left[\frac{1}{2} + \frac{1}{4}\right]$$

$$= \frac{6}{13} \cdot \frac{3}{4}$$

$$= \frac{18}{52} = \frac{9}{26}$$

The marginal distribution of x is

$$f_X(x) = \int_0^1 \frac{6}{13}(3y + 2x^2)\, dy$$

$$= \frac{6}{13}\left[\frac{3}{2}y^2 + 2yx^2\right]_0^1$$

$$= \frac{6}{13}\left[\frac{3}{2} + 2x^2\right] = \frac{12}{13}x^2 + \frac{18}{26} = \frac{12}{13}x^2 + \frac{9}{13}$$

(1)(c) con't

$$E(X_i^2) = \int_0^1 x^2 \left(\frac{12}{13}x^2 + \frac{9}{13}\right) dx$$

$$= \int_0^1 \frac{12}{13}x^4 + \frac{9}{13}x^2 \, dx$$

$$= \left[\frac{12}{13}\frac{x^5}{5} + \frac{9}{13}\frac{x^3}{3}\right]_0^1$$

$$= \frac{12}{65} + \frac{9}{39}$$

$$= \frac{36}{195} + \frac{45}{195}$$

$$= \frac{81}{195}$$

Thus, $\text{plim}(\hat{\gamma}) = \frac{9/26}{81/195} = \frac{9 \cdot 195}{26 \cdot 81} = \frac{1755}{2106}$

$$= \boxed{\frac{5}{6}}$$

② (d) $R^2 = \dfrac{\sum_i (\hat{y}_i - \bar{y})^2}{\sum_i (y_i - \bar{y})^2}$

$= \dfrac{\frac{1}{n} \sum_i (\hat{y}_i - \bar{y})^2}{\frac{1}{n} \sum_i (y_i - \bar{y})^2}$

$\xrightarrow{p} \dfrac{E((\hat{y}_i - \bar{y})^2)}{E((y_i - \bar{y})^2)}$

$E[(\hat{y} - \bar{y})^2] = E[(\gamma x_i - \bar{y})^2]$

$= E[\gamma^2 x_i^2 - 2\gamma x_i \bar{y} + \bar{y}^2]$

$= \gamma^2 E[x_i^2] - 2\gamma E(x_i \bar{y}) + E(\bar{y}^2)$

$E[(y_i - \bar{y})^2] = E[y_i^2] - 2E[y_i \bar{y}] + E(\bar{y}^2]$

$=$

$$② (e) \quad E(\varepsilon_i) = \int_0^1 \int_0^1 (y_i - x_i \gamma) f_{XY}(x,y)\,dy\,dx$$

$$= \int_0^1 \int_0^1 y_i f_{XY}(x,y)\,dy\,dx - \gamma \int_0^1 \int_0^1 x_i f_{XY}(x,y)\,dy\,dx$$

$$= E(y_i) - \gamma E(x_i)$$

$$= \frac{30}{39} - \frac{5}{6} E(x_i)$$

$$E(x_i) = \int_0^1 x \left[\frac{12}{13}x^2 + \frac{9}{13}\right] dx$$

$$= \int_0^1 \frac{12}{13} x^3 + \frac{9}{13} x \, dx$$

$$= \left[\frac{12}{13}\frac{x^4}{4} + \frac{9}{13}\frac{x^2}{2}\right]_0^1$$

$$= \frac{12}{13}\frac{1}{4} + \frac{9}{13}\frac{1}{2}$$

$$= \frac{12}{52} + \frac{9}{26}$$

$$= \frac{12}{52} + \frac{18}{52}$$

$$= \frac{30}{52}$$

$$= \frac{15}{26}$$

$$= \frac{30}{39} - \frac{5}{6}\frac{15}{26}$$

$$= \frac{30}{39} - \frac{75}{156} = \frac{15}{52}$$

(2)(e) con't

$$E(x_i \varepsilon_i) = \int_0^1 \int_0^1 x_i (y_i - \gamma x_i) \, f_{XY}(x_i, y_i) \, dy \, dx$$

$$= \int_0^1 \int_0^1 x_i y_i f_{XY}(x_i, y_i) \, dx \, dy$$

$$- \gamma \int_0^1 \int_0^1 x_i^2 \, dx \, dy$$

$$= E[x_i y_i] - \gamma E[x_i^2]$$

$$= 9/26 - \frac{5}{6} \frac{81}{195}$$

$$= 9/26 - 9/26$$

$$= 0$$

$$E[\varepsilon_i | x_i] = E[y_i - \gamma x_i | x_i]$$

$$= E[y_i | x_i] - \gamma E[x_i | x_i]$$

$$= E[y_i | x_i] - \gamma x_i$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$= \left[\frac{6}{13}(3y + 2x^2)\right] / \left[\frac{12}{13} x^2 + \frac{9}{13}\right] = \frac{4x^2 + 6y}{4x^2 + 3}$$

(2)(e) cont'd $E[y_i|x_i] = \int_0^1 y_i \, f_{Y|X}(y_i|x_i)\, dy_i$

$\cancel{\int_0^1 y_i \left[\frac{\frac{6}{13}(3y+2x^2)}{\frac{12}{13}}\right]}$

$$= \int_0^1 y_i \left[\frac{4x_i^2 + 6y_i}{4x_i^2+3}\right] dy_i$$

$$= \frac{1}{4x_i^2+3} \int_0^1 4x_i^2 y_i + 6y_i^2 \, dy_i$$

$$= \frac{1}{4x_i^2+3} \left[\frac{4x^2y^2}{2} + \frac{6y^3}{3}\right]_0^1$$

$$= \frac{1}{4x^2+3}\left[2x^2+2\right]$$

$$= \frac{2x^2+2}{4x^2+3}$$

$$\Rightarrow E[\varepsilon_i | x_i] = \frac{2x^2+2}{4x^2+3} - \frac{5}{6}x$$

$$= \frac{6(2x^2+2)}{6(4x^2+3)} - \frac{5x(4x^2+3)}{6(4x^2+3)}$$

$$= \frac{12x^2+12-20x^3-15x}{24x^2+18}$$

$$= \boxed{\frac{-20x^3+12x^2-15x+12}{24x^2+18}}$$

From (c) $E(x_i^2) = \frac{81}{195}$ From (e) $E(x_i \varepsilon_i) = 0$

IID from the problem setup

(2) (f) We have OLS 0, OLS 1, OLS 2'.

We can show that OLS 3' holds:

$$V(x_i \varepsilon_i) = E(x_i^2 \varepsilon_i^2) - [E(x_i \varepsilon_i)]^2$$

$$= E(x_i^2 \varepsilon_i^2)$$

$$\cancel{= \int_0^1 \int_0^1 x_i^2 (y_i - \gamma x_i)^2 f_{x,y}(x,y)\, dy\, dx}$$

$$\cancel{= \int_0^1 \int_0^1 x}$$

$$= E(x_i^2 (y_i - \gamma x_i)^2)$$

$$= E[x_i^2 (y_i^2 - 2\gamma x_i y_i + \gamma^2 x_i^2)]$$

We show that each component is finite

$$= E[x_i^2 y_i^2] - 2\gamma E[x_i^3 y_i] + \gamma^2 E[x_i^4]$$

$$E[x_i^4] = \int_0^1 x_i^4 \left(\frac{12}{13} x_i^2 + \frac{9}{13}\right) dx$$

$$= \left[\frac{12}{13} \frac{x_i^6}{6} + \frac{9}{13} \frac{x_i^5}{4}\right]_0^1$$

$$= \frac{12}{13} \frac{1}{6} + \frac{9}{13} \frac{1}{4} = \frac{17}{52} < \infty$$

$$E[x_i^3 y_i] = \int_0^1 \int_0^1 x_i^3 y_i \left(\frac{6}{13}(3y_i + 2x_i^2)\right) dy\, dx$$

$$= \frac{6}{13} \int_0^1 \int_0^1 3y^2 x^3 + 2x^5 y \, dx\, dy$$

$$= \frac{6}{13} \int_0^1 \left[\frac{3}{4} y^2 x^4 + \frac{2}{6} x^6 y\right]_0^1 dy$$

$$= \frac{6}{13} \int_0^1 \frac{3}{4} y^2 + \frac{1}{3} y \, dy$$

① (f) Cont'd

$$= \frac{6}{13}\left[\frac{1}{4}y^3 + \frac{1}{6}y^2\right]_0^1$$

$$= \frac{6}{13}\left[\frac{1}{4} + \frac{1}{6}\right] = \frac{5}{26} < \infty$$

$$E\left[x_i^2 y_i^2\right] = \int_0^1\int_0^1 x^2 y^2 \left(\tfrac{6}{13}(3y + 2x^2)\right) dy\,dx$$

$$= \frac{6}{13}\int_0^1\int_0^1 3x^2y^3 + 2x^4y^2 \, dy\,dx$$

$$= \frac{6}{13}\int_0^1 \left[\frac{3}{4}x^2y^4 + \frac{2}{3}x^4y^3\right]_0^1 dx$$

$$= \frac{6}{13}\int_0^1 \frac{3}{4}x^2 + \frac{2}{3}x^4 \, dx$$

$$= \frac{6}{13}\left[\frac{1}{4}x^3 + \frac{2}{15}x^5\right]_0^1$$

$$= \frac{6}{13}\left[\frac{1}{4} + \frac{2}{15}\right] = \frac{23}{130} < \infty$$

Thus, $V(x_i \varepsilon_i)$ is finite. So,

$$\cancel{\sqrt{n}} = \frac{23}{130} - 2\left(\frac{5}{6}\right)\left(\frac{5}{26}\right) + \left(\frac{5}{6}\right)^2\left(\frac{17}{52}\right) = \frac{781}{9360}$$

$$\sqrt{n}(\hat{\gamma} - \gamma) \xrightarrow{d} N(0, V)$$

where $V = E(X_i^2)^{-1} V(x_i\varepsilon_i) E(x_i^2)^{-1}$

$$= \frac{195}{81} \cdot \frac{781}{9360} \cdot \frac{195}{81}$$

$$= \frac{50765}{104976}.$$