ECON 711B - Problem Set 3

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1. (Arms race with market power) Two (expected payoff maximizing) gangs are competing in an arms race. Each of them likes having more weapons w_i but dislikes having a different amount than the other agent. Obtaining weapons is costly, with the price of weapons increasing in the average quantity of weapons purchased. The price of weapons is $P(\bar{w}) = \rho + \alpha\bar{w}$ where $\bar{w}$ is the average amount of weapons purchased. Each gang's payoff is $u_i(w_i, w_j) = \gamma w_i - \beta(w_i - w_j)^2 - P(\bar{w})w_i$. All parameters α, β, γ and ρ are strictly positive, and $\gamma > \rho$.

(a) Explain the economic intuition of the assumption that $\gamma > \rho$. What does this assumption guarantee?

Substituting in the price function and $\bar{w} = \frac{w_i + w_j}{2}$ into the payoff function:

$$
\begin{aligned}
u_i(w_i, w_j) &= \gamma w_i - \beta(w_i - w_j)^2 - (\rho + \alpha(w_i + w_j)/2)w_i \\
&= (\gamma - \rho)w_i - \beta(w_i - w_j)^2 - (\alpha/2)w_i^2 - (\alpha/2)w_i w_j
\end{aligned}
$$

Assuming that $\gamma > \rho$ insures that there is positive payoff for a gang to buy a positive amount of weapons. If $\gamma \leq \rho$, neither gang would buy any weapons; it pushes us away from a corner solution.

(b) Under what condition(s) is this game supermodular?

The game is supermodular if payoffs $u_i(w_i, w_j)$ has increasing differences for i and j.

$$\frac{\partial u_i}{\partial w_i} = (\gamma - \rho) - 2\beta(w_i - w_j) - \alpha w_i - (\alpha/2)w_j$$

$$\frac{\partial^2 u_i}{\partial w_i \partial w_j} = 2\beta - \alpha/2$$

If $4\beta \geq \alpha$, the game is supermodular.

(c) Find the symmetric pure-strategy Nash equilibrium.

FOC $[w_i]$:

$$0 = (\gamma - \rho) - 2\beta(w_i^* - w_j) - \alpha w_i^* - (\alpha/2)w_j$$

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For a symmetric pure-strategy Nash equilibrium $w_j = w_i^*$:

$$\begin{aligned} 0 &= (\gamma - \rho) - 2\beta(w_i^* - w_i^*) - \alpha w_i^* - (\alpha/2)w_i^* \\ \implies \gamma - \rho &= \alpha w_i^* + (\alpha/2)w_i^* \\ \implies w_i^* &= \frac{2\gamma - 2\rho}{3\alpha} \end{aligned}$$

Thus, the symmetric pure-strategy Nash equilibrium is both gangs buying $\frac{2\gamma-2\rho}{3\alpha}$ weapons.

(d) In the Nash equilibrium you found in part (c), how does the equilibrium quantity of weapons change with each parameter? Provide intuition for each effect.

$$\begin{aligned} &\uparrow \gamma \rightarrow \uparrow w_i^* \\ &\uparrow \rho \rightarrow \downarrow w_i^* \\ &\uparrow \alpha \rightarrow \downarrow w_i^* \\ &\uparrow \beta \rightarrow \text{ no change in } w_i^* \end{aligned}$$

γ is the inherent marginal benefit to the gang of additional weapons (maybe it's the additional revenue associated with additional weapons). If weapons are inherently more valuable to the gang, they'll consume more.

ρ is the reservation price for the weapon supplier. It is the intercept of the supply curve. If the reservation price for the weapon supplier is higher, the gang will buy fewer weapons because they would be more expensive.

α measures the market power of the weapons supplier. It is twice the slope of the supply curve. If the weapon supplier have more market power, the gang will buy fewer weapons because they would be more expensive.

β measures the disutility associated with having a different amount of weapons as the other gang. In the symmetric equilibrium, the gangs choose the same amount of weapons, so the disutility of having different amounts of weapons does not factor in.

(e) Does there exist an equilibrium in which one or both gangs choose to have no weapons? If so, find such an equilibrium; if not, show why not.

First, let us consider the candidate equilibrium in which both gangs choose to have no weapons. For this candidate to be an equilibrium, it cannot be profitable for gang i to deviate to buying $\varepsilon > 0$ amount of weapons:

$$\begin{aligned} u_i(0,0) &\geq u_i(\varepsilon, 0) \\ (\gamma-\rho)(0) - \beta(0-0)^2 - (\alpha/2)(0)^2 - (\alpha/2)(0)(0) &\geq (\gamma-\rho)\varepsilon - \beta(\varepsilon - (0))^2 - (\alpha/2)\varepsilon^2 - (\alpha/2)\varepsilon(0) \\ 0 &\geq (\gamma-\rho)\varepsilon - \beta\varepsilon^2 - (\alpha/2)\varepsilon^2 \\ \varepsilon(\beta + \alpha/2) &\geq \gamma - \rho \end{aligned}$$

We know that the RHS is strictly positive. For any β and α, we can choose ε small enough to make the LHS arbitrary close to zero and break the inequality. Thus, both gang having zero weapons is not an equilibrium because there's an incentive for each gang to deviate and buy some weapons.

Second, let us consider the candidate equilibrium in which one gang has weapons and the other does not. For this candidate to be an equilibrium, it cannot be profitable for the gang without weapons to buy $\varepsilon > 0$ weapons.

$$\begin{aligned}
u_i(0, w_j) &\geq u_i(\varepsilon, w_j) \\
(\gamma - \rho)(0) - \beta((0) - w_j)^2 - (\alpha/2)(0)^2 - (\alpha/2)(0)w_j &\geq (\gamma - \rho)\varepsilon - \beta(\varepsilon - w_j)^2 - (\alpha/2)\varepsilon^2 - (\alpha/2)\varepsilon w_j \\
-\beta w_j^2 &\geq (\gamma - \rho)\varepsilon - \beta(\varepsilon - w_j)^2 - (\alpha/2)\varepsilon^2 - (\alpha/2)\varepsilon w_j \\
-\beta w_j^2 &\geq (\gamma - \rho)\varepsilon - \beta(\varepsilon^2 - 2w_j\varepsilon + w_j^2) - (\alpha/2)\varepsilon^2 - (\alpha/2)\varepsilon w_j \\
0 &\geq (\gamma - \rho)\varepsilon - \beta(\varepsilon^2 - 2w_j\varepsilon) - (\alpha/2)\varepsilon^2 - (\alpha/2)\varepsilon w_j \\
0 &\geq (\gamma - \rho) - \beta(\varepsilon - 2w_j) - (\alpha/2)\varepsilon - (\alpha/2)w_j \\
0 &\geq (\gamma - \rho) - \beta\varepsilon + 2\beta w_j - (\alpha/2)\varepsilon - (\alpha/2)w_j \\
(\alpha/2)w_j - 2\beta w_j &\geq (\gamma - \rho) - \beta\varepsilon - (\alpha/2)\varepsilon \\
w_j &\geq \frac{(\gamma - \rho) - \beta\varepsilon - (\alpha/2)\varepsilon}{(\alpha/2) - 2\beta}
\end{aligned}$$

Thus, if the game is not supermodular (i.e., $\alpha > 4\beta$), it's profitable for gang i to have zero weapons if gang j has $\tilde{w_j}$ weapons:

$$\tilde{w_j} = \frac{2\gamma - 2\rho}{\alpha - 4\beta}$$

(f) Can this game support a mixed strategy Nash equilibrium? If so, find such an equilibrium. If not, explain why it cannot exist.

If the game is supermodular ($\alpha \leq 4\beta$), then the only Nash equilibrium is symmetric pure-strategy Nash equilibrium $(\frac{2\gamma-2\rho}{3\alpha}, \frac{2\gamma-2\rho}{3\alpha})$.

If the game is not supermodular ($\alpha > 4\beta$), then there are three pure-strategy Nash equilibriums:

1. The symmetric equilibrium $(\frac{2\gamma-2\rho}{3\alpha}, \frac{2\gamma-2\rho}{3\alpha})$.
2. Gang 1 with $\frac{2\gamma-2\rho}{\alpha-4\beta}$ weapons and gang 2 with zero.
3. Gang 2 with $\frac{2\gamma-2\rho}{\alpha-4\beta}$ weapons and gang 1 with zero.

Thus, there are four mixed strategy Nash equilibriums:

a. Mixing between 1. and 2.
b. Mixing between 2. and 3.
c. Mixing between 1. and 3.
d. Mixing between 1., 2., and 3.

Let us find the Nash equilibrium that mixes 2. and 3.

$$u_i\left(0, \frac{2\gamma - 2\rho}{\alpha - 4\beta}\right) = 0$$

$$u_i\left(\frac{2\gamma - 2\rho}{\alpha - 4\beta}, 0\right) = (\gamma - \rho)\left(\frac{2\gamma - 2\rho}{\alpha - 4\beta}\right) - \beta\left(\left(\frac{2\gamma - 2\rho}{\alpha - 4\beta}\right) - (0)\right)^2 - (\alpha/2)\left(\frac{2\gamma - 2\rho}{\alpha - 4\beta}\right)^2 - (\alpha/2)\left(\frac{2\gamma - 2\rho}{\alpha - 4\beta}\right)(0) = (\gamma - \rho)\left(\frac{2\gamma - 2\rho}{\alpha - 4\beta}\right)$$

(g) Suppose both gangs have the equilibrium quantity of weapons from part (c). A horde of goblins suddenly invades the area. Because weapons can be used to fight goblins, the inherent value γ of weapons increases. However, the goblins also steal all the money of gang 2, leaving that gang unable to purchase new weapons. How does gang 1's weapons quantity respond to this shock? How does the magnitude of this response compare to the magnitude if both gangs were able to respond?

2. Consider the game from question 1, but now with a continuum of gangs. Let the payoffs now depend on the average weapons quantity $\bar{q}$, rather than that of any single other gang: $u_i(w_i, \bar{w}) = \gamma w_i - \beta(w_i - \bar{w})^2 - P(\bar{q})w_i$. Find the symmetric pure strategy Nash equilibrium. How and why does the equilibrium quantity of weapons differ from the equilibrium quantity in the two-gang game?

3. A continuum of agents plays a guessing game, in which each agent i guesses a number $x_i \in [0, 1]$. Agents prefer to guess numbers closer to some commonly known constant $\alpha \in (0, 1)$, and dislike guessing further from the average value $\bar{x}$. Each agent's payoff is $u_i(x_i; \bar{x}, \alpha) = (x_i - \alpha)^2 - (x_i - \bar{x})^2$.

(a) Find all symmetric pure strategy Nash equilibria.

(b) Recall that with a continuum of agents, a Nash equilibrium can be characterized by a quantile function, regardless of whether it is achieved using asymmetric pure strategies or with randomization. Describe all non-degenerate quantile functions that are Nash equilibria in this game, excluding the equilibria found in part (a).

(c) Now suppose agents can choose any $x \in \mathbb{R}$. Describe all Nash equilibria.

4. Two players choose numbers in $\mathbb{R}$. Their payoffs are $u_i(q_i, q_j) = q_i + q_i(q_j - 1)^{1/3} - \frac{1}{2}q_i^2$. Find all Nash equilibria of this game.

5. Two players play a game of Rock, Paper, Scissors. In this game, each player simultaneously chooses a strategy from the set fRock, Paper, Scissorsg. Paper beats Rock: if one player chooses Paper and the other chooses Rock, the player choosing Paper receives a payoff of 10, and the other player receives a payoff of 10. Similarly, Scissors beats Paper, and Rock beats Scissors. If both players choose the same pure strategy, each receives a payoff of 0. Players can also choose mixed strategies. However, in order to implement a mixed strategy, a player must rent a randomizing device. The rental costs 1 unit of payoffs. Find all Nash equilibria of this game.