ECON 712 - PS 2

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Problem 1: Two-dimensional non-linear system

Consider the Ramsey model of consumption c_t and capital k_t:

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \tag{1}$$
$$\beta u'(c_{t+1}) = \frac{u'(c_t)}{1-\delta+f'(k_{t+1})} \tag{2}$$

parametrized by: $f(k) = zk^\alpha, z = 1, \alpha = 0.3, \delta = 0.1, \beta = 0.97, u(c) = \log(c)$.

1. Solve for steady state $(\bar{k}, \bar{c})$.

The functional forms provided imply:

$$f(k) = zk^\alpha \implies f'(k) = z\alpha k^{\alpha-1}$$
$$u(c) = \log(c) \implies u(c) = c^{-1}$$

Setting $\bar{c} := c_t = c_{t+1}, \bar{k} := k_t = k_{t+1}$:

$$\begin{aligned}
(2) &\implies c_{t+1} = \beta c_t(1-\delta+z\alpha k_{t+1}^{\alpha-1}) \\
&\implies 1 = \beta(1-\delta+z\alpha\bar{k}^{\alpha-1}) \\
&\implies \bar{k} = \left(\frac{\beta^{-1}-1+\delta}{z\alpha}\right)^{\frac{1}{\alpha-1}} \\
&\implies \bar{k} \approx 3.2690
\end{aligned}$$

$$\begin{aligned}
(1) &\implies \bar{c} = z\bar{k}^\alpha + (1-\delta)\bar{k} - \bar{k} \\
&\implies \bar{c} = z\left(\frac{\beta^{-1}-1+\delta}{z\alpha}\right)^{\frac{\alpha}{\alpha-1}} + \delta\left(\frac{\beta^{-1}-1+\delta}{z\alpha}\right)^{\frac{1}{\alpha-1}} \\
&\implies \bar{c} \approx 1.0998
\end{aligned}$$

The steady state is $(\bar{k}, \bar{c}) = (3.2690, 1.0998)$.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

2. Linearize the system around its steady state.

(a) Rewrite equations (1) and (2) as

$$
\begin{aligned}
k_{t+1} &= g(k_t, c_t) \\
(1) \implies k_{t+1} &= z k_t^{\alpha} + (1-\delta)k_t - c_t \\
c_{t+1} &= h(k_t, c_t) \\
(2) \implies c_{t+1} &= \beta c_t (1 - \delta + z\alpha k_{t+1}^{\alpha-1}) \\
\implies c_{t+1} &= \beta c_t (1 - \delta + z\alpha (z k_t^{\alpha} + (1-\delta)k_t - c_t)^{\alpha-1})
\end{aligned}
$$

(b) Analytically calculate Jacobian $J = \begin{pmatrix} dk_{t+1}/dk_t & dk_{t+1}/dc_t \\ dc_{t+1}/dk_t & dc_{t+1}/dc_t \end{pmatrix}$ (use provided functional forms, but don't plug in parameters yet).

$$
\begin{aligned}
dk_{t+1}/dk_t &= z\alpha k_t^{\alpha-1} + 1 - \delta \\
dk_{t+1}/dc_t &= -1 \\
dc_{t+1}/dk_t &= z\alpha\beta c_t (\alpha - 1)(z k_t^{\alpha} + (1-\delta)k_t - c_t)^{\alpha-2}(z\alpha k_t^{\alpha-1} + 1 - \delta) \\
dc_{t+1}/dc_t &= (1-\delta)\beta + z\alpha\beta[(z k_t^{\alpha} + (1-\delta)k_t - c_t)^{\alpha-1} - c_t(\alpha-1)(z k_t^{\alpha} + (1-\delta)k_t - c_t)^{\alpha-2}]
\end{aligned}
$$

(c) Using Taylor expansion (first-order approximation here), systems can be written in terms of deviations from steady state $\tilde{k}_t = k_t - \bar{k}$ and $\tilde{c}_t = c_t - \bar{c}$:

$$
\begin{pmatrix} \tilde{k}_{t+1} \\ \tilde{c}_{t+1} \end{pmatrix} = J \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix}
$$

3. Compute numerically eigenvalues and eigenvectors of the Jacobian at the steady state. Verify that the system has a saddle path. What is the slope of the saddle path at the steady state?

At $(\bar{k}, \bar{c}) = (3.269, 1.100)$ and the above parameters (from Matlab).

$$
J = \begin{pmatrix} 1.0309 & -1 \\ -0.0308 & 1.0299 \end{pmatrix}
$$

From Matlab, the eigenvectors and eigenvalues for J are:

$$
\begin{aligned}
\Lambda &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1.2060 & 0 \\ 0 & 0.8548 \end{pmatrix} \\
E &= \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 0.9850 & 0.9848 \\ -0.1725 & 0.1734 \end{pmatrix} \\
\begin{pmatrix} k_t \\ c_t \end{pmatrix} &= \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} x_1 \lambda_1^t \\ x_2 \lambda_2^t \end{pmatrix}
\end{aligned}
$$

The system has a saddle path because the absolute value of one eignevalue is greater than one and the absolute value of the other eigenvalue is less than one. The saddle path is (k_t, c_t) where $x_1 = 0$.

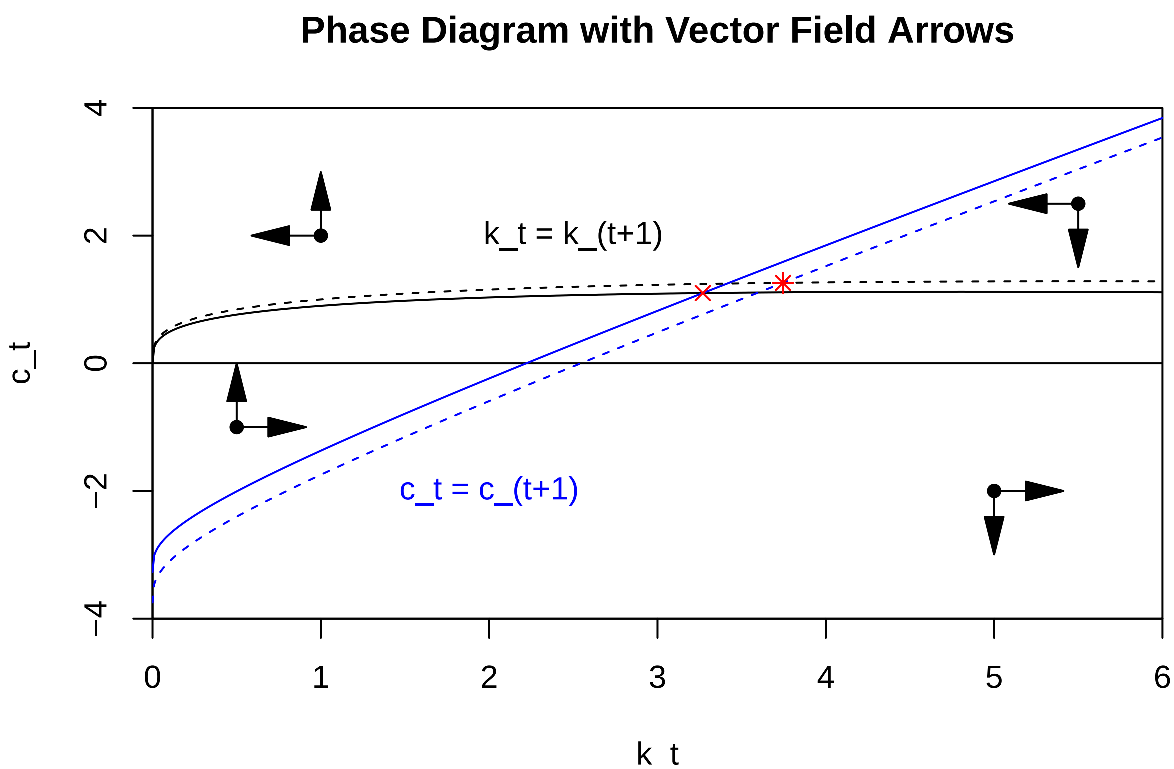
$$
\begin{pmatrix} k_t \\ c_t \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \lambda_2^t \end{pmatrix} = \begin{pmatrix} e_{12} x_2 \lambda_2^t \\ e_{22} x_2 \lambda_2^t \end{pmatrix}
$$

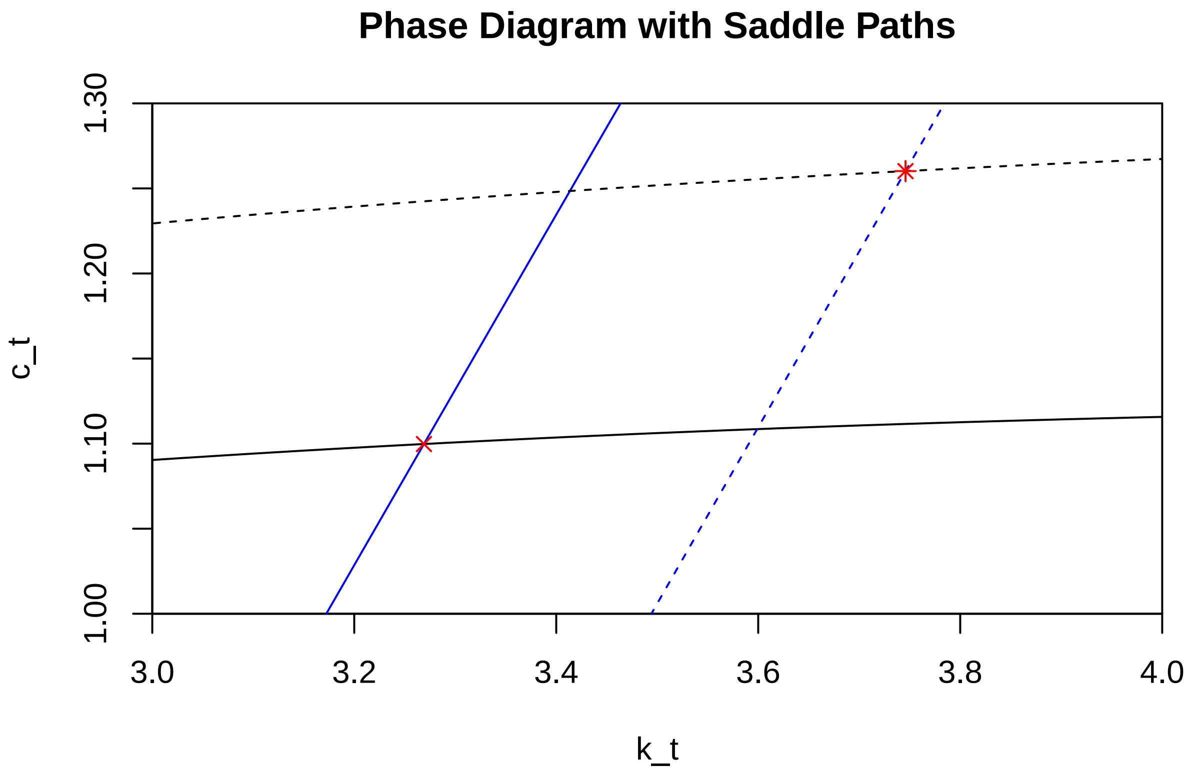
So, the slope of the saddle path at the steady state is $\frac{e_{22} x_2 \lambda_2^t}{e_{12} x_2 \lambda_2^t} = \frac{e_{22}}{e_{12}} = \frac{0.1734}{0.9848} = 0.1761$.

4. On a phase diagram in (k_t, c_t) show how the system evolves after an unexpected permanent positive productivity shock at $t_0, z' > z$. (You don't need to plot lines precisely - do this by hand, but pay attention to vector field (arrows), relative position of old and new steady states, directions of saddle paths and system trajectory after the shock.)

$$
\begin{aligned}
\Delta k_{t+1} &= 0 \\
\implies k_{t+1} - k_t &= 0 \\
\implies (f(k_t) + (1-\delta)k_t - c_t) - k_t &= 0 \\
\implies c_t &= f(k_t) - \delta k_t
\end{aligned}
$$

$$
\begin{aligned}
\Delta c_{t+1} &= 0 \\
\implies c_{t+1} &= c_t \\
\implies \beta u'(c_{t+1}) &= \beta u'(c_t) \\
\implies \frac{u'(c_t)}{1-\delta+f'(k_{t+1})} &= \beta u'(c_t) \\
\implies f'(k_{t+1}) &= \beta^{-1} - 1 + \delta \\
\implies k_{t+1} &= \bar{k} \\
\implies c_t &= f(k_t) + (1-\delta)k_t - \bar{k}
\end{aligned}
$$





5. (continuing from 4) Compute numerically and plot trajectories of k_t and c_t for $t = 1, 2, ..., 20$ if the productivity shock occurs at $t_0 = 5$ and $z = z + 0.1$. For this question, we will be looking at the linearized version of the nonlinear system around the new steady state.

(a) Compute the new steady state $(\bar{k}', \bar{c}')$ and Jacobian matrix at that point.

From Matlab:

$$(\bar{k}', \bar{c}') \approx (3.7458, 1.2424)$$

The new Jacobian equals

$$J = \begin{pmatrix} 1.0440 & -1 \\ -0.0319 & 1.0392 \end{pmatrix}$$

(b) Diagonalize the system using eigenvectors and rewrite it in terms of $\hat{k}_t$ and $\hat{c}_t$.

From Matlab:

The new matrix of eigenvalues is:

$$\Lambda \approx \begin{pmatrix} 1.2203 & 0 \\ 0 & 0.8629 \end{pmatrix}$$

The new matrix of eignvectors is:

$$E \approx \begin{pmatrix} 0.9848 & 0.9840 \\ -0.1736 & 0.1783 \end{pmatrix}$$

Define

$$\begin{pmatrix}\hat{k}_t\\ \hat{c}_t\end{pmatrix} = E^{-1}\begin{pmatrix}k_t\\ c_t\end{pmatrix} \implies \begin{pmatrix}\hat{k}_{t+1}\\ \hat{c}_{t+1}\end{pmatrix} = \Lambda\begin{pmatrix}\hat{k}_t\\ \hat{c}_t\end{pmatrix}$$

(c) Write down non-explosive solution for $(\hat{k}_t, \hat{c}_t)$, rewrite in terms of original variables (k_t, c_t).

(d) Pin down a particular saddle path trajectory using a boundary condition $k_{t_0} = \bar{k}$ (capital can't jump from the old steady state at the time of the stock, so pick suitable c_{t_0}).

(e) Use the particular solution to compute and graph k_t and c_t after the shock.

6. For this question, we explore the nonlinear nature of the system and numerically solve the actual transition path using the "shooting method".

(a) In the previous question, you solve c_{t_0} under the linear system. Put $(k_{t_0}, c_{t_0}$ into the nonlinear system (1) and (2). Compute and graph how the system evolves. Does it converge to a steady state?

(b) Use "shooting method" to find the actual c_{t_0} needed. The method is to try different values of c_{t_0} such that after long enough time, the system will converge to the new steady state.

Problem 2: Setting up a model

- For the problems below, state the Social Planner Problem (SPP), the Consumer Problem (CP), and define the Competitive Equilibrium (CE). (Don't solve).

1. Consider an overlapping generations economy of 2-period-lived agents. There is a constant measure of N agents in each generation. New young agents enter the economy at each date $t \geq 1$. Half of the young agents are endowed with w_1 when young and 0 when old. The other half are endowed with 0 when young and w_2 when old. There is no savings technology. Agents order their consumption stead by $U(c_t^t, c_{t+1}^t) = \ln c_t^t + \ln c_{t+1}^t$. There is a measure N of initial old agents. Half of them are endowed with w_2 and the other half endowed with 0. Each old agent order their consumption by c_1^0. Each old agent is endowed with M units of fiat currency. No other generation is endowed with fiat currency, and the stock of fiat currency is fixed over time.

2. Consider an overlapping generations economy with 3-period-lived agents. Denote these periods as young, mid, and old. At each date $t \geq 1$, N_t new young agents enter the economy, each endowed with w_1 units of the consumption good when young, w_2 units when mid, and w_3 units when old. The consumption good is non-storable. The population is described by $N_{t+1} = n * N_t$, where $n > 0$. Consumption preference is described by $\ln c_t^t + \ln c_{t+1}^t + \ln c_{t+2}^t$. At time $t = 1$, there is a measure N_{-1} of old agents, each endowed with w_3 units of the consumption good, and a measure of N_0 mid agents, each endowed with w_2 units of the consumption good at $t = 1$ and w_3 units at $t = 2$. Additionally, each initial old agent is endowed with 1 unit of a fiat currency.

- (Cake eating problem) Consider a single infinitely lived agent with preference over their consumption stream $\mathbf{c} = \{c_t\}$, given by $U(\mathbf{c}) = \sum_{t=1}^{\infty} \beta^t u(c_t)$, where $\beta < 1$ and $u(\cdot)$ is increasing and concave. Consumption cannot be negative in any period. The agent is endowed with k_1 units of the consumption good in period $t = 1$. There is a perfect storage technology, such that the consumption good is effectively infinity durable. State the agent's problem (don't solve).