# ECON 714B - Problem Set 1

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**Exercise 8.1 - Existence of representative consumer**

Suppose households 1 and 2 have one-period utility functions $u(c_1)$ and $w(c_2)$, respectively, where $u$ and $w$ are both increasing, strictly concave, twice differentiable functions of a scalar consumption rate. Consider the Pareto problem:

$$v_\theta(c) = \max_{\{c_1, c_2\}} [\theta u(c_1) + (1-\theta)w(c_2)]$$

subject to the constraint $c_1 + c_2 = c$. Show that the solution of this problem has the form of a concave utility function $v_\theta(c)$, which depends on the Pareto weight $\theta$. Show that $v'_\theta(c) = \theta u'(c_1) = (1-\theta)w'(c_2)$.The function $v_\theta(c)$ is the utility function of the representative consumer. Such a representative consumer always lurks within a complete markets competitive equilibrium even with heterogeneous preferences. At a competitive equilibrium, the marginal utilities of the representative agent and each and every agent are proportional.

First, notice that $v_\theta$ is increasing, strictly concave, and twice differentiable because $\theta, 1-\theta \geq 0$ and $u$ and $w$ are both increasing, strictly concave, twice differentiable functions.

$$v_\theta(c) = \max_{\{c_1, c_2\}} [\theta u(c_1) + (1-\theta)w(c_2)] \text{ s.t. } c = c_1 + c_2$$

$$\implies v_\theta(c) = \max_{c_1} [\theta u(c_1) + (1-\theta)w(c - c_1)]$$

Since $u$ and $w$ are continuously differentiable functions, we can find the envelope condition:

$$\implies v'_\theta(c) = (1-\theta)w'(c - c_1) = (1-\theta)w'(c_2)$$

FOC $[c_1]$:

$$\theta u'(c_1) = (1-\theta)w'(c - c_1) \implies \theta u'(c_1) = (1-\theta)w'(c_2)$$

Therefore,

$$v'_\theta(c) = \theta u'(c_1) = (1-\theta)w'(c_2)$$

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**Exercise 8.3**

An economy consists of two infinitely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer $i$ consumes $c_{it}$ at time $t$. Consumer $i$ ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_{it})$$

where $\beta \in (0, 1)$ and $u(c)$ is increasing, strictly concave, and twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_{1t} = 1, 0, 0, 1, 0, 0, 1, ...$ Consumer 2 is endowed with a stream of the consumption good $y_{2t} = 0, 1, 1, 0, 1, 1, 0, ....$ Assume that there are complete markets with time 0 trading.

a. Define a competitive equilibrium.

First, notice that there is no stochastic states (i.e., endowment process is deterministic). The agent $i$'s problem is:

$$\max_{\{c_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{it}) \text{ s.t. } \sum_{t=0}^{\infty} Q_t c_{it} \leq \sum_{t=0}^{\infty} Q_t y_{it}$$

A competitive equilibrium is a feasible allocation $\{\{c_{it}\}_{t=0}^{\infty}\}_{i=1,2}$ and a price system $\{Q_t\}_{t=0}^{\infty}$ such that given the price system, the allocation maximized the agent's utility subject to their budget constraint.

b. Compute a competitive equilibrium.

Let $\mu_i$ be the multipler on the BC for agent $i$:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_{it}) - \mu_i \left[ \sum_{t=0}^{\infty} Q_t y_{it} - \sum_{t=0}^{\infty} Q_t c_{it} \right]$$

FOC $[c_{it}]$:

$$\beta^t u'(c_{it}) = \mu_i Q_t$$

Thus, the ratio of the FOC for 1 and the FOC for 2 is:

$$\frac{u'(c_{1t})}{u'(c_{2t})} = \frac{\mu_1}{\mu_2} \implies c_{1t} = u'^{-1}\left( \frac{\mu_1}{\mu_2} u'(c_{2t}) \right)$$

Plugging into the resource constraint:

$$c_{1t} + c_{2t} = 1 \implies u'^{-1}\left( \frac{\mu_1}{\mu_2} u'(c_{2t}) \right) + c_{2t} = 1$$

This implies that $c_{2t}$ a function of the aggregate endowment for all $t$. Since the aggregate endowment is constant, $c_{2t} = c_{2,t+1} = c_2$. Thus, $c_{1t}$ is also constant: $c_{1t} = c - c_2 \equiv c_1$. Let the numeraire be date 0 consumption $Q_0 = 1$:

$$\frac{\beta^t u'(c_{1,t})}{\beta^0 u'(c_{1,0})} = \frac{\mu_1 Q_t}{\mu_1} \implies \frac{\beta^t u'(c_1)}{u'(c_1)} = Q_t \implies Q_t = \beta^t$$

The budget constraint implies

$$\sum_{t=0}^{\infty} \beta^t c_{1t} = \sum_{t=0}^{\infty} \beta^t y_{1t} \implies c_1 \frac{1}{1-\beta} = \sum_{t=0}^{\infty} \beta^{3t} \implies c_1 = \frac{1-\beta}{1-\beta^3}$$

Furthermore,

$$c_2 = 1 - \frac{1-\beta}{1-\beta^3} = \frac{\beta-\beta^3}{1-\beta^3}$$

c. Suppose that one of the consumers markets a derivative asset that promises to pay .05 units of consumption each period. What would the price of that asset be?

The price of the derivative asset in period 0 with promises $\{d_t\}_{t=0}^{\infty} = \{0.05\}_{t=0}^{\infty}$ is:

$$P_0^0 = \sum_{t=0}^{\infty} Q_t d_t = \sum_{t=0}^{\infty} \beta^t 0.05 = \frac{0.05}{1-\beta}$$

**Exercise 8.4**

Consider a pure endowment economy with a single representative consumer; $\{c_t, d_t\}_{t=0}^{\infty}$ are the consumption and endowment processes, respectively. Feasible allocations satisfy $c_t \leq d_t$. The endowment process is described by $d_{t+1} = \lambda_{t+1} d_t$. The growth rate $\lambda_{t+1}$ is described by a two-state Markov process with transition probabilities $P_{ij} = Prob(\lambda_{t+1} = \bar{\lambda}_j | \lambda_t = \bar{\lambda}_i)$. Assume that

$$P = \begin{bmatrix} .8 & .2 \\ .1 & .9 \end{bmatrix}, \bar{\lambda} = \begin{bmatrix} .97 \\ 1.03 \end{bmatrix}$$

In addition, $\lambda_0 = .97$ and $d_0 = 1$ are both known at date 0. The consumer has preferences over consumption ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

where $E_0$ is the mathematical expectation operator, conditioned on information known at time 0, $\gamma = 2, \beta = .95$.

**Part I**

At time 0, after $d_0$ and $\lambda_0$ are known, there are complete markets in date- and history-contingent claims. The market prices are denominated in units of time 0 consumption goods.

a. Define a competitive equilibrium, being careful to specify all the objects composing an equilibrium.

The household problem is

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{\lambda^t} \pi(\lambda^t|\lambda_0) \beta^t \frac{c_t(\lambda^t)^{1-\gamma}}{1-\gamma} \text{ s.t. } \sum_{t=0}^{\infty} \sum_{\lambda^t} Q_t(\lambda^t) c_t(\lambda^t) \leq \sum_{t=0}^{\infty} \sum_{\lambda^t} Q_t(\lambda^t) d_t(\lambda^t)$$

A competitive equilibrium is an allocation $\{c_t(\lambda^t)\}_{t=0}^{\infty}$ and a price system $\{Q_t(\lambda^t)\}_{t=0}^{\infty}$ such that the allocation solves the household problem and markets clear (i.e., $c_t(\lambda^t) \leq d_t(\lambda^t)$).

Denote $\mu$ as the legrangian multipler on the budget constrant:

$$\mathcal{L} = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{\lambda^t} \pi(\lambda^t|\lambda_0) \beta^t \frac{(c_t(\lambda^t))^{1-\gamma}}{1-\gamma} + \mu \left[ \sum_{t=0}^{\infty} \sum_{\lambda^t} Q_t(\lambda^t) d_t(\lambda^t) - \sum_{t=0}^{\infty} \sum_{\lambda^t} Q_t(\lambda^t) c_t(\lambda^t) \right]$$

FOC $[c_t(\lambda^t)]$:

$$\beta^t \pi_t(\lambda^t|\lambda_0)(c_t(\lambda^t))^{-\gamma} = \mu Q_t(\lambda^t)$$

Market clearing requires $c_t(\lambda^t) = d_t(\lambda^t) = \lambda_1 ... \lambda_{t-1} \lambda_t$. For $t = 0$, $Q_0(\lambda^0) = \pi_0(\lambda^0|\lambda_0) = 1$ and $c_0(\lambda^0) = d_0 = 1$:

$$\beta^0 \pi_0(\lambda^0|\lambda_0)(c_0(\lambda^0))^{-\gamma} = \mu Q_0(\lambda^0) \implies \mu = 1$$

Thus, for any $t$, the FOC and market clearing implies:

$$Q_t(\lambda^t) = \beta^t \pi_t(\lambda^t|\lambda_0)(\lambda_1 ... \lambda_t)^{-\gamma}$$

b. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, ..., \lambda_5) = (0.97, 0.97, 1.03, 0.97, 1.03)$. Please give a numerical answer.

For $(\lambda_1, \lambda_2, ..., \lambda_5) = (0.97, 0.97, 1.03, 0.97, 1.03)$,

$$\beta^5 = (0.95)^5 = 0.77378$$

$$\pi_5(\lambda^5 = (0.97, 0.97, 1.03, 0.97, 1.03)|\lambda_0 = 0.97) = 0.8 * 0.8 * 0.2 * 0.1 * 0.2 = 0.00256$$

$$d_5(\lambda^5 = (0.97, 0.97, 1.03, 0.97, 1.03)) = 1 * 0.97 * 0.97 * 1.03 * 0.97 * 1.03 = 0.96825$$

Thus, $Q_5(\lambda^5 = (0.97, 0.97, 1.03, 0.97, 1.03)) = (0.77378)(0.00256)(0.96825)^{-2} = 0.00211$.

c. Compute the equilibrium price of a claim to one unit of consumption at date 5, denominated in units of time 0 consumption, contingent on the following history of growth rates: $(\lambda_1, \lambda_2, ..., \lambda_5) = (1.03, 1.03, 1.03, 1.03, .97)$.

$$\pi_5(\lambda^5 = (1.03, 1.03, 1.03, 1.03, .97)|\lambda_0 = 0.97) = 0.2 * 0.9 * 0.9 * 0.9 * 0.1 = 0.01458$$

$$d_5(\lambda^5 = (1.03, 1.03, 1.03, 1.03, .97)) = 1 * 1.03 * 1.03 * 1.03 * 1.03 * .97 = 1.09174$$

Thus, $Q_5(\lambda^5 = (1.03, 1.03, 1.03, 1.03, .97)) = (0.77378)(0.01458)(1.09174)^{-2} = 0.00947$

d. Give a formula for the price at time 0 of a claim on the entire endowment sequence.

The price at time 0 of an asset that promises $\{d_t(\lambda^t)\}_{t=0}^{\infty}$ is:

$$P^0(\lambda_0) = \sum_{t=0}^{\infty} \sum_{\lambda^t} Q_t(\lambda^t) d_t(\lambda^t) = \sum_{t=0}^{\infty} \sum_{\lambda^t} \beta^t \pi_t(\lambda^t|\lambda_0)(d_t(\lambda^t))^{-\gamma} d_t(\lambda^t) = \sum_{t=0}^{\infty} \sum_{\lambda^t} \beta^t \pi_t(\lambda^t|\lambda_0)(\lambda_0 ... \lambda_t)^{1-\gamma}$$

e. Give a formula for the price at time 0 of a claim on consumption in period 5, contingent on the growth rate $\lambda_5$ being 0.97 (regardless of the intervening growth rates).

Define $A$ as a matrix of one-period growth rates weighted by their conditional probability:

$$A = \begin{pmatrix} P_{11}\bar{\lambda}_1^{-\gamma} & P_{12}\bar{\lambda}_2^{-\gamma} \\ P_{21}\bar{\lambda}_1^{-\gamma} & P_{22}\bar{\lambda}_2^{-\gamma} \end{pmatrix} = \begin{pmatrix} 0.8(0.97)^{-2} & 0.2(1.03)^{-2} \\ 0.1(0.97)^{-2} & 0.9(1.03)^{-2} \end{pmatrix} = \begin{pmatrix} 0.8502 & 0.1885 \\ 0.1062 & 0.8483 \end{pmatrix}$$

Thus, $A^x$ is the price at time 0 (before discounting) of a claim on consumption in period $x$ contingent on the final growth rate.

$$A^5 = \begin{pmatrix} 0.5689 & 0.5177 \\ 0.2919 & 0.5637 \end{pmatrix} \implies P_0^5 = \beta^5 [A^5]_{11} = (0.95)^5 (0.5689) = 0.4402$$

**Part II**

Now assume a different market structure. Assume that at each date $t \geq 0$ there is a complete set of one-period forward Arrow securities.

f. Define a (recursive) competitive equilibrium with Arrow securities, being careful to define all of the objects that compose such an equilibrium.

Define $a_t(\lambda^t)$ as the household's holding of Arrow securities in period $t$ for history $\lambda^t$. The recursive formulation of the household problem is:

$$v(a, \lambda) = \max_{c,a'} \{u(c) + \beta E[v(a', \lambda')]\} \text{ s.t. } c + \sum_{\lambda'} q(\lambda, \lambda')a'(\lambda') \leq a + d$$

A recursive competitve equilibrium are decision rules $c(\lambda, a)$ and $a'(\lambda')$ and a pricing kernal $q(\lambda, \lambda')$ such that the decision rules solve the household problem and markets clear (i.e., $c = d$ and $a = 0$). In addition, there is a natural debt limit: $-a'(\lambda) \leq A(\lambda)$.

g. For the representative consumer in this economy, for each state compute the "natural debt limits" that constrain state-contingent borrowing.

The natural borrowing limit prevents Ponzi schemes and is the maximum the household can eventually payback. We can define it recursively as the present discounted value of future income:

$$A(\lambda) = d + \beta \sum_{\lambda'} q(\lambda, \lambda')A(\lambda')$$

h. Compute a competitive equilibrium with Arrow securities. In particular, compute both the pricing kernel and the allocation.

We can rewrite the household problem as:

$$v(a, \lambda) = \max_{a'} \left\{ u\left( a + d - \sum_{\lambda'} q(\lambda, \lambda')a'(\lambda') \right) + \beta E[v(a', \lambda')] \right\}$$

Envelope condition:

$$v'(a, \lambda) = u'\left( a + d - \sum_{\lambda'} q(\lambda, \lambda')a'(\lambda') \right) = u'(c)$$

FOC $[a']$:

$$u'\left( a + d - \sum_{\lambda'} q(\lambda, \lambda')a'(\lambda') \right) q(\lambda, \lambda') = \beta\pi(\lambda'|\lambda)v'(a', \lambda')$$

$$\implies u'(c)q(\lambda, \lambda') = \beta\pi(\lambda'|\lambda)u'(c')$$

$$\implies q(\lambda, \lambda') = \beta\pi(\lambda'|\lambda)\frac{u'(c')}{u'(c)} = \beta\pi(\lambda'|\lambda)\left(\frac{c(\lambda')}{c(\lambda)}\right)^{-\gamma} = \beta\pi(\lambda'|\lambda)\left(\frac{d\lambda'}{d}\right)^{-\gamma} = \beta\pi(\lambda'|\lambda)(\lambda')^{-\gamma}$$

Furthermore, market clearing implies that $a'(\lambda') = 0$ for all $\lambda'$.

i. An entrepreneur enters this economy and proposes to issue a new security each period, namely, a risk-free two-period bond. Such a bond issued in period $t$ promises to pay one unit of consumption at time $t+2$ for sure. Find the price of this new security in period $t$, contingent on $\lambda_t$.

The price for a two-period risk-free is the price of buying two concurrent one-period risk-free bonds:

$$
\begin{aligned}
P_t^{t+2}(\lambda) &= \beta^2 \sum_{\lambda'} \sum_{\lambda''} \pi(\lambda''|\lambda')\pi(\lambda'|\lambda)(\lambda')^{-2}(\lambda'')^{-2} \\
&= \begin{cases} (.95)^2[(.2)(.1)(1.03)^{-2}(.97)^{-2} + (.8)(.8)(.97)^{-2}(.97)^{-2}+ & \\ (.2)(.9)(1.03)^{-2}(1.03)^{-2} + (.8)(.2)(.97)^{-2}(1.03)^{-2}] & \text{if } \lambda = 0.97 \\ (.95)^2[(.9)(.1)(1.03)^{-2}(.97)^{-2} + (.1)(.8)(.97)^{-2}(.97)^{-2}+ & \\ (.9)(.9)(1.03)^{-2}(1.03)^{-2} + (.1)(.2)(.97)^{-2}(1.03)^{-2}] & \text{if } \lambda = 1.03 \end{cases} \\
&= \begin{cases} 0.9595 & \text{if } \lambda = 0.97 \\ 0.8305 & \text{if } \lambda = 1.03 \end{cases}
\end{aligned}
$$