ECON 709 - PS 3

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1. A random point (X, Y) is distributed uniformly on the square with vertices $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$. That is, the joint PDF is $f(x, y) = 1/4$ on the square and $f(x, y) = 0$ outside the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$

Case 1: $X = 1$ or $X = -1$ (left and right edge of the box). $X^2 + Y^2 < 1 \implies 1 + Y^2 < 1$. Since $Y^2 \geq 0 \implies P(1 + Y^2 < 1) = 0$.

Case 2: $Y = 1$ or $Y = -1$ (top and bottom edge of the box). $X^2 + Y^2 < 1 \implies X^2 + 1 < 1$. Since $X^2 \geq 0 \implies P(X^2 + 1 < 1) = 0$.

Therefore, $P(X^2 + Y^2 < 1) = 0$.

(b) $|X + Y| < 2$

Notice that only two points on the box do not meet $|X+Y| < 2$. At $(1, 1)$ and $(-1, -1)$, $|1+1| = |-1+(-1)| = 2$. At all other points, $|X + Y| \in [0, 2)$. Since X and Y are continuous random variables, the probability that they equal a given point is zero, so $P(|X + Y| < 2) = 1$.

2. Let the joint PDF of X and Y be given by $f(x, y) = g(x)h(y)\ \forall x, y \in \mathbb{R}$ for some functions $g(x)$ and $h(y)$. Let a denote $\int_{-\infty}^{\infty} g(x)dx$ and b denote $\int_{-\infty}^{\infty} h(x)dx$

(a) What conditions a and b should satisfy in order for $f(x, y)$ to be a bivariate PDF?

For $f(x, y)$ to be a PDF, it should integrate to one:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= 1 \\
\implies \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)dxdy &= 1 \\
\implies \int_{-\infty}^{\infty} g(x)dx \int_{-\infty}^{\infty} h(y)dy &= 1 \\
\implies ab &= 1 \\
\implies a &= b^{-1}
\end{aligned}$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

(b) Find the marginal PDF of X and Y.

The marginal PDF of X:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\infty}^{\infty} g(x)h(y)dy \\ &= g(x)\int_{-\infty}^{\infty} h(y)dy \\ &= b \cdot g(x) \end{aligned}$$

The marginal PDF of Y:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_{-\infty}^{\infty} g(x)h(y)dx \\ &= h(y)\int_{-\infty}^{\infty} g(x)dx \\ &= a \cdot h(y) \end{aligned}$$

(c) Show that X and Y are independent.

Proof: X and Y are independent if the product of their marginal distributions is their joint distribution:

$$\begin{aligned} f_X(x) \cdot f_Y(y) &= b \cdot g(x) \cdot a \cdot h(y) \\ &= b \cdot g(x) \cdot b^{-1} \cdot h(y) \\ &= g(x) \cdot h(y) \\ &= f(x,y) \end{aligned}$$

□

3. Let the joint PDF of X and Y be given by

$$f(x,y) = \begin{cases} cxy & \text{if } x, y \in [0,1], x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c such that $f(x,y)$ is a joint PDF.

$$\begin{aligned}
\int_0^1 \int_0^{1-x} f(x,y)dydx &= 1 \\
\implies \int_0^1 \int_0^{1-x} cxydydx &= 1 \\
\implies c \int_0^1 \left[\frac{xy^2}{2}\right]_{y=0}^{1-x} dx &= 1 \\
\implies \frac{c}{2} \int_0^1 x(1-x)^2 dx &= 1 \\
\implies \frac{c}{2} \int_0^1 x - 2x^2 + x^3 dx &= 1 \\
\implies \frac{c}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{x=0}^{1} &= 1 \\
\implies \frac{c}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] &= 1 \\
\implies \frac{c}{2}\left(\frac{1}{12}\right) &= 1 \\
\implies c &= 24
\end{aligned}$$

(b) Find the marginal distributions of X and Y.

$$f_X(x) = \int_0^{1-x} 24xydy$$

$$f_Y(y) = \int_0^{1-y} 24xydx$$

(c) Are X and Y independent? Compare your answer to Problem 2 and discuss.

4. Show that any random variable is uncorrelated with a constant.

5. Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for the correlation of XY and Y in terms of these means and variances.

6. Prove the following: For any random vector $(X_1, X_2, ..., X_n)$,

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \leq i < j \leq n} Cov(X_i, Y_i).$$

7. Suppose that X and Y are joint normal, i.e. they have the joint PDF:

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp(-(2(1-\rho^2))^{-1}(x^2/\sigma_X - 2xy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))$$

(a) Derive the marginal distributions of X and Y, and observe that both normal distributions.

(b) Derive the conditional distribution of Y given $X = x$. Observe that it is also a nromal distribution.

(c) Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X and Z are independent.

8. Consider a function $g : \mathbb{R} \to \mathbb{R}$. Recall that the inverse image of a set A, denoted $g^{-1}(A)$ is $g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$. Let there be functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$. Let X and Y be two random variables that are independent. Suppose that g_1 and g_2 are both Borel-measurable, which means that $g_1^{-1}(A)$ and $g_2^{-1}(A)$ are both in the Borel σ-field whenever A is in the Borel σ-field. Show that the two random variables $Z := g_1(X)$ and $W := g_2(Y)$ are independent. (Hint: use the 1st or the 2nd definition of independence.)