ECON 711 - PS 4

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Question 1. Choice rules from preferences

Let X be a choice set and $\succsim$ a complete and transitive preference relation on X. Show that the choice rule induced by $\succsim$, $C(A, \succsim) = \{x \in A : x \succsim y \ \forall y \in A\}$, must satisfy the Weak Axiom of Revealed Preference (WARP).

Proof: $C(\cdot)$ satisfies WARP if for any sets $A, B \subset X$ and any $x, y \in A \cap B$, if $x \in C(A)$ and $y \in C(B)$, then $x \in C(B)$ and $y \in C(A)$. Since $x \in C(A)$ and $y \in C(B)$, $x \succsim y$ and $y \succsim x$. For an arbitrary $w \in B$, $y \succsim w$ because $y \in C(B)$. By transitivity, $x \succsim w$, so $x \in C(B)$. For arbitrary $z \in A$, $x \succsim z$ because $x \in C(A)$. By transitivity, $y \succsim z$, so $y \in C(A)$. $\square$

Question 2. Preferences from choice rules

Let X be a choice set and $C : \mathcal{P}(X) \to \mathcal{P}(X)$ a nonempty choice rule. Show that if C satisfies WARP, then the preference relation $\succsim_C$ defined on X by "$x \succsim_C y$ iff there exists a set $A \subseteq X$ such that $x, y \in A$ and $x \in C(A)$" is complete and transitive, and that the choice rule it induces, $C(\cdot, \succsim_C)$, is equal to C.

Proof: For completeness, choose $x, y \in X$. Construct $A := \{x, y\}$. Since C is nonempty, we know that $x \in C(A)$ and/or $y \in C(A)$. If $x \in C(A)$, then $x \succsim_C y$. If $y \in C(A)$, then $y \succsim_C x$. Thus, $\succsim_C$ is complete.

For transitivity, choose $x, y, z \in X$ such that $x \succsim_C y$ and $y \succsim_C z$. This setup implies that there exists $A, B \subset X$ such that $x, y \in A$, $y, z \in B$, $x \in C(A)$, and $y \in C(B)$. Assume for sake of a contradiction that $x \notin C(A \cup B)$ and $z \in C(A \cup B)$. By WARP, $z \in C(A \cup B)$ and $y \in C(B)$ implies that $y \in C(A \cup B)$, By WARP, $y \in C(A \cup B)$ and $x \in C(A)$ implies that $x \in C(A \cup B) \Rightarrow\Leftarrow$. This is a contradiction, so $x \in C(A \cup B) \implies x \succsim_C z$.

For equality of $C(\cdot, \succsim_C)$ and C, fix nonempty $A \subset X$. Choose $x \in C(A)$. For an arbitrary $y \in A$, $x \succsim_C y$. Thus, $x \in C(A, \succsim)$. Choose $x \in C(A, \succsim_C)$, then $x \succsim_C y$ for all $y \in A$. Thus, $x \in C(A)$. Therefore, $C(\cdot, \succsim_C)$ is equal to C. $\square$

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Question 3. Choice over finite sets

Let X be a finite set, and $\succsim$ a complete and transitive preference relation on X.

(a) Show that the induced choice rule $C(\cdot, \succsim)$ is nonempty - that is $C(A, \succsim) \neq \emptyset$ if $A = \emptyset$.

Proof (by induction): Let nonempty $A, B \subset X$ such that $A := \{x\}$ for some $x \in X \setminus B$ and $|B| = n$ for some $n \in \mathbb{N}$. Notice that $|A| = 1$. Because $\succsim$ is complete, $x \succsim x$. Thus, x is weakly preferred to all elements of A. Thus, $x \in C(A, \succsim) \neq \emptyset$.

Assume $C(B, \succsim) \neq \emptyset$. Notice that $|A \cup B| = n + 1$. Choose arbitrary y from $C(B, \succsim)$, so by definition $y \succsim z$ for all $z \in B$. By completeness, $x \succsim y$ and/or $y \succsim x$. If $x \succsim y$, x is weakly preferred to all elements in B by transitivity, so $x \in C(A \cup B, \succsim)$. If $y \succsim x$, then y is weakly preferred to all elements in $A \cup B$, so $y \in C(A \cup B, \succsim)$. Thus, $C(A \cup B, \succsim) \neq \emptyset$. $\square$

(b) Show that a utility representation exists.

Proof (by induction): We will prove the stronger result that a utility representation exists with range $\{1, 2, ..., |X|\}$. Let nonempty $A, B \subset X$ such that $A := \{x\}$ for some $x \in X \setminus B$ and $|B| = n$ for some $n \in \mathbb{N}$. Define utility function $u_A : A \to \{1\}$. Trivially, $x \succsim x \iff u_A(x) = u_A(x) = 1$.

Assume that there exists such a utility function u_B for B such that $y \succsim z$ iff $u_B(y) \geq u_B(z)$ for $y, z \in B$. Construct a set of the elements of B that are strictly preferred to x, or $B_0 = \{y \in B : y \succ x\}$ where $A := \{x\}$. Let $a := \min_{z \in B_0}\{u_B(z)\}$. Define utility function $v : A \cup B \to \{1, 2, ..., |X|\}$ such that, for $w \in A \cup B$,

$$v(w) = \begin{cases} u_B(w) + 1, & w \in B_0 \\ a, & w \in A \\ u_B(w), & w \in B \setminus B_0 \end{cases}$$

To see that v is a valid utility function for $A \cup B$, notice that $A \cup B$ is composed of three disjoint sets B_0, A, and $B \setminus B_0$. When picking $y, z \in A \cup B$, there are six possibilities:

- If $y, z \in B_0$, $y \succsim z \iff u_B(y) \geq u_B(z) \iff u_B(y) + 1 \geq u_B(z) + 1 \iff v(y) \geq v(z)$ and/or $z \succsim y \iff u_B(z) \geq u_B(y) \iff u_B(z) + 1 \geq u_B(y) + 1 \iff v(z) \geq v(y)$.
- If $y, z \in A$, $y \succsim z$ and $z \succsim y$ then $v(y) = v(z) = a$, so $v(y) \geq v(z)$ and $v(y) \leq v(z)$.
- If $y, z \in B \setminus B_0$, $y \succsim z \iff u_B(y) \geq u_B(z) \iff v(y) \geq v(z)$ and/or $z \succsim y \iff u_B(z) \geq u_B(y) \iff v(z) \geq v(y)$.
- If $y \in B_0$ and $z \in A$, $y \succsim z$. Assume, for sake of a contradiction, that $v(z) > v(y) \implies a > u_B(y) > \implies \min_{w \in B_0}\{u_B(w)\} > u_B(y) \Rightarrow\Leftarrow$. $v(y) \geq v(z)$.
- If $y \in B \setminus B_0$ and $z \in A$, $z \succsim y$. Assume, for sake of a contradiction, that $v(y) > v(z) \implies u_B(y) > a \implies u_B(y) > \min_{w \in B_0}\{u_B(w)\} \implies y \succ w_0$ for some $w_0 \in B_0$. By transitivity, $y \succ z \Rightarrow\Leftarrow$. $v(y) \leq v(z)$.
- If $y \in B_0$ and $z \in B \setminus B_0$, $y \succ x$ and $x \succsim z$. By transitivity, $y \succ z$.

Thus, v is valid utility function for $A \cup B$. $\square$