

# ECON 703 - PS 4

Alex von Hafften\*

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- (1) Let  $X, Y$  be two vector spaces such that  $\dim X = n$ ,  $\dim Y = m$ . Construct a basis of  $L(X, Y)$ .
- (2) Suppose that  $T \in L(X, X)$  and  $\lambda$  is  $T$ 's eigenvalue.
- (a) Prove that  $\lambda^k$  is an eigenvalue of  $T^k$ ,  $k \in \mathbb{N}$ .

Proof: If  $\lambda$  is  $T$ 's eigenvalue  $\implies T(v) = \lambda v$  for eigenvector  $v \neq \bar{0}$ . Applying  $T$  again, we get  $T(T(v)) = \lambda(\lambda v) \implies T^2(v) = \lambda^2 v$ . Similarly, applying  $T$   $k$  times to  $v$ , we get  $T^k(v) = \lambda^k v$ . Thus,  $\lambda^k$  is an eigenvalue for  $T^k$  where  $k \in \mathbb{N}$ .

- (b) Prove that if  $T$  is invertible, then  $\lambda^{-1}$  is an eigenvalue of  $T^{-1}$ .

Proof: If  $T$  is invertible,  $T^{-1} \in L(X, X)$ . Thus, for eigenvector  $v$  of  $T$ ,

$$\begin{aligned} T(v) = \lambda v &\implies T^{-1}(T(v)) = T^{-1}(\lambda v) \\ &\implies v = \lambda T^{-1}(v) \\ &\implies T^{-1}(v) = \lambda^{-1} v. \end{aligned}$$

Therefore,  $\lambda^{-1}$  is an eigenvalue of  $T^{-1}$ .

- (c) Define an operator  $S : X \rightarrow X$ , such that  $S(x) = T(x) - \lambda x$  for all  $x \in X$ . Is  $S$  linear? Prove that  $\ker S := \{x \in X | S(x) = \bar{0}\}$  is a vector space.

For  $x_1, x_2 \in X$  and  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,

$$\begin{aligned} S(\alpha_1 x_1 + \alpha_2 x_2) &= T(\alpha_1 x_1 + \alpha_2 x_2) - \lambda(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \alpha_1 T(x_1) + \alpha_2 T(x_2) - \lambda \alpha_1 x_1 - \lambda \alpha_2 x_2 \\ &= \alpha_1 (T(x_1) - \lambda x_1) + \alpha_2 (T(x_2) - \lambda x_2) \\ &= \alpha_1 S(x_1) + \alpha_2 S(x_2) \end{aligned}$$

Yes,  $S$  is linear.

Proof: Let  $x, y \in \ker S$  and  $\alpha, \beta \in \mathbb{R}$ . We know that  $S(x) = S(y) = \bar{0}$ . As a linear transformation,  $S(\alpha x + \beta y) = \alpha S(x) + \beta S(y) = \alpha \bar{0} + \beta \bar{0} = \bar{0}$ , so properties 1, 2, 5, 6, 7 of the definition of a vector space are satisfied.

For property 3, note that  $S(\bar{0}) = T(\bar{0}) - \lambda \bar{0} = T(\bar{0}) = \bar{0}$ , so  $\bar{0} \in \ker S$ .<sup>1</sup> Furthermore,  $x + \bar{0} = \bar{0} + x = x$  for  $x \in \ker S$ .

For property 4, for  $x \in \ker S$ ,  $S(-x) = (-1)S(x) = (-1)\bar{0} = \bar{0}$  where  $x + (-x) = \bar{0}$ .

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<sup>1</sup>For any linear transformation  $T : X \rightarrow Y$ ,  $T(\bar{0}_X) = T(\bar{0}_X + \bar{0}_X) = T(\bar{0}_X) + T(\bar{0}_X) \implies T(\bar{0}_X) = T(\bar{0}_X) - T(\bar{0}_X) = \bar{0}_Y$ .

For property 8, for  $x \in \ker S$ ,  $S(1 \cdot x) = 1 \cdot S(x) = 1 \cdot \bar{0} = \bar{0}$ .

Thus  $\ker S$  is a vector space.  $\square$

- (3) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by  $T(x, y) = (x - y, 2x + 3y)$ . Let  $W$  be the standard basis of  $\mathbb{R}^2$  and let  $V$  be another basis of  $\mathbb{R}^2$ ,  $V = \{(1, -4), (-2, 7)\}$  in the coordinates of  $W$ .

- (a) Find  $\text{mtx}_W(T)$ .

$$\text{mtx}_W(T) = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

- (b) Find  $\text{mtx}_V(T)$ .

$$\text{mtx}_{W,V}(id) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- (c) Find  $T(1, -2)$  in the basis  $V$ .

- (4) In this exercise you will learn to solve first order linear difference equations in  $n$  variables. We want to find an  $n$ -dimensional process  $\{\mathbf{x}_1, \mathbf{x}_2, \dots\}$  such that each  $\mathbf{x}_i$  is an  $n$ -dimensional vector and

$$\mathbf{x}_t = A\mathbf{x}_{t-1}, t = 1, 2, \dots, \quad (1)$$

where  $A \in M_{n \times n}$  and  $\mathbf{x}_0 \in \mathbb{R}^n$  are given. Then

$$\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0, \mathbf{x}_t = A^t\mathbf{x}_0 \forall t \in \mathbb{N},$$

where  $A^t = A \cdot A \cdot \dots \cdot A$  ( $t$  times). Thus, we need to calculate  $A^t$ .

To do this, we diagonalize  $A$ ,  $A = PDP^{-1}$ , where  $D$  is diagonal,  $D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ .

Hence we can rewrite

$$A^t = PDP^{-1}PDP^{-1} \dots PDP^{-1} = PD^tP^{-1} = P\text{diag}\{\lambda_1, \dots, \lambda_n\}P^{-1},$$

which is now easy to compute. Thus, what you is

Step 1: Calculate  $A$ 's eigenvalues  $\lambda_1, \dots, \lambda_n$  and eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

Remember that we need to independent eigenvectors (this holds if all eigenvalues are distinct).

Step 2: Set  $D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$  and  $P = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  (eigenvectors are columns of  $P$ ).

Step 3: Calculate  $P^{-1}$  and  $P\text{diag}\{\lambda_1^t, \dots, \lambda_n^t\}P^{-1}$ .

Step 4: Plug  $A^t$  from Step 3 to get  $\mathbf{x}_t = A^t\mathbf{x}_0$ .

Implement the above approach to solve for  $\mathbf{x}_t \in \mathbb{R}^2$ :

$$\mathbf{x}_t = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} \mathbf{x}_{t-1}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Simplify your answer as much as possible.

- (5) In this exercise you will learn to solve  $n$ th order linear difference equations in one variable. We want to find a sequence of real numbers  $\{z_t\}_{t=1}^\infty$ , which satisfies

$$z_t = a_1 z_{t-1} + a_2 z_{t-2} + \dots + a_n z_{t-n}, \quad (2)$$

where  $a_1, \dots, a_n \in \mathbb{R}$  and  $z_0, z_{-1}, \dots, z_{-n+1} \in \mathbb{R}$  are given.

- (a) Define  $\mathbf{x}_t := (z_t, z_{t-1}, \dots, z_{t-n+1})'$  and rewrite Eq. (2) in the form of Eq. (1). What is  $A$ ?
- (b) Notice that if you find the function form of  $z_t = f(t)$ , then you do not need to find a similar form for  $z_{t-1}, \dots, z_{t-n+1}$  (you use the same function  $f(\cdot)$  and evaluate it at a different time). Thus, you actually do not need to calculate  $P \text{diag}\{\lambda_1^t, \dots, \lambda_n^t\} P^{-1} \mathbf{x}_0$ . You only need the first coordinate of that  $n$ -dimensional vector. The first coordinate takes the form

$$\mathbf{x}_{t1} \equiv z_t = c_1 \lambda_1^t + c_2 \lambda_2^t + \dots + c_n \lambda_n^t, \quad (3)$$

where coefficient  $c_1, \dots, c_n$  depend on  $P$  and  $\mathbf{x}_0$ .

Given Eq. (3) which holds for any  $t$  and initial values  $z_0, \dots, z_{-n+1}$ , which equations must  $c_1, \dots, c_n$  solve?

- (c) Suppose that  $n = 3$ ,  $a_1 = 2$ ,  $a_2 = 1$ ,  $a_3 = -2$ , and  $z_0 = 2$ ,  $z_{-1} = 2$ ,  $z_{-2} = 1$ . Find the expression for  $a_t$  as a function of  $t$ .