FIN 970: Homework 2

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March 27, 2022

1 Cash-Flow and Return Predictability

The goal for this exercise is to examine the evidence on cash flow and return predictability in the data.

Get annual data from 1930 to most recent period for log nominal market return $r^{\$}_{d,t}$, nominal dividends $\Delta d^{\$}_t$, and log price-dividend ratio pd_t from CRSP. We also want to use log inflation rate π_t (for example, from FRED) to help us convert nominal series into real.

1. To get real market returns, we can just subtract inflation from nominal market returns: $r_{d,t} = r^{\$}_{d,t} - \pi_t$. Do similar adjustment to remove inflation from nominal dividends.

   Solution: See `data.xlsx`.

2. Let us consider h-horizon forecasts of future cumulative returns and dividends:

$$\frac{1}{h}\sum_{j=1}^{h} \Delta d_{t+j} = const + \beta_d x_t + error$$

$$\frac{1}{h}\sum_{j=1}^{h} \Delta r_{t+j} = const + \beta_r x_t + error$$

   where x_t is a predictive variable, and h is the forecast horizon. We can compute slope coefficients and R^2 using standard OLS regressions. Note the overlapping observations on the LHS - how would that affect the computation of the standard errors?

   Solution: Since we're average future dividends and returns, the observations on LHS are now mechanically correlated we each other. For example, if $h = 3$, the dividends in 1980 are included in the LHS variable for 1977, 1978, and 1979. Asymptotic standard errors are only valid if observations are a random sample. Not accounting for this correlation will bias the standard errors upward. For valid standard errors, we use bootstrap or use Newey-West standard errors.

3. Run these regressions for the horizons of h 1 year to 5 years, using price dividend ratio as a predictive variable, i.e. set $x_t = pd_t$. What are the slope coefficients and R^2s? Do you find the signs of slope coefficients reasonable? What is the statistical significance of the slope coefficients and the R^2s? What does this evidence suggest in terms of the drivers of the aggregate equity prices in the data?

   Solution: My estimates for slope coefficients and R^2s are within the confidence intervals for such regression in the lecture slides. My regressions feature the same pattern for dividends the R^2s starts relatively high and then drops as you extend the forecasting window and the R^2s for returns starts relatively low and increases. The significance and magnitude of the slope coefficients for the dividend regressions grow whereas the slope coefficient for the return regressions stay about the same magnitude and become more significant. For dividend, the sign of the slope coefficients indicate that higher *pd* ratios predict higher future dividends and lower market returns. This suggests that *pd* ratios are better predictors of returns than dividends.

Table 1: Dividends

	Dependent variable:				
	delta_d_1	delta_d_2	delta_d_3	delta_d_4	delta_d_5
	(1)	(2)	(3)	(4)	(5)
pd	0.077***	0.055***	0.039***	0.029**	0.024**
	(0.024)	(0.019)	(0.015)	(0.011)	(0.009)
Constant	−0.246***	−0.169**	−0.115**	−0.077*	−0.058*
	(0.084)	(0.067)	(0.052)	(0.039)	(0.031)
Observations	91	90	89	88	87
R^2	0.100	0.083	0.074	0.071	0.074
Note:			*p<0.1; **p<0.05; ***p<0.01		

Table 2: Market Returns

	Dependent variable:				
	r_1	r_2	r_3	r_4	r_5
	(1)	(2)	(3)	(4)	(5)
pd	−0.054	−0.068**	−0.061***	−0.058***	−0.057***
	(0.043)	(0.028)	(0.020)	(0.017)	(0.015)
Constant	0.252*	0.302***	0.279***	0.267***	0.266***
	(0.148)	(0.098)	(0.070)	(0.058)	(0.051)
Observations	91	90	89	88	87
R^2	0.017	0.061	0.092	0.119	0.151
Note:			*p<0.1; **p<0.05; ***p<0.01		

See `data.R` for regressions.

2 Twist on Log-Linearization

The return log-linearization formula is typically written as

$$r_{t+1} \approx \kappa_0 + \kappa_1 pd_{t+1} - pd_t + \Delta d_{t+1} \tag{1}$$

where κ_0 and κ_1 are log-linearization constants which depend on the mean of price-dividend ratio. Use their solution to show that you can rewrite this formula in the following way:

$$r_{t+1} \approx -\log \kappa_1 + \kappa_1 \tilde{pd}_{t+1} - \tilde{pd}_t + \Delta d_{t+1} \tag{2}$$

where $\tilde{pd}$ is the demeaned price-dividend ratio, $\tilde{pd}_t = pd_t - E(pd_t)$. There is nothing deep about this result. However, it's oftentimes more convenient to use (e.g., in equilibrium model solutions), as we do not need to keep track of κ_0 and the (unimportant) intercepts in the solution of price-dividend ratios.

Solution: Following the lecture notes from class,

$$\begin{aligned}
R_{t+1} &= \frac{P_{t+1} + D_{t+1}}{P_t} \\
&= \frac{D_{t+1}(\frac{P_{t+1}}{D_{t+1}} + 1)}{D_t \frac{P_t}{D_t}} \\
&= \frac{D_{t+1}}{D_t} \frac{PD_{t+1} + 1}{PD_t} \\
\text{where } PD_t &:= P_t / D_t \\
\implies \log(R_t) &= \log(\frac{D_{t+1}}{D_t}) + \log(PD_{t+1} + 1) - \log(PD_t) \\
r_t &= \Delta d_{t+1} - pd_t + \log(\exp(pd_t) + 1) \\
\text{where } r_t &:= \log(R_t) \\
\text{and } pd_t &:= \log(PD_t) \\
f(x) &:= \log(\exp(x) + 1) \\
\implies f'(x) &= \frac{\exp(x)}{\exp(x) + 1} \\
f(x) &\approx f(\bar{x}) + f'(\bar{x})(x - \bar{x}) \\
&= \log(\exp(\bar{x}) + 1) + \frac{\exp(\bar{x})}{\exp(\bar{x}) + 1}(x - \bar{x}) \\
\implies \log(\exp(pd_t) + 1) &\approx \log(\exp(E(pd_t)) + 1) + \frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}(pd_t - E(pd_t)) \\
\implies r_{t+1} &\approx \Delta d_{t+1} - pd_t + \log(\exp(E(pd_t)) + 1) + \frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}(pd_t - E(pd_t)) \\
&= \kappa_0 + \kappa_1 pd_{t+1} + \Delta d_{t+1} - pd_t \\
\text{where } \kappa_0 &= \log(\exp(E(pd_t)) + 1) - \frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1} E(pd_t) \\
\text{and } \kappa_1 &= \frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}
\end{aligned}$$

(1) and (2) are equivalent iff

$$
\begin{aligned}
\kappa_0 + \kappa_1 pd_{t+1} - pd_t &= -\log k_1 + \kappa_1 \tilde{pd}_{t+1} - \tilde{pd}_t \\
\iff \left[\log(\exp(E(pd_t)) + 1) - \frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1} E(pd_t)\right] + \left[\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}\right] pd_{t+1} - pd_t \\
= -\log\left[\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}\right] + \left[\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}\right][pd_{t+1} - E(pd_{t+1})] - [pd_t - E(pd_t)] \\
\iff \log(\exp(E(pd_t)) + 1) - \frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1} E(pd_t) \\
= -\log\left[\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}\right] - \left[\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}\right] E(pd_{t+1}) + E(pd_t) \\
\iff \log(\exp(E(pd_t)) + 1) - \frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1} E(pd_t) \\
= -\log(\exp(E(pd_t))) + \log(\exp(E(pd_t)) + 1) - \left[\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}\right] E(pd_{t+1}) + E(pd_t) \\
\iff -\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1} E(pd_t) &= -E(pd_t) - \left[\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}\right] E(pd_{t+1}) + E(pd_t) \\
\iff -\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1} E(pd_t) &= -\left[\frac{\exp(E(pd_t))}{\exp(E(pd_t)) + 1}\right] E(pd_{t+1}) \\
E(pd_t) &= E(pd_{t+1})
\end{aligned}
$$

$E(pd_t) = E(pd_{t+1})$ holds iff pd_t is a martingale.

3 External Habits

Consider an environment very similar to the external habits model of Campbell-Cochrane, 1999.

- Investors have utility over consumption relative to a reference point X_t:

$$u_t = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

- Consumption growth is iid Normal:

$$\Delta c_{t+1} = g + v_{t+1}, \quad Var(v_{t+1}) = \sigma_v^2$$

- Define surplus consumption S_t:

$$S_t = \frac{C_t - X_t}{C_t}$$

- Log surplus consumption is driven by the consumption news:

$$s_{t+1} = \bar{s} + \phi(s_t - \bar{s}) + \lambda(s_t)v_{t+1}$$

  where the sensitivity function is specified as in CC, 99:

$$\lambda(s) = \frac{1}{\bar{S}}\sqrt{1 - 2(s - \bar{s})} - 1,$$

  when $s_t < s_{max}$ and 0 otherwise.

- The only difference between CC, 99 is the specification of $\bar{s} = \log \bar{S}$

$$\bar{S} = \sigma_v \sqrt{\frac{\gamma}{1 - \phi - b/\gamma}}$$

  where b is a preference parameter.

1. Show that the one-period real risk-free rates are now time-varying, and are linear in the consumption surplus s_t.

   Solution: Manipulating consumption growth:

$$\Delta c_{t+1} = g + v_{t+1} \implies \frac{C_{t+1}}{C_t} = \exp(g + v_{t+1})$$

   Manipulating the surplus consumption law of motion:

$$s_{t+1} = \bar{s} + \phi(s_t - \bar{s}) + \lambda(s_t)v_{t+1} \implies \frac{S_{t+1}}{S_t} = \exp((\phi - 1)(s_t - \bar{s}) + (1 + \lambda(s_t))v_{1+t})$$

   Plugging consumption surplus into the utility function:

$$u_t = \frac{(C_t S_t)^{1-\gamma} - 1}{1-\gamma} \implies u_{c,t} = (C_t S_t)^{-\gamma}$$

Thus, the stochastic discount factor is:

$$
\begin{aligned}
M_{t+1} &= \beta \frac{u_{c,t+1}}{u_{c,t}} \\
&= \beta \frac{(C_{t+1}S_{t+1})^{-\gamma}}{(C_t S_t)^{-\gamma}} \\
&= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{S_{t+1}}{S_t} \right)^{-\gamma} \\
&= \beta \exp(-\gamma(g + v_{t+1})) \exp(-\gamma[(\phi - 1)(s_t - \bar{s}) + (1 + \lambda(s_t))v_{1+t})]) \\
&= \beta \exp(-\gamma g) \exp(\gamma(1-\phi)(s_t - \bar{s})) \exp(\underbrace{-\gamma(1+\lambda(s_t))v_{t+1}}_{\sim Normal(0, \gamma^2(1+\lambda(s_t))^2\sigma_v^2)})
\end{aligned}
$$

The expected value of the stochastic discount factor is

$$
\begin{aligned}
E_t[M_{t+1}] &= E_t[\beta \exp(-\gamma g) \exp(\gamma(1-\phi)(s_t - \bar{s})) \exp(-\gamma(1+\lambda(s_t))v_{t+1})] \\
&= \beta \exp(-\gamma g) \exp(\gamma(1-\phi)(s_t - \bar{s})) E_t[\exp(-\gamma(1+\lambda(s_t))v_{t+1})] \\
&= \beta \exp(-\gamma g) \exp(\gamma(1-\phi)(s_t - \bar{s})) \exp\left(\frac{1}{2}\gamma^2(1+\lambda(s_t))^2\sigma_v^2 \right)
\end{aligned}
$$

using the MGF of the normal distribution (i.e., if $X \sim N(\mu, \sigma^2)$ then $E[\exp(tX)] = \exp(\mu t + \sigma^2 t^2/2)$). The gross risk free rate is:

$$
\begin{aligned}
R_t^f &= \frac{1}{E_t[M_{t+1}]} \\
&= \frac{1}{\beta} \exp(\gamma g) \exp(-\gamma(1-\phi)(s_t - \bar{s})) \exp\left(-\frac{1}{2}\gamma^2(1+\lambda(s_t))^2\sigma_v^2 \right)
\end{aligned}
$$

Taking logs,

$$
\begin{aligned}
r_t^f &= -\log(\beta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{1}{2}\gamma^2(1+\lambda(s_t))^2\sigma_v^2 \\
&= -\log(\beta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{1}{2}\gamma^2 \left(\frac{1}{\bar{S}}\sqrt{1 - 2(s_t - \bar{s})} \right)^2 \sigma_v^2 \\
&= -\log(\beta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{1}{2}\gamma^2 \frac{1}{\left(\sigma_v \sqrt{\frac{\gamma}{1-\phi-b/\gamma}} \right)^2} (1 - 2(s_t - \bar{s}))\sigma_v^2 \\
&= -\log(\beta) + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \frac{1}{2}\gamma(1 - \phi - b/\gamma)(1 - 2(s_t - \bar{s})) \\
&= -\log(\beta) + \gamma g - \gamma s_t + \gamma \bar{s} + \gamma\phi s_t - \gamma\phi\bar{s} - \frac{1}{2}(\gamma - \gamma\phi - b) + s_t\gamma - s_t\gamma\phi - s_t b - \bar{s}\gamma + \bar{s}\gamma\phi + \bar{s}b \\
&= -\log(\beta) + \gamma g - \frac{1}{2}(\gamma - \gamma\phi - b) - b(s_t - \bar{s})
\end{aligned}
$$

by substituting in for $\lambda(s_t)$ and $\bar{S}$. Clearly, r_t^f is linear in s_t and time varying through s_t.

2. Consider the case of $b > 0$ - this is the case studied in Wachter, 2006. How do the interest rates vary with the consumption surplus ratio? Are they low or high in “good times”? Are the real bonds risky in this model, or do they hedge aggregate risks?

   Solution: As discussed in Wachter (2006), if $b < 0$, intertemporal smoothing dominates precautionary savings, so an increase in surplus consumptions drives down interest rates. Formally, $b > 0$, $\frac{\partial r_t^f}{\partial s_t} = -b < 0$. Interest rates are low for in times with high consumption surplus ratios. High consumption surpluses correspond to “good times”.

   Consider any (possibly risky) bond with real return R_{t+1}. If the bond is traded, then the Euler equation must hold:

$$\begin{aligned}
E_t[M_{t+1}R_{t+1}] &= 1 \\
\implies E_t[M_{t+1}]E_t[R_{t+1}] + Cov_t(M_{t+1}, R_{t+1}) &= 1 \\
\implies E_t[R_{t+1}] + \frac{Cov_t(M_{t+1}, R_{t+1})}{E_t[M_{t+1}]} &= \frac{1}{E_t[M_{t+1}]} \\
\implies E_t[R_{t+1}] + \frac{Cov_t(M_{t+1}, R_{t+1})}{E_t[M_{t+1}]} &= R_{t+1}^f \\
\implies E_t[R_{t+1}] - R_{t+1}^f &= -\frac{Cov_t(M_{t+1}, R_{t+1})}{E_t[M_{t+1}]} \\
&= -\frac{\rho_t(M_{t+1}, R_{t+1})\sigma_t(M_{t+1})\sigma_t(R_{t+1})}{E[M_{t+1}]} \\
\implies \frac{E_t[R_{t+1}] - R_{t+1}^f}{\sigma_t(R_{t+1})} &= -\rho_t(M_{t+1}, R_{t+1})\frac{\sigma_t(M_{t+1})}{E_t[M_{t+1}]} \\
&\approx -\rho_t(M_{t+1}, R_{t+1})\gamma\sigma_v^2(1 + \lambda(s_t))
\end{aligned}$$

   with the last approximation following from the log-normality of M_{t+1} conditional on time-t information (see Wachter 2006; still need to verify the logic on my own). Since $\lambda(s_t)$ is decreasing in s, risk premia move counter-cyclically (i.e. risk premia are high when s_t is low). So yes, real bonds are risky and they do hedge aggregate risks because the expected excess returns are higher in “bad times.”

3. Compare the behavior of interest rates in this model with the predictions in the Long-Run Risk model. In the long-run risks model, do real rates fall or rise in “good” times (think about “good” times as high expected growth and/or low conditional volatility).

   Solution: In the Long-Run Risk model, the risk-free rate is

$$r_{f,t} = -\log\delta + \frac{1}{\psi}(\mu + x_t) + r_\sigma\sigma_t^2$$

   (see Long-Run Risk question or lecture notes) where $\psi > 1$ is the intertemporal elasticity of substitution and $r_q < 0$ is the loading on the conditional volatility. In the Long-Run risk model, the risk-free rate increases is good times (i.e. x_t is high and σ_t is low): $\frac{\partial r_{f,t}}{\partial x_t} = \frac{1}{\psi} > 0$ and $\frac{\partial r_{f,t}}{\partial \sigma_t} = r_q < 0$.

4. Can you use these model predictions to test the two asset-pricing theories? How would you do that?

   Solution: Yes, we can look at good times (i.e., with high consumption growth and low volatility) and bad times (i.e., with low consumption growth and high volatility). If interest rates are higher in good times, this evidence is supportive of the long-run risk models, but if interest rates are higher in bad times, this evidence is supportive of the habits models.

4 Long-Run Risks Model

Consider the following specification of the long-run risks model. Consumption and dividend dynamics are given by

$$\begin{aligned}
\Delta c_{t+1} &= \mu_g + x_t + \sigma_t \eta_{t+1}, \\
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1}, \\
\sigma^2_{t+1} &= \sigma^2_0 + \nu(\sigma^2_t - \sigma^2_0) + \sigma_w w_{t+1}, \\
\Delta d_{t+1} &= \mu_d + \phi x_t + \pi_d \sigma_t \eta_{t+1} + \varphi_d \sigma_t u_{d,t+1}
\end{aligned}$$

where all shocks are iid uncorrelated standard Normal.

The investor has recursive Epstein-Zin preferences over the future consumption,

$$U_t = \left[(1-\delta)C_t^{1-\frac{1}{\psi}} + \delta(E_t U_{t+1}^{1-\gamma})^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

Theoretical Model Solution

1. Conjecture that price-consumption ratio is linear in expected growth,

$$pc_t = A_0 + A_x x_t + A_\sigma \sigma^2_t,$$

and use Euler equation on consumption asset and log-linearization of consumption return to solve for A_0, A_x, A_σ and log-linearization coefficient κ_1 in terms of the fundamental model parameters. Under what conditions asset valuations respond positively to expected growth? negatively to consumption volatility? What do those conditions mean, economically?

Solution: The stochastic discount factor for Epstein-Zin preferences (without labor income) is:

$$\begin{aligned}
M_{t+1} &= \frac{U_{c,t+1}}{U_{c,t}} \\
&= \delta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{C,t+1}^{\theta-1}
\end{aligned}$$

where $R_{C,t+1} = \frac{W_{t+1}}{W_t - C_t}$ is the gross return on the aggregate consumption portfolio and $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$. Taking logs,

$$\begin{aligned}
m_{t+1} &= \log M_{t+1} \\
&= \theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1) r_{c,t+1}
\end{aligned}$$

where $\Delta c_{t+1} = \log(C_{t+1}/C_t)$ and $r_{c,t+1} = \log R_{C,t+1}$.

Using the Campbell-Schiller approximation to log-linearization the consumption return:

$$
\begin{aligned}
R_{C,t+1} &= \frac{P_{C,t+1} + C_{t+1}}{P_{C,t}} \\
&= \frac{\frac{P_{C,t+1}}{C_{t+1}} + 1}{\frac{P_{C,t}}{C_t}} \frac{C_{t+1}}{C_t} \\
r_{c,t+1} &= \log(\exp(pc_{t+1}) + 1) - pc_t + \Delta c_{t+1} \\
&\approx \left[\log(\exp(\bar{pc}) + 1) + \frac{\exp(\bar{pc})}{\exp(\bar{pc}) + 1}(pc_{t+1} - \bar{pc}) \right] - pc_t + \Delta c_{t+1} \\
&= \underbrace{\log(\exp(\bar{pc}) + 1) - \frac{\exp(\bar{pc})}{\exp(\bar{pc}) + 1}\bar{pc}}_{\equiv \kappa_0} + \underbrace{\frac{\exp(\bar{pc})}{\exp(\bar{pc}) + 1}}_{\equiv \kappa_1} pc_{t+1} - pc_t + \Delta c_{t+1}
\end{aligned}
$$

where $pc_t = \log(P_{C,t}/C_t)$. Using the result from problem 2,

$$
\begin{aligned}
r_{c,t+1} &= \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1} \\
&= -\log \kappa_1 + \kappa_1 \underbrace{\tilde{pc}_{t+1}}_{\equiv pc_{t+1} - \bar{pc}} - \tilde{pc}_t + \Delta c_{t+1}
\end{aligned}
$$

Given the guess for pc_t, its unconditional expected value is $\bar{pc} = A_0 + A_\sigma \sigma_0^2$, so $\tilde{pc}_t = A_x x_t + A_\sigma(\sigma_t^2 - \sigma_0^2)$. Plugging the dynamics for volatility and consumption:

$$
\begin{aligned}
\tilde{pc}_{t+1} &= A_x x_{t+1} + A_\sigma(\sigma_{t+1}^2 - \sigma_0^2) \\
&= A_x[\rho x_t + \varphi_e \sigma_t e_{t+1}] + A_\sigma[\nu(\sigma_t^2 - \sigma_0^2) + \sigma_w w_{t+1}] \\
&= A_x \rho x_t + A_\sigma \nu(\sigma_t^2 - \sigma_0^2) + A_x \varphi_e \sigma_t e_{t+1} + A_\sigma \sigma_w w_{t+1}
\end{aligned}
$$

Plugging into the consumption return:

$$
\begin{aligned}
r_{c,t+1} &= -\log \kappa_1 + \kappa_1[A_x \rho x_t + A_\sigma \nu(\sigma_t^2 - \sigma_0^2) + A_x \varphi_e \sigma_t e_{t+1} + A_\sigma \sigma_w w_{t+1}] \\
&\quad - [A_x x_t + A_\sigma(\sigma_t^2 - \sigma_0^2)] + [\mu_g + x_t + \sigma_t \eta_{t+1}] \\
&= [-\log \kappa_1 + \mu_g] + [\kappa_1 A_x \rho - A_x + 1]x_t + [\kappa_1 A_\sigma \nu - A_\sigma](\sigma_t^2 - \sigma_0^2) \\
&\quad + \kappa_1 A_x \varphi_e \sigma_t e_{t+1} + \kappa_1 A_\sigma \sigma_w w_{t+1} + \sigma_t \eta_{t+1}
\end{aligned}
$$

For any asset with return $R_{i,t+1}$, the Euler equation holds and if the return is log-normal:

$$
\begin{aligned}
1 &= E_t[M_{t+1} R_{i,t+1}] \\
&= E_t[\exp(m_{t+1} + r_{i,t+1})] \\
&= \exp(E_t[m_{t+1} + r_{i,t+1}] + \frac{1}{2} Var_t[m_{t+1} + r_{i,t+1}]) \\
\implies 0 &= E_t[m_{t+1} + r_{i,t+1}] + \frac{1}{2} Var_t[m_{t+1} + r_{i,t+1}]
\end{aligned}
$$

In particular for the consumption assets, the Euler equation holds.

$$
\begin{aligned}
m_{t+1} + r_{c,t+1} &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta r_{c,t+1} \\
&= \theta \log \delta - \frac{\theta}{\psi} [\mu_g + x_t + \sigma_t \eta_{t+1}] \\
&+ \theta[-\log \kappa_1 + \mu_g] + \theta[\kappa_1 A_x \rho - A_x + 1] x_t + \theta[\kappa_1 A_\sigma \nu - A_\sigma](\sigma_t^2 - \sigma_0^2) \\
&+ \theta \kappa_1 A_x \varphi_e \sigma_t e_{t+1} + \theta \kappa_1 A_\sigma \sigma_w w_{t+1} + \theta \sigma_t \eta_{t+1} \\
&= \theta \log \delta - \frac{\theta}{\psi} [\mu_g + x_t + \sigma_t \eta_{t+1}] \\
&+ \theta[-\log \kappa_1 + \mu_g] + \theta[\kappa_1 A_x \rho - A_x + 1] x_t + \theta[\kappa_1 A_\sigma \nu - A_\sigma](\sigma_t^2 - \sigma_0^2) \\
&+ \theta \kappa_1 A_x \varphi_e \sigma_t e_{t+1} + \theta \kappa_1 A_\sigma \sigma_w w_{t+1} + \theta \sigma_t \eta_{t+1} \\
&= \theta[\log \delta - \log \kappa_1 + (1 - \frac{1}{\psi}) \mu_g] + \theta[\kappa_1 A_x \rho - A_x + 1 - \frac{1}{\psi}] x_t \\
&+ \theta[\kappa_1 A_\sigma \nu - A_\sigma](\sigma_t^2 - \sigma_0^2) + \theta \kappa_1 A_x \varphi_e \sigma_t e_{t+1} \\
&+ \theta \kappa_1 A_\sigma \sigma_w w_{t+1} + \theta(1 - \frac{1}{\psi}) \sigma_t \eta_{t+1}
\end{aligned}
$$

The expected value and variance of $m_{t+1} + r_{c,t+1}$ is

$$
\begin{aligned}
E_t[m_{t+1} + r_{c,t+1}] &= \theta[\log \delta - \log \kappa_1 + (1 - \frac{1}{\psi}) \mu_g] + \theta[\kappa_1 A_x \rho - A_x + 1 - \frac{1}{\psi}] x_t \\
&+ \theta[\kappa_1 A_\sigma \nu - A_\sigma](\sigma_t^2 - \sigma_0^2) \\
Var_t[m_{t+1} + r_{c,t+1}] &= \theta^2 \kappa_1^2 A_x^2 \varphi_e^2 \sigma_t^2 + \theta^2 \kappa_1^2 A_\sigma^2 \sigma_w^2 + \theta^2 (1 - \frac{1}{\psi})^2 \sigma_t^2
\end{aligned}
$$

Thus, we can plug the expected value and variance back in:

$$
\begin{aligned}
& 0 = E_t[m_{t+1} + r_{i,t+1}] + \frac{1}{2} Var_t[m_{t+1} + r_{i,t+1}] \\
\implies & 0 = \left[\theta[\log \delta - \log \kappa_1 + (1 - \frac{1}{\psi}) \mu_g] + \theta[\kappa_1 A_x \rho - A_x + 1 - \frac{1}{\psi}] x_t + \theta[\kappa_1 A_\sigma \nu - A_\sigma](\sigma_t^2 - \sigma_0^2) \right] \\
& + \frac{1}{2} \left[\theta^2 \kappa_1^2 A_x^2 \varphi_e^2 \sigma_t^2 + \theta^2 \kappa_1^2 A_\sigma^2 \sigma_w^2 + \theta^2 (1 - \frac{1}{\psi})^2 \sigma_t^2 \right] \\
& 0 = \left[\theta[\log \delta - \log \kappa_1 + (1 - \frac{1}{\psi}) \mu_g] + \frac{1}{2} \theta^2 \kappa_1^2 A_\sigma^2 \sigma_w^2 - \theta[\kappa_1 A_\sigma \nu - A_\sigma] \sigma_0^2 \right] \\
& + \left[\frac{1}{2} \theta^2 \kappa_1^2 A_x^2 \varphi_e^2 + \frac{1}{2} \theta^2 (1 - \frac{1}{\psi})^2 + \theta[\kappa_1 A_\sigma \nu - A_\sigma] \right] \sigma_t^2 \\
& + \theta[\kappa_1 A_x \rho - A_x + 1 - \frac{1}{\psi}] x_t \\
\implies & \begin{cases} 0 & = \log \delta - \log \kappa_1 + (1 - \frac{1}{\psi}) \mu_g + \frac{1}{2} \theta \kappa_1^2 A_\sigma^2 \sigma_w^2 - [\kappa_1 A_\sigma \nu - A_\sigma] \sigma_0^2 \\ 0 & = \kappa_1 A_x \rho - A_x + 1 - \frac{1}{\psi} \\ 0 & = \frac{1}{2} \theta \kappa_1^2 A_x^2 \varphi_e^2 + \frac{1}{2} (1 - \gamma)(1 - \frac{1}{\psi}) + [\kappa_1 A_\sigma \nu - A_\sigma] \end{cases}
\end{aligned}
$$

Thus, we now have three equations in three unknowns $(A_x, A_\sigma, \kappa_1)$:

$$A_x = \frac{1-1/\psi}{1-\kappa_1\rho}$$
$$A_\sigma = \frac{1}{2}\frac{(1-\gamma)(1-\frac{1}{\psi})}{1-\kappa_1\nu}\left[1+\varphi_e^2\Big(\frac{\kappa_1}{1-\kappa_1\rho}\Big)^2\right]$$

We can solve for κ_1 using a fixed point algorithm as described in lecture:

$$\kappa_1^{(n)} = \delta\exp((1-1/\psi)\mu_g + A_\sigma^{(n-1)}(1-\kappa_1^{(n-1)}\nu)\sigma_0^2 + \frac{1}{2}\theta(\kappa_1^{(n-1)})^2(A_\sigma^{(n-1)})^2\sigma_w^2)$$

where $\kappa_1^{(n)}$ is the nth guess for κ_1 and $A_\sigma^{(n)}$ is given by the formula above with $\kappa_1^{(n)}$.
Thus, asset valuations respond positive to expected growth if

$$\begin{aligned} & A_x > 0 \\ \iff & \frac{1-1/\psi}{1-\kappa_1\rho} > 0 \\ \iff & 1 > 1/\psi \\ \iff & \psi > 1 \end{aligned}$$

Economically, this parameter restriction means that the substitution effect dominates the wealth effect. Asset valuations respond negatively to consumption volatility if

$$\begin{aligned} & A_\sigma < 0 \\ \iff & \frac{1}{2}\frac{(1-\gamma)(1-\frac{1}{\psi})}{1-\kappa_1\nu}\left[1+\varphi_e^2\Big(\frac{\kappa_1}{1-\kappa_1\rho}\Big)^2\right] < 0 \\ \iff & (1-\gamma)(1-\frac{1}{\psi}) < 0 \end{aligned}$$

Thus, if $\psi > 1 \implies \gamma > 1$. Economically, this parameter restriction means that agents are risk averse. The combination of $\psi > 1$ and $\gamma > 1$ means that agents prefer early resolution of uncertainty. Notice that $\psi > 1$ and $\gamma > 1 \implies \theta < 1$.

In general, we can express $r_{c,t+1}$ as:

$$\begin{aligned} r_{c,t+1} &= \underbrace{[-\log\kappa_1 + \mu_g - A_\sigma(\kappa_1\nu - 1)\sigma_0^2]}_{\equiv r_{0,c}} + \underbrace{[A_x(\kappa_1\rho - 1) + 1]}_{=1/\psi} x_t + \underbrace{A_\sigma(\kappa_1\nu - 1)}_{=-\frac{1}{2}(1-\gamma)(1-\frac{1}{\psi})\left[1+\varphi_e^2\left(\frac{\kappa_1}{1-\kappa_1\rho}\right)^2\right]} \sigma_t^2 \\ &\quad + \underbrace{\kappa_1 A_x\varphi_e}_{\equiv B_x}\sigma_t e_{t+1} + \underbrace{\kappa_1 A_\sigma\sigma_w}_{\equiv B_\sigma} w_{t+1} + \sigma_t\eta_{t+1} \\ &= r_{0,c} + \frac{1}{\psi}x_t - \frac{1}{2}(1-\gamma)(1-\frac{1}{\psi})\left[1+\Big(\frac{\kappa_1\varphi_2}{1-\kappa_1\rho}\Big)^2\right]\sigma_t^2 + B_c\sigma_t\eta_t + B_x\sigma_t e_{t+1} + B_\sigma w_{t+1} \end{aligned}$$

where $B_c \equiv 1$.

2. Express the stochastic discount factor in terms of the primitive state variables and parameters of the model:

$$m_{t+1} = m_0 + m_x x_t + m_\sigma \sigma_t^2 - \lambda_c \sigma_t \eta_{t+1} - \lambda_x \varphi_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1}.$$

What are the signs of market prices of risks? Under what conditions the market prices of expected growth and volatility risk are equal to zero?

<u>Solution:</u> From part 1, we know that

$$\begin{aligned}
m_{t+1} &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1} \\
&= \theta \log \delta - \frac{\theta}{\psi} [\mu_g + x_t + \sigma \eta_{t+1}] \\
&+ (\theta - 1) \left[r_{0,c} + \frac{1}{\psi} x_t - \frac{1}{2}(1-\gamma)(1 - \frac{1}{\psi}) \left[1 + \Big(\frac{\kappa_1 \varphi_2}{1 - \kappa_1 \rho} \Big)^2 \right] \sigma_t^2 + B_c \sigma_t \eta_t + B_x \sigma_t e_{t+1} + B_\sigma w_{t+1} \right] \\
&= \left[\theta \log \delta - \frac{\theta}{\psi} \mu_g - (1-\theta) r_{0,c} \right] - \frac{1}{\psi} x_t + (1-\theta)(1-\gamma)(1 - \frac{1}{\psi}) \left[1 + \Big(\frac{\kappa_1 \varphi_2}{1 - \kappa_1 \rho} \Big)^2 \right] \sigma_t^2 \\
&- ((1-\theta) B_c + \frac{\theta}{\psi}) \sigma^t \eta_{t+1} - (1-\theta) B_x \sigma_t e_{t+1} - (1-\theta) B_\sigma w_{t+1} \\
&= m_0 + m_x x_t + m_\sigma \sigma_t^2 - \lambda_c \sigma_t \eta_{t+1} - \lambda_x \varphi_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1}
\end{aligned}$$

where
$$\begin{aligned}
m_0 &\equiv \theta \log \delta - \frac{\theta}{\psi} \mu_g - (1-\theta) r_{0,c} \\
&= \theta \log \delta - (\theta - 1) \log \kappa_1 - \gamma \mu_g - m_\sigma \sigma_0^2 \\
m_x &\equiv -\frac{1}{\psi} \\
m_\sigma &\equiv \frac{1}{2}(1-\theta)(1-\gamma)(1 - \frac{1}{\psi}) \left[1 + \Big(\frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \Big)^2 \right] \\
&= (1-\theta)(1 - \kappa_1 \nu) A_\sigma \\
\lambda_c &\equiv (1-\theta) B_c + \frac{\theta}{\psi} \\
&= 1 - \theta + \frac{\theta}{\psi} \\
&= -\gamma < 0 \\
\lambda_x &\equiv (1-\theta) \kappa_1 A_x > 0 \\
\lambda_w &\equiv (1-\theta) \kappa_1 A_\sigma < 0
\end{aligned}$$

For the market price of expected growth to be equal to zero,

$$m_x = 0 \iff -\frac{1}{\psi} = 0 \iff \psi \to \infty$$

For the market price of volatility risk to be equal to zero,

$$m_\sigma = 0 \iff (1-\theta)(1-\gamma)(1 - \frac{1}{\psi}) \left[1 + \Big(\frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \Big)^2 \right] = 0 \iff \varphi_e + \rho = \frac{1}{\kappa_1}$$

3. Conjecture that the log price of an n-period zero-coupon risk-free bond satisfies

$$p_{n,t} = -B_{0,n} - B_{x,n}x_t - B_{\sigma,n}\sigma_t^2,$$

so that (monthly) yields are given by $y_{n,t} = -p_{n,t}/n$. Set up equations for $B_{x,n}$ and $B_{\sigma,n}$. How do risk-free rates respond to expected growth and volatility shocks?

<u>Solution:</u> The Euler equation must hold for the n-period zero-coupon risk-free bond in each period:

$$\begin{aligned}
1 &= E_t[M_{t+1}R_{n-1,t+1}] \\
&= E_t\left[M_{t+1}\frac{P_{n_1,t+1}}{P_{n,t}}\right] \\
\implies P_{n,t} &= E_t[M_{t+1}P_{n-1,t+1}] \\
\implies \exp(p_{n,t}) &= E_t[\exp(m_{t+1} + p_{n-1,t+1}] \\
\implies p_{n,t} &= E_t[m_{t+1} + p_{n-1,t+1}] + \frac{1}{2}Var_t[m_{t+1} + p_{n-1,t+1}]
\end{aligned}$$

because our guess implies log-normality. Using our result from part 2 and our guess:

$$\begin{aligned}
m_{t+1} + p_{n-1,t+1} &= m_0 + m_x x_t + m_\sigma \sigma_t^2 - \lambda_c \eta_{t+1} - \lambda_x \varphi_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1} \\
&- B_{0,n-1} - B_{x,n-1}x_{t+1} - B_{\sigma,n-1}\sigma_{t+1}^2 \\
&= m_0 + m_x x_t + m_\sigma \sigma_t^2 - \lambda_c \sigma_t \eta_{t+1} - \lambda_x \varphi_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1} \\
&- B_{0,n-1} - B_{x,n-1}[\rho x_t + \varphi_e \sigma_t e_{t+1}] - B_{\sigma,n-1}[\sigma_0^2 + \nu(\sigma_t^2 - \sigma_0^2) + \sigma_w w_{t+1}] \\
&= [m_0 - B_{0,n-1} - B_{\sigma,n-1}\sigma_0^2(1+\nu)] + [m_x - B_{x,n-1}\rho]x_t + [m_\sigma - B_{\sigma,n-1}\nu]\sigma_t^2 \\
&- \lambda_c \sigma_t \eta_{t+1} - [\lambda_x + B_{x,n-1}]\varphi_e \sigma_t e_{t+1} - [\lambda_w + B_{\sigma,n-1}]\sigma_w w_{t+1}
\end{aligned}$$

The expected value and variance are:

$$\begin{aligned}
E_t[m_{t+1} + p_{n-1,t+1}] &= [m_0 - B_{0,n-1} - B_{\sigma,n-1}\sigma_0^2(1+\nu)] + [m_x - B_{x,n-1}\rho]x_t + [m_\sigma - B_{\sigma,n-1}\nu]\sigma_t^2 \\
Var_t[m_{t+1} + p_{n-1,t+1}] &= \lambda_c^2\sigma_t^2 + [\lambda_x + B_{x,n-1}]^2\varphi_e^2\sigma_t^2 + [\lambda_w + B_{\sigma,n-1}]^2\sigma_w^2
\end{aligned}$$

Plugging into the expression above:

$$\begin{aligned}
p_{n,t} &= E_t[m_{t+1} + p_{n-1,t+1}] + \frac{1}{2}Var_t[m_{t+1} + p_{n-1,t+1}] \\
&= [m_0 - B_{0,n-1} - B_{\sigma,n-1}\sigma_0^2(1+\nu)] + [m_x - B_{x,n-1}\rho]x_t + [m_\sigma - B_{\sigma,n-1}\nu]\sigma_t^2 \\
&+ \frac{1}{2}\left[\lambda_c^2\sigma_t^2 + [\lambda_x + B_{x,n-1}]^2\varphi_e^2\sigma_t^2 + [\lambda_w + B_{\sigma,n-1}]^2\sigma_w^2\right]
\end{aligned}$$

Thus, we can recursively define the coefficients of interest:

$$\begin{aligned}
B_{0,n} &= -m_0 + B_{0,n-1} + B_{\sigma,n-1}\sigma_0^2(1+\nu) - \frac{1}{2}[\lambda_w + B_{\sigma,n-1}]^2\sigma_w^2 \\
B_{x,n} &= -m_x + B_{x,n-1}\rho \\
B_{\sigma,n} &= -m_\sigma + B_{\sigma,n-1} - \frac{1}{2}[\lambda_x + B_{x,n-1}]^2\varphi_e^2 - \frac{1}{2}\lambda_c^2
\end{aligned}$$

4. Define $rx_{n,t+1}$ a one-period log excess return on an n-period bond. That is, it is the excess return on buying an n-period bond now and selling it tomorrow as an $(n-1)$ period bond:

$$rx_{n,t+1} = -p_{n,t} + p_{n-1,t+1} - y_{t,1}.$$

Show that the risk-premia on bonds is time-varying and driven by stochastic volatility:

$$E_t[rx_{n,t+1}] + \frac{1}{2}Var_t[rx_{n,t+1}] = r_{n,0} + r_{n,1}\sigma_t^2.$$

What is the sign of $r_{n,1}$? Show that the real bond risk premium *decreases* in high uncertainty times. Why does it happen in the model, from an economic point of view?

<u>Solution:</u> ....

5. Conjecture that price-dividend ratio is linear in expected growth,

$$pd_t = H_0 + H_x x_t + H_\sigma \sigma_t^2,$$

and use Euler equation on a dividend-paying asset and log-linearization of the return to solve for H_0, H_x, H_σ and log-linearization coefficient $\kappa_{1,d}$ in terms of the fundamental model parameters.

<u>Solution:</u> We can follow a similar process as in part 1 but with dividends instead of the consumption asset:

$$r_{d,t+1} \approx -\log(\kappa_{d,1}) + \kappa_{d,1}\tilde{pd}_{t+1} - \tilde{pd}_t + \Delta d_{t+1}$$

where $\tilde{pd}_t \equiv pd_t - E[pd_t] = H_x x_t + H_\sigma(\sigma_t^2 - \sigma_0^2)$. Pluggin dividend dynamics into the log-linearized return:

$$\begin{aligned}
r_{d,t+1} &= -\log(\kappa_{d,1}) + \kappa_{d,1}[H_x x_{t+1} + H_\sigma(\sigma_{t+1}^2 - \sigma_0^2)] - [H_x x_t + H_\sigma(\sigma_t^2 - \sigma_0^2)] \\
&\quad + [\mu_d + \phi x_t + \pi_d \sigma_t \eta_{t+1} + \varphi_d \sigma_t u_{d,t+1}] \\
&= -\log(\kappa_{d,1}) + \kappa_{d,1}[H_x[\rho x_t + \varphi_e \sigma_t e_{t+1}] + H_\sigma(\nu(\sigma_t^2 - \sigma_0^2) + \sigma_w w_{t+1})] - [H_x x_t + H_\sigma(\sigma_t^2 - \sigma_0^2)] \\
&\quad + [\mu_d + \phi x_t + \pi_d \sigma_t \eta_{t+1} + \varphi_d \sigma_t u_{d,t+1}] \\
&= -\log(k_{d,1}) + \mu_d - H_\sigma(\kappa_{d,1}\nu - 1)\sigma_0^2 + (H_x(\kappa_{d,t}\rho - 1) + \phi)x_t + H_\sigma(\kappa_{d,1}\nu - 1)\sigma_t^2 \\
&\quad + \pi_d \sigma_t \eta_{t+1} + \varphi_d \sigma_t u_{d,t+1} + \kappa_{d,1} H_x \varphi_e \sigma_t e_{t+1} + \kappa_{d,1} H_\sigma \sigma_w w_{t+1}
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
m_{t+1} + r_{d,t+1} &= m_0 + m_x x_t + m_\sigma \sigma_t^2 - \lambda_c \sigma_t \eta_{t+1} - \lambda_x \sigma_t e_{t+1} - \lambda_w w_{t+1} \\
&- \log(k_{d,1}) + \mu_d - H_\sigma(\kappa_{d,1}\nu - 1)\sigma_0^2 + (H_x(\kappa_{d,t}\rho - 1) + \phi)x_t + H_\sigma(\kappa_{d,1}\nu - 1)\sigma_t^2 \\
&+ \pi_d \sigma_t \eta_{t+1} + \varphi_d \sigma_t u_{d,t+1} + \kappa_{d,1} H_x \varphi_e \sigma_t e_{t+1} + \kappa_{d,1} H_\sigma \sigma_w w_{t+1} \\
&= m_0 - \log(\kappa_{d,1}) + \mu_d - H_\sigma(\kappa_{d,1}\nu - 1)\sigma_0^2 \\
&+ (m_x + H_x(\kappa_{d,1}\rho - 1) + \phi)x_t + [m_\sigma + H_\sigma(\kappa_{d,1}\nu - 1)]\sigma_t^2 \\
&- [\lambda_c - \pi_d]\sigma_t \eta_{t+1} - [\lambda_x - \kappa_{d,1} H_x \varphi_e]\sigma_t e_{t+1} \\
&- [\lambda_w - \kappa_{d,1} H_\sigma \sigma_w] w_{t+1} + \varphi_d \sigma_t u_{d,t+1}
\end{aligned}
$$

The expected value and variance are

$$
\begin{aligned}
E_t[m_{t+1} + r_{d,t+1}] &= m_0 - \log(\kappa_{d,1} + \mu_d - H_\sigma(\kappa_{d,1}\nu - 1)\sigma_0^2 + (m_x + H_x(\kappa_{d,1}\rho - 1) + \phi)x_t \\
&+ [m_\sigma + H - \sigma(\kappa_1 \nu - 1)]\sigma_t^2 \\
Var_t[m_{t+1} + r_{d,t+1}] &= [\lambda_c - \pi_d]^2 \sigma_t^2 + [\lambda_x - \kappa_{d,1} H_x \varphi_e]^2 \sigma_t^2 + [\lambda_w - \kappa_{d,1} H_\sigma \sigma_w]^2 + \varphi_d^2 \sigma_t^2
\end{aligned}
$$

By the Euler equation,

$$
\begin{aligned}
& 1 = E_t[\exp(m_{t+1} + r_{d,t+1})] \\
\implies & 0 = E[m_{t+1} + r_{d,t+1}] + \frac{1}{2} Var_t[m_{t+1} + r_{d,t+1}] \\
\implies & 0 = \left[m_0 - \log(\kappa_{d,1} + \mu_d - H_\sigma(\kappa_{d,1}\nu - 1)\sigma_0^2 + (m_x + H_x(\kappa_{d,1}\rho - 1) + \phi)x_t + [m_\sigma + H - \sigma(\kappa_1 \nu - 1)]\sigma_t^2 \right] \\
& \quad + \frac{1}{2}\left[[\lambda_c - \pi_d]^2 \sigma_t^2 + [\lambda_x - \kappa_{d,1} H_x \varphi_e]^2 \sigma_t^2 + [\lambda_w - \kappa_{d,1} H_\sigma \sigma_w]^2 + \varphi_d^2 \sigma_t^2 \right] \\
\implies & 0 = [m_0 - \log(\kappa_{d,1} + \mu_d - H_\sigma(\kappa_{d,1}\nu - 1)\sigma_0^2 + \frac{1}{2}[\lambda_w - \kappa_{d,1} H_\sigma \sigma_w]^2] \\
& \quad + [m_x + H_w(\kappa_{d,1}\rho - 1) + \phi]x_t \\
& \quad + [m_\sigma + H_\sigma(\kappa_{d,1}\nu - 1) + (\lambda_c - \pi_d)^2 + (\lambda_x - \kappa_{d,1} H_x \varphi_e)^2 + \varphi_d^2]\sigma_t^2 \\
\implies & \begin{cases} 0 & = m_0 - \log(\kappa_{d,1} + \mu_d - H_\sigma(\kappa_{d,1}\nu - 1)\sigma_0^2 + \frac{1}{2}[\lambda_w - \kappa_{d,1} H_\sigma \sigma_w]^2 \\ 0 & = m_x + H_w(\kappa_{d,1}\rho - 1) + \phi \\ 0 & = m_\sigma + H_\sigma(\kappa_{d,1}\nu - 1) + \frac{1}{2}(\lambda_c - \pi_d)^2 + \frac{1}{2}(\lambda_x - \kappa_{d,1} H_x \varphi_e)^2 + \frac{1}{2}\varphi_d^2 \end{cases}
\end{aligned}
$$

Three equations in three unknowns $(\kappa_{d,1}, H_x, H_\sigma)$. We can express H_x and H_σ in terms of $\kappa_{d,1}$ and solve for $\kappa_{d,1}$ using a fixed point algorithm:

$$
\begin{aligned}
H_x &= \frac{m_x + \phi}{1 - \kappa_1 \rho} \\
H_\sigma &= \frac{m_\sigma + \frac{1}{2}[(\lambda_c - \pi_d)^2 + (\lambda_x - \kappa_{d,1} H_x \varphi_e)^2 + \varphi_d^2]}{1 - \kappa_{d,1}\nu}
\end{aligned}
$$

6. Show that the risk-premia on the stock market is time-varying and driven by stochastic volatility:

$$E_t[r_{d,t+1}] + \frac{1}{2}Var_t[r_{d,t+1}] = r_{d,0} + r_{d,1}\sigma_t^2$$

What is the sign of $r_{d,1}$? Under what conditions do market premia go up in high volatility times?

Solution: ....

Calibration and Asset-Pricing Implications:

1. The Matlab code on Canvas contains a calibration and solution of the long-run risk model. The calibration is very similar, though it might not be exactly identical to Bansal Yaron 2004 and BKY, 2009.[1] It computes equilibrium solutions to the price-consumption ratio, SDF, and one-period risk-free rate. After you run this code it should display unconditional log consumption return and average one-period risk-free rate.

   Solution: ...

2. What is the average conditional volatility of the stochastic discount factor? Decompose the volatility into components related to short-run, long-run, and volatility news.

   Solution: ...

3. Extend the code to solve for the term-structure of real rates, and the price-dividend ratio on the dividend-paying asset.

   Solution: ...

4. What are the model implications for the levels of the risk-free rates from 1 month to 5 years in maturity?

   Solution: ...

5. What are the average log price-consumption and log price-dividend ratios? How does the market pd ratio compare to the data?

   Solution: ...

6. What is the average equity premium on consumption asset? on dividend-paying asset? Decompose the premia into components related to short-run, long-run and volatility risks.

   Solution: ...

7. What is the volatility of equity returns and risk-free rates in the model? How do they compare to the data?

   Solution: ...

8. Comment on the ability of the model to solve equity premium, risk-free rate, and return volatility puzzles.

   Solution: ...

[1] The model produces output on monthly frequency, which is then annualized for an easier comparison to the data. For mean equity returns and risk premia, it's acceptable to multiply monthly numbers by 12 to convert to per annum numbers. Standard deviations of equity returns are multiplied by the square root of 12 (equity returns are approximately i.i.d., so variance of the sum of 12 returns is approximately 12 times the variance of a single return). The interest rates are typically converted to annual yields first: e.g., multiply 1-month interest rate by 12 to get an annualized yield on an 1-month bond. Then we can report the mean and standard deviation of the yield directly. For price-dividend ratios, $PD_{annual} = P/D_{annual}$, where annual D is the sum over 12 past dividends. Approximately, $D_{annual} = 12D_{monthly}$, so that $PD_{annual} = P/D_{annual} = PD_{monthly}/12$. So, to convert monthly PD (levels) to annual PD, divide it by 12 - in logs, subtract $log(12)$ from average log monthly pd to get to annual log pd. Annualization of consumption growth is more delicate. It is NOT the case that log annual consumption growth is the sum of log monthly consumption growth rates (why?). A proper way to handle it is in simulations, get monthly consumption growth rates, convert them to monthly (un-log) levels of consumption, sum up monthly consumption over the year to get annual consumption, and then compute log annual consumption growth and its statistics directly in simulations.