ECON 703 - PS 3

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(1) Let (X, d) be a nonempty complete metric space. Suppose an operator $T : X \to X$ satisfies $d(T(x), T(y)) < d(x, y)$ for all $x \neq y, x, y \in X$. Prove or disprove that T has a fixed point. Compare with the Contraction Mapping Theorem.

I prove that T has a fixed point.

Proof: Define $\beta_{xy} := \frac{d(T(x),T(y))}{d(x,y)}$ and $\beta := \max \beta_{xy}$ for all $x \neq y, x, y \in X$. Notice that $d(T(x), T(y)) < d(x, y) \implies \beta_{xy} < 1$. Thus, $d(T(x), T(y)) = \beta_{xy} d(x, y) \leq \beta d(x, y)$. By the convergence mapping theorem, T has a fixed point.

(3) Prove that the function $f(x) = \cos^2(x)e^{5-x-x^2}$ has a maximum on $\mathbb{R}$.

Proof: Define $g(x) = \cos^2(x)$ and $h(x) = e^{5-x-x^2}$. Notice that, since $-1 \leq \cos(x) \leq 1$, $0 \leq g(x) \leq 1$. Thus, $f(x) \leq h(x)$. $h(x) < 1$ on $(-\infty, -\sqrt{21}/2 - 1/2) \cup (\sqrt{21}/2 - 1/2, \infty)$, so $\max h(x)$ is at $x \in A = [\sqrt{21}/2 - 1/2, \sqrt{21}/2 - 1/2]$. Similarly, $\max f(x)$ is at an $y \in A$. A is a closed subset of $\mathbb{R}$. Since g and h are continuous, f is continuous. Since A is closed and f is continous, $f(x)$ is bounded on A. Thus, A is compact. By the extreme value theorem, f reaches its maximum on A and therefore on $\mathbb{R}$. $\square$

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