ECON 714A - Problem Set 3

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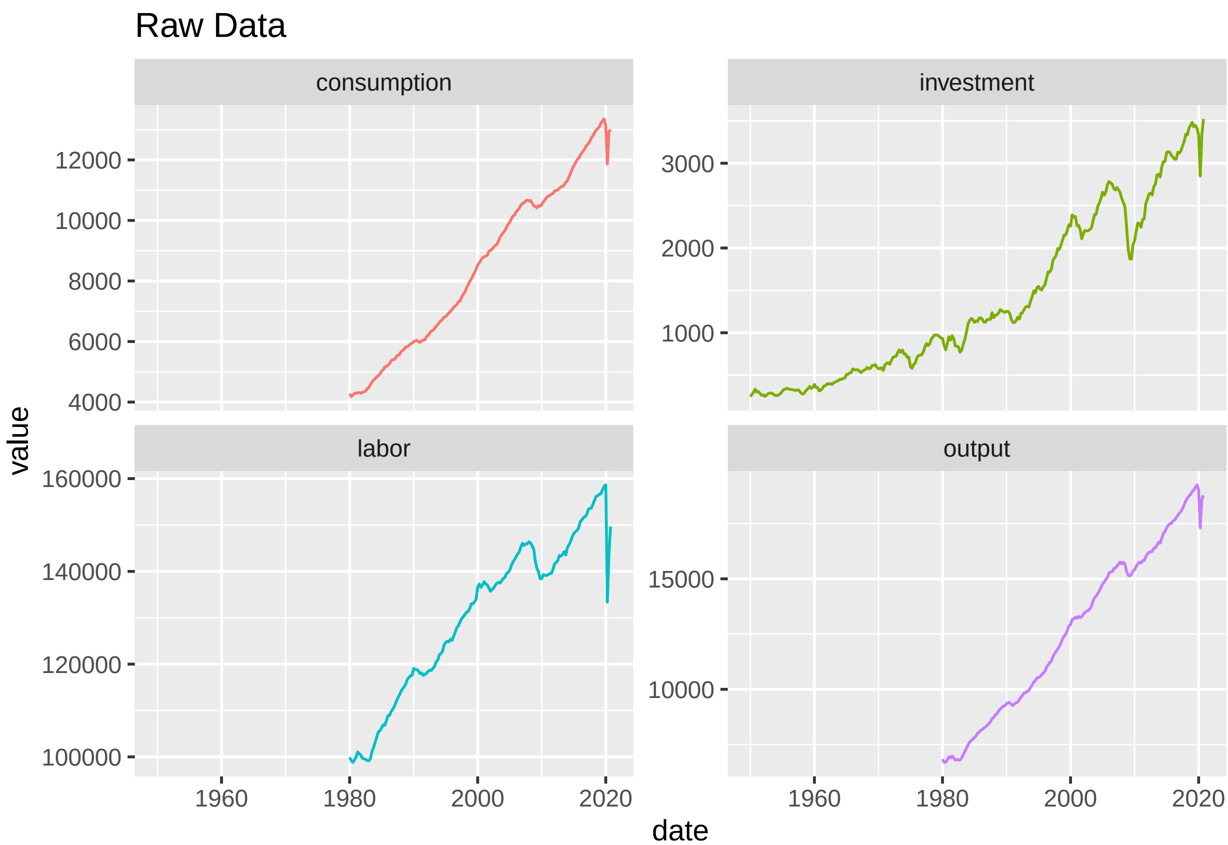
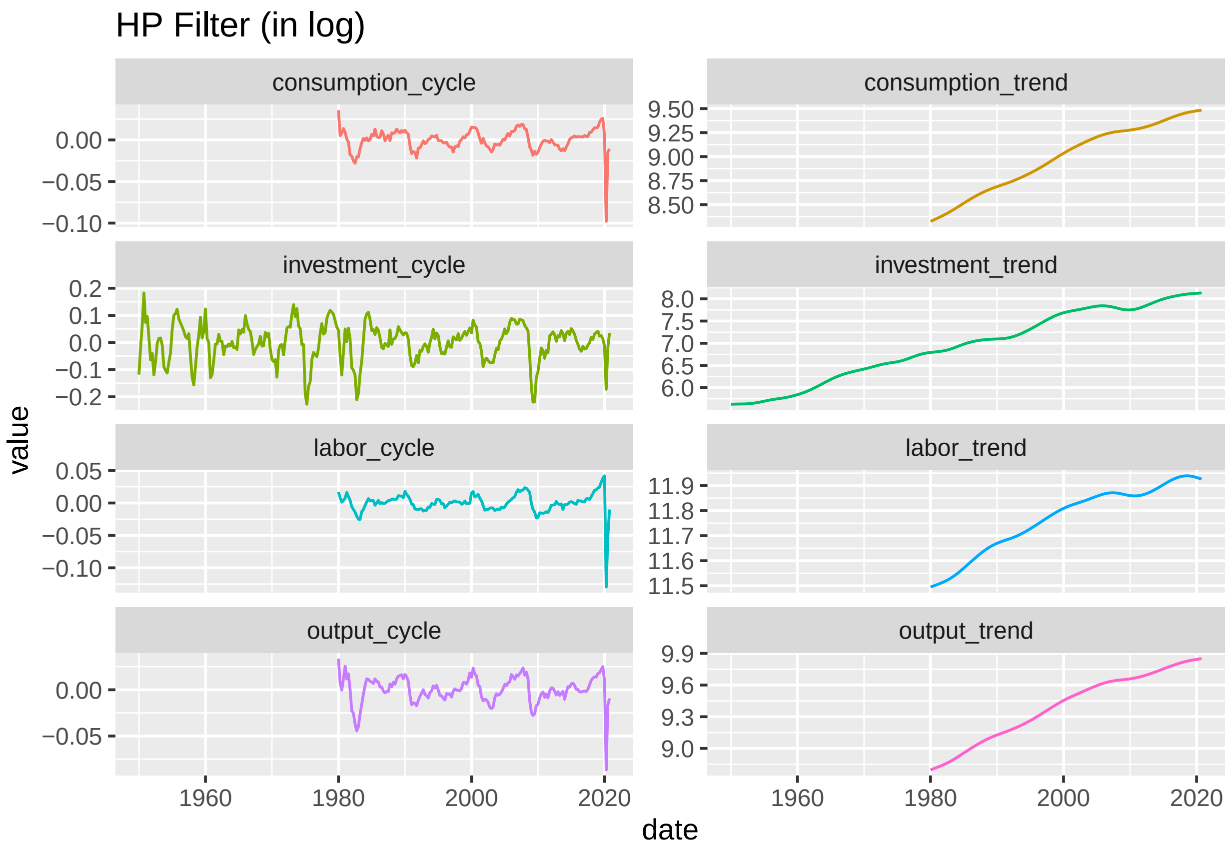
This problem asks you to update the CKM (2007) wedge accounting using more recent data.You are encouraged to use Matlab for the computations. Consider a standard RBC model with the CRRA preferences $E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$, $U(C, L) = \frac{C^{1-\sigma}-1}{1-\sigma} - \frac{L^{1+\phi}}{1+\phi}$, a Cobb-Douglas production function $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, a standard capital law of motion $K_{t+1} = (1-\delta)K_t + I_t$, and four wedges $\tau_t = \{a_t, g_t, \tau_{Lt}, \tau_{It}\}$. Each wedge τ_{it} follows an AR(1) process $\tau_{it} = \rho_i \tau_{it-1} + \varepsilon_{it}$ with innovations ε_{it} potentially correlated across i. One period corresponds to a quarter.

1. Download quarterly data for real seasonally adjusted consumption, employment, and output in the U.S. from 1980–2020 from FRED database. The series for capital are not readily available, but can be constructed using the "perpetual inventory method". To this end, download the series for (real seasonally adjusted) investment from 1950-2020.



*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

2. Convert all variables into logs and de-trend using the Hodrick-Prescott filter.

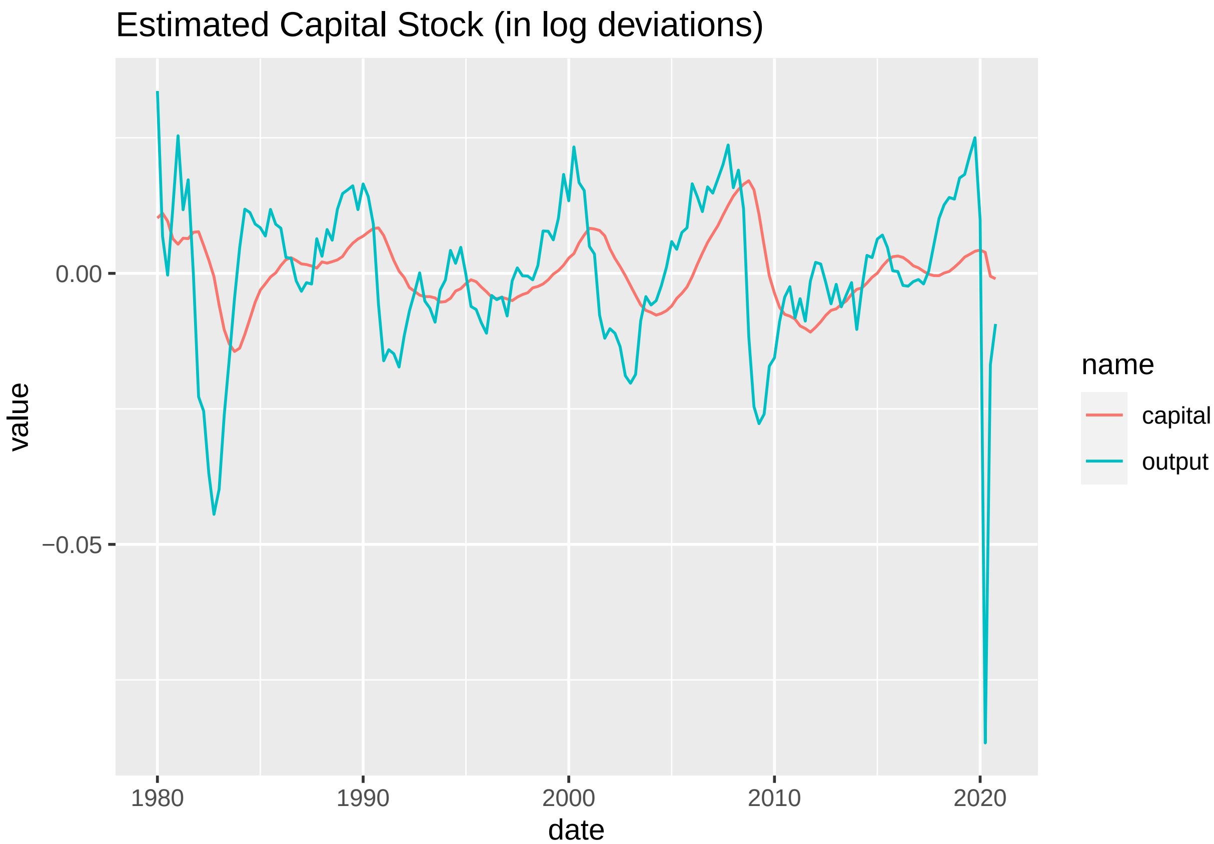


3. Assume that capital was at the steady-state level in 1950 and the rate of depreciation is $\delta = 0.025$ and use the linearized capital law of motion and the series for investment to estimate the capital stock (in log deviations) in 1980-2020. Justify this approach.

Log-linearizing the capital law of motion,

$$
\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
\implies (1+k_{t+1})\bar{K} &= (1-\delta)(1+k_t)\bar{K} + \bar{I}(1+i_t) \\
\implies \bar{K} + k_{t+1}\bar{K} &= (1-\delta)\bar{K} + (1-\delta)\bar{K}k_t + \bar{I} + \bar{I}i_t \\
\implies k_{t+1} &= (1-\delta)k_t + \frac{\bar{I}}{\bar{K}}i_t \\
\implies k_{t+1} &= (1-\delta)k_t + \delta i_t
\end{aligned}
$$

If we assume that we're in the steady state in 1950, then $k_t = 0$. Thus, we can interate forward using i_t from the data over the next thirty years until 1980.



4. Linearize the equilibrium conditions. Assuming $\beta = 0.99$, $\alpha = 1/3$, $\sigma = 1$, $\phi = 1$ and the steady-state share of government spendings in GDP equal 1/3, estimate a_t, g_t and τ_{Lt} for 1980-2020. Run the OLS regression for each of these wedges to compute their persistence parameters ρ_i.

The equilibrium conditions from the lecture 3 notes are:

$$\begin{aligned}
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
A_t F(K_t, L_t) &= C_t + I_t + G_t \\
-\frac{U_{L_t}}{U_{C_t}} &= (1 - \tau_{L_t}) A_t F_{L_t} \\
U_{C_t}(1 + \tau_{I_t}) &= \beta E_t U_{C_{t+1}} (A_{t+1} F_{K_{t+1}} + (1 - \delta)(1 + \tau_{I_{t+1}})) \\
K_{t+1} &= (1 - \delta) K_t + I_t
\end{aligned}$$

The functional forms from the prompt:

$$\begin{aligned}
U(C_t, L_t) &= \ln(C_t) - \frac{L_t^2}{2} \\
\implies U_{C_t} &= \frac{1}{C_t} \\
\implies U_{L_t} &= -L_t \\
F(K_t, L_t) &= K_t^{\alpha} L_t^{1-\alpha} \\
\implies F_{L_t} &= (1 - \alpha) K_t^{\alpha} L_t^{-\alpha} \\
\implies F_{K_t} &= \alpha K_t^{\alpha-1} L_t^{1-\alpha}
\end{aligned}$$

Thus, the equilibrium conditions become:

$$
\begin{aligned}
A_t K_t^{\alpha} L_t^{1-\alpha} &= C_t + I_t + G_t \\
C_t L_t &= (1-\tau_{L_t}) A_t (1-\alpha) K_t^{\alpha} L_t^{-\alpha} \\
\frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t \frac{1}{C_{t+1}} (A_{t+1}\alpha K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}}))
\end{aligned}
$$

In the steady state, $K_t = K_{t+1} = \bar{K}$, $C_t = C_{t+1} = \bar{C}$, $L_t = L_{t+1} = \bar{L}$, $I_t = I_{t+1} = \delta\bar{K}$, $G_t = G_{t+1} = \bar{G} = \bar{Y}/3$, $A_t = A_{t+1} = \bar{A} = 1$, and $\bar{\tau_L} = \bar{\tau_I} = 0$.

$$
\begin{aligned}
(2/3)\bar{K}^{\alpha}\bar{L}^{1-\alpha} &= \bar{C} + \delta\bar{K} \\
\bar{C} &= (1-\alpha)\bar{K}^{\alpha}\bar{L}^{-1-\alpha} \\
1 &= \beta(\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha} + (1-\delta))
\end{aligned}
$$

We can solve for the labor steady state:

$$
\begin{aligned}
(2/3)\bar{K}^{\alpha}\bar{L}^{1-\alpha} - \delta\bar{K} &= (1-\alpha)\bar{K}^{\alpha}\bar{L}^{-1-\alpha} \\
\implies \delta &= (2/3)\bar{K}^{\alpha-1}\bar{L}^{1-\alpha} - (1-\alpha)\bar{K}^{\alpha-1}\bar{L}^{-1-\alpha} \\
\implies \bar{K} &= (\frac{\delta}{(2/3)\bar{L}^{1-\alpha} - (1-\alpha)\bar{L}^{-1-\alpha}})^{1/(\alpha-1)} \\
\implies 1 &= \beta(\alpha\frac{\delta}{(2/3)L^{1-\alpha} - (1-\alpha)\bar{L}^{-1-\alpha}}\bar{L}^{1-\alpha} + (1-\delta) \\
\implies 1/\beta - 1 + \delta &= \frac{\alpha\delta}{(2/3) - (1-\alpha)\bar{L}^{-2}} \\
\implies \bar{L} &= \sqrt{\frac{(1-\alpha)}{(2/3) - \frac{\alpha\delta}{(1/\beta - 1 + \delta)}}}
\end{aligned}
$$

The steady state values of $\bar{Y}, \bar{K}, \bar{C}$, and $\bar{G}$ are implied by the equations above:

variable	value
L_bar	1.246
K_bar	36.470
I_bar	0.912
Y_bar	3.840
C_bar	1.649
G_bar	1.280

We now log-linearize around the steady state. The log-linearized Euler equation is:

$$
\begin{aligned}
\frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t \frac{1}{C_{t+1}}(A_{t+1}\alpha K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}})) \\
X_{t+1} &:= A_{t+1}\alpha K_{t+1}^{\alpha-1}L_{t+1}^{1-\alpha} + (1-\delta)(1+\tau_{I_{t+1}}) \\
\bar{X} &= \bar{A}\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha} + 1 - \delta \\
&= \frac{1}{\beta} \\
\implies x_{t+1} &= \frac{\bar{A}\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}}{\bar{X}}(a_{t+1} + (\alpha-1)k_{t+1} + (1-\alpha)l_{t+1}) + \frac{(1-\delta)}{\bar{X}}\hat{\tau}_{I_{t+1}} \\
&= \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + \beta(1-\delta)\hat{\tau}_{I_{t+1}} \\
\implies \frac{1}{C_t}(1+\tau_{I_t}) &= \beta E_t \frac{1}{C_{t+1}} X_{t+1} \\
\implies \hat{\tau}_{I_t} - c_t &= E_t[x_{t+1}] - E_t[c_{t+1}] \\
\implies \hat{\tau}_{I_t} - c_t &= E_t[\beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + \beta(1-\delta)\hat{\tau}_{I_{t+1}}] - E_t[c_{t+1}] \\
\implies E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta E_t[\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}(a_{t+1} + (1-\alpha)(l_{t+1}-k_{t+1})) + (1-\delta)\hat{\tau}_{I_{t+1}}]
\end{aligned}
$$

The log-linearized law of motion of capital is:

$$
\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
\implies k_{t+1} &= \frac{(1-\delta)\bar{K}}{(1-\delta)\bar{K} + \bar{I}}k_t + \frac{\bar{I}}{(1-\delta)\bar{K} + \bar{I}}i_t \\
&= \frac{(1-\delta)\bar{K}}{\bar{K}}k_t + \frac{\bar{I}}{\bar{K}}i_t \\
&= (1-\delta)k_t + \delta i_t
\end{aligned}
$$

The log-linearized consumption-labor equation is:

$$
\begin{aligned}
C_t L_t^{1+\alpha} &= (1-\tau_{L_t})A_t(1-\alpha)K_t^{\alpha} \\
\implies c_t + (1+\alpha)l_t &= -\hat{\tau}_{L_t} + a_t + \alpha k_t
\end{aligned}
$$

The log-linearized output equations are:

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
\implies y_t &= a_t + \alpha k_t + (1-\alpha)l_t \\
Y_t &= C_t + I_t + G_t \\
\implies \bar{Y}(1+y_t) &= \bar{C}(1+c_t) + \bar{I}(1+i_t) + \bar{G}(1+g_t) \\
\implies \bar{Y}y_t &= \bar{C}c_t + \bar{I}i_t + \bar{G}g_t
\end{aligned}
$$

The efficiency wedge can be derived from the first log-linearized output equation:

$$a_t = y_t - \alpha k_t - (1-\alpha)l_t$$

The government consumption wedge can be derived from the second log-linearized output equation:

$$g_t = \frac{\bar{Y}}{\bar{G}} y_t - \frac{\bar{C}}{\bar{G}} c_t - \frac{\bar{I}}{\bar{G}} i_t$$

The labor wedge can be derived from the log-linearized consumption-labor equation:

$$\hat{\tau_{L_t}} = a_t + \alpha k_t - c_t - (1+\alpha)l_t$$

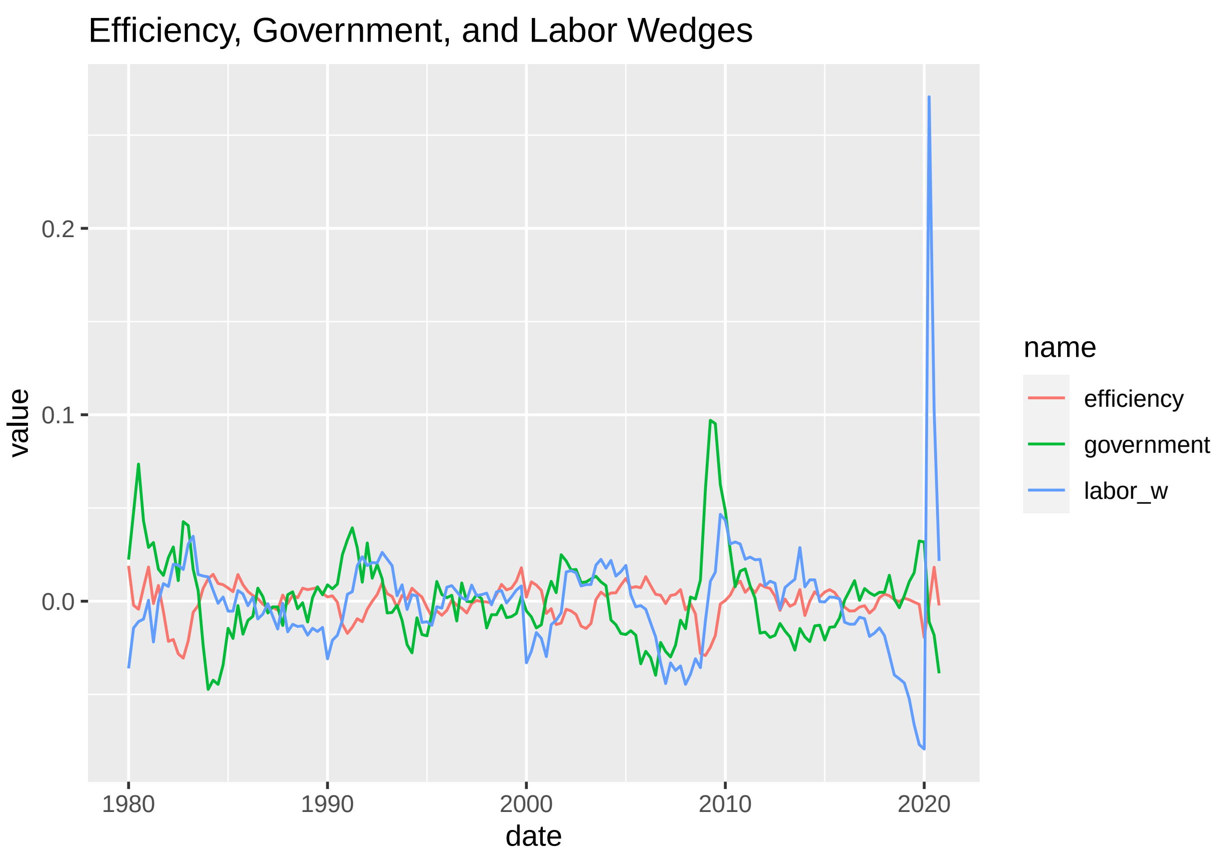


Table 2: Efficiency, Government, and Labor Wedge Persistance (OLS)

	Dependent variable:		
	efficiency	government	labor_w
	(1)	(2)	(3)
lag(efficiency)	0.711***		
	(0.054)		
lag(government)		0.856***	
		(0.042)	
lag(labor_w)			0.451***
			(0.070)
Observations	163	163	163
Adjusted R^2	0.517	0.722	0.200
Note:	*p<0.1; **p<0.05; ***p<0.01		

5. Write down a code that implements the Blanchard-Kahn method to solve the model. Use the values of parameters, including ρ_a, ρ_g, and ρ_{τ_L}, obtained above, and assume $\rho_{\tau_I} = 0$ for now.

The log-linearized consumption-labor equation implies

$$l_t = \frac{\alpha}{1+\alpha}k_t + \frac{-1}{1+\alpha}c_t + \frac{1}{1+\alpha}a_t + \frac{-1}{1+\alpha}\hat{\tau}_{L_t}$$

The log-linearized output equations implies

$$\begin{aligned}
i_t &= (1-\alpha)\frac{\bar{Y}}{\bar{I}}l_t + \alpha\frac{\bar{Y}}{\bar{I}}k_t + \frac{-\bar{C}}{\bar{I}}c_t + \frac{\bar{Y}}{\bar{I}}a_t + \frac{-\bar{G}}{\bar{I}}g_t \\
&= (1-\alpha)\frac{\bar{Y}}{\bar{I}}\left[\frac{\alpha}{1+\alpha}k_t + \frac{-1}{1+\alpha}c_t + \frac{1}{1+\alpha}a_t + \frac{-1}{1+\alpha}\hat{\tau}_{L_t}\right] + \alpha\frac{\bar{Y}}{\bar{I}}k_t + \frac{-\bar{C}}{\bar{I}}c_t + \frac{\bar{Y}}{\bar{I}}a_t + \frac{-\bar{G}}{\bar{I}}g_t \\
&= \frac{2\bar{Y}\alpha}{\bar{I}(1+\alpha)}k_t + \frac{-\bar{Y}(1-\alpha)-\bar{C}(1+\alpha)}{\bar{I}(1+\alpha)}c_t + \frac{2\bar{Y}}{(1+\alpha)\bar{I}}a_t + \frac{-\bar{Y}(1-\alpha)}{\bar{I}(1+\alpha)}\hat{\tau}_{L_t} + \frac{-\bar{G}}{\bar{I}}g_t
\end{aligned}$$

The law of motion of capital implies

$$\begin{aligned}
k_{t+1} &= (1-\delta)k_t + \delta i_t \\
&= (1-\delta)k_t + \delta\left[\frac{2\bar{Y}\alpha}{\bar{I}(1+\alpha)}k_t + \frac{-\bar{Y}(1-\alpha)-\bar{C}(1+\alpha)}{\bar{I}(1+\alpha)}c_t + \frac{2\bar{Y}}{(1+\alpha)\bar{I}}a_t + \frac{-\bar{Y}(1-\alpha)}{\bar{I}(1+\alpha)}\hat{\tau}_{L_t} + \frac{-\bar{G}}{\bar{I}}g_t\right] \\
&= \left[1-\delta+\frac{2\alpha\delta\bar{Y}}{(1+\alpha)\bar{I}}\right]k_t + \frac{-\delta\bar{Y}(1-\alpha)-\bar{C}(1+\alpha)}{\bar{I}(1+\alpha)}c_t + \frac{2\delta\bar{Y}}{(1+\alpha)\bar{I}}a_t + \frac{-\delta\bar{Y}(1-\alpha)}{\bar{I}(1+\alpha)}\hat{\tau}_{L_t} + \frac{-\delta\bar{G}}{\bar{I}}g_t
\end{aligned}$$

Based on the AR(1) process for a_t, g_t, $\hat{\tau}_{L_t}$, and $\hat{\tau}_{I_t}$:

$$\begin{aligned}
E_t[a_{t+1}] &= \rho_a a_t \\
E_t[g_{t+1}] &= \rho_g g_t \\
E_t[\hat{\tau}_{L_{t+1}}] &= \rho_{\tau_L} \hat{\tau}_{L_t} \\
E_t[\hat{\tau}_{I_{t+1}}] &= \rho_{\tau_I} \hat{\tau}_{I_t}
\end{aligned}$$

The log-linearized Euler equation implies:

$$\begin{aligned}
E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta E_t\Bigg[\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\Bigg(a_{t+1} + (1-\alpha)\Bigg[\frac{\alpha}{1+\alpha}k_{t+1} + \frac{-1}{1+\alpha}c_{t+1} \\
&\quad + \frac{1}{1+\alpha}a_{t+1} + \frac{-1}{1+\alpha}\hat{\tau}_{L_{t+1}}\Bigg] - (1-\alpha)k_{t+1}\Bigg) + (1-\delta)\hat{\tau}_{I_{t+1}}\Bigg] \\
\implies E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta E_t\Bigg[\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\Bigg(\frac{\alpha-1}{1+\alpha}k_{t+1} + \frac{\alpha-1}{1+\alpha}c_{t+1} \\
&\quad + \frac{2}{1+\alpha}a_{t+1} + \frac{\alpha-1}{1+\alpha}\hat{\tau}_{L_{t+1}}\Bigg)\Bigg] + (1-\delta)E_t[\hat{\tau}_{I_{t+1}}] \\
\implies E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\Bigg(\frac{\alpha-1}{1+\alpha}E_t[k_{t+1}] + \frac{\alpha-1}{1+\alpha}E_t[c_{t+1}] \\
&\quad + \frac{2}{1+\alpha}E_t[a_{t+1}] + \frac{\alpha-1}{1+\alpha}E_t[\hat{\tau}_{L_{t+1}}]\Bigg) + (1-\delta)\rho_{\tau_I}\hat{\tau}_{I_t} \\
\implies E_t[c_{t+1}] - c_t + \hat{\tau}_{I_t} &= \beta\alpha \bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\Bigg(\frac{\alpha-1}{1+\alpha}E_t[k_{t+1}] + \frac{\alpha-1}{1+\alpha}E_t[c_{t+1}] \\
&\quad + \frac{2}{1+\alpha}\rho_a a_t + \frac{\alpha-1}{1+\alpha}\rho_{\tau_L}\hat{\tau}_{L_t}\Bigg) + (1-\delta)\rho_{\tau_I}\hat{\tau}_{I_t} \\
\implies \Bigg[1 - \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha}\Bigg]E_t[c_{t+1}] &- \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha}E_t[k_{t+1}] \\
&= c_t + \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{2}{1+\alpha}\rho_a a_t + \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha}\rho_{\tau_L}\hat{\tau}_{L_t} \\
&\quad + ((1-\delta)\rho_{\tau_I} - 1)\hat{\tau}_{I_t}
\end{aligned}$$

Thus, we can represent the model with a matrix:

$$\begin{aligned}
x_t &:= \begin{pmatrix} k_t \\ c_t \end{pmatrix} \\
z_t &:= \begin{pmatrix} a_t \\ g_t \\ \hat{\tau}_{L_t} \\ \hat{\tau}_{I_t} \end{pmatrix} \\
\tilde{C}E[x_{t+1}] &= \tilde{A}x_t + \tilde{B}z_t \\
\tilde{C} &:= \begin{pmatrix} 1 & 0 \\ -\beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha} & 1-\beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha} \end{pmatrix} \\
\tilde{A} &:= \begin{pmatrix} 1-\delta+\frac{2\alpha\delta\bar{Y}}{(1+\alpha)\bar{I}} & \frac{-\delta\bar{Y}(1-\alpha)-\bar{C}(1+\alpha)}{\bar{I}(1+\alpha)} \\ 0 & 1 \end{pmatrix} \\
\tilde{B} &:= \begin{pmatrix} \frac{2\delta\bar{Y}}{(1+\alpha)\bar{I}} & \frac{-\delta\bar{G}}{\bar{I}} & \frac{-\delta\bar{Y}(1-\alpha)}{\bar{I}(1+\alpha)} & 0 \\ \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{2}{1+\alpha}\rho_a & 0 & \beta\alpha\bar{K}^{\alpha-1}\bar{L}^{1-\alpha}\frac{\alpha-1}{1+\alpha}\rho_{\tau_L} & (1-\delta)\rho_{\tau_I}-1 \end{pmatrix} \\
A &:= \tilde{C}^{-1}\tilde{A} \\
B &:= \tilde{C}^{-1}\tilde{B} \\
A &:= Q^{-1}\Lambda Q \\
E_t[x_{t+1}] &= Q\Lambda Q^{-1}x_t + Bz_t \\
\implies Q^{-1}E_t[x_{t+1}] &= \Lambda Q^{-1}x_t + Q^{-1}Bz_t \\
\implies E_t[y_{t+1}] &= \Lambda y_t + Dz_t \\
D &:= Q^{-1}B
\end{aligned}$$

Below are these matrices estimated with our parameters. Notice that $|\lambda_1| > 1$ and $|\lambda_2| < 1$.

Table 3: A

1.028	-1.861
-0.018	1.051

Table 4: B

0.192	-0.035	-0.053	0.000
0.034	0.001	-0.007	-1.018

Table 5: Lambda

1.223	0.000
0.000	0.855

Table 6: Q

0.995	-0.996
-0.105	-0.092

Table 7: Q-Inverse

0.471	-5.078
-0.533	-5.072

With $|\lambda_1| > 1$, we can iterate y_{1t} forward:

$$\begin{aligned}
\implies E_t[y_{1,t+1}]] &= \lambda_1 y_{1,t} + D_1 z_t \\
\implies y_{1,t} &= \lambda_1^{-1} E_t[y_{1,t+1}] + \lambda_1^{-1} D_1 z_t \\
&= \lim_{j\to\infty} \lambda_1^{-j} y_{1,t+j} + \lambda_1^{-1} D_1 \sum_{j=0}^{\infty} \lambda_1^{-j} E_t[z_{t+j}] \\
&= \lambda_1^{-1} D_1 \sum_{j=0}^{\infty} \lambda_1^{-j} \rho^j z_t \\
&= \lambda_1^{-1} D_1 (I_4 - \lambda_1^{-1}\rho)^{-1} z_t \\
&= \Theta z_t \\
\rho &:= diag(\rho_a, \rho_g, \rho_{\tau_L}, \rho_{\tau_I}) \\
\Theta &:= \lambda_1^{-1} D_1 (I_4 - \lambda_1^{-1}\rho)^{-1}
\end{aligned}$$

Table 8: Theta

-0.164	-0.054	0.014	4.224

Thus, we can solve for $\hat{\tau_{I_t}}$:

$$\begin{aligned}
\implies y_{1,t} &= \Theta z_t \\
\implies q_{11} k_t + q_{12} c_t &= \Theta_1 a_t + \Theta_2 g_t + \Theta_3 \hat{\tau_{L_t}} + \Theta_4 \hat{\tau_{I_t}} \\
\hat{\tau_{I_t}} &= \frac{q_{11} k_t + q_{12} c_t - \Theta_1 a_t - \Theta_2 g_t - \Theta_3 \hat{\tau_{L_t}}}{\Theta_4} \\
Q^{-1} &:= \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}
\end{aligned}$$

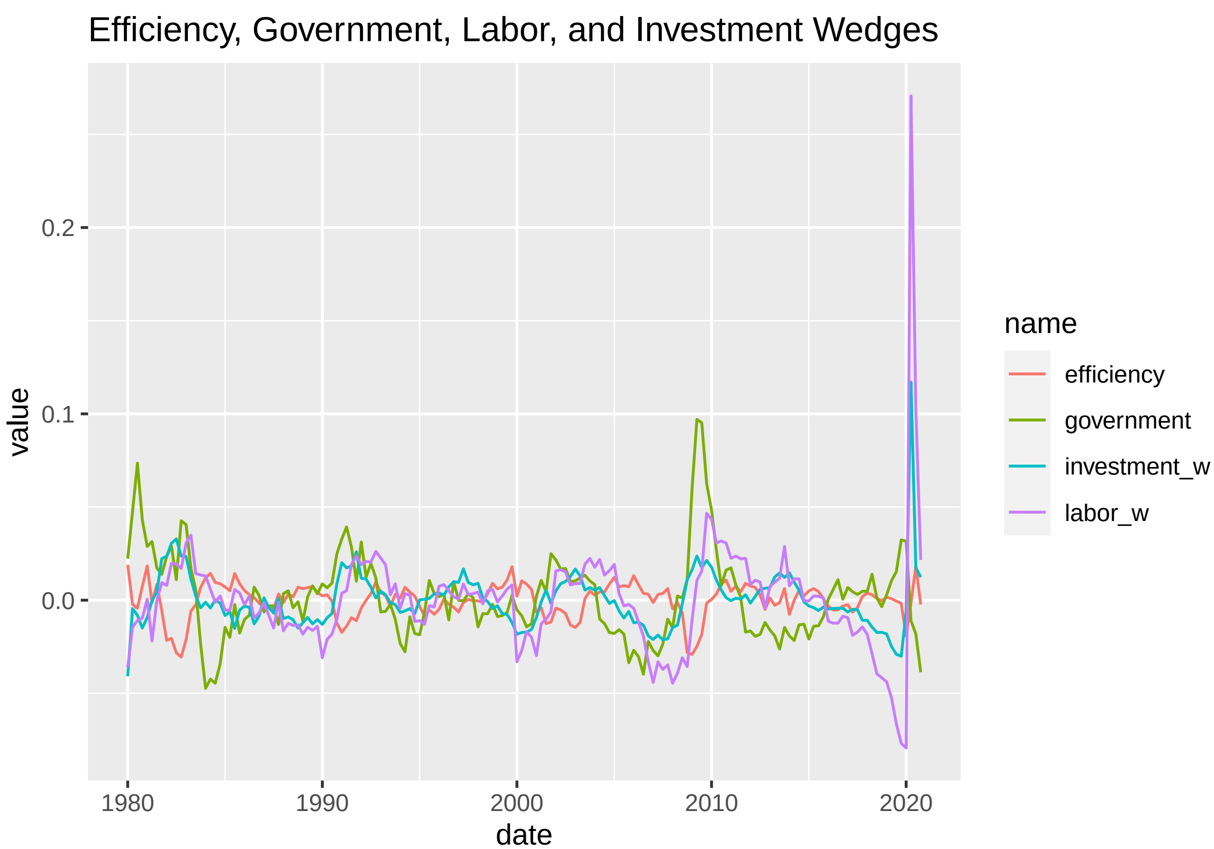


Table 9: Investment Wedge Persistance (OLS)

	Dependent variable:
	investment_w
lag(investment_w)	0.572***
	(0.062)
Observations	163
Adjusted R^2	0.337
Note:	*p<0.1; **p<0.05; ***p<0.01

6. Solve the fixed-point problem to estimate τ_{I_t}: conjecture a value of ρ_{τ_I}, solve numerically the model for consumption as a function of capital and wedges, use the estimated values of consumption and other wedges to infer the series of τ_{I_t}, run AR(1) regression and estimate ρ_{τ_I}, iterate until convergence.

Solving for the fixed-point takes the following number of iterations and the value of ρ_{τ_I} does not change much:

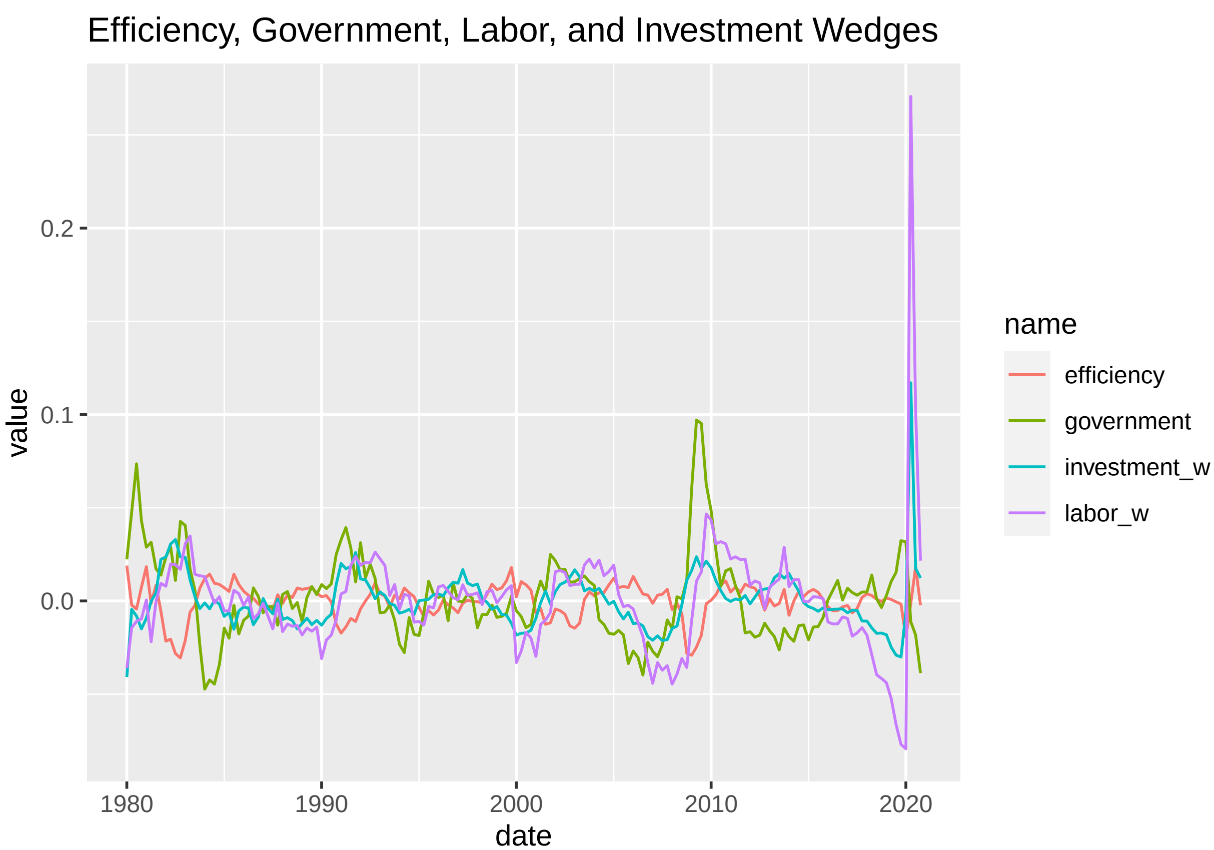
[1] 17

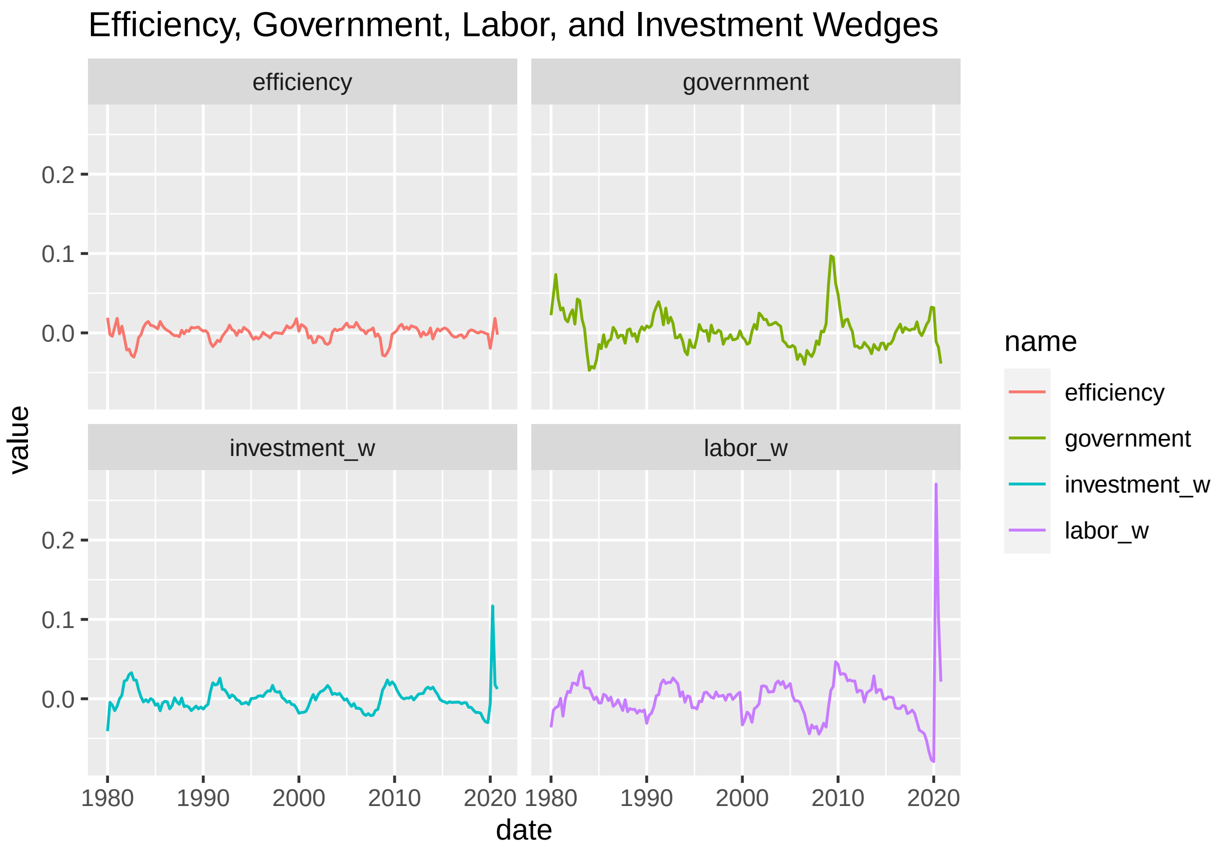
Table 10: Investment Wedge Persistance (OLS)

	Dependent variable:
	investment_w
lag(investment_w)	0.572***
	(0.062)
Observations	163
Adjusted R^2	0.337
Note:	*p<0.1; **p<0.05; ***p<0.01

7. Draw one large figure that shows dynamics of all wedges during the period.

Two figures are below. The first has all wedges on the same panel and the second separates them out into different panels.



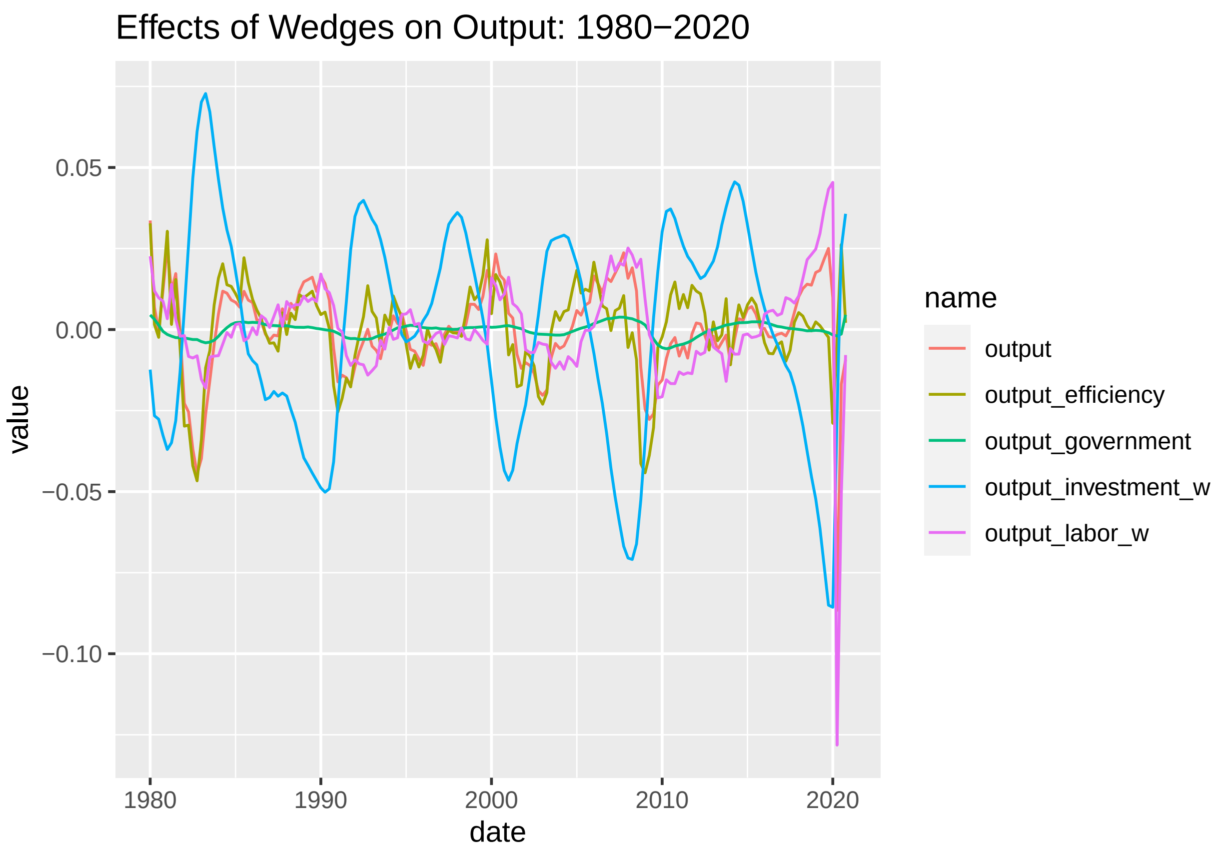

Efficiency, Government, Labor, and Investment Wedges
efficiency
government
investment_w
labor_w
value
0.2
0.1
0.0
1980
1990
2000
2010
2020
date
name
efficiency
government
investment_w
labor_w

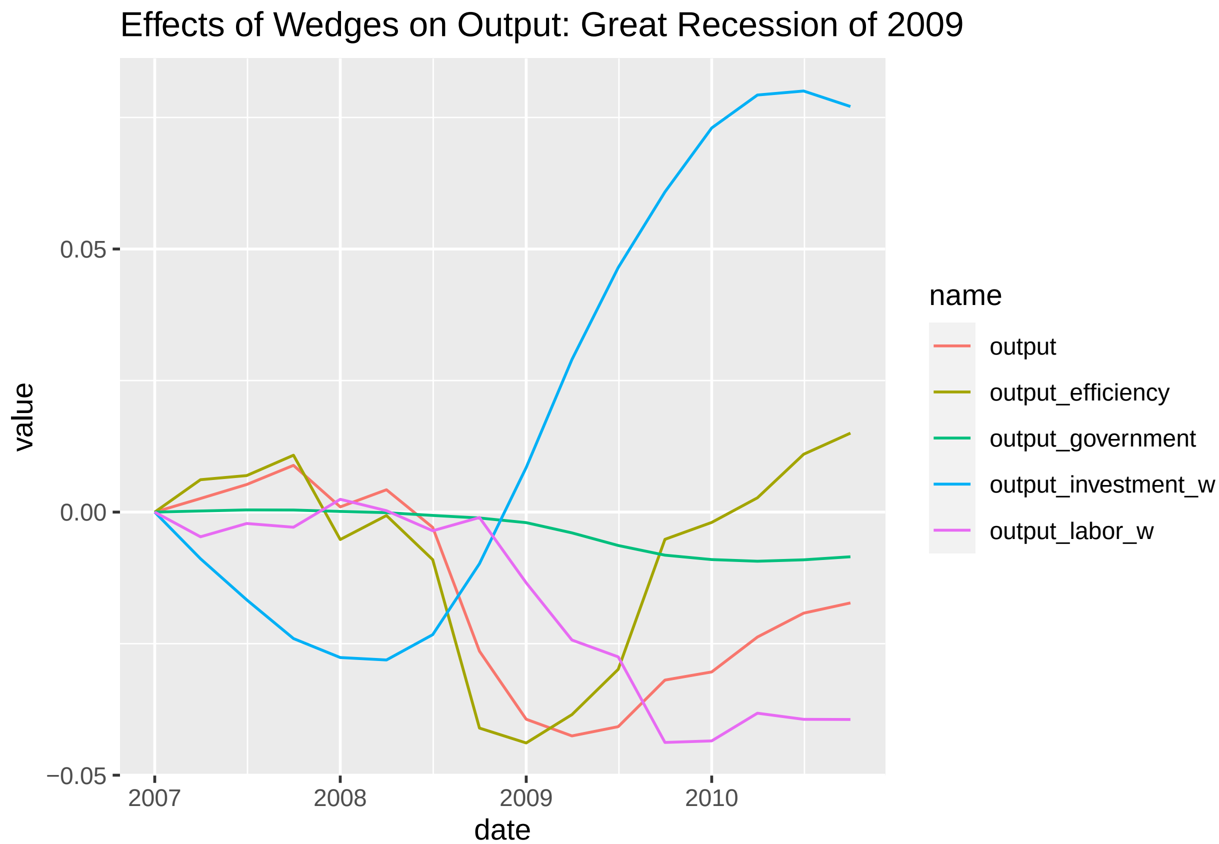
8. Solve the model separately for each wedge. Show a figure with the actual GDP and the four counterfactual series of output. Which wedge explains most of the contraction during the Great Recession of 2009? During the Great Lockdown of 2020? Explain.

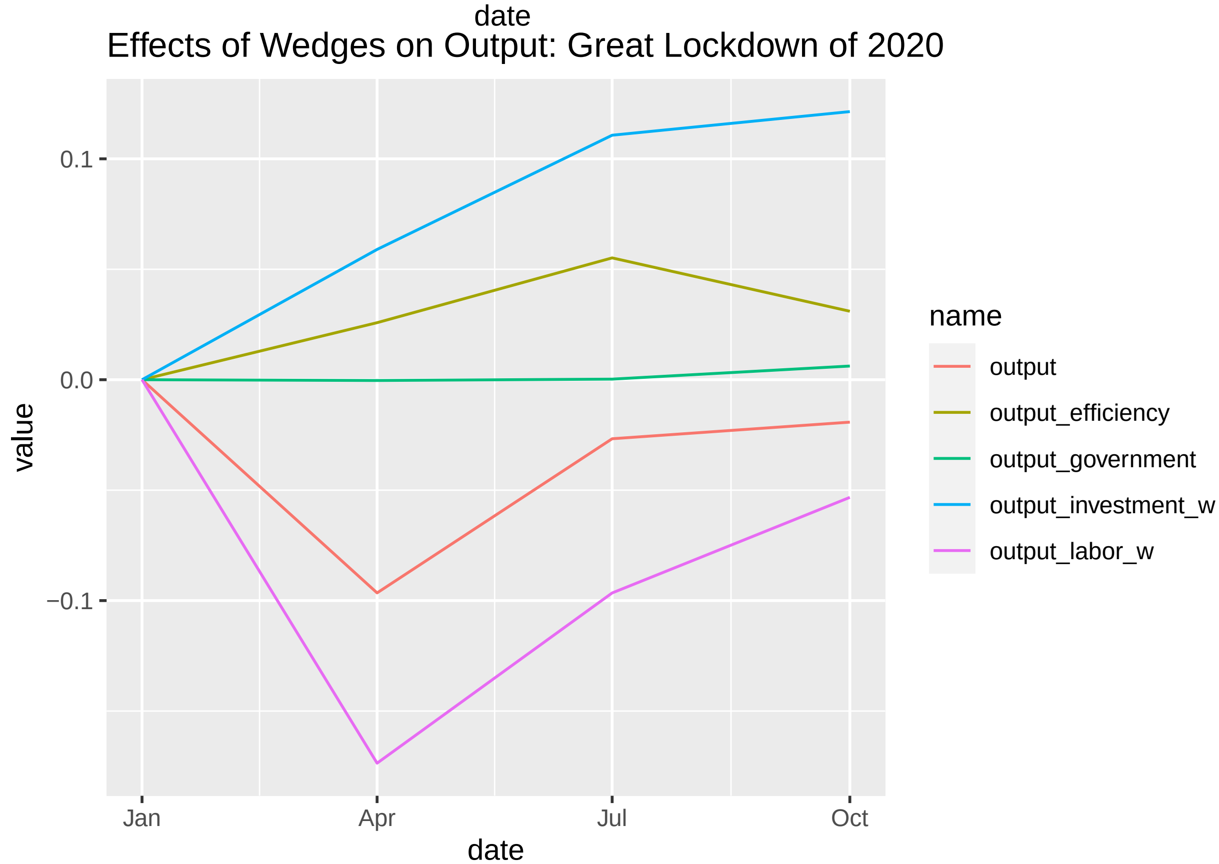
There are three figures below. The first is the entire period with output (red) and the four counterfactual series of output (yellow, green, blue, purple). Most notably, the counterfactual output series based on the investment wedge (blue) is the most volatile.

The second shows 2007-2009 with the first period normalized to zero. Output (red) hovers around zero and then drops in mid-2008 and stays low until mid-2009 where it slowly rebounds. A combination of the efficiency wedge (yellow) and the labor wedge (purple) seems to most affect output. The series based on the efficiency wedge drops between the third and fourth quarter of 2008, but quickly return to about zero in mid-2009. The series based on the labor wedge drops thoroughout 2009 and stays low for the rest of 2010.

The third shows 2020 with the first period normalized to zero. Output (red) drops from the first quarter to the second quarter of 2008. The labor wedge (purple) seems to be the main driver of this drop with the counterfactual output series increasing or remaining flat between the first and second quarters.




Effects of Wedges on Output: Great Recession of 2009
0.05
0.00
−0.05
value
2007
2008
2009
2010
date
name
output
output_efficiency
output_government
output_investment_w
output_labor_w


Effects of Wedges on Output: Great Lockdown of 2020
0.1
0.0
−0.1
value
Jan
Apr
Jul
Oct
date
name
output
output_efficiency
output_government
output_investment_w
output_labor_w

Appendix

```
# PCECC96 Real Personal Consumption Expenditures  Billions of Chained 2012 Dollars,
#                                                 Seasonally Adjusted Annual Rate
# GDPC1   Real Gross Domestic Product             Billions of Chained 2012 Dollars
#                                                 Seasonally Adjusted Annual Rate
# CE16OV  Employment Level                        Thousands of Persons
#                                                 Seasonally Adjusted
# GPDIC1  Real Gross Private Domestic Investment  Billions of Chained 2012 Dollars
#                                                 Seasonally Adjusted Annual Rate
```

```
knitr::opts_chunk$set(echo = TRUE)
library(tidyverse)
library(lubridate)
library(quantmod)
library(mFilter)
library(stargazer)
library(nleqslv)
library(knitr)
library(reticulate)

loadSymbols(c("PCECC96","GDPC1", "CE16OV", "GPDIC1"),
      src = "FRED",
      auto.assign = TRUE)
```

Problem 1

```
temp1 <- cbind(PCECC96, GDPC1, CE16OV, GPDIC1)

temp2 <- temp1 %>%
  as_tibble() %>%
  transmute(date = ymd(index(temp1)),
            consumption = PCECC96,
            output = GDPC1,
            labor = CE16OV,
            investment = GPDIC1) %>%
  filter(!is.na(consumption + output + labor + investment))

data <- temp2 %>%
  mutate(estimation_period = date >= ymd("1980-01-01"),
         consumption = case_when( estimation_period ~ consumption),
         output = case_when( estimation_period ~ output),
         labor = case_when( estimation_period ~ labor)) %>%
  filter(date >= ymd("1950-01-01"),
         date <= ymd("2020-10-01"))
```

Problem 2

```
data_hp_raw <- data %>%
  filter(estimation_period) %>%
```

```
  transmute(date,
            consumption = hpfilter(log(consumption), freq = 1600)$cycle,
            output = hpfilter(log(output), freq = 1600)$cycle,
            labor = hpfilter(log(labor), freq = 1600)$cycle) %>%
  full_join(data %>%
              transmute(date,
                        estimation_period,
                        investment = hpfilter(log(investment),
                                              freq = 1600)$cycle),
            by = "date") %>%
  arrange(date)

data_hp_detailed <- data %>%
  filter(estimation_period) %>%
  transmute(date,
            consumption_cycle = hpfilter(log(consumption), freq = 1600)$cycle,
            consumption_trend = hpfilter(log(consumption), freq = 1600)$trend,
            output_cycle = hpfilter(log(output), freq = 1600)$cycle,
            output_trend = hpfilter(log(output), freq = 1600)$trend,
            labor_cycle = hpfilter(log(labor), freq = 1600)$cycle,
            labor_trend = hpfilter(log(labor), freq = 1600)$trend) %>%
  full_join(data %>%
              transmute(date,
                        estimation_period,
                        investment_cycle = hpfilter(log(investment),
                                              freq = 1600)$cycle,
                        investment_trend = hpfilter(log(investment),
                                              freq = 1600)$trend),
            by = "date") %>%
  arrange(date)
```

Problem 3

```
delta <- 0.025
capital <- 0
for (i in 2:nrow(data_hp_raw)) {
  capital <- c(capital,
               (1-delta)*capital[i-1] + delta * data_hp_raw$investment[i-1])
}

data_hp <- data_hp_raw %>%
  mutate(capital = capital) %>%
  filter(estimation_period) %>%
  select(-estimation_period)
```

Problem 4

```
alpha <- 1/3
sigma <- 1
phi <- 1
```

```r
alpha <- 1/3
beta <- .99
sigma <- 1
phi <- 1

labor_ss <- sqrt(((1-alpha))/((2/3) - (alpha*delta)/((1/beta - 1 + delta))))
capital_ss <- (delta / ((2/3)*labor_ss^(1-alpha) -
                          (1 - alpha) * labor_ss^(-1-alpha)))^(1/(alpha - 1))
investment_ss <- capital_ss *delta
output_ss <- capital_ss ^ alpha *labor_ss ^(1-alpha)
government_ss <- (1/3)*output_ss
consumption_ss <- output_ss - investment_ss - government_ss
```

```r
data_wedge <- data_hp %>%
  transmute(date,
            efficiency = output - alpha * capital - (1 - alpha) * labor,
            government = (output_ss * output - consumption_ss * consumption -
                            investment_ss * investment) / government_ss,
            labor_w = efficiency + alpha * capital - consumption - (1 + alpha) * labor)

efficiency_wedge_ols <- lm(efficiency ~ lag(efficiency) - 1, data = data_wedge)
government_wedge_ols <- lm(government ~ lag(government)- 1, data = data_wedge)
labor_wedge_ols <- lm(labor_w ~ lag(labor_w)- 1, data = data_wedge)
```

Problem 5

```r
# Additional estimated parameters
rho_a <- as.numeric(efficiency_wedge_ols$coef[1])
rho_g <- as.numeric(government_wedge_ols$coef[1])
rho_tau_l <- as.numeric(labor_wedge_ols$coef[1])
rho_tau_i <- 0

# Create C_tilde
c_tilde_11 <- 1
c_tilde_12 <- 0
c_tilde_21 <- - beta * alpha * capital_ss^(alpha - 1)*labor_ss ^(1 - alpha) *
  (alpha -1) / (1+alpha)
c_tilde_22 <- 1 - c_tilde_21

c_tilde <- matrix(c(c_tilde_11, c_tilde_21,c_tilde_12, c_tilde_22), nrow = 2, ncol = 2)

# Create A_tilde
a_tilde_11 <- 1 - delta + 2 *alpha *delta * output_ss / (1+alpha) / investment_ss
a_tilde_12 <- (-delta * output_ss * (1 - alpha) - consumption_ss * (1+alpha))/
  (investment_ss * (1+alpha))
a_tilde_21 <- 0
a_tilde_22 <- 1

a_tilde <- matrix(c(a_tilde_11, a_tilde_21,a_tilde_12, a_tilde_22), nrow = 2, ncol = 2)

# Create B_tilde
b_tilde_11 <- 2 *delta *output_ss
```

```r
b_tilde_12 <- - delta * government_ss / investment_ss
b_tilde_13 <- - delta * output_ss *(1-alpha) / investment_ss / (1+alpha)
b_tilde_14 <- 0
b_tilde_21 <- beta * alpha * capital_ss^(alpha - 1) *
  labor_ss^(1 - alpha) * 2 * rho_a / (1+alpha)
b_tilde_22 <- 0
b_tilde_23 <- beta * alpha * capital_ss^(alpha - 1) *
  labor_ss^(1 - alpha) *(alpha - 1) / (1 + alpha) * rho_tau_l
b_tilde_24 <- (1-delta) * rho_tau_i -1

b_tilde <- matrix(c(b_tilde_11, b_tilde_21,b_tilde_12, b_tilde_22,
                    b_tilde_13, b_tilde_23,b_tilde_14, b_tilde_24),
                  nrow = 2, ncol = 4)

# Create A and B
a <- solve(c_tilde) %*% a_tilde
b <- solve(c_tilde) %*% b_tilde

# Eigen decomposition of A
lambda <- diag(eigen(a)$values)
q <- eigen(a)$vector
q_inv <- solve(q)
d <- q_inv %*% b
```

```r
rho <- diag(c(rho_a, rho_g, rho_tau_l, rho_tau_i))
i_4 <- diag(rep(1, 4))
lambda_1 <- as.numeric(lambda[1, 1])
theta <- (1/lambda_1) * d[1,] %*% solve(i_4 - (1/lambda_1) * rho)
```

```r
data_wedge_2 <- data_wedge %>%
  right_join( data_hp %>% select(date, consumption, capital), by="date") %>%
  mutate(investment_w = (q_inv[1, 1] * capital + q_inv[1, 2] * consumption -
                           theta[1] * efficiency - theta[2] * government -
                           theta[3] * labor_w)/ theta[4]) %>%
  select(-capital, -consumption)

investment_wedge_ols <- lm(investment_w ~ lag(investment_w)- 1, data = data_wedge_2)
```

Problem 6

```r
rho_tau_i <- 0
iter <- 1
max_iter <- 1000
tolerance <- 0.00001

while(TRUE) {
  # print(iter)
  # print(rho_tau_i)

  b_tilde_24 <- (1-delta) * rho_tau_i -1

  b_tilde <- matrix(c(b_tilde_11, b_tilde_21,b_tilde_12, b_tilde_22,
```

```
                      b_tilde_13,b_tilde_23,b_tilde_14, b_tilde_24),
                    nrow = 2, ncol = 4)

  b <- solve(c_tilde) %*% b_tilde

  d <- q_inv %*% b

  data_wedge_3 <- data_wedge %>%
    right_join( data_hp %>% select(date, consumption, capital), by="date") %>%
    mutate(investment_w = (q_inv[1, 1] * capital + q_inv[1, 2] * consumption -
                              theta[1] * efficiency - theta[2] * government -
                              theta[3] * labor_w)/ theta[4]) %>%
    select(-capital, -consumption)

  investment_wedge_ols <- lm(investment_w ~ lag(investment_w)- 1, data = data_wedge_3)

  rho_tau_i_new <- as.numeric(investment_wedge_ols$coefficients[1])

  if (abs(rho_tau_i - rho_tau_i_new) < tolerance) break
  if (iter > max_iter) {
    print("Error in problem 6 code")
    break
  }

  rho_tau_i <-(rho_tau_i +rho_tau_i_new)/ 2
  iter <- iter + 1
}
```

Problem 8

```
simulate_model <- function(df, zero_efficiency = TRUE, zero_government = TRUE,
                           zero_labor_w = TRUE, zero_investment_w = TRUE) {
  if (zero_efficiency) {df$efficiency = 0}
  if (zero_government) {df$government = 0}
  if (zero_labor_w) {df$labor_w = 0}
  if (zero_investment_w) {df$investment_w = 0}

  for (i in 2:nrow(df)) {
    df$consumption[i-1] <- (-q_inv[1, 1] * df$capital[i-1] +
                              theta[1] * df$efficiency[i-1] +
                              theta[2] * df$government[i-1] +
                              theta[3] * df$labor_w[i-1] +
                              theta[4] * df$investment_w[i-1])/ q_inv[1, 2]

    df$capital[i] <-
      (1-delta+(2*alpha*delta*output_ss)/((1+alpha)*investment_ss))*df$capital[i-1] +
      (-delta*output_ss*(1-alpha)-consumption_ss*(1+alpha))/
      (investment_ss*(1+alpha))*df$consumption[i-1] +
      2*delta*output_ss/ ((1 +alpha)*investment_ss) * df$efficiency[i-1] +
      (-delta* government_ss) / investment_ss * df$government[i-1] +
      (-delta*output_ss * (1-alpha))/(investment_ss * (1+alpha)) * df$labor_w[i-1]
  }
```

```
  df <- df %>%
    mutate(labor = alpha/(1+alpha)*capital +
             (-1)/(1+alpha) * consumption +
             1/(1+alpha) * efficiency +
             (-1)/(1+alpha) * labor_w,
           output = efficiency + alpha * capital + (1-alpha) * labor)
  return(df$output)
}
```