# ECON 736A: Problem Set 3

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## 1 Atkeson, Chari, and Kehoe (2007)

1. *Prove Propositions 2 and 3.*

   **Proposition 2** (Determinacy of Equilibrium and the Taylor Principle)**.** *The linear equilibria with interest rate rules of the Taylor rule form $i_t = \bar{\imath} + ax_t$ have outcomes of the form*

$$\begin{aligned} x_{t+1} &= i_t + c\eta_t \\ \pi_t &= x_t + (1+\sigma c)\eta_t \\ y_t &= (1+\sigma c)\eta_t \end{aligned}$$

   *For every $a < 1$ and $\bar{i}$, the economy has a continuum of equilibria indexed by the parameter $c$. For every $a \geq 1$ and $\bar{i}$, within the class of bounded linear equilibria, the economy has a unique equilibrium with $c = 0$.*

   **Proof:** First, verify that the candidate equilibrium satisfies the equilibrium conditions in the lemma:

$$\begin{aligned} E[y_t|h_{t-1}] &= E[(1+\sigma c)\eta_t|h_{t-1}] \\ &= (1+\sigma c)E[\eta_t|h_{t-1}] \\ &= 0 \\ E[\pi_{t+1}|h_{t-1}] &= E[x_{t+1} + (1+\sigma c)\eta_{t+1}|h_{t-1}] \\ &= E[i_t + c\eta_t|h_{t-1}] + (1+\sigma c)E[\eta_{t+1}|h_{t-1}] \\ &= i_t + cE[\eta_t|h_{t-1}] \\ &= i_t \end{aligned}$$

   Second, verify that the candidate equilibrium satisfies the Phillips curve and the Euler equation:

$$
\begin{aligned}
y_t &= \pi_t - x_t \\
\implies [(1+\sigma c)\eta_t] &= [x_t + (1+\sigma c)\eta_t] - x_t \\
\implies (1+\sigma c)\eta_t &= (1+\sigma c)\eta_t \\
y_t &= E[y_{t+1}|h_t] - \sigma(i_t - E[\pi_{t+1}|h_t]) + \eta_t \\
\implies (1+\sigma c)\eta_t &= -\sigma(i_t - E[\pi_{t+1}|h_t]) + \eta_t \\
&= -\sigma(i_t - E[x_{t+1} + (1+\sigma c)\eta_{t+1}|h_t]) + \eta_t \\
&= -\sigma(i_t - E[x_{t+1}|h_t]) + \eta_t \\
&= -\sigma(i_t - E[i_t + c\eta_t|h_t]) + \eta_t \\
&= -\sigma(-c\eta_t) + \eta_t \\
&= (1+\sigma c)\eta_t
\end{aligned}
$$

Finally, we can derive a first order difference equation for $x$ using lemma, the Taylor rule, and agent optimize and representativeness $x_t(h_{t-1}) = z_t(h_{t-1}) = E[\pi_t|h_{t-1}]$:

$$
\begin{aligned}
E[\pi_{t+1}|h_{t-1}] &= i_t \\
&= \bar{\iota} + a x_t(h_{t-1}) \\
\implies x_{t+1}(h_t) &= \bar{\iota} + a x_t(h_{t-1})
\end{aligned}
$$

A solution to the first order difference equation for $x$ is

$$
x_t = a^{(t-s)}\left(x_s - \frac{\bar{\iota}}{1-a}\right) + \frac{\bar{\iota}}{1-a}
$$

for some $s < t$. For an equilibrium to exist, $x_t$ must not diverge. If $a < 1$, $x_t \to \frac{\bar{\iota}}{1-a}$, so for any $c$, an equilibrium exists. If $a > 1$, for $x_t$ not to diverge, the following must hold:

$$
x_s - \frac{\bar{\iota}}{1-a} = 0
$$

for all $s$. Thus, $x_{s+1} = x_s$, so

$$
\begin{aligned}
x_{s+1} &= \bar{\iota} + a x_{s+1} \\
&= \bar{\iota} + a x_s \\
&= i_s \\
\implies c &= 0
\end{aligned}
$$

**Proposition 3** (Rules Satisfying the Taylor Principle are Inefficient)**.** *The outcomes under a Taylor rule of the form $i_t = \bar{\iota} + ax_t$ with $a > 1$ are dominated by the outcomes of an equilibrium with $a = 0$ and $\bar{\iota} = 0$.*

**Proof:** By Proposition 2, the outcome under the Taylor rule of the form $i_t = \bar{\iota} + ax_t$ with $a > 1$ is unique:

$$\begin{aligned} x_{t+1} = i_t &= \frac{\bar{\iota}}{1-a} \\ \pi_t &= x_t + \eta_t \\ y_t &= \eta_t \end{aligned}$$

Thus, representative agent expected payoff is

$$\begin{aligned} E\left[r^A\left(\frac{\bar{\iota}}{1-a}, \frac{\bar{\iota}}{1-a}, \frac{\bar{\iota}}{1-a} + \eta_t\right)\right] &= -\frac{1}{2}E\left[\left(\frac{\bar{\iota}}{1-a} - \left(\frac{\bar{\iota}}{1-a} + \eta_t\right)\right)^2 + (\eta_t - \bar{y})^2 + \left(\frac{\bar{\iota}}{1-a} + \eta_t\right)^2\right] \\ &= -\frac{1}{2}E\left[\eta_t^2 + (\eta_t^2 - 2\bar{y}\eta_t + \bar{y}^2) + \left(\frac{\bar{\iota}}{1-a}\right)^2 + 2\frac{\bar{\iota}}{1-a}\eta_t + \eta_t^2)\right] \\ &= -\frac{1}{2}\left[2E[\eta_t^2] - 2\bar{y}E[\eta_t] + \bar{y}^2 + \left(\frac{\bar{\iota}}{1-a}\right)^2 + 2\left(\frac{\bar{\iota}}{1-a}\right)E[\eta_t] + E[\eta_t^2]\right] \\ &= -\frac{1}{2}\left[3\sigma_\eta^2 + \bar{y}^2 + \left(\frac{\bar{\iota}}{1-a}\right)^2\right] \end{aligned}$$

If $\bar{\iota} = a = 0$, then $i_t = 0$, but the equilibrium is not unique and for every $c$, the following is an equilibrium. Thus, there is an equilibrium for $c = 0$ where

$$\begin{aligned} x_{t+1} &= 0 \\ \pi_t &= \eta_t \\ y_t &= \eta_t \end{aligned}$$

And the representative agent expected payoff in that equilibrium is:

$$\begin{aligned} E[r^A(0, 0, \eta_t)] &= -\frac{1}{2}E[\eta_t^2 + (\eta_t - \bar{y})^2 + \eta_t^2] \\ &= -\frac{1}{2}\left[E[\eta_t^2] + E[\eta_t^2] - 2\bar{y}E[\eta_t] + \bar{y}^2 + E[\eta_t^2]\right] \\ &= -\frac{1}{2}\left[3\sigma_\eta^2 + \bar{y}^2\right] \end{aligned}$$

The equilibrium associated Taylor principle is not optimal, if there exists a $c$ such that the representative agent is better off:

$$
\begin{aligned}
E[r^A(0,0,\eta_t)] &> E\left[r^A\left(\frac{\bar{\iota}}{1-a}, \frac{\bar{\iota}}{1-a}, \frac{\bar{\iota}}{1-a} + \eta_t\right)\right] \\
\iff -\frac{1}{2}\left[3\sigma_\eta^2 + \bar{y}^2\right] &> -\frac{1}{2}\left[3\sigma_\eta^2 + \bar{y}^2 + \left(\frac{\bar{\iota}}{1-a}\right)^2\right] \\
\iff 0 &< \left(\frac{\bar{\iota}}{1-a}\right)^2
\end{aligned}
$$

Which holds for $a > 1$ and $\bar{\iota} > 0$.

2. *Consider a slight extension of the environment we saw in class. There, we assumed that the agents did not observe the money growth rate but only observed the inflation rate (which was a noisy signal of the money growth rate). Now suppose that in addition to observing the inflation, agents observe the true money growth rate choice but with a lag. In particular, suppose that* $\mu_{t-1}$ *is observable in period* $t$ *after wages are chosen. Is transparency valuable in this case? More specifically, prove the analog of Proposition 6 in the paper for this environment.*