# ECON 709 - PS 5

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10/11/2020

1. For the following sequences, show $a_n \to 0$ as $n \to \infty$:

(a) $a_n = 1/n$

Fix $\varepsilon > 0$. Choose $\bar{n} > \frac{1}{\varepsilon}$. For all $n \geq \bar{n}$,

$$|1/n - 0| = |1/n| = \varepsilon.$$

Thus, $a_n = 1/n \to 0$ as $n \to \infty$.

(b) $a_n = \frac{1}{n}\sin(\frac{n\pi}{2})$

Fix $\varepsilon > 0$. Notice that $|\sin(x)| \leq 1 \ \forall x$. Choose $\bar{n} > \frac{1}{\varepsilon}$. For all $n \geq \bar{n}$,

$$\left|\frac{1}{n}\sin\left(\frac{n\pi}{2}\right) - 0\right| = \left|\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right| \leq |1|\left|\frac{1}{n}\right| = \left|\frac{1}{n}\right| \leq \varepsilon$$

Thus, $a_n = \frac{1}{n}\sin(\frac{n\pi}{2}) \to 0$ as $n \to \infty$.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

2. Consider a random variable $X^n$ with the probability function

$$X_n = \begin{cases} -n, & \text{with probability } 1/n \\ 0, & \text{with probability } 1-2/n \\ n, & \text{with probability } 1/n \end{cases}$$

(a) Does $X_n \to_p 0$ as $n \to \infty$?

Fix $\varepsilon > 0$. Choose $\bar{n} > \varepsilon$. For $n \geq \bar{n}$,

$$P(|X_n| \geq \varepsilon) \leq P(|X_n| \geq n) = P(X_n = -n) + P(X_n = n) = 1/n + 1/n = 2/n$$

Since $1/n \to 0$, $2/n \to 0$. Thus, $X_n \to_p 0$ as $n \to \infty$.

(b) Calculate $E(X_n)$.

$$E(X_n) = \sum_{x \in \text{Supp}(X)} \pi(x)x = (1/n) * (-n) + (1-2/n)(0) + (1/n)(n) = -1 + 1 = 0.$$

(c) Calculate $Var(X_n)$.

$$Var(X_n) = E(X_n^2) - E(X_n)^2 = E(X_n^2) = \sum_{x \in \text{Supp}(X)} \pi(x)x^2 = (1/n)*(-n)^2 + (1-2/n)(0)^2 + (1/n)(n)^2 = n+n = 2n.$$

(d) Now suppose the distribution is

$$X_n = \begin{cases} 0, & \text{with probability } 1-1/n \\ n, & \text{with probability } 1/n \end{cases}$$

Calculate $E(X_n)$.

$$E(X_n) = \sum_{x \in \text{Supp}(X)} \pi(x)x = (1-1/n)(0) + (1/n)(n) = 0 + 1 = 1$$

(e) Conclude that $X_n \to_p 0$ is not sufficient for $E(X_n) \to 0$.

Fix $\varepsilon > 0$. Choose $\bar{n} > \varepsilon$. For $n > \bar{n}$

$$P(|X_n| \geq \varepsilon) \leq P(|X_n| \geq n) = P(X_n = n) = 1/n$$

Since $1/n \to 0$, $X_n \to_p 0$ as $n \to \infty$. Thus, $X_n \to_p 0$ is not sufficient for $E(X_n) \to 0$.

3. A weighted sample mean takes the form $\bar{Y}^* = \frac{1}{n}\sum_{i=1}^{n} w_i Y_i$ for some non-negative constants $w_i$ satisfying $\frac{1}{n}\sum_{i=1}^{n} w_i = 1$. Assume that $Y_i : i = 1, ..., n$ are i.i.d.

(a) Show that $\bar{Y}^*$ is unbiased for $\mu = E(Y_i)$.

$$E(\bar{Y}^*) = E\left(\frac{1}{n}\sum_{i=1}^{n} w_i Y_i\right) = \frac{1}{n}\sum_{i=1}^{n} w_i E(Y_i) = \frac{1}{n}\sum_{i=1}^{n} w_i \mu = (1)\mu = \mu$$

(b) Calculate $Var(\bar{Y}^*)$.

$$Var(\bar{Y}^*) = Var\left(\frac{1}{n}\sum_{i=1}^{n} w_i Y_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} w_i^2 Var(Y_i)$$

(c) Show that a sufficient condition for $\bar{Y}^* \to_p \mu$ is that $\frac{1}{n^2}\sum_{i=1}^{n} w_i^2 \to 0$. (Hint: use the Markov's or Chebyshev's Inequality).

$P(|\bar{Y}^*|)$

(d) Show that the sufficient condition for the condition in part (c) is $\max_{i \le n} w_i/n \to 0$.

4. Take a random sample $\{X_1, ..., X_n\}$. Which statistic converges in probability by the weak law of large numbers and continuous mapping theorem, assuming the moment exists?

(a) $\frac{1}{n}\sum_{i=1}^{n} X_i^2$

(b) $\frac{1}{n}\sum_{i=1}^{n} X_i^3$

(c) $\max_{i\leq n} X_i$

(d) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2$

(e) $\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} X_i}$ assuming $\mu = E(X_i) > 0$.

(f) $1(\frac{1}{n}\sum_{i=1}^{n} X_i > 0)$ where

$$1(a) = \begin{cases} 1 & \text{if } a \text{ is true} \\ 0 & \text{if } a \text{ is not true} \end{cases}$$

is called the indicator function of event $a$.