FIN 970: Final Exam

Alex von Hafften

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1 Problem 1a: Term Structure and No Arbitrage Models

1. Use SDF approach.

   Solution: Conjecture $P_t^n = \exp(A_n + B_n' X_t)$. Proof by induction.

   For $n = 1$,

$$
\begin{aligned}
P_t^1 &= E_t[M_{t+1} \cdot 1] \\
\implies \exp(A_1 + B_1' X_t) &= E_t\left[\exp\left(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\right)\right] \\
\implies E_t\left[-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\right] &= -\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t \\
Var_t\left[-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\right] &= \lambda_t'\lambda_t \\
\implies \exp(A_1 + B_1' X_t) &= \exp(-\delta_0 - \delta_1 X_t) \\
\implies &\begin{cases} A_1 = -\delta_0 \\ B_1 = -\delta_1' \end{cases}
\end{aligned}
$$

   For some $n > 1$, the Euler equation holds:

$$
\begin{aligned}
P_t^n &= E_t[M_{t+1} P_{t+1}^{n-1}] \\
\exp(A_n + B_n' X_t) &= E_t\left[\exp\left(-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\right)\exp(A_{n-1} + B_{n-1}' X_{t+1})\right] \\
&= E_t\left[\exp\left(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1} + A_{n-1} + B_{n-1}'(\mu + \Phi X_t + \Sigma\varepsilon_{t+1})\right)\right] \\
&= E_t\left[\exp\left(-\delta_0 - \delta_1 X_t - \frac{1}{2}\lambda_t'\lambda_t + A_{n-1} + B_{n-1}'\mu + B_{n-1}'\Phi X_t + [B_{n-1}'\Sigma - \lambda_t']\varepsilon_{t+1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&E_t\Bigg[-\delta_0-\delta_1 X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t+[B_{n-1}'\Sigma-\lambda_t']\varepsilon_{t+1}\Bigg]\\
&=-\delta_0-\delta_1 X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t\\
&Var_t\Bigg[-\delta_0-\delta_1 X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t+[B_{n-1}'\Sigma-\lambda_t']\varepsilon_{t+1}\Bigg]\\
&=[B_{n-1}'\Sigma-\lambda_t'][B_{n-1}'\Sigma-\lambda_t']'\\
&=B_{n-1}'\Sigma\Sigma' B_{n-1}+\lambda_t'\lambda_t-2B_{n-1}'\Sigma\lambda_t
\end{aligned}
$$

$$
\begin{aligned}
\exp(A_n+B_n'X_t)&=\exp\Bigg(-\delta_0-\delta_1 X_t-\frac{1}{2}\lambda_t'\lambda_t+A_{n-1}+B_{n-1}'\mu+B_{n-1}'\Phi X_t\\
&\quad+\frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1}+\frac{1}{2}\lambda_t'\lambda_t-B_n'\Sigma(\lambda_0+\lambda_1 X_t)\Bigg)\\
&=\exp\Bigg(-\delta_0+A_{n-1}+B_{n-1}'\mu-B_{n-1}'\Sigma\lambda_0+\frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1}+(-\delta_1+B_{n-1}'\Phi-B_{n-1}'\Sigma\lambda_1)X_t)\Bigg)\\
&\implies\begin{cases}A_n=-\delta_0+A_{n-1}+B_{n-1}'(\mu-\Sigma\lambda_0)+\frac{1}{2}B_{n-1}'\Sigma\Sigma' B_{n-1}\\ B_n=-\delta_1+(\Phi-\Sigma\lambda_1)'B_{n-1}\end{cases}
\end{aligned}
$$

2. Use risk-neutral density approach.

Solution: Conjecture $P_t^n=\exp(C_n+D_n'X_t)$. Proof by induction.

For $n=0$,

$$
\begin{aligned}
P_t^1&=e^{-r_t}E_t^Q[1]\\
\exp(C_1+D_1'X_t)&=\exp(-\delta_0-\delta_1 X_t)\\
&\implies\begin{cases}C_1=-\delta_0\\ D_1=-\delta_1'\end{cases}
\end{aligned}
$$

For some $n>1$, the Euler equation holds:

$$
\begin{aligned}
P_t^n&=e^{-r_t}E_t^Q[P_{t+1}^{n-1}]\\
\exp(C_n+D_n'X_t)&=e^{-r_t}E_t^Q[\exp(C_{n-1}+D_{n-1}'X_{t+1})]\\
&=e^{-r_t}E_t^Q[\exp(C_{n-1}+D_{n-1}'(\mu^Q+\Phi^Q X_t+\Sigma\varepsilon_{t+1}^Q))]\\
&=e^{-r_t}E_t^Q[\exp(C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t+D_{n-1}'\Sigma\varepsilon_{t+1}^Q)]
\end{aligned}
$$

$$
\begin{aligned}
E_t^Q[C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t+D_{n-1}'\Sigma\varepsilon_{t+1}^Q]&=E_t[C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t]\\
Var_t^Q[C_{n-1}+D_{n-1}'\mu^Q+D_{n-1}'\Phi^Q X_t+D_{n-1}'\Sigma\varepsilon_{t+1}^Q]&=D_{n-1}'\Sigma\Sigma' D_{n-1}
\end{aligned}
$$

$$\exp(C_n + D_n' X_t) = \exp\left(-\delta_0 - \delta_1 X_t + C_{n-1} + D_{n-1}'\mu^Q + D_{n-1}'\Phi^Q X_t + \frac{1}{2} D_{n-1}'\Sigma\Sigma' D_{n-1} \right)$$

$$\begin{cases} C_n = -\delta_0 + C_{n-1} + D_{n-1}'\mu^Q + \frac{1}{2} D_{n-1}'\Sigma\Sigma' D_{n-1} \\ D_n = -\delta_1' + \Phi^{Q'} D_{n-1} \end{cases}$$

3. Show that this is a one-to-one mapping between the risk-neutral parameters (μ^Q, Φ^Q) and the market prices of risk (λ_0, λ_1).

   Solution: Clearly, parts (1) and (2) are equivalent iff

$$\mu^Q = \mu - \Sigma\lambda_0 \iff \lambda_0 = \Sigma^{-1}(\mu - \mu^Q)$$
$$\Phi^Q = \Phi - \Sigma\lambda_1 \iff \lambda_1 = \Sigma^{-1}(\Phi - \Phi^Q)$$

   with $A_n = C_n$ and $B_n = D_n$. Thus, we can go back and forth from SDF to risk-neutral densities to price bonds of any maturity.

2 Problem 2h

1. Conjecture pc_t is linear in state variables. Solve for pc_t and $r_{c,t+1}$. Explain how risk exposures depend on preferences and consumption dynamics.

   Solutions: Conjecture that $pc_t = A_0 + A_x x_t$. Using the Campbell-Schiller approximation to log-linearization the consumption return:

$$
\begin{aligned}
R_{C,t+1} &= \frac{P_{C,t+1} + C_{t+1}}{P_{C,t}} \\
&= \frac{\frac{P_{C,t+1}}{C_{t+1}} + 1}{\frac{P_{C,t}}{C_t}} \frac{C_{t+1}}{C_t} \\
r_{c,t+1} &= \log(\exp(pc_{t+1}) + 1) - pc_t + \Delta c_{t+1} \\
&\approx \left[\log(\exp(\bar{pc}) + 1) + \frac{\exp(\bar{pc})}{\exp(\bar{pc}) + 1}(pc_{t+1} - \bar{pc}) \right] - pc_t + \Delta c_{t+1} \\
&= \underbrace{\log(\exp(\bar{pc}) + 1) - \frac{\exp(\bar{pc})}{\exp(\bar{pc}) + 1}\bar{pc}}_{\equiv \kappa_0} + \underbrace{\frac{\exp(\bar{pc})}{\exp(\bar{pc}) + 1}}_{\equiv \kappa_1} pc_{t+1} - pc_t + \Delta c_{t+1}
\end{aligned}
$$

   where $pc_t = \log(P_{C,t}/C_t)$. Alternatively, we can express the return in terms of the demeaned price-consumption ratio,

$$
\begin{aligned}
r_{c,t+1} &= \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1} \\
&= -\log \kappa_1 + \kappa_1 \underbrace{\tilde{pc}_{t+1}}_{\equiv pc_{t+1} - \bar{pc}} - \tilde{pc}_t + \Delta c_{t+1}
\end{aligned}
$$

   Given the guess for pc_t, its unconditional expected value is $\bar{pc} = A_0$, so $\tilde{pc}_t = A_x x_t$. Plugging the dynamics for volatility and consumption:

$$
\begin{aligned}
\tilde{pc}_{t+1} &= A_x x_{t+1} \\
&= A_x[\rho x_t + \varphi_e \sigma e_{t+1}] \\
&= A_x \rho x_t + A_x \varphi_e \sigma e_{t+1}
\end{aligned}
$$

   Plugging into the consumption return:

$$
\begin{aligned}
r_{c,t+1} &= -\log \kappa_1 + \kappa_1 \tilde{pc}_{t+1} - \tilde{pc}_t + \Delta c_{t+1} \\
&= -\log \kappa_1 + \kappa_1[A_x \rho x_t + A_x \varphi_e \sigma e_{t+1}] - [A_x x_t] + [\mu + x_t + \sigma \varepsilon_{t+1}] \\
&= [-\log \kappa_1 + \mu] + [\kappa_1 A_x \rho - A_x + 1] x_t + \kappa_1 A_x \varphi_e \sigma e_{t+1} + \sigma \varepsilon_{t+1}
\end{aligned}
$$

   For any asset with return $R_{i,t+1}$, the Euler equation holds and if the return is log-normal:

$$
\begin{aligned}
1 &= E_t[M_{t+1} R_{i,t+1}] \\
&= E_t[\exp(m_{t+1} + r_{i,t+1})] \\
&= \exp(E_t[m_{t+1} + r_{i,t+1}] + \frac{1}{2} Var_t[m_{t+1} + r_{i,t+1}]) \\
\implies 0 &= E_t[m_{t+1} + r_{i,t+1}] + \frac{1}{2} Var_t[m_{t+1} + r_{i,t+1}]
\end{aligned}
$$

In particular for the consumption assets, the Euler equation holds.

$$
\begin{aligned}
m_{t+1}+r_{c,t+1} &= \theta\log\delta - \frac{\theta}{\psi}\Delta c_{t+1} + \theta r_{c,t+1} \\
&= \theta\log\delta - \frac{\theta}{\psi}[\mu + x_t + \sigma\varepsilon_{t+1}] + \theta[(-\log\kappa_1+\mu) + (\kappa_1 A_x\rho - A_x + 1)x_t + \kappa_1 A_x\varphi_e\sigma e_{t+1} + \sigma\varepsilon_{t+1}] \\
&= \left[\theta\log\delta - \frac{\theta}{\psi}\mu + \theta(-\log\kappa_1+\mu)\right] + \left[\theta(\kappa_1 A_x\rho - A_x + 1) - \frac{\theta}{\psi}\right]x_t + \theta\left[1-\frac{1}{\psi}\right]\sigma\varepsilon_{t+1} + \theta\kappa_1 A_x\varphi_e\sigma e_{t+1}
\end{aligned}
$$

The expected value and variance of $m_{t+1}+r_{c,t+1}$ is

$$
\begin{aligned}
E_t[m_{t+1}+r_{c,t+1}] &= \left[\theta\log\delta - \frac{\theta}{\psi}\mu + \theta(-\log\kappa_1+\mu)\right] + \left[\theta(\kappa_1 A_x\rho - A_x + 1) - \frac{\theta}{\psi}\right]x_t \\
Var_t[m_{t+1}+r_{c,t+1}] &= \theta^2\left[1-\frac{1}{\psi}\right]^2\sigma^2 + \theta^2\kappa_1^2 A_x^2\varphi_e^2\sigma^2
\end{aligned}
$$

Thus, we can plug the expected value and variance back in:

$$
\begin{aligned}
&0 = E_t[m_{t+1}+r_{i,t+1}] + \frac{1}{2}Var_t[m_{t+1}+r_{i,t+1}] \\
\implies\ &0 = \left[\theta\log\delta - \frac{\theta}{\psi}\mu + \theta(-\log\kappa_1+\mu)\right] + \left[\theta(\kappa_1 A_x\rho - A_x + 1) - \frac{\theta}{\psi}\right]x_t + \frac{1}{2}\theta^2\left[1-\frac{1}{\psi}\right]^2\sigma^2 + \frac{1}{2}\theta^2\kappa_1^2 A_x^2\varphi_e^2\sigma^2 \\
\implies\ &\begin{cases} 0 &= \left[\theta\log\delta - \frac{\theta}{\psi}\mu + \theta(-\log\kappa_1+\mu)\right] + \frac{1}{2}\theta^2\left[1-\frac{1}{\psi}\right]^2\sigma^2 + \frac{1}{2}\theta^2\kappa_1^2 A_x^2\varphi_e^2\sigma^2 \\ 0 &= \left[\theta(\kappa_1 A_x\rho - A_x + 1) - \frac{\theta}{\psi}\right] \end{cases} \\
\implies\ &A_x = \frac{1-1/\psi}{1-\kappa_1\rho}
\end{aligned}
$$

Thus, asset valuations respond positive to expected growth if

$$
A_x > 0 \iff \frac{1-1/\psi}{1-\kappa_1\rho} > 0 \iff 1 > 1/\psi \iff \psi > 1
$$

Economically, this parameter restriction means that the substitution effect dominates the wealth effect. Using the solution for A_x, we can express $r_{c,t+1}$ as

$$
\begin{aligned}
r_{c,t+1} &= [-\log\kappa_1+\mu] + [\kappa_1 A_x\rho - A_x + 1]x_t + \kappa_1 A_x\varphi_e\sigma e_{t+1} + \sigma\varepsilon_{t+1} \\
&= [-\log\kappa_1+\mu] + \frac{1}{\psi}x_t + \frac{(1-1/\psi)\kappa_1\varphi_e}{1-\kappa_1\rho}\sigma e_{t+1} + \sigma\varepsilon_{t+1} \\
&= r_{c,0} + \frac{1}{\psi}x_t + B_x\varphi_e\sigma e_{t+1} + B_c\sigma\varepsilon_{t+1}
\end{aligned}
$$

where $r_{c,0} \equiv -\log\kappa_1+\mu$, $B_x \equiv \frac{(1-1/\psi)\kappa_1}{1-\kappa_1\rho}$, and $B_c \equiv 1$.

2. Solve for M_{t+1}.

Solution:

$$
\begin{aligned}
m_{t+1} &= \theta \log \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1) r_{c,t+1} \\
&= \theta \log \delta - \frac{\theta}{\psi}\left[\mu + x_t + \sigma\varepsilon_{t+1}\right] + (\theta-1)\left[r_{c,0} + \frac{1}{\psi}x_t + B_x\varphi_e\sigma e_{t+1} + B_c\sigma\varepsilon_{t+1}\right] \\
&= \left[\theta \log\delta - \frac{\theta}{\psi}\mu + (\theta-1)r_{c,0}\right] + \left[-\frac{\theta}{\psi} + \frac{\theta-1}{\psi}\right]x_t + (\theta-1)B_x\varphi_e\sigma e_{t+1} + (-\theta/\psi + (\theta-1)B_c)\sigma\varepsilon_{t+1} \\
&= m_0 - m_x x_t - \lambda_x\varphi_e\sigma e_{t+1} - \lambda_c\sigma\varepsilon_{t+1}
\end{aligned}
$$

where

$$
\begin{aligned}
m_0 &\equiv \left[\theta\log\delta - \frac{\theta}{\psi}\mu + (\theta-1)r_{c,0}\right] \\
m_x &\equiv -1/\psi \\
\lambda_x &\equiv (1-\theta)B_x \\
&= (1-\theta)\frac{(1-1/\psi)\kappa_1}{1-\kappa_1\rho} \\
\lambda_c &\equiv \theta/\psi - \theta + 1 \\
&= -\gamma
\end{aligned}
$$

If $\psi > 1 \implies \lambda_x > 0$ and $\lambda_c < 0$ by assumption.

3. Consumption strip at maturity $n = 1$.

Solution:

$$
\begin{aligned}
P_{t,1} &= E_t[M_{t+1}C_{t+1}] \\
\implies \frac{P_{t,1}}{C_t} &= E_t\left[M_{t+1}\frac{C_{t+1}}{C_t}\right] \\
&= E_t[\exp(m_{t+1} + \Delta c_{t+1})] \\
&= \exp(E_t[m_{t+1} + \Delta c_{t+1}] + \frac{1}{2}V_t[m_{t+1} + \Delta c_{t+1}]) \\
\implies pc_{t,1} &= E_t[m_{t+1} + \Delta c_{t+1}] + \frac{1}{2}V_t[m_{t+1} + \Delta c_{t+1}] \\
\implies m_{t+1} + \Delta c_{t+1} &= m_0 - m_x x_t - \lambda_x\varphi_e\sigma e_{t+1} - \lambda_c\sigma\varepsilon_{t+1} + \mu + x_t + \sigma\varepsilon_{t+1} \\
&= m_0 + \mu - (m_x - 1)x_t - \lambda_x\varphi_e\sigma e_{t+1} - (\lambda_c - 1)\sigma\varepsilon_{t+1} \\
\implies E_t[m_{t+1} + \Delta c_{t+1}] &= m_0 + \mu - (m_x - 1)x_t \\
\implies V_t[m_{t+1} + \Delta c_{t+1}] &= \lambda_x^2\varphi_e^2\sigma^2 + (\lambda_c - 1)^2\sigma^2 \\
pc_{t,1} &= m_0 + \mu + (1 - m_x)x_t + \frac{1}{2}\lambda_x^2\varphi_e^2\sigma^2 + \frac{1}{2}(\lambda_c - 1)^2\sigma^2
\end{aligned}
$$

4. Solve for the return on the consumption strip and risk premium

Solution: The return on the consumption strip is:

$$\begin{aligned}
\log r_{t+1,1} &= \Delta c_{t+1} - pc_{t,1} \\
&= \mu + x_t + \sigma\varepsilon_{t+1} - m_0 - \mu - (1-m_x)x_t - \frac{1}{2}\lambda_x^2\varphi_e^2\sigma^2 - \frac{1}{2}(\lambda_c - 1)^2\sigma^2 \\
&= -m_0 + m_x x_t + \sigma\varepsilon_{t+1} - \frac{1}{2}\lambda_x^2\varphi_e^2\sigma^2 - \frac{1}{2}(\lambda_c - 1)^2\sigma^2
\end{aligned}$$

The risk premium of the consumption strip is

$$\begin{aligned}
-Cov_t(r_{t+1,1}, m_{t+1}) = -Cov_t(&-m_0 + m_x x_t + \sigma\varepsilon_{t+1} - \frac{1}{2}\lambda_x^2\varphi_e^2\sigma^2 - \frac{1}{2}(\lambda_c - 1)^2\sigma^2, \\
&m_0 - m_x x_t - \lambda_x\varphi_e\sigma e_{t+1} - \lambda_c\sigma\varepsilon_{t+1}) \\
&= -Cov_t(\sigma\varepsilon_{t+1}, -\lambda_x\varphi_e\sigma e_{t+1} - \lambda_c\sigma\varepsilon_{t+1}) \\
&= \lambda_c\sigma^2 \\
&= -\gamma\sigma^2
\end{aligned}$$

The risk premium on the consumption claim is

$$\begin{aligned}
-Cov_t(r_{c,t+1}, m_{t+1}) &= -Cov_t(r_{c,0} + \frac{1}{\psi}x_t + B_x\varphi_e\sigma e_{t+1} + B_c\sigma\varepsilon_{t+1}, m_0 - m_x x_t - \lambda_x\varphi_e\sigma e_{t+1} - \lambda_c\sigma\varepsilon_{t+1}) \\
&= -Cov_t(B_x\varphi_e\sigma e_{t+1} + B_c\sigma\varepsilon_{t+1}, -\lambda_x\varphi_e\sigma e_{t+1} - \lambda_c\sigma\varepsilon_{t+1}) \\
&= B_x\lambda_x\varphi_e^2\sigma^2 + B_c\lambda_c\sigma^2 \\
&= (1-\theta)B_x^2\varphi_e^2\sigma^2 - \gamma\sigma^2
\end{aligned}$$

The consumption strip has a negative risk premium and the consumption claim has a higher risk premium, so this model is inconsistent with the evidence that short-term consumptions strips have higher average excess returns than claim on all future cash-flows.

$$\begin{aligned}
-Cov_t(r_{t+1,1}, m_{t+1}) &< -Cov_t(r_{c,t+1}, m_{t+1}) \\
\iff -\gamma\sigma^2 &< (1-\theta)B_x^2\varphi_e^2\sigma^2 - \gamma\sigma^2 \\
\iff 0 &< (1-\theta)B_x^2\varphi_e^2\sigma^2
\end{aligned}$$

5. Time-varying risk premium on consumption strips.

Solution: No, this model implies a constant risk premium on consumption strips. We can introduce time-varying volatility σ_t. With time-varying volatility, the risk premium of the consumption strip would be $-\gamma\sigma_t^2$, which would vary over time.

3 Problem 3

4 Problem 4