Methods of proof: direct, contradiction, contraposition, induction.

Set operations ($A, B \subset X$):

- Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- Intersection: $A \cap B = \{x | x \in A \text{ and } x \in B\}$
- Difference: $A \setminus B = \{x \in A | x \notin B\}$
- Complement: $A^C = \{x \in X | x \notin A\}$

DeMorgan's Laws: $(A \cap B)^C = A^C \cup B^C$ and $(A \cup B)^C = A^C \cap B^C$.

Cardinality is the size of the set. Sets A and B are numerically equivalent (have the same cardinality) if their elements can be uniquely matched up and paired off.

Set A is finite if it is numerically equivalent to $1, ..., n$ for some n. Then A's cardinality $= n$. A set that is not finite is infinite.

An infinite set is either countable (it is numerically equivalent to $\mathbb{N}$) or uncountable.

Metric (distance) on a set X is a function $d : X \times X \to \mathbb{R}_+$ s.t. $\forall x, y, z \in X$,

- $d(x, y) \geq 0$, with equality iff $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) + d(y, x)$

Metric space is a pair (X, d), where X is a set and d is a metric on X.

Euclidean space $(\mathbb{R}^m, d_E(x, y)) = (\{(x_1, ..., x_m) | x_i \in \mathbb{R}, i = 1, ..., m\}, \sqrt{\sum_{i=1}^m (x_i, y_i)^2})$.

In metric space (X, d), open ball with center x and radius ε is $B_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}$.

In metric space (X, d), closed ball with center x and radius ε is $B_\varepsilon(x) = \{y \in X | d(x, y) \leq \varepsilon\}$.

Sequence in a set X is a function $s : \mathbb{N} \to X$, which we write as $\{s_n\}$, where $s_n = s(n)$.

Sequence $\{x_n\}$ in a metric space (X, d) converges to $x \in X$ if $\forall \varepsilon > 0 \exists N(\varepsilon) > 0$ s.t. $\forall n > N(\varepsilon)$, $d(x_n, x) < \varepsilon$ (written as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$).

A sequence $\{x_n\}$ in a metric space (X, d) has at most one limit.

Consider a sequence $\{x_n\}$ and a rule that assigns to each $k \in \mathbb{N}$ a value $n_k \in \mathbb{N}$ s.t. $n_k < n_{k+1} \forall k$. Then $\{x_{n_k}\}$ is called a subsequence.

If $\{x_n\}$ converges to x as $n \to \infty$, then any subsequence $\{x_{n_k}\}$ also converges to x as $k \to \infty$.

A subset $S \subset X$ in a metric space (X, d) is bounded if $\exists x \in X, \beta \in \mathbb{R}$ s.t. $\forall s \in S, d(x, s) < \beta$.

Every convergent sequence in a metric space is bounded.

In $(\mathbb{R}, d_E)$, if $x_n \to x \in \mathbb{R}, y_n \to y \in \mathbb{R}$, and $x_n \leq y_n \forall n \in \mathbb{N}$, then $x \leq y$ (limits preserve weak inequality).

In $(\mathbb{R}, d_E)$, if $x_n \to x \in \mathbb{R}$ and $y_n \to y \in \mathbb{R}$, then $x_n + y_n \to x + y$, $x_n - y_n \to x - y$, $x_n y_n \to xy$, and $x_n / y_n \to x/y$ if $y \neq 0$ and $y_n \neq 0 \forall n$ (limits preserve algebraic operations).

Bolzano-Weierstrass Theorem: Every bounded real sequence contains at least one convergent subsequence.

Monotone Convergence Theorem: Every increasing sequence of real numbers that is bounded above converges. Every decreasing sequence of real numbers that is bounded below converges.

Every real sequence contains either a decreasing subsequence or increasing subsequence (and possibly both).

Given a real sequence $\{x_n\}$, the infinite sum of its terms is well-defined if the sequence of partial sums $\{S_n\}$ converges, $S_n = \sum_{i=1}^n x_i$. If $S_n \to S$, we write $\sum_{i=1}^\infty x_i = S$.

Let (X, d) be a metric space. A set $A \subset X$ is **open** if $\forall x \in A \exists \varepsilon > 0$ s.t. $B_\varepsilon \subset A$.

A set $C \subset X$ is **closed** id its complement, $C^c = X \setminus C$, is open.

Open ball $B_\varepsilon(x)$ is an open set. Closed ball $B_\varepsilon[x]$ is a closed set.

Let (X, d) be a metric space. Then

- $\emptyset$ and X are simultaneously open and closed in X,
- the union of an arbitrary collection of open sets is open,
- the intersection of a finite collection of open sets is open,
- the union of a finite collection of closed sets is closed,
- the intersection of an arbitrary collection of closed sets is closed.

A set A in a metric space (X, d) is closed iff every convergent sequence $\{x_n\}$ contained in A has its limit in A.

Let (X, d) be a metric space and A a set in X. A point $x_L \in X$ is a **limit point** of A if $\forall \varepsilon > 0$, $(B_\varepsilon(x_L) \setminus \{x_l\}) \cap A \neq \emptyset$. ($A$ has points that are arbitrarily close to x_L)

Let (X, d) and (Y, ρ) be two metric spaces, $A \subset X$, $f : A \to Y$, \$x^0 = \$ limit point of A. A function f has a limit y^0 as x approaches x^0 if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $x \in A$ and $0 < d(x, x^0) < \delta$, then $\rho(f(X), y^0) < \varepsilon$ (written as $\lim_{x \to x^0} f(x) = y^0$).

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y$, \$x^0 = \$ limit point of X. Then $\lim_{x\to x^0} f(x) = y^0$ iff for any sequence $\{x_n\} \in X$ s.t. $x_n \to x^0$ and $x_n \neq x^0$, the sequence $\{f_n\}$ converges to y^0.

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y$, \$x^0 = \$ limit point of X. Then the limit of f as $x \to x^0$, when it exists, is unique.

Let (X, d) and (Y, ρ) be two metric spaces. A function $f : X \to Y$ is **continuous** at a point x^0 if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $d(x, x^0) < \delta$, then $\rho(f(x), f(x^0)) < \varepsilon$.

Continuity at x^0 requires $f(x^0)$ is defined and either x^0 is an isolated point of X ($\exists x^0$ s.t. $B_\varepsilon(x^0) = \{x^0\}$) or $\lim_{x\to x^0} f(x)$ exists and equals $f(x^0)$.

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y$. Then f is continuous at x^0 iff either (1) $f(x)$ is defined and either x^0 is an isolated point or x^0 is a limit point of X and $\lim_{x\to x^0} = f(X^0)$ or (2) for any sequence $\{x_n\}$ s.t. $x_n \to x^0$, the sequence $\{f(x_n)\}$ converges to $f(x^0)$.

A function f is continuous if it is continuous at every point of its domain.

$f^{-1}(A) = \{x \in X | f(x) \to A\}$

Let (X, d) and (Y, ρ) be two metric spaces, $f : x \to Y$. Then f is continuous iff for any closed set C in (Y, ρ), the set $f^{-1}(C)$ is closed in (X, d).

Let (X, d) and (Y, ρ) be two metric spaces, $f : x \to Y$. Then f is continuous iff for any open set C in (Y, ρ), the set $f^{-1}(C)$ is open in (X, d).

Let (X, d) and (Y, ρ) be two metric spaces. A function $f : X \to Y$ is uniformly continuous if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $d(x, x^0) < \delta$, then $\rho(f(x), f(x^0)) < \varepsilon$ (δ depends only on ε not on x^0).

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y, E \subset X$. Then f is Lipschitz on E if $\exists K > 0$ s.t. $\rho(f(x), f(y)) \leq Kd(x, y) \forall x, y \in E$.

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y, E \subset X$. Then f is locally Lipschitz on E if $\forall x \in E \exists \varepsilon > 0$ s.t. f is Lipschitz on $B_\varepsilon(x) \cap E$.

Lipschitz continuity $\implies$ uniform continuity $\implies$ continuity

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Let $X \subset \mathbb{R}$. Then $u \in \mathbb{R}$ is an upper bound for X if $x \leq u$ for all $x \in X$.

Let $X \subset \mathbb{R}$. Then $l \in \mathbb{R}$ is an lower bound for X if $x \geq l$ for all $x \in X$.

Suppose X is bounded above. The **supremum** of X, $\sup X$, is the smallest upper bound for X. That is, $\sup X$ satisfies $\sup X \geq x \forall x \in X$ and $\forall y < \sup X \exists x \in X$ s.t. $x > y$.

Suppose X is bounded below. The **infimum** of X, $\inf X$, is the largest lower bound for X. That is, $\inf X$ satisfies $\inf X \leq x \forall x \in X$ and $\forall y > \inf X \exists x \in X$ s.t. $x < y$.

Supremum Property: Every nonempty set of real numbers that is bounded above has a supremum. This supremum is a real number.

EVT: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains its maximum and minimum on $[a, b]$: $f(x_M) = \sup_{x \in [a,b]} f(x), f(x_m) = inf_{x \in [a,b]} f(x), x_M, x_m \in [a, b]$.

IVT: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any $\gamma \in [f(a), f(b)]$ there exists $c \in [a, b]$ s.t. $f(c) = \gamma$.

$f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing if $\forall x, y, x < y$ implies $f(x) < f(y)$.

Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Then one-sided limits $f(x^+) := \lim_{x \to x^+} f(y)$ and $f(x^-) := \lim_{x \to x^-} f(y)$ exist $\forall x \in (a, b)$. Moreover, $\sup\{f(s) | a < s < x\} = f(x^-) \leq f(x) \leq f(x^+) = \inf\{f(s) | x < s < b\}$.

A sequence $\{x_n\}$ in a metric space (X, d) is **Cauchy** if $\forall \varepsilon > 0 \exists N > 0$ s.t. if $m, n > N$, then $d(x_n, x_m) < \varepsilon$.

Every **convergent** sequence in a metric space is **Cauchy**.

A metric space (X, d) is **complete** if every Cauchy sequence contained in X converges to some point in X.

Euclidean space $(\mathbb{R}^m, d_E)$ is complete for any m.

If (X, d) is a complete metric space and $Y \subset X$, then (Y, d) is complete iff Y is closed.

A function $T : X \to X$ from a metric space to itself is called and **operator**.

An operator $T : X \to X$ is a **contraction of modulus** β if $\beta < 1$ and $d(T(x), T(y)) \leq \beta d(x, y) \forall x, y \in X$.

Every contraction is uniformly continuous.

A fixed point of an operator T is an element $x^* \in X$ s.t. $T(x^*) = x^*$.

Contraction Mapping Theorem: Let (X, d) be a nonempty complete metric space and $T : X \to X$ a contraction with modulus $\beta < 1$. Then T has a unique fixed point x^* and $\forall x_0 \in X$ the sequence $\{x_n\}$, where $x_n = T^n(x_0) = T(T(...T(x_0)))$ converges to x^*.

Continuous Dependence of the Fixed Point on Parameters: Let (X, d) and (Ω, ρ) be two metric spaces and $T : X \times \Omega \to X$. For each $\omega \in \Omega$, let $T_\omega : X \to X$ be defined by $T_\omega(x) = T(x, \omega)$. Suppose (X, d) is complete, T is continuous in ω, and $\exists \beta < 1$ s.t. T_ω is a contraction of modulus β for all $\omega \in \Omega$. Then the fixed point function $x^* : \Omega \to X$ defined by $x^*(\omega) = T_\omega(x^*(\omega))$ is continuous.

Blackwell's Sufficient Conditions: Let $B(X)$ be the set of all bounded functions from X to $\mathbb{R}$ with metric $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$. Let $T : B(X) \to B(X)$ satisfy monotonicity (if $f(x) \leq g(x) \forall x \in X$, then $(T(f))(x) \leq (T(g))(x)$ for all $x \in X$) and discounting ($\exists \beta \in (0, 1)$ s.t. for every $\alpha \geq 0$ and $x \in X$, $(T(f + a))(x) \leq (T(f))(x) + \beta\alpha)$, then T is a contraction with modulus β.

A collection of sets $\mathcal{U} = \{U_\lambda | \lambda \in \Lambda\}$ in a metric space (X, d) is an **open cover** of the set A if U_λ is open for all $\lambda \in \Lambda$ and $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$.

A set A in a metric space is **compact** if every open cover of A contains a **finite subcover** of A. That is, if $\{U_\lambda | \lambda \in \Lambda\}$ is an open cover of A, then $\exists n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_n \in \Lambda$ such that $A \subset U_{\lambda_1} \cup ... \cup U_{\lambda_n}$.

Any closed subset of a compact space is compact.

If A is a compact subset of a metric space, then A is closed and bounded.

Heine-Borel Theorem: If $A \subset \mathbb{R}^m$, then A is compact iff A is closed and bounded.

Closed interval $[a, b] = \{a \in \mathbb{R}^m | a_i \leq x_i \leq b_i, i = 1, ..., m\}$ is compact in $(\mathbb{R}^m, d_E)$ for any $a, b, \in \mathbb{R}^m$.

Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and C is a compact set in (X, d), then $f(C)$ is compact in (Y, ρ).

EVT: If C is a compact set in a metric space (X, d) and $f : C \to \mathbb{R}$ is continuous, then f is bounded on C and attains its maximum and minimum.

Let (X, d) and (Y, ρ) be metric spaces, $C \subset X$ compact, $f : C \to Y$ continuous. Then f is uniformly continuous on C.

A **vector space** V is a collection of objects called vectors, which may be added together and multiplied by real numbers, called scalars, satisfying $\forall x, y, z \in V$, $\forall \alpha, \beta \in \mathbb{R}$:

- $(x+y)+z = x+(y+z)$,
- $x+y = y+x$,
- $\exists \bar{0} \in V$ s.t. $x+\bar{0} = \bar{0}+x = x$,
- $\exists(-x) \in V$ s.t. $x+(-x) = \bar{0}$,
- $\alpha(x+y) = \alpha x + \alpha y$,
- $(\alpha+\beta)x = \alpha x + \beta x$,
- $(\alpha \cdot \beta)x = \alpha(\beta \cdot x)$,
- $1 \cdot x = x$.

Let V be a vector space. A linear combination of $x_1, ..., x_n \in V$ equals $y = \sum_{i=1}^n \alpha_i x_i$, where $\alpha_i \in \mathbb{R}$. α_i is called the coefficient of x_i in the linear combination.

Let W be a subset of V. A span of W is the set of all linear combinations of elements of W, span $W = \{\sum_{i=1} n\alpha_i x_i | n \in \mathbb{N}, \alpha_i \in \mathbb{R}, x_i \in W\}$. The set $W \subset V$ spans V if $V =$ span W.

A set $X \subset V$ is linearly dependent if $\exists x_1, ..., x_n \in X, \alpha_1, ..., \alpha_n \in \mathbb{R}$, s.t. $\sum_{i=1}^n \alpha_i^2 \neq 0$ and $\sum_{i=1}^n \alpha_i x_i \bar{0}$.

A set $X \subset V$ is linearly independent if $\nexists x_1, ..., x_n \in X, \alpha_1, ..., \alpha_n \in \mathbb{R}$, s.t. $\sum_{i=1}^n \alpha_i^2 \neq 0$ and $\sum_{i=1}^n \alpha_i x_i \bar{0}$ $(\alpha_1 = ... = \alpha_n = 0)$.

A basis of a vector space V is a linearly independent set of vectors in V that spans V.

Let $B = \{v_\lambda | \lambda \in \Lambda\}$ be a basis for V. Then every vector $x \in V$ has a unique representation as a linear combination of elements of B with finitely many nonzero coefficients.

Every vector space has a basis. Any two bases of a vector space V have the same cardinality.

If V is a vector space and $W \subset V$ is linearly independent, then there exists a linearly independent set B s.t. $W \subset B \subset$ span$B = V$.

Let V be a vector space. The dimension of V, denoted dim$V$, is the cardinality of any basis of V. If dim$V = n$ for some $n \in \mathbb{N}$ then V is finite-dimenional. Otherwise V is infinite-dimensional.

Suppose dim$V = n \in \mathbb{N}$. If $W \subset V$ and $|W| > n$, then W is linearly dependent.

Suppose dim$V = n$ and $W \subset V$, $|W| = n$. Then

- If W is linearly independent, then span$W = V$, so W is a basis of V.
- If span$W = V$, then W is linearly independent, so W is a basis of V.

Let X and Y be two vector spaces. We say that $T : X \to Y$ is a linear transformation if for all $x_1, x_2 \in X$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$.

$L(X, Y)$ is the set of all linear transformations from X to Y.

$L(X, Y)$ is a vector space.

If $R : X \to Y$ and $S : Y \to Z$ are linear transformations, then $S \circ R : X \to Z$ is a linear transformation.

Let $X \in L(X, Y)$. The image of T is Im$T := T(X) = \{T(x) | x \in X\}$, the kernal of T is ker$T := \{x \in X | T(x) = \bar{0}\}$, and the rank of T is rank$T :=$ dim(ImT).

If $T \in L(X, Y)$, then ImT and ker$T$ are vector subspaces of Y and X, respectively.

Let X be a finite-dimenional vector space and $T \in L(X, Y)$. Then dim $X =$ dim(ker$T$) + rank$T$ = dim(ker$T$) + dim(ImT).

$T \in L(X, Y)$ is invertible if there exists a function $S : Y \to X$ s.t. $S(T(x)) = x \forall x \in X$ and $T(S(y)) = y \forall y \in Y$. The transformation S is called the inverse of T and is denoted T^{-1}.

T is invertible means (1) T is one-to-one $(\forall x_1 \neq x_2, T(x_1) \neq T(x_2))$ and (2) T is onto $(\forall y \in Y \exists x \in X$ s.t. $T(x) = y)$.

If $T \in L(X, Y)$ is invertible, then $T^{-1} \in L(Y, X)$.

If $T \in L(X, Y)$ is one-to-one iff ker$T \equiv \{\bar{0}\}$.

Two vector spaces X and Y are isomorphic if there exists an invertible linear function from X to Y. A function with these properties is called an isomorphism.

Let X and Y be two vector spaces, and let $V = \{v_\lambda | \lambda \in \Lambda\}$ be a basis for X. Then a linear transformation $T : X \to Y$ is completely defined by its value on V, that is:

- Given any set $\{y_\lambda | \lambda \in \Lambda\} \subset Y$, $\exists T \in L(X, Y)$ s.t. $T(v_\lambda) = y_\lambda$ for all $\lambda \in \Lambda$.
- If $S, T \in L(X, Y)$ and $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$, then $S = T$.

Two vector spaces X and Y are isomorphic iff dim$X$ =dim$Y$.

$V = \{v_1, ..., v_n\} \in X$ is a basis of $X \implies \forall x \in X$ has a unique representation $x = \sum_{i=1}^n \alpha_i v_i$.

$$\text{crd}_V(x) = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}$$

$V = \{v_1, ..., v_n\} \in X$ is a basis of X and $W = \{w_1, ..., w_n\} \in Y$ is a basis of Y. $\forall y \in Y$ has a unique representation $y = \sum_{i=1}^m \alpha_i w_i$.

$$\text{mtx}_{W,V}(T) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & ... & \alpha_{mn} \end{bmatrix} \in M_{m \times n}$$

$U \subset X$ is a basis of X, $V \subset Y$ is a basis of Y, and $W \subset Z$ is a basis of Z. $S \in L(X, Y), T \in L(Y, Z)$. $\text{mtx}_{W,V}(T) \cdot \text{mtx}_{V,U}(S) = \text{mtx}_{W,U}(T \circ S)$.

dim $X = n, T \in L(X, X)$. $\text{mtx}_V(T) \equiv \text{mtx}_{V,V}(T)$. To change basis from V to W, $\text{mtx}_V(T) = P^{-1} \cdot \text{mtx}_W(T) \cdot P$ where $P = \text{mtx}_{W,V}(id)$.

$A, B \in M_{n \times n}$ are similar if $A = P^{-1}BP$ for some invertible matrix P.

If dim$X = n$, then

- If $T \in L(X, X)$, then any two matrix representations of T are similar.
- Two similar matrices represent the same linear transformation T, relative to suitable bases.