# ECON 703 - PS 7

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(1) Let $X \subset \mathbb{R}^n$ be a convex set, and $\lambda_1, ..., \lambda_k \geq 0$ with $\sum_{i=1}^k \lambda_i = 1$. Prove that if $x_1, ..., x_k \in X$, then $\sum_{i=1}^k \lambda_i x_i \in X$.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

(2) The sum $\sum_{i=1}^{k} \lambda_i x_i$ defined in Problem (1) is called a convex combination. The convex hull of a set $S$, denoted by $\mathrm{co}(S)$, is the intersection of all convex sets which contain $S$. Prove that the set of all convex combinations of the elements of $S$ is exactly $\mathrm{co}(S)$.

(3) For any set $X \subset \mathbb{R}^n$, let its closure be $\mathrm{cl}X = X \cup \{\text{all limit points of } X\}$. Show that the closure of a convex set is convex.

(4) The function $f : X \to \mathbb{R}$, where $X$ is a convex set in $\mathbb{R}^n$, is concave if $\forall \lambda \in [0, 1], x', x'' \in X f((1 - \lambda)x' + \lambda x'') \geq (1 - \lambda)f(x') + \lambda f(x'')$. Given a function $f : X \to \mathbb{R}$, its hypograph is the set of points $(y, x)$ lying on or below the graph of the function: $\text{hyp}f = \{(y, x)|x \in X, y \leq f(x)\}$. Show that the function $f$ is concave if and only if its hypograph is a convex set.

(5) Let $X$ and $Y$ be disjoint, closed, and convex sets in $\mathbb{R}^n$, one of which is compact. Show that there exists a hyperplane $H(p, \alpha)$ that strictly separates $X$ and $Y$.

(6) Call a vector $\pi \in \mathbb{R}^n$ a probability vector if $\sum_{i=1}^n \pi_i = 1$ and $\pi_i \geq 0$ for all $i = 1, ..., n$. Interpretation is that there are $n$ states of the world and $\pi_i$ is the probability that state $i$ occurs. Suppose that Alice and Bob each have a set of probability distributions ($\Pi_A$ and $\Pi_B$) which are nonempty, convex, and compact. They propose bids on each state of the world. A vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$, where $x_i$ denotes the net transfer Alice recieves from Bob in state $i$, is called a trade (Thus, $-x$ is the net transfer Bob receives in each state of the world.) A trade is agreeable if $\inf_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i x_i > 0$ and $\inf_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i (-x_i) > 0$. The above means that both Alice and Bob expect to strictly gain from the trade. Prove that there exists an agreeable trade iff there is no common prior (i.e., $\Pi_A \cap \Pi_B = \emptyset$).