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Final - Econ 703

(1) a. $T(x) = (b \cdot x)\, b$ where $b = e_1 + 2e_2 + 3e_3$

T is a linear operator if for all $x_1, x_2 \in \mathbb{R}^3$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$.

$$T(\alpha_1 x_1 + \alpha_2 x_2) = [b \cdot (\alpha_1 x_1 + \alpha_2 x_2)]\, b$$

$$= [b \cdot (\alpha_1 x_1) + b \cdot (\alpha_2 x_2)]\, b$$

$$= [(b \cdot (\alpha_1 x_1)) \cdot b + (b \cdot (\alpha_2 x_2)) \cdot b]$$

$$= \alpha_1 [b \cdot x_1] \cdot b + \alpha_2 [b \cdot x_2] \cdot b$$

$$= \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

Yes, T is a linear operator.

$$b = e_1 + 2e_2 + 3e_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$\forall\ x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, so $b \cdot x = (x_1 + 2x_2 + 3x_3)$ and

$$[b \cdot x]\, b = (x_1 + 2x_2 + 3x_3)\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}x_1 + 2x_2 + 3x_3\\ 2x_1 + 4x_2 + 6x_3\\ 3x_1 + 6x_2 + 9x_3\end{pmatrix}$$

Clear $A = \begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{pmatrix}$ represents T.

$$A \cdot x = \begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1 + 2x_2 + 3x_3\\ 2x_1 + 4x_2 + 6x_3\\ 3x_1 + 6x_2 + 9x_3\end{pmatrix}.$$

(1) b. $T(x) = (b \cdot x)x$

For generic $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $T(x) = (b \cdot x)x = (x_1 + 2x_2 + 3x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$= \begin{pmatrix} x_1^2 + 2x_1x_2 + 3x_1x_3 \\ x_1x_2 + 2x_2^2 + 3x_2x_3 \\ x_1x_3 + 2x_2x_3 + 3x_3^2 \end{pmatrix}$. T is clearly not a linear operator.

(2) $p > 0 \quad \theta > 0$ * $\max_{q} (pq - \theta c(q))$

Assume $c(\cdot)$ nonnegative, twice continuously diff, increasing and strictly convex $c' > 0, c'' > 0$)

a. FOC & Hessian

$$\frac{\partial \Pi}{\partial q} = 0 \Rightarrow p - \theta c'(q^*) = 0 \Rightarrow p = \theta c'(q^*).$$

↳ Price = Marginal cost.

$$\frac{\partial^2 \Pi}{\partial^2 q} = -\theta c''(q^*) < 0 \Rightarrow \text{local Maximum at } q^*$$

b. FOC $\Rightarrow p - \theta c'(q^*) = 0 \qquad$ Let $q^* = q(p, \theta)$

$$\Rightarrow p - \theta c'(q(p,\theta)) = 0$$

Using the implicit function theorem, $\frac{\partial}{\partial p}\left[p - \theta c'(q(p,\theta))\right] = \frac{\partial}{\partial p}[0]$

$$1 - \theta c'(q(p,\theta)) \frac{\partial q^*}{\partial p} = 0$$

$$\frac{1}{\theta c'(q(p,\theta))} = \frac{\partial q^*}{\partial p}$$

(+) (+)

$$\Rightarrow \frac{\partial q^*}{\partial p} > 0.$$

Apply the implicit function theorem again,

$$\frac{\partial}{\partial \theta}\left[p - \theta\, c'(q(p,\theta))\right] = \frac{\partial}{\partial \theta}\left[\boxed{0}\right]$$

$$-\left[c'(q(p,\theta)) + \theta\, c''(q(p,\theta))\frac{\partial q^*}{\partial \theta}\right] = 0$$

$$\frac{\partial q^*}{\partial \theta} = \frac{-\,c'(q(p,\theta))}{\theta\, c''(q(p,\theta))}$$

$+$ $+$ $+$

$$\Rightarrow \frac{\partial q^*}{\partial \theta} < 0.$$

③ $x^2 + y^2 + z^2 - 3xyz = 0$

$f(x,y,z) = xy^2z^3$

a. Find $\frac{\partial f}{\partial x}(1,1,1)$, where $z = z(x,y)$

$$f(x,y,z) = xy^2 z(x,y)^3$$

$$\frac{\partial f}{\partial x} = y^2 z(x,y)^3 + 3xy^2 z(x,y)^2 \frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x}\left[x^2 + y^2 + z^2 - 3xyz\right] = \frac{\partial}{\partial x}[0]$$

$$2x + 2z\frac{dz}{dx} - 3y\left[z + x\frac{dz}{dx}\right] = 0$$

$$2x + 2z\frac{dz}{dx} - 3yz - 3xy\frac{dz}{dx} = 0$$

$$2z\frac{dz}{dx} - 3xy\frac{dz}{dx} = 3yz - 2x$$

$$\frac{dz}{dx} = \frac{3yz - 2x}{2z - 3xy}$$

$$\frac{\partial f}{\partial x} = y^2 z(x,y)^3 + 3xy^2 z(x,y)^2 \left[\frac{3yz - 2x}{2z - 3xy}\right]$$

$$\frac{\partial f}{\partial x}(1,1,1) = 1^2 1^3 + 3(1)(1)^2(1)^2\left[\frac{3-2}{2-3}\right]$$

$$= 1 + 3\left(\frac{1}{-1}\right) = \boxed{-2}$$

3b) $\frac{\partial f}{\partial x}(1,1,1)$ where $y = y(x,z)$

$\cancel{\frac{\partial f}{\partial x}}$ $f(x, y(x,z), z) = x\,(y(x,z))^2 z^3$

$$\frac{\partial f}{\partial x} = z^3\left[(y(x,z))^2 + 2x\,y(x,z)\frac{dy}{dx}\right]$$

$$\frac{\partial}{\partial x}\left[x^2 + y^2 + z^2 - 3xyz\right] = \frac{d}{dy}[0]$$

$$2x + 2y\frac{dy}{dx} - 3z\left[y + x\frac{dy}{dx}\right] = 0$$

$$2x + 2y\frac{dy}{dx} - 3zy - 3xz\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3zy - 2x}{2y - 3xz}$$

$$\frac{\partial f}{\partial x} = z^3\left[(y(x,z))^2 + 2xy(x,z)\left[\frac{3zy - 2x}{2y - 3xz}\right]\right]$$

$$\frac{\partial f}{\partial x}(1,1,1) = 1^3\left[1^2 + 2(1)(1)\left(\frac{3-2}{2-3}\right)\right]$$

$$= 1 + 2(-1)$$

$$= \boxed{-1}$$

(3) c. The answers in (a) and (b) are different because in (a) y is effectly treated as a constant and z is a function of x based on Eq.(1). In (b), it's vice versa. The answers are different because y is squared in f and z is cubed in P.

④ $X = \{(x,y) \in \mathbb{R}^2 \mid x+y \leq 4,\ 2x-y \geq 1\ \ x-2y \leq -1\}$

(a) Notice that $X = W \cap Y \cap Z$ where

$W = \{(x,y) \in \mathbb{R}^2 \mid x+y \leq 4\}$,

$Y = \{(x,y) \in \mathbb{R}^2 \mid 2x-y \geq 1\}$,

and $Z = \{(x,y) \in \mathbb{R}^2 \mid x-2y \leq -1\}$.

W is convex: (x_1, y_1) $(x_2, y_2) \in W$
$t \in [0,1]$. ~~$tx_1 + (1-t)x_2 + ty_1$~~ Consider
$(tx_1 + (1-t)x_2,\ ty_1 + (1-t)y_2)$:

$$\begin{aligned}
&tx_1 + (1-t)x_2 + ty_1 + (1-t)y_2 \\
&= t(x_1+y_1) + (1-t)(x_2+y_2) \\
&\leq t(4) + (1-t)(4) \\
&= 4.
\end{aligned}$$

Y is convex: $(x_1, y_1), (x_2, y_2) \in Y$, $t \in [0,1]$
Consider $(tx_1 + (1-t)x_2,\ ty_1 + (1-t)y_2)$:

$$\begin{aligned}
&2[tx_1 + (1-t)x_2] - [ty_1 + (1-t)y_2] \\
&= t(2x_1 - y_1) + (1-t)(2x_2 - y_1) \\
&\geq t(1) + (1-t)(1) \\
&= 1
\end{aligned}$$

Z is convex: $(x_1, y_1), (x_2, y_2) \in Z$, $t \in [0,1]$
Consider $(tx_1 + (t-1)x_2,\ ty_1 + (1-t)y_2)$:

$$\begin{aligned}
&[tx_1 + (t-1)x_2] - 2[ty_1 + (1-t)y_2] \\
&= t[x_1 - 2y_1] + (1-t)(x_2 - 2y_2) \\
&\leq t(-1) + (1-t)(-1) \\
&= -1
\end{aligned}$$

X is convex bc the intersection of convex sets is convex.

(4) b. Construct hyperplane that strictly separates X and $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Consider $H((1,1), 1.5\}$.

Proof Define $Y = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Consider arbitrary $(x,y) \in Y$. We show that $x + y < 1.5$. Notice that the $\max\limits_{(x,y)\in Y} \{x+y\}$ occurs at $x^2 + y^2 = 1$ when both $x, y > 0$. Then $\max\{x+y\}$ occurs at $(\sqrt{2}/2, \sqrt{2}/2)$. Since $\sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2} < 1.5 \Rightarrow x + y < 1.5$ for all $(x,y) \in X$.

Now consider $(x,y) \in X$. We show that ~~[illegible]~~ $x + y > 1.5$. Notice that $x + y \geq \min\limits_{(x,y)\in X} (x+y)$. $\min\limits_{(x,y)\in X} (x+y)$ occurs at $(1,1)$.

$1 + 1 = 2 > 1.5$. Thus, $x + y > 1.5 \ \forall (x,y) \in X$.

Thus $H((1,1), 1.5)$ strictly separate X & Y.

$\blacksquare$

(5) Proof: First notice that the identity operator is invertible. Its inverse is itself:

$$I(I(x)) = I(x) = x \quad \forall x \in X.$$

Since $T^n(x) = \bar{0}$, T^n is not invertible because $\ker T^n = X$. Thus, T is not invertible. Suppose for sake of contradiction T is invertible. Then $T^{-1}(T^n(x)) = T^{-1}(\bar{0}) = \bar{0}$ because the $\ker T^{-1} = \{\bar{0}\}$. This a contradiction. Since X is finite dimensional, $\exists$ an isomorphism between X and $\mathbb{R}^m$. So T can be represented by a non-invertible matrix. The sum of a non-invertible matrix and an invertible matrix is invertible. Thus

e.i. I.

$T+I$ is an invertible linear transformation. ∎