ECON 709 - PS 2

Alex von Hafften*

9/14/2020

1. Suppose that $Y = X^3$ and $f_X(x) = 42x^5(1-x), x \in (0,1)$. Find the PDF of Y, and show that the PDF integrates to 1.

Notice that $Y = X^3$ is a monotone transformation, so we can use the following theorem from the lecture notes:

$$
\begin{aligned}
f_Y(y) &= \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|, y \in Y \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} 42(y^{1/3})^5(1-y^{1/3})|(1/3)y^{-2/3}|, y \in (0,1) \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} 14y(1-y^{1/3}), y \in (0,1) \\ 0, \text{ otherwise} \end{cases}
\end{aligned}
$$

where $g^{-1}(y) = y^{1/3}$ and $Y = \{0^3, 1^3\} = \{0, 1\}$.

$f_Y(y)$ integrates to 1:

$$
\begin{aligned}
\int_0^1 14t(1-t^{1/3})dt &= 14\left[y^2/2 - \frac{y^{7/3}}{7/3}\right]_0^1 \\
&= 14\left[\frac{1}{2} - \frac{3}{7}\right] \\
&= 1
\end{aligned}
$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

2. For the following CDF and PDF, show that f_X is the density function of F_X as long as $a \geq 0$. That is, show that for all $x \in [0,1], F_X(x) = \int_0^x f_X(t)dt$.

$$F_X(x) = \begin{cases} 1.2x, x \in [0, 0.5) \\ 0.2 + 0.8x, x \in [0.5, 1] \end{cases}$$

$$f_X(x) = \begin{cases} 1.2, x \in [0, 0.5) \\ a, x = 0.5 \\ 0.8, x \in (0.5, 1] \end{cases}$$

Case 1: $x < 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^x 1.2dt \\ &= 1.2x \\ &= F_X(x) \end{aligned}$$

Case 2: $x = 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} adt \\ &= 1.2(0.5) + 0 \\ &= 0.6 \\ &= 0.2 + 0.8(0.5) \\ &= F_X(0.5) \end{aligned}$$

Case 3: $x > 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} adt + \int_{0.5}^{x} 0.8dt \\ &= 1.2(0.5) + 0 + 0.8x - 0.8(0.5) \\ &= 0.6 + 0.8x - 0.4 \\ &= 0.2 + 0.8x \\ &= F_X(x) \end{aligned}$$

3. Let X have PDF $f_X(x) = \frac{2}{9}(x+1), x \in [-1, 2]$. Find the PDF of $Y = X^2$.

For $x \in [-1, 2]$

$$\begin{aligned}
F_X(x) &= \int_{-1}^{x} \frac{2}{9}(t+1)dt \\
&= \frac{2}{9}\left[\frac{t^2}{2} + t\right]_{-1}^{x} \\
&= \frac{2}{9}\left[\frac{x^2}{2} + x - \left(\frac{1}{2} - 1\right)\right] \\
&= \frac{x^2}{9} + \frac{2x}{9} + \frac{1}{9}
\end{aligned}$$

Thus,

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{x^2}{9} + \frac{2x}{9} + \frac{1}{9}, & x \in [-1, 2] \\ 1, & x > 2 \end{cases}$$

Consider $Y = X^2$. First, notice that $y \in [0, 4]$. I consider two cases $y \in [0, 1]$ and $y \in (1, 4]$

Case 1: $y \in [0, 1]$

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y}) \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\
&= \left[\frac{y}{9} + \frac{2\sqrt{y}}{9} + \frac{1}{9}\right] - \left[\frac{y}{9} - \frac{2\sqrt{y}}{9} + \frac{1}{9}\right] \\
&= \frac{4\sqrt{y}}{9}
\end{aligned}$$

Case 2: $y \in (1, 4]$

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(X^2 \leq y) \\
&= P(X \leq \sqrt{y}) \\
&= F_X(\sqrt{y}) \\
&= \frac{y}{9} + \frac{2\sqrt{y}}{9} + \frac{1}{9}
\end{aligned}$$

Thus, the CDF and PDF of Y is:

$$F_X(x) = \begin{cases} 0, & y < 0 \\ \frac{4\sqrt{y}}{9}, & y \in [0,1] \\ \frac{y}{9} + \frac{2\sqrt{y}}{9} + \frac{1}{9}, & y \in (1,4] \\ 1, & y > 4 \end{cases}$$

$$f_X(x) = \begin{cases} \frac{2}{0\sqrt{y}}, & y \in [0,1] \\ \frac{1}{9} + \frac{1}{9\sqrt{y}}, & y \in (1,4] \\ 0, & \text{otherwise.} \end{cases}$$

4. A median of a distribution is a value m such that $P(X \le m) \ge 1/2$ and $P(X \ge m) \ge 1/2$. Find the median of the distribution $f(x) = \frac{1}{\pi(1+x^2)}, x \in \mathbb{R}$.

The CDF of X is

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+t^2} dt \\ &= \frac{1}{\pi} \left[\tan^{-1}(t) \right]_{-\infty}^{x} \\ &= \frac{1}{\pi} \left[\tan^{-1}(x) - \lim_{t \to -\infty} \tan^{-1}(t) \right] \\ &= \frac{1}{\pi} \left[\tan^{-1}(x) - \frac{\pi}{2} \right] \end{aligned}$$

Now, notice that the distribution is symmetric around 0, so we will consider $m = 0$

$$\begin{aligned} P(X \le 0) &= F(0) \\ &= \frac{1}{\pi} \left[\tan^{-1}(0) - \frac{\pi}{2} \right] \\ &= \frac{1}{\pi} \left[0 - \frac{\pi}{2} \right] \\ &= \frac{1}{2} \\ P(X \ge 0) &= 1 - P(X \le 0) \\ &= 1 - F(0) \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Thus, $m = 0$.

5. Show that if X is a continuous random variable, then $\min_a E|X-a| = E|x-m|$, where m is the median of X.

$$
\begin{aligned}
E|X-a| &= \int_{-\infty}^{\infty} |t-a| f(t)dt \\
&= \int_{-\infty}^{a} (a-t)f(t)dt + \int_{a}^{\infty} (t-a)f(t)dt \\
&= \int_{-\infty}^{a} af(t)dt - \int_{-\infty}^{a} tf(t)dt + \int_{a}^{\infty} tf(t)dt - \int_{a}^{\infty} af(t)dt \\
&= a\Bigg(\int_{-\infty}^{a} f(t)dt - \int_{a}^{\infty} f(t)dt\Bigg) - \Bigg(\int_{-\infty}^{a} tf(t)dt - \int_{a}^{\infty} tf(t)dt\Bigg) \\
&= a\Bigg(F(a) - (1-F(a))\Bigg) - \Bigg(\int_{-\infty}^{a} tf(t)dt - \int_{a}^{\infty} tf(t)dt\Bigg) \\
&= a\Bigg(2F(a) - 1\Bigg) - \Bigg(\int_{-\infty}^{a} tf(t)dt - \int_{a}^{\infty} tf(t)dt\Bigg)
\end{aligned}
$$

Notice that this expression for $E|X-a|$ is differentiable by a. Consider first the second half:

$$
\frac{d}{da}\Bigg(\int_{-\infty}^{a} tf(t)dt - \int_{a}^{\infty} tf(t)dt\Bigg) = af(a) - af(a) = 0
$$

Then the full expression:

$$
\begin{aligned}
\frac{d}{da}E|X-a| &= \frac{d}{da}a\Bigg(2F(a)-1\Bigg) + 0 \\
&= \Bigg(2F(a)-1\Bigg)
\end{aligned}
$$

Setting the derivative equal to zero:

$$
\begin{aligned}
2F(a) - 1 &= 0 \\
F(a) &= \frac{1}{2}
\end{aligned}
$$

Thus, $a = m$ where $P(X \geq m) = P(X \leq m) = F(m) = \frac{1}{2}$.

6. Let μ_n denote the nth central moment of a random variable X. Two quantities of interest in addition to the mean and variance are $\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}}$ and $\alpha_4 = \frac{\mu_4}{\mu_2^2}$. The value α_3 is called the skewness and α_4 is called the kurtosis. The skewness measures the lack of symmetry in the density function. The kurtosis, although harder to interpret, measures the peakedness or flatness of the density function.

(a) Show that if a density function is symmetric about a point a, then $\alpha_3 = 0$.

Proof: Define $Y = X - a$. Y has a symmetric distribution about zero $\implies E[Y^3] = E[(-Y)^3] = -E[Y^3] = 0$ and $E[Y] = 0$. Thus, the 3rd central moment of X is zero:

$$\begin{aligned}
\mu_3 &= E[(X - E(X))^3] \\
&= E[(Y + a - E(Y + a))^3] \\
&= E[(Y + a - E(Y) - a)^3] \\
&= E[(Y - E(Y))^3] \\
&= E[Y^3] \\
&= 0
\end{aligned}$$

Therefore, the skewness of X is zero: $\alpha_3 = \frac{0}{\mu_2^{3/2}} = 0$. $\square$

(b) Calculate α_3 for $f(x) = \exp(-x), x \geq 0$, a density function that is skewed to the right.

$$\begin{aligned}
M_X(t) &= E[e^{tx}] \\
&= \int_0^\infty e^{tx} e^{-x} dx \\
&= \int_0^\infty e^{-x(1-t)} dx \\
&= \left[\frac{e^{-x(1-t)}}{1-t}(-1) \right]_0^\infty \\
&= \frac{0}{1-t}(-1) - \frac{1}{1-t}(-1) \\
&= (1-t)^{-1}
\end{aligned}$$

where $0 \leq t < 1$.

$$\begin{aligned}
M_X^{(1)}(t) &= (-1)(1-t)^{-2}(-1) \\
&= (1-t)^{-2} \\
M_X^{(2)}(t) &= (-2)(1-t)^{-3}(-1) \\
&= 2(1-t)^{-3} \\
M_X^{(3)}(t) &= (-3)2(1-t)^{-4}(-1) \\
&= 6(1-t)^{-4}
\end{aligned}$$

The first, second, and third moments of X are:

$$\begin{aligned} E[X] &= M_X^{(1)}(0) = 1 \\ E[X^2] &= M_X^{(2)}(0) = 2 \\ E[X^3] &= M_X^{(3)}(0) = 6 \end{aligned}$$

The second and third central moments of X are:

$$\begin{aligned} \mu_2 &= E[(X - E(X))^2] \\ &= E(X^2) - E(X)^2 \\ &= 2 - 1 \\ &= 1 \\ \mu_3 &= E[(X - E(X))^3] \\ &= E[(X - E(X))(X^2 - 2XE(X) + E(X)^2)] \\ &= E[X^3 - 2X^2E(X) + XE(X)^2 - E(X)X^2 + 2XE(X)^2 - E(X)^3] \\ &= E(X^3) - 2E(X^2)E(X) + E(X)E(X)^2 - E(X)E(X^2) + 2E(X)E(X)^2 - E(X)^3 \\ &= (6) - 2(2)(1) + (1)(1)^2 - (1)(2) + 2(1)(1)^2 - (1)^3 \\ &= 6 - 4 + 1 - 2 + 2 - 1 \\ &= 2 \end{aligned}$$

Skewness of X is

$$\begin{aligned} \alpha_3 &= \frac{\mu_3}{\mu_2^{3/2}} \\ &= \frac{2}{1^{3/2}} \\ &= 2 \end{aligned}$$

(c) Calculate α_4 for the following density functions and comment on the peakedness of each:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2), x \in \mathbb{R} \tag{1}$$
$$f(x) = 1/2, x \in (-1, 1) \tag{2}$$
$$f(x) = \frac{1}{2} \exp(-|x|), x \in \mathbb{R} \tag{3}$$

For (1), notice that $X \sim N(0, 1)$, so $M_X(t) = \exp\left(\frac{1}{2}t^2\right)$

$$\begin{aligned}
M_X^{(1)}(t) &= \exp\left(\frac{1}{2}t^2\right)t \\
M_X^{(2)}(t) &= \exp\left(\frac{1}{2}t^2\right) + \exp\left(\frac{1}{2}t^2\right)t^2 \\
M_X^{(3)}(t) &= \exp\left(\frac{1}{2}t^2\right)t + \exp\left(\frac{1}{2}t^2\right)t^3 + 2\exp\left(\frac{1}{2}t^2\right)t \\
&= 3\exp\left(\frac{1}{2}t^2\right)t + \exp\left(\frac{1}{2}t^2\right)t^3 \\
M_X^{(4)}(t) &= 3\exp\left(\frac{1}{2}t^2\right) + 3\exp\left(\frac{1}{2}t^2\right)t^2 + \exp\left(\frac{1}{2}t^2\right)t^4 + 3\exp\left(\frac{1}{2}t^2\right)t^2 \\
&= \exp\left(\frac{1}{2}t^2\right)t^4 + 6\exp\left(\frac{1}{2}t^2\right)t^2 + 3\exp\left(\frac{1}{2}t^2\right)
\end{aligned}$$

$$\begin{aligned}
E[X] &= M_X^{(1)}(0) = 0 \\
E[X^2] &= M_X^{(2)}(0) = 1 \\
E[X^3] &= M_X^{(3)}(0) = 0 \\
E[X^4] &= M_X^{(4)}(0) = 3
\end{aligned}$$

$E(X) = 0 \implies \mu_2 = E((X - E(x))^4) = E(X^4)$ and $\mu_4 = E((X - E(x))^4) = E(X^4)$.

$$\begin{aligned}
\alpha_4 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{3}{1} \\
&= 3
\end{aligned}$$

For (2), notice that $X \sim \text{Uniform}(-1, 1)$. $E(X) = 0$ by symmetry around 0.

$$\begin{aligned}
E[X^2] &= \int_{-1}^{1} \frac{1}{2} x^2 dx \\
&= \left[\frac{x^3}{6} \right]_{-1}^{1} \\
&= \frac{1}{6} - \frac{-1}{6} \\
&= \frac{1}{3} \\
E[X^4] &= \int_{-1}^{1} \frac{1}{2} x^4 dx \\
&= \left[\frac{x^5}{10} \right]_{-1}^{1} \\
&= \frac{1}{10} - \frac{-1}{10} \\
&= \frac{1}{5}
\end{aligned}$$

$E(X) = 0 \implies \mu_2 = E((X - E(x))^4) = E(X^4)$ and $\mu_4 = E((X - E(x))^4) = E(X^4)$.

$$\begin{aligned}
\alpha_4 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{1/5}{(1/3)^2} \\
&= \frac{9}{5}
\end{aligned}$$

Compared to the standard normal distribution, the uniform distribution is less peaked. This is backed up by the standard normal distribution having a higher kurtosis than the uniform distribution.

For (3), $E(X) = 0$ by symmetry. Furthermore, notice for even moments:

$$\begin{aligned}
E(X^{2k}) &= \int_{-\infty}^{\infty} x^{2k} \frac{1}{2} \exp(-|x|) dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} x^{2k} \exp(-|x|) dx \\
&= \int_{0}^{\infty} x^{2k} \exp(-x) dx
\end{aligned}$$

This expression matches the functional form for finding moments of an exponential distribution with $\lambda = 1$. Thus, we can use the moment generating function from 6(b):

$$\begin{aligned}
M_X^{(2)}(t) &= 2(1-t)^{-3} \\
M_X^{(4)}(t) &= 6(1-t)^{-5}(-4)(-1) \\
&= 24(1-t)^{-5}
\end{aligned}$$

Thus $\mu_2 = E((X - E(X))^2) = E(X^2) = M_X^{(2)}(0) = 2$ and $\mu_4 = E((X - E(X))^4) = E(X^4) = M_X^{(4)}(0) = 24$.

$$\begin{aligned}
\alpha_4 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{24}{(2)^2} \\
&= \frac{24}{4} \\
&= 6
\end{aligned}$$

Thus, this distribution is more peaked than both the standard normal distribution and uniform distribution.