ECON 899A - Problem Set 7

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1. *Derive the following asymptotic moments associated with $m_3(x)$: mean, variance, first order autocorrelation. Furthermore, compute $\nabla_b g(b_0)$. Which moments are informative for estimating b?*

With $|\rho_0| < 1$, the stochastic process $\{x_t\}$ is stationary, so $E[x_t] = E[x_{t-1}]$ and $Var[x_t] = Var[x_{t-1}]$:[1]

$$E[x_t] = E[\rho_0 x_{t-1} + \varepsilon_t] = \rho_0 E[x_{t-1}] = \rho_0 E[x_t]$$

$$\implies E[x_t] = 0$$

$$Var[x_t] = Var[\rho_0 x_{t-1} + \varepsilon_t] = \rho_0^2 Var[x_{t-1}] + \sigma_0^2 = \rho_0^2 Var[x_t] + \sigma_0^2$$

$$\implies Var[x_t] = \frac{\sigma_0^2}{1 - \rho_0^2}$$

$$Cov[x_t, x_{t-1}] = Cov[\rho_0 x_{t-1} + \varepsilon_t, x_{t-1}] = \rho_0 Cov[x_{t-1}, x_{t-1}] + Cov[\varepsilon_t, x_{t-1}] = \rho_0 Var[x_t] = \frac{\sigma_0^2 \rho_0}{1 - \rho_0^2}$$

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[1] Alternatively, we can write x_t in terms of x_0, $\{\varepsilon_i\}_{i=1}^t$, and ρ_0:

$$\begin{aligned}
x_t &= \rho_0 x_{t-1} + \varepsilon_t \\
&= \rho_0(\rho_0 x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \rho_0^2 x_{t-2} + \varepsilon_t + \rho_0 \varepsilon_{t-1} \\
&= \rho_0^2(\rho_0 x_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + \rho_0 \varepsilon_{t-1} \\
&= \rho_0^3 x_{t-3} + \varepsilon_t + \rho_0 \varepsilon_{t-1} + \rho_0^2 \varepsilon_{t-2} \\
&= \rho_0^t x_0 + \sum_{i=1}^t \rho_0^{t-i} \varepsilon_i \\
&= \sum_{i=1}^t \rho_0^{t-i} \varepsilon_i
\end{aligned}$$

$$E[x_t] = E\left[\sum_{i=1}^t \rho_0^{t-i} \varepsilon_i\right] = \sum_{i=1}^t \rho_0^{t-i} E[\varepsilon_i] = 0$$

Thus, the asymptotic moments associated with $m_3(x)$ are:

$$\mu(x) = E[m_3(x)] = \begin{bmatrix} 0 \\ \frac{\sigma_0^2}{1-\rho_0^2} \\ \frac{\sigma_0^2 \rho_0}{1-\rho_0^2} \end{bmatrix}$$

To calculate the Jacobian, we compute the derivative of the moment conditions with respect to each parameter:

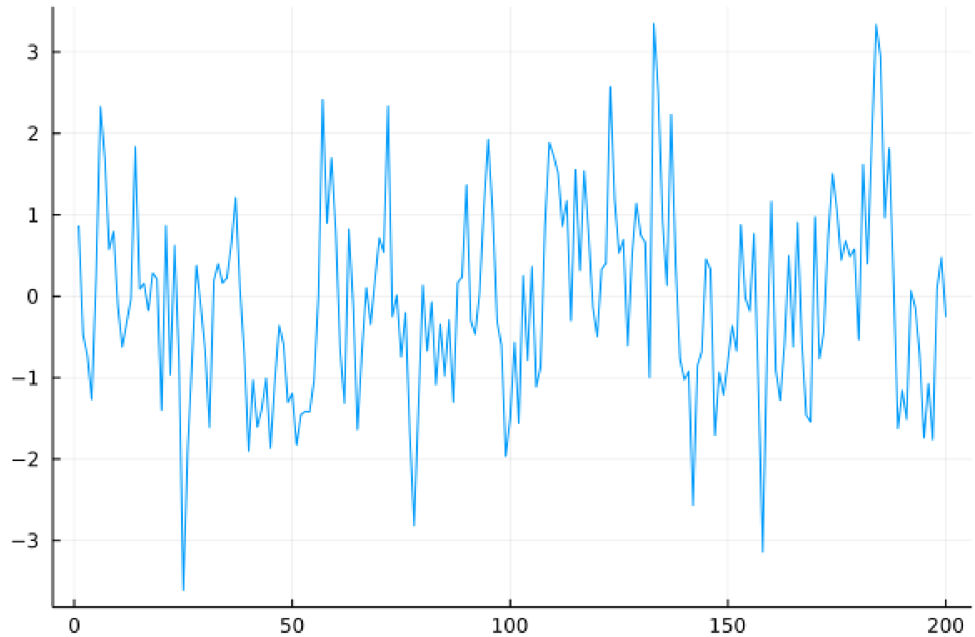
$$\begin{aligned}
\frac{\partial}{\partial \rho}\left(\frac{\sigma^2}{1-\rho^2}\right) &= \frac{\sigma^2(-2\rho) - (1-\rho^2)(0)}{(1-\rho^2)^2} \\
&= \frac{-2\rho\sigma^2}{(1-\rho^2)^2} \\
\frac{\partial}{\partial \sigma}\left(\frac{\sigma^2}{1-\rho^2}\right) &= \frac{2\sigma}{1-\rho^2} \\
\frac{\partial}{\partial \rho}\left(\frac{\sigma^2\rho}{1-\rho^2}\right) &= \frac{\sigma^2\rho(-2\rho) - (1-\rho^2)\sigma^2}{(1-\rho^2)^2} \\
&= \frac{-\sigma^2(1+\rho^2)}{(1-\rho^2)^2} \\
\frac{\partial}{\partial \sigma}\left(\frac{\sigma^2\rho}{1-\rho^2}\right) &= \frac{2\sigma\rho}{1-\rho^2}
\end{aligned}$$

Each cell of the Jacobian is the negative of the derivative of the moment condition:

$$\implies \nabla_b g(b_0) = \begin{bmatrix} 0 & 0 \\ \frac{2\rho_0\sigma_0^2}{(1-\rho_0^2)^2} & \frac{-2\sigma_0}{1-\rho_0^2} \\ \frac{\sigma_0^2(1+\rho_0^2)}{(1-\rho_0^2)^2} & \frac{-2\sigma_0\rho_0}{1-\rho_0^2} \end{bmatrix}$$

Variance and first order autocorrelation are informative for estimating b.

2. *Simulate a series of "true" data of length* $T = 200$ *using (1). We will use this to compute* $M_T(x)$.



$$M_T(x) = \begin{bmatrix} -0.0764 \\ 1.3953 \\ 0.6916 \end{bmatrix}$$

3. *Set* $H = 10$ *and simulate* H *vectors of length* $T = 200$ *random variables* e_t *from* $N(0,1)$. *We will use this to compute* $M_{TH}(y(b))$. *Store these vectors. You will use the same vector of random variables throughout the entire exercise. Since this exercise requires you to estimate* σ^2, *you want to change the variance of* e_t *during the estimation. You can simply use* σe_t *when the variance is* σ^2.

4-6. *Different moments. Given what you found in part (1), do you think there will be a problem? In general we would not know whether this case would be a problem, so hopefully the standard error of the estimate of* b *as well as the* J *test will tell us something.*

(a) *Set* $W = I$ *and graph in three dimensions, the objective function (3) over* $\rho \in [0.35, 0.65]$ *and* $\sigma \in [0.8, 1.2]$. *Obtain an estimate of* b *by using* $W = I$ *in (4) using fminsearch. Report* $\hat{b}^1_{TH}$.

(b) *Set* $i(T) = 4$. *Obtain an estimate of* W^*. *Using* $\hat{W}^*{}_{TH} = \hat{S}^{-1}_{TH}$ *in (4), obtain an estimate of* $\hat{b}^2_{TH}$. *Report* $\hat{b}^2_{TH}$.

(c) *To obtain standard errors, compute numerically* $\nabla_b g_T(\hat{b}^2_{TH})$ *defined in (6). Report the values of* $\nabla_b g_T(\hat{b}^2_{TH})$. *Next, obtain the* $\ell \times \ell$ *variance-covariance matrix of* $\hat{b}^2_{TH}$ *as in (7). Finally, what are the standard errors defined in (8)? How can we use the information on* $\nabla_b g_T(\hat{b}^2_{TH})$ *to think about local identification?*

(d) *Since we are in the just identified case, the* J *test should be zero (on a computer this may be not be exact). However, given the identification issues in this particular case where we use mean and variance, the* J *test may not be zero. Compute the value of the* J *test:*

$$T\frac{H}{1+H} \times J_{TH}(\hat{b}^2_{TH}) \to \chi^2$$

noting that in this just identified case $n - \ell = 0$ *degrees of freedom recognizing that there really is not distribution.*

- From part (1), I found that in expectation the mean is zero, so the problem is that the mean is not informative for estimating b. Both the variance and the first order covariance were non-zero, so they are likely informative for estimating b. Thus, I think there will be a problem with (4), but not (5) and (6)
- I plot the objective function for both the identity weighting matrix and the estimated optimal weighting matrices for each case on the following pages.
- Table 1 shows the first stage estimates, second stage estimates, and standard errors for ρ and σ.

Table 1: Estimates and Standard Errors

moments	rho_0	rho_hat_1	rho_hat_2	rho_se	sigma_0	sigma_hat_1	sigma_hat_2	sigma_se
Mean and Variance	0.5	0.532	0.534	1.895	1	1.011	1.010	0.932
Variance and Covariance	0.5	0.488	0.488	0.044	1	1.043	1.043	0.210
Mean, Variance and Covariance	0.5	0.488	0.488	0.084	1	1.043	1.042	0.099

- To obtain the standard errors, I computed the following Jacobians:

Table 2: Jacobian (Mean and Variance)

d/(d rho)	d/(d sigma)
0.25	0.1875
-2.00	-4.0000

Table 3: Jacobian (Variance and Covariance)

d/(d rho)	d/(d sigma)
-2.0	-2.0
-3.5	-1.5

Table 4: Jacobian (Mean, Variance, and Covariance)

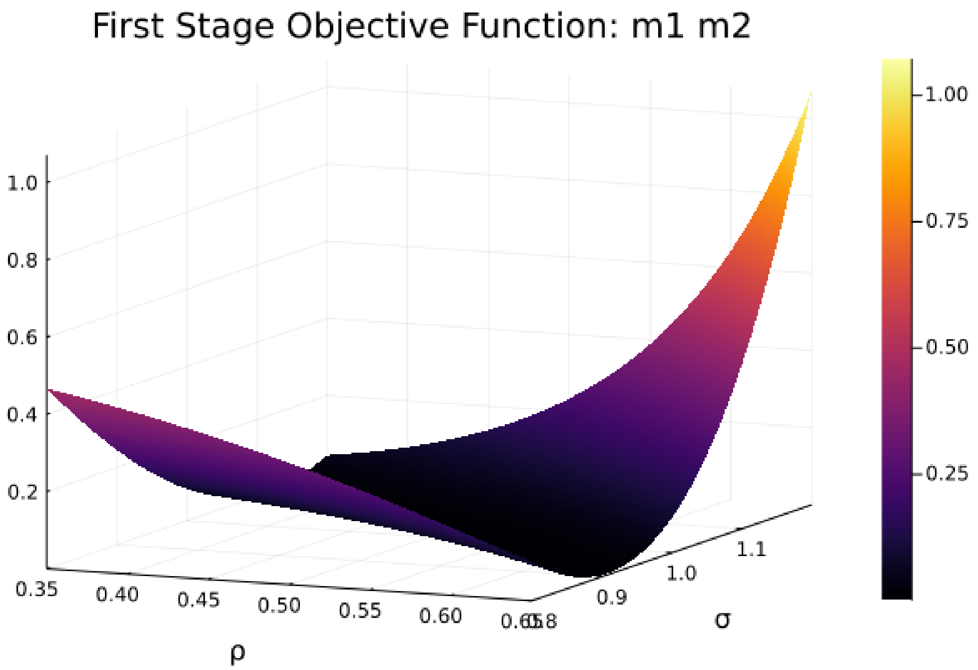
d/(d rho)	d/(d sigma)
0.125	0.125
-2.000	-3.000
-2.500	-1.500

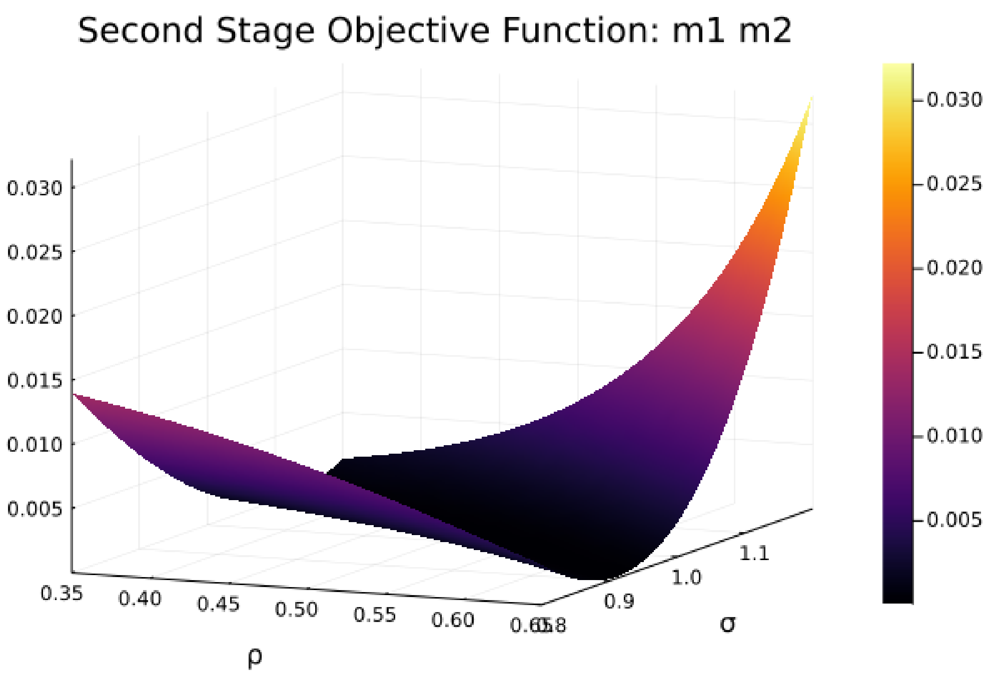
- I found very low j-test statistics and p-values.

Table 5: J-Test Statistics and P-Values

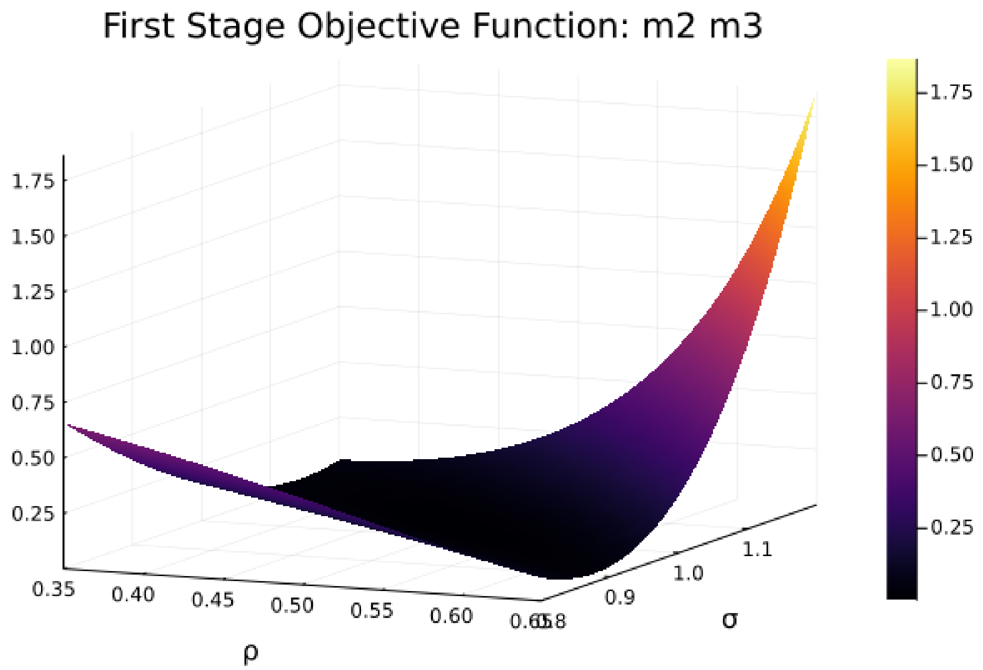
moments	j_test_stat	j_test_p_value
Mean and Variance	0.00000	0.00089
Variance and Covariance	0.00000	0.00022
Mean, Variance and Covariance	0.00021	0.01146

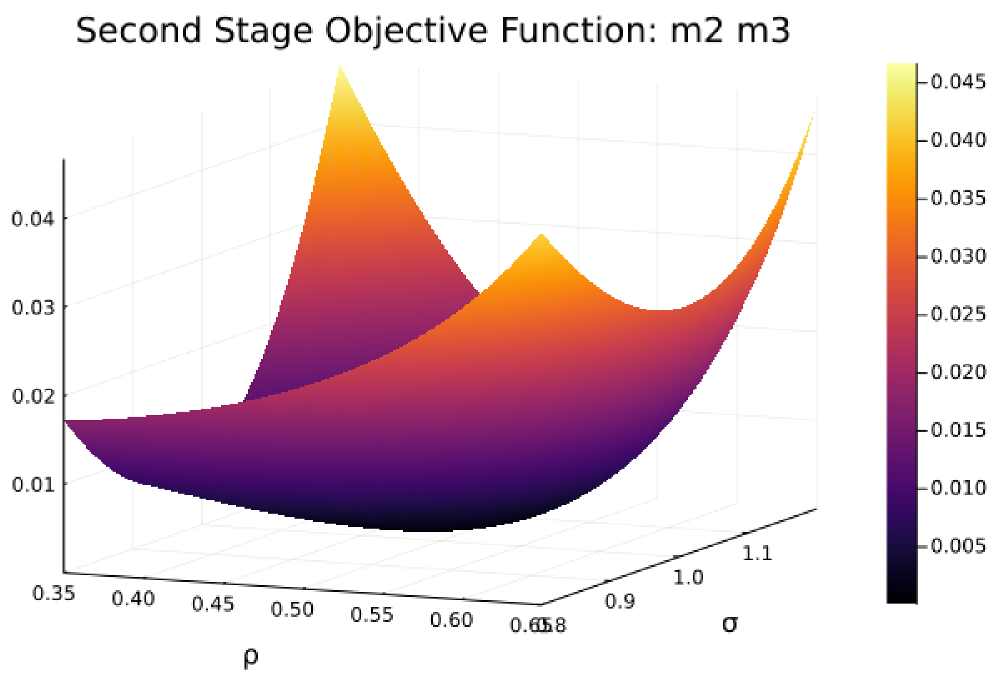
4. The just identified case where m_2 uses mean and variance.



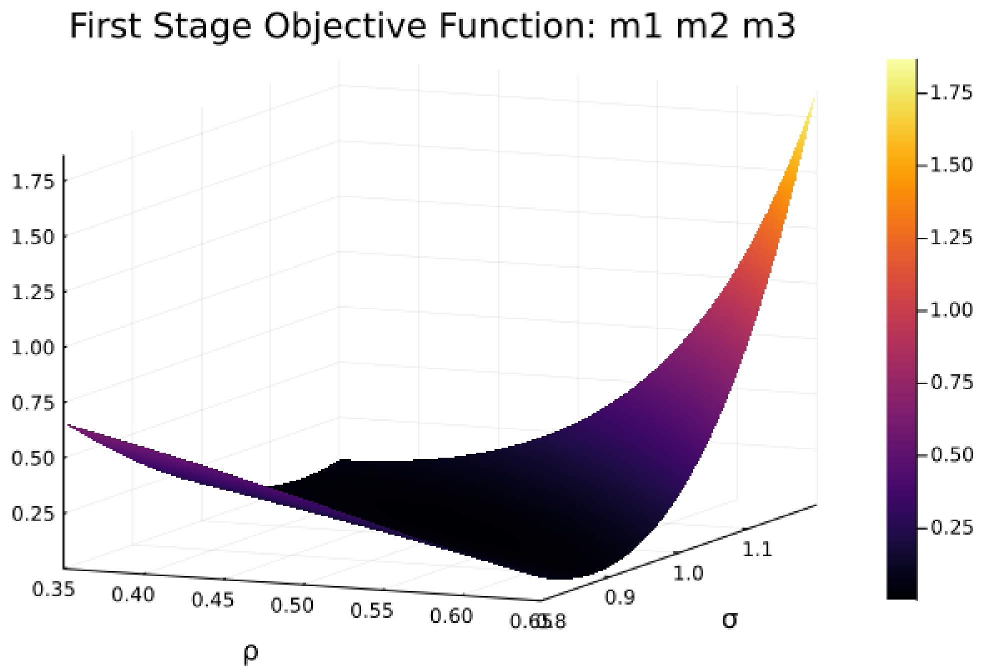


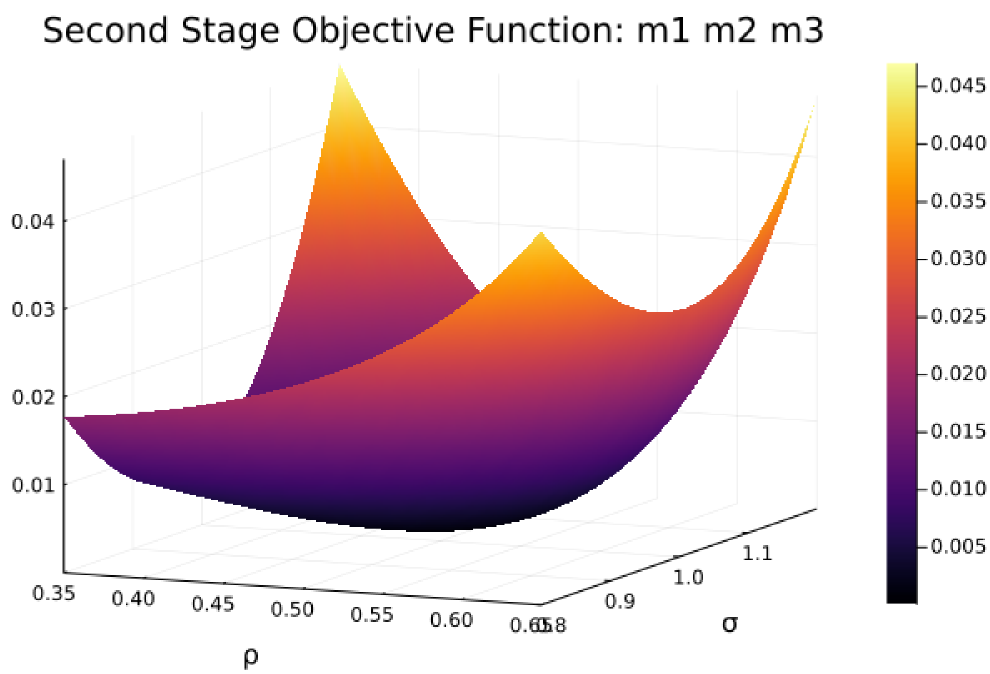
5. The just identified case where m_2 uses the variance and autocorrelation.



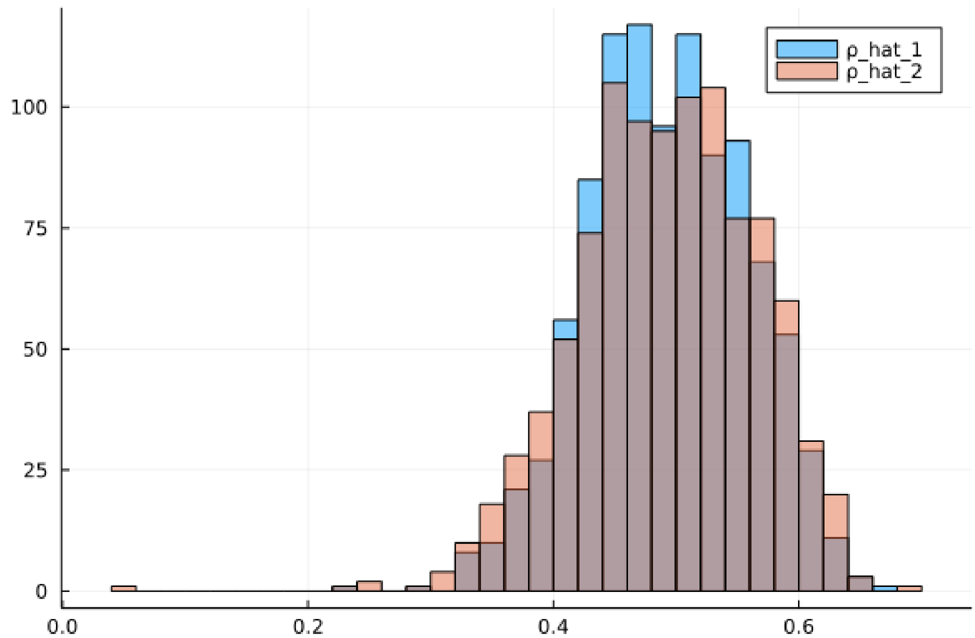


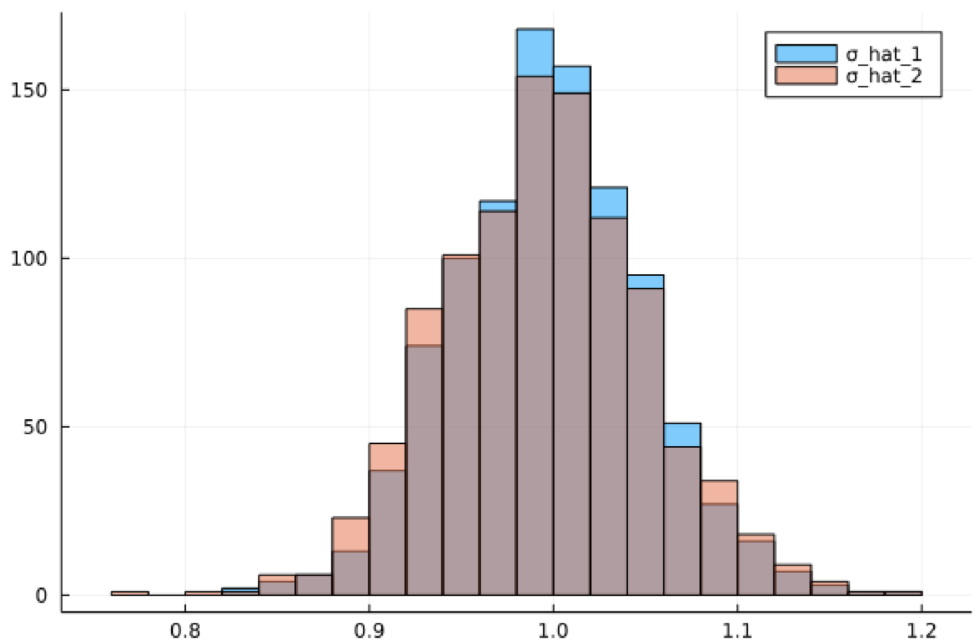
6. The overidentified case where m_3 uses the mean, variance and autocorrelation.





6. (e) Bootstrap the the finite sample distribution of the estimators by repeatedly drawing ε_t and e_t^h from $N(0,1)$ for $t = 1, ..., T$ and $h = 1, ..., H$.





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