ECON 710A - Problem Set 5

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1. Suppose that $\{\varepsilon_t\}_{t=0}^T$ are iid random variables with mean zero, variance σ^2 and $E[\varepsilon_t^8] < \infty$. Let $U_t = \varepsilon_t \varepsilon_{t-1}$, $W_t = \varepsilon_t \varepsilon_0$, and $V_t = \varepsilon_t^2 \varepsilon_{t-1}$ where $t = 1, ..., T$.

(i) Show that $\{U_t\}_{t=1}^T$, $\{W_t\}_{t=1}^T$, and $\{V_t\}_{t=1}^T$ are covariance stationary.

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(ii) Argue that the following three sample means $\bar{U}$, $\bar{W}$, $\bar{V}$ converge in probability to their expectations.

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(iii) Determine whether the following three sample second moments $\hat{\gamma}_U(0) = \frac{1}{T}\sum_{t=1}^T U_t^2$, $\hat{\gamma}_W(0) = \frac{1}{T}\sum_{t=1}^T W_t^2$, and $\hat{\gamma}_V(0) = \frac{1}{T}\sum_{t=1}^T V_t^2$ converge in probability to their expectations.

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(iv) Determine whether the scaled sample means $\sqrt{T}\bar{U}$, $\sqrt{T}\bar{W}$, and $\sqrt{T}\bar{V}$ are asymptotically normal.

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*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

2. Consider a time series of length T from the model

$$Y_t = \alpha_0 + t\beta_0 + X_t\delta_0 + Y_{t-1}\rho_1 + U_t$$

where Y_0 and $\{U_t\}_{t=1}^T$ are iid $N(0,1)$, and

$$X_t = X_{t-1} \cdot 0.3 + V_t$$

where X_0 and $\{V_t\}_{t=1}^T$ are iid $N(0,1)$ and independent of Y_0 and $\{U_t\}_{t=1}^T$. We will let $\alpha_0 = \delta_0 = 100$, $\beta_0 = 1$ and consider all combinations of $T \in \{50, 150, 250\}$ and $\rho_1 \in \{0.7, 0.9, 0.95\}$.

(i) In a statistical software of your choice, generate data from (1), estimate the coefficients by OLS, and calculate heteroscedasticity robust two-sided 95% confidence intervals for α_0, δ_0, and ρ_1.

```
tees <- c(50, 150, 250)
rhos <- c(0.7, 0.9, 0.95)
alpha <- 100
delta <- 100
beta <- 1

results <- NULL

for (t in tees) {
  for (rho in rhos) {
    x_t <- rnorm(1)
    y_t <- rnorm(1)
    v_t <- rnorm(t)
    u_t <- rnorm(t)

    for (i in 1:t) x_t[i+1] <- 0.3 * x_t[i] + v_t[i]
    for (i in 1:t) y_t[i+1] <- alpha + i * beta + x_t[i+1] * delta + y_t[i] * rho + u_t[i]

    x <- cbind(rep(1, t),
               1:t,
               x_t[2:(t+1)],
               y_t[1:t])
    y <- y_t[2:(t+1)]

    ols <- solve(t(x) %*% x) %*% (t(x) %*% y)

    e_hat <- as.numeric(y - x %*% ols)
    omega <- crossprod(x * e_hat)
    varcov <- solve(t(x) %*% x) %*% omega %*% solve(t(x) %*% x)
    se_robust <- sqrt(diag(varcov))

    results <- tibble(t = t,
           rho = rho,
           name = c("alpha", "beta", "delta", "rho"),
           ols = as.numeric(ols),
           se = se_robust) %>%
      bind_rows(results)
  }
}
```

```r
results %>%
  mutate(upper_bound = ols + se * 1.96,
         lower_bound = ols - se * 1.96) %>%
  kable(digits = 3)
```

t	rho	name	ols	se	upper_bound	lower_bound
250	0.95	alpha	100.222	0.217	100.647	99.796
250	0.95	beta	1.003	0.002	1.008	0.998
250	0.95	delta	99.994	0.066	100.124	99.865
250	0.95	rho	0.950	0.000	0.950	0.950
250	0.90	alpha	99.875	0.180	100.228	99.523
250	0.90	beta	1.001	0.003	1.006	0.996
250	0.90	delta	99.977	0.055	100.085	99.870
250	0.90	rho	0.900	0.000	0.900	0.900
250	0.70	alpha	99.866	0.169	100.197	99.536
250	0.70	beta	1.000	0.001	1.002	0.998
250	0.70	delta	99.983	0.052	100.086	99.880
250	0.70	rho	0.700	0.000	0.701	0.700
150	0.95	alpha	100.320	0.229	100.768	99.871
150	0.95	beta	1.003	0.006	1.014	0.992
150	0.95	delta	100.035	0.076	100.184	99.886
150	0.95	rho	0.950	0.000	0.950	0.949
150	0.90	alpha	100.068	0.180	100.422	99.715
150	0.90	beta	1.003	0.005	1.012	0.994
150	0.90	delta	99.804	0.090	99.980	99.629
150	0.90	rho	0.900	0.000	0.900	0.899
150	0.70	alpha	99.903	0.194	100.283	99.522
150	0.70	beta	0.999	0.003	1.005	0.993
150	0.70	delta	99.997	0.084	100.162	99.833
150	0.70	rho	0.700	0.001	0.701	0.699
50	0.95	alpha	99.763	0.387	100.523	99.004
50	0.95	beta	1.026	0.047	1.118	0.934
50	0.95	delta	99.884	0.154	100.186	99.582
50	0.95	rho	0.950	0.001	0.951	0.948
50	0.90	alpha	100.036	0.270	100.566	99.506
50	0.90	beta	1.008	0.014	1.036	0.980
50	0.90	delta	100.055	0.151	100.351	99.758
50	0.90	rho	0.900	0.000	0.901	0.899
50	0.70	alpha	99.922	0.329	100.567	99.277
50	0.70	beta	0.995	0.014	1.022	0.967
50	0.70	delta	100.042	0.122	100.280	99.804
50	0.70	rho	0.701	0.001	0.703	0.699

(ii) Across 10000 simulated repetitions of the above, report the simulated mean of the point estimators for α_0, δ_0, and ρ_1 and the simulated coverage rate of the confidence intervals.

```
ntrials <- 10000
results2 <- NULL

for (t in tees) {
  for (rho in rhos) {
    for (trial in 1:ntrials) {
      print(trial)

      x_t <- rnorm(1)
      y_t <- rnorm(1)
      v_t <- rnorm(t)
      u_t <- rnorm(t)

      for (i in 1:t) x_t[i+1] <- 0.3 * x_t[i] + v_t[i]
      for (i in 1:t) y_t[i+1] <- alpha + i * beta + x_t[i+1] * delta +
        y_t[i] * rho + u_t[i]

      x <- cbind(rep(1, t),
                 1:t,
                 x_t[2:(t+1)],
                 y_t[1:t])
      y <- y_t[2:(t+1)]

      ols <- solve(t(x) %*% x) %*% (t(x) %*% y)

      e_hat <- as.numeric(y - x %*% ols)
      omega <- crossprod(x * e_hat)
      varcov <- solve(t(x) %*% x) %*% omega %*% solve(t(x) %*% x)
      se_robust <- sqrt(diag(varcov))

      results2 <- tibble(t = t,
                         rho = rho,
                         trial = trial,
                         name = c("alpha", "beta", "delta", "rho"),
                         ols = as.numeric(ols),
                         se = se_robust) %>%
        bind_rows(results2)
    }
  }
}

save(results2, file = "ps5_vonhafften.RData")
```

```
load("ps5_vonhafften.RData")

results2 %>%
  group_by(t, rho, name) %>%
  summarise(mean = mean(ols),
            lower_bound = quantile(ols, probs = .05),
            upper_bound = quantile(ols, probs = .95),
            .groups = "keep") %>%
  kable(digits = 3)
```

t	rho	name	mean	lower_bound	upper_bound
50	0.70	alpha	100.007	99.374	100.629
50	0.70	beta	1.000	0.980	1.020
50	0.70	delta	100.001	99.758	100.243
50	0.70	rho	0.700	0.698	0.702
50	0.90	alpha	99.999	99.363	100.640
50	0.90	beta	1.000	0.967	1.034
50	0.90	delta	100.003	99.760	100.245
50	0.90	rho	0.900	0.899	0.901
50	0.95	alpha	100.006	99.389	100.618
50	0.95	beta	1.000	0.945	1.056
50	0.95	delta	99.999	99.760	100.237
50	0.95	rho	0.950	0.949	0.951
150	0.70	alpha	99.999	99.637	100.365
150	0.70	beta	1.000	0.996	1.004
150	0.70	delta	100.000	99.867	100.135
150	0.70	rho	0.700	0.699	0.701
150	0.90	alpha	100.000	99.575	100.420
150	0.90	beta	1.000	0.993	1.007
150	0.90	delta	100.000	99.869	100.130
150	0.90	rho	0.900	0.900	0.900
150	0.95	alpha	100.001	99.575	100.420
150	0.95	beta	1.000	0.990	1.010
150	0.95	delta	99.999	99.867	100.127
150	0.95	rho	0.950	0.950	0.950
250	0.70	alpha	100.000	99.719	100.278
250	0.70	beta	1.000	0.997	1.003
250	0.70	delta	100.001	99.900	100.105
250	0.70	rho	0.700	0.699	0.701
250	0.90	alpha	100.002	99.665	100.343
250	0.90	beta	1.000	0.996	1.004
250	0.90	delta	100.001	99.900	100.103
250	0.90	rho	0.900	0.900	0.900
250	0.95	alpha	100.002	99.652	100.356
250	0.95	beta	1.000	0.994	1.006
250	0.95	delta	100.000	99.899	100.101
250	0.95	rho	0.950	0.950	0.950

(iii) How does sample size and the degree of persistence in Y_t affect the results of the simulations.

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