FIN 971: Corporate Finance
Fall 2021, University of Wisconsin
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Problem set #2 - Goal Due 11/11/21

This problem asks you to estimate the parameters of a linearized version of Hayashi's frictionless investment demand function and graph impulse response functions.

1 Exogenous Stochastic Discount Factor

The dynamic programming formulation of the model is:

$$\begin{aligned} V(k_t, z_t) = \max_{k_{t+1}\geq 0} \; & \pi(z_t, k_t) - \psi(k_{t+1} - (1-\delta)k_t, k_t) - (k_{t+1} - (1-\delta)k_t) \\ & + E_t\left[M_{t+1} V(k_{t+1}, z_{t+1})\right] \end{aligned} \tag{1}$$

where z_{t+1} follows an AR(1) process and M_{t+1} is the stochastic discount factor, defined as a function of consumption growth g^c_{t+1}

$$z_{t+1} = (1-\rho_z) + \rho_z z_t + \epsilon_{t+1}, \quad \mathrm{Var}(\epsilon_{t+1}) = \sigma^2_\epsilon$$
$$M_{t+1} = \frac{1}{1+r}\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{1+r}\left(g^c_{t+1}\right)^{-\gamma}$$

In this problem, we assume that consumption growth follows an AR(1) process of the form

$$g^c_{t+1} = (1-\rho_c) + \rho_c g^c_t + \eta\epsilon_{t+1}$$

where η represents the correlation between the consumption growth shock and the productivity shock. This is a convenient way to incorporate the correlation in the impulse response analysis of Dynare.

In Dynare, we need to input the following equations:

$$
\begin{aligned}
q_t &= 1 + \psi_I(I_t, k_t), \\
q_t &= E_t[M_{t+1}\{\pi_k(k_{t+1}, z_{t+1}) - \psi_k(I_{t+1}, k_{t+1}) + q_{t+1}(1-\delta)\}], \\
I_t &= k_{t+1} - k_t + \delta k_t, \\
z_t &= (1-\rho_z) + \rho_z z_{t-1} + \epsilon_t, \\
M_t &= \frac{1}{1+r}(g_t^c)^{-\gamma}, \\
g_t^c &= (1-\rho_c) + \rho_c g_{t-1}^c + \eta\epsilon_t
\end{aligned}
$$

where we assume the following functional forms

$$
\begin{aligned}
\pi(k_t, z_t) &= z_t k_t^\theta \\
\psi(k_{t+1} - (1-\delta)k_t, k_t) &= \frac{\psi_0(k_{t+1} - k_t)^2}{2k_t}.
\end{aligned}
$$

2 Endogenous Stochastic Discount Factor

Suppose that the firm is owned by a representative household. For each share, the household will receive a dividend

$$
d_t = \pi(z_t, k_t) - \psi(I_t, k_t) - I_t
$$

and use it to purchase the share. The budget constraint is

$$
c_t + p_t s_{t+1} = (p_t + d_t)s_t
$$

where p_t is the after-dividend stock price. The household solves

$$
\max_{\{c_t, s_{t+1}\}_{t=0}^\infty} \mathrm{E}_0 \left[\sum_{t=0}^\infty \left(\frac{1}{1+r}\right)^t u(c_t)\right], \quad \text{subject to}
$$
$$
c_t + p_t s_{t+1} = (p_t + d_t)s_t
$$

The first order condition is

$$
p_t u'(c_t) = \frac{1}{1+r} E_t[(p_{t+1} + d_{t+1})u'(c_{t+1})]
$$

Using the stochastic discount factor, this condition can be written as

$$
p_t = E_t[M_{t+1}(p_{t+1} + d_{t+1})].
$$

In equilibrium, $s_t = 1$ and $c_t = d_t$. In addition,

$$p_t = V(k_t, z_t) - d_t$$

To see this, we can substitute it into the Euler equation of the household to obtain

$$\begin{aligned}V(k_t, z_t) - d_t &= E_t[M_{t+1}V(k_{t+1}, z_{t+1})] \\ &\iff V(k_t, z_t) = d_t + E_t[M_{t+1}V(k_{t+1}, z_{t+1})]\end{aligned}$$

and $V(k_t, z_t)$ satisfies this equation.

The equilibrium is a sequence of quantities and prices $(c_t^*, s_{t+1}^*, I_t^*, k_{t+1}^*, d_t^*, v_t^*, M_{t+1}^*, p_t^*, q_t^*)_{t=0}^{\infty}$ such that

1. Households maximize their utility:

$$\begin{aligned}p_t^* u'(c_t^*) &= \frac{1}{1+r} E_t[(p_{t+1}^* + d_{t+1}^*)u'(c_{t+1}^*)] \\ c_t^* + p_t^* s_{t+1}^* &= (p_t^* + d_t^*)s_t^* \\ M_{t+1}^* &= \frac{1}{1+r}\frac{u'(c_{t+1}^*)}{u'(c_t^*)}\end{aligned}$$

2. Firms maximize the expected discounted present value of their profits (these are the necessary conditions (4) and (5) we found in the handout):

$$\begin{aligned}q_t^* &= 1 + \psi_I(I_t^*, k_t^*) \\ q_t^* &= E_t[M_{t+1}^*\{\pi_k(k_{t+1}^*, z_{t+1}) - \psi_k(I_{t+1}^*, k_{t+1}^*) + q_{t+1}^*(1-\delta)\}]\end{aligned}$$

and the expected present value of their profits (remember Dynare thinks in sequence form) is given by

$$\begin{aligned}v_t^* &= d_t^* + E_t\left[\sum_{j=0}^{\infty}\left(\prod_{s=t}^{j+t} M_{s+1}^*\right) d_{j+t+1}^*\right] \\ &= d_t^* + E_t[M_{t+1}^* v_{t+1}^*] \\ d_t^* &= \pi(k_t^*, z_t) - \psi(I_t^*, k_t^*) - I_t^*\end{aligned}$$

3. Markets clear:

$$
\begin{aligned}
s_t^* &= 1 \\
c_t^* &= d_t^* \\
p_t^* &= v_t^* - d_t^*
\end{aligned}
$$

The difference between this model and the model with exogenous stochastic discount factor is that consumption is endogenously determined by the market clearing condition

$$c_t = d_t = \pi(k_t, z_t) - \psi(I_t, k_t) - I_t.$$

In Dynare, we need to input the following equations:

$$
\begin{aligned}
q_t &= 1 + \psi_I(I_t, k_t), \\
q_t &= E_t[M_{t+1}\{\pi_k(k_{t+1}, z_{t+1}) - \psi_k(I_{t+1}, k_{t+1}) + q_{t+1}(1-\delta)\}], \\
I_t &= k_{t+1} - k_t + \delta k_t, \\
c_t &= \pi(k_t, z_t) - \psi(I_t, k_t) - I_t, \\
z_t &= (1-\rho_z) + \rho_z z_{t-1} + \epsilon_t, \\
M_t &= \frac{1}{1+r}\left(\frac{c_t}{c_{t-1}}\right)^{-\gamma}.
\end{aligned}
$$

Problems

In order to illustrate Dynare, we will solve the model backwards. In particular, in any new application, you may not know parameter values of the environment. Hence you would start with estimation which is part 5 of this problem. Here we will simply start with parameter values that come from Hennesy and Whited as in the handout.

1. For the parameterization in Table 1 find the steady state value of endogenous variables since Dynare requires you to give an initial value for the steady state.

2. Dynare automatically linearizes the first order condition around the steady state. However, finding the linear decision rule requires either

Parameter	Description	Value
θ	curvature of profit function	0.7
r	discount rate	0.04
δ	depreciation rate	0.15
ψ_0	adjustment cost	0.01
ρ_z	persistency of shock	0.7
σ_ϵ	standard deviation of shock	0.1
η	Corr btw c growth shock and z shock	0.5
ρ_c	persistency of consumption growth	0.2
γ	CRRA utility parameter	2.0

Table 1: Parameter values

using the method of undetermined coefficients (as in the "Frictionless Benchmark" notes on my website or alternatively directly solving the linear difference equation by the method proposed by Blanchard and Khan (1980). Below you will find a description of that method. On the website, we have put Dynare code which solves the case without adjustment costs (noadjustment.mod). You simply need to change that code to include adjustment costs. Then solve for decision rules and generate the impulse response functions of investment and capital in response to 1 standard deviation shock to productivity. Since Dynare produces $x_t - x_{ss}$, not the percentage deviation from the steady state, $(x_t - x_{ss})/x_{ss}$, you have to divide the impulse response (stored in `oo_.irfs`) by their steady state values (stored in `oo_.steady_state`) to get the impulse response in terms of the percentage deviation from the steady state. What happens if you set $\gamma = 0$?

3. Once you run the Dynare file, it will produce `filename_results.mat`. If you specify the length of the simulation period in `stoch_simul` command, then dynare generates the simulated time series in this mat file (stored in `oo_.endo_simul`). Using this simulated data, estimate the following equation

$$\frac{I_t}{k_t} = \alpha + \beta_1 q_{t-1} + \beta_2 \frac{\pi_{t-1}}{k_{t-1}}$$

and report the estimated values of $(\alpha, \beta_1, \beta_2)$. What is the sign of β_2?

4. Next, to understand how Bayesian estimation works in Dynare, we will start with the case where the times series is actually generated by the above parameters. Fix $r = 0.04$, $\delta = 0.15$, $\rho_z = 0.7$ and $\gamma = 2.0$. Using the simulated data of capital and marginal q, we can estimate the parameters using dynare. In order to implement Bayesian estimation in dynare, we need as many unobservable shock as observable variables. We have two observables (q_t, k_t) and one unobservable shock ϵ_t, so we need one more unobservable shock. We assume that we observe the marginal q with measurement error

$$\tilde{q}_t = q_t + \epsilon_t^q, \quad \epsilon_t^q \sim N(0, \sigma_q^2)$$

where ϵ_t^q is uncorrelated with the productivity shock ϵ_t. To do so, you need to add this measurement error to the simulated data. Set $\sigma_q = 0.01$. Use the following prior distribution to estimate the parameters $(\theta, \psi_0, \sigma_\epsilon, \sigma_q)$.

$$\begin{aligned} \theta &\sim U(0,1) \\ \psi_0 &\sim U(0,1) \\ \sigma_\epsilon &\sim \text{Inv-Gamma}(0.007, 1) \\ \sigma_q &\sim \text{Inv-Gamma}(0.007, 1) \end{aligned}$$

On the website, we have put Dynare code which estimates the case without adjustment costs (noadjustment_est.mod). A matlab file run.m is also included. The following figure shows the priors and posteriors of the above parameters. Since the posterior is very different from the prior, this is favorable evidence that the model is well-identified. In fact, in Dynare, there is a command named "identification" to check identification.

5. Finally, we start with data on the real capital stock from the Penn World Table and balance sheets from the Federal Reserve Board (available on FRED). We have placed that data file on the website. With this data, Dynare finds the Hessian of the likelihood function of the posterior mode is not positive definite. Hence to estimate the model, we need to change some settings which is now in noadjustment_est_D.mod on the website. Why do you think the estimated parameter values are so different?

Blanchard-Kahn method

The linearized Euler equation is

$$
\begin{aligned}
(1+r)\psi_0(\hat{k}_{t+1}-\hat{k}_t) &= E_t\left[\theta\bar{k}^{\theta-1}[(\theta-1)\hat{k}_{t+1}+\hat{z}_{t+1}]+\psi_0(\hat{k}_{t+2}-\hat{k}_{t+1})\right] \\
\hat{z}_{t+1} &= \rho\hat{z}_t+\epsilon_{t+1}
\end{aligned}
$$

or using $\hat{i}_t-\delta\hat{k}_t=\hat{k}_{t+1}-\hat{k}_t$,

$$
\begin{aligned}
E_t\left[\theta\bar{k}^{\theta-1}[(\theta-1)\hat{k}_{t+1}+\hat{z}_{t+1}]+\psi_0(\hat{i}_{t+1}-\delta\hat{k}_{t+1})\right] &= (1+r)\psi_0(\hat{i}_t-\delta\hat{k}_t) \\
\hat{k}_{t+1} &= \hat{i}_t+(1-\delta)\hat{k}_t \\
\hat{z}_{t+1} &= \rho\hat{z}_t+\epsilon_{t+1}
\end{aligned}
$$

In vector form, this system can be written as

$$
A\begin{bmatrix} E_t[\hat{i}_{t+1}] \\ \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = B\begin{bmatrix} \hat{i}_t \\ \hat{k}_t \\ \hat{z}_t \end{bmatrix} + C\epsilon_{t+1}
$$

where

$$
\begin{aligned}
A &= \begin{bmatrix} \psi_0 & \theta(\theta-1)\bar{k}^{\theta-1}-\psi_0\delta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\tilde{B} &= \begin{bmatrix} (1+r)\psi_0 & -(1+r)\psi_0\delta & -\theta\rho\bar{k}^{\theta-1} \\ 1 & 1-\delta & 0 \\ 0 & 0 & \rho \end{bmatrix} \\
\tilde{C} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

We want to solve this linear expectational difference equation.

We can multiply both sides by A^{-1} from left to obtain[1]

$$
\begin{bmatrix} E_t[\hat{i}_{t+1}] \\ \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = B\begin{bmatrix} \hat{i}_t \\ \hat{k}_t \\ \hat{z}_t \end{bmatrix} + C\epsilon_{t+1}
$$

[1] Here we assume that A^{-1} exists. Klein(2000,JEDC) developed an algorithm which can be applied even if A is singular.

where $B \equiv A^{-1}\tilde{B}$ and $C = A^{-1}\tilde{C}$. Under the parameter values we use, it is possible to obtain the Jordan Canonical form of B;

$$B = VJV^{-1} = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}^{-1}$$

where d_i is the eigenvector of B associated with eigenvalue λ_i, and λ_i are ordered such that $|\lambda_1| > |\lambda_2| > |\lambda_3|$.

In this system, we have 1 control variable, $\hat{i}_t$, and 2 state variables, $\hat{k}_t$ and $\hat{z}_t$. Blanchard and Kahn(1980) proved that this system has a unique non-explosive solution if and only if the number of control variables is equal to the number of eigenvalues greater than 1. So if this system has a unique solution, then it should be the case that $|\lambda_1| > 1 > |\lambda_2| > |\lambda_3|$.

To solve this difference equation, consider the following transformation

$$\mathbf{x_t} = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = V^{-1} \begin{bmatrix} \hat{i}_t \\ \hat{k}_t \\ \hat{z}_t \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} \hat{i}_t \\ \hat{k}_t \\ \hat{z}_t \end{bmatrix}$$

Then if we take expectations, the system in period $t+i$ can be written as

$$E_t \mathbf{x}_{t+i+1} = JE_t \mathbf{x_{t+i}} + V^{-1}CE_t[\epsilon_{t+i+1}]$$

where

$$V^{-1}C = \begin{bmatrix} vc_1 \\ vc_2 \\ vc_3 \end{bmatrix}$$

Since J is a diagonal matrix, this system consists of single variable AR(1) processes:

$$\begin{aligned} E_t[x_{1,t+i+1}] &= \lambda_1 E_t[x_{1,t+i}] + vc_1 E_t[\epsilon_{t+i+1}] \\ E_t[x_{2,t+i+1}] &= \lambda_2 E_t[x_{2,t+i}] + vc_2 E_t[\epsilon_{t+i+1}] \\ E_t[x_{3,t+i+1}] &= \lambda_3 E_t[x_{3,t+i}] + vc_3 E_t[\epsilon_{t+i+1}] \end{aligned}$$

Because of $|\lambda_1| > 1$, the first equation should be solved forward[2]

$$\begin{aligned} x_{1,t} = E_t[x_{1,t}] &= \lambda_1^{-1} E_t[x_{1,t+1}] - \lambda_1^{-1} v c_1 E_t[\epsilon_{t+1}] \\ &= \lambda_1^{-2} E_t[x_{1,t+2}] - \lambda_1^{-1} v c_1 E_t[\epsilon_{t+1}] - \lambda_1^{-2} v c_1 E_t[\epsilon_{t+2}] \\ &= \cdots \\ &= -\sum_{i=1}^{\infty} \lambda_1^{-i} v c_1 E_t[\epsilon_{t+i}] + \lim_{i\to\infty} \lambda_1^{-i} E_t[x_{1,t+i+1}] = 0 \end{aligned}$$

Since $x_{1,t} = v_{11}\hat{i}_t + v_{12}\hat{k}_t + v_{13}\hat{z}_t$, we obtain the decision rule

$$\hat{i}_t = -\frac{v_{12}}{v_{11}}\hat{k}_t - \frac{v_{13}}{v_{11}}\hat{z}_t$$

Once we have a decision rule for the control variables, the law of motion for the endogenous state variable can be obtained by plugging the decision rule into the original linear difference equation.

[2]Here $E_t[\epsilon_{t+i}] = 0$ from the assumption in the model. If this assumption is not satisfied, $x_{1,t}$ depends on the expectation of future shocks.

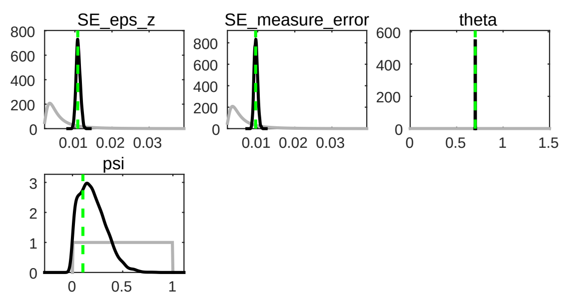

Figure 1: Prior (grey) vs. Posterior mode (green) and Distribution (black)