

# ECON 711 - PS 2

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## Question 1. Convex production sets, concave production functions, convex costs

Consider a production function  $f : \mathbb{R}_+^m \rightarrow \mathbb{R}_+$  for a single-output firm.

- (a) Prove that if the production set  $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$  is convex, the production function  $f$  is concave.

Proof: Choose  $(q, -z), (q', -z') \in Y$  such that  $f(z) = q$  and  $f(z') = q'$ . The convexity of  $Y$  implies that  $t(q, -z) + (1-t)(q', -z') \in Y$  for  $t \in (0, 1)$ . Thus,  $f(tz + (1-t)z') \geq tq + (1-t)q'$  by the definition of  $Y$ . Our choice of  $(q, -z), (q', -z') \implies f(tz + (1-t)z') \geq tf(z) + (1-t)f(z')$ . Therefore,  $f$  is concave.  $\square$

- (b) Prove that if  $f$  concave, the cost function

$$c(q, w) = \min w \cdot z \text{ subject to } f(z) \geq q$$

is convex in  $q$ .

Proof: Fixing  $w \in \mathbb{R}_+^k$ , choose  $q, q' \in \mathbb{R}$ . Define  $z \in Z^*(q, w)$ ,  $z' \in Z^*(q', w)$ , and  $\tilde{z} \in Z^*(tq + (1-t)q', w)$  for  $t \in (0, 1)$ . By the concavity of  $f$ ,

$$\begin{aligned} \tilde{z} &\leq tz + (1-t)z' \\ \implies w\tilde{z} &\leq w(tz + (1-t)z') \\ \implies w\tilde{z} &\leq twz + (1-t)wz' \\ \implies c(f(\tilde{z}), w) &\leq tc(f(z), w) + (1-t)c(f(z'), w) \\ \implies c(tq + (1-t)q', w) &\leq tc(q, w) + (1-t)c(q', w) \end{aligned}$$

Therefore,  $c$  is convex.  $\square$

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\*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, Tyler Welch, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

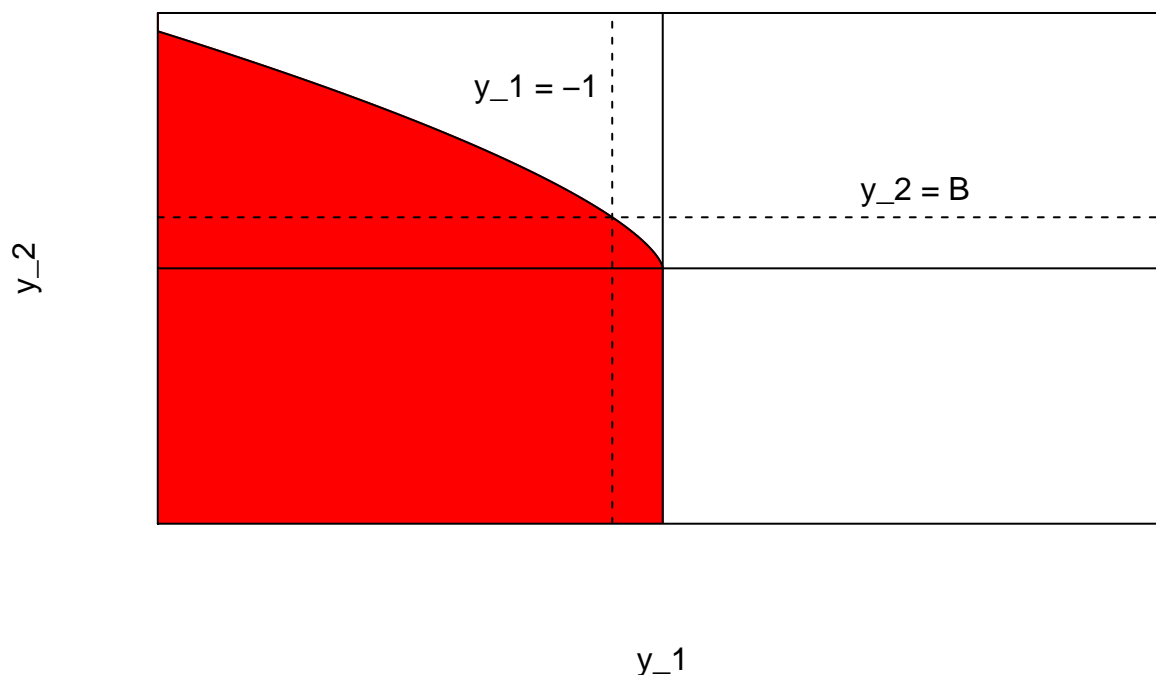
## Question 2. Solving for the profit function given technology...

Let  $k = 2$ , and let the production set be

$$Y = \{(y_1, y_2) : y_1 \leq 0 \text{ and } y_2 \leq B(-y_1)^{\frac{2}{3}}\}$$

where  $B > 0$  is a known constant. Assume both prices are strictly positive.

(a) Draw  $Y$ , or describe it clearly.



(b) Solve the firm's profit maximization problem to find  $\pi(p)$  and  $Y^*(p)$ .

The firm's profit is

$$\pi(p) = \max_{y_1, y_2 \in Y} \{p_1 y_1 + p_2 y_2\}$$

Define  $z = -y_1$ . Notice that the firm will produce  $y_2 = Bz^{2/3}$  because it is the maximum output given  $z$  units of input. Thus, we can rewrite the firm's profit function as

$$\pi(p) = \max_z \{p_1(-z) + p_2 B z^{2/3}\} = \max_z \{p_2 B z^{2/3} - p_1 z\}$$

Setting the first order condition of the profit function to zero:

$$\begin{aligned} \frac{\partial \pi}{\partial z} &= p_2 (2/3) B z^{-1/3} - p_1 \\ z^* &= \left( \frac{2p_2 B}{3p_1} \right)^3 \end{aligned}$$

Plugging  $z^*$  into transformations for  $y_1, y_2$ :

$$\begin{aligned} y_1^* &= -\left(\frac{2p_2 B}{3p_1}\right)^3 \\ y_2^* &= B\left(\left(\frac{2p_2 B}{3p_1}\right)^3\right)^{2/3} \\ &= B^3\left(\frac{2p_2}{3p_1}\right)^2 \end{aligned}$$

Notice that  $Y^*(p)$  is single-valued:

$$Y^*(p) = \left\{ (y_1, y_2) : y_1 = -\left(\frac{2p_2 B}{3p_1}\right)^3, y_2 = B^3\left(\frac{2p_2}{3p_1}\right)^2 \right\} \implies y(p) = \left( -\left(\frac{2p_2 B}{3p_1}\right)^3, B^3\left(\frac{2p_2}{3p_1}\right)^2 \right)$$

Thus, the profit function is

$$\begin{aligned} \pi(p) &= p_1 \left( -\left(\frac{2p_2 B}{3p_1}\right)^3 \right) + p_2 \left( B^3\left(\frac{2p_2}{3p_1}\right)^2 \right) \\ &= \frac{B^3 p_2^3}{p_1^2} \left( \frac{2^2}{3^2} - \frac{2^3}{3^3} \right) \\ &= \frac{4B^3 p_2^3}{27p_1^2} \end{aligned}$$

- (c) Since  $Y^*(p)$  is single-valued, I'll refer to it below as  $y(p)$ . Verify that  $\pi(\cdot)$  is homogeneous of degree 1, and  $y(\cdot)$  is homogeneous of degree 0.

For  $\alpha \in \mathbb{R}$ :

$$\begin{aligned} \pi(\alpha p) &= \frac{4B^3(\alpha p_2)^3}{27(\alpha p_1)^2} \\ &= \alpha \frac{4B^3(p_2)^3}{27(p_1)^2} \\ y(\alpha p) &= \left( -\left(\frac{2(\alpha p_2)B}{3(\alpha p_1)}\right)^3, B^3\left(\frac{2(\alpha p_2)}{3(\alpha p_1)}\right)^2 \right) \\ &= \left( -\left(\frac{2p_2 B}{3p_1}\right)^3, B^3\left(\frac{2p_2}{3p_1}\right)^2 \right) \end{aligned}$$

(d) Verify that  $y_1(p) = \frac{\partial \pi}{\partial p_1}(p)$  and  $y_2(p) = \frac{\partial \pi}{\partial p_2}(p)$ .

$$\begin{aligned}
\frac{\partial \pi}{\partial p_1}(p) &= \frac{4B^3 p_2^3}{27p_1^3}(-2) \\
&= -\frac{8B^3 p_2^3}{27p_1^3} \\
&= -\left(\frac{2p_2 B}{3p_1}\right)^3 \\
&= y_1(p) \\
\frac{\partial \pi}{\partial p_2}(p) &= \frac{4B^3 p_2^2}{27p_1^2}(3) \\
&= \frac{4B^3 p_2^2}{9p_1^2} \\
&= B^3 \left(\frac{2p_2}{3p_1}\right)^2 \\
&= y_2(p)
\end{aligned}$$

(e) Calculate  $D_p y(p)$ , and verify it is symmetric, positive semidefinite, and  $[D_p y]p = 0$

$$\begin{aligned}
D_p y(p) &= \begin{pmatrix} \frac{\partial y_1}{\partial p_1}(p) & \frac{\partial y_2}{\partial p_1}(p) \\ \frac{\partial y_1}{\partial p_2}(p) & \frac{\partial y_2}{\partial p_2}(p) \end{pmatrix} \\
&= \begin{pmatrix} \frac{8p_2^3 B^3}{9p_1^4} & \frac{-8p_2^2 B^3}{9p_1^3} \\ \frac{-8p_2^2 B^3}{9p_1^3} & \frac{8p_2 B^3}{9p_1^2} \end{pmatrix}
\end{aligned}$$

Since both off diagonal elements of  $D_p y(p)$  equal  $\frac{-8p_2^2 B^3}{9p_1^3}$ ,  $D_p y(p)$  is symmetric.

$$B > 0, p_1 > 0, p_2 > 0 \implies \frac{8p_2^3 B^3}{9p_1^4} > 0$$

$$\begin{aligned}
\det D_p y(p) &= \frac{8p_2^3 B^3}{9p_1^4} \frac{8p_2 B^3}{9p_1^2} - \frac{-8p_2^2 B^3}{9p_1^3} \frac{-8p_2^2 B^3}{9p_1^3} \\
&= \frac{64p_2^4 B^6}{81p_1^6} - \frac{64p_2^4 B^6}{81p_1^6} \\
&= 0
\end{aligned}$$

Therefore,  $D_p y(p)$  is positive semidefinite.

$$[D_p y]p = \begin{pmatrix} \frac{8p_2^3 B^3}{9p_1^4} & \frac{-8p_2^2 B^3}{9p_1^3} \\ \frac{-8p_2^2 B^3}{9p_1^3} & \frac{8p_2 B^3}{9p_1^2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{8p_2^3 B^3}{9p_1^3} + \frac{-8p_2^3 B^3}{9p_1^3} \\ \frac{-8p_2^2 B^3}{9p_1^2} + \frac{8p_2^2 B^3}{9p_1^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

### Question 3 ... and recovering technology from the profit function

Finally, suppose we didn't know a firm's production set  $Y$ , but did know its profit function was  $\pi(p) = Ap_1^{-2}p_2^3$  for all  $p_1, p_2 > 0$  and  $A > 0$  a known constant.

(a) What conditions must hold for this profit function to be rationalizable? (You don't need to check them.)

Since  $\pi$  is differentiable, it is rationalizable if and only if it is homogeneous of degree 1 and convex (Lecture 3 Notes).

(b) Recall that the outer bound was defined as  $Y^O = \{y : p \cdot y \leq \pi(p) \text{ for all } p \in P\}$ . In this case, this is  $Y^O = \{(y_1, y_2) : p_1 y_1 + p_2 y_2 \leq Ap_1^{-2}p_2^3 \text{ for all } (p_1, p_2) \in \mathbb{R}_{++}^2\}$ . Show that any  $y \in Y^O$  must have  $y_1 \leq 0$ , i.e., that good 1 must be an input only.

Proof: Assume for sake of a contradiction that  $y_1 > 0$ . Fixing  $y_1, y_2$ , and  $p_3$ , notice that increasing  $p_1$  increases the right-hand side of the inequality and decreases the left-hand side. Thus, there exists a sufficiently large  $p'_1$  such that  $p_1 y_1 + p_2 y_2 > Ap_1^{-2}p_2^3$ .  $\Rightarrow \Leftarrow$  Therefore,  $y_1$  must be nonpositive.  $\square$

(c) Dividing both sides by  $p_2$  and moving  $\frac{p_1}{p_2}y_1$  to the right-hand side, we can rewrite  $Y^O$  as  $Y^O = \{(y_1, y_2) : y_2 \leq Ap_1^{-2}p_2^3 - \frac{p_1}{p_2}y_1 \text{ for all } (p_1, p_2) \in \mathbb{R}_{++}^2\}$ . Since the expression on the right depends only on the price ratio  $\frac{p_2}{p_1}$  rather than the two individual prices, we can let  $r \equiv \frac{p_2}{p_1} > 0$  denote this ratio, and write  $Y^O$  as  $Y_O = \{(y_1, y_2) : y_2 \leq Ar^2 - \frac{y_1}{r} \text{ for all } r \in \mathbb{R}_{++}\} = \{(y_1, y_2) : y_2 \leq \min_{r>0}(Ar^2 - \frac{y_1}{r})\}$ . Solve this minimization problem, and describe the production set  $Y^O$ .

Setting the first order condition of  $Ar^2 - \frac{y_1}{r}$  equal to zero:

$$\begin{aligned} 2Ar - \frac{y_1}{r^2}(-1) &= 0 \\ 2Ar &= \frac{-y_1}{r^2} \\ 2Ar^3 &= -y_1 \\ r^* &= \left(\frac{-y_1}{2A}\right)^{\frac{1}{3}} \end{aligned}$$

Plugging it back into  $Ar^2 - \frac{y_1}{r}$ :

$$\begin{aligned} A\left(\left(\frac{-y_1}{2A}\right)^{\frac{1}{3}}\right)^2 - y_1\left(\frac{2A}{-y_1}\right)^{\frac{1}{3}} &= \frac{A^{1/3}(-y_1)^{2/3}}{2^{2/3}} + (-y_1)^{2/3}2^{1/3}A^{1/3} \\ &= A^{1/3}(-y_1)^{2/3}(2^{-2/3} + 2^{1/3}) \end{aligned}$$

Thus,

$$Y_O = \left\{ (y_1, y_2) : y_2 \leq (2^{-2/3} + 2^{1/3})A^{1/3}(-y_1)^{2/3} \right\}$$

- (d) Verify that a production set  $Y$  equal to the set  $Y^O$  you just calculated would generate the “data”  $\pi(p) = Ap_1^{-2}p_2^3$  that we started with. [Hint:  $(2^{-2/3} + 2^{1/3})^3 = \frac{27}{4}$ ].

Fix  $p_1, p_2 \in \mathbb{R}_{++}$ . Since the profit function are increase in  $y_2$ , we know that the firm will produce at the boundary of its production set, so  $y_2 = (2^{-2/3} + 2^{1/3})A^{1/3}(-y_1)^{2/3}$ . Notice that  $Y_0$  is equivalent to the production set outlined in Question 2 with  $B = (2^{-2/3} + 2^{1/3})A^{1/3}$ . Thus, from my answer to Question 2, we know that

$$\begin{aligned}\pi(p) &= \frac{4B^3p_2^3}{27p_1^2} \\ &= \frac{4((2^{-2/3} + 2^{1/3})A^{1/3})^3p_2^3}{27p_1^2} \\ &= \frac{4(27/4)Ap_2^3}{27p_1^2} \\ &= \frac{Ap_2^3}{p_1^2}\end{aligned}$$