ECON 711 - PS 2

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Question 1. Convex production sets, concave production functions, convex costs

Consider a production function $f : \mathbb{R}_+^m \to \mathbb{R}_+$ for a single-output firm.

(a) Prove that if the production set $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$ is convex, the production function f is concave.

Proof: Choose $(q, -z), (q', -z') \in Y$ such that $f(z) = q$ and $f(z') = q'$. The convexity of Y implies that $t(q, -z) + (1-t)(q', -z') \in Y$ for $t \in (0, 1)$. Thus, $f(tz + (1-t)z') \geq tq + (1-t)q'$ by the definition of Y. Our choice of $(q, -z), (q', -z') \implies f(tz + (1-t)z') \geq tf(z) + (1-t)f(z')$. Therefore, f is concave. □

(b) Prove that if f concave, the cost function

$$c(q, w) = \min w \cdot z \text{ subject to } f(z) \geq q$$

is convex in q.

Proof: Fixing w, choose q, q' from the domain of c. Define $z \in Z^*(q)$, $z' \in Z^*(q')$, and $\tilde{z} \in Z^*(tq + (1-t)q')$ for $t \in (0, 1)$. By the concavity of f,

$$\begin{aligned}
\tilde{z} &\leq tz + (1-t)z' \\
\implies w\tilde{z} &\leq w(tz + (1-t)z') \\
\implies w\tilde{z} &\leq twz + (1-t)wz') \\
\implies c(f(\tilde{z}), w) &\leq tc(f(z), w) + (1-t)c(f(z'), w) \\
\implies c(tq + (1-t)q', w) &\leq tc(q, w) + (1-t)c(q', w)
\end{aligned}$$

Therefore, c is concave. □

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, Tyler Welch, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

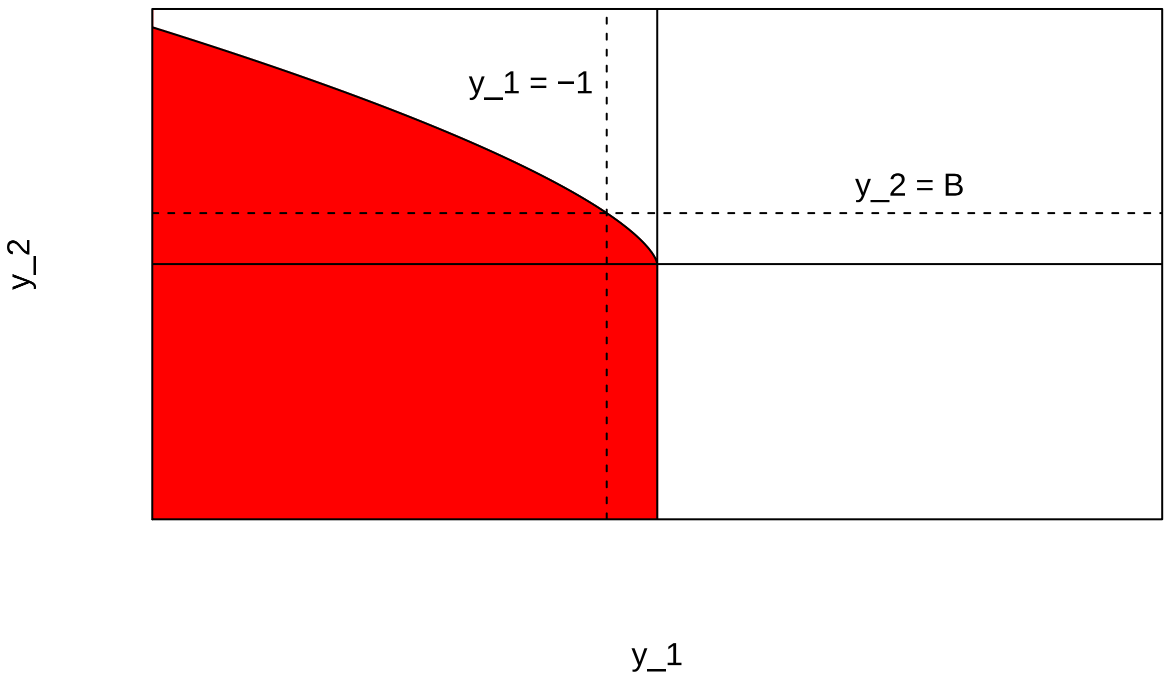
Question 2. Solving for the profit function given technology...

Let $k = 2$, and let the production set be

$$Y = \{(y_1, y_2) : y_1 \leq 0 \text{ and } y_2 \leq B(-y_1)^{\frac{2}{3}}\}$$

where $B > 0$ is a known constant. Assume both prices are strictly positive.

(a) Draw Y, or describe it clearly.



(b) Solve the firm's profit maximization problem to find $\pi(p)$ and $Y^*(p)$. (It may help to set $z = -y_1$ as the amount of input used, explain why a profit-maximizing firm will set $y_2 = Bz^{\frac{2}{3}}$, and solve a single-dimensional maximization problem for z, but be sure to state your solution $Y^*(p) \in R^2$.)

(c) Since $Y * (p)$ is single-valued, I'll refer to it below as $y(p)$. Verify that $\pi(\cdot)$ is homogeneous of degree 1, and $y(\cdot)$ is homogeneous of degree 0.

(d) Verify that $y_1(p) = \frac{\partial \pi}{\partial p_1}(p)$ and $y_2(p) = \frac{\partial \pi}{\partial p_2}(p)$.

(e) Calculate $D_p y(p)$, and verify it is symmetric, positive semidefinite, and $[D_p y]p = 0$

Question 3 . . . and recovering technology from the profit function

Finally, suppose we didn't know a firm's production set Y, but did know its profit function was

$$\pi(p) A p_1^{-2} p_2^3$$

for all $p_1, p_2 > 0$ and $A > 0$ a known constant.