Notes on Equilibrium Models of Default©

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Intro

- In Huggett (1993), there was a commitment technology whereby households could not strategically default on their noncontingent debt and the natural borrowing constraint assured us that there were no histories of shocks which would lead to an empty budget set for the household.
- With 2 states $s \in \{e, u\}$, if the only asset is a noncontingent discount bond with payoffs $[1\ 1]'$, there is no way to perfectly smooth consumption. Zame (1993) illustrates how a default option can complete the incomplete market (i.e. it adds an option so the payoff matrix is $[1\ 1;\ 1\ 0]'$.

Endogenous Incomplete Markets: Kehoe and Levine (1993)

- Solve for eqm. alln. which satisfy an individual rationality (or endogenous borrowing) constraint: the alln. is such that people prefer it in all possible states and dates to defaulting and entering financial autarky.
- Unlike Huggett, contracts are contingent on future states of the world but subject to a commitment problem.
- The state space can be expanded from (s, a) in Huggett, to (s, a, h) where $h = 0$ means the agent has not defaulted and $h = 1$ means they have defaulted (and enter financial autarky).

Endogenous Incomplete Markets: cont.

The IR constraint is given by

$$\begin{aligned} V(s,a,0) &= \max_{\{a'(s',s),\forall s'\}} u(y(s)+a(s)-\sum_{s'} q(s',s)a'(s',s)) \\ &\quad +\beta E_{s'|s} V(s',a'(s',s),0) \\ &\geq u(y(s))+\beta E_{s'|s} V(s',0,1), \forall s. \end{aligned}$$

1. By virtue of the constraint, there is no default in equilibrium.
2. If one interprets a binding constraint as states in which default occurs (since agents are indifferent), then the theory predicts default in high income states.
   - Since agents are risk averse, punishment to autarky implies $E_{s'|s} V(s',0,1) < E_{s'|s} V(s',a'(s',s),0)$, then only when $y(s)$ is high will constraint bind (i.e. when the agent must transfer goods to the principal).
3. Both predictions are inconsistent with data.

Endogenous Incomplete Markets: Krueger and Perri (2006)

EIM useful for understanding Cross-sectional consumption volatility:

- Complete markets (which implies zero cross-sectional dispersion) is inconsistent with the data.
- Financial autarky implies the cross-sectional dispersion in consumption should equal the cross-sectional dispersion in earnings. Also inconsistent with data.
- The IR constraint implies *incomplete* transfers: consumption is higher when income is higher and consumption is lower when income is lower. Consistent qualitatively with the data.

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- An increase in earnings dispersion may actually improve risk sharing since it makes punishment harsher (i.e. $E_{s'|s}V(s',0,1)$ drops with high variance earnings loosening the constraint $\longrightarrow$ more consumption smoothing. Consistent with data that consumption inequality has risen less than earnings inequality. A nice endogenous result.

Noncontingent contracts with equilibrium default

- Assume non-contingent contracts as in Huggett, but weakens the commitment assumption to allow strategic default.
- One of the first papers in this genre was on sovereign default by Eaton and Gersovitz (1981) where the price of the contract depended on how much the sovereign borrows, but not its endowment.
- In consumer finance, one of the first quantitative models of strategic default was Athreya (2002). He assumed noncontingent "pooling" contracts as in Huggett (i.e. the contracts do not depend even on how much an agent borrows).

Noncontingent contracts with equilibrium default - cont.

- Since an individual's likelihood of default is increasing in the amount of debt they hold, Athreya's "pooling" contracts may not survive competition.
  - Cross-subsidization from those with little debt (less risky) to those with lots of debt (more risky).
  - Lenders would try to "cream-skim" the low risk borrowers from the "pooling" contract by offering a lower interest rate if one borrows less (i.e. a nonlinear contract).
  - But if those low risk agents leave the pooling contract, then the lender offering the pooling contract will earn negative profits at the original (subsidized) interest rate leading to non-existence of eqm.
- One way to deal with non-existence is to offer "separating" contracts as in Chatterjee, et. al. (2007). Differentiated contracts whose price depends on observable characteristics like earnings and the amount borrowed.

Methodological additions to Huggett

- Discrete-Continuous Choice methods.
- Lenders use borrower decision rules to predict break-even prices. Loans of different sizes imply different probabilities of default so like a differentiated goods model.
- Can be applied in/with:
  - International Finance (Arellano (2008, AER))
  - Corporate Finance (Corbae and D'Erasmo (2021, RESTUD))
  - Mortgage Finance (Corbae and Quintin (2015, JPE))
  - Exclusive vs. Non-exclusive contracts and Perfect vs. imperfect competition (Herkenhoff and Raveendranathan (2021))
  - Persistent hidden information (credit scoring) (Chatterjee, et. al. (2021))
  - Pre-Employment Credit Screening with Labor Search (Corbae and Glover (2021))

Environment

These are as in Huggett:

- Population: unit measure of agents.
- Preferences: $E_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t)\right]$. Assume $U:\mathbb{R}_+\to\mathbb{R}$ is ctsly differentiable, strictly increasing, strictly concave, and bounded.
- Endowments (nonstorable): In any period t, hhs face two earnings shocks $s_t \in S = \{e, u\}$ where e denotes employed and u denotes unemployed. These shocks are i.i.d. across agents. The employment process is Markov with transition matrix denoted $\pi(s'|s) = prob(s_{t+1} = s'|s_t = s)$. If employed, hh earns income $y(e) = 1$. If unemployed, receives income $y(u) = b < 1$.

Market Structure: "Pooling"

As in Athreya:

- Save at riskfree interest rate (i.e. if $a_{t+1} > 0$, $q_t = 1/(1+r)$).
- Borrow up to $\underline{a} < 0$ at zero profit price (i.e. if $a_{t+1} \in [\underline{a}, 0)$, $q_t = (1 - \Delta_{t+1})/(1+r)$ where Δ_{t+1} is the economywide loss rate at $t+1$ on the debt agents took at t.

Market Structure: "Separating"

Separating contracts as in Chatterjee, et. al.:

- Save at riskfree interest rate (i.e. if $a_{t+1} > 0$, $q = 1/(1+r)$).
- Borrow $a_{t+1} < 0$ at price which depends on observable characteristics s_t at zero profit price (i.e. if $a_{t+1} < 0$, $q_t(a_{t+1}, s_t) = (1 - \delta_{t+1}(a_{t+1}, s_t))/(1+r)$ where $\delta_{t+1}(a_{t+1}, s_t)$ is the fraction of households who take on debt a_{t+1} in state s_t at t and default at $t+1$.
- These contracts have endogenous borrowing constraints that depend on household characteristics; specifically $a_{t+1} \geq \underline{a}(s_t)$ where $\underline{a}(s_t)$ is defined implicitly by $q_t(\underline{a}(s_t), s_t) = 0$, which determines a very weak borrowing constraint.
- A tighter constraint depends on the debt Laffer curve.
- Since prices depend on the bankruptcy rules, a change in bankruptcy policy can change the borrowing constraints.

Legal Structure: stylized way to model Ch. 7

- If the household does not have a bankruptcy flag on its history (i.e. $h_t = 0$) and has debt (i.e. $a_t < 0$), it can default $d_t = 1$ in which case the debt a_t is discharged (wiped clean).
- In the period of default, no borrowing or saving (i.e. $c_t = y_t$).
- A bankruptcy flag is placed on one's credit history (i.e. $h_{t+1} = 1$).
- A bankruptcy flag leaves one's history with probability $(1 - \rho)$. This is an approximation to the Fair Credit Reporting Act requirement that adverse events leave a household's credit record after 10 years. Hence in an annual model if $\rho = 0.9$, then the duration that a flag is on a household's credit history is $1/(1 - \rho)$ leaves a household's credit record on average over 10 years.

Timing

- enter period t with a_t and h_t,
- realize earnings shock $y(s_t)$,
- if $h_t = 0$ and $a_t < 0$ make default decision $d_t \in \{0, 1\}$,
  - if $d_t = 1$, $a_{t+1} = 0$ and $h_{t+1} = 1$.
  - if $d_t = 0$, make asset decision $a_{t+1} \in \Gamma(s_t, a_t, h_t)$ and $h_{t+1} = 0$.
- if $h_t = 1$, make savings decision $a_{t+1} \in [0, (y(s_t) + a_t)(1 + r)]$ and $h_{t+1} = 0$ with probability $(1 - \rho)$.

Household Problem

- Households solve a mixed discrete, continuous choice problem.
- Let I denote the type of contracting environment we are assuming: pooling $I = \varnothing$ or separating $I = (a', s)$.
- In contracting environment I, the value function of an agent with endowment state s, assets a, who is in bankruptcy flag status h is denoted $V(s, a, h; I)$. Agents maximize lifetime utility by choosing (d, a').

HH Problem with good credit history

- If $h = 0$, the agent will make the default decision based on whether default or repayment yields higher lifetime utility:

$$V(s, a, 0; I) = \max_{d}\{V^{d=0}(s, a, 0; I), V^{d=1}(s, a, 0; I)\}.$$

where

$$V^{d=1}(s, a, 0; I) = u(y(s)) + \beta \sum_{s'} \pi(s'|s) V(s', 0, 1; I)$$

and

$$\begin{aligned} V^{d=0}(s, a, 0; I) &= \max_{a'} u(y(s) + a - q(I)a') \\ &+ \beta \sum_{s'} \pi(s'|s) V(s', a', 0; I) \end{aligned}$$

with $y(s) + a - q(I)a' \geq 0$.

- Note that the prices depend on the type of contract.

HH Problem with bad credit history

- If $h = 1$, the household value function is given by

$$V(s, a, 1; I) = \max_{a' \geq 0} u\left(y(s) + a - a'/(1+r)\right)$$
$$+\beta \sum_{s'} \pi(s'|s)\left[\rho \cdot V(s', a', 1; I) + (1-\rho) \cdot V(s', a', 0; I)\right].$$

- The solution to these problems gives decision rules: $d(s, a, 0; I)$ and $a' = g(s, a, h; I)$.
- Note that certain restrictions on the decision rules are inherent in the environment (e.g. if you don't have debt, you cannot default - i.e. $a > 0$, then $d(s, a, 0; I) = 0$).

Default Interval

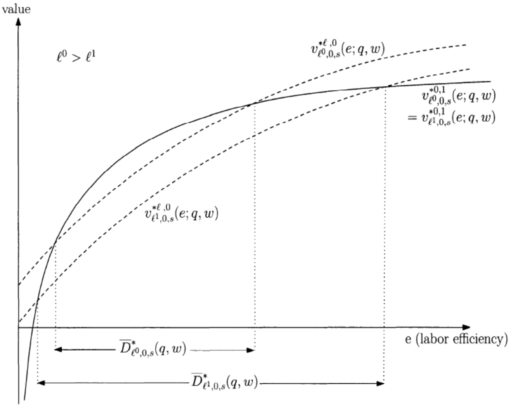

FIGURE 1.—Typical default sets conditional on household type.

- What are the properties of $d(s, a, 0; I)$? It is decreasing in s (i.e. more earnings, less default (almost everywhere)and a (i.e. lower levels of debt, less default).

Discrete Choice Models

- Given the discrete default decision rule $d(s, a, 0; I)$, the probability of default on a loan of size $a' < 0$ tomorrow depends on your stochastic earnings shock tomorrow s'.
- i.e. an agent's future default probability is stochastic.

Discrete Choice Models

- Given the discrete default decision rule $d(s, a, 0; I)$, the probability of default on a loan of size $a' < 0$ tomorrow depends on your stochastic earnings shock tomorrow s'.
- i.e. an agent's future default probability is stochastic.
- Another way to make default decisions stochastic is to use methods from the discrete choice literature (e.g. McFadden (1973) and Rust (1987)); developed to understand how 2 people with the same observable characteristics (say s) may make different choices.
- Applied to our problem, the decision to default $d = 1$ or repay $d = 0$ on debt a in earnings state s is perturbed by extreme value shocks $(\epsilon^1, \epsilon^0) \sim EV1\left(\frac{1}{\alpha}\right)$ where the higher is α, the lower is the variance of the shocks (i.e. median choice probability gets closer to the default decision rule without shocks).

Relation to Discrete Choice Models - cont.

If we let $p(s, a, 0; I)$ denote the probability of repayment $d = 0$ on debt of size $a < 0$ in earnings state s, then the probability of repayment solves

$$\begin{aligned} E_{\epsilon^1,\epsilon^0} \max_{p \in \{0,1\}} \quad & p \cdot \left\{ V^{d=0}(s, a, 0; I) + \epsilon^0 \right\} \\ + \quad & (1-p) \cdot \left\{ V^{d=1}(s, a, 0; I) + \epsilon^1 \right\} \end{aligned}$$

and is given by

$$p(s, a, 0; I) = \frac{\exp\left\{\alpha V^{d=0}(s, a, 0; I)\right\}}{\exp\left\{\alpha V^{d=0}(s, a, 0; I)\right\} + \exp\left\{\alpha V^{d=1}(s, a, 0; I)\right\}}.$$

- One can show $p(s, a, 0; I)$ is increasing in s and a (i.e. less debt) while it is decreasing in the punishment $V^{d=1}(s, a, 0; I)$.

Law of Motion for the Cross-sectional Distribution

The measure of households without bankruptcy flags include those who do not default and who have bankruptcy flags but receive the opportunity to remove the adverse event from their history:

$$\mu'(s', A, 0; I)$$
$$= \sum_s \left[\int_a 1_{\{a'=g(s,a,0;I)\in A\}}(s, a, 0) \left[(1 - d(s, a, 0; I))\right] \mu(s, da, 0; I) \right.$$
$$\left. + (1-\rho) \int_a 1_{\{a'=g(s,a,1;I)\in A\}}(s, a, 1)\mu(s, da, 1; I) \right] \pi(s'|s)$$

- Recall again restrictions implied by the environment that, for instance, if $d(s, a, 0; I) = 1$, then $g(s, a, 0; I) = 0$ as part of the bankruptcy law.

Law of Motion for the Cross-sectional Distribution -cont.

The measure of households with bankruptcy flags include those who default or who have bankruptcy flags and do not receive the chance to erase the default history.

$$\mu'(s', A, 1; I)$$
$$= \sum_{s} \Bigg[\int_{a} 1_{\{a'=g(s,a,0;I)\in A\}}(s, a, 0) d(s, a, 0; I) \mu(s, da, 0; I)$$
$$+ \rho \int_{a} 1_{\{a'=g(s,a,1;I)\in A\}}(s, a, 1) \mu(s, da, 1; I) \Bigg] \pi(s'|s)$$

Intermediary's Problem

- If L is the amount of loans made today and D' is the amount of those loans that will be defaulted on tomorrow, then the intermediary's profit on L loans is given by

$$\frac{L - D'}{1 + r} - q(I)L. \tag{1}$$

- Intermediaries will use next period's decision rules to forecast the fraction of loans that will default next period. This is where the IO part comes in; firms use household decision rules to price their products.

Pooling Prices

With pooling contracts (i.e. $I = \varnothing$), the zero profit condition implies (just divide (1) by L) the loan price with $a' < 0$ is

$$q(\varnothing) = \frac{1 - \Delta'}{1 + r} \tag{2}$$

where the loss rate $\Delta' = D'/L$ is given by the ratio of the total loan amount agents default on in period $t + 1$:

$$D' = \sum_{s',s} \int_a \quad 1_{\{g(s,a,0;\varnothing)<0\}} g(s, a, 0; \varnothing) \pi(s'|s) \\ \cdot d(s', g(s, a, 0; \varnothing), 0; \varnothing) \mu(s, da, 0; \varnothing)$$

to total borrowing in period t :

$$L = \sum_s \int_a 1_{\{g(s,a,0;\varnothing)<0\}} g(s, a, 0; \varnothing) \mu(s, da, 0; \varnothing).$$

Separating Prices

With separating contracts (i.e. $I = (a', s)$), the zero profit condition implies loan prices with $a' < 0$ are

$$q(a', s) = \frac{1 - \delta'(a', s)}{1 + r}$$

where

$$\delta'(a', s) = \sum_{s'} \pi(s'|s) d(s', a', 0).$$

- Notice that while there are many more separating prices to compute, you don't need the distribution to compute them as in the pooling case.
- The pooling case needs the distribution because there will be pooling of high and low risk agents associated with their place in the cross-sectional distribution.

Definition of Equilibrium

A steady state equilibrium is a list of value functions, decision rules, bond prices, and cross-sectional distribution such that

- Taking prices $\mathbf{q}$ and contract structure I as given, $\{V(s,a,h;\mathbf{q},\mathbf{I}), d(s,a,h;\mathbf{q},\mathbf{I})\}$ satisfy HH optimization,
- Prices are such that intermediaries earn zero profits,
- There is a fixed point of the law of motion of the cross-sectional distribution (i.e. $\mu = T^*_{\mathbf{q}}\mu$).

- As in other models without commitment (e.g. monetary models), there is always a nonborrowing (autarkic) equilibrium where $q(a', s) = 0$ for all $a' < 0$.
  - Suppose people take $q(a', s) = 0$ for all $a' < 0$ as given and find themselves in an off-the-equilibrium path position where they have debt (i.e. where $a < 0$).
  - Then it is a dominant strategy to default, so that it is optimal to set $q(a', s) = 0$.

- As stated above, there are always theoretical questions about existence of active "pooling" equilibria in the presence of competition.[1]

[1] An existence proof of a "separating" equilibrium for a prodn. economy in Chatterjee, et. al. (2007) uses Theorem 12.13 of Stokey and Lucas in our Lemma A12 on page 1578.