ECON 710A - Problem Set 5

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1. Suppose that $\{\varepsilon_t\}_{t=0}^T$ are iid random variables with mean zero, variance σ^2 and $E[\varepsilon_t^8] < \infty$. Let $U_t = \varepsilon_t \varepsilon_{t-1}$, $W_t = \varepsilon_t \varepsilon_0$, and $V_t = \varepsilon_t^2 \varepsilon_{t-1}$ where $t = 1, ..., T$.

(i) Show that $\{U_t\}_{t=1}^T$, $\{W_t\}_{t=1}^T$, and $\{V_t\}_{t=1}^T$ are covariance stationary.

For each time series, we check that (1) the second moment is finite, (2) the mean does not depend on t, and (3) the variance does not depend on t.

$\{U_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$\begin{aligned}
E[U_t^2] &= E[(\varepsilon_t \varepsilon_{t-1})^2] \\
&= E[\varepsilon_t^2 \varepsilon_{t-1}^2] \\
&= E[\varepsilon_t^2] E[\varepsilon_{t-1}^2] \\
&= E[\varepsilon_t^2]^2 \\
&= \sigma^4 \\
&< \infty
\end{aligned}$$

For (2), $E[U_t] = E[\varepsilon_t \varepsilon_{t-1}] = E[\varepsilon_t] E[\varepsilon_{t-1}] = 0$. For (3),

$$\begin{aligned}
\gamma(0) &= Cov(U_t, U_t) \\
&= Var(U_t) \\
&= Var(\varepsilon_t \varepsilon_{t-1}) \\
&= Var(\varepsilon_t) Var(\varepsilon_{t-1}) \\
&= \sigma^4
\end{aligned}$$

$$\begin{aligned}
\gamma(1) &= Cov(U_t, U_{t+1}) \\
&= E[U_t U_{t+1}] \\
&= E[(\varepsilon_t \varepsilon_{t-1})(\varepsilon_{t+1} \varepsilon_t)] \\
&= E[\varepsilon_t^2] E[\varepsilon_{t-1}] E[\varepsilon_{t+1}] \\
&= 0
\end{aligned}$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

$$
\begin{aligned}
\gamma(2) &= Cov(U_t, U_{t+2}) \\
&= E[U_t U_{t+2}] \\
&= E[(\varepsilon_t \varepsilon_{t-1})(\varepsilon_{t+2}\varepsilon_{t+1})] \\
&= E[\varepsilon_{t-1}]E[\varepsilon_t]E[\varepsilon_{t+1}]E[\varepsilon_{t+2}] \\
&= 0
\end{aligned}
$$

Thus, $\gamma(k) = \sigma^4$ if $k = 0$ and zero otherwise.

$\{W_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$
\begin{aligned}
E[W_t^2] &= E[(\varepsilon_t \varepsilon_0)^2] \\
&= E[\varepsilon_t^2 \varepsilon_0^2] \\
&= E[\varepsilon_t^2]E[\varepsilon_0^2] \\
&= E[\varepsilon_t^2]^2 \\
&= \sigma^4 \\
&< \infty
\end{aligned}
$$

For (2), $E[W_t] = E[\varepsilon_t \varepsilon_0] = E[\varepsilon_t]E[\varepsilon_0] = 0$. For (3),

$$
\begin{aligned}
\gamma(0) &= Cov(W_t, W_t) \\
&= Var(W_t) \\
&= Var(\varepsilon_t \varepsilon_0) \\
&= Var(\varepsilon_t)Var(\varepsilon_0) \\
&= \sigma^4
\end{aligned}
$$

$$
\begin{aligned}
\gamma(1) &= Cov(W_t, W_{t+1}) \\
&= E[W_t W_{t+1}] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+1}\varepsilon_0)] \\
&= E[\varepsilon_0^2]E[\varepsilon_t]E[\varepsilon_{t+1}] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma(2) &= Cov(W_t, W_{t+2}) \\
&= E[W_t W_{t+2}] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+2}\varepsilon_0)] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+2}\varepsilon_0)] \\
&= E[\varepsilon_0^2]E[\varepsilon_t]E[\varepsilon_{t+2}] \\
&= 0
\end{aligned}
$$

Thus, $\gamma(k) = \sigma^4$ if $k = 0$ and zero otherwise.

$\{V_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$\begin{aligned}
E[V_t^2] &= E[(\varepsilon_t^2 \varepsilon_{t-1})^2] \\
&= E[\varepsilon_t^4 \varepsilon_{t-1}^2] \\
&= E[\varepsilon_t^4] E[\varepsilon_{t-1}^2] \\
&= E[\varepsilon_t^4] \sigma^2 \\
&< \infty
\end{aligned}$$

For (2), $E[V_t] = E[\varepsilon_t^2 \varepsilon_{t-1}] = E[\varepsilon_t^2] E[\varepsilon_{t-1}] = 0$. For (3),

$$\begin{aligned}
\gamma(0) &= Cov(V_t, V_t) \\
&= Var(V_t) \\
&= Var(\varepsilon_t^2 \varepsilon_{t-1}) \\
&= Var(\varepsilon_t^2) Var(\varepsilon_{t-1}) \\
&= E[(\varepsilon_t^2 - E[\varepsilon_t^2])^2] \sigma^2 \\
&= E[(\varepsilon_t^2 - \sigma^2)^2] \sigma^2 \\
&= E[\varepsilon_t^4 - 2\sigma^2 \varepsilon_t^2 + \sigma^4] \sigma^2 \\
&= (E[\varepsilon_t^4] - 2\sigma^2 \sigma^2 + \sigma^4) \sigma^2 \\
&= (E[\varepsilon_t^4] - \sigma^4) \sigma^2 \\
&= \sigma^2 E[\varepsilon_t^4] - \sigma^6
\end{aligned}$$

$$\begin{aligned}
\gamma(1) &= Cov(V_t, V_{t+1}) \\
&= E[V_t V_{t+1}] \\
&= E[(\varepsilon_t^2 \varepsilon_{t-1})(\varepsilon_{t+1}^2 \varepsilon_t)] \\
&= E[\varepsilon_t^3 \varepsilon_{t-1} \varepsilon_{t+1}^2] \\
&= E[\varepsilon_t^3] E[\varepsilon_{t-1}] E[\varepsilon_{t+1}^2] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\gamma(2) &= Cov(V_t, V_{t+2}) \\
&= E[V_t V_{t+2}] \\
&= E[(\varepsilon_t^2 \varepsilon_{t-1})(\varepsilon_{t+2}^2 \varepsilon_{t+1})] \\
&= E[\varepsilon_t^2 \varepsilon_{t-1} \varepsilon_{t+2}^2 \varepsilon_{t+1}] \\
&= E[\varepsilon_t^2] E[\varepsilon_{t-1}] E[\varepsilon_{t+2}^2] E[\varepsilon_{t+1}] \\
&= 0
\end{aligned}$$

Thus, $\gamma(k) = \sigma^2 E[\varepsilon_t^4] - \sigma^6$ if $k = 0$ and zero otherwise.

(ii) Argue that the following three sample means $\bar{U}$, $\bar{W}$, $\bar{V}$ converge in probability to their expectations.

In (i), we found that $E[U_t] = E[W_t] = E[V_t] = 0 \implies E[\bar{U}] = E[\bar{W}] = E[\bar{V}] = 0$. Below I show that $Var(\bar{U}) \to 0$, $Var(\bar{V}) \to 0$, and $Var(\bar{W}) \to 0$, so by Chebyshev's inequality $\bar{U} \to_p E[\bar{U}]$, $\bar{W} \to_p E[\bar{W}]$, and $\bar{V} \to_p E[\bar{V}]$.

$$\begin{aligned}
Var(\bar{U}) &= Var\left(\frac{1}{T}\sum_{t=1}^{T} U_t\right) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} Cov(U_t, U_s) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma(t-s) \\
&= \frac{1}{T^2} T\gamma(0) \\
&= \frac{\gamma(0)}{T} \\
&= \frac{\sigma^2}{T} \\
&\to 0
\end{aligned}$$

As $T \to \infty$. Because V_t and W_t have the same autocovariance function, the variances of $\bar{W}$ and $\bar{V}$ similarly converge to zero.

(iii) Determine whether the following three sample second moments converge in probability to their expectations:

$$\hat{\gamma}_U(0) = \frac{1}{T}\sum_{t=1}^{T} U_t^2, \quad \hat{\gamma}_W(0) = \frac{1}{T}\sum_{t=1}^{T} W_t^2, \quad \hat{\gamma}_V(0) = \frac{1}{T}\sum_{t=1}^{T} V_t^2$$

Similar to (ii), we proceed by applying Chebyshev's inequality to show convergence. For $\hat{\gamma}_U(0)$,

$$E[\hat{\gamma}_U(0)] = E[\frac{1}{T}\sum_{t=1}^{T} U_t^2] = \frac{1}{T}\sum_{t=1}^{T} E[U_t^2] = \sigma^4$$

Now, let us consider the autocorrelation function for $\{U_t^2\}_{t=0}^{T}$:

$$\begin{aligned}
\gamma_{U^2}(0) &= Var(U_t^2) \\
&= E[U_t^4] - (\sigma^4)^2 \\
&= E[\varepsilon_t^4 \varepsilon_{t-1}^4] - \sigma^8 \\
&= E[\varepsilon_t^4]E[\varepsilon_{t-1}^4] - \sigma^8 \\
&= E[\varepsilon_t^4]^2 - \sigma^8
\end{aligned}$$

$$\begin{aligned}
\gamma_{U^2}(1) &= Cov(U_t^2, U_{t+1}^2) \\
&= E[U_t^2 U_{t+1}^2] - E[U_t^2]E[U_{t+1}^2] \\
&= E[(\varepsilon_t \varepsilon_{t-1})^2(\varepsilon_{t+1}\varepsilon_t)^2] - \sigma^4\sigma^4 \\
&= E[\varepsilon_t^4 \varepsilon_{t-1}^2 \varepsilon_{t+1}^2] - \sigma^8 \\
&= E[\varepsilon_t^4]\sigma^2\sigma^2 - \sigma^8 \\
&= E[\varepsilon_t^4]\sigma^4 - \sigma^8
\end{aligned}$$

$$\begin{aligned}
\gamma_{U^2}(2) &= Cov(U_t^2, U_{t+2}^2) \\
&= E[U_t^2 U_{t+2}^2] - E[U_t^2]E[U_{t+2}^2] \\
&= E[(\varepsilon_t \varepsilon_{t-1})^2(\varepsilon_{t+2}\varepsilon_{t+1})^2] - \sigma^8 \\
&= E[\varepsilon_t^2]E[\varepsilon_{t-1}^2]E[\varepsilon_{t+2}^2]E[\varepsilon_{t+1}^2] - \sigma^8 \\
&= (\sigma^2)^4 - \sigma^8 \\
&= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var\left(\frac{1}{T}\sum_{t=1}^{T} U_t^2\right) &= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} Cov(U_t^2, U_s^2) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma_{U^2}(t-s) \\
&= \frac{1}{T^2} T(E[\varepsilon_t^4]^2 - \sigma^8 + E[\varepsilon_t^4]\sigma^4 - \sigma^8) \\
&= \frac{E[\varepsilon_t^4]^2 - 2\sigma^8 + E[\varepsilon_t^4]\sigma^4}{T} \\
&\to 0
\end{aligned}$$

As $T \to \infty$. For $\hat{\gamma}_W(0)$,

$$E[\hat{\gamma}_W(0)] = E[\frac{1}{T}\sum_{t=1}^{T} W_t^2] = \frac{1}{T}\sum_{t=1}^{T} E[W_t^2] = \sigma^4$$

Now, let us consider the autocorrelation function for $\{W_t^2\}_{t=0}^{T}$:

$$\begin{aligned}
\gamma_{W^2}(0) &= Var(W_t^2) \\
&= E[W_t^4] - (\sigma^4)^2 \\
&= E[\varepsilon_t^4 \varepsilon_0^4] - \sigma^8 \\
&= E[\varepsilon_t^4]E[\varepsilon_0^4] - \sigma^8 \\
&= E[\varepsilon_t^4]^2 - \sigma^8
\end{aligned}$$

$$\begin{aligned}
\gamma_{W^2}(1) &= Cov(W_t^2, W_{t+1}^2) \\
&= E[W_t^2 W_{t+1}^2] - E[W_t^2]E[W_{t+1}^2] \\
&= E[(\varepsilon_t\varepsilon_0)^2(\varepsilon_{t+1}\varepsilon_0)^2] - \sigma^4\sigma^4 \\
&= E[(\varepsilon_t^2\varepsilon_{t+1}^2\varepsilon_0^4] - \sigma^8 \\
&= E[\varepsilon_0^4]\sigma^2\sigma^2 - \sigma^8 \\
&= E[\varepsilon_t^4]\sigma^4 - \sigma^8
\end{aligned}$$

$$\begin{aligned}
\gamma_{W^2}(2) &= Cov(W_t^2, W_{t+2}^2) \\
&= E[W_t^2 W_{t+2}^2] - E[W_t^2]E[W_{t+2}^2] \\
&= E[(\varepsilon_t\varepsilon_0)^2(\varepsilon_{t+2}\varepsilon_0)^2] - \sigma^8 \\
&= E[\varepsilon_t^2]E[\varepsilon_{t+2}^2]E[\varepsilon_0^4] - \sigma^8 \\
&= E[\varepsilon_t^4]\sigma^4 - \sigma^8
\end{aligned}$$

Thus, for $k \geq 2$, $\gamma_{W^2}(k) > 0$, so $\hat{\gamma}_W(0)$ does not converge to its expectation.

For $\hat{\gamma}_V(0)$,

$$E[\hat{\gamma}_V(0)] = E[\frac{1}{T}\sum_{t=1}^{T} V_t^2] = \frac{1}{T}\sum_{t=1}^{T} E[V_t^2] = \sigma^2 E[\varepsilon_t^4]$$

Now, let us consider the autocorrelation function for $\{V_t^2\}_{t=0}^T$:

$$\begin{aligned}
\gamma_{V^2}(0) &= Var(V_t^2) \\
&= E[V_t^4] - E[V_t^2]^2 \\
&= E[(\varepsilon_t^2\varepsilon_{t-1})^4] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= E[\varepsilon_t^8\varepsilon_{t-1}^4] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= E[\varepsilon_t^8]E[\varepsilon_t^4] - \sigma^4 E[\varepsilon_t^4]^2
\end{aligned}$$

$$\begin{aligned}
\gamma_{V^2}(1) &= Cov(V_t^2, V_{t+1}^2) \\
&= E[V_t^2 V_{t+1}^2] - E[V_t^2]E[V_{t+1}^2] \\
&= E[(\varepsilon_t^2\varepsilon_{t-1})^2(\varepsilon_{t+1}^2\varepsilon_t)^2] - \sigma^2 E[\varepsilon_t^4]\sigma^2 E[\varepsilon_t^4] \\
&= E[\varepsilon_t^6\varepsilon_{t-1}^2\varepsilon_{t+1}^4] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= E[\varepsilon_t^6]E[\varepsilon_t^4]\sigma^2 - \sigma^4 E[\varepsilon_t^4]^2
\end{aligned}$$

$$\begin{aligned}
\gamma_{V^2}(1) &= Cov(V_t^2, V_{t+2}^2) \\
&= E[V_t^2 V_{t+2}^2] - E[V_t^2]E[V_{t+2}^2] \\
&= E[(\varepsilon_t^2 \varepsilon_{t-1})^2 (\varepsilon_{t+2}^2 \varepsilon_{t+1})^2] - \sigma^2 E[\varepsilon_t^4]\sigma^2 E[\varepsilon_t^4] \\
&= E[\varepsilon_t^4 \varepsilon_{t-1}^2 \varepsilon_{t+2}^4 \varepsilon_{t+1}^2] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= E[\varepsilon_t^4]E[\varepsilon_{t-1}^2]E[\varepsilon_{t+2}^4]E[\varepsilon_{t+1}^2] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var\left(\frac{1}{T}\sum_{t=1}^{T} V_t^2\right) &= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} Cov(V_t^2, V_s^2) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma_{V^2}(t-s) \\
&= \frac{1}{T^2} T(E[\varepsilon_t^8]E[\varepsilon_t^4] - \sigma^4 E[\varepsilon_t^4]^2 + E[\varepsilon_t^6]E[\varepsilon_t^4]\sigma^2 - \sigma^4 E[\varepsilon_t^4]^2) \\
&= \frac{E[\varepsilon_t^8]E[\varepsilon_t^4] + E[\varepsilon_t^6]E[\varepsilon_t^4]\sigma^2 - 2\sigma^4 E[\varepsilon_t^4]^2}{T} \\
&\to 0
\end{aligned}$$

(iv) Determine whether the scaled sample means $\sqrt{T}\bar{U}$, $\sqrt{T}\bar{W}$, and $\sqrt{T}\bar{V}$ are asymptotically normal.

$\sqrt{T}\bar{W}$ is not asymptotically normal because $\frac{1}{T}\sum_{t=1}^{T} W_t^2$ does not converge in probability to its expectation.

We have shown that all but the martingale condition of the martingale central limit theorem hold for $\sqrt{T}\bar{U}$ and $\sqrt{T}\bar{V}$. For $\sqrt{T}\bar{U}$:

$$\begin{aligned}
E[U_t|U_{t-1}, U_{t-2}, ..., U_1] &= E[E[\varepsilon_t\varepsilon_{t-1}|\varepsilon_{t-1}, ..., \varepsilon_0]|U_{t-1}, U_{t-2}, ..., U_1] \\
&= E[\varepsilon_{t-1}E[\varepsilon_t|\varepsilon_{t-1}, ..., \varepsilon_0]|U_{t-1}, U_{t-2}, ..., U_1] \\
&= E[\varepsilon_{t-1} * 0|U_{t-1}, U_{t-2}, ..., U_1] \\
&= 0
\end{aligned}$$

Thus, by the martingale CLT, $\sqrt{T}\bar{U}$ is asymptotically normal. For $\sqrt{T}\bar{V}$:

$$\begin{aligned}
E[V_t|V_{t-1}, V_{t-2}, ..., V_1] &= E[E[\varepsilon_t^2\varepsilon_{t-1}|\varepsilon_{t-1}, ..., \varepsilon_0]|V_{t-1}, V_{t-2}, ..., V_1] \\
&= E[\varepsilon_{t-1}E[\varepsilon_t^2|\varepsilon_{t-1}, ..., \varepsilon_0]|V_{t-1}, V_{t-2}, ..., V_1] \\
&= E[\varepsilon_{t-1} * \sigma^2|V_{t-1}, V_{t-2}, ..., V_1] \\
&= \sigma^2 E[\varepsilon_{t-1}|V_{t-1}, V_{t-2}, ..., V_1] \\
&\neq 0
\end{aligned}$$

Thus, $\sqrt{T}\bar{V}$ is not asymptotically normal.

2. Consider a time series of length T from the model

$$Y_t = \alpha_0 + t\beta_0 + X_t\delta_0 + Y_{t-1}\rho_1 + U_t$$

where Y_0 and $\{U_t\}_{t=1}^{T}$ are iid $N(0,1)$, and

$$X_t = X_{t-1} \cdot 0.3 + V_t$$

where X_0 and $\{V_t\}_{t=1}^{T}$ are iid $N(0,1)$ and independent of Y_0 and $\{U_t\}_{t=1}^{T}$. We will let $\alpha_0 = \delta_0 = 100$, $\beta_0 = 1$ and consider all combinations of $T \in \{50, 150, 250\}$ and $\rho_1 \in \{0.7, 0.9, 0.95\}$.

(i) In a statistical software of your choice, generate data from (1), estimate the coefficients by OLS, and calculate heteroscedasticity robust two-sided 95% confidence intervals for α_0, δ_0, and ρ_1.

```r
tees <- c(50, 150, 250)
rhos <- c(0.7, 0.9, 0.95)
alpha <- 100
delta <- 100
beta <- 1

results <- NULL

for (t in tees) {
  for (rho in rhos) {
    x_t <- rnorm(1)
    y_t <- rnorm(1)
    v_t <- rnorm(t)
    u_t <- rnorm(t)

    for (i in 1:t) x_t[i+1] <- 0.3 * x_t[i] + v_t[i]
    for (i in 1:t) y_t[i+1] <- alpha + i * beta + x_t[i+1] * delta + y_t[i] * rho + u_t[i]

    x <- cbind(rep(1, t),
               1:t,
               x_t[2:(t+1)],
               y_t[1:t])
    y <- y_t[2:(t+1)]

    ols <- solve(t(x) %*% x) %*% (t(x) %*% y)

    e_hat <- as.numeric(y - x %*% ols)
    # omega <- crossprod(x * e_hat)
    omega <- t(x) %*% diag(e_hat^2) %*% x
    varcov <- solve(t(x) %*% x) %*% omega %*% solve(t(x) %*% x)
    se_robust <- sqrt(diag(varcov))

    results <- tibble(t = t,
           rho = rho,
           name = c("alpha", "beta", "delta", "rho"),
           ols = as.numeric(ols),
           se = se_robust) %>%
      bind_rows(results)
  }
}
```

```r
results %>%
  mutate(upper_bound = ols + se * 1.96,
         lower_bound = ols - se * 1.96) %>%
  kable(digits = 3)
```

t	rho	name	ols	se	upper_bound	lower_bound
250	0.95	alpha	100.175	0.204	100.574	99.775
250	0.95	beta	0.998	0.003	1.004	0.993
250	0.95	delta	100.007	0.060	100.125	99.888
250	0.95	rho	0.950	0.000	0.950	0.950
250	0.90	alpha	100.053	0.190	100.426	99.681
250	0.90	beta	1.000	0.002	1.004	0.995
250	0.90	delta	100.052	0.060	100.169	99.935
250	0.90	rho	0.900	0.000	0.900	0.900
250	0.70	alpha	99.924	0.166	100.250	99.598
250	0.70	beta	1.005	0.002	1.009	1.001
250	0.70	delta	100.008	0.069	100.143	99.872
250	0.70	rho	0.699	0.000	0.700	0.698
150	0.95	alpha	99.752	0.197	100.138	99.367
150	0.95	beta	1.009	0.007	1.023	0.996
150	0.95	delta	99.929	0.086	100.098	99.760
150	0.95	rho	0.950	0.000	0.950	0.950
150	0.90	alpha	99.798	0.170	100.131	99.466
150	0.90	beta	0.994	0.006	1.006	0.982
150	0.90	delta	100.135	0.072	100.277	99.994
150	0.90	rho	0.900	0.000	0.901	0.900
150	0.70	alpha	100.225	0.228	100.673	99.777
150	0.70	beta	1.002	0.003	1.007	0.996
150	0.70	delta	99.914	0.087	100.085	99.742
150	0.70	rho	0.699	0.000	0.700	0.699
50	0.95	alpha	99.885	0.352	100.575	99.195
50	0.95	beta	1.025	0.029	1.081	0.969
50	0.95	delta	99.984	0.139	100.256	99.711
50	0.95	rho	0.950	0.001	0.951	0.948
50	0.90	alpha	99.668	0.299	100.254	99.083
50	0.90	beta	1.002	0.016	1.033	0.970
50	0.90	delta	100.054	0.095	100.241	99.867
50	0.90	rho	0.900	0.000	0.901	0.899
50	0.70	alpha	100.016	0.299	100.602	99.429
50	0.70	beta	1.007	0.010	1.027	0.988
50	0.70	delta	100.091	0.123	100.333	99.849
50	0.70	rho	0.700	0.001	0.701	0.698

(ii) Across 10000 simulated repetitions of the above, report the simulated mean of the point estimators for α_0, δ_0, and ρ_1 and the simulated coverage rate of the confidence intervals.

```
ntrials <- 10000
results2 <- NULL

for (t in tees) {
  for (rho in rhos) {
    for (trial in 1:ntrials) {
      print(trial)

      x_t <- rnorm(1)
      y_t <- rnorm(1)
      v_t <- rnorm(t)
      u_t <- rnorm(t)

      for (i in 1:t) x_t[i+1] <- 0.3 * x_t[i] + v_t[i]
      for (i in 1:t) y_t[i+1] <- alpha + i * beta + x_t[i+1] * delta +
        y_t[i] * rho + u_t[i]

      x <- cbind(rep(1, t),
                 1:t,
                 x_t[2:(t+1)],
                 y_t[1:t])
      y <- y_t[2:(t+1)]

      ols <- solve(t(x) %*% x) %*% (t(x) %*% y)

      results2 <- tibble(t = t,
                         rho = rho,
                         trial = trial,
                         name = c("alpha", "beta", "delta", "rho"),
                         ols = as.numeric(ols)) %>%
        bind_rows(results2)
    }
  }
}

save(results2, file = "ps5_vonhafften_temp.RData")
```

```r
ntrials <- 10000

load("ps5_vonhafften.RData")

coverage_df <- results %>%
  full_join(results2, by=c("t", "rho", "name")) %>%
  mutate(is_covered = ols.x - 1.96* se < ols.y & ols.x + 1.96*se > ols.y,
         rho = as.factor(rho)) %>%
  group_by(t, rho, name) %>%
  summarise(mean = mean(ols.y),
            coverage_rate = sum(is_covered)/ntrials,
            .groups = "keep")

coverage_df %>%
  kable(digits = 3)
```

t	rho	name	mean	coverage_rate
50	0.7	alpha	100.007	0.877
50	0.7	beta	1.000	0.840
50	0.7	delta	100.001	0.836
50	0.7	rho	0.700	0.731
50	0.9	alpha	99.999	0.742
50	0.9	beta	1.000	0.883
50	0.9	delta	100.003	0.765
50	0.9	rho	0.900	0.852
50	0.95	alpha	100.006	0.919
50	0.95	beta	1.000	0.828
50	0.95	delta	99.999	0.936
50	0.95	rho	0.950	0.941
150	0.7	alpha	99.999	0.844
150	0.7	beta	1.000	0.911
150	0.7	delta	100.000	0.849
150	0.7	rho	0.700	0.734
150	0.9	alpha	100.000	0.680
150	0.9	beta	1.000	0.932
150	0.9	delta	100.000	0.532
150	0.9	rho	0.900	0.870
150	0.95	alpha	100.001	0.701
150	0.95	beta	1.000	0.757
150	0.95	delta	99.999	0.896
150	0.95	rho	0.950	0.818
250	0.7	alpha	100.000	0.918
250	0.7	beta	1.000	0.238
250	0.7	delta	100.001	0.971
250	0.7	rho	0.700	0.627
250	0.9	alpha	100.002	0.919
250	0.9	beta	1.000	0.926
250	0.9	delta	100.001	0.857
250	0.9	rho	0.900	0.931
250	0.95	alpha	100.002	0.858
250	0.95	beta	1.000	0.869
250	0.95	delta	100.000	0.946
250	0.95	rho	0.950	0.878

(iii) How does sample size and the degree of persistence in Y_t affect the results of the simulations.

The figure below plot the simulated points estimates from part ii that fall in the confidence intervals from part i differing by sample size (horizontal) and degree of persistence (line color). The point estimate for part i is the OLS estimate based on a single trial of simulated data and the confidence interval is the heteroskedastic robust standard error. The point estimate for part ii is the mean of OLS estimates over 10,000 trials of simulated data. Large sample sizes result in point estimates that are closer to the true value and tighter confidence intervals. For β, we see that higher degrees of persistence dramatically expand confidence intervals particularly for small samples. For δ and α, we that the confidence intervals are similarly sized across degrees of persistence and shrink with larger samples.