ECON 709B - Problem Set 1

Alex von Hafften*

11/11/2020

1. 2.1 - 2.2[1]

Find $E[E[E[Y|X_1, X_2, X_3]|X_1, X_2]|X_1]$.

Using the law of iterated expectations,

$$E[E[E[Y|X_1, X_2, X_3]|X_1, X_2]|X_1] = E[E[Y|X_1, X_2]|X_1] = E[Y|X_1]$$

If $E[Y|X] = a + bX$, find $E[YX]$ as a function of moments of X.

Using the law of iterated expectations,

$$E[YX] = E[E[YX|X]] = E[XE[Y|X]] = E[X(a + bX)] = E[aX + bX^2] = aE[X] + bE[X^2]$$

2. 2.3 Prove conclusion (4) of Theorem 2.4.

If $E|Y| < \infty$ then for any function $h(x)$ such that $E|h(X)e| < \infty$ then $E[h(X)e] = 0$.

Proof: Using the law of iterated expectations, Theorem 2.3, and conclusion (1) (i.e., $E[e|X] = 0$),

$$E[h(X)e] = E[E[h(X)e|X]] = E[h(X)E[e|X]] = E[h(X)(0)] = E[0] = 0$$

$\square$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

[1]These problems come from *Econometrics* by Bruce Hansen, revised on October 23, 2020.

3. 2.4

Suppose that the random variables Y and X only take the values 0 and 1, and have the following joint probability distribution

	$X=0$	$X=1$
$Y=0$	.1	.2
$Y=1$	.4	.3

Find $E[Y|X]$, $E[Y^2|X]$ and $\text{var}[Y|X]$ for $X=0$, $X=1$.

$$\begin{aligned}
E[Y|X=0] &= (1)P[Y=1|X=0] + (0)P[Y=0|X=0] = (1)(.4)/(.5) = .8 \\
E[Y|X=1] &= (1)P[Y=1|X=1] + (0)P[Y=0|X=1] = (1)(.3)/(.5) = .6 \\
E[Y^2|X=0] &= (1)^2P[Y=1|X=0] + (0)^2P[Y=0|X=0] = (1)^2(.4)/(.5) = .8 \\
E[Y^2|X=1] &= (1)^2P[Y=1|X=1] + (0)^2P[Y=0|X=1] = (1)^2(.3)/(.5) = .6 \\
\text{var}[Y|X=0] &= E[Y^2|X=0] - (E[Y|X=0])^2 = (.8) - (.8)^2 = 0.16 \\
\text{var}[Y|X=1] &= E[Y^2|X=1] - (E[Y|X=1])^2 = (.6) - (.6)^2 = 0.24
\end{aligned}$$

4. 2.5 (c)

Show that $\sigma^2(X)$ is the best predictor of e^2 given X.

Show that $\sigma^2(X)$ minimizes the mean-squared error and is thus the best predictor.

. . . .

5. 2.8

Suppose that Y is discrete-valued, taking values only on the non-negative integers, and the conditional distribution of Y given $X=x$ is Poisson:

$$P[Y=j|X=x] = \frac{\exp(-x'\beta)(x'\beta)^j}{j!}, j=0,1,2,...$$

Compute $E[Y|X]$ and $\text{var}[Y|X]$. Does this justify a linear regression model of the form $Y = X'\beta + e$?[2]

Using the hint, we know that $E[Y|X] = x'\beta$ and $\text{var}[Y|X] = x'\beta$.

Yes, this justifies a linear regression model because $E[e|X] = E[Y - X'\beta|X] = E[Y|X] - E[X'\beta|X] = x'\beta - x'\beta = 0$.

6. 2.10 - 2.14 Explain your answers.

If $Y = X\beta + e, X \in \mathbb{R}$, and $E[e|X] = 0$, then $E[X^2e] = 0$.

True, based on the law of iterated expectation:

$$E[X^2e] = E[E[X^2e|X]] = E[X^2E[e|X]] = E[X^2(0)] = E[0] = 0$$

If $Y = X\beta + e, X \in \mathbb{R}$, and $E[Xe] = 0$, then $E[X^2e] = 0$.

False, for a counter example, assume $X \sim N(0,1)$ and e is a degenerate random variable equal to 1. Notice that $E[Xe] = E[X] = 0$ and $E[X^2e] = E[X^2] = 1$.

[2]Hint: $P[Y=j] = \frac{\exp(-\lambda)(\lambda)^j}{j!}$, then $E[Y] = \lambda$ and $\text{var}[Y] = \lambda$.

If $Y = X'\beta + e$, and $E[e|X] = 0$, then e is independent of X.

False, for a counter example...

If $Y = X'\beta + e$, and $E[Xe] = 0$, then $E[e|X] = 0$.

False, for a counter example, assume $X \sim N(0, 1)$ and e is a degenerate random variable equal to 1. Notice that $E[Xe] = E[X] = 0$ and $E[e|X] = E[e] = 1$.

If $Y = X'\beta + e$, and $E[e|X] = 0$, and $E[e^2|X] = \sigma^2$, then e is independent of X.

False,

7. 2.16
8. 4.1 - 4.6