ECON 714B - Problem Set 3

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Problem 1 (50 points)

In the context of the environment studied in class, please prove the following proposition:[1]

Proposition 1. The allocations/price in a CE satisfy

$$c(s^t) + g(s^t) + k(s^t) = F(k(s^{t-1}), l(s^t), s_t) + (1-\delta)k(s^{t-1}) \tag{1}$$

$$\sum_{t,s^t} \beta^t \mu(s^t)[U_c(s^t)c(s^t) + U_\ell(s^t)l(s^t)] = U_c(s_0)[R_k(s_0)k_{-1} + R_b(s_0)b_{-1}] \tag{2}$$

Furthermore given allocations/prices that satisfy these equations we can construct allocations/prices that constitute a CE.

Recall from lecture: A CE is an allocation $x(s^t) = (c(s^t), \ell(s^t), k(s^t), b(s^t))$, a price system $(w(s^t), r(s^t), R_b(s^t))$, and a policy $\pi(s^t) = (\tau(s^t), \theta(s^t))$ such that

1. Given policy π and the price system, the allocation x maximizes HH utility s.t. their budget constraint:

$$\max \sum_{t,s^t} \beta^t \mu(s^t) U(c(s^t), l(s^t))$$

$$\text{s.t. } c(s^t) + k(s^t) + b(s^t) = [1-\tau(s^t)]w(s^t)\ell(s^t) + R_k(s^t)k(s^{t-1}) + R_b(s^t)b(s^{t-1})$$

where $R_k(s^t) = 1 + [1-\theta(s^t)][r(s^t) - \delta]$.

2. Firm's profits are maximized:

$$r(s^t) = F_k(k(s^{t-1}), \ell(s^t))$$
$$w(s^t) = F_\ell(k(s^{t-1}), \ell(s^t))$$

3. Government budget constraint holds:

$$b(s^t) = R_b(s^t)b(s^{t-1}) + g(s^t) - \tau(s^t)w(s^t)\ell(s^t) - \theta(s^t)[r(s^t) - \delta]k(s^{t-1})$$

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*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

[1] Please show all the steps in detail. In class we sketched out one direction of the proof.

Proof: ($\Rightarrow$)

Consider an allocation $x(s^t) = (c(s^t), \ell(s^t), k(s^t), b(s^t))$, a price system $(w(s^t), r(s^t), R_b(s^t))$, and a policy $\pi(s^t) = (\tau(s^t), \theta(s^t))$ that constitute a CE.

Let's first consider the feasibility constraint. Thus, the HH and government budget constraints hold. Substituting the government budget constraint and the definition of $R_k(s^t)$ into the HH budget constraint:

$$\begin{aligned}
& c(s^t) + k(s^t) + [R_b(s^t)b(s^{t-1}) + g(s^t) - \tau(s^t)w(s^t)\ell(s^t) - \theta(s^t)[r(s^t) - \delta]k(s^{t-1})] \\
&\qquad = [1 - \tau(s^t)]w(s^t)\ell(s^t) + [1 + [1 - \theta(s^t)][r(s^t) - \delta]]k(s^{t-1}) + R_b(s^t)b(s^{t-1}) \\
\implies & c(s^t) + k(s^t) + g(s^t) = w(s^t)\ell(s^t) + r(s^t)k(s^{t-1}) + (1 - \delta)k(s^{t-1}) \\
\implies & c(s^t) + k(s^t) + g(s^t) = F(k(s^{t-1}), l(s^t), s_t) + (1 - \delta)k(s^{t-1})
\end{aligned}$$

Because firm profits are zero $\implies F(k(s^{t-1}), l(s^t), s_t) = w(s^t)\ell(s^t) + r(s^t)k(s^{t-1})$. Thus, (1) is satisfied.

Let's now consider the implementability constraint. Let $p(s^t)$ be the multiplier on the budget constraint in the HH problem. The FOCs are:

$$\beta^t \mu(s^t) U_c(s^t) = p(s^t) \qquad [c(s^t)] \tag{3}$$

$$\beta^t \mu(s^t) U_\ell(s^t) = -p(s^t)(1 - \tau(s^t))w(s^t) \qquad [\ell(s^t)] \tag{4}$$

$$[p(s^t) - \sum_{s^{t+1}} p(s^{t+1}) R_b(s^{t+1})]b(s^t) = 0 \qquad [b(s^t)] \tag{5}$$

$$[p(s^t) - \sum_{s^{t+1}} p(s^{t+1}) R_k(s^{t+1})]k(s^t) = 0 \qquad [k(s^t)] \tag{6}$$

Multiplying both sides of the HH budget constraint by $p(s^t)$ and sum up the these constraints for all t:

$$\sum_{t,s^t} p(s^t)[c(s^t) + k(s^t) + b(s^t)] = \sum_{t,s^t} p(s^t)[(1 - \tau(s^t))w(s^t)\ell(s^t) + R_k(s^t)k(s^{t-1}) + R_b(s^t)b(s^{t-1})]$$

Substituting in (5), we can cancel all but the initial bond holdings from the constraint:

$$\sum_{t,s^t} p(s^t)[c(s^t) + k(s^t)] = p(s_0)R_b(s_0)b_{-1} + \sum_{t,s^t} p(s^t)[(1 - \tau(s^t))w(s^t)\ell(s^t) + R_k(s^t)k(s^{t-1})]$$

Substituting in (6), we can cancel all but the initial capital holdings from the constraint:

$$\sum_{t,s^t} p(s^t)c(s^t) = p(s_0)[R_k(s_0)k_{-1} + R_b(s_0)b_{-1}] + \sum_{t,s^t} p(s^t)(1 - \tau(s^t))w(s^t)\ell(s^t)$$

Substituting in (3), the constraint becomes:

$$\sum_{t,s^t} \beta^t \mu(s^t) U_c(s^t) c(s^t) = p(s_0)[R_k(s_0)k_{-1} + R_b(s_0)b_{-1}] + \sum_{t,s^t} p(s^t)(1 - \tau(s^t))w(s^t)\ell(s^t)$$

Substituting in (4), the constraint becomes:

$$\sum_{t,s^t} \beta^t \mu(s^t) U_c(s^t) c(s^t) = p(s_0)[R_k(s_0)k_{-1} + R_b(s_0)b_{-1}] - \sum_{t,s^t} \beta^t \mu(s^t) U_\ell(s^t) \ell(s^t)$$

$$\implies \sum_{t,s^t} \beta^t \mu(s^t)[U_c(s^t)c(s^t) + U_\ell(s^t) l(s^t)] = U_c(s_0)[R_k(s_0)k_{-1} + R_b(s_0)b_{-1}]$$

Because $U_c(s_0) = p(s_0)$ by (3). Thus, (2) is satisfied.

($\Leftarrow$)

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Problem 2 (25 points)

Consider the previous environment and suppose that we also have proportional consumption taxes $\{\tau_{ct}\}$. Derive the implementability constraint.

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Problem 3 (25 points)

Consider the previous environment but suppose that the government only has access to consumption $\{\tau_{ct}\}$ and labor income taxes $\{\tau_{nt}\}$.

1. Define a competitive equilibrium for this setting

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2. Show that any allocation resulting in an equilibrium of this sort can also be realized as an equilibrium in a world where the government must finance the same sequence of expenditures, but can only use labor and capital income taxes.

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