ECON 712B - Problem Set 2

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1. In class, we only considered the growth model with inelastic labor supply. This problem relaxes that restriction. Consider the benchmark neoclassical growth model, with production function: $Y_t = F(K_t, A_t N_t)$ where Y_t is output, K_t is capital, A_t is technology, and N_t is labor, and F has constant returns to scale and satisfies the usual assumptions. Technology grows exogenously at rate g: $A_{t+1} = (1+g)A_t$. Capital depreciates at rate δ so (imposing the aggregate feasibility condition) we can write the law of motion for the capital stock as: $K_{t+1} = (1-\delta)K_t + Y_t - C_t$. The representative household has time additive preferences given by: $\sum_{t=0}^{\infty} \beta^t u(C_t, 1-N_t)$. The population size is fixed, but the labor input $N_t \in [0,1]$ is now endogenous. This problem will consider the existence of a balanced growth path, which is defined as an equilibrium allocation where consumption, capital, wages W_t, and output all grow at the same constant rate, while interest rates r_t and labor N_t are constant.

(a) From conditions characterizing the equilibrium, find a system of equations that the endogenous variables C_0, N_0, W_0, r_0 must solve in a balanced growth path. (Initial capital K_0 is given.)

In a competitive equilibrium, households optimize, firms optimize, and markets clear.[1]

The household problem is

$$\begin{aligned}
\max_{\{(C_t, N_t, K_t, I_t)\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(C_t, 1-N_t) \\
\text{s.t.} & \sum_{t=0}^{\infty} p_t (C_t + I_t) = \sum_{t=0}^{\infty} p_t (r_t K_t + W_t N_t) + \pi_t \\
& K_{t+1} = (1-\delta)K_t + I_t
\end{aligned}$$

The firm problem is

$$\max_{\{K_t^d, N_t^d\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t (F(K_t^d, A_t N_t^d) - r_t K_t^d - W_t N_t^d)$$

Transform consumption, capital, investment, and wages by dividing by technology: $c_t = \frac{C_t}{A_t}$, $k_t = \frac{K_t}{A_t}$, $i_t = \frac{I_t}{A_t}$, and $w_t = \frac{W_t}{A_t}$.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

[1]Note on notation: The original problem set specifies w_t as the wages that grow over time. Here, W_t is the real wage rate on labor N_t and w_t is the real wage rate on effective labor $A_t N_t$. In addition, r_t is the real rental rate on capital.

The firm problem becomes:

$$\max_{\{k_t^d, N_t^d\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(F(A_t k_t^d, A_t N_t^d) - r_t A_t k_t^d - A_t w_t N_t^d)$$

FOC $[k_t^d]$:[2]

$$p_t F_1(A_t k_t^d, A_t N_t^d) A_t = p_t A_t r_t \implies F_1(A_t k_t^d, A_t N_t^d) = r_t$$

FOC $[N_t^d]$:

$$p_t F_2(A_t k_t^d, A_t N_t^d) A_t = p_t A_t w_t \implies F_2(A_t k_t^d, A_t N_t^d) = w_t$$

Since F has CRS $\implies F(K,N) = KF_1(K,N) + NF_2(K,N)$ by Euler's Theorem. Thus, for all t,

$$\begin{aligned}
\pi_t &= p_t(F(A_t k_t^d, A_t N_t^d) - r_t A_t k_t^d - A_t w_t N_t^d) \\
&= p_t(A_t k_t^d F_1(A_t k_t^d, A_t N_t^d) + A_t N_t^d F_2(A_t k_t^d, A_t N_t^d) - r_t A_t k_t^d - A_t w_t N_t^d) \\
&= p_t(A_t k_t^d r_t + A_t N_t^d w_t - r_t A_t k_t^d - A_t w_t N_t^d) \\
&= 0
\end{aligned}$$

The household problem becomes:

$$\begin{aligned}
&\max_{\{(c_t, N_t, i_t, k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(A_t c_t, 1 - N_t) \\
&\qquad \text{s.t. } \sum_{t=0}^{\infty} p_t(A_t c_t + A_t i_t) = \sum_{t=0}^{\infty} p_t(r_t A_t k_t + w_t A_t N_t) + (0) \\
&\qquad\qquad A_{t+1} k_{t+1} = (1-\delta) A_t k_t + A_t i_t \\
\implies &\max_{\{(c_t, N_t, i_t, k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(A_t c_t, 1 - N_t) \\
&\qquad \text{s.t. } \sum_{t=0}^{\infty} p_t(c_t + (1+g)k_{t+1} - (1-\delta)k_t - r_t k_t - w_t N_t) = 0
\end{aligned}$$

Define Legrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(A_t c_t, 1 - N_t) + \lambda \left[\sum_{t=0}^{\infty} p_t(c_t + (1+g)k_{t+1} - (1-\delta)k_t - r_t k_t - w_t N_t) \right]$$

[2] Note on notation: F_1 is the derivative with respect to the first argument of F and F_2 is the derivative with respect to the second argument of F.

FOC $[c_t]$:[3]

$$
\begin{aligned}
0 &= \beta^t u_1(A_t c_t, 1 - N_t) A_t + \lambda p_t \\
\implies -\lambda &= \frac{\beta^t u_1(A_t c_t, 1 - N_t) A_t}{p_t}
\end{aligned}
$$

FOC $[N_t]$:

$$
\begin{aligned}
0 &= \beta^t u_2(A_t c_t, 1 - N_t)(-1) - \lambda p_t w_t \\
\implies -\lambda &= -\frac{\beta^t u_2(A_t c_t, 1 - N_t)}{p_t w_t}
\end{aligned}
$$

Combining the FOCs with respect to c_t and N_t, we get the following relationship:

$$
\begin{aligned}
\frac{\beta^t u_1(A_t c_t, 1 - N_t) A_t}{p_t} &= -\frac{\beta^t u_2(A_t c_t, 1 - N_t)}{p_t w_t} \\
\implies u_1(A_t c_t, 1 - N_t) &= -\frac{1}{A_t w_t} u_2(A_t c_t, 1 - N_t)
\end{aligned}
$$

In equilibrium, markets clear:

$$
\begin{aligned}
k_t &= k_t^d && \text{(Capital Market)} \\
N_t &= N_t^d && \text{(Labor Market)} \\
F(A_t k_t, A_t N_t) &= (c_t + k_t) A_t && \text{(Goods Market)}
\end{aligned}
$$

Finally, the law of capital can be rewritten with our transformed variables:

$$
\begin{aligned}
k_{t+1} A_{t+1} &= (1 - \delta) k_t A_t + F(A_t k_t, A_t N_t) A_t - c_t A_t \\
\implies k_{t+1}(1 + g) &= (1 - \delta) k_t + F(A_t k_t, A_t N_t) - c_t
\end{aligned}
$$

On a balanced growth path, we know that $k_{t+1} = k_t$:

$$
k_t(g + \delta) + c_t = F(A_t k_t, A_t N_t)
$$

At $t = 0$, there are four equations that characterize the four unknown variables (C_0, N_0, W_0, r_0):

$$
\begin{aligned}
F_1(K_0, A_0 N_0) &= r_0 && (1) \\
F_2(K_0, A_0 N_0) &= \frac{W_0}{A_0} && (2) \\
u_1(C_0, 1 - N_t) &= -\frac{1}{W_0} u_2(C_0, 1 - N_0) && (3) \\
K_0(g + \delta) + C_0 &= A_0 F(K_0, A_0 N_0) && (4)
\end{aligned}
$$

[3]Note on notation: u_1 is the derivative with respect to the first argument of u and u_2 is the derivative with respect to the second argument of u.

(b) Show that if preferences are of the form: $u(C, 1-N) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} h(1-N), \gamma > 0, \gamma \neq 1 \\ \log C + h(1-N), \gamma = 1 \end{cases}$ for some function h, then there will be a balanced growth path.

The FOCs with respect to c_t and c_{t+1} of the Legrangian to solve the household problem imply a consumption Euler equation:

$$\frac{\beta^t u_1(A_t c_t, 1-N_t) A_t}{p_t} = \frac{\beta^{t+1} u_1(A_{t+1} c_{t+1}, 1-N_{t+1}) A_{t+1}}{p_{t+1}}$$
$$\implies u_1(A_t c_t, 1-N_t) = \frac{\beta(1+g)}{q_{t+1}} u_1(A_{t+1} c_{t+1}, 1-N_{t+1})$$

where $q_{t+1} = \frac{p_{t+1}}{p_t}$.

The FOCs with respect to N_t and N_{t+1} of the Legrangian to solve the household problem imply a labor supply Euler equation:

$$\frac{\beta^t u_2(A_t c_t, 1-N_t)}{p_t w_t} = \frac{\beta^{t+1} u_2(A_{t+1} c_{t+1}, 1-N_{t+1})}{p_{t+1} w_{t+1}}$$
$$\implies u_2(A_t c_t, 1-N_t) = \frac{\beta}{q_{t+1} v_{t+1}} u_2(A_{t+1} c_{t+1}, 1-N_{t+1})$$

where $v_{t+1} = \frac{w_{t+1}}{w_t}$.

The Legrangian also implies a no arbitrage condition, by taking the derivative with respect to k_{t+1}:

$$0 = \lambda p_t(1+g) - \lambda p_{t+1}(1-\delta) - \lambda p_{t+1} r_{t+1}$$
$$\implies \frac{p_{t+1}}{p_t} = \frac{1+g}{(1-\delta) + r_{t+1}}$$

(c) Can we characterize the qualitative dynamics using a phase diagram in the same way that we did in the case of inelastic labor supply? For example, suppose $u(C, 1-N) = \log C + h(1-N)$, that we are on a balanced growth path and then there is an increase in the rate of depreciation δ. Can you say what happens both upon impact of the shock and in the long run?

(d) Now suppose that h is a constant function, so that labor is inelastically supplied, and suppose $\gamma > 1$. Show that we can summarize the equilibrium as a system of equations governing the evolution of consumption and capital per unit of effective labor: $c_t = C_t/A_t$ and $k_t = K_t/A_t$. Find the balanced growth path levels of c_t and k_t.

(e) Now suppose the economy is on the balanced growth path, and then there is a fall in the rate of technological change g. By analyzing the qualitative dynamics of the economy, discuss what happens to c_t and k_t at the time of the change and in the long run.

(f) For a marginal change in g, find an expression showing how the fraction of output saved on the balanced growth path changes. Does savings increase or decrease? Consider first a general production function, and then specialize to Cobb-Douglas production: $F(K, N) = K^\alpha N^{1-\alpha}$.

2. At any date t, a consumer has x_t units of a non-storable good. He can consume $c_t \in [0, x_t]$ of this stock, and plant the remaining $x_t - c_t$ units. He wants to maximize: $E \sum_{t=1}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$ where $0 < \gamma < 1$ and $0 < \beta < 1$. Goods planted at date t yield $A_t(x_t - c_t)$ as of the beginning of period $t+1$, where A_t is a sequence of i.i.d. random variables that take the values of $0 < A_h < 1/\beta$ with probability π and $A_l \in (0, A_h)$ with probability $1 - \pi$.

(a) Formulate the consumer's utility maximization problem in the space of shock contingent consumption sequences. Exactly what is this space? Exactly what does the expectations operator $E(\cdot)$ mean here? Be explicit.

$$E_0 \left[\sum_{t=0}^{\infty} \frac{\beta^t}{1-\gamma} (A_{t-1} S_{t-1} - S_t)^{1-\gamma} \right]$$

(b) State the Bellman equation for this problem. It is easiest to have the consumer choose savings $s_t = x_t - c_t$. Argue that the relevant state variable for the problem is the cum-return wealth $A_{t-1} s_{t-1}$. Prove that the optimal value function is continuous, increasing, and concave in this state. How can you handle the unboundedness of the utility function?

$$V(A_{t-1}, S_{t-1}) = \max_{s_t} \frac{(A_{t-1} s_{t-1} - s_t)^{1-\gamma}}{1-\gamma} + \beta[\pi V(A_h s_t) + (1-\pi) V(A_l s_t)]$$

(c) Solve the Bellman equation and obtain the corresponding optimal policy function. (Hint: guess that the optimal function consists of saving a constant fraction of wealth.)

(d) How do you know that the consumption sequence generated by this policy function is the unique solution of the original sequence problem?

3. This problem considers the computation of the optimal growth model. An infinitely lived representative household owns a stock of capital which it rents to firms. The household's capital stock K depreciates at rate δ. Households do not value leisure and are endowed with one unit of time each period with which they can supply labor N to firms. They have standard time additive expected utility preferences with discount factor β and period utility $u(c)$. Firms produce output according to the production function $zF(K,N)$ where z is the level of technology.

(a) First, write a computer program that solves the planners problem to determine the optimal allocation in the model. Set $\beta = 0.95$, $\delta = 0.1$, $z = 1$, $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma = 2$, and $F(K,N) = K^{0.35}N^{0.65}$. Plot the optimal policy function for K and the phase diagram with the $\Delta K = 0$ and $\Delta c = 0$ lines along with the saddle path (which is the decision rule $c(K)$).

The planners problem is very similar to the one discussed in lecture 4:

$$\begin{aligned}
\max_{\{c_t,n_t,k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty}\beta^t u(c_t) \\
\text{s.t. } & c_t + k_{t+1} - (1-\delta)k_t \leq zF(k_t,n_t) \\
& 0 \leq n_t \leq 1 \\
& c_t \geq 0
\end{aligned}$$

It is optimal for households to supply $n_t = 1$ for all t because leisure is not valued. Define $F(k_t,1) = f(k_t)$. The recursive formulation of the planners problem is

$$V(k) = \max_{k'}\{u(zf(k) + (1-\delta)k - k') + \beta V(k')\}$$

FOC $[k']$:

$$0 = u'(zf(k) + (1-\delta)k - k')(-1) + \beta V'(k')$$

Envelope condition:

$$V'(k) = u'(zf(k) + (1-\delta)k - k')(zf'(k) + (1-\delta))$$

Imply an Euler condition:

$$u'(zf(k) + (1-\delta)k - k') = \beta u'(zf(k') + (1-\delta)k' - k'')(zf'(k') + (1-\delta))$$

The Euler equation and the law of motion of capital are two difference equation for the two unknowns:

$$k' = zf(k) + (1-\delta)k - c \tag{5}$$
$$u'(c) = \beta u'(c')(zf'(k') + (1-\delta)) \tag{6}$$

In a steady state $\bar{k} = k = k'$ and $\bar{c} = c = c'$:

$$\begin{aligned}
u'(\bar{c}) &= \beta u'(\bar{c})(zf(\bar{k}) + (1-\delta)) \\
\implies f'(\bar{k}) &= \frac{\beta^{-1} - (1-\delta)}{z} \\
\bar{c} &= zf(\bar{k}) - \delta\bar{k}
\end{aligned}$$

Plugging in the provided functional forms:

$$\bar{k} = \left(\frac{\beta^{-1} - (1-\delta)}{0.35z}\right)^{(-1/0.65)}$$
$$\bar{c} = z\bar{k}^{0.35} - \delta\bar{k}$$

For $\Delta k = 0$, equation (5) implies:

$$k = zf(k) + (1-\delta)k - c \implies c = zf(k) - \delta k$$

Plugging in the provided functional forms:

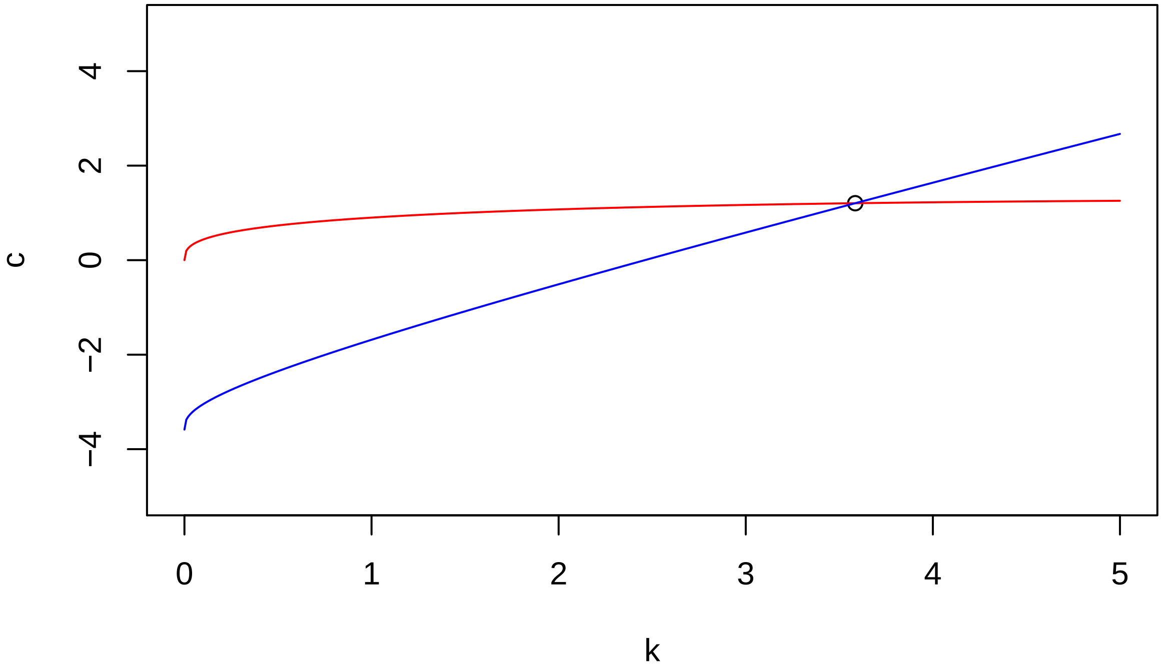
$$c = zk^{0.35} - \delta k$$

For $\Delta c = 0$, equations (5) and (6) imply:

$$f'(zf(k) + (1-\delta)k - c) = \frac{\frac{1}{\beta} - (1-\delta)}{z}$$

Plugging in the provided functional forms:

$$0.35(zk^{0.35} + (1-\delta)k - c)^{-0.65} = \frac{\frac{1}{\beta} - (1-\delta)}{z}$$
$$\implies c = zk^{0.35} + (1-\delta)k - \left(\frac{\frac{1}{\beta} - (1-\delta)}{0.35z}\right)^{1/(-0.65)}$$



(b) Re-do your calculations with $\gamma = 1.01$. What happens to the steady state? What happens to the saddle path? Interpret your answer.

(c) Now with $\gamma = 2$ assume that there is an unexpected permanent increase of 20% in total factor productivity, so now $z = 1.2$. What happens to the steady state levels of consumption and capital? Assuming the economy is initially in the steady state with $z = 1$, what happens to consumption and capital after the increase in z?