ECON 703 - PS 4

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(1) Let X, Y be two vector spaces such that dim $X = n$, dim $Y = m$. Construct a basis of $L(X, Y)$.

Let $B_X = \{u_1, ..., u_n\}$ be a basis for X. Consider $x \in X$ such that $x = c_1 u_1 + ... + c_n u_n$ for $c_1, ..., c_n \in \mathbb{R}$. Let $B_Y = \{v_1, ..., v_m\}$ be a basis for Y, so for any $T \in L(X, Y)$, $T(u_i) = b_{i1} v_1 + ... + b_{im} v_m$ for $i \in \{1, ..., n\}, b_{i1}, ..., b_{im} \in \mathbb{R}$. Thus,

$$\begin{aligned}
T(x) &= T(c_1 u_1 + ... + c_n u_n) \\
&= c_1 T(u_1) + ... + c_n T(u_n) \\
&= c_1(b_{11} v_1 + ... + b_{1m} v_m) + ... + c_n(b_{n1} v_1 + ... + b_{nm} v_m) \\
&= c_1 b_{11} v_1 + ... + c_1 b_{1m} v_m + ... + c_n b_{n1} v_1 + ... + c_n b_{nm} v_m \\
&= \begin{pmatrix} v_1 & ... & v_m \end{pmatrix} \begin{pmatrix} b_{11} & ... & b_{n1} \\ \vdots & & \vdots \\ b_{1m} & ... & b_{nm} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}
\end{aligned}$$

Therefore, $B_{L(X,Y)}$ forms a basis for $L(X, Y)$:

$$\begin{aligned}
B_{L(X,Y)} &= \left\{ \begin{pmatrix} 1 & ... & 0 \\ \vdots & & \vdots \\ 0 & ... & 0 \end{pmatrix}_{n \times m}, ..., \begin{pmatrix} 0 & ... & 1 \\ \vdots & & \vdots \\ 0 & ... & 0 \end{pmatrix}_{n \times m}, ..., \begin{pmatrix} 0 & ... & 0 \\ \vdots & & \vdots \\ 1 & ... & 0 \end{pmatrix}_{n \times m}, ..., \begin{pmatrix} 0 & ... & 0 \\ \vdots & & \vdots \\ 0 & ... & 1 \end{pmatrix}_{n \times m} \right\} \\
&= \{A_{n \times m} | a_{ij} = 1 \text{ for each } (i, j) \in \{1, ..., n\} \times \{1, ..., m\} \text{ and } a_{lk} = 0 \forall l \neq i, k \neq j\}
\end{aligned}$$

$B_{L(X,Y)}$ spans $L(X, Y)$ where b_{ij} is the coefficient on the element of $B_{L(X,Y)}$ with a one in the ith column and the jth row. $B_{L(X,Y)}$ is also linear independent because each element has a single nonzero cell whose location is to unique to the nonzero cells in other elements. Since no element of $B_{L(X,Y)}$ has the same nonzero cell as another element, any linear combination of the elements that equals an $n \times m$ matrix of zeros implies that all coefficients on elements of $B_{L(X,Y)}$ are zero.

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(2) Suppose that $T \in L(X, X)$ and λ is T's eigenvalue.

(a) Prove that λ^k is an eigenvalue of T^k, $k \in \mathbb{N}$.

Proof: If λ is T's eigenvalue $\implies T(v) = \lambda v$ for eigenvector $v \neq \bar{0}$. I prove that λ^k is an eigenvalue of T^k using induction. For $k = 1$, $T(v) = \lambda^1 v = \lambda v$. Assume λ^k is an eigenvalue for T^k, $T^{k+1}(v) = T(T^k(v)) = \lambda(\lambda^k v) = \lambda^{k+1} v$. Thus, λ^k is an eigenvalue for T^k where $k \in \mathbb{N}$. $\square$

(b) Prove that if T is invertible, then λ^{-1} is an eigenvalue of T^{-1}.

Proof: First, I show that if T is invertible then $\lambda \neq 0$. For sake of a contradiction, assume T is invertible and $\lambda = 0$. Then $T(v) = \lambda v = 0v = \bar{0} \implies T^{-1}(T(v)) = T^{-1}(\bar{0}) \implies v = \bar{0}$. This is a contradiction because eigenvectors cannot be $\bar{0}$. Thus, if T is invertible, $\lambda \neq 0$.

If T is invertible, $T^{-1} \in L(X, X)$. Thus, for eigvenvector v of T,

$$\begin{aligned} T(v) = \lambda v &\implies T^{-1}(T(v)) = T^{-1}(\lambda v) \\ &\implies v = \lambda T^{-1}(v) \\ &\implies T^{-1}(v) = \lambda^{-1} v. \end{aligned}$$

Therefore, λ^{-1} is an eigenvalue of T^{-1}.

(c) Define an operator $S : X \to X$, such that $S(x) = T(x) - \lambda x$ for all $x \in X$. Is S linear? Prove that ker $S := \{x \in X | S(x) = \bar{0}\}$ is a vector space.

For $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\begin{aligned} S(\alpha_1 x_1 + \alpha_2 x_2) &= T(\alpha_1 x_1 + \alpha_2 x_2) - \lambda(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \alpha_1 T(x_1) + \alpha_2 T(x_2) - \lambda \alpha_1 x_1 - \lambda \alpha_2 x_2 \\ &= \alpha_1 (T(x_1) - \lambda x_1) + \alpha_2 (T(x_2) - \lambda x_2) \\ &= \alpha_1 S(x_1) + \alpha_2 S(x_2) \end{aligned}$$

Yes, S is linear.

Proof: For a fixed λ, let $x, y \in \ker S$ and $\alpha, \beta \in \mathbb{R}$. We know that $S(x) = S(y) = \bar{0}$. As a linear transformation, $S(\alpha x + \beta y) = \alpha S(x) + \beta S(y) = \alpha \bar{0} + \beta \bar{0} = \bar{0}$, so properties 1, 2, 5, 6, 7 of the definition of a vector space are satisfied.

For property 3, note that $S(\bar{0}) = T(\bar{0}) - \lambda \bar{0} = T(\bar{0}) = \bar{0}$, so $\bar{0} \in \ker S$.[1] Furthermore, $x + \bar{0} = \bar{0} + x = x$ for $x \in \ker S$.

For property 4, for $x \in \ker S$, $S(-x) = (-1)S(x) = (-1)\bar{0} = \bar{0}$ where $x + (-x) = \bar{0}$.

For property 8, for $x \in \ker S$, $S(1 \cdot x) = 1 \cdot S(x) = 1 \cdot \bar{0} = \bar{0}$.

Thus $\ker S$ is a vector space. $\square$

[1] For any linear transformation $T : X \to Y$, $T(\bar{0}_X) = T(\bar{0}_X + \bar{0}_X) = T(\bar{0}_X) + T(\bar{0}_X) \implies T(\bar{0}_X) = T(\bar{0}_X) - T(\bar{0}_X) = \bar{0}_Y$.

(3) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x - y, 2x + 3y)$. Let W be the standard basis of $\mathbb{R}^2$ and let V be another basis of $\mathbb{R}^2, V = \{(1, -4), (-2, 7)\}$ in the coordinates of W.

(a) Find $\mathrm{mtx}_W(T)$.

$$\begin{aligned} T(x, y) &= (x - y)w_1 + (2x + 3y)w_2 \\ &= (w_1 + 2w_2)x + (-w_1 + 3w_2)y \end{aligned}$$

$\mathrm{mtx}_W(T) = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$.

(b) Find $\mathrm{mtx}_V(T)$.

$P = \mathrm{mtx}_{W,V}(id) = \begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix}$

$\mathrm{mtx}_V(T) = P^{-1}\mathrm{mtx}_W(T)P = \begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} -15 & 29 \\ -10 & 19 \end{pmatrix}$

(c) Find $T(1, -2)$ in the basis V.

$\mathrm{mtx}_V(T)\mathrm{mtx}_{V,W}(id) \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -15 & -10 \\ 29 & 19 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -15 & -10 \\ 29 & 19 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -13 \\ -8 \end{pmatrix}$

Check: $T(1, -2) = (1-(-2), 2(1)+3(-2)) = (3, -4)$ and $(-13)(1, -4)+(-8)(-2, 7) = (-13, 52)+(16, -56) = (3, -4)$.

(4) In this exercise you will learn to solve first order linear difference equations in n variables. We want to find an n-dimensional process $\{\mathbf{x}_1, \mathbf{x}_2, ...\}$ such that each $\mathbf{x}_i$ is an n-dimensional vector and

$$\mathbf{x}_t = A\mathbf{x}_{t-1}, t = 1, 2, ..., \tag{1}$$

where $A \in M_{n\times n}$ and $\mathbf{x}_0 \in \mathbb{R}^n$ are given. Then

$$\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0, \mathbf{x}_t = A^t\mathbf{x}_0 \forall t \in \mathbb{N},$$

where $A^t = A \cdot A \cdot ... \cdot A$ (t times). Thus, we need to calculate A^t.
To do this, we diagonalize A, $A = PDP^{-1}$, where D is diagonal, $D = diag\{\lambda_1, ..., \lambda_n\}$.
Hence we can rewrite

$$A^t = PDP^{-1}PDP^{-1}...PDP^{-1} = PD^tP^{-1} = Pdiag\{\lambda_1^t, ..., \lambda_n^t\}P^{-1},$$

which is now easy to compute. Thus, what you is

Step 1: Calculate A's eigenvalues $\lambda_1, ..., \lambda_n$ and eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$.
Remember that we need to independent eigenvectors (this holds if all eigenvalues are distinct).

Step 2: Set $D = diag\{\lambda_1, ..., \lambda_n\}$ and $P = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ (eigenvectors are columns of P).

Step 3: Calculate P^{-1} and $Pdiag\{\lambda_1^t, ..., \lambda_n^t\}P^{-1}$.

Step 4: Plug A^t from Step 3 to get $\mathbf{x}_t = A^t\mathbf{x}_0$.

Implement the above approach to solve for $\mathbf{x}_t \in \mathbb{R}^2$:

$$\mathbf{x}_t = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \mathbf{x}_{t-1}, \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Simplify your answer as much as possible.

The characteristic polynomial is $(1-\lambda)(-1-\lambda) - 4(2) = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3)$, so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$. The eigenvectors are thus solutions to:

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \mathbf{v}_1 = \bar{0}$$
$$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} \mathbf{v}_2 = \bar{0}$$

Thus, the eigenvectors are $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$. Thus,

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$$
$$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$
$$P^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

Thus,

$$
\begin{aligned}
\mathbf{x}_t &= PD^tP^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 * 3^{t-1} \\ -(-3)^{t-1} \end{pmatrix} \\
&= \begin{pmatrix} 4 * 3^{t-1} + (-3)^{t-1} \\ 2 * 3^{t-1} - (-3)^{t-1} \end{pmatrix}
\end{aligned}
$$

Below is R code verifying the answer:

```r
library(matlib)

A <- matrix(c(1, 2, 4, -1), ncol=2)
print(A)
```

```
##      [,1] [,2]
## [1,]    1    4
## [2,]    2   -1
```

```r
ev <- eigen(A)
p <- t(t(ev$vectors))
d <- diag(ev$values)
print(p)
```

```
##           [,1]       [,2]
## [1,] 0.8944272 -0.7071068
## [2,] 0.4472136  0.7071068
```

```r
print(d)
```

```
##      [,1] [,2]
## [1,]    3    0
## [2,]    0   -3
```

```r
print(p %*% d %*% inv(p))
```

```
##      [,1] [,2]
## [1,]    1    4
## [2,]    2   -1
```

```r
# for first ten x_t
x_0 <- c(1, 1)

for (t in 1:10) {
  print(paste("t =", t))
  print(p %*% d^t %*% inv(p) %*% x_0)
}
```

```
## [1] "t = 1"
##      [,1]
## [1,]    5
```

```
## [2,]    1
## [1] "t = 2"
##      [,1]
## [1,]    9
## [2,]    9
## [1] "t = 3"
##      [,1]
## [1,]   45
## [2,]    9
## [1] "t = 4"
##      [,1]
## [1,]   81
## [2,]   81
## [1] "t = 5"
##      [,1]
## [1,]  405
## [2,]   81
## [1] "t = 6"
##      [,1]
## [1,]  729
## [2,]  729
## [1] "t = 7"
##      [,1]
## [1,] 3645
## [2,]  729
## [1] "t = 8"
##      [,1]
## [1,] 6561
## [2,] 6561
## [1] "t = 9"
##       [,1]
## [1,] 32805
## [2,]  6561
## [1] "t = 10"
##       [,1]
## [1,] 59049
## [2,] 59049
```

(5) In this exercise you will learn to to solve nth order linear difference equations in one variable. We want to find a sequence of real numbers $\{z_t\}_{t=1}^{\infty}$, which satisfies

$$z_t = a_1 z_{t-1} + a_2 z_{t-2} + ... + a_n z_{t-n}, \tag{2}$$

where $a_1, ..., a_n \in \mathbb{R}$ and $z_0, z_{-1}, ..., z_{-n+1} \in \mathbb{R}$ are given.

(a) Define $\mathbf{x}_t := (z_t, z_{t-1}, ..., z_{t-n+1})'$ and rewrite Eq. (2) in the form of Eq. (1). What is A?

$$A = \begin{pmatrix} a_1 & a_2 & ... & a_{n-1} & a_n \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & ... & 1 & 0 \end{pmatrix}_{n \times n}$$

$$\begin{aligned} A\mathbf{x}_{t-1} &= \begin{pmatrix} a_1 & a_2 & a_3 & ... & a_{n-2} & a_{n-1} & a_n \\ 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_{t-1} \\ z_{t-2} \\ z_{t-3} \\ \vdots \\ z_{t-n+1} \\ z_{t-n} \end{pmatrix} \\ &= \begin{pmatrix} a_1 z_{t-1} + a_2 z_{t-2} + ... + a_{n-1} z_{t-n+1} + a_n z_{t-n} \\ z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-n+1} \end{pmatrix} \\ &= \begin{pmatrix} z_t \\ z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-n+1} \end{pmatrix} \\ &= \mathbf{x}_t \end{aligned}$$

(b) Notice that if you find the function form of $z_t = f(t)$, then you do not need to find a similar form for $z_{t-1}, ..., z_{t-n+1}$ (you use the same function $f(\cdot)$ and evaluate it at a different time). Thus, you actually do not need to calculate $Pdiag\{\lambda_1^t, ..., \lambda_n^t\}P^{-1}\mathbf{x}_0$. You only need the first coordinate of that n-dimensional vector. The first coordinate takes the form

$$\mathbf{x}_{t1} \equiv z_t = c_1\lambda_1^t + c_2\lambda_2^t + ... + c_n\lambda_n^t, \tag{3}$$

where coefficient $c_1, ..., c_n$ depend on P and $\mathbf{x}_0$. Given Eq. (3) which holds for any t and initial values $z_0, ..., z_{-n+1}$, which equations must $c_1, ..., c_n$ solve?

$$\begin{aligned}
\mathbf{x}_0 = \begin{pmatrix} z_0 \\ z_{-1} \\ \vdots \\ z_{-n+1} \end{pmatrix} &= \begin{pmatrix} c_1\lambda_1^0 + c_2\lambda_2^0 + ... + c_n\lambda_n^0 \\ c_1\lambda_1^{-1} + c_2\lambda_2^{-1} + ... + c_n\lambda_n^{-1} \\ \vdots \\ c_1\lambda_1^{-n+1} + c_2\lambda_2^{-n+1} + ... + c_n\lambda_n^{-n+1} \end{pmatrix} \\
\implies \begin{pmatrix} z_0 \\ z_{-1} \\ \vdots \\ z_{-n+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & ... & 1 \\ \lambda_1^{-1} & \lambda_2^{-1} & ... & \lambda_n^{-1} \\ \vdots & & & \vdots \\ \lambda_1^{-n+1} & \lambda_2^{-n+1} & ... & \lambda_n^{-n+1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}
\end{aligned}$$

(c) Suppose that $n = 3$, $a_1 = 2$, $a_2 = 1$, $a_3 = -2$, and $z_0 = 2$, $z_{-1} = 2$, $z_{-2} = 1$. Find the expression for z_t as a function of t.

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned}
& 0 = \det(A - \lambda I) \\
\implies & 0 = \det \begin{pmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} \\
\implies & 0 = (2-\lambda)(-\lambda)(-\lambda) + (-2)(1)(1) - (1)(1)(-\lambda) \\
\implies & 0 = -\lambda^3 + 2\lambda^2 + \lambda - 2
\end{aligned}$$

Roots at $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$. Thus,

$$\begin{aligned}
\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ (-1)^{-1} & (1)^{-1} & (2)^{-1} \\ (-1)^{-2} & (1)^{-2} & (2)^{-2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} c_1 + c_2 + c_3 \\ -c_1 + c_2 + (1/2)c_3 \\ c_1 + c_2 + (1/4)c_3 \end{pmatrix}
\end{aligned}$$

Thus, $c_1 = -1/3, c_2 = 1, c_3 = 4/3 \implies z_t = (-1)^t(-1/3) + 1 + 2^t(4/3)$.