ECON 711 - PS 2

Alex von Hafften*

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Question 1. Convex production sets, concave production functions, convex costs

Consider a production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ for a single-output firm.

(a) Prove that if the production set $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$ is convex, the production function f is concave.

Proof: Choose $(q, -z), (q', -z') \in Y$ such that $f(z) = q$ and $f(z') = q'$. The convexity of Y implies that $t(q, -z) + (1-t)(q', -z') \in Y$ for $t \in (0, 1)$. Thus, $f(tz + (1-t)z') \geq tq + (1-t)q'$ by the definition of Y. Our choice of $(q, -z), (q', -z') \implies f(tz + (1-t)z') \geq tf(z) + (1-t)f(z')$. Therefore, f is concave. $\square$

(b) Prove that if f concave, the cost function

$$c(q, w) = \min w \cdot z \text{ subject to } f(z) \geq q$$

is convex in q.

Proof: Fixing $w \in \mathbb{R}^k_+$, choose $q, q' \in \mathbb{R}$. Define $z \in Z^*(q, w)$, $z' \in Z^*(q', w)$, and $\tilde{z} \in Z^*(tq + (1-t)q', w)$ for $t \in (0, 1)$. By the concavity of f,

$$\begin{aligned}
&\tilde{z} \leq tz + (1-t)z' \\
\implies &w\tilde{z} \leq w(tz + (1-t)z') \\
\implies &w\tilde{z} \leq twz + (1-t)wz') \\
\implies &c(f(\tilde{z}), w) \leq tc(f(z), w) + (1-t)c(f(z'), w) \\
\implies &c(tq + (1-t)q', w) \leq tc(q, w) + (1-t)c(q', w)
\end{aligned}$$

Therefore, c is convex. $\square$

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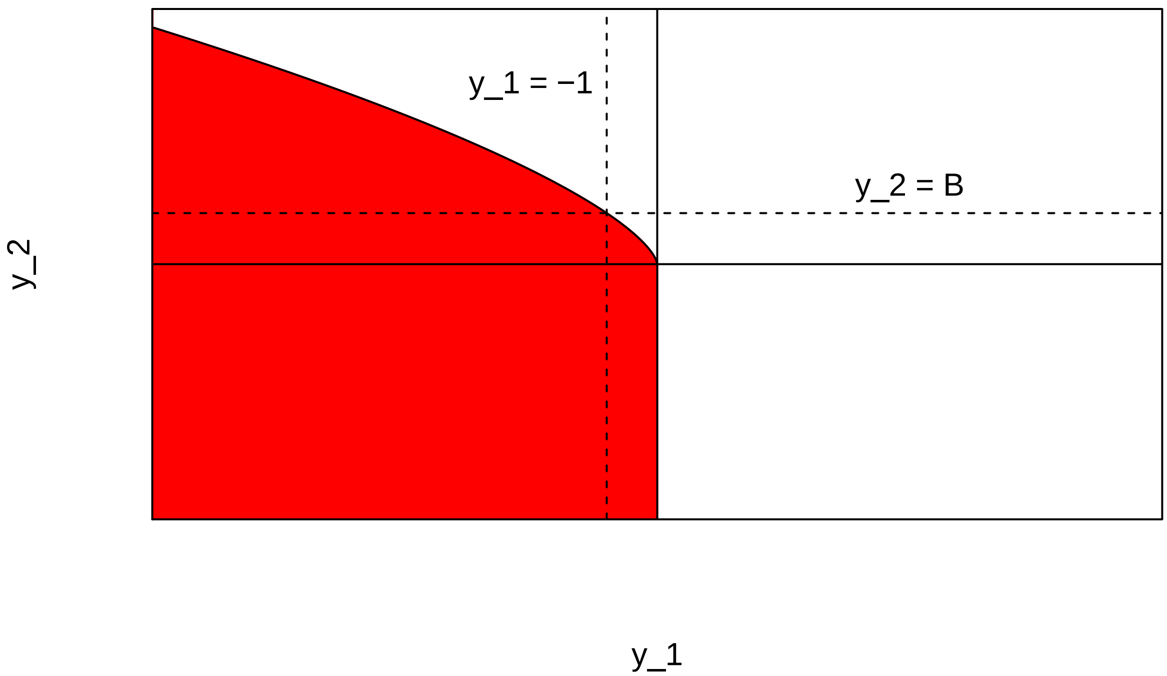
Question 2. Solving for the profit function given technology...

Let $k = 2$, and let the production set be

$$Y = \{(y_1, y_2) : y_1 \leq 0 \text{ and } y_2 \leq B(-y_1)^{\frac{2}{3}}\}$$

where $B > 0$ is a known constant. Assume both prices are strictly positive.

(a) Draw Y, or describe it clearly.



(b) Solve the firm's profit maximization problem to find $\pi(p)$ and $Y^*(p)$.

The firm's profit is

$$\pi(p) = \max_{y_1, y_2 \in Y} \{p_1 y_1 + p_2 y_2\}$$

Define $z = -y_1$. Notice that the firm will produce $y_2 = Bz^{2/3}$ because it is the maximum output given z units of input. Thus, we can rewrite the firm's profit function as

$$\pi(p) = \max_z \{p_1(-z) + p_2 B z^{2/3}\} = \max_z \{p_2 B z^{2/3} - p_1 z\}$$

Setting the first order condition of the profit function to zero:

$$\frac{\partial \pi}{\partial z} = p_2 (2/3) B z^{-1/3} - p_1$$
$$z^* = \left(\frac{2p_2 B}{3p_1}\right)^3$$

Plugging z^* into transformations for y_1, y_2:

$$\begin{aligned}
y_1^* &= -\left(\frac{2p_2B}{3p_1}\right)^3 \\
y_2^* &= B\left(\left(\frac{2p_2B}{3p_1}\right)^3\right)^{2/3} \\
&= B^3\left(\frac{2p_2}{3p_1}\right)^2
\end{aligned}$$

Notice that $Y^*(p)$ is single-valued:

$$Y^*(p) = \left\{(y_1, y_2) : y_1 = -\left(\frac{2p_2B}{3p_1}\right)^3, y_2 = B^3\left(\frac{2p_2}{3p_1}\right)^2\right\} \implies y(p) = \left(-\left(\frac{2p_2B}{3p_1}\right)^3, B^3\left(\frac{2p_2}{3p_1}\right)^2\right)$$

Thus, the profit function is

$$\begin{aligned}
\pi(p) &= p_1\left(-\left(\frac{2p_2B}{3p_1}\right)^3\right) + p_2\left(B^3\left(\frac{2p_2}{3p_1}\right)^2\right) \\
&= \frac{B^3p_2^3}{p_1^2}\left(\frac{2^2}{3^2} - \frac{2^3}{3^3}\right) \\
&= \frac{4B^3p_2^3}{27p_1^2}
\end{aligned}$$

(c) Since $Y^*(p)$ is single-valued, I'll refer to it below as $y(p)$. Verify that $\pi(\cdot)$ is homogeneous of degree 1, and $y(\cdot)$ is homogeneous of degree 0.

For $\alpha \in \mathbb{R}$:

$$\begin{aligned}
\pi(\alpha p) &= \frac{4B^3(\alpha p_2)^3}{27(\alpha p_1)^2} \\
&= \alpha\frac{4B^3(p_2)^3}{27(p_1)^2} \\
y(\alpha p) &= \left(-\left(\frac{2(\alpha p_2)B}{3(\alpha p_1)}\right)^3, B^3\left(\frac{2(\alpha p_2)}{3(\alpha p_1)}\right)^2\right) \\
&= \left(-\left(\frac{2p_2B}{3p_1}\right)^3, B^3\left(\frac{2p_2}{3p_1}\right)^2\right)
\end{aligned}$$

(d) Verify that $y_1(p) = \frac{\partial \pi}{\partial p_1}(p)$ and $y_2(p) = \frac{\partial \pi}{\partial p_2}(p)$.

$$\begin{aligned}
\frac{\partial \pi}{\partial p_1}(p) &= \frac{4B^3p_2^3}{27p_1^3}(-2) \\
&= -\frac{8B^3p_2^3}{27p_1^3} \\
&= -\left(\frac{2p_2B}{3p_1}\right)^3 \\
&= y_1(p) \\
\frac{\partial \pi}{\partial p_2}(p) &= \frac{4B^3p_2^2}{27p_1^2}(3) \\
&= \frac{4B^3p_2^2}{9p_1^2} \\
&= B^3\left(\frac{2p_2}{3p_1}\right)^2 \\
&= y_2(p)
\end{aligned}$$

(e) Calculate $D_p y(p)$, and verify it is symmetric, positive semidefinite, and $[D_p y]p = 0$

$$\begin{aligned}
D_p y(p) &= \begin{pmatrix} \frac{\partial y_1}{\partial p_1}(p) & \frac{\partial y_2}{\partial p_1}(p) \\ \frac{\partial y_1}{\partial p_2}(p) & \frac{\partial y_2}{\partial p_2}(p) \end{pmatrix} \\
&= \begin{pmatrix} \frac{8p_2^3B^3}{9p_1^4} & \frac{-8p_2^2B^3}{9p_1^3} \\ \frac{-8p_2^2B^3}{9p_1^3} & \frac{8p_2B^3}{9p_1^2} \end{pmatrix}
\end{aligned}$$

Since both off diagonal elements of $D_p y(p)$ equal $\frac{-8p_2^2B^3}{9p_1^3}$, $D_p y(p)$ is symmetric.

$$B > 0, p_1 > 0, p_2 > 0 \implies \frac{8p_2^3B^3}{9p_1^4} > 0$$

$$\begin{aligned}
\det D_p y(p) &= \frac{8p_2^3B^3}{9p_1^4}\frac{8p_2B^3}{9p_1^2} - \frac{-8p_2^2B^3}{9p_1^3}\frac{-8p_2^2B^3}{9p_1^3} \\
&= \frac{64p_2^4B^6}{81p_1^6} - \frac{64p_2^4B^6}{81p_1^6} \\
&= 0
\end{aligned}$$

Therefore, $D_p y(p)$ is positive semidefinite.

$$[D_p y]p = \begin{pmatrix} \frac{8p_2^3B^3}{9p_1^4} & \frac{-8p_2^2B^3}{9p_1^3} \\ \frac{-8p_2^2B^3}{9p_1^3} & \frac{8p_2B^3}{9p_1^2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{8p_2^3B^3}{9p_1^3} + \frac{-8p_2^3B^3}{9p_1^3} \\ \frac{-8p_2^2B^3}{9p_1^2} + \frac{8p_2^2B^3}{9p_1^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Question 3 ... and recovering technology from the profit function

Finally, suppose we didn't know a firm's production set Y, but did know its profit function was $\pi(p)Ap_1^{-2}p_2^3$ for all $p_1, p_2 > 0$ and $A > 0$ a known constant.

(a) What conditions must hold for this profit function to be rationalizable? (You don't need to check them.)

Since π is differentiable, it is rationalizable if and only if it is homogeneous of degree 1 and convex (Lecture 3 Notes).

(b) Recall that the outer bound was defined as $Y^O = \{y : p \cdot y \leq \pi(p) \text{ for all } p \in P\}$. In this case, this is $Y^O = \{(y_1, y_2) : p_1 y_1 + p_2 y_2 \leq Ap_1^{-2}p_2^3 \text{ for all } (p_1, p_2) \in \mathbb{R}^2_{++}\}$. Show that any $y \in Y^O$ must have $y_1 \leq 0$, i.e., that good 1 must be an input only.

(c) Dividing both sides by p_2 and moving $\frac{p_1}{p_2}y_1$ to the right-hand side, we can rewrite Y^O as $Y^O = \{(y_1, y_2) : y_2 \leq Ap_1^{-2}p_2^3 - \frac{p_1}{p_2}y_1 \text{ for all } (p_1, p_2) \in \mathbb{R}^2_{++}\}$. Since the expression on the right depends only on the price ratio $\frac{p_2}{p_1}$ rather than the two individual prices, we can let $r \equiv \frac{p_2}{p_1} > 0$ denote this ratio, and write Y^O as $Y_O = \{(y_1, y_2) : y_2 \leq Ar^2 - \frac{y_1}{r} \text{ for all } r \in \mathbb{R}_{++}\} = \{(y_1, y_2) : y_2 \leq \min_{r>0}(Ar^2 - \frac{y_1}{r})\}$. Solve this minimization problem, and describe the production set Y^O.

Setting the first order condition of $Ar^2 - \frac{y_1}{r}$ equal to zero:

$$\begin{aligned} 2Ar - \frac{y_1}{r^2}(-1) &= 0 \\ 2Ar &= \frac{-y_1}{r^2} \\ 2Ar^3 &= -y_1 \\ r^* &= \left(\frac{-y_1}{2A}\right)^{\frac{1}{3}} \end{aligned}$$

Plugging it back into $Ar^2 - \frac{y_1}{r}$:

$$\begin{aligned} A\left(\left(\frac{-y_1}{2A}\right)^{\frac{1}{3}}\right)^2 - y_1\left(\frac{2A}{-y_1}\right)^{\frac{1}{3}} &= \frac{A^{1/3}(-y_1)^{2/3}}{2^{2/3}} + (-y_1)^{2/3}2^{1/3}A^{1/3} \\ &= A^{1/3}(-y_1)^{2/3}(2^{-2/3} + 2^{1/3}) \\ &= \frac{27}{4}A^{1/3}(-y_1)^{2/3} \end{aligned}$$

Thus,

$$Y_O = \{(y_1, y_2) : y_2 \leq \frac{27}{4}A^{1/3}(-y_1)^{2/3}\}$$

(d) Verify that a production set Y equal to the set Y^O you just calculated would generate the "data" $\pi(p) = Ap_1^{-2}p_2^3$ that we started with. [Hint: $2^{-2/3} + 2^{1/3} = \frac{27}{4}$].