

ECON 899A - Problem Set 7

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1. *Derive the following asymptotic moments associated with $m_3(x)$: mean, variance, first order autocorrelation. Furthermore, compute $\nabla_b g(b_0)$. Which moments are informative for estimating b ?*

With $|\rho_0| < 1$, the stochastic process $\{x_t\}$ is stationary, so $E[x_t] = E[x_{t-1}]$ and $Var[x_t] = Var[x_{t-1}]$:¹

$$E[x_t] = E[\rho_0 x_{t-1} + \varepsilon_t] = \rho_0 E[x_{t-1}] = \rho_0 E[x_t]$$

$$\implies E[x_t] = 0$$

$$Var[x_t] = Var[\rho_0 x_{t-1} + \varepsilon_t] = \rho_0^2 Var[x_{t-1}] + \sigma_0^2 = \rho_0^2 Var[x_t] + \sigma_0^2$$

$$\implies Var[x_t] = \frac{\sigma_0^2}{1 - \rho_0^2}$$

$$Cov[x_t, x_{t-1}] = Cov[\rho_0 x_{t-1} + \varepsilon_t, x_{t-1}] = \rho_0 Cov[x_{t-1}, x_{t-1}] + Cov[\varepsilon_t, x_{t-1}] = \rho_0 Var[x_t] = \frac{\sigma_0^2 \rho_0}{1 - \rho_0^2}$$

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¹Alternatively, we can write x_t in terms of x_0 , $\{\varepsilon_i\}_{i=1}^t$, and ρ_0 :

$$\begin{aligned} x_t &= \rho_0 x_{t-1} + \varepsilon_t \\ &= \rho_0(\rho_0 x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \rho_0^2 x_{t-2} + \varepsilon_t + \rho_0 \varepsilon_{t-1} \\ &= \rho_0^2(\rho_0 x_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + \rho_0 \varepsilon_{t-1} \\ &= \rho_0^3 x_{t-3} + \varepsilon_t + \rho_0 \varepsilon_{t-1} + \rho_0^2 \varepsilon_{t-2} \\ &= \rho_0^t x_0 + \sum_{i=1}^t \rho_0^{t-i} \varepsilon_i \\ &= \sum_{i=1}^t \rho_0^{t-i} \varepsilon_i \\ E[x_t] &= E\left[\sum_{i=1}^t \rho_0^{t-i} \varepsilon_i\right] = \sum_{i=1}^t \rho_0^{t-i} E[\varepsilon_i] = 0 \end{aligned}$$

Thus, the asymptotic moments associated with $m_3(x)$ are:

$$\mu(x) = E[m_3(x)] = \begin{bmatrix} 0 \\ \frac{\sigma_0^2}{1-\rho_0^2} \\ \frac{\sigma_0^2 \rho_0}{1-\rho_0^2} \end{bmatrix}$$

To calculate the Jacobian, we compute the derivative of the moment conditions with respect to each parameter:

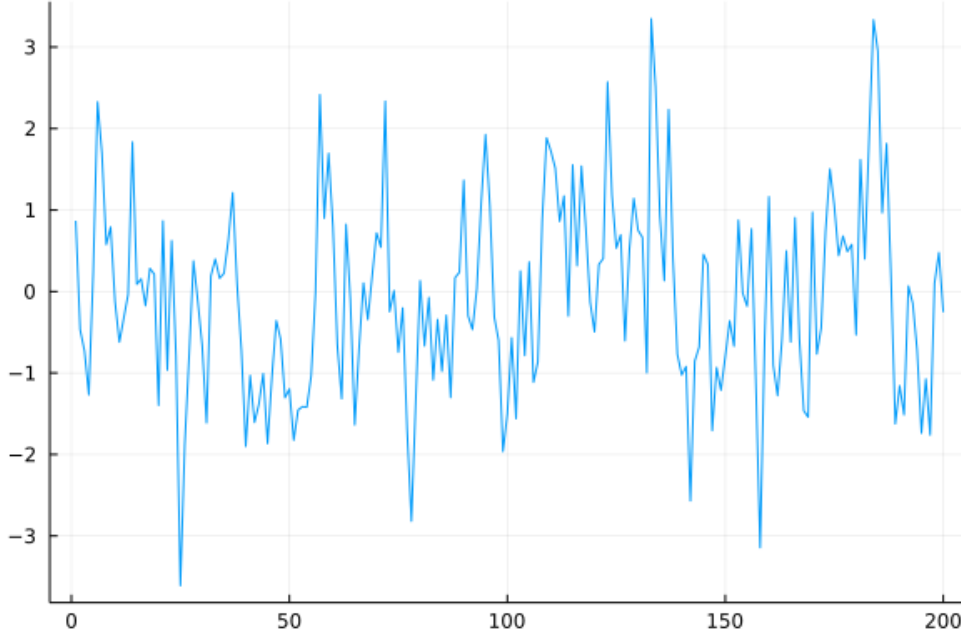
$$\begin{aligned} \frac{\partial}{\partial \rho} \left(\frac{\sigma^2}{1-\rho^2} \right) &= \frac{\sigma^2(-2\rho) - (1-\rho^2)(0)}{(1-\rho^2)^2} \\ &= \frac{-2\rho\sigma^2}{(1-\rho^2)^2} \\ \frac{\partial}{\partial \sigma} \left(\frac{\sigma^2}{1-\rho^2} \right) &= \frac{2\sigma}{1-\rho^2} \\ \frac{\partial}{\partial \rho} \left(\frac{\sigma^2 \rho}{1-\rho^2} \right) &= \frac{\sigma^2 \rho(-2\rho) - (1-\rho^2)\sigma^2}{(1-\rho^2)^2} \\ &= \frac{-\sigma^2(1+\rho^2)}{(1-\rho^2)^2} \\ \frac{\partial}{\partial \sigma} \left(\frac{\sigma^2 \rho}{1-\rho^2} \right) &= \frac{2\sigma \rho}{1-\rho^2} \end{aligned}$$

Each cell of the Jacobian is the negative of the derivative of the moment condition:

$$\Rightarrow \nabla_b g(b_0) = \begin{bmatrix} 0 & 0 \\ \frac{2\rho_0 \sigma_0^2}{(1-\rho_0^2)^2} & \frac{-2\sigma_0}{1-\rho_0^2} \\ \frac{\sigma_0^2(1+\rho_0^2)}{(1-\rho_0^2)^2} & \frac{-2\sigma_0 \rho_0}{1-\rho_0^2} \end{bmatrix}$$

Variance and first order autocorrelation are informative for estimating b .

2. Simulate a series of “true” data of length $T = 200$ using (1). We will use this to compute $M_T(x)$.



$$M_T(x) = \begin{bmatrix} -0.0764 \\ 1.3953 \\ 0.6916 \end{bmatrix}$$

3. Set $H = 10$ and simulate H vectors of length $T = 200$ random variables e_t from $N(0, 1)$. We will use this to compute $M_{TH}(y(b))$. Store these vectors. You will use the same vector of random variables throughout the entire exercise. Since this exercise requires you to estimate σ^2 , you want to change the variance of e_t during the estimation. You can simply use σe_t when the variance is σ^2 .

4-6. Different moments. Given what you found in part (1), do you think there will be a problem? In general we would not know whether this case would be a problem, so hopefully the standard error of the estimate of b as well as the J test will tell us something.

- Set $W = I$ and graph in three dimensions, the objective function (3) over $\rho \in [0.35, 0.65]$ and $\sigma \in [0.8, 1.2]$. Obtain an estimate of b by using $W = I$ in (4) using `fminsearch`. Report \hat{b}_{TH}^1 .
- Set $i(T) = 4$. Obtain an estimate of W^* . Using $\hat{W}_{TH}^* = \hat{S}_{TH}^{-1}$ in (4), obtain an estimate of \hat{b}_{TH}^2 . Report \hat{b}_{TH}^2 .
- To obtain standard errors, compute numerically $\nabla_b g_T(\hat{b}_{TH}^2)$ defined in (6). Report the values of $\nabla_b g_T(\hat{b}_{TH}^2)$. Next, obtain the $\ell \times \ell$ variance-covariance matrix of \hat{b}_{TH}^2 as in (7). Finally, what are the standard errors defined in (8)? How can we use the information on $\nabla_b g_T(\hat{b}_{TH}^2)$ to think about local identification?
- Since we are in the just identified case, the J test should be zero (on a computer this may be not be exact). However, given the identification issues in this particular case where we use mean and variance, the J test may not be zero. Compute the value of the J test:

$$T \frac{H}{1+H} \times J_{TH}(\hat{b}_{TH}^2) \rightarrow \chi^2$$

noting that in this just identified case $n - \ell = 0$ degrees of freedom recognizing that there really is not distribution.

- From part (1), I found that in expectation the mean is zero, so the problem is that the mean is not informative for estimating b . Both the variance and the first order covariance were non-zero, so they are likely informative for estimating b . Thus, I think there will be a problem with (4), but not (5) and (6)
- I plot the objective function for both the identity weighting matrix and the estimated optimal weighting matrices for each case on the following pages.
- Table 1 shows the first stage estimates, first stage standard errors, second stage estimates, and second stage standard errors for ρ and σ as well as j-test statistics and p-values.

Table 1: Estimates and Standard Errors

| | Mean and Variance | Variance and Covariance | Mean, Variance and Covariance |
|----------------|-------------------|-------------------------|-------------------------------|
| rho_hat_1 | 0.707 | 0.513 | 0.509 |
| rho_se_1 | 0.464 | 0.055 | 0.055 |
| sigma_hat_1 | -0.832 | 1.015 | 1.018 |
| sigma_se_1 | 0.557 | 0.055 | 0.054 |
| rho_hat_2 | -0.995 | 0.513 | 0.519 |
| rho_se_2 | 8.693 | 0.185 | 0.183 |
| sigma_hat_2 | 0.094 | 1.015 | 1.009 |
| sigma_se_2 | 78.177 | 0.123 | 0.124 |
| j_test_stat | 0.036 | 0.000 | 0.184 |
| j_test_p_value | 0.151 | 0.000 | 0.332 |

- Using the mean and variance, the first-stage coefficients estimates are way off. This makes sense, in part 1, I argued that the mean doesn't contain relevant information for estimating the parameters. The first stage standard errors are similarly quite large (about ten times larger than the variance and auto covariance case).
- The second-stage coefficients don't really get much better and the standard errors actually get much larger.
- The j-test statistic is around 0.04, which corresponds to a p-value of around 0.15, so we cannot reject the hypothesis that the model is true.
- Using the mean and variance, the first-stage coefficients estimates are way off. This makes sense, in part 1, I argued that the mean doesn't contain relevant information for estimating the parameters. The first stage standard errors are similarly quite large (about ten times larger than the variance and auto covariance case).
- To obtain the standard errors, I computed the following Jacobians:

Table 2: Jacobian (Mean and Variance): First-Stage

| | |
|--------|--------|
| 0.258 | -0.089 |
| -3.975 | 3.317 |

Table 3: Jacobian (Variance and Covariance): First-Stage

| | |
|--------|--------|
| -1.902 | -2.738 |
| -2.385 | -1.359 |

Table 4: Jacobian (Mean, Variance, and Covariance): First-Stage

| | |
|--------|--------|
| -0.109 | -0.053 |
| -1.872 | -2.730 |
| -2.361 | -1.342 |

Table 5: Jacobian (Mean and Variance): Second-Stage

| | |
|---------|---------|
| 0.102 | -0.016 |
| 159.659 | -17.752 |

Table 6: Jacobian (Variance and Covariance): Second-Stage

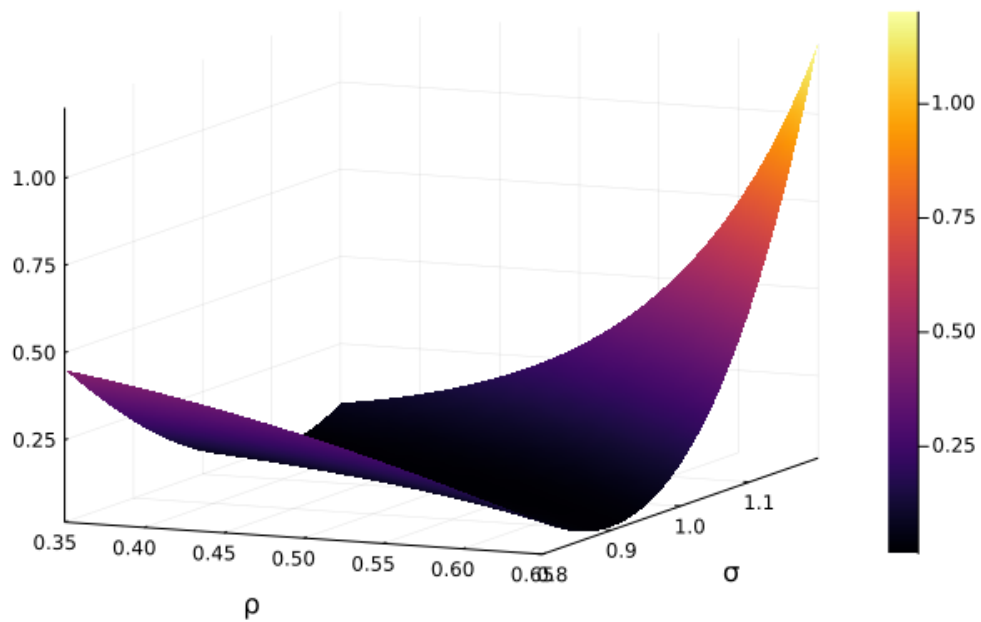
| | |
|--------|--------|
| -1.901 | -2.737 |
| -2.384 | -1.359 |

Table 7: Jacobian (Mean, Variance, and Covariance): Second-Stage

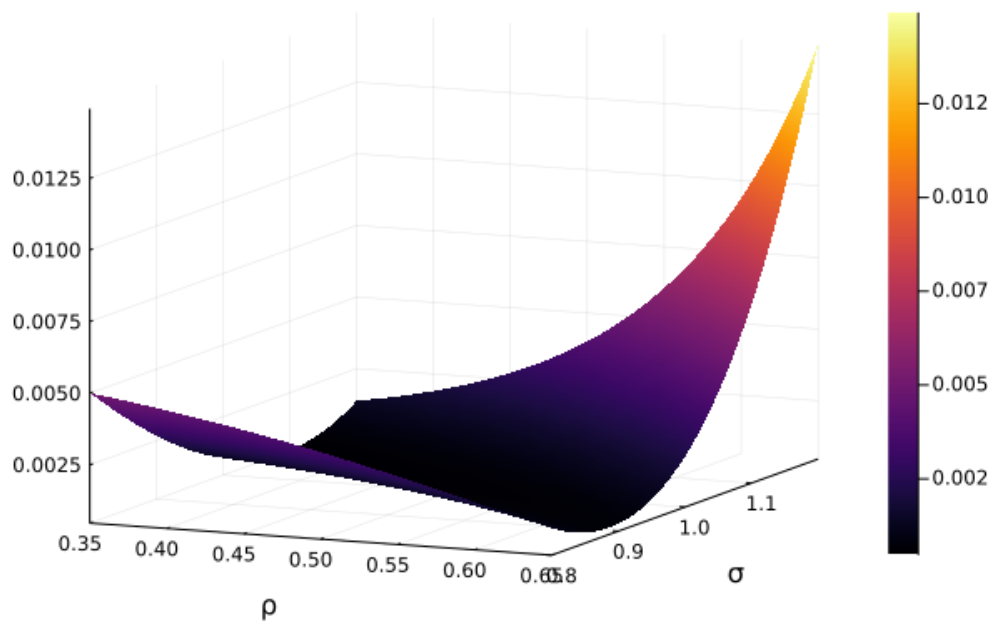
| | |
|--------|--------|
| -0.112 | -0.054 |
| -1.931 | -2.742 |
| -2.405 | -1.377 |

4. The just identified case where m_2 uses mean and variance.

First Stage Objective Function: m1 m2

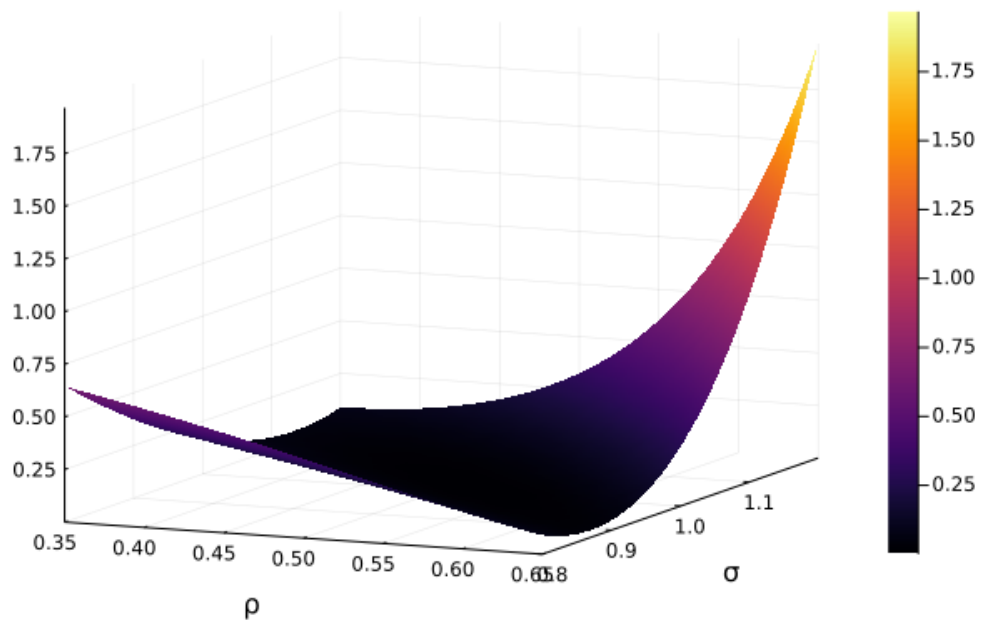


Second Stage Objective Function: m1 m2

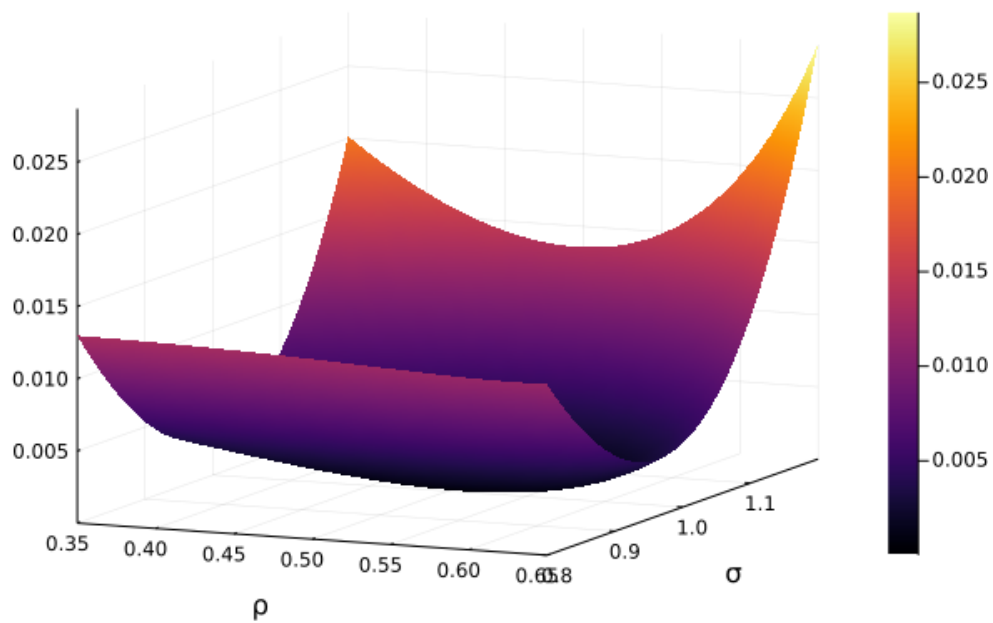


5. The just identified case where m_2 uses the variance and autocorrelation.

First Stage Objective Function: m2 m3

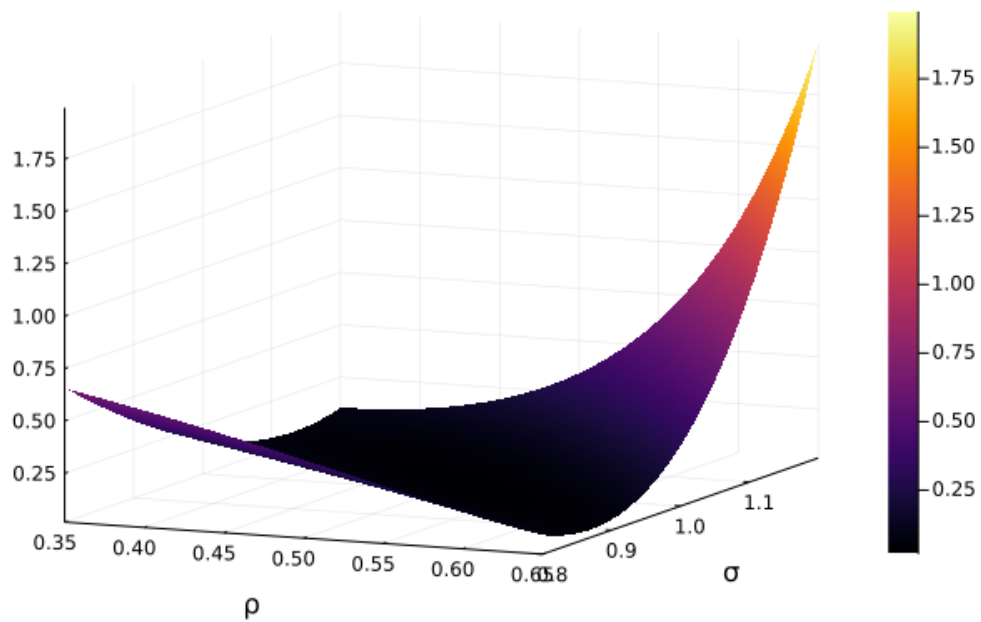


Second Stage Objective Function: m2 m3

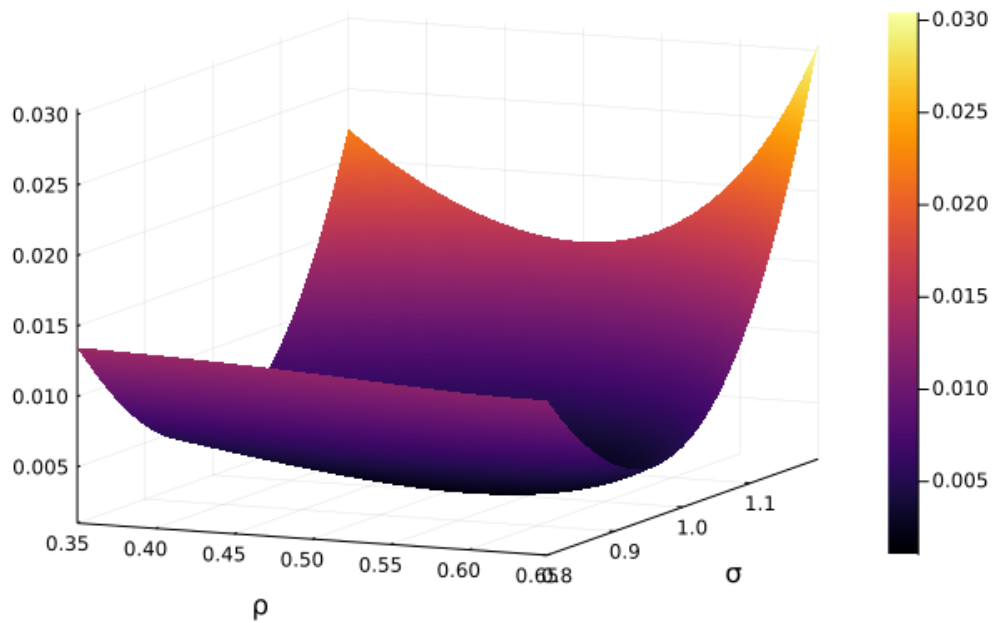


6. The overidentified case where m_3 uses the mean, variance and autocorrelation.

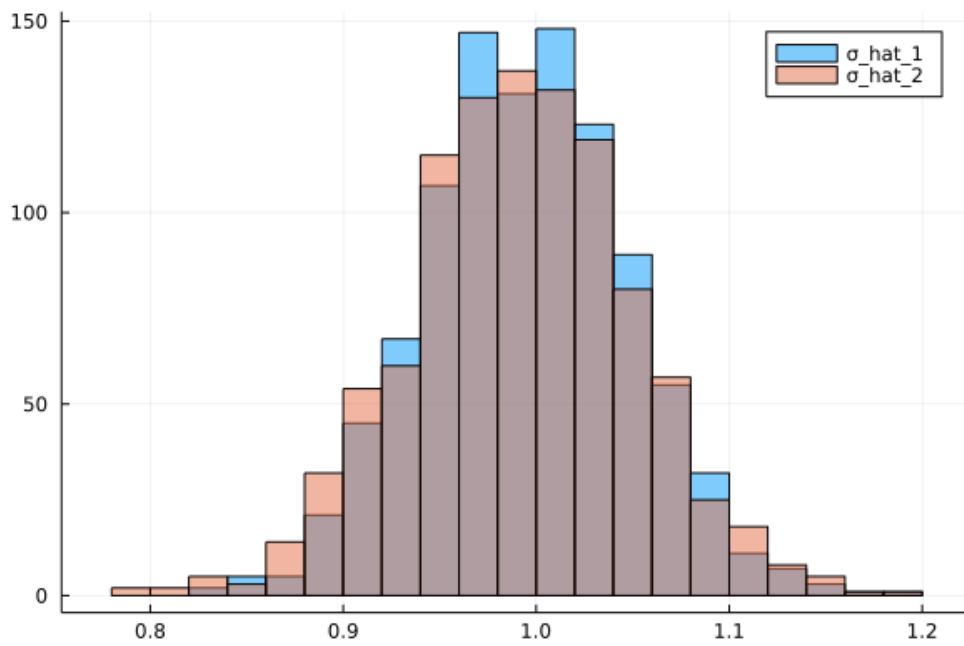
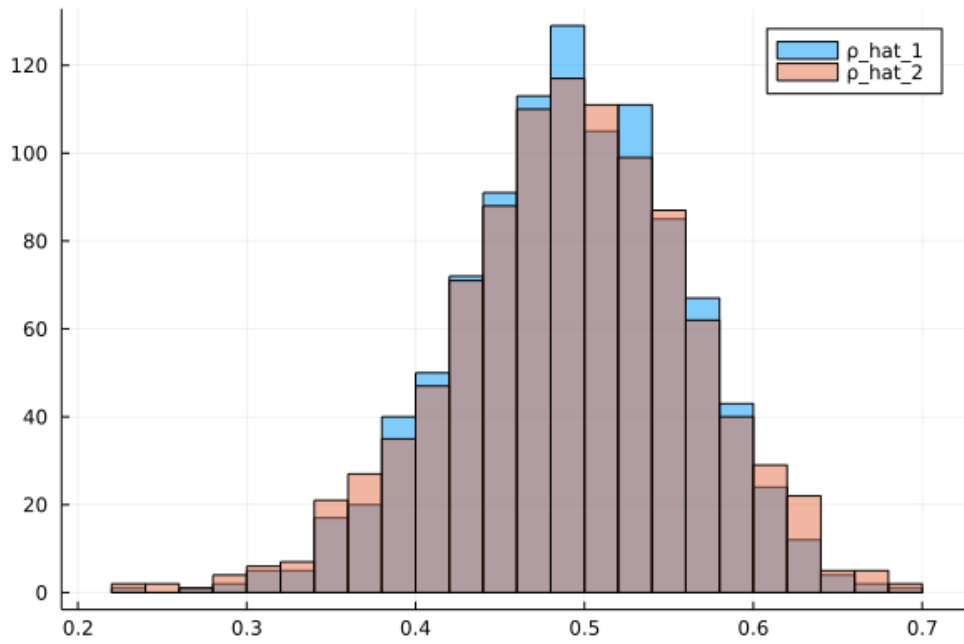
First Stage Objective Function: m1 m2 m3



Second Stage Objective Function: m1 m2 m3



6. (e) Bootstrap the the finite sample distribution of the estimators by repeatedly drawing ε_t and e_t^h from $N(0, 1)$ for $t = 1, \dots, T$ and $h = 1, \dots, H$.



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