ECON 703 - PS 7

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(1) Let $X \subset \mathbb{R}^n$ be a convex set, and $\lambda_1, ..., \lambda_k \geq 0$ with $\sum_{i=1}^k \lambda_i = 1$. Prove that if $x_1, ..., x_k \in X$, then $\sum_{i=1}^k \lambda_i x_i \in X$.

Proof (by induction): For the base step, choose $\lambda_1^2, \lambda_2^2 \geq 0$ such that $\lambda_1^2 + \lambda_2^2 = 1$.[1] For any $x_1, x_2 \in X \subset \mathbb{R}^n$, $\lambda_1^2 x_1 + \lambda_2^2 x_2 \in X$ because X is convex. For some k, assume that $\sum_{i=1}^k \lambda_i^k x_i \in X$ for $x_1, ..., x_k \in X$ with $\lambda_1^k, ..., \lambda_k^k \geq 0$ and $\sum_{i=1}^k \lambda_i^k = 1$. Consider $k+1$. Choose $\lambda_1^{k+1}, ..., \lambda_{k+1}^{k+1} \geq 0$ such that $\sum_{i=1}^{k+1} \lambda_i^{k+1} = 1$:

$$\sum_{i=1}^{k+1} \lambda_i^{k+1} x_i = \sum_{i=1}^{k} \lambda_i^{k+1} x_i + \lambda_{k+1}^{k+1} x_{k+1} = \left(\sum_{i=1}^{k} \lambda_i^{k+1} \right) \sum_{i=1}^{k} \left(\frac{\lambda_i^{k+1}}{\sum_{i=1}^k \lambda_i^{k+1}} x_i \right) + \lambda_{k+1}^{k+1} x_{k+1}$$

By the induction hypothesis, $y := \sum_{i=1}^k \left(\frac{\lambda_i^{k+1}}{\sum_{i=1}^k \lambda_i^{k+1}} x_i \right) \in X$ because $\sum_{i=1}^k \frac{\lambda_i^{k+1}}{\sum_{i=1}^k \lambda_i^{k+1}} = 1$. Thus,

$$\sum_{i=1}^{k+1} \lambda_i^{k+1} x_i = \left(\sum_{i=1}^{k} \lambda_i^{k+1} \right) y + \lambda_{k+1}^{k+1} x_{k+1}$$

By the definition of convexity, $\sum_{i=1}^{k+1} \lambda_i^{k+1} x_i \in X$ because $\sum_{i=1}^k \lambda_i^{k+1} + \lambda_{k+1}^{k+1} = 1$. $\square$

(2) The sum $\sum_{i=1}^k \lambda_i x_i$ defined in Problem (1) is called a convex combination. The convex hull of a set S, denoted by $\text{co}(S)$, is the intersection of all convex sets which contain S. Prove that the set of all convex combinations of the elements of S is exactly $\text{co}(S)$.

Proof: We show that an arbitrary convex combination of elements of S is in $\text{co}(S)$ and an arbitrary point in $\text{co}(S)$ can be represented by a convex combination of elements of S. First, notice that $S \subset \text{co}(S)$ and $\text{co}(S)$ is convex because it is the intersection of convex sets.

Consider an arbitrary convex combination of elements of S, $\sum_{i=1}^k \lambda_i s_i$ with $s_1, ..., s_k \in S$. Since $s_i \in S$, $s_i \in \text{co}(S)$ for $i \in \{1, ..., k\}$. Since $\text{co}(S)$ is convex, $\sum_{i=1}^k \lambda_i s_i \in \text{co}(S)$.

Consider $x \in \text{co}(S)$. Assume for the sake of a contradiction that x cannot be represented as a convex combination of elements of S. Then there exists a convex set Y such that $S \subset Y$ and $x \notin Y$. This is a contradiction because $\text{co}(S)$ is the intersection of all convex sets which contain S. Thus, x can be represented as a convex combination of elements of S. $\square$

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[1] A note on notation; λ_i^j denotes the coefficient on x_i when the convex combination is composed of j elements. For example, λ_i^2 pertains to the base step, λ_i^k pertains to the induction hypothesis, and λ_i^{k+1} pertains to the induction step.

(3) For any set $X \subset \mathbb{R}^n$, let its closure be $\text{cl}X = X \cup \{\text{all limit points of } X\}$. Show that the closure of a convex set is convex.

Proof: Let X be a convex set. Choose two points $x, y \text{cl}X$. Thus, there exists sequences $\{x_n\}, \{y_n\} \in X$ such that $\{x_n\} \to x$ and $\{y_n\} \to y$. Since X is convex, $\lambda x_n + (1-\lambda) y_n \in X$ for all n with $\lambda \in [0,1]$. Because $\text{cl}X$ contains all limit points of X, $\lambda x + (1-\lambda) y = \lim_{n\to\infty}(\lambda x_n + (1-\lambda) y_n) \in \text{cl}X$. $\square$

(4) The function $f : X \to \mathbb{R}$, where X is a convex set in $\mathbb{R}^n$, is concave if $\forall \lambda \in [0,1], x', x'' \in X$, $f((1-\lambda)x' + \lambda x'') \geq (1-\lambda) f(x') + \lambda f(x'')$. Given a function $f : X \to \mathbb{R}$, its hypograph is the set of points (y, x) lying on or below the graph of the function: $\text{hyp}f = \{(y,x) \in \mathbb{R}^{n+1} | x \in X, y \leq f(x)\}$. Show that the function f is concave if and only if its hypograph is a convex set.

Proof: Assume a function $f : X \to \mathbb{R}$ is concave where X is a convex set in $\mathbb{R}^n$. To show that its hypograph is a convex set, we need to show that, for any $\lambda \in [0,1]$ and $(y', x'), (y'', x'') \in \text{hyp}f$, $\lambda(y', x') + (1-\lambda)(y'', x'') = (\lambda y' + (1-\lambda) y'', \lambda x' + (1-\lambda) x'') \in \text{hyp}f$. First, notice that since X is convex, $\lambda x' + (1-\lambda) x'' \in X$. Since f is concave, $f(\lambda x' + (1-\lambda) x'') \leq (1-\lambda) f(x') + \lambda f(x'') \leq (1-\lambda) y' + \lambda y''$. Thus, $\lambda(y', x') + (1-\lambda)(y'', x'') \in \text{hyp}f$.

Assume that the hypograph of a function $f : X \to \mathbb{R}$ is convex. Choose $(x', y'), (x'', y'') \in \text{hyp}f$. To show that X is convex, we need to show that, for any $\lambda \in [0,1]$, $f((1-\lambda)x' + \lambda x'') \geq (1-\lambda) f(x') + \lambda f(x'')$. Since $\text{hyp}f$ is convex, we know that $\lambda f(x') + (1-\lambda) f(x'') \leq \lambda y' + (1-\lambda) y'' \leq f((1-\lambda)x' + \lambda x'')$. Thus, f is concave.

(5) Let X and Y be disjoint, closed, and convex sets in $\mathbb{R}^n$, one of which is compact. Show that there exists a hyperplane $H(p, \alpha)$ that strictly separates X and Y.

(6) Call a vector $\pi \in \mathbb{R}^n$ a probability vector if $\sum_{i=1}^n \pi_i = 1$ and $\pi_i \geq 0$ for all $i = 1, ..., n$. Interpretation is that there are n states of the world and π_i is the probability that state i occurs. Suppose that Alice and Bob each have a set of probability distributions (Π_A and Π_B) which are nonempty, convex, and compact. They propose bids on each state of the world. A vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$, where x_i denotes the net transfer Alice receives from Bob in state i, is called a trade (Thus, $-x$ is the net transfer Bob receives in each state of the world.) A trade is agreeable if $\inf_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i x_i > 0$ and $\inf_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i(-x_i) > 0$. The above means that both Alice and Bob expect to strictly gain from the trade. Prove that there exists an agreeable trade iff there is no common prior (i.e., $\Pi_A \cap \Pi_B = \emptyset$).