ECON 713B - Problem Set 1

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1 All Pay Auction

Consider a symmetric IPV (independent private values) setting with N bidders. Find an equilibrium of the all-pay auction when each bidder's valuation is an iid draw from $F(x) = x^a$ for $a \in (0, \infty)$ and $x \in [0, 1]$.

(a) Define this auction as a Bayesian game.

A Bayesian game is a five-tuple $(I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot))$:

- The set of players is $I = \{1, ..., N\}$.
- The action set of player $i \in I$ is $S_i = [0, \infty)$.
- The payoff for player $i \in I$ is

$$u_i(b_1, ..., b_N; v_1, ..., v_N) = u_i(b_1, ..., b_N; v_i) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{1}{2}(v_i - b_i) + \frac{1}{2}(-b_i) & \text{if } b_i = \max_{j \neq i} b_j \\ -b_i & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

- $\Theta = [0, 1] \times ... \times [0, 1]$.
- $F(x) = x^a$ for $a \in (0, \infty)$

(b) Find equilibrium strategies of all players.

Focus on BNE with symmetric, strictly increasing, and differentiable bids $b(v_i)$. Since the F is continuous and $b(v_i)$ is strictly increasing, the probability of a tie is zero. The expected payoff for bidder i is:

$$\begin{aligned} E[u_i(b_1, ..., b_N; v_i)] &= (v_i - b_i) \Pr(b_i > b_j, \forall j \neq i) + (-b_i) \Pr(b_i < b_j, \forall j \neq i) \\ &= v_i \Pr(b_i > b_j, \forall j \neq i) - b_i \end{aligned}$$

Suppose bidder $j \neq i$ submit $b(v_j)$:

$$\begin{aligned} \Pr(b_i > b_j, \forall j \neq i) &= \Pr(b(v_i) > b(v_j), \forall j \neq i) \\ &= \Pr(b^{-1}(b(v_i)) > v_j, \forall j \neq i) \\ &=^{iid} F(b^{-1}(b(v_i)))^{N-1} \\ &= ((b^{-1}(b(v_i)))^a)^{N-1} \\ &= (b^{-1}(b(v_i)))^{aN-a} \end{aligned}$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

Thus, the expected payoff of bidder i is:

$$E[u_i(b_1, ..., b_N; v_i)] = v_i(b^{-1}(b(v_i)))^{aN-a} - b(v_i)$$

FOC $[b(v_i)]$:

$$\begin{aligned}
0 &= (aN-a)v_i(b^{-1}(b(v_i)))^{aN-a-1}\frac{1}{b'(v_i)} - 1 \\
\implies b'(v_i) &= (aN-a)v_i^{aN-a} \\
\implies b(v_i) &= \frac{aN-a}{aN-a+1}v_i^{aN-a+1} + c_i
\end{aligned}$$

For $v_i = 0$, bidder i would bid zero: $b(v_i) = 0 \implies c_i = 0$.

$$\implies b(v_i) = \frac{aN-a}{aN-a+1}v_i^{aN-a+1}$$

Since $aN - a + 1 > 0$ and $\frac{aN-a}{aN-a+1}$, b is strictly increasing.

(c) Verify that the strategies that you have found do constitute an equilibrium.

We can verify that b is an equilibrium strategy by verifying that $b(v_i)$ is the best response for player i when bidders $j \neq i$ bid $b(v_j)$.

$$\begin{aligned}
E[u_i(b_1, ..., b_N; v_i)] &= v_i \Pr\left(b_i > \frac{aN-a}{aN-a+1}v_j^{aN-a+1}, \forall j \neq i\right) - b_i \\
&= v_i \Pr\left(\left(\frac{aN-a+1}{aN-a}b_i\right)^{\frac{1}{aN-a+1}} > v_j, \forall j \neq i\right) - b_i \\
&= v_i\left(\frac{aN-a+1}{aN-a}b_i\right)^{\frac{aN-a}{aN-a+1}} - b_i
\end{aligned}$$

FOC $[b_i]$:

$$\begin{aligned}
0 &= v_i\frac{aN-a}{aN-a+1}\left(\frac{aN-a+1}{aN-a}b_i\right)^{\frac{aN-a}{aN-a+1}-1}\frac{aN-a+1}{aN-a} - 1 \\
\implies \frac{1}{v_i} &= \left(\frac{aN-a+1}{aN-a}b_i\right)^{\frac{-1}{aN-a+1}} \\
\implies b_i = b(v_i) &= \frac{aN-a}{aN-a+1}v_i^{aN-a+1}
\end{aligned}$$

Thus, all bidders playing $b(\cdot)$ is an equilibrium.

(d) Does the bidding become more competitive when a increases? Explain.

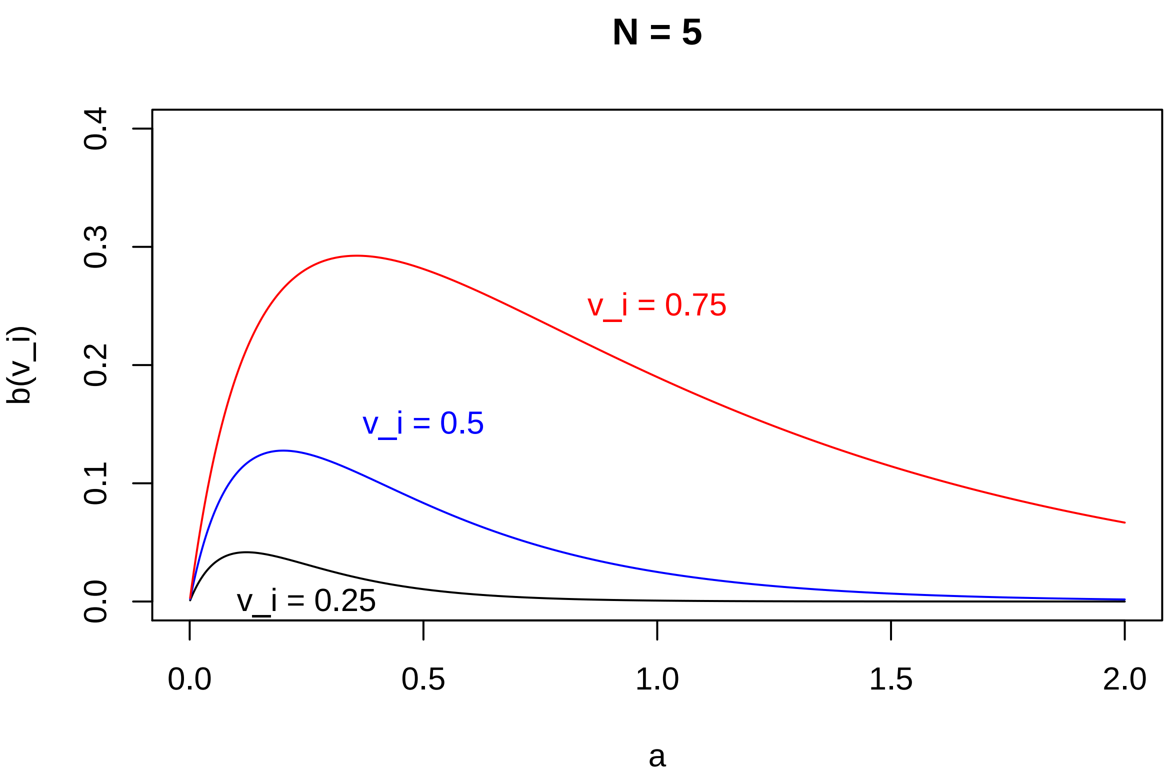
Conditional on v_i and N, a higher a results in more mass closer to 1. Thus, the probability that bidder i wins the auction decreases, so bidder i should decrease her bid. This makes the bidding less competitive. Unconditional on v_i, a higher a results in higher realizations of v_i, so the bids are correspondingly larger.

Consider the derivative of the bidder function with respect to a

$$
\begin{aligned}
\frac{\partial}{\partial a} b(v_i) &= \frac{\partial}{\partial a}\left(\frac{aN-a}{aN-a+1} v_i^{aN-a+1}\right) \\
&= \frac{N-1}{aN-a+1} v_i^{aN-a+1} - \frac{(aN-a)(N-1)}{(aN-a+1)^2} v_i^{aN-a+1} + \frac{(N-1)(aN-a)}{aN-a+1} v_i^{aN-a+1} \log(v_i)
\end{aligned}
$$

The derivative is not strictly positive or negative, so increasing a may increase competition and may decrease competition. But for a sufficiently large a, the derivative is negative, so an increase in a decreases a bid conditional on v_i.

We can see that a higher a reduces bids for a sufficiently large a in the figure below $(N = 5, a \in (0, 2), v_i \in \{0.25, 0.5, 0.75\})$:



(e) Compute the expected payment from each bidder before and after she learns her value.

The expected payment from bidder i conditional on v_i is their bid:

$$b(v_i) = \frac{aN - a}{aN - a + 1} v_i^{aN-a+1}$$

The expected payment from bidder i unconditionally is:

$$\begin{aligned}
b(v_i) &= \int_0^1 \frac{aN - a}{aN - a + 1} v_i^{aN-a+1} a v_i^{a-1} dv_i \\
&= a\frac{aN - a}{aN - a + 1} \int_0^1 v_i^{aN} dv_i \\
&= a\frac{aN - a}{aN - a + 1} \left[\frac{1}{aN + 1} v_i^{aN+1} \right]_0^1 \\
&= \frac{a^2 N - a^2}{(aN - a + 1)(aN + 1)}
\end{aligned}$$

The expected payoff of bidder i conditional on v_i is:

$$\begin{aligned}
E[u_i(v_i)] &= v_i^{aN-a+1} - b(v_i) \\
&= v_i^{aN-a+1} - \frac{aN - a}{aN - a + 1} v_i^{aN-a+1} \\
&= \frac{v_i^{aN-a+1}}{aN - a + 1}
\end{aligned}$$

The unconditional expected payoff of bidder i is:

$$\begin{aligned}
\int_0^1 E[u_i(v_i)] f(v_i) dv_i &= \int_0^1 \frac{v_i^{aN-a+1}}{aN - a + 1} a v_t^{a-1} dv_i \\
&= \int_0^1 \frac{a v_i^{aN}}{aN - a + 1} dv_i \\
&= \left[\frac{a v_i^{aN+1}}{(aN - a + 1)(aN + 1)} \right]_0^1 \\
&= \frac{a}{(aN - a + 1)(aN + 1)}
\end{aligned}$$

2 Tricky Seller

Two people are interested in one object. Their valuations are drawn independently from $F(x) = x$ and $F(x) = x^2$, respectively, with $x \in [0, 1]$. The seller's value (a cost, perhaps) for the object is known, $c \in [0, 1]$.

(a) Describe outcome of the First-Price Auction with a reserve price r.

Let us disregard the reserve price for the moment and consider the auction as Bayesian game $(I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot))$:

- The set of players is $I = \{1, 2\}$.
- The action set of player $i \in I$ is $B_i = [0, \infty)$.
- The payoff for player $i \in I$ is

$$u_i(b_1, b_2; x_1, x_2) = u_i(b_1, b_2; x_i) = \begin{cases} x_i - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(x_i - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

- $\Theta = [0, 1] \times [0, 1]$.
- $F_1(x) = x$ and $F_2(x) = x^2$.

The expected payoff of bidder 1 when bidder 2 plays $b_2(x_2) = a_2 x_2$ (linear bid function and bids zero when value is zero):

$$\begin{aligned} E[u_1(b_1, b_2; x_1)] &= (x_1 - b_1) \Pr(b_1 > b_2(x_2)) \\ &= (x_1 - b_1) \Pr(b_2^{-1}(b_1) > x_2) \\ &= (x_1 - b_1) F_2(b_2^{-1}(b_1)) \\ &= (x_1 - b_1)(b_2^{-1}(b_1))^2 \\ &= (x_1 - b_1)(\frac{b_1}{a_2})^2 \\ &= \frac{x_1 b_1^2}{a_2^2} - \frac{b_1^3}{a_2^2} \end{aligned}$$

FOC:

$$2\frac{x_1 b_1}{a_2^2} - 3\frac{b_1^2}{a_2^2} = 0 \implies b_1(x_1) = \frac{2}{3} x_1$$

The expected payoff of bidder 2 when bidder 1 plays $b_1(x_1) = a_1 x_1$ (linear bid function and bids zero when value is zero):

$$\begin{aligned} E[u_2(b_1, b_2; x_2)] &= (x_2 - b_2) \Pr(b_2 > b_1(x_1)) \\ &= (x_2 - b_2) \Pr(b_1^{-1}(b_2) > x_1) \\ &= (x_2 - b_2) F_1(b_1^{-1}(b_2)) \\ &= (x_2 - b_2) b_1^{-1}(b_2)) \\ &= (x_2 - b_2)\frac{b_2}{a_2} \\ &= \frac{b_2 x_2}{a_2} - \frac{b_2^2}{a_2} \end{aligned}$$

FOC:

$$\frac{x_2}{a_2} - 2\frac{b_2}{a_2} = 0 \implies b_2(x_2) = \frac{x_2}{2}$$

Thus, bidder 1 bids $b_1(x_1) = \frac{2}{3}x_1$ and $b_2(x_2) = \frac{x_2}{2}$, so both bidders underbid their values.

The seller will only sell the good if the revenue weakly exceeds c, so the reservation price $r = c$ and it is common knowledge. The bidders know this, so r is a floor on the amount that they will bid:

$$b_1(x) = \begin{cases} \frac{2}{3}x, & \frac{2}{3}x \geq r \\ r, & x_1 \geq r > \frac{2}{3}x \\ 0, & r > x \end{cases}$$

$$b_2(x) = \begin{cases} \frac{1}{2}x, & \frac{1}{2}x \geq r \\ r, & x \geq r > \frac{1}{2}x \\ 0, & r > x \end{cases}$$

- If both bids are below r, then the both bidders submit zero bids and the seller doesn't sell.
- If one or both bids is above r, then a bid at least as large as r is submitted.
- Bidders still would have positive surplus if r is between their value and bid absent a reserve price, so they would bid the reserve price.

(b) Describe outcome of the Second-Price Auction with a reserve price r.

As discussed in lecture, bidding $b(v) = v$ is a weakly dominate strategy. Similar to FPA, the seller will only sell the good if the revenue weakly exceeds c, so the reservation price $r = c$ and it is common knowledge. The bidders know this, so r is a floor on the amount that they will bid:

$$b_1(x) = b_2(x) = \begin{cases} x, & x \geq r \\ 0, & x < r \end{cases}$$

- If both bids or one bid are below r, then the auction revenue is below the reserve price and the seller doesn't sell.
- If both bids is above r, then a bid at least as large as r is submitted.

(c) What auction and what r will the seller choose? Which player wins more often?

As mentioned above, the seller will choose $r = c$. If $r > c$, the seller may turn down trades that they would be better off if they accepted. If $r < c$, they may be selling the good for less than they value it.

The seller will choose the auction format that maximizes their revenue...

Bidder 2 is more likely to have higher valuation, so they are more likely to submit a higher bid.

(d) Suppose now that $c = 0$ and there is no reserve price. Suppose that a seller can offer discount of α to one of the bidders in the second-price auction. If a bidder is offered a discount $\alpha \in [0; 1]$, then, if she wins, she pays only a fraction α of what she had to pay otherwise. Who should be offered a discount? Compute the optimal discount and expected revenues.

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3 Third Price Auction

Consider a third-price auction with three players: an auction in which bidder with the highest value wins, but pays only the third highest bid. Assume that valuation of players are iid from the uniform distribution on [0, 1].

(a) Define the auction as a Bayesian game.

For this part, I consider a third-price auction with only three players. A Bayesian game is a five-tuple $(I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot))$:

- The set of players is $I = \{1, 2, 3\}$.
- The action set of player $i \in I$ is $B_i = [0, \infty)$.
- The payoff for player $i \in I$ is

$$u_i(b_1, b_2, b_3; v_1, v_2, v_3) = u_i(b_1, b_2, b_3; v_i) = \begin{cases} v_i - b_k & \text{if } b_i > b_j \geq b_k, \\ \frac{1}{3}(v_i - b_k) & \text{if } b_i = b_j = b_k, \\ \frac{1}{2}(v_i - b_k) & \text{if } b_i = b_j > b_k, \\ 0 & \text{otherwise.} \end{cases}$$

- $\Theta = [0, 1] \times [0, 1] \times [0, 1]$.
- $F(v) = v$.

(b) Prove that a bid of $b_i(v_i) = \frac{n-1}{n-2} v_i$ is a symmetric Bayes Nash equilibrium of the third-price auction.

In this part, I consider a third-price auction with n bidders. I show that bidder i's best response to $b(v_{-i}) = \frac{n-1}{n-2} v_{-i}$ is to play $b(v_i) = \frac{n-1}{n-2} v_i$ below and thus it is a symmetric Bayes Nash equilibrium. The expected payoff of bidder i is

$$\begin{aligned}
E[u_i(b_1, ..., b_n; v_i)] &= (v_i - E[b_{(n-2)} | b_i > b_j, j \neq i]) \Pr(b_i > b_j, j \neq i) \\
&= \left(v_i - \frac{n-1}{n-2} E\left[v_{(n-2)} | b_i > \frac{n-1}{n-2} v_j, j \neq i \right] \right) \Pr\left(b_i > \frac{n-1}{n-2} v_j, j \neq i \right) \\
&= \left(v_i - \frac{n-1}{n-2} E\left[v_{(n-2)} \middle| \frac{n-2}{n-1} b_i > v_j, j \neq i \right] \right) \Pr\left(\frac{n-2}{n-1} b_i > v_j, j \neq i \right) \\
&= \left(v_i - \frac{n-1}{n-2} E[w_{(n-2)}] \right) F\left(\frac{n-2}{n-1} b_i \right)^{n-1} \\
&= \left(v_i - \frac{n-1}{n-2} \frac{n-2}{n-1} b_i \frac{n-2}{n} \right) \left(\frac{n-2}{n-1} b_i \right)^{n-1} \\
&= \left(v_i - \frac{n-2}{n} b_i \right) \left(\frac{n-2}{n-1} b_i \right)^{n-1} \\
&= \left(\frac{n-2}{n-1} \right)^{n-1} v_i b_i^{n-1} - \frac{n-2}{n} \left(\frac{n-2}{n-1} \right)^{n-1} b_i^n
\end{aligned}$$

where $w_j \sim U(0, \frac{n-2}{n-1} b_i)$ for $j \neq i$. Generally, note that if $X_1, ..., X_n \sim U(0, 1)$, then the kth order statistic $X_{(k)} \sim Beta(k, n-k+1) \implies E[X_{(k)}] = \frac{k}{n+1}$. So, $E[w_{(n-2)}] = \frac{n-2}{n-1} b_i \frac{n-2}{n}$.

FOC $[b_i]$:

$$(n-1)\left(\frac{n-2}{n-1}\right)^{n-1} v_i b_i^{n-2} = n\frac{n-2}{n}\left(\frac{n-2}{n-1}\right)^{n-1} b_i^{n-1} \implies b_i(v_i) = \frac{n-1}{n-2}v_i$$

Thus, $b_i(v_i) = \frac{n-1}{n-2}v_i$ is a best response.

(c) Show that the expected revenue of a seller in the third-price auction is $R_3 = \frac{n-1}{n+1}$.

The expected seller revenue is the expected value of the third highest bid:

$$\begin{aligned} R_3 &= E[b(v_{(n-2)})] \\ &= E\left[\frac{n-1}{n-2}v_{(n-2)}\right] \\ &= \frac{n-1}{n-2}E[v_{(n-2)}] \\ &= \frac{n-1}{n-2}\frac{n-2}{n+1} \\ &= \frac{n-1}{n+1} \end{aligned}$$

(d) What is the symmetric Bayes-Nash equilibrium strategy in a kth price auction? (You need only state how each bidder bids; you need not provide a detailed analysis.)

From lecture notes and this problem, we know the bidding function in symmetric BNEs for $k \in \{1, 2, 3\}$:

$$b(v_i) = \begin{cases} \frac{n-1}{n}v_i & k = 1 \\ v_i & k = 2 \\ \frac{n-1}{n-2}v_i & k = 3 \end{cases}$$

These findings suggest that $b(v_i) = \frac{n-1}{n-k+1}$ for all $k \in \mathbb{N}$ is a candidate.