ECON 736A: Problem Set 2

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1 Acharya and Dogra (2020)

1. *Carefully prove Proposition 1 in the paper*

   Proposition 1. *Individual decision problem: Given a sequence of real interest rates, aggregate output, and idiosyncratic risk $\{r_t, y_t, \sigma_{y,t}\}$, household i's consumption decision can be expressed as*

$$c_t^i = \mathcal{C}_t + \mu_t(a_t^i + y_t^i) \tag{1}$$

   where $a_t^i = A_t^i / P_t$ is real net worth at the state of date t and $\mathcal{C}_t$ and μ_t solve the following recursions:

$$\mathcal{C}_t[1 + \mu_{t+1}(1 + r_t)] = -\frac{1}{\gamma} \ln \beta(1 + r_t) + \mathcal{C}_{t+1} + \mu_{t+1}\bar{y}_{t+1} - \frac{\gamma \mu_{t+1}^2 \sigma_{y,t+1}^2}{2} \tag{2}$$

$$\mu_t = \frac{\mu_{t+1}(1 + r_t)}{1 + \mu_{t+1}(1 + r_t)} \tag{3}$$

   We prove with guess and verify. With CARA utility, the household Euler equation is

$$e^{-\gamma c_t^i} = \beta(1 + r_t) E_t e^{-\gamma c_{t+1}^i}$$

   Taking logs of both sides

$$-\gamma c_t^i = \ln \beta(1 + r_t) + \ln E_t e^{-\gamma c_{t+1}^i}$$

   Guess consumption decision rule is

$$c_t^i = \mathcal{C}_t + \mu_t(a_t^i + y_t^i)$$

   where $\mathcal{C}_t$ and μ_t are deterministic. Using the HH budget constraint and the guessed consumption decision rule, we can get c_{t+1}^i:

$$\begin{aligned}
a^i_{t+1} &= (1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t \\
\implies c^i_{t+1} &= \mathcal{C}_{t+1} + \mu_{t+1}[a^i_{t+1} + y^i_{t+1}] \\
&= \mathcal{C}_{t+1} + \mu_{t+1}[(1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t + y^i_{t+1}]
\end{aligned}$$

All terms in c^i_{t+1} are known at t except for y^i_{t+1} which is normal, so c^i_{t+1} is normal with

$$\begin{aligned}
E_t[c^i_{t+1}] &= \mathcal{C}_{t+1} + \mu_{t+1}[(1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t + \bar{y}_{t+1}] \\
Var_t[c^i_{t+1}] &= \mu^2_{t+1}\sigma^2_{y,t+1}
\end{aligned}$$

Applying MGF of normal distributions,

$$\begin{aligned}
\ln E_t[e^{-\gamma c^i_{t+1}}] &= E_t[-\gamma c^i_{t+1}] + \frac{Var_t[-\gamma c^i_{t+1}]}{2} \\
&= -\gamma E_t[c^i_{t+1}] + \frac{\gamma^2 Var_t[c^i_{t+1}]}{2} \\
&= -\gamma\mathcal{C}_{t+1} - \gamma\mu_{t+1}[(1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t + \bar{y}_{t+1}] + \frac{\gamma^2\mu^2_{t+1}\sigma^2_{y,t+1}}{2}
\end{aligned}$$

Substituting into the logged Euler equation,

$$-\gamma[\mathcal{C}_t + \mu_t(a^i_t + y^i_t)] = \ln\beta(1+r_t) - \gamma\mathcal{C}_{t+1} - \gamma\mu_{t+1}[(1+r_t)(1-\mu_t)(a^i_t + y^i_t) - (1+r_t)\mathcal{C}_t + \bar{y}_{t+1}] + \frac{\gamma^2\mu^2_{t+1}\sigma^2_{y,t+1}}{2}$$

Matching coefficients on $(a^i_t + y^i_t)$:

$$\begin{aligned}
-\gamma\mu_t &= -\gamma\mu_{t+1}(1+r_t)(1-\mu_t) \\
\mu_t(1 + \mu_{t+1}(1+r_t)) &= \mu_{t+1}(1+r_t) \\
\mu_t &= \frac{\mu_{t+1}(1+r_t)}{1+\mu_{t+1}(1+r_t)}
\end{aligned}$$

Matching constant coefficients:

$$\begin{aligned}
-\gamma\mathcal{C}_t &= \ln\beta(1+r_t) - \gamma\mathcal{C}_{t+1} + \gamma\mu_{t+1}(1+r_t)\mathcal{C}_t + \gamma\bar{y}_{t+1} + \frac{\gamma^2\mu^2_{t+1}\sigma^2_{y,t+1}}{2} \\
\mathcal{C}_t[1 + \mu_{t+1}(1+r_t)] &= -\frac{1}{\gamma}\ln\beta(1+r_t) + \mathcal{C}_{t+1} + \mu_{t+1}\bar{y}_{t+1} - \frac{\gamma\mu^2_{t+1}\sigma^2_{y,t+1}}{2}
\end{aligned}$$

2. *Carefully derive equations 4.1-4.4 in the paper.*

Linearizing (3.3), (3.5), (3.6), and (2.2)

$$\mu_t = \frac{\mu_{t+1}(1+r_t)}{1+\mu_{t+1}(1+r_t)} \tag{4}$$

$$y_t = y_{t+1} - \frac{\ln\beta(1+r_t)}{\gamma} - \frac{\gamma\mu_{t+1}^2\sigma_{y,t+1}^2}{2} + g_t - g_{t+1} \tag{5}$$

$$\Psi\Pi_t(\Pi_t - 1) = 1 - \theta(1 - mc_t) + \Psi(\Pi_{t+1} - 1)\Pi_{t+1}\left[\frac{1}{1+r_t}\frac{x_{t+1}}{x_t}\right] \tag{6}$$

$$1 + i_t = (1+r)\Pi_t^{\Phi_\pi} \geq 1 \tag{7}$$

around the flexible price level of output $y^* = 1$, $\mu = \frac{r}{1+r}$, and $\Pi = 1$, we get

$$\hat{y}_t = \Theta\hat{y}_{t+1} - \frac{1}{\gamma}(i_t - \pi_{t+1}) - \Lambda\hat{\mu}_{t+1} \tag{8}$$

$$\hat{\mu}_t = \tilde{\beta}\hat{\mu}_{t+1} + \tilde{\beta}(i_t - \pi_{t+1}) \tag{9}$$

$$\pi_t = \tilde{\beta}\pi_{t+1} + \kappa\hat{y}_t \tag{10}$$

$$i_t = \Phi_\pi\pi_t \tag{11}$$

where

$$\Theta = 1 - \frac{\gamma\mu^2}{2}\frac{d\boldsymbol{\sigma}^2(y^*)}{dy}$$

$$\Lambda = \gamma\mu^2\boldsymbol{\sigma}_y^2(y^*)$$

$$\tilde{\beta} = \frac{1}{1+r}$$

κ denotes the slope of the linearized Phillips curve, and $\hat{y}_t$, $\hat{\mu}_t$, i_t, and π_t denote the log deviation of y_t, μ_t, $1+i_t$, and Π_t from their steady state values.

2 Werning (2015)

1. Carefully derive Proposition 5 in the paper.

   Proposition 5. *Suppose utilities satisfy (14) with $\sigma = 1$, household income satisfies (15) and borrowing constraints satisfy (17). In addition, suppose initial bond holdings are zero $b_0^i = 0$ for all households. Then $\{C_t, R_t\}$ is part of an equilibrium if and only if*

   $$U'(C_t) = \beta_t R_t U'(C_{t+1})$$

   for some sequence of discount factors $\{\beta_t\}$, independent of both $\{R_t\}$ and $\{C_t\}$