FIN 920: Midterm Cheatsheet

Arbitrage

- Type 1 arbitrage exists if $\exists\ x$ s.t. $P(x \geq 0) = 1$ and $P(x > 0) > 0$ with $p(x) \leq 0$.
- Type 2 arbitrage exists if $\exists\ x$ s.t. $P(x \geq 0) = 1$ and $p(x) < 0$.
- Type 1 arbitrage is "free money later" and type 2 arbitrage is "free money now."
- Steimke's Lemma: For any matrix A either: (1) $\exists\ \beta >> 0$ s.t. $A\beta = 0$ or (2) $\exists\ N$ s.t. $N'A > 0$.
- (1) corresponds to no arbitrage and (2) corresponds to arbitrage.

Minimum Variance Frontier

- Portfolios on MVF satisfy

$$\min_{\alpha} \frac{1}{2}\alpha' V \alpha + \lambda(\alpha'\mu - E[r_p]) + \theta(\alpha'\mathbb{1} - 1)$$

$$\begin{aligned}
A &= \mu' V^{-1}\mathbb{1} = \mathbb{1}' V^{-1}\mu \\
B &= \mu' V^{-1}\mu \\
C &= \mathbb{1} V^{-1}\mathbb{1} \\
D &= BC - A^2 \\
\lambda_p &= \frac{CE[r_p] - A}{D} \\
\theta_p &= \frac{B - AE[r_p]}{D} \\
\alpha_p &= \lambda_p V^{-1}\mu + \theta_p V^{-1}\mathbb{1} \\
&= A\lambda_p \alpha_{mvp} + C\theta_p \alpha_t \\
\alpha_{mvp} &= \frac{V^{-1}\mathbb{1}}{\mathbb{1}' V^{-1}\mathbb{1}} \\
\alpha_t &= \frac{V^{-1}\mu}{\mathbb{1}' V^{-1}\mu} \\
\sigma^2_{mvp} &= \frac{1}{C} \\
E[r_{mvp}] &= \frac{A}{C} \\
\sigma^2_t &= \frac{B}{A^2} \\
E[r_t] &= \frac{B}{A} \\
\sigma_{p,q} &= \frac{C}{D}\Big(E[r_p] - \frac{A}{C}\Big)\Big(E[r_q] - \frac{A}{C}\Big) + \frac{1}{C}
\end{aligned}$$

- For any portfolio q and the tangency portfolio t: $r_q = r_{p(q)} + \varepsilon_q = r_f + \beta_{qt}(r_t - r_f) + \varepsilon_q$, where $\beta_{qt} = \frac{Cov(r_q, r_t)}{Var(r_t)}$, $Cov(r_f, \varepsilon_q) = E[\varepsilon_q] = 0$ [$E[\varepsilon_q | r_t] = 0$ is sufficient.]

Stochastic Discount Factors

- If an investor is risk-neutral, their SDF is one.
- If a SDF m exists and $P(m > 0) = 1$, no type 1 or type 2 arbitrage exists.
- pf: Consider x s.t. $P(x \geq 0) = 1$ and $P(x > 0) > 0$, then $P(m > 0) = 1 \implies p(x) = E[mx] = E[mx | x > 0]P(x > 0) > 0$ (no type 1 arbitrage). Consider x s.t. $p(x \geq 0) = 1$. $P(m > 0) = 1 \implies p(x) \geq 0$ (no type 2 arbitrage).

Utility Functions

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Risk Aversion

- Coef. of absolute risk aversion: $R_A(w, u(\cdot)) = \frac{-u''(w)}{u'(w)}$.
- Coef. of relative risk aversion: $R_R(w, u(\cdot)) = wR_A(w, u(\cdot)) = \frac{-u''(w)}{u'(w)}w$.

Stochastic Dominance

- Two gambles X and Y with CDFs F and G, respectively.
- First-degree stochastic dominance:
  - $X \succsim Y$ (i.e. everyone with increasing utility prefers X to Y).
  - $F(z) \leq G(z)\ \forall z$
  - $Y =^{dist} X + \varepsilon$ where $\varepsilon \geq 0$
- Second-degree stochastic dominance:
  - $X \succsim_2 Y$ (i.e. everyone with increasing and concave utility prefers X to Y)
  - $\int_{\alpha}^{y} F(z) - G(z) dz \leq 0\ \forall y$ or $E[X] \geq E[Y]$
  - $Y =^{dist} X + \varepsilon$ where $E[\varepsilon | X] \geq 0$

CAPM

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Equilibrium Asset Pricing

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Normal Distribution

- If $x \sim N(\mu, \sigma^2)$, then $E[e^x] = e^{\mu + \sigma^2/2}$
- MGF: $m(t) = E[e^{tx}] = \exp(t\mu + t^2\sigma^2/2)$

Options Stuff

- Put-call parity: $C - P = S - PV(k)$