ECON 710A - Problem Set 5

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1. Suppose that $\{\varepsilon_t\}_{t=0}^T$ are iid random variables with mean zero, variance σ^2 and $E[\varepsilon_t^8] < \infty$. Let $U_t = \varepsilon_t \varepsilon_{t-1}$, $W_t = \varepsilon_t \varepsilon_0$, and $V_t = \varepsilon_t^2 \varepsilon_{t-1}$ where $t = 1, ..., T$.

(i) Show that $\{U_t\}_{t=1}^T$, $\{W_t\}_{t=1}^T$, and $\{V_t\}_{t=1}^T$ are covariance stationary.

For each time series, we check that (1) the second moment is finite, (2) the mean does not depend on t, and (3) the variance does not depend on t.

$\{U_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$\begin{aligned} E[U_t^2] &= E[(\varepsilon_t \varepsilon_{t-1})^2] \\ &= E[\varepsilon_t^2 \varepsilon_{t-1}^2] \\ &= E[\varepsilon_t^2] E[\varepsilon_{t-1}^2] \\ &= E[\varepsilon_t^2]^2 \\ &= \sigma^4 \\ &< \infty \end{aligned}$$

For (2), $E[U_t] = E[\varepsilon_t \varepsilon_{t-1}] = E[\varepsilon_t] E[\varepsilon_{t-1}] = 0$. For (3),

$$\begin{aligned} \gamma(0) &= Cov(U_t, U_t) \\ &= Var(U_t) \\ &= Var(\varepsilon_t \varepsilon_{t-1}) \\ &= Var(\varepsilon_t) Var(\varepsilon_{t-1}) \\ &= \sigma^4 \end{aligned}$$

$$\begin{aligned} \gamma(1) &= Cov(U_t, U_{t+1}) \\ &= E[U_t U_{t+1}] \\ &= E[(\varepsilon_t \varepsilon_{t-1})(\varepsilon_{t+1} \varepsilon_t)] \\ &= E[\varepsilon_t^2] E[\varepsilon_{t-1}] E[\varepsilon_{t+1}] \\ &= 0 \end{aligned}$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

$$
\begin{aligned}
\gamma(2) &= Cov(U_t, U_{t+2}) \\
&= E[U_t U_{t+2}] \\
&= E[(\varepsilon_t \varepsilon_{t-1})(\varepsilon_{t+2} \varepsilon_{t+1})] \\
&= E[\varepsilon_{t-1}]E[\varepsilon_t]E[\varepsilon_{t+1}]E[\varepsilon_{t+2}] \\
&= 0
\end{aligned}
$$

Thus, $\gamma(k) = \sigma^4$ if $k = 0$ and zero otherwise.

$\{W_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$
\begin{aligned}
E[W_t^2] &= E[(\varepsilon_t \varepsilon_0)^2] \\
&= E[\varepsilon_t^2 \varepsilon_0^2] \\
&= E[\varepsilon_t^2]E[\varepsilon_0^2] \\
&= E[\varepsilon_t^2]^2 \\
&= \sigma^4 \\
&< \infty
\end{aligned}
$$

For (2), $E[W_t] = E[\varepsilon_t \varepsilon_0] = E[\varepsilon_t]E[\varepsilon_0] = 0$. For (3),

$$
\begin{aligned}
\gamma(0) &= Cov(W_t, W_t) \\
&= Var(W_t) \\
&= Var(\varepsilon_t \varepsilon_0) \\
&= Var(\varepsilon_t)Var(\varepsilon_0) \\
&= \sigma^4
\end{aligned}
$$

$$
\begin{aligned}
\gamma(1) &= Cov(W_t, W_{t+1}) \\
&= E[W_t W_{t+1}] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+1} \varepsilon_0)] \\
&= E[\varepsilon_0^2]E[\varepsilon_t]E[\varepsilon_{t+1}] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\gamma(2) &= Cov(W_t, W_{t+2}) \\
&= E[W_t W_{t+2}] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+2} \varepsilon_0)] \\
&= E[(\varepsilon_t \varepsilon_0)(\varepsilon_{t+2} \varepsilon_0)] \\
&= E[\varepsilon_0^2]E[\varepsilon_t]E[\varepsilon_{t+2}] \\
&= 0
\end{aligned}
$$

Thus, $\gamma(k) = \sigma^4$ if $k = 0$ and zero otherwise.

$\{V_t\}_{t=1}^T$: For (1), because $E[\varepsilon_t^8] < \infty$ and $\{\varepsilon_t\}_{t=0}^T$ are iid,

$$\begin{aligned} E[V_t^2] &= E[(\varepsilon_t^2 \varepsilon_{t-1})^2] \\ &= E[\varepsilon_t^4 \varepsilon_{t-1}^2] \\ &= E[\varepsilon_t^4] E[\varepsilon_{t-1}^2] \\ &= E[\varepsilon_t^4] \sigma^2 \\ &< \infty \end{aligned}$$

For (2), $E[V_t] = E[\varepsilon_t^2 \varepsilon_{t-1}] = E[\varepsilon_t^2] E[\varepsilon_{t-1}] = 0$. For (3),

$$\begin{aligned} \gamma(0) &= Cov(V_t, V_t) \\ &= Var(V_t) \\ &= Var(\varepsilon_t^2 \varepsilon_{t-1}) \\ &= Var(\varepsilon_t^2) Var(\varepsilon_{t-1}) \\ &= E[(\varepsilon_t^2 - E[\varepsilon_t^2])^2] \sigma^2 \\ &= E[(\varepsilon_t^2 - \sigma^2)^2] \sigma^2 \\ &= E[\varepsilon_t^4 - 2\sigma^2 \varepsilon_t^2 + \sigma^4] \sigma^2 \\ &= (E[\varepsilon_t^4] - 2\sigma^2 \sigma^2 + \sigma^4) \sigma^2 \\ &= (E[\varepsilon_t^4] - \sigma^4) \sigma^2 \\ &= \sigma^2 E[\varepsilon_t^4] - \sigma^6 \end{aligned}$$

$$\begin{aligned} \gamma(1) &= Cov(V_t, V_{t+1}) \\ &= E[V_t V_{t+1}] \\ &= E[(\varepsilon_t^2 \varepsilon_{t-1})(\varepsilon_{t+1}^2 \varepsilon_t)] \\ &= E[\varepsilon_t^3 \varepsilon_{t-1} \varepsilon_{t+1}^2] \\ &= E[\varepsilon_t^3] E[\varepsilon_{t-1}] E[\varepsilon_{t+1}^2] \\ &= 0 \end{aligned}$$

$$\begin{aligned} \gamma(2) &= Cov(V_t, V_{t+2}) \\ &= E[V_t V_{t+2}] \\ &= E[(\varepsilon_t^2 \varepsilon_{t-1})(\varepsilon_{t+2}^2 \varepsilon_{t+1})] \\ &= E[\varepsilon_t^2 \varepsilon_{t-1} \varepsilon_{t+2}^2 \varepsilon_{t+1}] \\ &= E[\varepsilon_t^2] E[\varepsilon_{t-1}] E[\varepsilon_{t+2}^2] E[\varepsilon_{t+1}] \\ &= 0 \end{aligned}$$

Thus, $\gamma(k) = \sigma^2 E[\varepsilon_t^4] - \sigma^6$ if $k = 0$ and zero otherwise.

(ii) Argue that the following three sample means $\bar{U}$, $\bar{W}$, $\bar{V}$ converge in probability to their expectations.

In (i), we found that $E[U_t] = E[W_t] = E[V_t] = 0 \implies E[\bar{U}] = E[\bar{W}] = E[\bar{V}] = 0$. Below I show that $Var(\bar{U}) \to 0$, $Var(\bar{V}) \to 0$, and $Var(\bar{W}) \to 0$, so by Chebyshev's inequality $\bar{U} \to_p E[\bar{U}]$, $\bar{W} \to_p E[\bar{W}]$, and $\bar{V} \to_p E[\bar{V}]$.

$$\begin{aligned}
Var(\bar{U}) &= Var\left(\frac{1}{T}\sum_{t=1}^{T} U_t\right) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} Cov(U_t, U_s) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma(t-s) \\
&= \frac{1}{T^2} T\gamma(0) \\
&= \frac{\gamma(0)}{T} \\
&= \frac{\sigma^2}{T} \\
&\to 0
\end{aligned}$$

As $T \to \infty$. Because V_t and W_t have the same autocovariance function, the variances of $\bar{W}$ and $\bar{V}$ similarly converge to zero.

(iii) Determine whether the following three sample second moments converge in probability to their expectations:

$$\hat{\gamma}_U(0) = \frac{1}{T}\sum_{t=1}^{T} U_t^2, \quad \hat{\gamma}_W(0) = \frac{1}{T}\sum_{t=1}^{T} W_t^2, \quad \hat{\gamma}_V(0) = \frac{1}{T}\sum_{t=1}^{T} V_t^2$$

Similar to (ii), we proceed by applying Chebyshev's inequality to show convergence. For $\hat{\gamma}_U(0)$,

$$E[\hat{\gamma}_U(0)] = E[\frac{1}{T}\sum_{t=1}^{T} U_t^2] = \frac{1}{T}\sum_{t=1}^{T} E[U_t^2] = \sigma^4$$

Now, let us consider the autocorrelation function for $\{U_t^2\}_{t=0}^{T}$:

$$\begin{aligned}
\gamma_{U^2}(0) &= Var(U_t^2) \\
&= E[U_t^4] - (\sigma^4)^2 \\
&= E[\varepsilon_t^4 \varepsilon_{t-1}^4] - \sigma^8 \\
&= E[\varepsilon_t^4]E[\varepsilon_{t-1}^4] - \sigma^8 \\
&= E[\varepsilon_t^4]^2 - \sigma^8
\end{aligned}$$

$$\begin{aligned}
\gamma_{U^2}(1) &= Cov(U_t^2, U_{t+1}^2) \\
&= E[U_t^2 U_{t+1}^2] - E[U_t^2]E[U_{t+1}^2] \\
&= E[(\varepsilon_t \varepsilon_{t-1})^2(\varepsilon_{t+1}\varepsilon_t)^2] - \sigma^4\sigma^4 \\
&= E[\varepsilon_t^4 \varepsilon_{t-1}^2 \varepsilon_{t+1}^2] - \sigma^8 \\
&= E[\varepsilon_t^4]\sigma^2\sigma^2 - \sigma^8 \\
&= E[\varepsilon_t^4]\sigma^4 - \sigma^8
\end{aligned}$$

$$\begin{aligned}
\gamma_{U^2}(2) &= Cov(U_t^2, U_{t+2}^2) \\
&= E[U_t^2 U_{t+2}^2] - E[U_t^2]E[U_{t+2}^2] \\
&= E[(\varepsilon_t \varepsilon_{t-1})^2(\varepsilon_{t+2}\varepsilon_{t+1})^2] - \sigma^8 \\
&= E[\varepsilon_t^2]E[\varepsilon_{t-1}^2]E[\varepsilon_{t+2}^2]E[\varepsilon_{t+1}^2] - \sigma^8 \\
&= (\sigma^2)^4 - \sigma^8 \\
&= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var\left(\frac{1}{T}\sum_{t=1}^{T} U_t^2\right) &= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} Cov(U_t^2, U_s^2) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma_{U^2}(t-s) \\
&= \frac{1}{T^2} T(E[\varepsilon_t^4]^2 - \sigma^8 + E[\varepsilon_t^4]\sigma^4 - \sigma^8) \\
&= \frac{E[\varepsilon_t^4]^2 - 2\sigma^8 + E[\varepsilon_t^4]\sigma^4}{T} \\
&\to 0
\end{aligned}$$

As $T \to \infty$. For $\hat{\gamma}_W(0)$,

$$E[\hat{\gamma}_W(0)] = E[\frac{1}{T}\sum_{t=1}^{T} W_t^2] = \frac{1}{T}\sum_{t=1}^{T} E[W_t^2] = \sigma^4$$

Now, let us consider the autocorrelation function for $\{W_t^2\}_{t=0}^{T}$:

$$\begin{aligned}
\gamma_{W^2}(0) &= Var(W_t^2) \\
&= E[W_t^4] - (\sigma^4)^2 \\
&= E[\varepsilon_t^4 \varepsilon_0^4] - \sigma^8 \\
&= E[\varepsilon_t^4]E[\varepsilon_0^4] - \sigma^8 \\
&= E[\varepsilon_t^4]^2 - \sigma^8
\end{aligned}$$

$$\begin{aligned}
\gamma_{W^2}(1) &= Cov(W_t^2, W_{t+1}^2) \\
&= E[W_t^2 W_{t+1}^2] - E[W_t^2]E[W_{t+1}^2] \\
&= E[(\varepsilon_t \varepsilon_0)^2 (\varepsilon_{t+1}\varepsilon_0)^2] - \sigma^4\sigma^4 \\
&= E[(\varepsilon_t^2 \varepsilon_{t+1}^2 \varepsilon_0^4] - \sigma^8 \\
&= E[\varepsilon_0^4]\sigma^2\sigma^2 - \sigma^8 \\
&= E[\varepsilon_t^4]\sigma^4 - \sigma^8
\end{aligned}$$

$$\begin{aligned}
\gamma_{W^2}(2) &= Cov(W_t^2, W_{t+2}^2) \\
&= E[W_t^2 W_{t+2}^2] - E[W_t^2]E[W_{t+2}^2] \\
&= E[(\varepsilon_t \varepsilon_0)^2 (\varepsilon_{t+2}\varepsilon_0)^2] - \sigma^8 \\
&= E[\varepsilon_t^2]E[\varepsilon_{t+2}^2]E[\varepsilon_0^4] - \sigma^8 \\
&= E[\varepsilon_t^4]\sigma^4 - \sigma^8
\end{aligned}$$

Thus, for $k \geq 2$, $\gamma_{W^2}(k) > 0$, so $\hat{\gamma}_W(0)$ does not converge to its expectation.

For $\hat{\gamma}_V(0)$,

$$E[\hat{\gamma}_V(0)] = E[\frac{1}{T}\sum_{t=1}^{T} V_t^2] = \frac{1}{T}\sum_{t=1}^{T} E[V_t^2] = \sigma^2 E[\varepsilon_t^4]$$

Now, let us consider the autocorrelation function for $\{V_t^2\}_{t=0}^T$:

$$\begin{aligned}
\gamma_{V^2}(0) &= Var(V_t^2) \\
&= E[V_t^4] - E[V_t^2]^2 \\
&= E[(\varepsilon_t^2 \varepsilon_{t-1})^4] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= E[\varepsilon_t^8 \varepsilon_{t-1}^4] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= E[\varepsilon_t^8]E[\varepsilon_t^4] - \sigma^4 E[\varepsilon_t^4]^2
\end{aligned}$$

$$\begin{aligned}
\gamma_{V^2}(1) &= Cov(V_t^2, V_{t+1}^2) \\
&= E[V_t^2 V_{t+1}^2] - E[V_t^2]E[V_{t+1}^2] \\
&= E[(\varepsilon_t^2 \varepsilon_{t-1})^2 (\varepsilon_{t+1}^2 \varepsilon_t)^2] - \sigma^2 E[\varepsilon_t^4]\sigma^2 E[\varepsilon_t^4] \\
&= E[\varepsilon_t^6 \varepsilon_{t-1}^2 \varepsilon_{t+1}^4] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= E[\varepsilon_t^6]E[\varepsilon_t^4]\sigma^2 - \sigma^4 E[\varepsilon_t^4]^2
\end{aligned}$$

$$
\begin{aligned}
\gamma_{V^2}(1) &= Cov(V_t^2, V_{t+2}^2) \\
&= E[V_t^2 V_{t+2}^2] - E[V_t^2]E[V_{t+2}^2] \\
&= E[(\varepsilon_t^2 \varepsilon_{t-1})^2(\varepsilon_{t+2}^2 \varepsilon_{t+1})^2] - \sigma^2 E[\varepsilon_t^4]\sigma^2 E[\varepsilon_t^4] \\
&= E[\varepsilon_t^4 \varepsilon_{t-1}^2 \varepsilon_{t+2}^4 \varepsilon_{t+1}^2] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= E[\varepsilon_t^4]E[\varepsilon_{t-1}^2]E[\varepsilon_{t+2}^4]E[\varepsilon_{t+1}^2] - \sigma^4 E[\varepsilon_t^4]^2 \\
&= 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Var\left(\frac{1}{T}\sum_{t=1}^{T} V_t^2\right) &= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} Cov(V_t^2, V_s^2) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} \gamma_{V^2}(t-s) \\
&= \frac{1}{T^2} T(E[\varepsilon_t^8]E[\varepsilon_t^4] - \sigma^4 E[\varepsilon_t^4]^2 + E[\varepsilon_t^6]E[\varepsilon_t^4]\sigma^2 - \sigma^4 E[\varepsilon_t^4]^2) \\
&= \frac{E[\varepsilon_t^8]E[\varepsilon_t^4] + E[\varepsilon_t^6]E[\varepsilon_t^4]\sigma^2 - 2\sigma^4 E[\varepsilon_t^4]^2}{T} \\
&\to 0
\end{aligned}
$$

(iv) Determine whether the scaled sample means $\sqrt{T}\bar{U}$, $\sqrt{T}\bar{W}$, and $\sqrt{T}\bar{V}$ are asymptotically normal.

$\sqrt{T}\bar{W}$ is not asymptotically normal because $\frac{1}{T}\sum_{t=1}^{T} W_t^2$ does not converge in probability to its expectation.

We have shown that all but the martingale condition of the martingale central limit theorem hold for $\sqrt{T}\bar{U}$ and $\sqrt{T}\bar{V}$. For $\sqrt{T}\bar{U}$:

$$
\begin{aligned}
E[U_t|U_{t-1}, U_{t-2}, ..., U_1] &= E[E[\varepsilon_t \varepsilon_{t-1}|\varepsilon_{t-1}, ..., \varepsilon_0]|U_{t-1}, U_{t-2}, ..., U_1] \\
&= E[\varepsilon_{t-1} E[\varepsilon_t|\varepsilon_{t-1}, ..., \varepsilon_0]|U_{t-1}, U_{t-2}, ..., U_1] \\
&= E[\varepsilon_{t-1} * 0|U_{t-1}, U_{t-2}, ..., U_1] \\
&= 0
\end{aligned}
$$

Thus, by the martingale CLT, $\sqrt{T}\bar{U}$ is asymptotically normal. For $\sqrt{T}\bar{V}$:

$$
\begin{aligned}
E[V_t|V_{t-1}, V_{t-2}, ..., V_1] &= E[E[\varepsilon_t^2 \varepsilon_{t-1}|\varepsilon_{t-1}, ..., \varepsilon_0]|V_{t-1}, V_{t-2}, ..., V_1] \\
&= E[\varepsilon_{t-1} E[\varepsilon_t^2|\varepsilon_{t-1}, ..., \varepsilon_0]|V_{t-1}, V_{t-2}, ..., V_1] \\
&= E[\varepsilon_{t-1} * \sigma^2|V_{t-1}, V_{t-2}, ..., V_1] \\
&= \sigma^2 E[\varepsilon_{t-1}|V_{t-1}, V_{t-2}, ..., V_1] \\
&\neq 0
\end{aligned}
$$

Thus, $\sqrt{T}\bar{V}$ is not asymptotically normal.

2. Consider a time series of length T from the model

$$Y_t = \alpha_0 + t\beta_0 + X_t\delta_0 + Y_{t-1}\rho_1 + U_t$$

where Y_0 and $\{U_t\}_{t=1}^T$ are iid $N(0,1)$, and

$$X_t = X_{t-1} \cdot 0.3 + V_t$$

where X_0 and $\{V_t\}_{t=1}^T$ are iid $N(0,1)$ and independent of Y_0 and $\{U_t\}_{t=1}^T$. We will let $\alpha_0 = \delta_0 = 100$, $\beta_0 = 1$ and consider all combinations of $T \in \{50, 150, 250\}$ and $\rho_1 \in \{0.7, 0.9, 0.95\}$.

(i) In a statistical software of your choice, generate data from (1), estimate the coefficients by OLS, and calculate heteroscedasticity robust two-sided 95% confidence intervals for α_0, δ_0, and ρ_1.

```r
tees <- c(50, 150, 250)
rhos <- c(0.7, 0.9, 0.95)
alpha <- 100
delta <- 100
beta <- 1

results <- NULL

for (t in tees) {
  for (rho in rhos) {
    x_t <- rnorm(1)
    y_t <- rnorm(1)
    v_t <- rnorm(t)
    u_t <- rnorm(t)

    for (i in 1:t) x_t[i+1] <- 0.3 * x_t[i] + v_t[i]
    for (i in 1:t) y_t[i+1] <- alpha + i * beta + x_t[i+1] * delta + y_t[i] * rho + u_t[i]

    x <- cbind(rep(1, t),
               1:t,
               x_t[2:(t+1)],
               y_t[1:t])
    y <- y_t[2:(t+1)]

    ols <- solve(t(x) %*% x) %*% (t(x) %*% y)

    e_hat <- as.numeric(y - x %*% ols)
    # omega <- crossprod(x * e_hat)
    omega <- t(x) %*% diag(e_hat^2) %*% x
    varcov <- solve(t(x) %*% x) %*% omega %*% solve(t(x) %*% x)
    se_robust <- sqrt(diag(varcov))

    results <- tibble(t = t,
           rho = rho,
           name = c("alpha", "beta", "delta", "rho"),
           ols = as.numeric(ols),
           se = se_robust) %>%
      bind_rows(results)
  }
}
```

```r
results %>%
  mutate(upper_bound = ols + se * 1.96,
         lower_bound = ols - se * 1.96) %>%
  kable(digits = 3)
```

t	rho	name	ols	se	upper_bound	lower_bound
250	0.95	alpha	100.097	0.187	100.463	99.731
250	0.95	beta	1.000	0.004	1.007	0.993
250	0.95	delta	99.930	0.066	100.059	99.801
250	0.95	rho	0.950	0.000	0.950	0.950
250	0.90	alpha	100.427	0.161	100.742	100.112
250	0.90	beta	1.003	0.002	1.007	0.998
250	0.90	delta	100.007	0.059	100.122	99.892
250	0.90	rho	0.900	0.000	0.900	0.899
250	0.70	alpha	100.141	0.177	100.488	99.794
250	0.70	beta	1.002	0.002	1.005	0.999
250	0.70	delta	100.004	0.067	100.135	99.873
250	0.70	rho	0.699	0.000	0.700	0.699
150	0.95	alpha	99.943	0.268	100.468	99.419
150	0.95	beta	0.997	0.004	1.005	0.988
150	0.95	delta	100.071	0.081	100.229	99.914
150	0.95	rho	0.950	0.000	0.950	0.950
150	0.90	alpha	100.117	0.241	100.590	99.645
150	0.90	beta	0.999	0.003	1.005	0.993
150	0.90	delta	100.102	0.078	100.255	99.948
150	0.90	rho	0.900	0.000	0.901	0.900
150	0.70	alpha	99.905	0.203	100.304	99.507
150	0.70	beta	0.996	0.003	1.002	0.990
150	0.70	delta	100.005	0.088	100.177	99.833
150	0.70	rho	0.701	0.001	0.702	0.700
50	0.95	alpha	99.675	0.424	100.505	98.844
50	0.95	beta	0.997	0.037	1.070	0.924
50	0.95	delta	99.993	0.120	100.228	99.758
50	0.95	rho	0.950	0.001	0.952	0.949
50	0.90	alpha	100.219	0.442	101.085	99.353
50	0.90	beta	0.994	0.028	1.049	0.940
50	0.90	delta	99.813	0.170	100.146	99.479
50	0.90	rho	0.900	0.001	0.902	0.899
50	0.70	alpha	99.861	0.335	100.517	99.204
50	0.70	beta	0.999	0.011	1.020	0.978
50	0.70	delta	100.025	0.175	100.367	99.683
50	0.70	rho	0.700	0.001	0.702	0.699

(ii) Across 10000 simulated repetitions of the above, report the simulated mean of the point estimators for α_0, δ_0, and ρ_1 and the simulated coverage rate of the confidence intervals.

```
ntrials <- 10000
results2 <- NULL

for (t in tees) {
  for (rho in rhos) {
    for (trial in 1:ntrials) {
      print(trial)

      x_t <- rnorm(1)
      y_t <- rnorm(1)
      v_t <- rnorm(t)
      u_t <- rnorm(t)

      for (i in 1:t) x_t[i+1] <- 0.3 * x_t[i] + v_t[i]
      for (i in 1:t) y_t[i+1] <- alpha + i * beta + x_t[i+1] * delta +
        y_t[i] * rho + u_t[i]

      x <- cbind(rep(1, t),
                 1:t,
                 x_t[2:(t+1)],
                 y_t[1:t])
      y <- y_t[2:(t+1)]

      ols <- solve(t(x) %*% x) %*% (t(x) %*% y)

      results2 <- tibble(t = t,
                         rho = rho,
                         trial = trial,
                         name = c("alpha", "beta", "delta", "rho"),
                         ols = as.numeric(ols)) %>%
        bind_rows(results2)
    }
  }
}

save(results2, file = "ps5_vonhafften_temp.RData")
```

```r
ntrials <- 10000

load("ps5_vonhafften.RData")

coverage_df <- results %>%
  full_join(results2, by=c("t", "rho", "name")) %>%
  mutate(is_covered = ols.x - 1.96* se < ols.y & ols.x + 1.96*se > ols.y,
         rho = as.factor(rho)) %>%
  group_by(t, rho, name) %>%
  summarise(mean = mean(ols.y),
            coverage_rate = sum(is_covered)/ntrials,
            .groups = "keep")

coverage_df %>%
  kable(digits = 3)
```

t	rho	name	mean	coverage_rate
50	0.7	alpha	100.007	0.894
50	0.7	beta	1.000	0.921
50	0.7	delta	100.001	0.977
50	0.7	rho	0.700	0.882
50	0.9	alpha	99.999	0.948
50	0.9	beta	1.000	0.979
50	0.9	delta	100.003	0.830
50	0.9	rho	0.900	0.955
50	0.95	alpha	100.006	0.908
50	0.95	beta	1.000	0.951
50	0.95	delta	99.999	0.894
50	0.95	rho	0.950	0.959
150	0.7	alpha	99.999	0.902
150	0.7	beta	1.000	0.750
150	0.7	delta	100.000	0.962
150	0.7	rho	0.700	0.750
150	0.9	alpha	100.000	0.907
150	0.9	beta	1.000	0.859
150	0.9	delta	100.000	0.746
150	0.9	rho	0.900	0.897
150	0.95	alpha	100.001	0.952
150	0.95	beta	1.000	0.774
150	0.95	delta	99.999	0.857
150	0.95	rho	0.950	0.825
250	0.7	alpha	100.000	0.887
250	0.7	beta	1.000	0.791
250	0.7	delta	100.001	0.965
250	0.7	rho	0.700	0.759
250	0.9	alpha	100.002	0.290
250	0.9	beta	1.000	0.762
250	0.9	delta	100.001	0.935
250	0.9	rho	0.900	0.627
250	0.95	alpha	100.002	0.885
250	0.95	beta	1.000	0.955
250	0.95	delta	100.000	0.828
250	0.95	rho	0.950	0.935

(iii) How does sample size and the degree of persistence in Y_t affect the results of the simulations.

The figure below plot the simulated points estimates from part ii that fall in the confidence intervals from part i differing by sample size (horizontal) and degree of persistence (line color). The point estimate for part i is the OLS estimate based on a single trial of simulated data and the confidence interval is the heteroskedastic robust standard error. The point estimate for part ii is the mean of OLS estimates over 10,000 trials of simulated data. Large sample sizes result in point estimates that are closer to the true value and tighter confidence intervals. For β, we see that higher degrees of persistence dramatically expand confidence intervals particularly for small samples. For δ and α, we that the confidence intervals are similarly sized across degrees of persistence and shrink with larger samples.