ECON 714A - Problem Set 4

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This problem asks you to solve a model of oligopolistic competition from Atkeson and Burstein (AER 2008), which extends the Dixit-Stiglitz setup and is widely used to analyze heterogeneous markups and incomplete pass-through.

Consider a static model with a continuum of sectors $k \in [0, 1]$ and $i = 1, ..N_k$ firms in sector k, each producing a unique variety of the good. Households supply inelastically one unit of labor and have nested-CES preferences:

$$C = \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}}, C_k = \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}, \theta > \rho \geq 1.$$

Production function of firm i in sector k is given by $Y_{ik} = A_{ik} L_{ik}$.

1. Solve household cost minimization problem for the optimal demand C_{ik}, the sectoral price index P_k, and the aggregate price index P as functions of producers' prices.

Notice that labor is inelastically supplied. The household cost minization problem is:

$$\min_{\{C_{ik}\}} \int \sum_{i=1}^{N_k} P_{ik} C_{ik} dk$$

$$\text{s.t. } C = \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}}$$

$$\text{and } C_k = \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$$

Define the legrange multiplers with P and P_k:

$$\mathcal{L} = \int \sum_{i=1}^{N_k} P_{ik} C_{ik} dk + P \left[C - \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} \right] + \int P_k \left[C_k - \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \right] dk$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

FOC $[C_k]$:

$$
\begin{aligned}
P_k &= P\frac{\rho}{\rho-1}\left(\int C_k^{\frac{\rho-1}{\rho}}dk\right)^{\frac{1}{\rho-1}}\frac{\rho-1}{\rho}C_k^{\frac{-1}{\rho}} \\
\implies P_k &= P\left(\int C_k^{\frac{\rho-1}{\rho}}dk\right)^{\frac{1}{\rho-1}}C_k^{\frac{-1}{\rho}} \\
\implies P_k &= PC^{\frac{1}{\rho}}C_k^{\frac{-1}{\rho}} \\
\implies C_k &= \left(\frac{P_k}{P}\right)^{-\rho}C
\end{aligned}
$$

Substituting into the constraint, we get the aggregate price index in terms of the sectoral price indexes:

$$
\begin{aligned}
C &= \left(\int\left(\left(\frac{P_k}{P}\right)^{-\rho}C\right)^{\frac{\rho-1}{\rho}}dk\right)^{\frac{\rho}{\rho-1}} \\
\implies 1 &= \left(\int\left(\frac{P_k}{P}\right)^{1-\rho}dk\right)^{\frac{\rho}{\rho-1}} \\
\implies 1 &= P^{-\rho}\left(\int P_k^{1-\rho}dk\right)^{\frac{\rho}{\rho-1}} \\
\implies P &= \left(\int P_k^{1-\rho}dk\right)^{\frac{1}{1-\rho}}
\end{aligned}
$$

FOC $[C_{ik}]$:

$$
\begin{aligned}
P_{ik} &= P_k\frac{\theta}{\theta-1}\left(\sum_{i=1}^{N_k}C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}-1}\frac{\theta-1}{\theta}C_{ik}^{\frac{\theta-1}{\theta}-1} \\
\implies P_{ik} &= P_k\left(\sum_{i=1}^{N_k}C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{1}{\theta-1}}C_{ik}^{\frac{-1}{\theta}} \\
\implies P_{ik} &= P_kC_k^{\frac{1}{\theta}}C_{ik}^{\frac{-1}{\theta}} \\
\implies C_{ik} &= \left(\frac{P_{ik}}{P_k}\right)^{-\theta}C_k
\end{aligned}
$$

Substituting into the constraint, we get the sectoral price index in terms of the producers' prices:

$$
\begin{aligned}
C_k &= \left(\sum_{i=1}^{N_k} \left(\left(\frac{P_{ik}}{P_k} \right)^{-\theta} C_k \right)^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \\
\implies 1 &= \sum_{i=1}^{N_k} P_{ik}^{1-\theta} P_k^{\theta-1} \\
\implies P_k &= \left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta} \right)^{\frac{1}{1-\theta}}
\end{aligned}
$$

Thus, the aggregate price index P as a function of producers' prices is:

$$
P = \left(\int \left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta} \right)^{\frac{1-\rho}{1-\theta}} dk \right)^{\frac{1}{1-\rho}}
$$

And optimal demand C_{ik} as a function of producer's prices and aggregate demand is:

$$
C_{ik} = \left(\frac{P_{ik}}{P_k} \right)^{-\theta} \left(\frac{P_k}{P} \right)^{-\rho} C
$$

2. Assume that firms compete a la Bertrand, i.e. choose P_{ik} taking the prices of other firms in a sector $P_{jk}, j \neq i$ as given. Derive demand elasticity for a given firm and the optimal price.

We get rewrite demand for firm i in sector k as:

$$
C_{ik} = \frac{P_{ik}^{-\theta}}{P^{-\rho}} \left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta} \right)^{\frac{\theta-\rho}{1-\theta}} C
$$

The demand elasticity for firm i in sector k is:

$$
\begin{aligned}
\frac{dC_{ik}/C_{ik}}{dP_{ik}/P_{ik}} &= \frac{C}{P^{-\rho}} \left[\frac{\theta-\rho}{1-\theta} \left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta} \right)^{\frac{\theta-\rho}{1-\theta}-1} (1-\theta) P_{ik}^{-\theta} P_{ik}^{-\theta} + \left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta} \right)^{\frac{\theta-\rho}{1-\theta}} (-\theta) P_{ik}^{-\theta-1} \right] \frac{P_{ik}}{C_{ik}} \\
&= \frac{C}{P^{-\rho}} \left[(\theta-\rho) \left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta} \right)^{\frac{2\theta-\rho-1}{1-\theta}} P_{ik}^{-2\theta} + \left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta} \right)^{\frac{\theta-\rho}{1-\theta}} (-\theta) P_{ik}^{-\theta-1} \right] \frac{P_{ik}}{C_{ik}} \\
&= \frac{C}{P^{-\rho}} \left[(\theta-\rho) P_k^{2\theta-\rho-1} P_{ik}^{-2\theta} - \theta P_k^{\theta-\rho} P_{ik}^{-\theta-1} \right] \frac{P_{ik}}{C_{ik}} \\
&= P_{ik} \frac{C}{P^{-\rho}} \left[(\theta-\rho) P_k^{2\theta-\rho-1} P_{ik}^{-2\theta} - \theta P_k^{\theta-\rho} P_{ik}^{-\theta-1} \right] \left(\frac{P_{ik}}{P_k} \right)^{\theta} \left(\frac{P_k}{P} \right)^{\rho} C^{-1} \\
&= \left[(\theta-\rho) P_k^{2\theta-\rho-1} P_{ik}^{-2\theta} - \theta P_k^{\theta-\rho} P_{ik}^{-\theta-1} \right] P_{ik}^{1+\theta} P_k^{\rho-\theta} \\
&= (\theta-\rho) s_{ik} - \theta
\end{aligned}
$$

Where $s_{ik} := (\frac{P_{ik}}{P_k})^{1-\theta}$. The firms' problem is:

$$\begin{aligned}
&\max_{\{P_{ik},L_{ik},C_{ik}\}} P_{ik}C_{ik} - WL_{ik} \\
&\quad \text{s.t. } C_{ik} = A_{ik}L_{ik} \\
&\quad \text{and } C_{ik} = \frac{P_{ik}^{-\theta}}{P^{-\rho}}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} C \\
&\implies \max_{P_{ik}} P_{ik}\frac{P_{ik}^{-\theta}}{P^{-\rho}}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} C - \frac{W}{A_{ik}}\frac{P_{ik}^{-\theta}}{P^{-\rho}}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} C \\
&\implies \max_{P_{ik}} P_{ik}^{1-\theta}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} - \frac{W}{A_{ik}}P_{ik}^{-\theta}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}}
\end{aligned}$$

FOC $[P_{ik}]$:

$$\begin{aligned}
(1-\theta)P_{ik}^{-\theta}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} + \frac{\theta-\rho}{1-\theta}P_{ik}^{1-\theta}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}-1}(1-\theta)P_{ik}^{-\theta} \\
= \frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-\theta-1}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} + P_{ik}^{-\theta}\frac{\theta-\rho}{1-\theta}\left(\sum_{j=1}^{N_k} P_{jk}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}-1}(1-\theta)P_{ik}^{-\theta}\right] \\
(1-\theta)P_{ik}^{-\theta}P_k^{\theta-\rho} + (\theta-\rho)P_{ik}^{1-2\theta}P_k^{2\theta-\rho-1} = \frac{W}{A_{ik}}[(-\theta)P_{ik}^{-\theta-1}P_k^{\theta-\rho} + (\theta-\rho)P_{ik}^{-2\theta}P_k^{2\theta-\rho-1}] \\
(1-\theta) + (\theta-\rho)P_{ik}^{1-\theta}P_k^{\theta-1} = \frac{W}{A_{ik}}[(-\theta)P_{ik}^{-1} + (\theta-\rho)P_{ik}^{-\theta}P_k^{\theta-1}] \\
P_{ik}[(1-\theta) + (\theta-\rho)s_{ik}] = \frac{W}{A_{ik}}[(-\theta) + (\theta-\rho)s_{ik}] \\
\implies P_{ik} = \frac{W}{A_{ik}}\left[\frac{(\theta-\rho)s_{ik}-\theta}{(\theta-\rho)s_{ik}+1-\theta}\right]
\end{aligned}$$

3. Prove that other things equal, firms with higher A_{ik} set higher markups.

The total cost for firm i in sector k is $WL_{ik} = \frac{WC_{ik}}{A_{ik}}$, which implies that the marginal cost is $\frac{W}{A_{ik}}$. Firm i's mark-up, X_{ik} is the ratio of their price to their marginal cost:

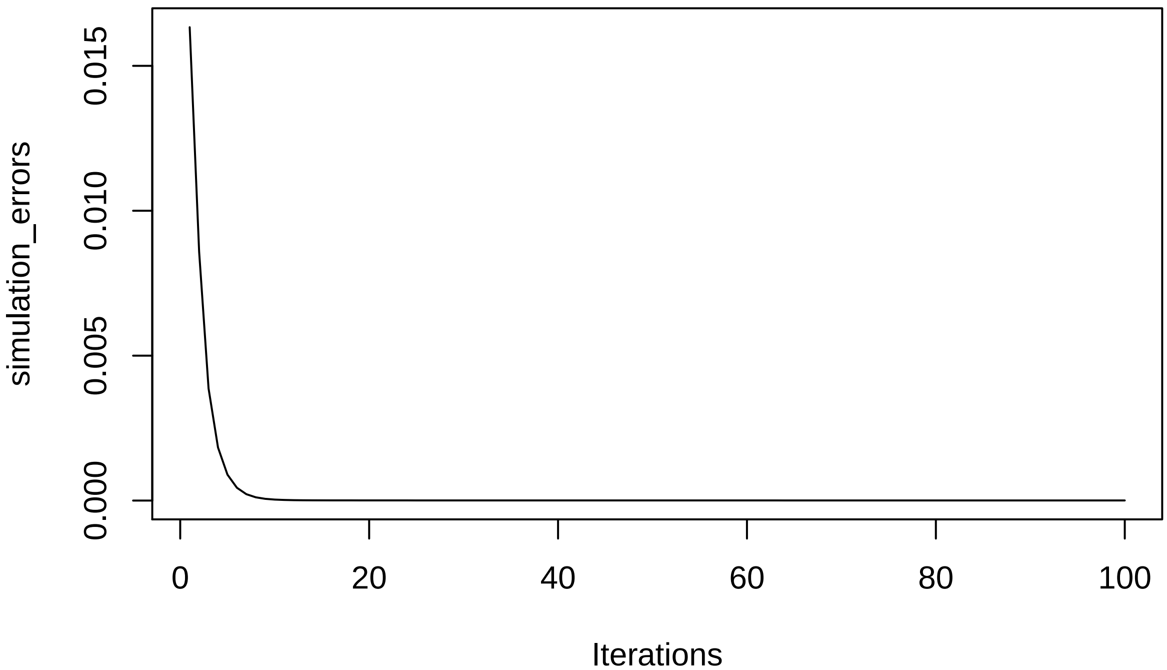
$$\begin{aligned}
X_{ik} &= \frac{(\theta-\rho)s_{ik}-\theta}{(\theta-\rho)s_{ik}+1-\theta} \\
&= 1 - \frac{1}{(\theta-\rho)s_{ik}+1-\theta} \\
&= 1 - [(\theta-\rho)(\frac{P_{ik}}{P_k})^{1-\theta} + 1 - \theta]^{-1} \\
&= 1 - [(\theta-\rho)W^{1-\theta}A_{ik}^{\theta-1}X_{ik}^{1-\theta}P_k^{\theta-1} + 1 - \theta]^{-1}
\end{aligned}$$

Thus, the first order change in the mark-up with a higher productivity is positive:

$$\frac{\partial X_{ik}}{\partial A_{ik}} \approx [(\theta-\rho)W^{1-\theta}A_{ik}^{\theta-1}X^{1-\theta}P_k^{\theta-1} + 1 - \theta]^{-2}(\theta-1)W^{1-\theta}A_{ik}^{\theta}X^{1-\theta}P_k^{\theta-1} > 0$$

4. Assume that $\rho = 2, \theta = 5, N_k = 20$, and $\log A_{ik} \sim i.i.d.N(0,1)$. Solve the model numerically by approximating the number of sectors with $K = 100,000$. You will need an efficient algorithm to compute a sectoral equilibrium (search for a fixed point, do not use "solve") nested in a general equilibrium loop solving for real wages.

```
# parameters
rho <- 1+1e-6
theta <- 5
n_k <- 20
k <- 100000
a <- matrix(exp(rnorm(n = k * n_k)), ncol = n_k, nrow = k)
w <- 1

# Initialize price matrices
# p_ik_0 starts at Dixit-Stiglitz
p_ik_0 <- w / a * theta / (theta - 1)
p_ik_1 <- p_ik_0

# Loop objects
iter <- 1
max_iter <- 100
simulation_errors <- NULL
tolerence <- 1e-6

# p_ik_0 -> p_k -> s_ik -> p_ik_1
while (TRUE) {
  # find sector prices and shares
  p_k <- apply(p_ik_0^(1-theta), 1, sum)^(1/(1-theta))
  s_ik <- (p_ik_0 / p_k %*% t(rep(1, times = n_k)))^(1-theta)

  # find prices based on shares and sector prices
  p_ik_1 <- (w/a) * ((theta-rho)*s_ik - theta)/((theta-rho)*s_ik + 1 - theta)

  # check for convergence
  simulation_errors[iter] <- sum(abs(p_ik_0 - p_ik_1)/sum(p_ik_0))
  if (simulation_errors[iter] < tolerence) break

  # update price guess
  p_ik_0 <- (1/2)*p_ik_0 + (1/2)*p_ik_1

  # Increment counter
  iter <- iter + 1
  if (iter > max_iter ) break
}

plot(simulation_errors, type = "l", xlab = "Iterations")
```



5. Compute the aggregate output C of the economy and compare it to the first-best value.

The simulation above implies a aggregate price level P and thus a real wage W/P. The household consuming their entire real wage, so $C = W/P$:

```
p <- ((1/k)*sum(p_k^(1-rho)))^(1/(1-rho))
print(w/p)
```

```
## [1] 4.599863
```

The first best outcome is when firms charge their marginal cost $P_{ik} = \frac{W}{A_{ik}}$:

```
p_ik_fb <- w/a
p_k_fb <- apply(p_ik_fb^(1-theta), 1, sum)^(1/(1-theta))
p_fb <- ((1/k)*sum(p_k_fb^(1-rho)))^(1/(1-rho))
print(w/p_fb)
```

```
## [1] 7.213827
```

Thus, the aggregate consumption from this model is about 2/3 of the first best aggregate consumption.

6. Bonus task: Does the sectoral equilibrium converge to the one under Betrand competition with homogeneous goods in the limit $\theta \to \infty$?

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