This note is based on the lecture slide by Professor Wouter J. Den Haan[1].

Case 1: Only capital is observable

The first order condition of the investment model with adjustment cost we discussed in class is

$$
\begin{aligned}
1+\psi_I(I_t,k_t) &= \frac{1}{1+r}E_t[\pi_k(k_{t+1},z_{t+1})-\psi_k(I_{t+1},k_{t+1})+[1+\psi_I(I_{t+1},k_{t+1})](1-\delta)] \\
k_{t+1} &= (1-\delta)k_t+I_t \\
z_t &= (1-\rho_z)+\rho_z z_{t-1}+\epsilon_t
\end{aligned}
$$

After linearizing the FOCs and applying the method of undetermined coefficient, we can obtain the decision rule about k_t as

$$
\hat{k}_{t+1} = \chi_k \hat{k}_t + \chi_z \hat{z}_t.
$$

Here χ_k and χ_z is a function of the structural parameters $\boldsymbol{\theta} = (\theta, \psi_0, \delta, \rho_z, r)$. Since $z_{ss} = 1$,

$$
\begin{aligned}
\frac{z_t - z_{ss}}{z_{ss}} &= \frac{(1-\rho)+\rho_z z_{t-1}+\epsilon_t - z_{ss}}{z_{ss}} \\
\iff \hat{z}_t &= \rho_z \hat{z}_{t-1} + \epsilon_t
\end{aligned}
$$

So $\hat{z}_t$ also follows an AR(1) process with mean zero and variance $\sigma_z^2 = \sigma_\epsilon^2/(1-\rho_z^2)$.

Let $\{k_t^*\}_{t=1}^T$ denote the observed data. We can apply the HP filter to obtain the de-trended data $\{\hat{k}_t^*\}_{t=1}^T$. Given the parameter values, we can compute implied productivity as

$$
\hat{z}_t^* = \frac{\hat{k}_{t+1}^* - \chi_k \hat{k}_t^*}{\chi_z} \tag{1}
$$

for $t = 1, 2, \cdots, T-1$. Note that since $\hat{z}_t$ follows an AR(1) process with normal shock, the likelihood function of $\mathbf{z} = \{\hat{z}_t^*\}$ can be written as

$$
L(\mathbf{z}|\boldsymbol{\theta}) = f_{\hat{Z}_1}(\hat{z}_1^*|\boldsymbol{\theta}) \prod_{t=2}^{T-1} f_{\hat{Z}_t|\hat{Z}_{t-1}}(\hat{z}_t^*|\boldsymbol{\theta}, \hat{z}_{t-1}^*)
$$

[1] http://www.wouterdenhaan.com/numerical/slidesbayesian.pdf

where

$$f_{\hat{Z}_t|\hat{Z}_{t-1}}(\hat{z}_t^*|\boldsymbol{\theta}, \hat{z}_{t-1}^*) = \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left[-\frac{\{\hat{z}_t^* - \rho_z \hat{z}_{t-1}^*\}^2}{\sigma_z^2}\right]$$

and

$$f_{\hat{Z}_1}(\hat{z}_1^*|\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma_z^2}} \exp\left[-\left(\frac{\hat{z}_1^*}{\sigma_z}\right)^2\right].$$

Suppose that the prior distribution of $\boldsymbol{\theta}$ is $\pi(\boldsymbol{\theta})$. Then from Bayes theorem, the posterior distribution can be written as

$$\begin{aligned}\pi(\boldsymbol{\theta}|\mathbf{z}) &= \frac{L(\mathbf{z}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{f(\mathbf{z})} \\ &= \frac{L(\mathbf{z}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int L(\mathbf{z}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}} \qquad (2)\end{aligned}$$

From this equation, we can see see the relationship between prior, likelihood, and posterior.

1. $\hat{\boldsymbol{\theta}}_{MLE} = \arg\max_{\boldsymbol{\theta}} \pi(\boldsymbol{\theta}|\mathbf{z})$ if the parameter space is bounded and the prior is uniform (or uninformative).

2. $\pi(\boldsymbol{\theta}|\mathbf{z}) = \pi(\boldsymbol{\theta})$ if likelihood function contain no information about the parameter $\boldsymbol{\theta}$.

If the parameter space is bounded and $\pi(\boldsymbol{\theta})$ is uniform (= doesn't depend on $\boldsymbol{\theta}$),

$$\begin{aligned}\pi(\boldsymbol{\theta}|\mathbf{z}) &= \frac{L(\mathbf{z}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int L(\mathbf{z}|\boldsymbol{\theta})d\boldsymbol{\theta}\pi(\boldsymbol{\theta})} \\ &= \frac{L(\mathbf{z}|\boldsymbol{\theta})}{\int L(\mathbf{z}|\boldsymbol{\theta})d\boldsymbol{\theta}} \propto L(\mathbf{z}|\boldsymbol{\theta}).\end{aligned}$$

So the posterior distribution is identical to the likelihood function multiplied by some constant, and maximizer of the right hand side, which is the maximum likelihood estimator, coincides with the maximizer of the left hand side, which is Maximum a posteriori (MAP) estimator.

On the other hand, if the likelihood function is independent of $\boldsymbol{\theta}$,

$$\begin{aligned}\pi(\boldsymbol{\theta}|\mathbf{z}) &= \frac{L(\mathbf{z}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int \pi(\boldsymbol{\theta})d\boldsymbol{\theta}L(\mathbf{z}|\boldsymbol{\theta})} \\ &= \frac{\pi(\boldsymbol{\theta})}{\int \pi(\boldsymbol{\theta})d\boldsymbol{\theta}} = \pi(\boldsymbol{\theta}).\end{aligned}$$

So if the model doesn't say anything about the parameter value, the researcher learn nothing from the data, and the prior will be the posterior.

In general, it is difficult to obtain the closed form expression for the posterior distribution, so dynare uses the Markov Chain Monte Carlo method to implement the Bayesian inference.

Identification

From equation (1), we can see that if we change (χ_k, χ_z), then $\hat{z}_t^*$ will change as well. This will lead to the different value of the likelihood. So if we observe k_t, we can identify $(\chi_k, \chi_z, \rho_z, \sigma_\epsilon)$. Since we have 4 moment $(\theta, \psi_0, r, \delta)$ which affect (χ_k, χ_z), essentially we have 2 equation for 4 parameters. So the parameter is not identified. We need to fix two parameter values (say, (r, δ)) if we want to identify the parameters.

Case 2: Marginal Q is also observable

In this section suppose that the data on marginal Q is observable. In the model, once we know capital $\{\hat{k}_t\}$, we can compute the marginal q $\{\hat{q}_t\}$ by linearizing the first order condition with respect to I_t around the steady state:

$$q_t = 1 + \psi_I(k_{t+1} - (1-\delta)k_t, k_t).$$

The linearized FOC is

$$\begin{aligned}\hat{q}_t &= \psi_0(\hat{k}_{t+1} - \hat{k}_t) \\ &= \psi_0(\chi_k\hat{k}_t + \chi_z\hat{z}_t - \hat{k}_t) \\ &= \psi_0(\chi_k - 1)\hat{k}_t + \psi_0\chi_z\hat{z}_t.\end{aligned}$$

Then, if we observe the data on marginal q_t, $\{q_t^*\}$, we can compute

$$\hat{z}_t^q = \frac{\hat{q}_t^* - \psi_0(\chi_k - 1)\hat{k}_t^*}{\psi_0\chi_z} \tag{3}$$

Note that we already computed $\hat{z}_t^*$ in (1) using the capita data. Obviously, in this model,

$$\hat{z}_t^* = \hat{z}_t^q \tag{4}$$

or in probabilistic term

$$P(\hat{z}_t^* = \hat{z}_t^q) = 1.$$

However, in the actual data, the condition (4) is rarely satisfied. As a result, $P(\hat{z}_t^* \neq \hat{z}_t^q) = 0$, the likelihood function will be zero everywhere. This situation is called as stochastic singularity in the literature. In general, in order to avoid the stochastic singularity, we have to have as many unobservable shocks as the observable variables.

In the case without measurement error, we inverted the decision rule to compute the shocks and then evaluate the likelihood. In general, if we have measurement errors, we have to use the Kalman filter to evaluate the likelihood. Dynare automatically use the Kalman filter even if there is no measurement error.

Computational algorithm

For detail, see the section 8 of dynare user guide.

1. Dynare computes the posterior mode, $\boldsymbol{\theta}_{mode} = \arg\max_{\boldsymbol{\theta}} \pi(\boldsymbol{\theta}|\mathbf{z})$ using some optimization algorithm. In this process, dynare uses the Kalman filter to compute the likelihood.

2. Given $\boldsymbol{\theta}^1 = \boldsymbol{\theta}_{mode}$, draw a sequence of parameter values $\{\boldsymbol{\theta}^n\}$ using the random walk Metropolis-Hastings algorithm.

   (a) Given $\boldsymbol{\theta}^i$, draw a candidate $\boldsymbol{\theta}'$ from $N(\boldsymbol{\theta}^i, c\Sigma_m)$ where Σ_m is the inverse of the Hessian computed at the posterior mode. This means $\boldsymbol{\theta}' = \boldsymbol{\theta}^i + \epsilon$ with $\epsilon \sim N(0, c\Sigma_m)$ and this is why this algorithm is called random-walk.

   (b) Compute the acceptance ratio

$$q(\boldsymbol{\theta}'|\boldsymbol{\theta}^i) = \min\left\{1, \frac{\pi(\boldsymbol{\theta}'|\mathbf{z})}{\pi(\boldsymbol{\theta}^i|\mathbf{z})}\right\}.$$

Draw $w \sim U(0,1)$. Then set

$$\boldsymbol{\theta}^{i+1} = \begin{cases} \boldsymbol{\theta}' & \text{if } q(\boldsymbol{\theta}'|\boldsymbol{\theta}^i) \geq w, \\ \boldsymbol{\theta}^i & \text{othwewise.} \end{cases}$$

(c) Repeat until we get enough sample. Drop first several samples to get rid of the effect of the initial state.

3. Use $\{\boldsymbol{\theta}^n\}$ to compute the posterior distribution.