

ECON 709 - PS 3

Alex von Hafften*

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1. A random point (X, Y) is distributed uniformly on the square with vertices $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$. That is, the joint PDF is $f(x, y) = 1/4$ on the square and $f(x, y) = 0$ outside the square. Determine the probability of the following events:

(a) $X^2 + Y^2 < 1$

$$X^2 + Y^2 < 1 \implies -\sqrt{1 - X^2} < Y < \sqrt{1 - X^2}$$

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4} dy dx = \int_{-1}^1 \left[\frac{1}{4} y \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} dx$$

Define $x = \sin \theta \implies dx = \cos \theta d\theta$.

$$\begin{aligned} \frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} dx &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sqrt{1-(\sin \theta)^2} \cos \theta d\theta \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta \\ &= \frac{1}{4} \int_{-\pi/2}^{\pi/2} 1 + \cos(2\theta) d\theta \\ &= \frac{1}{4} \left[\theta + \frac{\sin(2\theta)}{2} \right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{4} \left[(\pi/2) + \frac{0}{2} - (-\pi/2) - \frac{0}{2} \right] \\ &= \frac{\pi}{4} \end{aligned}$$

(b) $|X + Y| < 2$

$$|X + Y| < 2 \implies -2 < X + Y < 2 \implies -2 - X < Y < 2 - X. \text{ Since } X \text{ ranges from } -1 \text{ to } 1, \\ -2 - X < Y < 2 - X \implies -1 < Y < 1$$

$$\int_{-1}^1 \int_{-1}^1 \frac{1}{4} dy dx = \frac{1}{4} \int_{-1}^1 [y]_{-1}^1 dx = \frac{1}{2} \int_{-1}^1 dx = \frac{1}{2} [x]_{-1}^1 = 1$$

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2. Let the joint PDF of X and Y be given by $f(x, y) = g(x)h(y) \forall x, y \in \mathbb{R}$ for some functions $g(x)$ and $h(y)$. Let a denote $\int_{-\infty}^{\infty} g(x)dx$ and b denote $\int_{-\infty}^{\infty} h(x)dx$

(a) What conditions a and b should satisfy in order for $f(x, y)$ to be a bivariate PDF?

For $f(x, y)$ to be a PDF, it should integrate to one:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= 1 \\ \implies \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) dx dy &= 1 \\ \implies \int_{-\infty}^{\infty} g(x) dx \int_{-\infty}^{\infty} h(y) dy &= 1 \\ &\implies ab = 1 \\ &\implies a = b^{-1} \end{aligned}$$

(b) Find the marginal PDF of X and Y .

The marginal PDF of X :

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\infty}^{\infty} g(x)h(y) dy = g(x) \int_{-\infty}^{\infty} h(y) dy = b \cdot g(x)$$

The marginal PDF of Y :

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_{-\infty}^{\infty} g(x)h(y) dx = h(y) \int_{-\infty}^{\infty} g(x) dx = a \cdot h(y)$$

(c) Show that X and Y are independent.

Proof: X and Y are independent if the product of their marginal distributions is their joint distribution:

$$\begin{aligned} f_X(x) \cdot f_Y(y) &= b \cdot g(x) \cdot a \cdot h(y) \\ &= b \cdot g(x) \cdot b^{-1} \cdot h(y) \\ &= g(x) \cdot h(y) \\ &= f(x, y) \end{aligned}$$

□

3. Let the joint PDF of X and Y be given by

$$f(x, y) = \begin{cases} cxy & \text{if } x, y \in [0, 1], x + y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c such that $f(x, y)$ is a joint PDF.

$$\begin{aligned} \int_0^1 \int_0^{1-x} f(x, y) dy dx &= 1 \\ \Rightarrow \int_0^1 \int_0^{1-x} cxy dy dx &= 1 \\ \Rightarrow c \int_0^1 \left[\frac{xy^2}{2} \right]_{y=0}^{1-x} dx &= 1 \\ \Rightarrow \frac{c}{2} \int_0^1 x(1-x)^2 dx &= 1 \\ \Rightarrow \frac{c}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_{x=0}^1 &= 1 \\ \Rightarrow \frac{c}{2} \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] &= 1 \\ \Rightarrow c &= 24 \end{aligned}$$

(b) Find the marginal distributions of X and Y .

$$f_X(x) = \int_0^{1-x} f(x, y) dy = \int_0^{1-x} 24xy dy = \left[12xy^2 \right]_{y=0}^{1-x} = \begin{cases} 12x(1-x)^2, & x \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_0^{1-y} f(x, y) dx = \int_0^{1-y} 24xy dx = \left[12x^2y \right]_{x=0}^{1-y} = \begin{cases} 12(1-y)^2y, & y \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

(c) Are X and Y independent? Compare your answer to Problem 2 and discuss.

X and Y independent if the product of the marginal distributions equals their joint distribution at all points in the support. If $x = y = 0.9$, $f(0.9, 0.9) = 0$ because $(0.9, 0.9)$ is not in the support, $x + y = 0.9 + 0.9 = 1.8 > 1$. But each marginal distribution is defined over $[0, 1]$, so the product of the marginals is positive at $(0.9, 0.9)$: $f_X(0.9)f_Y(0.9) = [12(0.9)(1 - (0.9))^2][12(1 - 0.9)^2(0.9)] = 0.0117$.

In (2), the support for the joint distribution is \mathbb{R}^2 , whereas the support for the joint distribution depends on the realization of the random variable.

4. Show that any random variable is uncorrelated with a constant.

Proof: Let $a \in \mathbb{R}$ and X be a random variable with distribution F_X . Define random variable Y as the degenerate random variable that equals a . Thus, the distribution Y is

$$F_Y(y) = \begin{cases} 0, & y < a \\ 1, & y \geq a \end{cases}$$

To show X is uncorrelated with a constant, I show that X and Y are independent and then, by a theorem in the Lecture 3 Notes, we know that X and Y are uncorrelated.

To find the joint distribution of X and Y , consider two cases: $y < a$ and $y \geq a$. For $y < a$,

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x \text{ and } Y \leq a) \\ &= 0 \end{aligned}$$

For $y \geq a$:

$$\begin{aligned} F(x, y) &= P(X \leq x \text{ and } Y \leq y) \\ &= P(X \leq x) \\ &= F_X(x) \end{aligned}$$

Thus, the joint distribution is

$$F(x, y) = \begin{cases} 0, & y < a \\ F_X(x), & y \geq a \end{cases}$$

The joint distribution equals the product of the marginals:

$$F(x, y) = \begin{cases} 0 * F_X(x), & y < a \\ 1 * F_X(x), & y \geq a \end{cases} = \begin{cases} F_Y(y) * F_X(x), & y < a \\ F_Y(y) * F_X(x), & y \geq a \end{cases}.$$

□

5. Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2 . Find an expression for the correlation of XY and Y in terms of these means and variances.

Note that $Var(X) = E(X^2) - E(X)^2 \implies E(X^2) = Var(X) + E(X)^2$.

$$\begin{aligned}
Corr(XY, Y) &= \frac{Cov(XY, Y)}{\sqrt{Var(XY)Var(Y)}} \\
&= \frac{E(XY^2) - E(XY)E(Y)}{\sigma_Y \sqrt{E((XY)^2) - E(XY)^2}} \\
&= \frac{E(X)E(Y^2) - E(X)E(Y)E(Y)}{\sigma_Y \sqrt{E(X^2)E(Y^2) - (E(X)E(Y))^2}} \\
&= \frac{\mu_X(Var(Y) + E(Y)^2) - \mu_X\mu_Y^2}{\sigma_Y \sqrt{(Var(X) + E(X)^2)(Var(Y) + E(Y)^2) - (\mu_X\mu_Y)^2}} \\
&= \frac{\mu_X\sigma_Y^2 + \mu_X\mu_Y^2 - \mu_X\mu_Y^2}{\sigma_Y \sqrt{\sigma_Y^2\sigma_X^2 + \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \mu_Y^2\mu_X^2 - \mu_X^2\mu_Y^2}} \\
&= \frac{\mu_X\sigma_Y}{\sqrt{\sigma_Y^2\sigma_X^2 + \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2}}
\end{aligned}$$

6. Prove the following: For any random vector (X_1, X_2, \dots, X_n) ,

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j).$$

Proof (by induction): For $n = 2$, $\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2)$ from lecture 3 notes. Assume that the formula holds for some n , then

$$\begin{aligned} & \text{Var}\left(\sum_{i=1}^{n+1} X_i\right) \\ &= \text{Var}\left(\sum_{i=1}^n X_i + X_{n+1}\right) \\ &= \text{Var}\left(\sum_{i=1}^n X_i\right) + \text{Var}(X_{n+1}) + 2\text{Cov}\left(\sum_{i=1}^n X_i, X_{n+1}\right) \\ &= \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j) + \text{Var}(X_{n+1}) + 2 \left[E\left(X_{n+1} \sum_{i=1}^n X_i\right) + E\left(X_{n+1}\right) E\left(\sum_{i=1}^n X_i\right) \right] \\ &= \sum_{i=1}^{n+1} \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j) + 2 \left[\sum_{i=1}^n E\left(X_{n+1} X_i\right) + E\left(X_{n+1}\right) \sum_{i=1}^n E\left(X_i\right) \right] \\ &= \sum_{i=1}^{n+1} \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j) + 2 \left[\sum_{i=1}^n E\left(X_{n+1} X_i\right) + E\left(X_{n+1}\right) E\left(X_i\right) \right] \\ &= \sum_{i=1}^{n+1} \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j) + 2 \sum_{i=1}^n \text{Cov}\left(X_{n+1} X_i\right) \\ &= \sum_{i=1}^{n+1} \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n+1} \text{Cov}(X_i, X_j) \end{aligned}$$

□

7. Suppose that X and Y are joint normal, i.e. they have the joint PDF:

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp(-(2(1-\rho^2))^{-1}(x^2/\sigma_X^2 - 2\rho xy/\sigma_X\sigma_Y + y^2/\sigma_Y^2))$$

(a) Derive the marginal distributions of X and Y , and observe that both normal distributions.

The marginal distribution of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right) dy \\ &= \frac{1}{\sigma_X\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2} + \frac{\rho^2 x^2}{\sigma_X^2} - \frac{\rho^2 x^2}{\sigma_X^2}\right)\right) dy \\ &= \frac{1}{\sigma_X\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{\rho^2 x^2}{\sigma_X^2}\right) - \frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{\rho^2 x^2}{\sigma_X^2}\right)\right) dy \\ &= \frac{1}{\sigma_X\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy(\sigma_Y/\sigma_X)}{\sigma_Y^2} + \frac{\rho^2 x^2(\sigma_Y^2/\sigma_X^2)}{\sigma_Y^2}\right) - \frac{(1-\rho^2)}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}\right)\right) dy \\ &= \frac{1}{\sigma_X\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{(y - \rho x(\sigma_Y/\sigma_X))^2}{2(1-\rho^2)\sigma_Y^2} - \frac{x^2}{2\sigma_X^2}\right) dy \\ &= \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{(y - \rho x(\sigma_Y/\sigma_X))^2}{2(1-\rho^2)\sigma_Y^2}\right) dy \\ &= \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \end{aligned}$$

Thus, $X \sim \text{Normal}(0, \sigma_X^2)$. By swapping x and y in the algebra above, we see that $Y \sim \text{Normal}(0, \sigma_Y^2)$.

(b) Derive the conditional distribution of Y given $X = x$. Observe that it is also a normal distribution.

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right) \left[\frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right)\right]^{-1} \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right) + \frac{x^2}{2\sigma_X^2}\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2} - \frac{x^2(1-\rho^2)}{\sigma_X^2}\right)\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{x^2\rho^2}{\sigma_X^2}\right)\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{y}{\sigma_Y} - \frac{x\rho}{\sigma_X}\right)^2\right) \\
&= \frac{1}{\sigma_Y\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{(y - \frac{\sigma_Y}{\sigma_X}\rho x)^2}{(1-\rho^2)\sigma_Y^2}\right)
\end{aligned}$$

Observe that $Y|X \sim \text{Normal}(\frac{\sigma_Y}{\sigma_X}\rho x, \sigma_Y(1-\rho^2))$.

- (c) Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X and Z are independent.

Define $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $g(x, y) = (x, (y/\sigma_Y) - (\rho x)/\sigma_X)$. Notice that g is one-to-one, so it is invertible. Define $h = g^{-1}$ such that $h(x, z) = (x, \sigma_Y(z + \rho x/\sigma_X))$. The determinant of the Jacobian of the tranformation is

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial h_1(x, z)}{\partial x} & \frac{\partial h_1(x, z)}{\partial z} \\ \frac{\partial h_2(x, z)}{\partial x} & \frac{\partial h_2(x, z)}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ \frac{\rho\sigma_Y}{\sigma_X} & \sigma_Y \end{vmatrix} \\ &= \sigma_Y \end{aligned}$$

Thus, from lecture 3 notes, we know that the joint distribution

$$\begin{aligned} f_{X,Z}(x, z) &= f_{X,Y}(h(x, z))|J| \\ &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho x(\sigma_Y(z + \rho x/\sigma_X))}{\sigma_X\sigma_Y} + \frac{(\sigma_Y(z + \rho x/\sigma_X))^2}{\sigma_Y^2}\right)\right)\sigma_Y \\ &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho x(z + \rho x/\sigma_X)}{\sigma_X} + (z + \rho x/\sigma_X)^2\right)\right) \\ &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xz}{\sigma_X} - \frac{2\rho^2 x^2}{\sigma_X^2} + z^2 + \frac{2z\rho x}{\sigma_X} + \frac{\rho^2 x^2}{\sigma_X^2}\right)\right) \\ &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{\rho^2 x^2}{\sigma_X^2} + z^2\right)\right) \\ &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(1-\rho^2)x^2}{\sigma_X^2} + z^2\right)\right) \\ &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2} + \frac{z^2}{(1-\rho^2)}\right)\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2}\right)\right) \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{z^2}{(1-\rho^2)}\right)\right) \\ &= f_X(x) \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{z^2}{(1-\rho^2)}\right)\right) \end{aligned}$$

Thus, X and Z are independent with $Z \sim \text{Normal}(0, 1 - \rho^2)$.

8. Consider a function $g : \mathbb{R} \rightarrow \mathbb{R}$. Recall that the inverse image of a set A , denoted $g^{-1}(A)$ is $g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$. Let there be functions $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $g_2 : \mathbb{R} \rightarrow \mathbb{R}$. Let X and Y be two random variables that are independent. Suppose that g_1 and g_2 are both Borel-measurable, which means that $g_1^{-1}(A)$ and $g_2^{-1}(A)$ are both in the Borel σ -field whenever A is in the Borel σ -field. Show that the two random variables $Z := g_1(X)$ and $W := g_2(Y)$ are independent. (Hint: use the 1st or the 2nd definition of independence.)