ECON 703 - PS 3

Alex von Hafften*

8/28/2020

(1) Let (X, d) be a nonempty complete metric space. Suppose an operator $T : X \to X$ satisfies $d(T(x), T(y)) < d(x, y)$ for all $x \neq y, x, y \in X$. Prove or disprove that T has a fixed point. Compare with the Contraction Mapping Theorem.

I prove that T has a fixed point.

Proof: Define $\beta_{xy} := \frac{d(T(x),T(y))}{d(x,y)}$ and $\beta := \max \beta_{xy}$ for all $x \neq y, x, y \in X$. Notice that $d(T(x), T(y)) < d(x, y) \implies \beta_{xy} < 1$. Thus, $d(T(x), T(y)) = \beta_{xy} d(x, y) \leq \beta d(x, y)$. By the convergence mapping theorem, T has a fixed point.

(3) Prove that the function $f(x) = \cos^2(x)e^{5-x-x^2}$ has a maximum on $\mathbb{R}$.

Proof: Define $g(x) = \cos^2(x)$ and $h(x) = e^{5-x-x^2}$. Notice that, since $-1 \leq \cos(x) \leq 1$, $0 \leq g(x) \leq 1$. Thus, $f(x) \leq h(x)$. $h(x) < 1$ on $(-\infty, -\sqrt{21}/2 - 1/2) \cup (\sqrt{21}/2 - 1/2, \infty)$, so $\max h(x)$ is at $x \in A = [\sqrt{21}/2 - 1/2, \sqrt{21}/2 - 1/2]$. Similarly, $\max f(x)$ is at an $y \in A$. A is a closed subset of $\mathbb{R}$. Since g and h are continuous, f is continuous. Since A is closed and f is continous, $f(x)$ is bounded on A. Thus, A is compact. By the extreme value theorem, f reaches its maximum on A and therefore on $\mathbb{R}$. $\square$

(4) Suppose you have two maps of Wisconsin: one large and one small. You put the large one on top of the small one, so that the small one is completely covered by the large one. Prove that it is possible to pierce the stack of those two maps in a way that the needle will go through exactly the same (geographical) points on both maps.

Proof: Draw Cartesian planes on both maps such that the Wisconsin Capital Building in Madison is at the origin. For the larger map, scale the x coordinate such that moving one unit right cooresponds to the point on the map that is one inch east of the Capital Building on the map and scale the y coordinate such that moving one unit up cooresponds to the point on the map that is one inch north of the Capital Building on the map. Scale the Cartesian plane on the smaller map similarly.

Define β as the the ratio of the miles between geographical points per inch on the larger map to the miles between geographical points per inch on the smaller map. Notice that $\beta < 1$. Define $A \subset \mathbb{R} \times \mathbb{R}$ as the closed set of points on the larger map that are on or within Wisconsin's borders. Notice that the metric space (A, d_E) is complete because A is closed. Define operator $T : A \to \mathbb{R} \times \mathbb{R}$ such that, for $x \in A$, (x_1, x_2) on the larger map and $(T(x_1), T(x_2))$ on the smaller map represent the same geographical point. Notice that, for all $x, y \in A, x \neq y$, $d_E(x, y) = \beta d_E(T(x), T(y))$. Thus, T is a contraction with modulus $\beta < 1$. Thus, T has a fixed point where $x^* = T(x^*)$, when you can pierce the stack of maps at the same geographical point.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, Tyler Welch, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.