ECON 709 - PS 6

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1. Let X be distributed Bernoulli $P(X = 1) = p$ and $P(X = 0) = 1 - p$ for some unknown parameter $0 < p < 1$.

(a) Verify the probability mass function can be written as $f(x) = p^x(1-p)^{(1-x)}$.

$$f(1) = p^1(1-p)^{(1-1)} = p = P(X = 1)$$
$$f(0) = p^0(1-p)^{(1-0)} = 1 - p = P(X = 0)$$

(b) Find the log-likelihood function $\ell_n(\theta)$.

$$\begin{aligned}
\ell_n(\theta) &= \sum_{i=1}^{n} \ln(f(x_i|\theta)) \\
&= \sum_{i=1}^{n} \ln(p^{x_i}(1-p)^{(1-x_i)}) \\
&= \sum_{i=1}^{n} [x_i \ln(p) + (1 - x_i)\ln(1-p)] \\
&= \ln(p)\sum_{i=1}^{n} x_i + \ln(1-p)\Big(n - \sum_{i=1}^{n} x_i\Big)
\end{aligned}$$

(c) Find the MLE $\hat{p}$ for p.

$$\begin{aligned}
\frac{\partial \ell_n}{\partial p} &= 0 \\
\frac{\partial}{\partial p}\left[\ln(p)\sum_{i=1}^{n} x_i + \ln(1-p)\Big(n - \sum_{i=1}^{n} x_i\Big)\right] &= 0 \\
\frac{\sum_{i=1}^{n} x_i}{p} - \frac{\Big(n - \sum_{i=1}^{n} x_i\Big)}{1-p} &= 0 \\
\sum_{i=1}^{n} x_i &= pn - p\sum_{i=1}^{n} x_i + p\sum_{i=1}^{n} x_i \\
\hat{p} &= \frac{1}{n}\sum_{i=1}^{n} x_i
\end{aligned}$$

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2. Let X be distributed Pareto with density $f(x) = \frac{\alpha}{x^{1+\alpha}}$ for $x \geq 1$. The unknown parameter is $\alpha > 0$.

(a) Find the log-likelihood function $\ell_n(\alpha)$.

$$\begin{aligned}
\ell_n(\alpha) &= \sum_{i=1}^{n} \ln(f(x_i|\alpha)) \\
&= \sum_{i=1}^{n} \ln\left(\frac{\alpha}{x_i^{1+\alpha}}\right) \\
&= \sum_{i=1}^{n} \ln \alpha - \sum_{i=1}^{n} \ln x_i^{1+\alpha} \\
&= n \ln \alpha - (1+\alpha) \sum_{i=1}^{n} \ln x_i
\end{aligned}$$

(b) Find the MLE $\hat{\alpha}_n$ for α.

$$\frac{\partial \ell_n}{\partial \alpha} = 0 \implies \frac{n}{\hat{\alpha}_n} - \sum_{i=1}^{n} \ln x_i = 0 \implies \hat{\alpha}_n = \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

3. Let X be distributed Cauchy with density $f(x) = \frac{1}{\pi(1+(x-\theta)^2)}$ for $x \in \mathbb{R}$. The unknown parameter is θ.

(a) Find the log-likelihood function $\ell_n(\theta)$.

$$\begin{aligned}
\ell_n(\theta) &= \sum_{i=1}^{n} \ln(f(x_i|\theta)) \\
&= \sum_{i=1}^{n} \ln\left(\frac{1}{\pi(1+(x_i-\theta)^2)}\right) \\
&= -\sum_{i=1}^{n} \ln(\pi) - \sum_{i=1}^{n} \ln\left(1+(x_i-\theta)^2\right) \\
&= -n\ln(\pi) - \sum_{i=1}^{n} \ln\left(1+(x_i-\theta)^2\right)
\end{aligned}$$

(b) Find the first-order condition for the MLE $\hat{\theta}$ for θ. You will not be able to solve for $\hat{\theta}$.

$$\frac{\partial \ell_n}{\partial \theta} = 0 \implies 0 - \sum_{i=1}^{n} \frac{2(x_i-\theta)(-1)}{1+(x_i-\theta)^2} \implies \sum_{i=1}^{n} \frac{2(x_i-\theta)}{1+(x_i-\theta)^2}$$

4. Let X be distributed double exponential (or Laplace) with density $f(x) = \frac{1}{2}\exp(-|x-\theta|)$ for $x \in \mathbb{R}$. The unknown parameter is θ.

(a) Find the log-likelihood function $\ell_n(\theta)$.

$$\begin{aligned}
\ell_n(\theta) &= \sum_{i=1}^{n} \ln(f(x_i|\theta)) \\
&= \sum_{i=1}^{n} \ln\left(\frac{1}{2}\exp(-|x_i-\theta|)\right) \\
&= -\sum_{i=1}^{n} \ln(2) + \sum_{i=1}^{n} \ln(\exp(-|x_i-\theta|)) \\
&= -n\ln(2) - \sum_{i=1}^{n} |x_i-\theta|
\end{aligned}$$

(b) Extra challenge: Find the MLE $\hat{\theta}_n$ for θ. This is challenging as it is not simply solving the FOC due to the nondifferentiability of the density function.

$$\ell_n(\theta) = -n\ln(2) - \sum_{i=1}^{n} |x_i-\theta| = -n\ln(2) - \sum_{i=1}^{n}((x_i-\theta)^2)^{1/2}$$

$\ell_n(\theta)$ is now differentiable.

$$\frac{\partial \ell_n}{\partial \theta} = -(1/2)\sum_{i=1}^{n}((x_i-\theta)^2)^{-1/2}(2(x_i-\theta))(-1) = \sum_{i=1}^{n} \frac{x_i-\theta}{|x_i-\theta|}$$

If $x_i > \theta$, $\frac{x_i-\theta}{|x_i-\theta|} = 1$ and if $x_i < \theta$, $\frac{x_i-\theta}{|x_i-\theta|} = -1$.

5. Take the Pareto model $f(x) = \alpha x^{-1-\alpha}, x \geq 1$. Calculate the information for α using the second derivative.

6. Take the model $f(x) = \theta\exp(-\theta x), x \geq 0, \theta > 0$.

(a) Find the Cramer-Rao lower bound for θ.

(b) Recall the MLE $\hat{\theta}_n$ for θ for Problem 1. Notice that this is a function of the sample mean. Use this formula and the delta method to find the asymptotic distribution for $\hat{\theta}_n$.

(c) Find the asymptotic distribution for $\hat{\theta}_n$ using the general formula for the asymptotic distribution of MLE introduced in Section 6. Do you find the same answer as in part (b)?

7. In the Bernoulli model, you found the asymptotic distribution of the MLE in Problem 2(c).

(a) Propose an estimator of V, the asymptotic variance.

(b) Show that this estimator is consistent for V as $n \to \infty$.

(c) Propose a standard error $s(\hat{p_n})$ for the MLE $\hat{p}_n$.[1]

[1] Recall that the standard error is supposed to approximate the variance of $\hat{p}_n$, not that of the variance of $\sqrt{n}(\hat{p}_n - p$. What would be a reasonable approximation of the variance of $\hat{p}_n$ once you have a reasonable approximation of the variance of $\sqrt{n}(\hat{p}_n - p)$ from part (b)?

8. Consider the MLE for the upper bound of the uniform distribution in the Uniform Boundary example in Section 3. Assume that $\{X_1, ..., X_n\}$ is a random sample from $Uniform[0, \theta]$. The general asymptotic distribution formula in Section 6 does not apply here because $\ell_n(\theta)$ is not differentiable at the MLE. But you can derive the asymptotic distribution using the definition of convergences in distribution. Do so by following the steps below.

(a) Let F_X denote the CDF of $Uniform[0, \theta]$. Calculate $F_X(c)$ for all $c \in \mathbb{R}$ based on the PDF of $Uniform[0, \theta]$.

(b) Show that the CDF of $n(\hat{\theta}_n - \theta) : F_{n(\hat{\theta}_n - \theta)}(x) = \Pr(\max_{i=1,\dots,n}(n(X_i - \theta)) \leq x) = (F_X(\theta + \frac{x}{n}))^n$.

(c) Recall that $\lim_{n\to\infty}(1 + \frac{y}{n})^n = e^y$ for any $y \in \mathbb{R}$. Derive the limit of $F_{n(\hat{\theta}_n - \theta)}(x)$ for all fixed $x \in \mathbb{R}$. (Hint: consider the case where $x < 0$ and the case where $x \geq 0$ separately).

(d) Conclude that $n(\hat{\theta}_n - \theta) \to_d -Z$ for Z being an exponential distribution with parameter θ.

9. Take the model $X \sim N(\mu, \sigma^2)$. Propose a test for $H_0 : \mu = 1$ against $H_1 : \mu \neq 1$.

Assuming that σ^2 is unknown, we can use a two-sided t-test by constructing the following t-statistic:

$$T = \frac{|\sqrt{n}(\bar{X}_n - 1)|}{S_X}$$

where $S_X^2 = \frac{1}{n-1}(X_i - \bar{X}_n)^2$. Under the $H_0 : \mu = 1$, $T \sim |t_{n-1}|$. Therefore, $\phi_n(\alpha) = 1(T > t_{\alpha/2,n-1})$ where $t_{\alpha/2,n-1}$ is the $(1 - \alpha/2)$ quantile of $t_{n=1}$.

If σ^2 is known, we can use a z-test by replacing S_X with σ in the test statistic:

$$T = \frac{|\sqrt{n}(\bar{X}_n - 1)|}{\sigma}$$

Under the $H_0 : \mu = 1$, $T \sim |N(0, 1)|$. Therefore, $\phi_n(\alpha) = 1(T > z_{\alpha/2})$ where $z_{\alpha/2}$ is the $(1 - \alpha/2)$ quantile of a standard normal.

10. Take the model $X \sim N(\mu, 1)$. Consider testing $H_0 : \mu \in \{0, 1\}$ against $H_1 : \mu \notin \{0, 1\}$. Consider the test statistic $T = \min\{|\sqrt{n}\bar{X}_n|, |\sqrt{n}(\bar{X}_n - 1)|\}$ Let the critical value be the $1 - \alpha$ quantile of the random variable $\min\{|Z|, |Z - \sqrt{n}|\}$, where $Z \sim N(0, 1)$. Show that $\Pr(T > c|\mu = 1) = \alpha$. Conclude that the size of the test $\phi_n = 1(T > c)$ is α.[2]

$$\Pr(T > c|\mu = 1) = \Pr(\min\{|\sqrt{n}\bar{X}_n|, |\sqrt{n}(\bar{X}_n - 1)|\} > c|\mu = 1) = \Pr(\min\{|\sqrt{n}\bar{X}_n|, |\sqrt{n}(\bar{X}_n - 1)|\} > c|\mu = 1)$$

[2] Use the fact that Z and $-Z$ have the same distribution. This is an example where the null distribution is the same under different points in a composite null. The test $\phi_n = 1(T > c)$ is called a similar test because $\inf_{\theta_0 \in \Theta_0} \Pr(T > c|\theta = \theta_0 = \sup_{\theta_0 \in \Theta_0} \Pr(T > c|\theta = \theta_0$.