ECON 736A: Problem Set 2

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Prompt: Write a **simple** model of something.

Idea: A simple three period model of firm bankruptcy and reorganziation with debt dilution from long and short-term debt.

General Timing:

- Period 0
  - Firm is born with funds I
  - Firm sells b_0 discount bonds at price q_0 to representative risk-neutral "long-term" investor who can also buy risk-free discount bonds with price $\tilde{q}$
- Period 1
  - First productivity shock $\varepsilon_1 \sim F$ realized
  - Firm sells b_1 discount bonds at price q_1 to representative risk-neutral "short-term" investor who can also buy risk-free discount bonds with price $\tilde{q}$
  - Firm buys capital $k \equiv I + q_0 b_0 + q_1 b_1$ and has outstanding debt $b \equiv b_0 + b_1$
- Period 2
  - Second productivity shock $\varepsilon_2 \sim G$ realized
  - Firm chooses to produce or default or reorganize
    - If firm produces, firm payoff is $f(\varepsilon_1, \varepsilon_2, k) - b$, long-term investor payoff is b_0, and long-term investor payoff is b_1
    - If firm defaults, firm payoff is 0, long-term investor's payoff is $m_0(b_0, b_1)(1-\xi)f(\varepsilon_1, \varepsilon_2, k)$, short-term investor's payoff is $m_1(b_0, b_1)(1-\xi)f(\varepsilon_1, \varepsilon_2, k)$ where $\xi \in [0, 1]$, $m_0(b_0, b_1) + m_1(b_0, b_1) \leq 1$, $m_0(b_0, b_1) \geq 0$, and $m_1(b_0, b_1) \geq 0$ for all b_0, b_1
    - If firm chooses to reorganize, firm produces and firm and lenders Nash bargain over $f(\varepsilon_1, \varepsilon_2, k)$.

With binary productivity shocks

Assumptions:

- In period 2, firm can only choose to produce or default (no reorganizing)
- Recovery value is zero: $\xi = 1$
- ε_1 and ε_2 are iid Bernoulli random variables that take $R > 1$ with probability p and 1 with probability $1 - p$
- One type of debt: $b_1 = 0$
- Production function $f(\varepsilon_1, \varepsilon_2, k) = \varepsilon_1 \varepsilon_2 k^\alpha$ where $\alpha \in (0, 1)$

With these assumptions, problem is equivalent to two period model with one productivity shock

$$\tilde{\varepsilon} = \begin{cases} R^2, & \text{with probability } p^2 \\ R, & \text{with probability } 2p(1-p) \\ 1, & \text{with probability } (1-p)^2 \end{cases}$$

Timing with simplifying assumptions:

- Period 0
  - Firm is born with funds I
  - Firm sells b discount bonds at price q to representative risk-neutral investor who can also buy risk-free discount bonds with price $\tilde{q}$
  - Firm buys capital $k \equiv I + qb$
- Period 1
  - Productivity shock $\tilde{\varepsilon}$ realized
  - Firm chooses to produce or default
    * If firm produces, firm payoff is $\tilde{\varepsilon} k^\alpha - b$, and investor payoff is b
    * If firm defaults, firm payoff is 0, investor's payoff is 0

Solution

- In period 1, firm produces iff $\tilde{\varepsilon} k^\alpha - b \geq 0 \implies \tilde{\varepsilon} \geq \frac{b}{k^\alpha}$. Notice that this threshold is increasing in b and decreasing in k. That is if the firm has more debt (or less capital), it defaults for a larger set of productivity shocks.
- Thus, the firm's profit is

$$\pi(\tilde{\varepsilon}, k, b) = \begin{cases} \tilde{\varepsilon} k^\alpha - b, & \text{if } \frac{b}{k^\alpha} \leq \tilde{\varepsilon} \leq 1 \\ 0, & \text{if } 0 \leq \tilde{\varepsilon} < \frac{b}{k^\alpha} \end{cases}$$

- Four cases for intermediate productivity shock: (1) firm always defaults, (2) firm only produces for $\tilde{\varepsilon} = R^2$, (3) firm only produces for $\tilde{\varepsilon} = R^2$ and $\tilde{\varepsilon} = R$ and (4) firm always produces. Let $\pi^{(j)}$ denote profit in case j.

Case 1: Firm always defaults

- Then, profit is zero for firm, $\pi^{(1)} = 0$. And $b = 0$. Would be better not default for at least $\varepsilon = R^2$ with expected profit at least $p^2 R^2 I^\alpha > 0 \Rightarrow\Leftarrow$. The firm never always defaults.

Case 2: Firm only produces for $\tilde{\varepsilon} = R^2$

- Expected profit for (k, b) is

$$\pi(k, b) = p^2(R^2 k^\alpha - b)$$

- Taking q as given, the firm chooses k and b to maximize expected profit

$$\begin{aligned}
&\max_{k,b} p^2 R^2 k^\alpha - p^2 b \\
&\text{s.t. } k = I + qb \\
&\quad \mathcal{L} = p^2 R^2 k^\alpha - p^2 b + \lambda(I + qb - k)
\end{aligned}$$

- FOCs

$$\begin{aligned}
\alpha p^2 R^2 k^{\alpha-1} &= \lambda \\
p^2 &= q\lambda \\
\implies q\alpha p^2 R^2 k^{\alpha-1} &= p^2 \\
\implies k &= q^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} R^{\frac{2}{1-\alpha}}
\end{aligned}$$

Higher q (or lower interest rate) means more capital. Makes sense.

- Debt demand is

$$\begin{aligned}
b &= \frac{1}{q}(k - I) \\
&= \frac{1}{q}[q^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} R^{\frac{2}{1-\alpha}} - I] \\
&= q^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} R^{\frac{2}{1-\alpha}} - \frac{I}{q}
\end{aligned}$$

Higher q (or lower interest rate) means more debt. Makes sense.

- Lender can buy b risk-free discount bonds with price $\tilde{q}$ or the b risky bonds that pay off with probability p^2. Zero profit condition for lender:

$$\begin{aligned} b - \tilde{q}b &= p^2 b - qb \\ \implies 1 - \tilde{q} &= p^2 - q \\ \implies q &= p^2 + \tilde{q} - 1 \end{aligned}$$

- Substituting in bond price to capital and debt:

$$\begin{aligned} k &= (p^2 + \tilde{q} - 1)^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} R^{\frac{2}{1-\alpha}} \\ b &= (p^2 + \tilde{q} - 1)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} R^{\frac{2}{1-\alpha}} - \frac{I}{p^2 + \tilde{q} - 1} \\ \pi^{(2)} &= p^2[R^2 k^\alpha - b] \\ &= p^2 \left[R^2 (p^2 + \tilde{q} - 1)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} R^{\frac{2\alpha}{1-\alpha}} - (p^2 + \tilde{q} - 1)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} R^{\frac{2}{1-\alpha}} + \frac{I}{p^2 + \tilde{q} - 1} \right] \\ &= p^2 \left[(p^2 + \tilde{q} - 1)^{\frac{\alpha}{1-\alpha}} R^{\frac{2}{1-\alpha}} [\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}] + \frac{I}{p^2 + \tilde{q} - 1} \right] \end{aligned}$$

Case 3: Firm produces for $\tilde{\varepsilon} = R^2$ and $\tilde{\varepsilon} = R$

- The firm's expected profit for (k, b) is

$$\begin{aligned} \pi(k, b) &= p^2(R^2 k^\alpha - b) + 2p(1-p)(Rk^\alpha - b) \\ &= [p^2 R^2 + 2p(1-p)R]k^\alpha - [p^2 + 2p(1-p)]b \end{aligned}$$

- Taking q as given, the firm chooses k and b to maximize expected profit

$$\begin{aligned} &\max_{k,b} [p^2 R^2 + 2p(1-p)R]k^\alpha - [p^2 + 2p(1-p)]b \\ &\text{s.t. } k = I + qb \\ &\quad \mathcal{L} = [p^2 R^2 + 2p(1-p)R]k^\alpha - [p^2 + 2p(1-p)]b + \lambda(I + qb - k) \end{aligned}$$

- FOCs wrt k and b

$$
\begin{aligned}
[p^2R^2 + 2p(1-p)R]\alpha k^{\alpha-1} &= \lambda \\
[p^2 + 2p(1-p)] &= q\lambda \\
\implies q[p^2R^2 + 2p(1-p)R]\alpha k^{\alpha-1} &= [p^2 + 2p(1-p)]
\end{aligned}
$$

$$
\begin{aligned}
\implies k &= q^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} R^{\frac{1}{1-\alpha}} \left(\frac{p^2R + 2p(1-p)}{p^2 + 2p(1-p)} \right)^{\frac{1}{1-\alpha}} \\
\implies b &= \frac{1}{q}(k - I) \\
&= q^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} R^{\frac{1}{1-\alpha}} \left(\frac{p^2R + 2p(1-p)}{p^2 + 2p(1-p)} \right)^{\frac{1}{1-\alpha}} - \frac{I}{q}
\end{aligned}
$$

- Lender zero-profit condition:

$$
\begin{aligned}
b - \tilde{q}b &= [p^2 + 2p(1-p)]b - qb \\
\implies 1 - \tilde{q} &= [p^2 + 2p(1-p)] - q \\
\implies q &= p^2 + 2p(1-p) + \tilde{q} - 1
\end{aligned}
$$

- Substituting into (k, b)

$$
\begin{aligned}
k &= [p^2+2p(1-p)+\tilde{q}-1]^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}R^{\frac{1}{1-\alpha}}\left(\frac{p^2R+2p(1-p)}{p^2+2p(1-p)}\right)^{\frac{1}{1-\alpha}} \\
b &= [p^2+2p(1-p)+\tilde{q}-1]^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}R^{\frac{1}{1-\alpha}}\left(\frac{p^2R+2p(1-p)}{p^2+2p(1-p)}\right)^{\frac{1}{1-\alpha}} - \frac{I}{p^2+2p(1-p)+\tilde{q}-1} \\
\pi^{(3)} &= [p^2R^2+2p(1-p)R]k^\alpha - [p^2+2p(1-p)]b \\
&= [p^2R^2+2p(1-p)R][p^2+2p(1-p)+\tilde{q}-1]^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}R^{\frac{\alpha}{1-\alpha}}\left(\frac{p^2R+2p(1-p)}{p^2+2p(1-p)}\right)^{\frac{\alpha}{1-\alpha}} \\
&- [p^2+2p(1-p)][p^2+2p(1-p)+\tilde{q}-1]^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}R^{\frac{1}{1-\alpha}}\left(\frac{p^2R+2p(1-p)}{p^2+2p(1-p)}\right)^{\frac{1}{1-\alpha}} \\
&+ [p^2+2p(1-p)]\frac{I}{p^2+2p(1-p)+\tilde{q}-1} \\
&= R^{\frac{1}{1-\alpha}}[p^2+2p(1-p)+\tilde{q}-1]^{\frac{\alpha}{1-\alpha}}\Bigg[[p^2R+2p(1-p)]\alpha^{\frac{\alpha}{1-\alpha}}\left(\frac{p^2R+2p(1-p)}{p^2+2p(1-p)}\right)^{\frac{\alpha}{1-\alpha}} \\
&- [p^2+2p(1-p)]\alpha^{\frac{1}{1-\alpha}}\left(\frac{p^2R+2p(1-p)}{p^2+2p(1-p)}\right)^{\frac{1}{1-\alpha}}\Bigg] \\
&+ [p^2+2p(1-p)]\frac{I}{p^2+2p(1-p)+\tilde{q}-1}
\end{aligned}
$$

Case 4: Firm always produces

- Thus, the firm's expected profit is

$$
\begin{aligned}
\pi(k,b) &= p^2(R^2k^\alpha-b)+2p(1-p)(Rk^\alpha-b)+(1-p)^2(k^\alpha-b) \\
&= [p^2R^2+2p(1-p)R+(1-p)^2]k^\alpha-b
\end{aligned}
$$

- Taking q as given, the firm chooses k and b to maximize expected profit

$$
\begin{aligned}
&\max_{k,b}[p^2R^2+2p(1-p)R+(1-p)^2]k^\alpha-b \\
&\text{s.t. } k=I+qb \\
&\quad \mathcal{L}=[p^2R^2+2p(1-p)R+(1-p)^2]k^\alpha-b+\lambda(I+qb-k)
\end{aligned}
$$

- FOCs wrt k and b

$$
\begin{aligned}
[p^2R^2 + 2p(1-p)R + (1-p)^2]\alpha k^{\alpha-1} &= \lambda \\
1 &= q\lambda \\
\implies q[p^2R^2 + 2p(1-p)R + (1-p)^2]\alpha k^{\alpha-1} &= 1 \\
\implies k &= q^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}[p^2R^2 + 2p(1-p)R + (1-p)^2]^{\frac{1}{1-\alpha}} \\
\implies b &= \frac{1}{q}(k - I) \\
&= q^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}[p^2R^2 + 2p(1-p)R + (1-p)^2]^{\frac{1}{1-\alpha}} - \frac{I}{q}
\end{aligned}
$$

- Lender zero-profit condition:

$$
b - \tilde{q}b = b - qb \implies q = \tilde{q}
$$

- Substituting into k, b

$$
\begin{aligned}
k &= \tilde{q}^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}[p^2R^2 + 2p(1-p)R + (1-p)^2]^{\frac{1}{1-\alpha}} \\
b &= \tilde{q}^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}[p^2R^2 + 2p(1-p)R + (1-p)^2]^{\frac{1}{1-\alpha}} - \frac{I}{\tilde{q}} \\
\pi^{(4)} &= [p^2R^2 + 2p(1-p)R + (1-p)^2]k^{\alpha} - b \\
&= [p^2R^2 + 2p(1-p)R + (1-p)^2]\tilde{q}^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}[p^2R^2 + 2p(1-p)R + (1-p)^2]^{\frac{\alpha}{1-\alpha}} \\
&- \tilde{q}^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}[p^2R^2 + 2p(1-p)R + (1-p)^2]^{\frac{1}{1-\alpha}} + \frac{I}{\tilde{q}} \\
&= [p^2R^2 + 2p(1-p)R + (1-p)^2]^{\frac{1}{1-\alpha}}\tilde{q}^{\frac{\alpha}{1-\alpha}}(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}) + \frac{I}{\tilde{q}}
\end{aligned}
$$

Summary of Cases

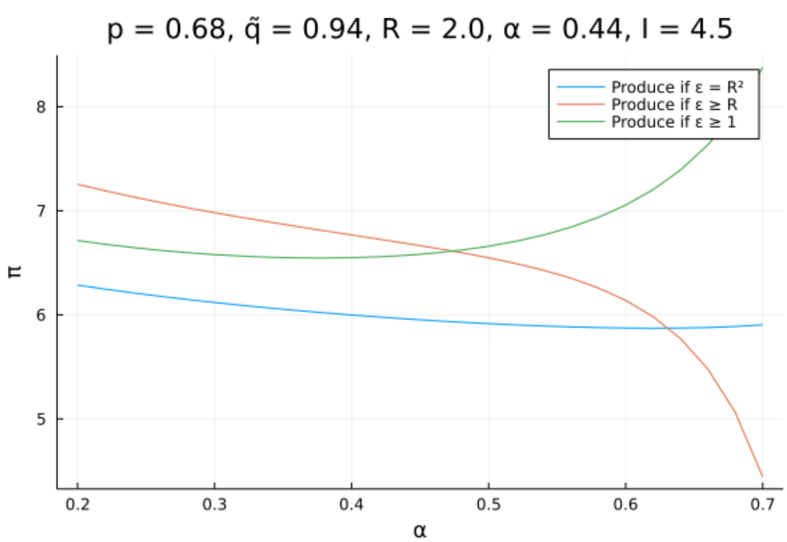
$$\pi^{(1)} = 0$$

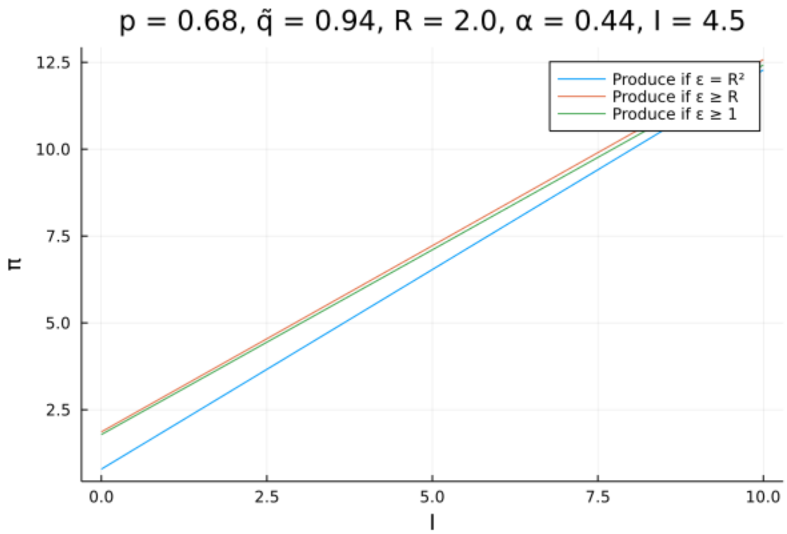
$$\pi^{(2)} = p^2\left[(p^2+\tilde{q}-1)^{\frac{\alpha}{1-\alpha}} R^{\frac{2}{1-\alpha}} [\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}] + \frac{I}{p^2+\tilde{q}-1}\right]$$

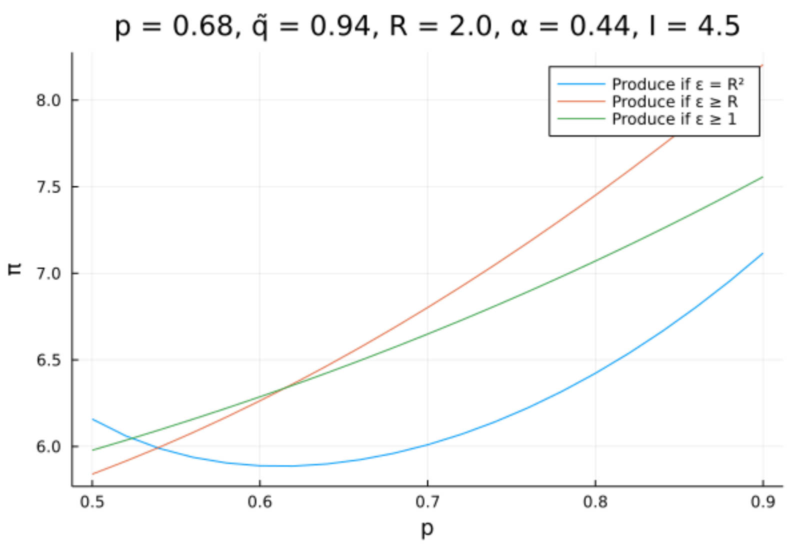
$$\begin{aligned}\pi^{(3)} &= R^{\frac{1}{1-\alpha}}[p^2+2p(1-p)+\tilde{q}-1]^{\frac{\alpha}{1-\alpha}}\left[[p^2R+2p(1-p)]\alpha^{\frac{\alpha}{1-\alpha}}\left(\frac{p^2R+2p(1-p)}{p^2+2p(1-p)}\right)^{\frac{\alpha}{1-\alpha}}\right.\\ &\quad \left. - [p^2+2p(1-p)]\alpha^{\frac{1}{1-\alpha}}\left(\frac{p^2R+2p(1-p)}{p^2+2p(1-p)}\right)^{\frac{1}{1-\alpha}}\right]\\ &\quad + [p^2+2p(1-p)]\frac{I}{p^2+2p(1-p)+\tilde{q}-1}\end{aligned}$$

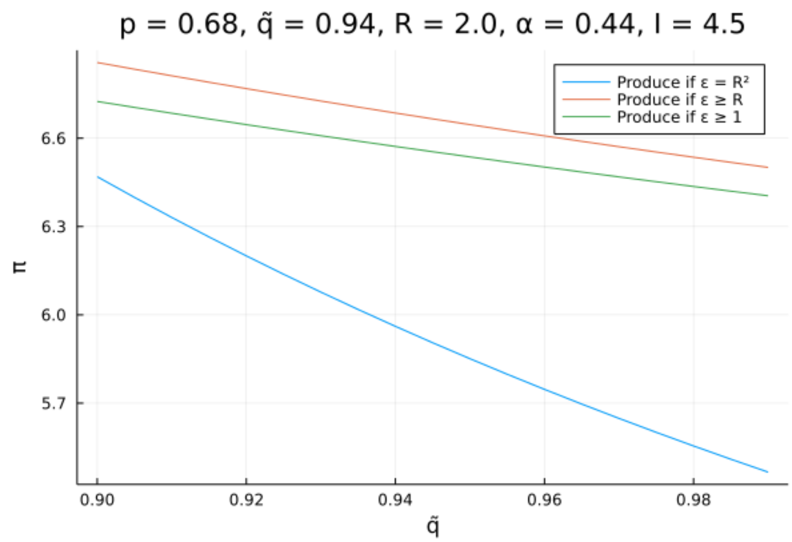
$$\pi^{(4)} = [p^2R^2+2p(1-p)R+(1-p)^2]^{\frac{1}{1-\alpha}}\tilde{q}^{\frac{\alpha}{1-\alpha}}(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}) + \frac{I}{\tilde{q}}$$

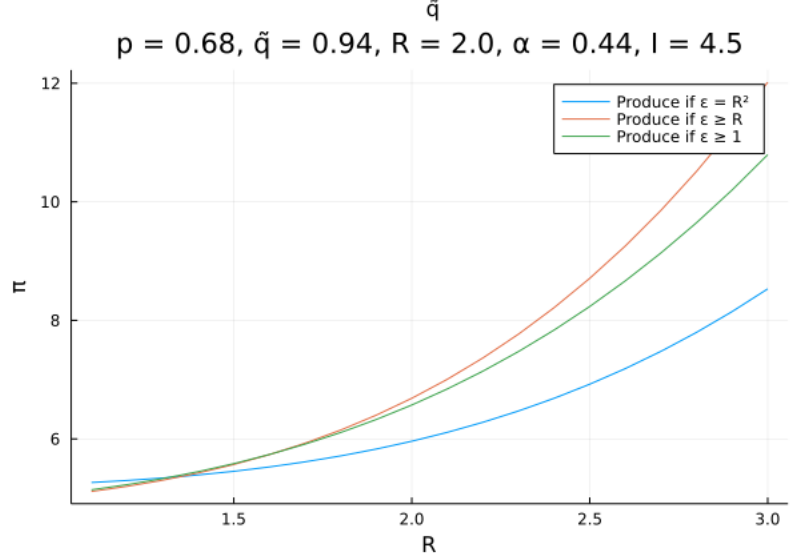
Figures




p = 0.68, q̃ = 0.94, R = 2.0, α = 0.44, I = 4.5
Produce if ε = R²
Produce if ε ≥ R
Produce if ε ≥ 1
π
I


p = 0.68, q̃ = 0.94, R = 2.0, α = 0.44, I = 4.5
Produce if ε = R²
Produce if ε ≥ R
Produce if ε ≥ 1
π
p


p = 0.68, q̃ = 0.94, R = 2.0, α = 0.44, I = 4.5
Produce if ε = R²
Produce if ε ≥ R
Produce if ε ≥ 1
π
6.6
6.3
6.0
5.7
0.90
0.92
0.94
0.96
0.98
q̃


p = 0.68, q̃ = 0.94, R = 2.0, α = 0.44, I = 4.5
Produce if ε = R²
Produce if ε ≥ R
Produce if ε ≥ 1
π
12
10
8
6
1.5
2.0
2.5
3.0
R

Uniform productivity shocks

Simplifying assumptions:

- In period 2, firm can only choose to produce or default (no reorganizing)
- Recovery value is zero: $\xi = 1$
- ε_1 and ε_2 are iid $U(0, 1/2)$
- One type of debt: $b_1 = 0$
- Production function $f(\varepsilon_1, \varepsilon_2, k) = (\varepsilon_1 + \varepsilon_2)k^\alpha$ for $\alpha \in (0, 1)$

With these assumptions, problem is equivalent to two period model with one productivity shock $\tilde{\varepsilon} \sim U(0, 1)$

Timing with simplifying assumptions:

- Period 0
  - Firm is born with funds I
  - Firm sells b discount bonds at price q to representative risk-neutral investor who can also buy risk-free discount bonds with price $\tilde{q}$
  - Firm buys capital $k \equiv I + qb$
- Period 1
  - Productivity shock $\tilde{\varepsilon} \sim U(0, 1)$ realized
  - Firm chooses to produce or default
    - If firm produces, firm payoff is $\tilde{\varepsilon}k^\alpha - b$, and investor payoff is b
    - If firm defaults, firm payoff is 0, investor's payoff is 0

Solution:

- In period 1, firm produces iff $\tilde{\varepsilon}k^\alpha - b \geq 0 \implies \tilde{\varepsilon} \geq \frac{b}{k^\alpha}$. Notice that this threshold is increasing in b and decreasing in k. That is if the firm has more debt (or less capital), it defaults for a larger measure of productivity shocks.
- Thus, the firm's profit is

$$\pi(\tilde{\varepsilon}, k, b) = \begin{cases} \tilde{\varepsilon}k^\alpha - b, & \text{if } \frac{b}{k^\alpha} \leq \tilde{\varepsilon} \leq 1 \\ 0, & \text{if } 0 \leq \tilde{\varepsilon} < \frac{b}{k^\alpha} \end{cases}$$

- Thus, the firm's expected profit is

$$
\begin{aligned}
\pi(k,b) &= \int_0^1 \pi(\tilde{\varepsilon},k,b)d\tilde{\varepsilon} \\
&= \int_0^{b/k^\alpha} \pi(\tilde{\varepsilon},k,b)d\tilde{\varepsilon} + \int_{b/k^\alpha}^1 \pi(\tilde{\varepsilon},k,b)d\tilde{\varepsilon} \\
&= 0 + \int_{b/k^\alpha}^1 (\tilde{\varepsilon}k^\alpha - b)d\tilde{\varepsilon} \\
&= \left[\frac{1}{2}\tilde{\varepsilon}^2 k^\alpha - \tilde{\varepsilon}b\right]_{b/k^\alpha}^1 \\
&= \frac{1}{2}k^\alpha - b - \frac{1}{2}(b/k^\alpha)^2 k^\alpha + (b/k^\alpha)b \\
&= \frac{1}{2}k^\alpha - b - \frac{1}{2}b^2 k^{-\alpha} + b^2 k^{-\alpha} \\
&= \frac{1}{2}k^\alpha - b + \frac{1}{2}b^2 k^{-\alpha}
\end{aligned}
$$

The first term is the expected output conditional on the firm producing, the second term is the (expected) debt repayment again conditional on the firm producing, and the third term is a positive term that accounts for the option value of defaulting. This term is increasing in debt b and decreasing in capital k because the measure of productivity shocks for which the firm defaults grows with more debt or less capital. So the option to default is more valuable.

- In period 0, taking q as given, firm chooses k and b to maximize expected profit in period 1:

$$
\begin{aligned}
&\max_{k,b}\ \pi(k,b) \\
&\text{s.t. } k = I + bq \\
\implies &\mathcal{L} = \frac{1}{2}k^\alpha - b + \frac{1}{2}b^2 k^{-\alpha} + \lambda(I + bq - k)
\end{aligned}
$$

- FOCs

$$
\begin{aligned}
\frac{\alpha}{2}k^{\alpha-1} &= \frac{\alpha}{2}b^2 k^{-\alpha-1} + \lambda && [k] \\
bk^{-\alpha} + \lambda q &= 1 && [b]
\end{aligned}
$$

λ can be interpreted as the value of relaxing the feasibility constraint that capital equals internal funds plus proceeds from the sale of discount bonds. Thus, the choice of k equates the marginal product of capital (marginal benefit of higher k; LHS) to the marginal decrease in the option value of defaulting plus the value of the feasibility constraint (marginal cost of higher k; RHS). And the choice of b equates the increase in the option value of default plus the value of loosening the feasibility constraint (marginal benefit of higher b; LHS) to one (the marginal cost of higher b; RHS)

- Combining FOCs:

$$bk^{-\alpha} + q\frac{\alpha}{2}k^{\alpha-1} = 1 + q\frac{\alpha}{2}b^2k^{-\alpha-1}$$

- The lender's zero profit condition pins down q. The risk-neutral lender buy risk-free bonds that is invest $\tilde{q}b$ and get b for certain. Or she can buy risky bonds that is invest qb and get get b with probability if the firm produces.

$$\begin{aligned}\int_0^1 bd\tilde{\varepsilon} - \tilde{q}b &= \int_{b/k^\alpha}^1 bd\tilde{\varepsilon} - qb \\ \implies b - \tilde{q}b &= b(1 - b/k^\alpha) - qb \\ \implies 1 - \tilde{q} &= 1 - b/k^\alpha - q \\ \implies q &= \tilde{q} - b/k^\alpha\end{aligned}$$

Thus, $q < \tilde{q}$. A lower discount bond price corresponds to a higher interest rate; the risky bonds pay premium over risk-free rate. The premium over the risk-free rate increases in b and decreases in k.

- Substitute in lender's zero profit condition into firm FOC and feasibility constraint

$$\begin{aligned}bk^{-\alpha} + (\tilde{q} - b/k^\alpha)\frac{\alpha}{2}k^{\alpha-1} &= 1 + (\tilde{q} - b/k^\alpha)\frac{\alpha}{2}b^2k^{-\alpha-1} \\ k &= I + b(\tilde{q} - b/k^\alpha)\end{aligned}$$

to get two nonlinear equations in two unknowns (k, b)

- Allow $b_1 \neq 0$
- Timing
  - Period 0
    - Firm is born with funds I
    - Firm sells b_0 discount bonds at price q_0 to representative risk-neutral "long-term" investor who can also buy risk-free discount bonds with price $\tilde{q}$
  - Period 1
    - First productivity shock $\varepsilon_1 \sim U(0, 1/2)$ realized
    - Firm sells b_1 discount bonds at price q_1 to representative risk-neutral "short-term" investor who can also buy risk-free discount bonds with price $\tilde{q}$
    - Firm buys capital $k \equiv I + q_0 b_0 + q_1 b_1$ and has outstanding debt $b \equiv b_0 + b_1$
  - Period 2
    - Second productivity shock $\varepsilon_2 \sim U(0, 1/2)$ realized
    - Firm chooses to produce or default
      - If firm produces, firm payoff is $f(\varepsilon_1, \varepsilon_2, k) - b$, long-term investor payoff is b_0, and long-term investor payoff is b_1
      - If firm defaults, firm payoff is 0, long-term investor's payoff is 0, short-term investor's payoff is 0
- In period 2, firm produces iff $(\varepsilon_1 + \varepsilon_2)k^\alpha - b_0 - b_1 \geq 0 \implies \varepsilon_2 \geq \frac{b_0+b_1}{k^\alpha} - \varepsilon_1$
- Firm profit is

$$\pi(\varepsilon_1, \varepsilon_2, k, b_0 + b_1) = \begin{cases} (\varepsilon_1 + \varepsilon_2)k^\alpha - b_0 - b_1, & \text{if } \frac{b_0+b_1}{k^\alpha} - \varepsilon_1 \leq \varepsilon_2 \leq 1 \\ 0, & \text{if } 0 \leq \varepsilon_2 < \frac{b_0+b_1}{k^\alpha} - \varepsilon_1 \end{cases}$$

- Thus, the firm's expected profit is

$$
\begin{aligned}
\pi(k,b) &= \int_0^1 \pi(\tilde{\varepsilon},k,b)d\tilde{\varepsilon} \\
&= \int_0^{b/k^\alpha} \pi(\tilde{\varepsilon},k,b)d\tilde{\varepsilon} + \int_{b/k^\alpha}^1 \pi(\tilde{\varepsilon},k,b)d\tilde{\varepsilon} \\
&= 0 + \int_{b/k^\alpha}^1 (\tilde{\varepsilon}k^\alpha - b)d\tilde{\varepsilon} \\
&= \left[\frac{1}{2}\tilde{\varepsilon}^2 k^\alpha - \tilde{\varepsilon}b\right]_{b/k^\alpha}^1 \\
&= \frac{1}{2}k^\alpha - b - \frac{1}{2}(b/k^\alpha)^2 k^\alpha + (b/k^\alpha)b \\
&= \frac{1}{2}k^\alpha - b - \frac{1}{2}b^2 k^{-\alpha} + b^2 k^{-\alpha} \\
&= \frac{1}{2}k^\alpha - b + \frac{1}{2}b^2 k^{-\alpha}
\end{aligned}
$$