ECON 899A - Problem Set 7

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1. *Derive the following asymptotic moments associated with $m_3(x)$: mean, variance, first order autocorrelation. Furthermore, compute $\nabla_b g(b_0)$. Which moments are informative for estimating b?*

With $|\rho_0| < 1$, the stochastic process $\{x_t\}$ is stationary, so $E[x_t] = E[x_{t-1}]$ and $Var[x_t] = Var[x_{t-1}]$:[1]

$$E[x_t] = E[\rho_0 x_{t-1} + \varepsilon_t] = \rho_0 E[x_{t-1}] = \rho_0 E[x_t]$$

$$\implies E[x_t] = 0$$

$$Var[x_t] = Var[\rho_0 x_{t-1} + \varepsilon_t] = \rho_0^2 Var[x_{t-1}] + \sigma_0^2 = \rho_0^2 Var[x_t] + \sigma_0^2$$

$$\implies Var[x_t] = \frac{\sigma_0^2}{1-\rho_0^2}$$

$$Cov[x_t, x_{t-1}] = Cov[\rho_0 x_{t-1} + \varepsilon_t, x_{t-1}] = \rho_0 Cov[x_{t-1}, x_{t-1}] + Cov[\varepsilon_t, x_{t-1}] = \rho_0 Var[x_t] = \frac{\sigma_0^2 \rho_0}{1-\rho_0^2}$$

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[1] Alternatively, we can write x_t in terms of x_0, $\{\varepsilon_i\}_{i=1}^t$, and ρ_0:

$$\begin{aligned}
x_t &= \rho_0 x_{t-1} + \varepsilon_t \\
&= \rho_0(\rho_0 x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \rho_0^2 x_{t-2} + \varepsilon_t + \rho_0 \varepsilon_{t-1} \\
&= \rho_0^2(\rho_0 x_{t-3} + \varepsilon_{t-2}) + \varepsilon_t + \rho_0 \varepsilon_{t-1} \\
&= \rho_0^3 x_{t-3} + \varepsilon_t + \rho_0 \varepsilon_{t-1} + \rho_0^2 \varepsilon_{t-2} \\
&= \rho_0^t x_0 + \sum_{i=1}^t \rho_0^{t-i} \varepsilon_i \\
&= \sum_{i=1}^t \rho_0^{t-i} \varepsilon_i
\end{aligned}$$

$$E[x_t] = E\left[\sum_{i=1}^t \rho_0^{t-i} \varepsilon_i\right] = \sum_{i=1}^t \rho_0^{t-i} E[\varepsilon_i] = 0$$

Thus, the asymptotic moments associated with $m_3(x)$ are:

$$\mu(x) = E[m_3(x)] = \begin{bmatrix} 0 \\ \frac{\sigma_0^2}{1-\rho_0^2} \\ \frac{\sigma_0^2\rho_0}{1-\rho_0^2} \end{bmatrix}$$

To calculate the Jacobian, we compute the derivative of the moment conditions with respect to each parameter:

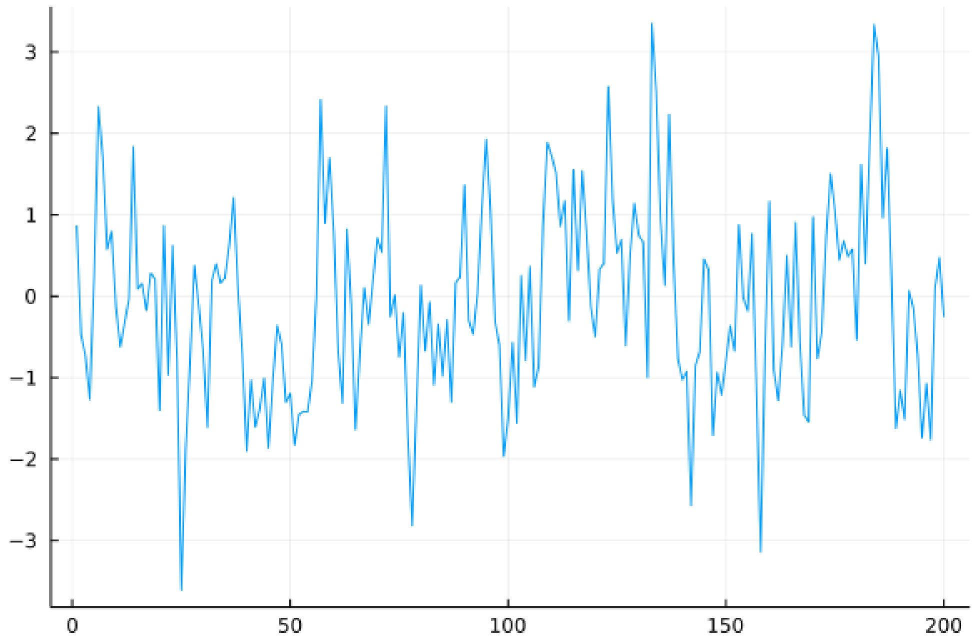
$$\begin{aligned}
\frac{\partial}{\partial\rho}\left(\frac{\sigma^2}{1-\rho^2}\right) &= \frac{\sigma^2(-2\rho)-(1-\rho^2)(0)}{(1-\rho^2)^2} \\
&= \frac{-2\rho\sigma^2}{(1-\rho^2)^2} \\
\frac{\partial}{\partial\sigma}\left(\frac{\sigma^2}{1-\rho^2}\right) &= \frac{2\sigma}{1-\rho^2} \\
\frac{\partial}{\partial\rho}\left(\frac{\sigma^2\rho}{1-\rho^2}\right) &= \frac{\sigma^2\rho(-2\rho)-(1-\rho^2)\sigma^2}{(1-\rho^2)^2} \\
&= \frac{-\sigma^2(1+\rho^2)}{(1-\rho^2)^2} \\
\frac{\partial}{\partial\sigma}\left(\frac{\sigma^2\rho}{1-\rho^2}\right) &= \frac{2\sigma\rho}{1-\rho^2}
\end{aligned}$$

Each cell of the Jacobian is the negative of the derivative of the moment condition:

$$\implies \nabla_b g(b_0) = \begin{bmatrix} 0 & 0 \\ \frac{2\rho_0\sigma_0^2}{(1-\rho_0^2)^2} & \frac{-2\sigma_0}{1-\rho_0^2} \\ \frac{\sigma_0^2(1+\rho_0^2)}{(1-\rho_0^2)^2} & \frac{-2\sigma_0\rho_0}{1-\rho_0^2} \end{bmatrix}$$

Variance and first order autocorrelation are informative for estimating b.

2. *Simulate a series of "true" data of length $T = 200$ using (1). We will use this to compute $M_T(x)$.*



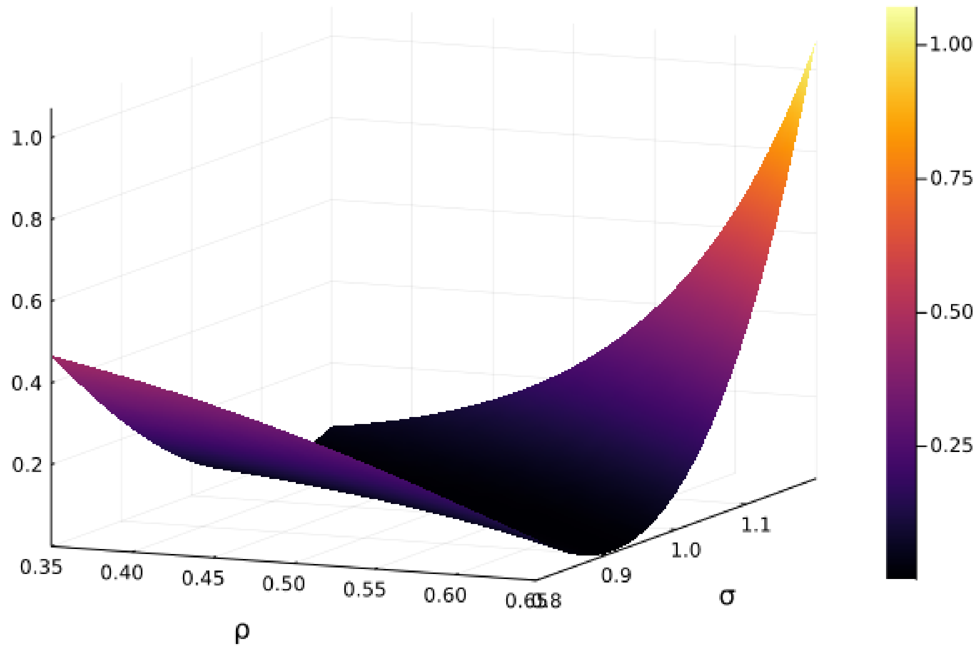
$$M_T(x) = \begin{bmatrix} -0.0764 \\ 1.3953 \\ 0.6916 \end{bmatrix}$$

3. *Set $H = 10$ and simulate H vectors of length $T = 200$ random variables e_t from $N(0, 1)$. We will use this to compute $M_{TH}(y(b))$. Store these vectors. You will use the same vector of random variables throughout the entire exercise. Since this exercise requires you to estimate σ^2, you want to change the variance of e_t during the estimation. You can simply use σe_t when the variance is σ^2.*

4. *We will start by estimating the $\ell = 2$ vector b for the just identified case where m_2 uses mean and variance. Given what you found in part (1), do you think there will be a problem? In general we would not know whether this case would be a problem, so hopefully the standard error of the estimate of b as well as the J test will tell us something.*

From part (1), I found that in expectation the mean is zero, so the problem is that the mean is not informative for estimating b.

(a) *Set $W = I$ and graph in three dimensions, the objective function (3) over $\rho \in [0.35, 0.65]$ and $\sigma \in [0.8, 1.2]$. Obtain an estimate of b by using $W = I$ in (4) using fminsearch. Report $\hat{b}^1_{TH}$.*



$$\hat{b}^1_{TH} = \begin{bmatrix} 0.5320 \\ 1.0115 \end{bmatrix}$$

(b) *Set $i(T) = 4$. Obtain an estimate of W^*. Using $\hat{W^*}_{TH} = \hat{S}^{-1}_{TH}$ in (4), obtain an estimate of $\hat{b}^2_{TH}$. Report $\hat{b}^2_{TH}$.*

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(c) *To obtain standard errors, compute numerically $\nabla_b g_T(\hat{b}^2_{TH})$ defined in (6). Report the values of $\nabla_b g_T(\hat{b}^2_{TH})$. Next, obtain the $\ell \times \ell$ variance-covariance matrix of $\hat{b}^2_{TH}$ as in (7). Finally, what are the standard errors defined in (8)? How can we use the information on $\nabla_b g_T(\hat{b}^2_{TH})$ to think about local identification?*

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(d) *Since we are in the just identified case, the J test should be zero (on a computer this may be not be exact). However, given the identification issues in this particular case where we use mean and variance, the J test may not be zero. Compute the value of the J test:*

$$T\frac{H}{1+H} \times J_{TH}(\hat{b}^2_{TH}) \to \chi^2$$

noting that in this just identified case $n - \ell = 0$ degrees of freedom recognizing that there really is not distribution.

...

5. Next we estimating the $\ell = 2$ vector b for the just identified case where m_2 uses the variance and autocorrelation. Given what you found in part (1), do you now think there will be a problem? If not, hopefully the standard error of the estimate of b as well as the J test will tell us something. For this case, perform steps (a)-(d) above.

(a)

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(b)

...

(c)

...

(d)

...

6. Next, we will consider the overidentified case where m_3 uses the mean, variance and autocorrelation. For this case, perform steps (a)-(d) above.

(a)

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(b)

...

(c)

...

(d)

...

(e) Bootstrap the the finite sample distribution of the estimators by repeatedly drawing ε_t and e_t^h from $N(0,1)$ for $t = 1, ..., T$ and $h = 1, ..., H$. Compute $(\hat{b}_{TH}^1, \hat{b}_{TH}^2)$.

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