

# ECON 703 - PS 4

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(1) Let  $X, Y$  be two vector spaces such that  $\dim X = n$ ,  $\dim Y = m$ . Construct a basis of  $L(X, Y)$ .

Let  $B_X = \{u_1, \dots, u_n\}$  be a basis for  $X$ . Consider  $x \in X$  such that  $x = c_1u_1 + \dots + c_nu_n$  for  $c_1, \dots, c_n \in \mathbb{R}$ . Let  $B_Y = \{v_1, \dots, v_m\}$  be a basis for  $Y$ , so for any  $T \in L(X, Y)$ ,  $T(u_i) = b_{i1}v_1 + \dots + b_{im}v_m$  for  $i \in \{1, \dots, n\}$ ,  $b_{i1}, \dots, b_{im} \in \mathbb{R}$ . Thus,

$$\begin{aligned} T(x) &= T(c_1u_1 + \dots + c_nu_n) \\ &= c_1T(u_1) + \dots + c_nT(u_n) \\ &= c_1(b_{11}v_1 + \dots + b_{1m}v_m) + \dots + c_n(b_{n1}v_1 + \dots + b_{nm}v_m) \\ &= c_1b_{11}v_1 + \dots + c_1b_{1m}v_m + \dots + c_nb_{n1}v_1 + \dots + c_nb_{nm}v_m \\ &= (v_1 \quad \dots \quad v_m) \begin{pmatrix} b_{11} & \dots & b_{n1} \\ \vdots & & \vdots \\ b_{1m} & \dots & b_{nm} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \end{aligned}$$

Therefore,  $B_{L(X, Y)}$  forms a basis for  $L(X, Y)$ :

$$\begin{aligned} B_{L(X, Y)} &= \left\{ \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}_{n \times m}, \dots, \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix}_{n \times m} \right\} \\ &= \{A_{n \times m} | a_{ij} = 1 \text{ and } a_{lk} = 0 \forall i \in \{1, \dots, n\}, j \in \{1, \dots, m\}, l \neq i, k \neq j\} \end{aligned}$$

$B_{L(X, Y)}$  spans  $L(X, Y)$  where  $b_{ij}$  is the coefficient on the element of  $B_{L(X, Y)}$  with a one in the  $i$ th column and the  $j$ th row.  $B_{L(X, Y)}$  is also linear independent because each element has a single nonzero cell whose location is unique to the nonzero cells in other elements. Since no element of  $B_{L(X, Y)}$  has the same nonzero cell as another element, any linear combination of the elements that equals an  $n \times m$  matrix of zeros implies that all coefficients on elements of  $B_{L(X, Y)}$  are zero.

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(2) Suppose that  $T \in L(X, X)$  and  $\lambda$  is  $T$ 's eigenvalue.

(a) Prove that  $\lambda^k$  is an eigenvalue of  $T^k$ ,  $k \in \mathbb{N}$ .

Proof: If  $\lambda$  is  $T$ 's eigenvalue  $\implies T(v) = \lambda v$  for eigenvector  $v \neq \bar{0}$ . Applying  $T$  again, we get  $T(T(v)) = \lambda(\lambda v) \implies T^2(v) = \lambda^2 v$ . Similarly, applying  $T$   $k$  times to  $v$ , we get  $T^k(v) = \lambda^k v$ . Thus,  $\lambda^k$  is an eigenvalue for  $T^k$  where  $k \in \mathbb{N}$ .

(b) Prove that if  $T$  is invertible, then  $\lambda^{-1}$  is an eigenvalue of  $T^{-1}$ .

Proof: If  $T$  is invertible,  $T^{-1} \in L(X, X)$ . Thus, for eigenvector  $v$  of  $T$ ,

$$\begin{aligned} T(v) = \lambda v &\implies T^{-1}(T(v)) = T^{-1}(\lambda v) \\ &\implies v = \lambda T^{-1}(v) \\ &\implies T^{-1}(v) = \lambda^{-1} v. \end{aligned}$$

Therefore,  $\lambda^{-1}$  is an eigenvalue of  $T^{-1}$ .

(c) Define an operator  $S : X \rightarrow X$ , such that  $S(x) = T(x) - \lambda x$  for all  $x \in X$ . Is  $S$  linear? Prove that  $\ker S := \{x \in X | S(x) = \bar{0}\}$  is a vector space.

For  $x_1, x_2 \in X$  and  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,

$$\begin{aligned} S(\alpha_1 x_1 + \alpha_2 x_2) &= T(\alpha_1 x_1 + \alpha_2 x_2) - \lambda(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \alpha_1 T(x_1) + \alpha_2 T(x_2) - \lambda \alpha_1 x_1 - \lambda \alpha_2 x_2 \\ &= \alpha_1 (T(x_1) - \lambda x_1) + \alpha_2 (T(x_2) - \lambda x_2) \\ &= \alpha_1 S(x_1) + \alpha_2 S(x_2) \end{aligned}$$

Yes,  $S$  is linear.

Proof: For a fixed  $\lambda$ , let  $x, y \in \ker S$  and  $\alpha, \beta \in \mathbb{R}$ . We know that  $S(x) = S(y) = \bar{0}$ . As a linear transformation,  $S(\alpha x + \beta y) = \alpha S(x) + \beta S(y) = \alpha \bar{0} + \beta \bar{0} = \bar{0}$ , so properties 1, 2, 5, 6, 7 of the definition of a vector space are satisfied.

For property 3, note that  $S(\bar{0}) = T(\bar{0}) - \lambda \bar{0} = T(\bar{0}) = \bar{0}$ , so  $\bar{0} \in \ker S$ .<sup>1</sup> Furthermore,  $x + \bar{0} = \bar{0} + x = x$  for  $x \in \ker S$ .

For property 4, for  $x \in \ker S$ ,  $S(-x) = (-1)S(x) = (-1)\bar{0} = \bar{0}$  where  $x + (-x) = \bar{0}$ .

For property 8, for  $x \in \ker S$ ,  $S(1 \cdot x) = 1 \cdot S(x) = 1 \cdot \bar{0} = \bar{0}$ .

Thus  $\ker S$  is a vector space.  $\square$

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<sup>1</sup>For any linear transformation  $T : X \rightarrow Y$ ,  $T(\bar{0}_X) = T(\bar{0}_X + \bar{0}_X) = T(\bar{0}_X) + T(\bar{0}_X) \implies T(\bar{0}_X) = T(\bar{0}_X) - T(\bar{0}_X) = \bar{0}_Y$ .

(3) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by  $T(x, y) = (x - y, 2x + 3y)$ . Let  $W$  be the standard basis of  $\mathbb{R}^2$  and let  $V$  be another basis of  $\mathbb{R}^2$ ,  $V = \{(1, -4), (-2, 7)\}$  in the coordinates of  $W$ .

(a) Find  $\text{mtx}_W(T)$ .

$$\begin{aligned} T(x, y) &= (x - y)w_1 + (2x + 3y)w_2 \\ &= (w_1 + 2w_2)x + (-w_1 + 3w_2)y \end{aligned}$$

$$\text{mtx}_W(T) = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}.$$

(b) Find  $\text{mtx}_V(T)$ .

$$P = \text{mtx}_{W,V}(id) = \begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix}$$

$$\text{mtx}_V(T) = P^{-1}\text{mtx}_W(T)P = \begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} -15 & 29 \\ -10 & 19 \end{pmatrix}$$

(c) Find  $T(1, -2)$  in the basis  $V$ .

$$\text{mtx}_V(T)\text{mtx}_{V,W}(id) \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -15 & -10 \\ 29 & 19 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -15 & -10 \\ 29 & 19 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -13 \\ -8 \end{pmatrix}$$

Check:  $T(1, -2) = (1 - (-2), 2(1) + 3(-2)) = (3, -4)$  and  $(-13)(1, -4) + (-8)(-2, 7) = (-13, 52) + (16, -56) = (3, -4)$ .

- (4) In this exercise you will learn to solve first order linear difference equations in  $n$  variables. We want to find an  $n$ -dimensional process  $\{\mathbf{x}_1, \mathbf{x}_2, \dots\}$  such that each  $\mathbf{x}_i$  is an  $n$ -dimensional vector and

$$\mathbf{x}_t = A\mathbf{x}_{t-1}, t = 1, 2, \dots, \quad (1)$$

where  $A \in M_{n \times n}$  and  $\mathbf{x}_0 \in \mathbb{R}^n$  are given. Then

$$\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0, \mathbf{x}_t = A^t\mathbf{x}_0 \forall t \in \mathbb{N},$$

where  $A^t = A \cdot A \cdot \dots \cdot A$  ( $t$  times). Thus, we need to calculate  $A^t$ .

To do this, we diagonalize  $A$ ,  $A = PDP^{-1}$ , where  $D$  is diagonal,  $D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ .

Hence we can rewrite

$$A^t = PDP^{-1}PDP^{-1} \dots PDP^{-1} = PD^tP^{-1} = P\text{diag}\{\lambda_1^t, \dots, \lambda_n^t\}P^{-1},$$

which is now easy to compute. Thus, what you is

Step 1: Calculate  $A$ 's eigenvalues  $\lambda_1, \dots, \lambda_n$  and eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

Remember that we need to independent eigenvectors (this holds if all eigenvalues are distinct).

Step 2: Set  $D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$  and  $P = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  (eigenvectors are columns of  $P$ ).

Step 3: Calculate  $P^{-1}$  and  $P\text{diag}\{\lambda_1^t, \dots, \lambda_n^t\}P^{-1}$ .

Step 4: Plug  $A^t$  from Step 3 to get  $\mathbf{x}_t = A^t\mathbf{x}_0$ .

Implement the above approach to solve for  $\mathbf{x}_t \in \mathbb{R}^2$ :

$$\mathbf{x}_t = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \mathbf{x}_{t-1}, \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Simplify your answer as much as possible.

The characteristic polynomial is  $(1 - \lambda)(-1 - \lambda) - 4(2) = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3)$ , so the eigenvalues are  $\lambda_1 = 3$  and  $\lambda_2 = -3$ . The eigenvectors are thus solutions to:

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \mathbf{v}_1 = \bar{0}$$

$$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} \mathbf{v}_2 = \bar{0}$$

Thus, the eigenvectors are  $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ . Thus,

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

Thus,

$$\begin{aligned}
 \mathbf{x}_t &= PD^t P^{-1} \mathbf{x}_0 \\
 &= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 * 3^{t-1} \\ -(-3)^{t-1} \end{pmatrix} \\
 &= \begin{pmatrix} 4 * 3^{t-1} + (-3)^{t-1} \\ 2 * 3^{t-1} - (-3)^{t-1} \end{pmatrix}
 \end{aligned}$$

Below is R code verifying the answer:

```

library(matlib)

A <- matrix(c(1, 2, 4, -1), ncol=2)
print(A)

##      [,1] [,2]
## [1,]    1    4
## [2,]    2   -1

ev <- eigen(A)
p <- t(t(ev$vectors))
d <- diag(ev$values)
print(p)

##      [,1]      [,2]
## [1,] 0.8944272 -0.7071068
## [2,] 0.4472136  0.7071068

print(d)

##      [,1] [,2]
## [1,]    3    0
## [2,]    0   -3

print(p %*% d %*% inv(p))

##      [,1] [,2]
## [1,]    1    4
## [2,]    2   -1

# for first ten x_t
x_0 <- c(1, 1)

for (t in 1:10) {
  print(paste("t =", t))
  print(p %*% d^t %*% inv(p) %*% x_0)
}

## [1] "t = 1"
##      [,1]
## [1,]    5

```

```

## [2,]      1
## [1] "t = 2"
##      [,1]
## [1,]      9
## [2,]      9
## [1] "t = 3"
##      [,1]
## [1,]     45
## [2,]      9
## [1] "t = 4"
##      [,1]
## [1,]     81
## [2,]     81
## [1] "t = 5"
##      [,1]
## [1,]    405
## [2,]     81
## [1] "t = 6"
##      [,1]
## [1,]    729
## [2,]    729
## [1] "t = 7"
##      [,1]
## [1,]   3645
## [2,]    729
## [1] "t = 8"
##      [,1]
## [1,]   6561
## [2,]   6561
## [1] "t = 9"
##      [,1]
## [1,]  32805
## [2,]   6561
## [1] "t = 10"
##      [,1]
## [1,]  59049
## [2,]  59049

```

- (5) In this exercise you will learn to solve  $n$ th order linear difference equations in one variable. We want to find a sequence of real numbers  $\{z_t\}_{t=1}^{\infty}$ , which satisfies

$$z_t = a_1 z_{t-1} + a_2 z_{t-2} + \dots + a_n z_{t-n}, \quad (2)$$

where  $a_1, \dots, a_n \in \mathbb{R}$  and  $z_0, z_{-1}, \dots, z_{-n+1} \in \mathbb{R}$  are given.

- (a) Define  $\mathbf{x}_t := (z_t, z_{t-1}, \dots, z_{t-n+1})'$  and rewrite Eq. (2) in the form of Eq. (1). What is  $A$ ?

$$A_{n \times n} = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} A\mathbf{x}_{t-1} &= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} & a_n \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_{t-1} \\ z_{t-2} \\ z_{t-3} \\ \vdots \\ z_{t-n+1} \\ z_{t-n} \end{pmatrix} \\ &= \begin{pmatrix} a_1 z_{t-1} + a_2 z_{t-2} + \dots + a_{n-1} z_{t-n+1} + a_n z_{t-n} \\ z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-n+1} \end{pmatrix} \\ &= \begin{pmatrix} z_t \\ z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-n+1} \end{pmatrix} \\ &= \mathbf{x}_t \end{aligned}$$

- (b) Notice that if you find the function form of  $z_t = f(t)$ , then you do not need to find a similar form for  $z_{t-1}, \dots, z_{t-n+1}$  (you use the same function  $f(\cdot)$  and evaluate it at a different time). Thus, you actually do not need to calculate  $Pdiag\{\lambda_1^t, \dots, \lambda_n^t\}P^{-1}\mathbf{x}_0$ . You only need the first coordinate of that  $n$ -dimensional vector. The first coordinate takes the form

$$\mathbf{x}_{t1} \equiv z_t = c_1\lambda_1^t + c_2\lambda_2^t + \dots + c_n\lambda_n^t, \quad (3)$$

where coefficient  $c_1, \dots, c_n$  depend on  $P$  and  $\mathbf{x}_0$ . Given Eq. (3) which holds for any  $t$  and initial values  $z_0, \dots, z_{-n+1}$ , which equations must  $c_1, \dots, c_n$  solve?

$$\begin{aligned} \mathbf{x}_0 = \begin{pmatrix} z_0 \\ z_{-1} \\ \vdots \\ z_{-n+1} \end{pmatrix} &= \begin{pmatrix} c_1\lambda_1^0 + c_2\lambda_2^0 + \dots + c_n\lambda_n^0 \\ c_1\lambda_1^{-1} + c_2\lambda_2^{-1} + \dots + c_n\lambda_n^{-1} \\ \vdots \\ c_1\lambda_1^{-n+1} + c_2\lambda_2^{-n+1} + \dots + c_n\lambda_n^{-n+1} \end{pmatrix} \\ \Rightarrow \begin{pmatrix} z_0 \\ z_{-1} \\ \vdots \\ z_{-n+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1^{-1} & \lambda_2^{-1} & \dots & \lambda_n^{-1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{-n+1} & \lambda_2^{-n+1} & \dots & \lambda_n^{-n+1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \end{aligned}$$

- (c) Suppose that  $n = 3$ ,  $a_1 = 2$ ,  $a_2 = 1$ ,  $a_3 = -2$ , and  $z_0 = 2$ ,  $z_{-1} = 2$ ,  $z_{-2} = 1$ . Find the expression for  $z_t$  as a function of  $t$ .

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \Rightarrow \det \begin{pmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} &= 0 \\ \Rightarrow (2-\lambda)(-\lambda)(-\lambda) + (-2)(1)(1) - (1)(1)(-\lambda) &= 0 \\ \Rightarrow -\lambda^3 + 2\lambda^2 + \lambda - 2 &= 0 \end{aligned}$$

Roots at  $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$ . Thus,

$$\begin{aligned} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ (-1)^{-1} & (1)^{-1} & (2)^{-1} \\ (-1)^{-2} & (1)^{-2} & (2)^{-2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} c_1 + c_2 + c_3 \\ -c_1 + c_2 + (1/2)c_3 \\ c_1 + c_2 + (1/4)c_3 \end{pmatrix} \end{aligned}$$

Thus,  $c_1 = -1/3, c_2 = 1, c_3 = 4/3 \Rightarrow z_t = (-1)^t(-1/3) + 1 + 2^t(4/3)$ .