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$\boxed{\text{ECON 710 A EXAM}}$

(1) (a) ~~Consider moment condition~~

$$\cancel{Cov\left(Z, \frac{Y-\beta_0}{X\beta}\right)}$$

Follow method of moment line of argument.

(1) Independence of Z implies:

$$\cancel{E[ZU] = E[Z]E[U] = E[Z]}$$

$$Cov(Z, U) = 0.$$

(2) ~~Use~~ $U = \dfrac{Y-\beta_0}{X\beta_1}$

$$Cov(Z, Y/X\beta) = Cov\left(Z, \frac{\beta_0 + X\beta_1 U}{X\beta}\right)$$

$$= Cov\left(Z, \frac{\beta_0}{X\beta}\right) + Cov\left(Z, \frac{\beta_1}{\beta} U\right)$$

$$= \frac{\beta_0}{\beta} Cov\left(Z, \frac{1}{X}\right) + \frac{\beta_1}{\beta} Cov(Z, U)$$

$$= \frac{\beta_0}{\beta} Cov\left(Z, \frac{1}{X}\right)$$

Thus, I'm having trouble coming up w/ a moment condition that is zero when $\beta_1 = \beta$.

(1) (b) $\hat{\Theta} = \{\hat{\theta}(h) : h \in H\}$ be set of unbiased parameters for θ_0.

Let $h^* \in H$ be $Var(\hat{\theta}(h^*)) \leq Var(\hat{\theta}(h))$ $\forall h \in H$.

$\Rightarrow$ For some $h \in H$,

$$Var(\hat{\theta}(h)) = Var(\hat{\theta}(h^*)) + X \quad \text{where } X > 0$$

. . .

$\Leftarrow$ For an arbitrary $h \in H$

$$\hat{\theta}(h) = \hat{\theta}(h^*) + Z(h)$$

where $Z(h)$ is mean zero & uncorrelated w/ $\hat{\theta}(h^*)$

$$\Rightarrow Var(\hat{\theta}(h)) = Var(\hat{\theta}(h^*) + Z(h))$$
$$= Var(\hat{\theta}(h^*)) + Var(Z(h))$$

~~Clearly~~ $Var(Z(h)) \geq 0$, so $\hat{\theta}(h^*)$ has weakly smaller variance that any $\hat{\theta}(h) \in \hat{\Theta}$.

False, I'm struggling to see why $\hat{\theta}(h)$ can't have a higher variance than $\hat{\theta}(h^*)$ in such a way that is correlated w/ $Var(\hat{\theta}(h^*))$.

(2) (a) Note that IID random variables are strictly stationary and functions (that do not depend on t) of strictly stationary random variables are strictly stationary.

$\{(U_t, Z_t^*, X_t)\}_{t=0}^T$ are iid. RVs, thus

~~$\Rightarrow$ X_t is strictly stationary~~

they are all strictly stationary.
$\{Y_t\}_{t=0}^T$ is a function of $\{X_t\}_{t=0}^T$ and $\{U_t\}_{t=0}^T$, thus $\{Y_t\}_{t=0}^T$ is strictly stationary.

$\{Z_t\}_{t=0}^T$ is a function of $\{Z_t^*\}_{t=0}^T$ al $\{U_t\}_{t=0}^T$. Thus $\{Z_t\}_{t=0}^T$ is strictly stationary.

2 (b) Since $\{u_t\}_{t=0}^{T}$ is iid, $E[u_t | z_t, u_{t-1}] = E[u_t | z_t]$. Thus

$$E[u_t | z_t] = E[u_t | z_t, u_{t-1}]$$

$$= E[u_t | z_t^*]$$

$$= 0.$$

(2)(c) No, ~~[illegible]~~ because Z_{t+1} is a function of u_t.

$$E[u_t \mid Z_1, \ldots, Z_T]$$

$$= E\left[\frac{Z_{t+1} - Z^*_{t+1}}{\gamma} \;\middle|\; Z_1, \ldots, Z_T\right]$$

$$= \frac{Z_{t+1}}{\gamma} - \frac{E[Z^*_{t+1} \mid Z_1, \ldots, Z_T]}{\gamma}$$

② (d) $E[Z_t(Y_t - X_t\beta)]$

$= E[Z_t(X_t\beta_1 + u_t - X_t\beta)]$

$= \beta_1 E[Z_t X_t] + \underbrace{E[Z_t u_t]}_{\substack{\parallel \\ 0 \text{ from (b)}}} - \beta E[Z_t X_t]$

$= (\beta_1 - \beta) E[Z_t X_t] = 0$

If $\beta = \beta_1 \Rightarrow (\beta - \beta) E[Z_t X_t] = 0.$

The moment function identifies β_1 iff

$\beta = \beta_1 \Leftrightarrow E[Z_t(Y_t - X_t\beta)] = 0$

Thus, we need $E[Z_t X_t] \neq 0$.

$E[Z_t X_t] = E[(Z_t^* + \gamma u_{t-1}) X_t]$

$= E[Z_t^* X_t] + \gamma E[u_{t-1} X_t]$

[$\{u_t\}$ and $\{X_t\}$ are independent]

$= E[Z_t^* X_t] + \gamma E[\underbrace{E[u_{t-1}|Z^*]}_{\substack{\parallel \\ 0}}] E[X_t]$

$= E[Z_t^* X_t]$

Thus, we need $E[Z_t^* X_t] \neq 0$. That is we need relavance for identification.

② (e) $E[S \mid H_0 \text{ is true}]$

$$= E\left[T^{-1/2} \sum_{t=1}^{T} W_t \mid \beta_1 = c\right]$$

$$= E\left[T^{-1/2} \sum_{t=1}^{T} Z_t (Y_t - X_t c) \mid \beta_1 = c\right]$$

$$= E\left[T^{-1/2} \sum_{t=1}^{T} Z_t (X_t \beta_1 + u_t - Xc) \mid \beta_1 = c\right]$$

$$= E\left[T^{-1/2} \sum_{t=1}^{T} Z_t u_t\right]$$

$$= T^{-1/2} \sum_{t=1}^{T} E[Z_t u_t]$$

Bound RVs mean we can apply LIE

$$= T^{-1/2} \sum_{t=1}^{T} E[Z_t \underbrace{E[u_t \mid Z_t]}_{=0}]$$

$$= 0$$

$\text{Var}(S \mid H_0 \text{ is true})$

$$= E[S^2 \mid \beta_1 = c] - [\underbrace{E[S \mid \beta_1 = c]}_{=0}]^2$$

$$= E[S^2 \mid \beta_1 = c]$$

$$= E\left[\left(T^{-1/2} \sum_{t=0}^{T} Z_t (Y_t - X_t c)\right)^2 \mid \beta_1 = c\right]$$

$$= T^{-1} E\left[\left(\sum_{t=0}^{T} Z_t u_t\right)^2\right]$$

$$= T^{-1} E\left[\sum_{t=0}^{T} Z_t^2 u_t^2 + \underbrace{\sum_{t \neq s}^{T} Z_t Z_s u_t u_s}_{=0}\right]$$

0, because u_t is iid and applying LIE w/ part (b)

$$= T^{-1} \sum_{t=0}^{T} E[Z_t^2 u_t^2]$$

$$= E[Z_t^2 u_t^2]$$

(2) (e) con't

$$\mathrm{Var}(S \mid H_0 \text{ is true}) = E[Z_t^2 U_t^2]$$

$$= E[(Z_t^* + \gamma U_{t-1})^2 U_t^2]$$

$$= E[((Z_t^*)^2 + \gamma Z_t^* U_{t-1} + \gamma^2 U_{t-1}^2) U_t^2]$$

$$= E[(Z_t^*)^2 U_t^2] + \gamma E[Z_t^* U_{t-1} U_t^2]$$

[Z_t^*, U_t are iid]

$$+ \gamma^2 E[U_{t-1}^2 U_t^2]$$

$$= E[(Z_t^*)^2] E[U_t^2]$$

$$+ \gamma E[Z_t^*] \underbrace{E[E[U_{t-1} \mid Z_t^*]]}_{=0} E[U_t^2]$$

$$+ \gamma^2 E[U_{t-1}^2] E[U_t^2]$$

$$= E[(Z_t^*)^2] E[U_t^2] + \gamma^2 [E[U_t^2]]^2$$

[Let $\sigma_u^2 = \mathrm{Var}(U_t) = E[U_t^2]$]

$$= E[(Z_t^*)^2] \sigma_u^2 + \gamma^2 \sigma_u^4$$

(2) (f)

From (e), we have $V = E[(Z_t^*)^2]\sigma_u^2 + \gamma^2\sigma_u^4$

We can estimate $\hat{\beta}_1^{IV} = \dfrac{\frac{1}{T}\sum_{t=0}^{T} Z_t Y_t}{\frac{1}{T}\sum_{t=0}^{T} Z_t X_t}$ based

on strict stationarity, contemporaneous exogeneity, and relevance: $\hat{\beta}_1^{IV} \xrightarrow{p} \beta_1$. Thus,

$\Rightarrow \hat{u}_t = Y_t - \hat{\beta}_1^{IV} X_t \rightarrow u_t$

It seems difficult to estimate γ, but we can use an earlier equation in (e) to get an estimator of V:

$$V = E[u_t^2 Z_t^2]$$

$$\hat{V} = \frac{1}{T}\sum_{t=0}^{T} \hat{u}_t^2 Z_t^2$$

[Moment exist, because vectors are bounded]

$\{\hat{u}_t\}$ and $\{Z_t\}$ are iid, so based on the standard LLN:

$$\hat{V} = \frac{1}{T}\sum_{t=0}^{T} \hat{u}_t^2 Z_t \rightarrow E[\hat{u}_t Z_t] = V$$

When H_0 holds.

(2) (g) From (e), we have

$$S = T^{-1/2} \sum_{t=0}^{T} Z_t U_t$$

* Note that Z_t & U_t are strictly stationary, So $Z_t U_t$ is strictly stationary.

* $E[Z_1^2 U_1^2] < \infty$ because the vectors are bounded.

* $E[Z_t U_t \mid Z_{t-1} U_{t-1}, \dots, Z_0 U_0]$

$$= E[Z_t^* U_t \mid Z_{t-1} U_{t-1}, \dots, Z_0 U_0]$$

$$+ \gamma [U_{t-1} U_t \mid Z_{t-1} U_{t-1}, \dots, Z_0 U_0]$$

$$= 0.$$

* By (f), $\frac{1}{T} \sum Z_t^2 U_t^2 \to E[Z_1^2 U_1^2]$,

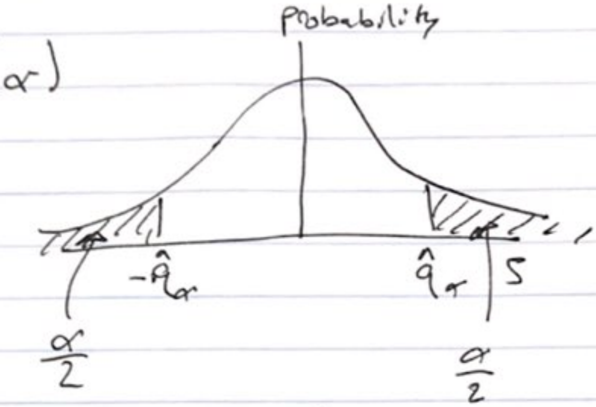
By the Martingale CLT

$$T^{-1/2} \sum_{t=0}^{T} Z_t U_t \xrightarrow{d} N\left(0, E(Z_1^2 U_1^2)\right)$$

$$= N(0, V).$$

(1)(h) Since S is assymptotically normal, we can use standard normal CVs.

$P(|S| > \hat{q}_\alpha)$



Let Φ be the CDF of a standard normal

Thus $\hat{q}_\alpha = \Phi(1 - \frac{\alpha}{2})$.