ECON 709B - Problem Set 4

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1. 7.28 Estimate the regression: $\log(\hat{wage}) = \beta_1 education + \beta_2 experience + \beta_3 experience^2/100 + \beta_4$.[1]

```
library(tidyverse)
```

```
cps09mar <- read_delim("cps09mar.txt",
                       delim = "\t",
                       col_names = c("age", "female", "hisp", "education", "earnings",
                                     "hours", "week", "union", "uncov", "region", "race",
                                     "maritial"),
                       col_types = "dddddddddddd") %>%
  mutate(experience = age - education - 6,
         experience_2 = (experience^2)/100,
         wage = earnings / (hours*week),
         l_wage = log(wage),
         constant = 1) %>%
  filter(race == 4,
         maritial == 7,
         female == 0,
         experience < 45)

y <- cps09mar$l_wage

x <- cps09mar %>%
  select(education, experience, experience_2, constant) %>%
  as.matrix() %>%
  unname()

n <- dim(x)[1]
i <- diag(nrow = n, ncol = n)
```

(a) Report the coefficient estimates and robust standard errors.

```
# OLS regression
beta <- solve(t(x) %*% x) %*% (t(x) %*% y)

# residuals
p_x <- x %*% solve(t(x) %*% x) %*% t(x)
m_x <- i - p_x
e_hat <- m_x %*% y
```

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

[1] Use the subsample of the CPS that you used for problems 3.24 and 3.25 (instead of the subsample requested in the problem)

```r
# heteroskedastic asymptotic variance
omega_hat <- solve(t(x) %*% x) %*%
  (t(x) %*% diag(as.numeric(e_hat^2)) %*% x) %*%
  solve(t(x) %*% x)
robust_se <- t(t(sqrt(diag(omega_hat))))
print(beta)
```

```
##             [,1]
## [1,]  0.14430729
## [2,]  0.04263326
## [3,] -0.09505636
## [4,]  0.53089068
```

```r
print(robust_se)
```

```
##            [,1]
## [1,] 0.01172552
## [2,] 0.01242217
## [3,] 0.03379572
## [4,] 0.20005051
```

(b) Let θ be the ratio of the return to one year of education to the return to one year of experience for $experience = 10$. Write θ as a function of the regression coefficients and variables. Compute $\hat{\theta}$ from the estimated model.

Taking partial derivatives of the regression equation with respect to *education* and *experience*, the return to one year of education is β_1 and the return to one year of experience is $\beta_2 + 2\beta_3 experience/100$. Thus the ratio at $experience = 10$ is $\theta = \frac{\beta_1}{\beta_2+\beta_3/5}$.

$$\hat{\theta} = \frac{\hat{\beta}_1}{\hat{\beta}_2 + \hat{\beta}_3/5} \approx \frac{0.1443}{0.0426 + (-0.0951)/5} \approx 6.1090$$

```r
theta_hat <- beta[1]/(beta[2]+beta[3]/5)
print(theta_hat)
```

```
## [1] 6.109023
```

(c) Write out the formula for the asymptotic standard error for $\hat{\theta}$ as a function of the covariance matrix for $\hat{\beta}$. Compute $s(\hat{\theta})$ from the estimated model.

Use the delta method. Define $h(\beta) = \frac{\beta_1}{\beta_2+\beta_3/5}$. Thus,

$$H(\beta) = \frac{\partial}{\partial \beta'} h(\beta) = \begin{pmatrix} \frac{1}{\beta_2+\beta_3/5} & \frac{-\beta_1}{(\beta_2+\beta_3/5)^2} & \frac{-\beta_1/5}{(\beta_2+\beta_3/5)^2} & 0 \end{pmatrix}$$

Thus, the asymptotic variance of $\theta = g(\beta)$ is $H(\beta)\Omega H(\beta)'$. We can estimate it with $H(\hat{\beta})\hat{\Omega}H(\hat{\beta})'$.

```r
H_hat_beta <- t(c(1/(beta[2]+beta[3]/5),
                -beta[1]/((beta[2]+beta[3]/5)^2),
                (-beta[1]/5)/((beta[2]+beta[3]/5)^2),
                0))
theta_se <- sqrt(H_hat_beta %*% omega_hat %*% t(H_hat_beta))
print(theta_se)
```

```
##          [,1]
## [1,] 1.617848
```

(d) Construct a 90% asymptotic confidence interval for θ from the estimated model.

The confidence interval is $[\hat{\theta} - CV_{\alpha}s(\hat{\theta}), \hat{\theta} + CV_{\alpha}s(\hat{\theta})]$:

```
cv <- qnorm(p =  .95)

theta_ci <- c(theta_hat - cv * theta_se, theta_hat + cv * theta_se)
print(theta_ci)
```

```
## [1] 3.447899 8.770147
```

2. 8.1 In the model $y = X_1'\beta_1 + X_2'\beta_2 + e$, show directly from definition (8.3) that the CLS estimate of $\beta = (\beta_1, \beta_2)$ subject to the constraint that $\beta_2 = 0$ is the OLS regression of y on X_1.

The CLS estimator is

$$\begin{aligned}\tilde{\beta} &= \arg\min_{\beta_2 = 0} SSE(\beta) \\ &= \arg\min_{\beta_2=0} (y - X_1\beta - X_2\beta_2)'(y - X_1\beta - X_2\beta_2)\end{aligned}$$

Define Legrangian:

$$\begin{aligned}\mathcal{L} &= (y - X_1\beta_1 - X_2\beta_2)'(y - X_1\beta_1 - X_2\beta_2) - \lambda'(\beta_2 - 0) \\ &= y'y - \beta_1'X_1'y - \beta_2'X_2'y - y'X_1\beta_1 + \beta_1'X_1'X_1\beta_1 + \beta_2'X_2'X_1\beta_1 - y'X_2\beta_2 + \beta_1'X_1'X_2\beta_2 + \beta_2'X_2'X_2\beta_2 + \lambda'\beta_2\end{aligned}$$

FOC $[\beta_1]$:

$$\begin{aligned}&0 = -X_1'y - X_1'y + 2X_1'X_1\tilde{\beta}_1 + X_1'X_2\tilde{\beta}_2' + X_1'X_2\tilde{\beta}_2 \\ \implies &0 = -2X_1'y + 2X_1'X_1\tilde{\beta}_1 + 2X_1'X_2\tilde{\beta}_2'\end{aligned}$$

FOC $[\lambda]$:

$$\tilde{\beta}_2 = 0$$

Combining FOCs:

$$\begin{aligned}&0 = -2X_1'y + 2X_1'X_1\tilde{\beta}_1 \\ \implies &\tilde{\beta}_1 = (X_1'X_1)^{-1}X_1'y\end{aligned}$$

3. 8.3 In the model $y = X_1'\beta_1 + X_2'\beta_2 + e$, with β_1 and β_2 each $k \times 1$, find the CLS estimate of $\beta = (\beta_1, \beta_2)$ subject to the constraint that $\beta_1 = -\beta_2$.

The CLS estimator is

$$\tilde{\beta} = \arg \min_{\beta_1 = -\beta_2} (y - X_1\beta - X_2\beta_2)'(y - X_1\beta - X_2\beta_2)$$

Define Legrangian:

$$\begin{aligned}
&\mathcal{L} = (y - X_1\beta_1 - X_2\beta_2)'(y - X_1\beta_1 - X_2\beta_2) - \lambda'(\beta_2 - \beta_1) \\
&\quad = y'y - \beta_1'X_1'y - \beta_2'X_2'y - y'X_1\beta_1 + \beta_1'X_1'X_1\beta_1 + \beta_2'X_2'X_1\beta_1 - y'X_2\beta_2 + \beta_1'X_1'X_2\beta_2 + \beta_2'X_2'X_2\beta_2 + \lambda'(\beta_2 + \beta_1)
\end{aligned}$$

FOC $[\beta_1]$:

$$\begin{aligned}
&0 = -X_1'y - X_1'y + 2X_1'X_1\tilde{\beta}_1 + X_1'X_2\tilde{\beta}_2' + X_1'X_2\tilde{\beta}_2 + \lambda \\
\implies &0 = -2X_1'y + 2X_1'X_1\tilde{\beta}_1 + 2X_1'X_2\tilde{\beta}_2' + \lambda
\end{aligned}$$

FOC $[\lambda]$:

$$\tilde{\beta}_1 = -\tilde{\beta}_2$$

These FOCs imply a value for lambda:

$$\begin{aligned}
0 &= -2X_1'y + 2X_1'X_1\tilde{\beta}_1 - 2X_1'X_2\tilde{\beta}_1' + \lambda \\
\lambda &= 2X_1'y - 2X_1'X_1\tilde{\beta}_1 + 2X_1'X_2\tilde{\beta}_1'
\end{aligned}$$

FOC $[\beta_2]$:

$$\begin{aligned}
&0 = -X_2'y - X_2'y + 2X_2'X_2\tilde{\beta}_2 + X_2'X_1\tilde{\beta}_1' + X_2'X_1\tilde{\beta}_1 + \lambda \\
\implies &0 = -2X_2'y + 2X_2'X_2\tilde{\beta}_2 + 2X_2'X_1\tilde{\beta}_1' + \lambda
\end{aligned}$$

Thus, the estimator is

$$\begin{aligned}
0 &= -2X_2'y - 2X_2'X_2\tilde{\beta}_1 + 2X_2'X_1\tilde{\beta}_1' + 2X_1'y - 2X_1'X_1\tilde{\beta}_1 + 2X_1'X_2\tilde{\beta}_1' \\
(X_1'X_1 - X_2'X_1 - X_1'X_2 + X_2'X_2)\tilde{\beta}_1 &= X_1'y - X_2'y \\
\tilde{\beta}_1 &= (X_1'X_1 - X_2'X_1 - X_1'X_2 + X_2'X_2)^{-1}(X_1'y - X_2'y) \\
&= ((X_1 - X_2)'(X_1 - X_2))^{-1}(X_1 - X_2)'y \\
&= -\tilde{\beta}_2
\end{aligned}$$

4. 8.4(a) In the linear projection model $y = \alpha + X'\beta + e$ consider the restriction $\beta = 0$. Find the constrained least squares (CLS) estimator of α under the restriction $\beta = 0$.

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \arg\min_{\beta=0} (y - \alpha - X\beta)'(y - \alpha - X\beta)$$

Define legrangian:

$$\mathcal{L} = (y - \alpha - X\beta)'(y - \alpha - X\beta) + \lambda'\beta$$

FOC $[\alpha]$:

$$0 = \vec{1}(y - \tilde{\alpha} - X\tilde{\beta})$$

FOC $[\lambda]$:

$$\tilde{\beta} = 0$$

$$\implies \tilde{\alpha} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

5. 8.22 Take the linear model $y = X_1\beta_1 + X_2\beta_2 + e$ with $E[Xe] = 0$. Consider the restriction $\beta_1/\beta_2 = 2$

(a) Find an explicit expression for the constrained least squares (CLS) estimator $\tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2)$ of $\beta = (\beta_1, \beta_2)$ under the restriction. Your answer should be specific to the restriction. It should not be a generic formula for an abstract general restriction.

We can rewrite the constraint as a linear constraint:

$$\beta_1/\beta_2 = 2 \implies = \beta_1 - 2\beta_2 = 0$$

Thus, the definition of the estimator is

$$\tilde{\beta} = \arg\min_{\beta_1 - 2\beta_2 = 0} (y - X_1\beta_1 - X_2\beta_2)'(y - X_1\beta_1 - X_2\beta_2)$$

Define a legrangian:

$$\begin{aligned}\mathcal{L} &= (y - X_1\beta_1 - X_2\beta_2)'(y - X_1\beta_1 - X_2\beta_2) + \lambda'(\beta_1 - 2\beta_2) \\ &= y'y + \beta_1^2 X_1'X_1 + \beta_2^2 X_2'X_2 - 2\beta_1 y'X_1 - 2\beta_2 y'X_2 + 2\beta_1\beta_2 X_1'X_2 + \lambda'(\beta_1 - 2\beta_2)\end{aligned}$$

FOC $[\theta]$:

$$\tilde{\beta}_1 = 2\tilde{\beta}_2$$

FOC $[\beta_1]$:

$$\begin{aligned} 0 &= -2X_1'y + 2X_1'X_1\tilde{\beta}_1 + 2X_1'X_2\tilde{\beta}_2 + \tilde{\lambda} \\ \implies \tilde{\lambda} &= 2X_1'y - 4X_1'X_1\tilde{\beta}_2 - 2X_1'X_2\tilde{\beta}_2 \end{aligned}$$

FOC $[\beta_2]$:

$$\begin{aligned}
0 &= -2X_2'y + 2X_2'X_2\tilde{\beta}_2 + 2X_2'X_1\tilde{\beta}_1 - 2\tilde{\lambda} \\
0 &= -X_2'y + X_2'X_2\tilde{\beta}_2 + 2X_2'X_1\tilde{\beta}_2 - (2X_1'y - 4X_1'X_1\tilde{\beta}_2 - 2X_1'X_2\tilde{\beta}_2) \\
2X_1'y + X_2'y &= X_2'X_2\tilde{\beta}_2 + 4X_2'X_1\tilde{\beta}_2 + 4X_1'X_1\tilde{\beta}_2 \\
(2X_1 + X_2)'y &= (X_2'X_2 + 4X_2'X_1 + 4X_1'X_1)\tilde{\beta}_2 \\
\tilde{\beta}_2 &= ((2X_1 + X_2)'(2X_1 + X_2))^{-1}(2X_1 + X_2)'y \\
\tilde{\beta}_1 &= 2\tilde{\beta}_2 = 2((2X_1 + X_2)'(2X_1 + X_2))^{-1}(2X_1 + X_2)'y
\end{aligned}$$

(b) Derive the asymptotic distribution of $\tilde{\beta}_1$ under the assumption that the restriction is true.

$$\sqrt{n}(\tilde{\beta}_1 - \beta_1) = \sqrt{n}(2\tilde{\beta}_2 - \beta_2) = 2\sqrt{n}(\tilde{\beta}_2 - \beta_2)$$

$$\sqrt{n}(\tilde{\beta}_2 - \beta_2) = \sqrt{n}((2X_1 + X_2)'(2X_1 + X_2))^{-1}(2X_1 + X_2)'e = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(2X_{i1} + X_{2i})e_i}{\frac{1}{n}\sum_{i=1}^{n}(2X_{i1} + X_{2i})^2}$$

By WLLN,

$$\frac{1}{n}\sum_{i=1}^{n}(2X_{i1} + X_{2i})^2 \to_p E[(2X_{i1} + X_{2i})^2]$$

By CLT,

$$\frac{1}{\sqrt{n}}(2X_{i1} + X_{2i})e_i \to_d N(0, E[(2X_{i1} + X_{2i})^2 e_i^2])$$

Thus,

$$\sqrt{n}(\tilde{\beta}_1 - \beta_1) \to N\left(0, \frac{E[(2X_{i1} + X_{2i})^2 e_i^2]}{E[(2X_{i1} + X_{2i})^2]^2}\right)$$

6. 9.1 Prove that if an additional regressor X_{k+1} is added to X, Theil's adjusted $\bar{R}^2$ increases if and only if $|T_{k+1}| > 1$, where $T_{k+1} = \hat{\beta}_{k+1}/s(\hat{\beta}_{k+1})$ is the t-ratio for $\hat{\beta}_{k+1}$ and $s(\hat{\beta}_{k+1}) = (s^2[(X'X)^{-1}]_{k+1,k+1})^{1/2}$ is the homoskedasticity-formula standard error.

Regressing y on X results in $\hat{\beta} = (X'X)^{-1}X'y$, $\hat{\varepsilon} = y - X\hat{\beta}$, and $\bar{R}^2_{k+1}$. Regressing y on X with the restriction that $\beta_{k+1} = 0$ results in

$$\tilde{\beta} = \hat{\beta} - (X'X)^{-1}[0_k\ 1]'([0_k\ 1](X'X)^{-1}[0_k\ 1]')^{-1}\hat{\beta}_{k+1}$$

The regression also results in $\tilde{\varepsilon} = y - X\tilde{\beta}$ and $\bar{R}^2_k$. We can rewrite $\tilde{\varepsilon}$ as the following:

$$\tilde{\varepsilon} = y - X\tilde{\beta} = y - X\hat{\beta} - X(\tilde{\beta} - \hat{\beta}) = \hat{\varepsilon} - X(\tilde{\beta} - \hat{\beta})$$

Thus, because $X\hat{\varepsilon} = 0$,

$$\tilde{\varepsilon}'\tilde{\varepsilon} = (\hat{\varepsilon} - X(\tilde{\beta} - \hat{\beta}))'(\hat{\varepsilon} - X(\tilde{\beta} - \hat{\beta})) = \hat{\varepsilon}'\hat{\varepsilon} + (\tilde{\beta} - \hat{\beta})'(X'X)(\tilde{\beta} - \hat{\beta})$$

Therefore, the difference between the squared residuals is

$$\begin{aligned}
\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon} &= (\tilde{\beta} - \hat{\beta})'(X'X)(\tilde{\beta} - \hat{\beta}) \\
&= \hat{\beta}_{k+1}([(X'X)^{-1}]_{k+1,k+1})^{-1}[0_k\ 1](X'X)^{-1}(X'X)(X'X)^{-1}[0_k\ 1]'([(X'X)^{-1}]_{k+1,k+1})^{-1}\hat{\beta}_{k+1} \\
&= \hat{\beta}_{k+1}([(X'X)^{-1}]_{k+1,k+1})^{-1}[0_k\ 1](X'X)^{-1}[0_k\ 1]'([(X'X)^{-1}]_{k+1,k+1})^{-1}\hat{\beta}_{k+1} \\
&= \hat{\beta}_{k+1}([(X'X)^{-1}]_{k+1,k+1})^{-1}[(X'X)^{-1}]_{k+1,k+1}([(X'X)^{-1}]_{k+1,k+1})^{-1}\hat{\beta}_{k+1} \\
&= \frac{\hat{\beta}^2_{k+1}}{[(X'X)^{-1}]_{k+1,k+1}}
\end{aligned}$$

Thus, the adjusted R-squared is higher iff the t-statistic is at least 1.

$$\begin{aligned}
\bar{R}^2_{k+1} &> \bar{R}^2_k \\
\iff 1 - \frac{(n-1)\hat{\varepsilon}'\hat{\varepsilon}}{(n-k-1)\sum_i(y_i - \bar{y})} &> 1 - \frac{(n-1)\tilde{\varepsilon}'\tilde{\varepsilon}}{(n-k)\sum_i(y_i - \bar{y})} \\
\iff \frac{1}{n-k}\tilde{\varepsilon}'\tilde{\varepsilon} &> \frac{1}{n-k-1}\hat{\varepsilon}'\hat{\varepsilon} \\
\iff (n-k-1)(\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon}) &> \hat{\varepsilon}'\hat{\varepsilon} \\
\iff \frac{\tilde{\varepsilon}'\tilde{\varepsilon} - \hat{\varepsilon}'\hat{\varepsilon}}{\frac{1}{n-k-1}\hat{\varepsilon}'\hat{\varepsilon}} &> 1 \\
\iff \frac{\hat{\beta}^2_{k+1}}{\frac{1}{n-k-1}\hat{\varepsilon}'\hat{\varepsilon}[(X'X)^{-1}]_{k+1,k+1}} &> 1 \\
\iff \frac{\hat{\beta}^2_{k+1}}{s(\hat{\beta}_{k+1})^2} &> 1 \\
\iff \left|\frac{\hat{\beta}_{k+1}}{s(\hat{\beta}_{k+1})}\right| &> 1 \\
\iff \left|T_{k+1}\right| &> 1
\end{aligned}$$

9.2 You have two independent samples (Y_{1i}, X_{1i}) and (Y_{2i}, X_{2i}) both with sample sizes n which satisfy $Y_1 = X_1\beta_1 + e_1$ and $Y_2 = X_2\beta_2 + e_2$, where $E[X_1e_1] = 0$ and $E[X_2e_2] = 0$. Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be the OLS estimates of $\beta_1 \in R_k$ and $\beta_2 \in R_k$.

(a) Find the asymptotic distribution of $\sqrt{n}((\hat{\beta}_2 - \hat{\beta}_1) - (\beta_2 - \beta_1))$ as $n \to \infty$.

$$\sqrt{n}((\hat{\beta}_2 - \hat{\beta}_1) - (\beta_2 - \beta_1)) = \sqrt{n}(\hat{\beta}_2 - \beta_2) - \sqrt{n}(\hat{\beta}_1 - \beta_1)$$

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) \to N(0, V_1)$$

$$V_1 = E(X'_{1i}X_{1i})^{-1}E(X'_{1i}X_{1i}e_1^2)E(X'_{1i}X_{1i})^{-1}$$

$$\sqrt{n}(\hat{\beta}_2 - \beta_2) \to_d N(0, V_2)$$

$$V_2 = E(X_{2i}'X_{2i})^{-1}E(X_{2i}'X_{2i}e_2^2)E(X_{2i}'X_{2i})^{-1}$$

By the continuous mapping theorem and the independence of the subsamples:

$$\sqrt{n}((\hat{\beta}_2 - \hat{\beta}_1) - (\beta_2 - \beta_1)) \to_d N(0, V_1 + V_2)$$

(b) Find an appropriate test statistic for $H_0 : \beta_2 = \beta_1$.

We can use a Wald test statistic:

$$\begin{aligned}
\hat{\theta} &= \hat{\beta}_2 - \hat{\beta}_1 \\
\theta_0 &= 0 \\
\hat{V}_{\hat{\theta}} &= \hat{V}_1 + \hat{V}_2 \\
\hat{V}_1 &= n(X_1'X_1)^{-1}(X_1' diag(\hat{e}_1^2)X_1)(X_1'X_1)^{-1} \\
\hat{V}_2 &= n(X_2'X_2)^{-1}(X_2' diag(\hat{e}_2^2)X_2)(X_2'X_2)^{-1} \\
W &= (\hat{\theta} - \theta_0)'\hat{V}_{\hat{\theta}}^{-1}(\hat{\theta} - \theta_0) \\
&= (\hat{\beta}_2 - \hat{\beta}_1)'(\hat{V}_1 + \hat{V}_2)^{-1}(\hat{\beta}_2 - \hat{\beta}_1)
\end{aligned}$$

(c) Find the asymptotic distribution of this statistic under H_0.

From (a), we know that

$$\sqrt{n}((\hat{\beta}_2 - \hat{\beta}_1) - (\beta_2 - \beta_1)) \to_d N(0, V_1 + V_2)$$

By the WLLN,

$$\begin{aligned}
\hat{V}_1 &\to_p V_1 \\
\hat{V}_2 &\to_p V_2 \\
\implies \hat{V}_1 + \hat{V}_2 &\to_p V_1 + V_2
\end{aligned}$$

Thus,

$$W \to_d \chi_k^2$$

7. 9.4 Let W be a Wald statistic for $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$, where θ is $q \times 1$. Since $W \to_d \chi^2_q$ under H_0, someone suggests the test "Reject H_0 if $W < c_1$ or $W > c_2$ where c_1 is the $\alpha/2$ quantile of χ^2_q and c_2 is the $1 - \alpha/2$ quantile of χ^2_q."

(a) Show that the asymptotic size of the test is α.

The asymptotic size of the test is

$$\lim_{n\to\infty} P(W < c_1 | H_0 \text{ true}) + P(W > c_2 | H_0 \text{ true}) = \alpha/2 + (1 - (1 - \alpha/2)) = \alpha$$

(b) Is this a good test of H_0 versus H_1? Why or why not?

No, a lower point estimate $\hat{\theta}$ will result in rejection even though it is closer to the null hypothesis. Thus, this test has low power.

8. 9.7 Take the model $y = X\beta_1 + X^2\beta_2 + e$ with $E[e|X] = 0$ where y is wages (dollars per hour) and X is age. Describe how you would test the hypothesis that the expected wage for a 40-year-old worker is \$20 an hour.

The expected wage for a 40-year-old worker is \$20 an hour $\implies 20 = 40\beta_1 + 1600\beta_2 \implies 0 = 2\beta_1 + 80\beta_2 - 1$. Define $\theta = 2\beta_1 + 80\beta_2 - 1$. We construct a test with $H_0 : \theta = 0$ and $H_1 : \theta \neq 0$. Let $\hat{\theta} = 2\hat{\beta}_1 + 80\hat{\beta}_2 - 1$. Now, we find the asymptotic variance of $\hat{\theta}$:

$$\begin{aligned}
\sqrt{n}(\hat{\theta} - \theta) &= \sqrt{n}((2\hat{\beta}_1 + 80\hat{\beta}_2 - 1) - (2\beta_1 + 80\beta_2 - 1)) \\
&= 2\sqrt{n}(\hat{\beta}_1 - \beta_1) + 80\sqrt{n}(\hat{\beta}_2 - \beta_2)
\end{aligned}$$

Thus, $\sqrt{n}(\hat{\theta} - \theta) \to_d N(0, V_\theta)$ where

$$\begin{aligned}
V_\theta &= \begin{pmatrix} 2 & 80 \end{pmatrix} V_\beta \begin{pmatrix} 2 \\ 80 \end{pmatrix} \\
&= \begin{pmatrix} 2V_\beta^{11} + 80V_\beta^{21} & 2V_\beta^{12} + 80V_\beta^{22} \end{pmatrix} \begin{pmatrix} 2 \\ 80 \end{pmatrix} \\
&= 4V_\beta^{11} + 160V_\beta^{21} + 160V_\beta^{12} + 6400V_\beta^{22}
\end{aligned}$$

Thus, we can estimate the standard error of θ as $s(\hat{\theta}) = \sqrt{4\hat{V}_\beta^{11} + 160\hat{V}_\beta^{21} + 160\hat{V}_\beta^{12} + 6400\hat{V}_\beta^{22}}$ where:

$$\hat{V}_\beta = n(X'X)^{-1}(X'diag(\hat{e}^2)X)(X'X)^{-1}$$

Define the test statistic as $T = \theta/s(\hat{\theta})$. Over the null hypothesis, the test statistic is asymptotically standard normal, so we reject if $|T| > c_{1-\alpha/2}$ where $c_{1-\alpha/2}$ is the $1 - \alpha/2$ percentile of a standard normal distribution a pre-specified α.