ECON 714A - Problem Set 4

Alex von Hafften[*]

2/22/2021

This problem asks you to solve a model of oligopolistic competition from Atkeson and Burstein (AER 2008), which extends the Dixit-Stiglitz setup and is widely used to analyze heterogeneous markups and incomplete pass-through.

Consider a static model with a continuum of sectors $k \in [0, 1]$ and $i = 1, ..N_k$ firms in sector k, each producing a unique variety of the good. Households supply inelastically one unit of labor and have nested-CES preferences:

$$C = \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}}, C_k = \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}, \theta > \rho \geq 1.$$

Production function of firm i in sector k is given by $Y_{ik} = A_{ik} L_{ik}$.

1. Solve household cost minimization problem for the optimal demand C_{ik}, the sectoral price index P_k, and the aggregate price index P as functions of producers' prices.

Notice that labor is inelastically supplied. The household cost minization problem is:

$$\min_{\{C_{ik}\}} \int \sum_{i=1}^{N_k} P_{ik} C_{ik} dk$$

$$\text{s.t. } C = \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}}$$

$$\text{and } C_k = \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$$

Define the legrange multipliers with P and P_k:

$$\mathcal{L} = \int \sum_{i=1}^{N_k} P_{ik} C_{ik} dk + P \left[C - \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} \right] + \int P_k \left[C_k - \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \right] dk$$

[*]I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Sarah Bass, Emily Case, Danny Edgel, and Katherine Kwok.

FOC $[C_k]$:

$$
\begin{aligned}
P_k &= P\frac{\rho}{\rho-1}\left(\int C_k^{\frac{\rho-1}{\rho}}dk\right)^{\frac{1}{\rho-1}}\frac{\rho-1}{\rho}C_k^{\frac{-1}{\rho}} \\
\implies P_k &= P\left(\int C_k^{\frac{\rho-1}{\rho}}dk\right)^{\frac{1}{\rho-1}}C_k^{\frac{-1}{\rho}} \\
\implies P_k &= PC^{\frac{1}{\rho}}C_k^{\frac{-1}{\rho}} \\
\implies C_k &= \left(\frac{P_k}{P}\right)^{-\rho}C
\end{aligned}
$$

Substituting into the constraint, we get the aggregate price index and aggregate consumption in terms of the sectoral price indexes:

$$
\begin{aligned}
C &= \left(\int\left(\left(\frac{P_k}{P}\right)^{-\rho}C\right)^{\frac{\rho-1}{\rho}}dk\right)^{\frac{\rho}{\rho-1}} \\
\implies 1 &= \left(\int\left(\frac{P_k}{P}\right)^{1-\rho}dk\right)^{\frac{\rho}{\rho-1}} \\
\implies 1 &= P^{-\rho}\left(\int P_k^{1-\rho}dk\right)^{\frac{\rho}{\rho-1}} \\
\implies P &= \left(\int P_k^{1-\rho}dk\right)^{\frac{1}{1-\rho}} \\
\implies C_k &= \left(\frac{P_k}{(\int P_k^{1-\rho}dk)^{\frac{1}{1-\rho}}}\right)^{-\rho}C
\end{aligned}
$$

We can rewrite the legrangian as:

$$
\mathcal{L} = \int\sum_{i=1}^{N_k}P_{ik}C_{ik}dk + P\left[C - \left(\int\left(\left(\sum_{i=1}^{N_k}C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}\right)^{\frac{\rho-1}{\rho}}dk\right)^{\frac{\rho}{\rho-1}}\right] + \int P_k\left[C_k - \left(\sum_{i=1}^{N_k}C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}\right]dk
$$

FOC $[C_{ik}]$:

$$
\begin{aligned}
P_{ik} = P\frac{\rho}{\rho-1}\left(\int\left(\left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}\right)^{\frac{\rho-1}{\rho}} dk\right)^{\frac{\rho}{\rho-1}-1} \frac{\rho-1}{\rho}\left(\left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}\right)^{\frac{\rho-1}{\rho}-1} \\
\times\frac{\theta}{\theta-1}\left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}-1}\frac{\theta-1}{\theta}C_{ik}^{\frac{\theta-1}{\theta}-1} + P_k\frac{\theta}{\theta-1}\left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}-1}\frac{\theta-1}{\theta}C_{ik}^{\frac{\theta-1}{\theta}-1} \\
\implies P_{ik} = P\left(\int C_k^{\frac{\rho-1}{\rho}} dk\right)^{\frac{\rho}{\rho-1}\frac{1}{\rho}}\left(C_k\right)^{\frac{-1}{\rho}}\left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}\frac{1}{\theta}} C_{ik}^{\frac{-1}{\theta}} + P_k\left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}\frac{1}{\theta}} C_{ik}^{\frac{-1}{\theta}} \\
\implies P_{ik} = PC^{\frac{1}{\rho}}C_k^{\frac{-1}{\rho}}C_k^{\frac{1}{\theta}}C_{ik}^{\frac{-1}{\theta}} + P_kC_k^{\frac{1}{\theta}}C_{ik}^{\frac{-1}{\theta}} \\
\implies P_{ik} = C_{ik}^{\frac{-1}{\theta}}C_k^{\frac{1}{\theta}}(PC^{\frac{1}{\rho}}C_k^{-\frac{1}{\rho}} + P_k) \\
\implies C_{ik} = \left(\frac{P_{ik}}{P(\frac{C_k}{C})^{-\frac{1}{\rho}} + P_k}\right)^{-\theta} C_k
\end{aligned}
$$

Substituting in the FOC based on C_k:

$$
C_{ik} = \left(\frac{P_{ik}}{P(\frac{(\frac{P_k}{P})^{-\rho}C}{C})^{-\frac{1}{\rho}} + P_k}\right)^{-\theta} C_k \implies C_{ik} = \left(\frac{P_{ik}}{2P_k}\right)^{-\theta} C_k
$$

Substituting into the constraint:

$$
\begin{aligned}
C_k &= \left(\sum_{i=1}^{N_k}\left(\left(\frac{P_{ik}}{2P_k}\right)^{-\theta} C_k\right)^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \\
\implies 1 &= \left(\sum_{i=1}^{N_k}\left(\left(\frac{P_{ik}}{2P_k}\right)^{-\theta}\right)^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} \\
\implies (2P_k)^{1-\theta} &= \sum_{i=1}^{N_k}\left(P_{ik}\right)^{1-\theta} \\
\implies P_k &= \frac{1}{2}\left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}}
\end{aligned}
$$

2. Assume that firms compete a la Bertrand, i.e. choose P_{ik} taking the prices of other firms in a sector $P_{jk}, j \neq i$ as given. Derive demand elasticity for a given firm and the optimal price.

...

3. Prove that other things equal, firms with higher A_{ik} set higher markups.

...

4. Assume that $\rho = 1, \theta = 5, N_k = 20, and \log A_{ik} \sim i.i.d. N(0, 1)$. Solve the model numerically by approximating the number of sectors with $K = 100,000$. You will need an efficient algorithm to compute a sectoral equilibrium (search for a fixed point, do not use "solve") nested in a general equilibrium loop solving for real wages.

...

5. Compute the aggregate output C of the economy and compare it to the first-best value.

...

6. Bonus task: Does the sectoral equilibrium converge to the one under Betrand competition with homogeneous goods in the limit $\infty \to \infty$?

...