FIN 920: Continuous-Time Diffusion Models Notes*

Alex von Hafften

December 13, 2021

1 Part I

(Discrete) Random Walks

- Random walk: $z_t = z_{t-1} + e_t = z_0 + \sum_{s=1}^t e_s$ (often $z_0 = 0$) with $E[e_t] = 0$, $\forall t$ and $e_t \perp e_s, t \neq s$.
- Random walk with drift: $z_t = \mu + z_{t-1} + e_t$.
- Geometric random walk with drift: $\ln(z_t) = \mu + \ln(z_{t-1}) + e_t$ or $z_t = z_{t-1} \exp(\mu + e_t)$.
- Normally distributed increments $e_t \sim N(0, \sigma^2)$.

Standard Brownian Motion

- A Brownian motion is a process $\{z_t\}_{t \geq 0}$ such that
  - $P(z_0 = 0) = 1$
  - $z_t - z_2 \sim N(0, t-s), t > s \geq 0$
  - $\lim_{e \to 0} z_{t-e} = z_t, t \geq 0$
  - $z_t - z_s \perp z_u - z_v, t > s > u > v \geq 0$
- Brownian motion is Markov: $E[f(z_t)|\{z_v\}_{v=0}^s] = E[f(z_t)|z_s] =: E_s[f(z_t)]$ for $t \geq s$.
- Paths are nowhere differentiable: $\lim_{t \to s} \frac{z_t - z_s}{t-s}$ is not defined.
- Paths have unbounded total variation: $\sum_{v=1}^N |z_{tv/N} - z_{t(v-1)/N}| \to \infty$ as $N \to \infty$.
- Paths have bounded quadratic variation: $\sum_{v=1}^N (z_{tv/N} - z_{t(v-1)/N})^2 \to t$ as $N \to \infty$.
- Conventional expressions:
  - $z_t - z_0 = \sum_{v=1}^N z_{tv/N} - z_{t(v-1)/N} \to \int_{v=0}^t dz_v$ as $N \to \infty$ where $dz_t \sim N(0, dt)$.
  - Rules for the product of dz and dt:

$$\begin{bmatrix} & dz & dt \\ dz & dt & 0 \\ dt & 0 & 0 \end{bmatrix}$$

  - For example, $\sum_{v=1}^N (z_{Tv/N} - z_{T(v-1)/N})(T/N) \to \int_{t=0}^T dz_t dt = 0$ when $N \to \infty$.

*These notes are based on four sets of notes on Continuous-Time Diffusion Models by David Brown. I've done my best to flesh arguments out and add detail.

Formal Construction of Brownian Motion

- Probability Space $(\Omega, \mathcal{F}, P)$ with set of states $\Omega = \{\omega\}$, tribe $\mathcal{F}$, probability measure $P : \mathcal{F} \to \mathbb{R}$.
- A Brownian motion is a measurable function $z(\omega, t) : \Omega \times [0, \infty) \to \mathbb{R}$, such that $\forall \omega \in \Omega$,
  - $z(\omega, 0) = 0$ almost surely,
  - $z(\omega, t) - z(\omega, s) \sim N(0, t - s)$ for $t > s$,
  - $z(\omega, t) - z(\omega, s) \perp z(\omega, u) - z(\omega, v), t > s > u > v \geq 0$
  - $\lim_{t \to s} z(\omega, t) = z(\omega, s)$
- The standard filtration $\{\mathcal{F}_t\}_{t \in [0,\infty)}$ is defined by the paths of the process together with the null sets of $\mathcal{F}$.

Scalar Diffusion Processes

- A diffusion (or Ito process) is an adapted process x_t with continuous paths,

$$\begin{aligned} x_t &= x_0 + \int_{v=0}^{t} \mu(x_v, v)dv + \int_{v=0}^{t} \sigma(x_v, v)dz_v \\ \iff dx_t &= \mu(x_t, t)dt + \sigma(x_t, t)dz_t \end{aligned}$$

  where $\mu(x_v, v)$ is a drift coefficient, $\sigma(x_v, v)$ is a diffusion coefficient, and z_t is a Brownian motion.
- The Ito integral is defined as

$$\int_{v=0}^{t} \sigma(x_v, v)dz_v := \lim_{N \to \infty} \sum_{v=1}^{N} \sigma(x_{(v-1)t/N}, (v-1)t/N)(z_{tv/N} - z_{t(v-1)/N})$$

- $E_t(dx_t) = E_t(\mu(x_t, t)dt + \sigma(x_t, t)dz_t) = \mu(x_t, t)dt + \sigma(x_t, t)E_t(dz_t) = \mu(x_t, t)dt = \mu_t dt$
- $E_t((dx_t)^2) = E_t(\mu(x_t, t)^2(dt)^2 + 2\mu(x_t, t)\sigma(x_t, t)dtdz_t + \sigma(x_t, t)^2(dz_t)^2) = E_t(\sigma(x_t, t)^2(dz_t)^2) = \sigma_t^2 dt$

Examples of Scalar Diffusion Processes

- Brownian motion with drift:
  - $Y_t = Y_0 + \mu t + \sigma z_t$
  - $dY_t = \mu dt + \sigma dz_t$
  - $Y_t - Y_s \sim N(\mu(t - s), \sigma^2(t - s))$ for $t > s$.
- Geometric Brownian Motion:
  - $dS_t = \mu S_t dt + \sigma S_t dz_t$, with constants μ, σ.
  - For example, stock price in Black and Scholes (JPE 1973).
- Ornstein-Uhlenbeck process (mean-reverting):
  - $dr_t = \kappa(\theta - r_t)dt + \sigma dz_t$ with constants $\kappa, \theta, \sigma > 0$.
  - Risk-free rate in Vasicek (JFE, 1977)
- Square root process (mean-reverting):
  - $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz_t$.
  - Risk-free rate in Cox, Ingersoll and Ross (ECTA, 1985)

Vector Diffusion Processes

- A vector of Brownian motions $\mathbf{z}_t$ is independent iff $z_{it} - z_{is} \perp z_{ju} - z_{jv}$ for all $i \neq j$ and all intervals $[t, s]$ and $[u, v]$.
- A diffusion (or Ito process) is an adapted random vector process $\mathbf{x}_t$ with continuous paths,

$$\begin{aligned} \mathbf{x}_t &= \mathbf{x}_0 + \int_{v=0}^{t} \boldsymbol{\mu}(\mathbf{x}_v, v)dv + \int_{v=0}^{t} \boldsymbol{\sigma}(\mathbf{x}_v, v)d\mathbf{z}_v \\ \iff d\mathbf{x}_t &= \boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t \end{aligned}$$

  where $\boldsymbol{\mu}(\mathbf{x}_t, t)$ is a vector of drift coefficients, $\boldsymbol{\sigma}(\mathbf{x}_t, t)$ is a diffusion coefficient, and $\mathbf{z}_t$ is a vector of independent Brownian motions.

- The Ito integral is defined as

$$\int_{v=0}^{t} \boldsymbol{\sigma}(\mathbf{x}_v, v)d\mathbf{z}_v := \lim_{N\to\infty} \sum_{v=1}^{N} \boldsymbol{\sigma}(\mathbf{x}_{(v-1)t/N}, (v-1)t/N)(\mathbf{z}_{tv/N} - \mathbf{z}_{t(v-1)/N})$$

$$\begin{aligned} E_t(d\mathbf{x}_t) &= E_t(\boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t) = \boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)E_t(d\mathbf{z}_t) = \boldsymbol{\mu}(\mathbf{x}_t, t)dt \\ E_t(d\mathbf{x}_t d\mathbf{x}^T) &= E_t((\boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t)(\boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t)^T) \\ &= E_t((dt)^2\boldsymbol{\mu}(\mathbf{x}_t, t)(\boldsymbol{\mu}(\mathbf{x}_t, t))^T + 2\boldsymbol{\mu}(\mathbf{x}_t, t)(dt d\mathbf{z}_t^T)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T + \boldsymbol{\sigma}(\mathbf{x}_t, t)(d\mathbf{z}_t d\mathbf{z}_t^T)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T) \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t)E_t(d\mathbf{z}_t d\mathbf{z}_t^T)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t)(dt \times I)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T \\ &= \boldsymbol{\sigma}(\mathbf{x}_t, t)\boldsymbol{\sigma}(\mathbf{x}_t, t)^T dt \end{aligned}$$

Examples of Vector Diffusion Processes

- Two Brownian motions with drift and correlation $\rho \in [-1, 1]$.

$$d\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} dt + \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}$$

- Multiperiod consumption-savings-portfolio choice in Merton (various 1970s)

$$\begin{aligned} dW_t &= W_t(\boldsymbol{\alpha}_t \cdot (\boldsymbol{\mu}(\mathbf{x}_t, t) - r(\mathbf{x}_t, t)\mathbb{1}) + r(\mathbf{x}_t, t))dt + W_t\boldsymbol{\alpha}_t^T\boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t - c_t dt + y_t dt \\ d\mathbf{x}_t &= \boldsymbol{\mu}_x(\mathbf{x}_t, t)dt + \sigma_x(\mathbf{x}_t, t)d\mathbf{z}_t \end{aligned}$$

  where $W_t \geq 0$ and W_0 and $\mathbf{x}_0$ is given.

- Constant returns-to-scale production and productivity in Ai (JF 2010).

$$\begin{aligned} dK_t &= x_t K_t dt - C_t dt + \sigma_c K_t dz_t^c \\ dx_t &= \kappa(\mu - x_t)dt + \sigma_x dz_t^x \\ dz_t^c dz_t^x &= \rho dt \end{aligned}$$

Convenient Facts

- For an adapted process γ_t (vector), we can express some functions of the expectation of the adapted process in terms of a change in time instead of a change in the Brownian motion value.
- For example, expectation of quadratic:

$$\begin{aligned} E_t\left(\left(\int_t^T \gamma_s d\mathbf{z}_s\right)^2\right) &= E_t \lim_{N\to\infty} \sum_{i=1}^N \sum_{j=1}^N K_i K_j \\ &= E_t \lim_{N\to\infty} 2\sum_{i=1}^N \sum_{i<j}^N K_j E_{(T-t)(i-1)/N} K_i + \sum_{j=1}^N E_{(T-t)(j-1)/N} K_j^2 \\ &= E_t \lim_{N\to\infty} \frac{T-t}{N} \sum_{j=1}^N \gamma_{(T-t)(j-1)/N} \cdot \gamma_{(T-t)(j-1)/N} \\ &= \int_t^T E_t(\gamma_s \cdot \gamma_s) ds \end{aligned} \tag{1}$$

where $K_j = \gamma_{T(j-1)/N} \cdot (\mathbf{z}_{(T-t)j/N} - \mathbf{z}_{(T-t)(j-1)/N})$.

- For example, expectation of exponential:

$$E_t\left(\exp\left(\int_t^T \gamma_s d\mathbf{z}_s\right)\right) = E_t\left(\exp\left(\frac{1}{2}\int_t^T (\gamma_s \cdot \gamma_s) ds\right)\right) \tag{2}$$

- Consider the square-root process,

$$\begin{aligned} dr_t &= \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz_t \\ \implies e^{\kappa t} dr_t &= e^{\kappa t}\kappa\theta dt - e^{\kappa t}\kappa r_t dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\ \implies e^{\kappa t} dr_t + e^{\kappa t}\kappa r_t dt &= e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\ \implies d(e^{\kappa t} r_t) &= e^{\kappa t}\kappa\theta dt + e^{\kappa t}\sigma\sqrt{r_t}dz_t \\ \implies \int_{-\infty}^t d(e^{\kappa s} r_s) &= \int_{-\infty}^t e^{\kappa s}\kappa\theta ds + \int_{-\infty}^t e^{\kappa s}\sigma\sqrt{r_s}dz_s \\ \implies e^{\kappa t} r_t &= e^{\kappa t}\theta + \sigma\int_{-\infty}^t e^{\kappa s}\sqrt{r_s}dz_s \\ \implies r_t &= \theta + \sigma\int_{-\infty}^t e^{\kappa(s-t)}\sqrt{r_s}dz_s \end{aligned}$$

- Using (**??**), we can find the unconditional variance (based on the unconditional expectation):

$$
\begin{aligned}
\implies E[r_t] &= \theta + E\left[\sigma \int_{-\infty}^{t} e^{\kappa(s-t)} \sqrt{r_s} dz_s\right] \\
&= \theta \\
\implies Var(r_t) &= E[r_t^2] - E[r_t]^2 \\
&= E\left[\left(\theta + \sigma \int_{-\infty}^{t} e^{\kappa(s-t)} \sqrt{r_s} dz_s\right)^2\right] - \theta^2 \\
&= E\left[2\theta\sigma \int_{-\infty}^{t} e^{\kappa(s-t)} \sqrt{r_s} dz_s\right] + E\left[\left(\sigma \int_{-\infty}^{t} e^{\kappa(s-t)} \sqrt{r_s} dz_s\right)^2\right] \\
&= \sigma^2 \int_{-\infty}^{t} e^{2\kappa(s-t)} E[r_s] dz_s \\
&= \sigma^2 \theta e^{-2\kappa t} \int_{-\infty}^{t} e^{2\kappa s} dz_s \\
&= \sigma^2 \theta e^{-2\kappa t} \left[\frac{1}{2\kappa} e^{2\kappa s}\right]_{-\infty}^{t} \\
&= \frac{\sigma^2 \theta}{2\kappa}
\end{aligned}
$$

Black and Scholes Structure

- Stock with price S_t: $dS_t = \mu S_t dt + \sigma S_t dz_t, \mu > 0, \sigma > 0$.
- Risk-free bond: $dB_t = B_t r dt, \mu > r > 0$.
- Option with strike price k: At the exercise date T, the payoff is $C(S_T, T) = \max\{0, S_t - K\}$
- Assumptions:
  - No dividend payments on stock.
  - Infinite depth in the stock and bond markets.
  - Constant drift and volatility in the stock return.
  - Constant rate of interest.
  - Frictionless markets (i.e. no transaction costs).
  - European call option (i.e. can only exercise at maturity date T).
- Goal is to find equation for $C(S_t, t), t < T$.

Future Values

- To get $d\ln B_t$ use Ito's lemma [where $\mu(B_t,t) = B_t r$, $\sigma = 0$ and $f(x) = \ln x \implies f_x(x) = \frac{1}{x}, f_{xx}(x) = \frac{-1}{x^2}, f_t(x) = 0$]:

$$\begin{aligned}
d\ln B_t &= \frac{1}{B_t}(0)dz_t + \frac{1}{B_t}B_t r dt + \frac{1}{2}\frac{-1}{x^2}(0)^2 dt + 0dt \\
&= rdt \\
\implies \int_0^t d\ln B_s &= \int_0^t rds \\
\implies \ln B_t - \ln B_0 &= r(t-0) \\
\implies B_t &= B_0 \exp(rt)
\end{aligned}$$

- To get $d\ln S_t$ use Ito's lemma [where $\mu(S_t,t) = \mu S_t$, $\sigma(S_t,t) = \sigma S_t$, and $f(x) = \ln x \implies f_x(x) = \frac{1}{x}, f_{xx}(x) = \frac{-1}{x^2}, f_t(x) = 0$]:

$$\begin{aligned}
d\ln S_t &= \frac{1}{S_t}\mu S_t dt + \frac{1}{S_t}\sigma S_t dz_t + \frac{1}{2}\frac{-1}{S_t^2}(\sigma S_t)^2 dt + (0)dt \\
&= \mu dt + \sigma dz_t - \frac{1}{2}\sigma^2 dt \\
\implies \int_0^t d\ln S_s &= \mu\int_0^t ds + \sigma\int_0^t dz_s - \frac{1}{2}\sigma^2\int_0^t dt \\
\implies \ln S_t - \ln S_0 &= \mu t + \sigma z_t - \frac{1}{2}\sigma^2 t \\
\implies S_t &= S_0 \exp(\mu t + \sigma z_t - \frac{1}{2}\sigma^2 t)
\end{aligned}$$

where $z_0 \equiv 0$.

$$\begin{aligned}
E[\ln S_t | \ln S_0] &= E[\ln S_0 + \mu t + \sigma z_t - \frac{1}{2}\sigma^2 t | \ln S_0] \\
&= \ln S_0 + \mu t + \sigma E[z_t | \ln S_0] - \frac{1}{2}\sigma^2 t \\
&= \ln S_0 + \mu t - \frac{1}{2}\sigma^2 t
\end{aligned}$$

Using (**??**),

$$\begin{aligned}
E[S_t|S_0] &= E[S_0 \exp(\mu t + \sigma z_t - \frac{1}{2}\sigma^2 t)] \\
&= S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t) E\left[\exp\left(\int_{v=0}^t \sigma dz_v\right)\right] \\
&= S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t) E\left[\exp\left(\frac{1}{2}\int_{v=0}^t \sigma^2 dv\right)\right] \\
&= S_0 \exp(\mu t - \frac{1}{2}\sigma^2 t) \exp\left(\frac{1}{2}\sigma^2 t\right) \\
&= S_0 \exp(\mu t)
\end{aligned}$$

Ito's Lemma (Scalar)

- Let $f(x,t)$ be twice differentiable in x and once in t. Let x be a (scalar) diffusion with $dx_t = \mu(x_t,t)dt + \sigma(x_t,t)dz_t$, then

$$\begin{aligned} f(x_t,t) - f(x_0,0) &= \int_{s=0}^{t} f_x(x_s,s)dx_s + \frac{1}{2}\int_{s=0}^{t} f_{xx}(x_s,s)\sigma(x_s,s)^2 ds + \int_{s=0}^{t} f_t(x_s,s)ds \\ df &= f_x\mu dt + f_x\sigma dz_t + \frac{1}{2}f_{xx}\sigma^2 dt + f_t dt \end{aligned}$$

  where $f_x = \frac{\partial f(x,s)}{\partial x}$ (f_{xx} and f_t similar).

- Examples
  - Consider $d(z_t^2)$. Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t,t) &= 0, \sigma(x_t,t) = 1 \ \forall x_t, t \\ \implies dx_t &= 0 * dt + 1 * dz_t = dz_t \\ f(x,t) &= x^2 \\ \implies f_x(x,t) &= 2x, f_{xx} = 2, f_t = 0 \end{aligned}$$

$$\implies d(z_t^2) = df = (2z_t)(1)dz_t + (2z_t)(0)dt + \frac{1}{2}(2)(1)^2 dt + (0)dt = 2z_t dz_t + dt$$

  - Consider $d\exp(z_t)$. Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t,t) &= 0, \sigma(x_t,t) = 1 \ \forall x_t, t \\ \implies dx_t &= 0 * dt + 1 * dz_t = dz_t \\ f(x,t) &= \exp(x) \\ \implies f_x(x,t) &= \exp(x), f_{xx} = \exp(x), f_t = 0 \end{aligned}$$

$$\begin{aligned} \implies d\exp(z_t) = df &= \exp(z_t)(0)dt + \exp(z_t)(1)dz_t + \frac{1}{2}\exp(z_t)(1)^2 dt + (0)dt \\ &= \exp(z_t)dz_t + \frac{1}{2}\exp(z_t)dt \end{aligned}$$

  - Consider $dx_t = \mu dt + \sigma dz_t$ and $d\exp(x_t)$. Mapping to Ito's lemma notation above:

$$\begin{aligned} \mu(x_t,t) &= \mu, \sigma(x_t,t) = \sigma \ \forall x_t, t \\ \implies dx_t &= \mu dt + \sigma dz_t \\ f(x,t) &= \exp(x) \\ \implies f_x(x,t) &= \exp(x), f_{xx} = \exp(x), f_t = 0 \end{aligned}$$

$$\begin{aligned} \implies d\exp(z_t) = df &= \exp(x_t)\mu dt + \exp(z_t)\sigma dz_t + \frac{1}{2}\exp(z_t)\sigma^2 dt + (0)dt \\ &= \exp(x_t)\mu dt + \exp(z_t)\sigma dz_t + \frac{1}{2}\exp(z_t)\sigma^2 dt \end{aligned}$$

No Instantaneous Arbitrage

- Bond increment is $dB_t = B_t r dt$
- Stock price increment is $dS_t = \mu S_t dt + \sigma S_t dz_t$
- Option price increment is [by Ito's Lemma where $\mu(S_t, t) = S_t\mu$, $\sigma(S_t, t) = \sigma S_t$, $f = C$, $f_x = C_s$, etc.]

$$\begin{aligned}
dC(S_t, t) &= C_s S_t \mu dt + C_s \sigma S_t dz_t + \frac{1}{2} C_{ss} (\sigma S_t)^2 dt + C_t dt \\
&= C_s (S_t \mu dt + \sigma S_t dz_t) + \frac{1}{2} C_{ss} (\sigma S_t)^2 dt + C_t dt \\
&= C_s dS_t + \frac{1}{2} C_{ss} (\sigma S_t)^2 dt + C_t dt
\end{aligned}$$

where $C_t = \frac{\partial C(S_t,t)}{\partial t}$ and similar for C_s and C_{ss}.

- Portfolio (value) increment: $dP_t = -dC(S_t, t) + C_s dS_t + (C - C_s S_t) r dt$

  This portfolio is where you sell one option (inflow of $C(S_t, t)$), buy C_s shares of stock at price S_t (outflow of $C_s S_t$), and invest $C(S_t, t) - C_s S_t$ dollars in the bond (outflow of $C(S_t, t) - C_s S_t$).

  Portfolio cost is zero [i.e. $C(S_t, t) - C_s S_t - (C(S_t, t) - C_s S_t) = 0$] and it is risk-free:

  No arbitrage $\implies dP_t = 0 \implies 0 = -dC(S_t, t) + C_s dS_t + (C(S_t, t) - C_s S_t) r dt$

  Substituting in option increment:

$$\begin{aligned}
\implies 0 &= -[C_s dS_t + \frac{1}{2} C_{ss} \sigma^2 S_t^2 dt + C_t dt] + C_s dS_t + (C(S_t, t) - C_s S_t) r dt \\
&= -\frac{1}{2} C_{ss} \sigma^2 S_t^2 dt - C_t dt + (C(S_t, t) - C_s S_t) r dt \\
&= -\frac{1}{2} C_{ss} \sigma^2 S_t^2 - C_t + (C(S_t, t) - C_s S_t) r
\end{aligned}$$

Black-Scholes Call Option Price

- The price of a European call option for $0 \le t \le T, 0 \le S_t$ satisfies:

$$\begin{aligned}
0 &= \frac{1}{2} C_{ss} \sigma^2 S_t^2 + C_t - (C(S_t, t) - C_s S_t) r && \text{[differential equation]} \\
C(S_T, T) &= \max[S_T - K, 0] && \text{[boundary condition]} \\
C(0, t) &= 0, && \forall 0 \le t < T
\end{aligned}$$

- A solution is:

$$\begin{aligned}
C(S_t, t) &= S_t \Phi(d_1(S_t)) - K \exp(-r(T - t)) \Phi(d_2(S_t)) \\
d_1(S) &:= \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\
d_2(S) &:= d_1(S) - \sigma\sqrt{T - t} \\
\Phi(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{v^2}{2}) dv && \text{[standard normal cdf]}
\end{aligned}$$

Ito's Lemma (Vector)

- Let scalar function $f(\mathbf{x}, t)$ be twice differentiable in vector $\mathbf{x}$ and once in t.
- Let $\mathbf{x}_t$ be a vector diffusion with increment:

$$d\mathbf{x}_t = \boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{z}_t$$

where $\mathbf{z}_t$ is a Brownian Motion vector. Then[1]

$$\begin{aligned} f(\mathbf{x}_t, t) - f(\mathbf{x}_0, 0) &= \int_{s=0}^{t} f_{\mathbf{x}}(\mathbf{x}_s, s)d\mathbf{x}_s + \frac{1}{2}\int_{s=0}^{t} tr[f_{\mathbf{xx}}(\mathbf{x}_s, s)\sigma(\mathbf{x}_s, s)\sigma(\mathbf{x}_s, s)^T]ds + \int_{s=0}^{t} f_t(\mathbf{x}_s, s)ds \\ \iff df &= f_{\mathbf{x}}^T \boldsymbol{\mu}_x dt + f_{\mathbf{x}}^T \boldsymbol{\sigma}_x d\mathbf{z}_t + \frac{1}{2}tr[f_{\mathbf{xx}}\boldsymbol{\sigma}\boldsymbol{\sigma}^T]dt + f_t dt \end{aligned}$$

Ito's Lemma Examples

$$d\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} dt + \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}$$

- Consider $d(x^2)$. We know that $dx = \mu_x dt + \sigma_x dz_1$. Mapping to Ito's Lemma notation, $f(x) = x^2, f_x = 2x, f_{xx} = 2, f_t = 0$:

$$d(x^2) = 2x\mu_x dt + 2x\sigma_x dz_1 + (0)dt + \frac{1}{2}(2)\sigma_x^2 dt = 2x\mu_x dt + 2x\sigma_x dz_1 + \sigma_x^2 dt$$

- Consider $d(xy)$. We know that $dy = \mu_y dt + \sigma_y \rho dz_1 + \sigma_y \sqrt{1-\rho^2} dz_2$

$$\begin{aligned} d(xy) &= xdy + ydx + dxdy \\ &= x[\mu_y dt + \sigma_y \rho dz_1 + \sigma_y \sqrt{1-\rho^2} dz_2] + y[\mu_x dt + \sigma_x dz_1] \\ &+ [\mu_x dt + \sigma_x dz_1][\mu_y dt + \sigma_y \rho dz_1 + \sigma_y \sqrt{1-\rho^2} dz_2] \\ &= x\mu_y dt + x\sigma_y \rho dz_1 + x\sigma_y \sqrt{1-\rho^2} dz_2 + y\mu_x dt + y\sigma_x dz_1 \\ &+ \mu_x dt \mu_y dt + \mu_x dt \sigma_y \rho dz_1 + \mu_x dt \sigma_y \sqrt{1-\rho^2} dz_2 \\ &+ \sigma_x dz_1 \mu_y dt + \sigma_x dz_1 \sigma_y \rho dz_1 + \sigma_x dz_1 \sigma_y \sqrt{1-\rho^2} dz_2 \\ &= x\mu_y dt + x\sigma_y \rho dz_1 + x\sigma_y \sqrt{1-\rho^2} dz_2 \\ &+ y\mu_x dt + y\sigma_x dz_1 + \sigma_x \sigma_y \rho dt + \sigma_x \sigma_y \sqrt{1-\rho^2} dt \\ &= (x\mu_y + y\mu_x + \sigma_x \sigma_y \rho + \sigma_x \sigma_y \sqrt{1-\rho^2})dt + (x\sigma_y \rho + y\sigma_x)dz_1 + x\sigma_y \sqrt{1-\rho^2} dz_2 \end{aligned}$$

Application of the Martingale Property

- Suppose X is Brownian motion, $dX = dz$.
- We know $X_T | X_t \sim N(X_t, T-t)$:

$$h(X_t, t) := Pr(X_T \leq A | X_t) = \Phi\left(\frac{A - X_t}{\sqrt{T-t}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{A-X_t}{\sqrt{T-t}}} \exp\left(-\frac{u^2}{2}\right) du$$

[1] $tr[\mathbf{B}]$ denotes the trace of matrix $\mathbf{B}$ i.e. the elements of the diagonal.

- By Ito's lemma, $dh = h_x dX + \frac{1}{2}h_{xx}dt + h_t dt$
  Notice that $E[dh] = h_x E[dX] + \frac{1}{2}h_{xx}dt + h_t dt = \frac{1}{2}h_{xx}dt + h_t dt$
- Also, probabilities are martingales:

$$\frac{1}{v}E_t(h(X_{t+v}, t+v) - h(X_t, t)) = \frac{1}{v}E_t(E_{t+v}(\mathbb{1}_{X_T \le A}) - E_t(\mathbb{1}_{X_T \le A})) = 0$$

  Taking v small, so $dt = v$:

$$\implies 0 = E_t(dh)/dt = \frac{1}{2}h_{xx} + h_t, \text{ subject to } h(X_T, T) = \begin{cases} 1 & \text{if } X_t \le A \\ 0 & \text{otherwise} \end{cases}$$

- Show that $0 = \frac{1}{2}h_{xx} + h_t$:

$$\begin{aligned}
\Phi'(x) &= \phi(x) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \\
\phi'(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)(-x) \\
&= -x * \phi(x) \\
h_t &= \frac{\partial h(X_t, t)}{\partial t} \\
&= \phi\left(\frac{A - X_t}{\sqrt{T-t}}\right) \frac{(A - X_t)\frac{1}{2}(T-t)^{-1/2} - (0)\sqrt{T-t}}{T-t} \\
&= \frac{1}{2}\phi\left(\frac{A - X_t}{\sqrt{T-t}}\right) \frac{A - X_t}{(T-t)^{3/2}} \\
h_x &= \frac{\partial h(X_t, t)}{\partial X_t} \\
&= \phi\left(\frac{A - X_t}{\sqrt{T-t}}\right) \frac{-1}{\sqrt{T-t}} \\
h_{xx} &= \frac{\partial^2 h(X_t, t)}{\partial^2 X_t} \\
&= \phi'\left(\frac{A - X_t}{\sqrt{T-t}}\right) \frac{1}{T-t} \\
&= \phi\left(\frac{A - X_t}{\sqrt{T-t}}\right) \frac{X_t - A}{(T-t)^{3/2}}
\end{aligned}$$

  Thus, $\frac{1}{2}h_{xx} + h_t = 0$.

Feynman-Kac I

- Suppose the risk-free rate r is constant (and technical conditions hold on functions μ and σ^2).
- The function $f(x,t)$ is a solution to the boundary-value problem:

$$\begin{aligned} 0 &= \frac{1}{2} f_{xx}(x,t)\sigma^2(x,t) + f_x(x,t)\mu(x,t) + f_t(x,t) - rf(x,t), 0 \le t < T \\ f(x,T) &= F(x) \end{aligned}$$

iff f is given by

$$\begin{aligned} f(x,t) &= \exp(-r(T-t))E[F(X_T)|X_t = x] \\ \text{where } dX_s &= \mu(X_s,s)ds + \sigma(X_s,s)dz_s; X_t = x \end{aligned}$$

- Intuition: Use the factor of integration $\exp(-rt)$.

$$\begin{aligned} f(X_T,T)\exp(-rT) - f(X_t,t)\exp(-rt) &= \int_{s=t}^{T} d(f(X_s,s)\exp(-rs)) \\ &= \int_{s=t}^{T} \exp(-rs) f_x(X_s,s)\sigma(X_s,s)dz_s \end{aligned}$$

Black-Scholes and Feynman-Kac

- The price of a European call option for $0 \le t \le T, 0 \le S_t$ satisfies:

$$\begin{aligned} 0 &= \frac{1}{2} C_{ss}\sigma^2 S_t^2 + C_t - (C(S_t,t) - C_s S_t)r \\ C(S_T,T) &= \max[S_T - K, 0] \\ C(0,t) &= 0 \end{aligned}$$

- A solution is:

$$\begin{aligned} C(s,t) &= \exp(-r(T-t))E[\max\{S_T^* - K, 0\}|S_t^* = s] \\ \text{where } \frac{dS_t^*}{S_t^*} &= rdt + \sigma dz_t \end{aligned}$$

Feynman-Kac II

- Under technical conditions on functions g, r, μ, and σ^2, the function $f(x,t)$ is a solution to the boundary-value problem:

$$\begin{aligned} 0 &= \frac{1}{2} f_{xx}(x,t)\sigma^2(x,t) + f_x(x,t)\mu(x,t) + f_t(x,t) + g(x,t) - r(x,t)f(x,t), 0 \le t < T \\ f(x,T) &= F(x) \end{aligned}$$

iff f is given by

$$f(x,t) = E\left[\int_{v=t}^{T} \exp\left(\int_{v=t}^{T} -r(X_v,v)dv\right) g(X_s,s)ds + \exp\left(\int_{s=t}^{T} -r(X_s,s)ds\right) F(X_T) \middle| X_t = x\right]$$

where $dX_s = \mu(X_s,s)ds + \sigma(X_s,s)dz_s; X_t = x$

2 Part II

Diffusion Processes

- In this section, any vector stochastic process S is an Ito process $dS_t = \mu(\omega,t)dt + \sigma(\omega,t)d\mathbf{z}_t$ or (shorthand) $dS_t = \mu_t dt + \sigma_t d\mathbf{z}_t$ (the subscripts remind us that the coefficient are functions of state and time.
- We casually speak of μ_t as the conditional expected growth rate (or drift) of S_t per unit of time, while the conditional covariance matrix of the increment dS_t is $\sigma_t \sigma^T$.

Continuous-Time Budget Constraint

- In continuous time, wealth is equal before and after a trade:

$$W_t = \underbrace{\theta_t \cdot S_t}_{\text{At time t, after trade}} = \underbrace{\theta_{t-\Delta} \cdot S_t + \theta_{t-\Delta} \cdot D_t\Delta - c_t\Delta + y_t\Delta}_{\text{At time t, before trade}}$$

where θ_t is the vector of positions, S_t is the price vector, D_t is the dividend vector, c_t is the consumption rate, and y_t is the labor income rate.

$$\begin{aligned} \Longrightarrow \underbrace{\theta_{t-\Delta} \cdot D_t\Delta}_{\text{dividend income}} - \underbrace{c_t\Delta}_{\text{consumption}} + \underbrace{y_t\Delta}_{\text{labor income}} &= \underbrace{(\theta_t - \theta_{t-\Delta}) \cdot S_t}_{\text{asset purchases}} \\ &= (\theta_t - \theta_{t-\Delta}) \cdot S_{t-\Delta} + (\theta_t - \theta_{t-\Delta}) \cdot (S_t - S_{t-\Delta}) \qquad (3) \end{aligned}$$

- Change in wealth is:

$$\begin{aligned} W_t - W_{t-\Delta} &= \theta_t \cdot S_t - \theta_{t-\Delta} \cdot S_{t-\Delta} \\ &= \theta_{t-\Delta} \cdot (S_t - S_{t-\Delta}) + (\theta_t - \theta_{t-\Delta}) \cdot S_{t-\Delta} + (\theta_t - \theta_{t-\Delta}) \cdot (S_t - S_{t-\Delta}) \qquad (4) \end{aligned}$$

- Combining (**??**) and (**??**):

$$\underbrace{W_t - W_{t-\Delta}}_{\text{Change in wealth}} = \underbrace{\theta_{t-\Delta} \cdot (S_t - S_{t-\Delta}) + \theta_{t-\Delta} \cdot D_t\Delta}_{\text{Portfolio gains or losses}} - \underbrace{c_t\Delta}_{\text{consumption}} + \underbrace{y_t\Delta}_{\text{labor income}}$$

- Let $\Delta \to 0$:

$$\begin{aligned} dW_t &= \theta_t \cdot dS_t + \theta_t \cdot D_t dt - c_t dt + y_t dt \\ &= W_t(\alpha_t \cdot ((dS_t + D_t dt)./S_t - r_t \mathbb{1} dt) + r_t dt) - c_t dt + y_t dt \end{aligned}$$

where α_t is the portfolio weight in each risky security, r_t is the instantaneous risk-free rate, and "./" denotes element-by-element division.

Normally Distributed Returns

- We write $(dS_t + D_t dt)./S_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{z}_t$ when prices are diffusion process, where expected rate of return is $\boldsymbol{\mu}_t$ is the sum of the expected capital gain and dividend yield.
- The instantaneous rate of return on a portfolio is distributed normal, with expected return and variance of return:

$$\mu_p = \alpha_t \cdot (\boldsymbol{\mu}_t - r_t \mathbb{1}) + r_t$$
$$\sigma_p^2 = \alpha_t^T \mathbf{V}_t \alpha_t = \alpha_t^T \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t^T \alpha_t$$

- Furthermore, if an investor has state-independent utility, an optimal portfolio is on the instantaneous minimum variance frontier.
- Given an instantaneous risk-free rate, the MVF is defined by the risk-free asset and the tangency portfolio of risky assets with the highest Sharpe ratio.
- The portfolio of (only) risky assets with the maximum Sharpe ratio is

$$\boldsymbol{\alpha}_t = \frac{\mathbf{V}_t^{-1}(\boldsymbol{\mu} - r_t \mathbb{1})}{\mathbb{1}^T \mathbf{V}_t^{-1}(\boldsymbol{\mu}_t - r_t \mathbb{1})}$$

Self-Financing Portfolio

- The process for wealth is $dW_t = \theta_t \cdot (dS_t + D_t dt) - c_t dt + y_t dt$.
- A **self-financing portfolio** has no inflows or outflows. Here this means that $c_t = y_t$ and all dividends are reinvested, so $dW_t = d(\theta_t \cdot S_t) = \theta_t \cdot (dS_t + D_t dt)$.
- Given processes S and D, the **gains process** is defined as $G_t := S_t + \int_{s=0}^t D_s ds$.
- Then, a self-financing strategy θ satisfies $dT_t = \theta_t \cdot dG_t$ and $W_t = W_0 + \int_{s=0}^t \theta_s \cdot dS_s + \int_{s=0}^t \theta_s \cdot D_s ds$.

Stochastic Discount Factor

- A **continuous-time stochastic discount factor** is a positive process $\pi > 0$, such that:

$$S_t = \frac{1}{\pi_t} E_t \int_{s=t}^{\infty} D_s \pi_s ds$$

- Define a **discounted gains process** $G_t^\pi = \pi_t S_t + \int_{s=0}^t \pi_s D_s ds$. Then, for $T > t$,

$$G_t^\pi = \pi_t S_t + \int_{s=0}^t \pi_s D_s ds = E_t \left(\int_{s=0}^T D_s \pi_s ds + \pi_T \frac{1}{\pi_T} \int_{s=T}^{\infty} D_s \pi_s ds \right) = E_t(G_t^\pi)$$

  Thus, the discounted gains process is a martingale.
- For a stock that first pays dividends at date $T > t$, $\pi_t S_t = E_t(\pi_T S_T)$.
- For a discount bond that pays \$1 at date $T > t$, $P(t,T) = \exp(-YTM(t,T)(T-t)) = E_t(\frac{\pi_T}{\pi_t})$.
- For a strategy of rolling over an instantaneously risk-free asset, $1 = E_t(\exp(\int_t^T r_s ds) \frac{\pi_T}{\pi_t})$

Martingale Representation Theorem

- Theorem: If M is a (local) martingale, then there exists an adapted process θ such that

$$M_t = M_0 + \int_0^t \theta_s dz_s, \text{ with } Pr\left(\int_0^t \theta_s \cdot \theta_s ds < \infty\right) = 1.$$

- The discounted process is an adapted process and a martingale. Thus, $dG_t^\pi = \sigma_G(\omega, t)d\mathbf{z}_t$, for some adapted $\sigma_G(\omega, t)$.

Risk Premiums

- Assume the stochastic discount factor π_t is a diffusion. Then,

$$S_t \pi_t = E_t\left(\int_{s=t}^{\infty} D_s \pi_s ds\right) = E_t\left(\int_{s=t}^{t+\Delta} D_s \pi_s ds + E_t S_{t+\Delta}\pi_{t+\Delta}\right)$$

Using element-by-element division

$$0 = \frac{1}{\Delta} E_t\left(\int_{s=t}^{t+\Delta} (D_s \pi_s./S_t \pi_t) ds + \frac{1}{\Delta}(E_t(S_{t+\Delta}\pi_{t+\Delta}) - S_t \pi_t)./S_t \pi_t\right)$$

- Taking $\Delta \to 0$:

$$\begin{aligned} 0 &= D_t dt./S_t + E_t[d(S_t \pi_t)./S_t \pi_t] \\ &= D_t dt./S_t + E[dS_t./S_t] + E_t\left[\frac{d\pi_t}{\pi_t}\mathbb{1}\right] + E_t\left[\frac{d\pi_t}{\pi_t} dS_t./S_t\right] \\ \implies \underbrace{E_t[dS_t./S_t + D_t dt./S_t]}_{\text{Expected Rate of Return}} &= -E_t\left[\frac{d\pi_t}{\pi_t}\mathbb{1}\right] - E_t\left[\frac{d\pi_t}{\pi_t} dS_t./S_t\right] \end{aligned}$$

- For the risk-free security:

$$\frac{d\pi_t}{\pi_t} r_t dt = 0 \implies r_t dt = -E_t\left[\frac{d\pi_t}{\pi_t}\right] \tag{5}$$

- For the risky securities,

$$\begin{aligned} \underbrace{E_t[dS_t./S_t + D_t dt./S_t] - r_t \mathbb{1} dt}_{\text{Risk Premium}} &= -E_t\left[\frac{d\pi_t}{\pi_t}\mathbb{1}\right] - E_t\left[\frac{d\pi_t}{\pi_t} dS_t./S_t\right] - E_t\left[\frac{d\pi_t}{\pi_t}\mathbb{1}\right] \\ &= \underbrace{-E_t\left[\frac{d\pi_t}{\pi_t} dS_t./S_t\right]}_{\text{Covariance of price and SDF changes}} \end{aligned} \tag{6}$$

Marginal Utility is a SDF

- Consider the lifetime expected utility $E_t[\int_{s=t}^{\infty} u(c_s, s)ds]$, where wealth follows the process $dW_t = \theta_t \cdot dS_t + \theta \cdot D_t dt - c_t dt + y_t dt$, $\forall t$; $W_t \geq 0$; and initial wealth W_0 is given.
- Pick an asset with price S_t and consider 2 perturbations:
  1. Increase the consumption rate by $\varepsilon S_t/\Delta$ for a brief interval of time $[t, t+\Delta]$, raising expected utility:

$$E_t\left(\int_{s=t}^{t+\Delta}[u(c_s + \varepsilon S_t/\Delta, s) - u(c_s, s)]ds\right) \approx E_t\left(\int_{s=t}^{t+\Delta} u'(c_s, s)(\varepsilon S_t/\Delta)ds\right) \to \varepsilon S_t u'(c_t, t)$$

  2. Increase and hold forever ε more shares of an asset, raising expected utility

$$E_t\left(\int_{s=t}^{\infty}[u(c_s + \varepsilon D_s, s) - u(c_s, s)]ds\right) \approx \varepsilon E_t\left(\int_{s=t}^{\infty} u'(c_s, s)D_s ds\right)$$

- Each perturbation costs εS_t, so optimality requires that

$$S_t u'(c_t, t) = E_t\left(\int_{s=t}^{\infty} u'(c_s, s)D_s ds\right)$$

- Consequently, we can use marginal utility as an SDF in (**??**) and (**??**):

$$E_t\left(\frac{du'(c_s, s)}{u'(c_s, s)}\right) = -r_t dt$$

$$\underbrace{\frac{D_t}{S_t}dt + E_t\frac{dS_t}{S-t} - r_t dt}_{\text{Risk Premium}} = \underbrace{-E_t\left(\frac{dS_t du'(c_s, s)}{S_t u'(c_s, s)}\right)}_{\text{Covariance}}$$

Consumption-Based Asset Pricing

- Let the process for per capita consumption be $\frac{dc_t}{c_t} = \mu_c dt + \boldsymbol{\sigma}_c d\mathbf{z}_t$.

  [Notice that consumption is a scalar, so $\boldsymbol{\sigma}_c$ is a row vector, which is different from almost all notion used here. Generally, $\boldsymbol{\sigma}$ is N by Q where N is length of the diffusion process vector and Q is the number of Brownian motions. Here $N = 1$ and Q is arbitrary.]
- Let the process for the vector of risky asset prices be $dS_t./S_t + D_t dt./S_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{z}_t$.

  Thus, the risk premium of these risky assets is $E_t[dS_t./S_t + D_t dt./S_t] - r_t \mathbb{1} = \boldsymbol{\mu}_t dt - r_t \mathbb{1} dt$
- Let time t utility be $\exp(-\beta t)u(c_t)$.
- Use Ito's Lemma to find $d(\exp(-\beta t)u'(c_t))$:
  - $\mu(c_t, t) = \mu_c c_t$ and $\boldsymbol{\sigma}(c_t, t) = \boldsymbol{\sigma}_c c_t$

– $f(c_t, t) = \exp(-\beta t)u'(c_t) \implies f_c = \exp(-\beta t)u''(c_t), f_{cc} = \exp(-\beta t)u'''(c_t), f_t = -\beta\exp(-\beta t)u'(c_t)$

$$\begin{aligned}
d(\exp(-\beta t)u'(c_t)) &= f_c\boldsymbol{\sigma}(c_t,t)d\mathbf{z}_t + \left[f_c\mu(c_t,t) + \frac{1}{2}f_{cc}\boldsymbol{\sigma}(c_t,t)\boldsymbol{\sigma}(c_t,t)^T + f_t\right]dt \\
&= \exp(-\beta t)u''(c_t)c_t\boldsymbol{\sigma}_c d\mathbf{z}_t + \Bigg[\exp(-\beta t)u''(c_t)\mu_c c_t \\
&+ \frac{1}{2}\exp(-\beta t)u'''(c_t)\boldsymbol{\sigma}_c\boldsymbol{\sigma}_c^T c_t^2 - \beta\exp(-\beta t)u'(c_t)\Bigg]dt \\
&= \exp(-\beta t)\left[u''(c_t)c_t\boldsymbol{\sigma}_c d\mathbf{z}_t + \left[u''(c_t)\mu_c c_t + \frac{1}{2}u'''(c_t)\boldsymbol{\sigma}_c\boldsymbol{\sigma}_c^T c_t^2 - \beta u'(c_t)\right]dt\right]
\end{aligned} \tag{7}$$

- Using marginal utility of a representative investor as the SDF in (**??**):

$$\begin{aligned}
\boldsymbol{\mu}_t dt - r_t\mathbb{1}dt &= -E_t\left[(dS_t./S_t)\frac{d(\exp(-\beta t)u'(c_t))}{\exp(-\beta t)u'(c_t)}\right] \\
\implies \boldsymbol{\mu}_t - r_t\mathbb{1} &= \frac{-1}{dt}E_t\left[(dS_t./S_t)\frac{d(\exp(-\beta t)u'(c_t))}{\exp(-\beta t)u'(c_t)}\right] \\
&= -E_t\left[(dS_t./S_t)\left(R_c\boldsymbol{\sigma}_c\frac{d\mathbf{z}_t}{dt} + R_c\mu_c + \frac{1}{2}\frac{u'''(c_t)}{u'(c_t)}\boldsymbol{\sigma}_c\boldsymbol{\sigma}_c^T c_t^2 - \beta\right)\right] \\
&= ... \\
&= R_c\boldsymbol{\sigma}_t\boldsymbol{\sigma}_c^T \\
&= R_c\mathbf{V}\mathbf{V}^{-1}\boldsymbol{\sigma}_t\boldsymbol{\sigma}_c^T \\
&= R_c\mathbf{V}\hat{\boldsymbol{\alpha}}_t
\end{aligned}$$

where $R_c = \frac{-c_t u''(c_t)}{u'(c_t)}$ is relative risk aversion, $\mathbf{V} = \boldsymbol{\sigma}_t\boldsymbol{\sigma}_t^T$, and $\hat{\boldsymbol{\alpha}}_t = \mathbf{V}^{-1}\boldsymbol{\sigma}_t\boldsymbol{\sigma}_c^T$ is a portfolio with maximum correlation with consumption.

- Thus,

$$\begin{aligned}
\boldsymbol{\mu}_t - r_t\mathbb{1} &= R_c\mathbf{V}\hat{\boldsymbol{\alpha}}_t \\
\implies \underbrace{\hat{\boldsymbol{\alpha}}_t^T\boldsymbol{\mu}_t}_{:=\hat{\mu}} - r_t\underbrace{\hat{\boldsymbol{\alpha}}_t^T\mathbb{1}}_{=1} &= R_c\hat{\boldsymbol{\alpha}}_t^T\mathbf{V}\hat{\boldsymbol{\alpha}}_t \\
\implies R_c &= \frac{\hat{\mu} - r_t}{\hat{\boldsymbol{\alpha}}_t^T\mathbf{V}\hat{\boldsymbol{\alpha}}_t} \\
\implies \boldsymbol{\mu}_t - r_t\mathbb{1} &= \underbrace{\frac{\mathbf{V}\hat{\boldsymbol{\alpha}}_t}{\hat{\boldsymbol{\alpha}}_t^T\mathbf{V}\hat{\boldsymbol{\alpha}}_t}}_{:=\hat{\boldsymbol{\beta}}}(\hat{\mu}_t - r_t)
\end{aligned}$$

The Instantaneous Risk-Free Rate

- Let per capita consumption follow process: $\frac{dc_t}{c_t} = \mu_c dt + \boldsymbol{\sigma}_c d\mathbf{z}_t$.
- Use the marginal utility of consumption as the discount factor and using (**??**),

$$
\begin{aligned}
r_t &= \frac{-1}{dt} E_t \left[\frac{d(\exp(-\beta t) u'(c_t)}{\exp(-\beta t) u'(c_t)} \right] \\
&= \beta - \frac{u''(c_t)}{u'(c_t)} \mu_c c_t - \frac{1}{2} \frac{u'''(c_t)}{u'(c_t)} \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T c_t^2
\end{aligned}
$$

- Example using CRRA utility

$$
\begin{aligned}
u(c_t) &= \frac{c_t^{1-B}}{1-B} \\
u'(c_t) &= c_t^{-B} \\
u''(c_t) &= -B c_t^{-B-1} \\
u'''(c_t) &= B(B-1) c_t^{-B-2} \\
r_t &= \beta - \frac{-B c_t^{-B-1}}{c_t^{-B}} \mu_c c_t - \frac{1}{2} \frac{B(B-1) c_t^{-B-2}}{c_t^{-B}} \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T c_t^2 \\
&= \beta + B\mu_c - \frac{1}{2} B(B-1) \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T \\
&= \beta + B\left(\mu_c + \frac{1}{2} \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T \right) - \frac{1}{2} B^2 \boldsymbol{\sigma}_c \boldsymbol{\sigma}_c^T
\end{aligned}
$$

The Risk-Free Rate is a State Variable

- Suppose a representative investor with time-additive logarithmic utility lives in an economy with one productive asset.
- Assume marginal utility is $u'(c_t, t) = \exp(-\beta t) c_t^{-1} = \exp(-\beta t - \ln c_t)$.
- Assume wealth in units of consumption follows $dW_t = W_t x_t dt + W_t \sigma \sqrt{x_t} dz_{1t} - c_t dt$.
- Assume the productivity rate x follows $dx_t = \kappa(\theta - x_t)dt + \sigma_x \sqrt{x_t} dz_{2t}$.
- In the equilibrium of CIR, consumption is $c_t = \beta W_t$ and it follows the subordinated process:

$$
d \ln c_t = -\beta dt + (1 - \frac{1}{2}\sigma^2) x_t dt + \sigma \sqrt{x_t} dz_{1t}
$$

As a consequence,

$$
r_t = \frac{-1}{dt} E_t \left[\frac{d(\exp(-\beta t) u'(c_t)}{\exp(-\beta t) u'(c_t)} \right] = (1 - \sigma^2) x_t
$$

and

$$
dr_t = \kappa(\theta_r - r_t)dt + \sigma_r \sqrt{r_t} dz_{2t}
$$

where $\theta_r = (1 - \sigma^2)\theta$ and $\sigma_r \sqrt{1 - \sigma^2} = \sigma_x$

3 Part III

Hamilton-Jacobi-Bellman Equation

- Consider the problem:

$$
\begin{aligned}
J(W_t, \mathbf{X}_t, t) = \max_{\{\boldsymbol{\alpha}_s, c_t\}} E_t\left[\int_{s=t}^{\infty} u(c_s, s)ds\right] \\
dW_t &= W_t(\boldsymbol{\alpha}_t^T(\boldsymbol{\mu}(\mathbf{X}_t, t) - r(\mathbf{X}_t, t)\mathbb{1}) + r(\mathbf{X}_t, t))dt + W_t\boldsymbol{\alpha}_t^T\boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{z}_t - c_t dt + y_t dt, \forall t; \\
d\mathbf{X}_t &= \boldsymbol{\mu}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}_x(\mathbf{X}_t, t)d\mathbf{z}_t
\end{aligned}
$$

where $W_t \geq 0$ and W_0 and $\mathbf{X}_0$ is given.

- Suppose the optimal policy for consumption and investment is followed for all times $s \geq t + \Delta$.

$$
\begin{aligned}
J(W_t, \mathbf{X}_t, t) &= \max_{\{\boldsymbol{\alpha}_s, c_t\}_{t \leq s < t+\Delta}} E_t\left[\int_{s=t}^{t+\Delta} u(c_s, s)ds + J(W_{t+\Delta}, \mathbf{X}_{t+\Delta}, t+\Delta)\right] \\
\implies 0 &= \max_{\{\boldsymbol{\alpha}_s, c_s\}_{t \leq s < t+\Delta}} \frac{1}{\Delta}\left[E_t[J(W_{t+\Delta}, \mathbf{X}_{t+\Delta}, t+\Delta)] - J(W_t, \mathbf{X}_t, t) + E_t\left[\int_{s=t}^{t+\Delta} u(c_s, s)ds\right]\right]
\end{aligned}
$$

- Taking the limit as $\Delta \to 0$, optimal policies satisfy:

$$
0 = \max_{\{\boldsymbol{\alpha}_t, c_t\}} E_t[dJ(W_t, \mathbf{X}_t, t)]/dt + u(c_t, t)
$$

$K + 2$ Fund Separation

- For simplicity, drop the explicit dependence of $\boldsymbol{\mu}, \boldsymbol{\mu}_x, r, \boldsymbol{\sigma}, \boldsymbol{\sigma}_x$ on $(\mathbf{X}_t, t)$.
- Applying Ito's lemma, the HJB is:

$$
0 = \max_{\{\boldsymbol{\alpha}, c\}} J_{\mathbf{x}}^T\boldsymbol{\mu}_x + J_W(W_t(\boldsymbol{\alpha}^T(\boldsymbol{\mu} - r\mathbb{1}) + r) - c + y_t + \frac{1}{2}tr\left(\begin{bmatrix} J_{\mathbf{xx}} & J_{\mathbf{x}W}^T \\ J_{\mathbf{x}W} & J_{WW}\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma}_x\sigma_x^T & W_t\boldsymbol{\alpha}^T\boldsymbol{\sigma}\boldsymbol{\sigma}_x^T \\ \boldsymbol{\sigma}_x\boldsymbol{\sigma}^T\boldsymbol{\alpha}W_t & \boldsymbol{\alpha}^T\boldsymbol{\sigma}\boldsymbol{\sigma}^T\boldsymbol{\alpha}W_t^2\end{bmatrix}\right) + J_t + u(c_t, t)
$$

The FOCs are:

$$
\begin{aligned}
0 &= u_c - J_w && [c] \\
\implies c_t^* &= u_c^{-1}(J_W(W_t, \mathbf{X}_t, t), t) \\
0 &= (\boldsymbol{\mu} - r\mathbb{1})J_W + \boldsymbol{\sigma}\boldsymbol{\sigma}_x^T J_{\mathbf{x}W} + \boldsymbol{\sigma}\boldsymbol{\sigma}^T\boldsymbol{\alpha}_t^* J_{WW}W_t && [\alpha] \\
\implies \boldsymbol{\alpha}_t^* &= T \underbrace{(\boldsymbol{\sigma}\boldsymbol{\sigma}^T)^{-1}(\boldsymbol{\mu} - r\mathbb{1})}_{\text{Maximum Sharpe Ratio portfolio}} + \underbrace{(\boldsymbol{\sigma}\boldsymbol{\sigma}^T)^{-1}\boldsymbol{\sigma}\boldsymbol{\sigma}_x^T}_{\text{Portfolios with maximum correlations with } \mathbf{X}_t} \mathbf{T_x} \\
\text{where } T &\equiv \frac{-J_W}{J_{WW}W_t} \\
\mathbf{T_x} &\equiv \frac{-J_{\mathbf{x}W}}{J_{WW}W_t}
\end{aligned}
$$

Maximum Correlation Portfolios

- Consider the portfolio with instantaneous return

$$\frac{dP_t}{P_t} = (\boldsymbol{\alpha}_t^T(\boldsymbol{\mu}(\mathbf{X}_t,t) - r(\mathbf{X}_t,t)\mathbb{1}) + r(\mathbf{X}_t,t))dt + \alpha_t^T\boldsymbol{\sigma}(\mathbf{X}_t,t)d\mathbf{z}_t$$

the state variables follow

$$d\mathbf{X}_t = \boldsymbol{\mu}_x(\mathbf{X}_t,t)dt + \boldsymbol{\sigma}_x(\mathbf{X}_t,t)d\mathbf{z}_t$$

where P_0 and X_0 are given.

- The correlation of the portfolio return with state variable j:

$$Cor\left(\frac{dP_t}{P_t}, dX_{jt}\right) = E\left(\frac{\frac{dP_t}{P_t}dX_{jt}}{\sqrt{\frac{(dP_t)^2}{P_t^2}}\sqrt{(dX_{jt})^2}}\right) = \frac{\boldsymbol{\alpha}_t^T\boldsymbol{\sigma}\boldsymbol{\sigma}_{jx}^T}{\sqrt{\boldsymbol{\alpha}_t^T\boldsymbol{\sigma}_t\boldsymbol{\sigma}^T\boldsymbol{\alpha}}\sqrt{\boldsymbol{\sigma}_{jx}\boldsymbol{\sigma}_{jx}^T}}$$

- The portfolio with maximum correlation is

$$\arg\max_{\boldsymbol{\alpha}} Cor\left(\frac{dP_t}{P_t}, dX_{jt}\right) \propto (\sigma\sigma^T)^{-1}\boldsymbol{\sigma}\boldsymbol{\sigma}_{jx}^T$$

The Value Function

- For function $J(w,\mathbf{x},t)$, policies c and $\boldsymbol{\alpha}$, and the processes for W_t and $\mathbf{X}_t$, the operator $D^{(c,\boldsymbol{\alpha})}$ is convenient shorthand, where:

$$D^{(c,\boldsymbol{\alpha})}J(w,\mathbf{x},t) \equiv J_{\mathbf{x}}^T\boldsymbol{\mu}_x + J_w(w(\boldsymbol{\alpha}^T(\boldsymbol{\mu}-r\mathbb{1})+r)-c+y_t + \frac{1}{2}tr\left(\begin{bmatrix} J_{\mathbf{xx}} & J_{\mathbf{x}w}^T \\ J_{\mathbf{x}w} & J_{ww}\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma}_x\sigma_x^T & w\boldsymbol{\alpha}^T\boldsymbol{\sigma}\boldsymbol{\sigma}_x^T \\ \boldsymbol{\sigma}_x\boldsymbol{\sigma}^T\boldsymbol{\alpha}w & \boldsymbol{\alpha}^T\boldsymbol{\sigma}\boldsymbol{\sigma}^T\boldsymbol{\alpha}w^2\end{bmatrix}\right) + J_t$$

Thus, the HJB is:

$$0 = \max_{\{\boldsymbol{\alpha},c\}} D^{(c,\boldsymbol{\alpha})}J(w,\mathbf{x},t) + u(c,t)$$

Given the optimal policies c_t^* and $\boldsymbol{\alpha}_t^*$, the value function solves:

$$0 = D^{(c_t^*,\boldsymbol{\alpha}_t^*)}J(W_t,\mathbf{X},t) + u(c_t^*,t)$$

Example - CRRA with no labor income, constant risk-free rate, and no risky assets

- Consider the problem:

$$J(W_t,t) \equiv \max_{\{c_s\}_{t\leq s}} E_t\left[\int_{s=t}^{\infty} e^{-\rho s}\frac{c_s^{1-B}}{1-B}ds\right]$$
$$dW_t = W_t r dt - c_t dt, \forall t$$

where $W_t \geq 0$ and W_0 is given.

- HJB: $0 = \max_{c_t} J_W(W_t r - c_t) + J_t + e^{-\rho t}\frac{c_t^{1-B}}{1-B}$
- FOC: $e^{-\rho t}c_t^{-B} = J_W$
- Guess: $J(W_t,t) = e^{-\rho t}H^B\frac{W_t^{1-B}}{1-B}$, where $H = W/C$ is the wealth-consumption ratio.
- Check:

$$
\begin{aligned}
J_W &= e^{-\rho t}H^B W_t^{-B} \\
\implies c_t &= H^{-1}W_t \text{ and } J_t = -\rho e^{-\rho t}H^B\frac{W_t^{1-B}}{1-B} \\
\implies 0 &= e^{-\rho t}H^B W_t^{-B}(W_t r - H^{-1}W_t) - \rho e^{-\rho t}H^B\frac{W_t^{1-B}}{1-B} + e^{-\rho t}\frac{H^{-1+B}W_t^{1-B}}{1-B} \\
\implies H^{-1} &= \frac{1}{B}\rho - \frac{1-B}{B}r
\end{aligned}
$$

Example II - CRRA with geometric labor income, constant risk-free rate, and no risky assets

- Consider the problem:

$$
\begin{aligned}
J(W_t, y_t t) &\equiv \max_{\{c_s\}_{t\leq s}} E_t\left[\int_{s=t}^{\infty} e^{-\rho s}\frac{c_s^{1-B}}{1-B}ds\right] \\
dW_t &= W_t r dt - c_t dt + y_t dt, \forall t \\
dy_t &= g y_t dt
\end{aligned}
$$

where $W_t \geq 0$ and W_0 and y_0 are given.

- Notice that $y_t = y_0 e^{gt}$.
- HJB: $0 = \max_{c_t} J_W(W_t r - c_t + y_t) + J_t + e^{-\rho t}\frac{c_t^{1-B}}{1-B}$
- FOC: $e^{-\rho t}c_t^{-B} = J_W$
- Guess: $J(W_t,t) = e^{-\rho t}H^B\frac{(W_t+Y_t)^{1-B}}{1-B}$, where $H = W/C$ and $Y_t = \frac{y_t}{r-g}$
- Check:

$$
\begin{aligned}
J_W &= e^{-\rho t}H^B(W_t+Y_t)^{-B} \\
\implies c_t &= H^{-1}(W_t+Y_t) \\
\text{and } J_t &= -\rho e^{-\rho t}H^B\frac{(W_t+Y_t)^{1-B}}{1-B} + \rho e^{-\rho t}H^B(W_t+Y_t)^{-B}gY_t \\
\implies 0 &= e^{-\rho t}H^B(W_t+Y_t)^{-B}(W_t r - H^{-1}(W_t+Y_t) + Y_t(r-g)) - \rho e^{-\rho t}H^B\frac{(W_t+Y_t)^{1-B}}{1-B} \\
&\quad + e^{-\rho t}H^B(W_t+Y_t)^{-B}gY_t + e^{-\rho t}\frac{H^{-1+B}(W_t+Y_t)^{1-B}}{1-B} \\
\implies H^{-1} &= \frac{1}{B}\rho - \frac{1-B}{B}r
\end{aligned}
$$

Example III - Log utility with one risky asset and mean-reverting state variable

- Consider the problem:

$$
\begin{aligned}
J(W_t, x_t, t) \equiv & \max_{\{c_s,\alpha_s\}_{t\le s}} E_t\left[\int_{s=t}^{\infty} e^{-\rho s}\log(c_s)ds\right] \\
dW_t &= W_t(\alpha_t(x_t - r(x_t)) + r(x_t))dt + W_t\alpha_t\sigma\sqrt{x_t}dz_{1t} - c_t dt, \forall t \\
dx_t &= \kappa(\theta - x_t) + \sigma_x\sqrt{x_t}dz_{2t}
\end{aligned}
$$

where $W_t \geq 0$, W_0 and x_0 is given, and z_1 independent of z_2.

- HJB:

$$
0 = \max_{c_t,\alpha_t} J_W(W_t(\alpha_t(x_t - r_t) + r_t) - c_t) + J_t + e^{-\rho t}\log(c_t) + \frac{1}{2}J_{WW}W_t^2\alpha_t^2\sigma^2 x_t + J_x\kappa(\theta - x_t) + \frac{1}{2}J_{xx}\sigma_x^2 x_t
$$

- Guess: $J(W_t, x_t, t) = e^{-\rho t}\frac{\log(W_t)}{\rho} + f(x_t)$
- FOC

$$
\begin{aligned}
& 0 = -e^{-\rho t}\frac{1}{\rho W_t} + e^{-\rho t}\frac{1}{c_t} && [c] \\
\implies & c_t = \rho W_t \\
& 0 = -e^{-\rho t}\frac{1}{\rho W_t}W_t(x_t - r_t) + \frac{1}{2}e^{-\rho t}\frac{1}{\rho W_t^2}2\alpha_t\sigma^2 x_t && [\alpha_t] \\
\implies & \alpha_t = \frac{x_t - r_t}{\sigma^2 x_t}
\end{aligned}
$$

- Thus, consumption is a constant wealth-consumption ratio because of log utility.
- HBJ:

$$
0 = \frac{1}{\rho}\left[\frac{(x_t - r_t)^2}{x_t^2\sigma^2} + r_t - \rho\right] + \log(\rho) - \frac{1}{2}\frac{1}{\rho}\frac{(x_t - r_t)^2}{x_t^2\sigma^2} + f_x\kappa(\theta - x_t) + \frac{1}{2}f_{xx}\sigma_x^2 x_t
$$

- One possible solution is $\alpha_t = 1 \implies r_t = x_t(1 - \sigma^2)$

Example IV - CRRA on terminal wealth with single risky asset and constant risk-free rate

- Consider the problem:

$$
\begin{aligned}
J(W_t, t) \equiv & \max_{\{\alpha_s\}_{t\le s\le T}} E_t\left[\frac{W_T^{1-B}}{1-B}\right] \\
dW_t &= W_t(\alpha_t(\mu - r) + r)dt + W_t\alpha_t\sigma dz_t, \forall t
\end{aligned}
$$

where $0 < B \neq 1$, $W_t \geq 0$, and W_0 is given.

- HJB: $0 = \max_{\alpha_t} J_W W_t(\alpha_t(\mu - r) + r) + J_t + \frac{1}{2} J_{WW} W_t^2 \alpha_t^2 \sigma^2$
- Guess: $J(W_t, t) = e^{-k(T-t)} \frac{W_t^{1-B}}{1-B}$
- FOC:

$$
\begin{aligned}
0 &= e^{-k(T-t)} W_t^{1-B}(\mu - r) - \frac{1}{2} B e^{-k(T-t)} W_t^{1-B} 2\alpha_t \sigma^2 \\
\implies \alpha_t &= \frac{\mu - r}{B\sigma^2}
\end{aligned}
$$

- HJB:

$$
\begin{aligned}
0 &= e^{-k(T-t)} W_t^{1-B} \left(\frac{(\mu - r)^2}{B\sigma^2} + r \right) + k e^{-k(T-t)} \frac{W_t^{1-B}}{1-B} - \frac{1}{2} B e^{-k(T-t)} W_t^{1-B} \left(\frac{\mu - r}{B\sigma^2} \right)^2 \sigma^2 \\
k &= -(1-B) \left(\frac{1}{2} \frac{(\mu - r)^2}{B\sigma^2} + r \right)
\end{aligned}
$$

Intertemporal CAPM (ICAPM)

- By $K+2$ fund separation, the optimal portfolio for individual i is:

$$
\boldsymbol{\alpha}_t^i W_t^i = \underbrace{(\boldsymbol{\sigma}\boldsymbol{\sigma}^T)^{-1}(\boldsymbol{\mu}_t - r_t \mathbb{1})}_{\text{Maximum Sharpe ratio portfolio}} T^i W_t^i + \underbrace{(\boldsymbol{\sigma}\boldsymbol{\sigma}^T)^{-1} \boldsymbol{\sigma}\boldsymbol{\sigma}_x^T}_{\text{Maximum correlation portfolios}} \mathbf{T}_{\mathbf{x}}^i W_t^i
$$

where $T \equiv \frac{-J_W}{J_{WW} W_t}$ and $\mathbf{T}_{\mathbf{x}} \equiv \frac{-J_{\mathbf{x}W}}{J_{WW} W_t}$.

- The market portfolio $\boldsymbol{\alpha}_t^m$ is:

$$
\begin{aligned}
\boldsymbol{\alpha}_t^m &\equiv \sum_i \boldsymbol{\alpha}_t^i W_t^i / W_t^m = (\boldsymbol{\sigma}\boldsymbol{\sigma}^T)^{-1}(\boldsymbol{\mu}_t - r_t \mathbb{1}) T^m + \boldsymbol{\alpha}_{MCP,t} \mathbf{T}_{\mathbf{x}}^m \\
\text{where } T^m &\equiv \sum_i T^i W^i / W^m \\
\mathbf{T}_{\mathbf{x}}^m &\equiv \sum_i \mathbf{T}_{\mathbf{x}}^i W^i / W^m \\
\boldsymbol{\alpha}_{MCP,t} &\equiv (\boldsymbol{\sigma}\boldsymbol{\sigma}^T)^{-1} \boldsymbol{\sigma}\boldsymbol{\sigma}_x^T
\end{aligned}
$$

- Expected returns are:

$$
\begin{aligned}
\boldsymbol{\mu}_t - r_t \mathbb{1} &= \boldsymbol{\sigma}\boldsymbol{\sigma}^T \begin{bmatrix} \boldsymbol{\alpha}_t^m & \boldsymbol{\alpha}_{MCP,t} \end{bmatrix} \begin{bmatrix} 1/T^m \\ -\mathbf{T}_{\mathbf{x}}^m / T^m \end{bmatrix} \\
&= \mathbf{Cov}(\mathbf{Var})^{-1} \begin{bmatrix} \mu_{mt} - r_t \\ \boldsymbol{\mu}_{MCP,t} - r_t \mathbb{1} \end{bmatrix} \\
\mathbf{Cov} &\equiv \boldsymbol{\sigma}\boldsymbol{\sigma}^T \begin{bmatrix} \boldsymbol{\alpha}_t^m & \boldsymbol{\alpha}_{MCP,t} \end{bmatrix} \\
\mathbf{Var} &\equiv \begin{bmatrix} \boldsymbol{\alpha}_t^m & \boldsymbol{\alpha}_{MCP,t} \end{bmatrix}^T \boldsymbol{\sigma}\boldsymbol{\sigma}^T \begin{bmatrix} \boldsymbol{\alpha}_t^m & \boldsymbol{\alpha}_{MCP,t} \end{bmatrix}
\end{aligned}
$$

Stochastic Differential Utility

- Consider the problem:

$$
\begin{aligned}
J(W_t, \mathbf{X}_t) \equiv \max_{\{\boldsymbol{\alpha}_s, c_t\}_{t\le s}} & E_t\left[\int_{s=t}^{\infty} f(c_s, J(W_s, \mathbf{X}_s)ds\right] \\
dW_t &= W_t(\boldsymbol{\alpha}_t^T(\boldsymbol{\mu}(\mathbf{X}_t, t) - r(\mathbf{X}_t, t)\mathbb{1}) + r(\mathbf{X}_t, t))dt + W_t\boldsymbol{\alpha}_t^T\boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{z}_t - c_t dt + y_t dt, \forall t; \\
d\mathbf{X}_t &= \boldsymbol{\mu}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}_x(\mathbf{X}_t, t)d\mathbf{z}_t
\end{aligned}
$$

where $W_t \geq 0$, W_0 and $\mathbf{X}_0$ is given, and where the felicity function is defined

$$f(c, J) = \phi\theta\left(\frac{c^{1-1/\psi}}{((1-\gamma)J)^{1/\theta}} - 1\right)J$$

with $\theta = \frac{1-\gamma}{1-1/\psi}$, rate of time preference if ϕ, risk aversion is γ, and elasticity of intertemporal substitution is ψ.

- The HJB equation now is:

$$0 = \max_{\boldsymbol{\alpha}_t, c_t} E_t dJ(W_t, \mathbf{X}_t)\frac{1}{dt} + f(c_t, J(W_t, \mathbf{X}_t)).$$

- Standard result:

$$
\begin{aligned}
J(W_t, \mathbf{X}_t) &= \frac{W_t^{1-\gamma}}{1-\gamma}(\phi H_t^{1/\psi})^\theta \\
\text{where } H_t = H(\mathbf{X}_t) &= \frac{W_t}{c_t}
\end{aligned}
$$

Special Case: Time Additive

- Set $\gamma = 1/\psi$ and $\theta = 1$, implying $f(c, J) = \phi(\frac{c^{1-\gamma}}{1-\gamma} - J)$.
- Under an optimal policy,

$$
\begin{aligned}
0 &= E_t\left[dJ_t + \phi\left(\frac{c_t^{1-\gamma}}{1-\gamma} - J_t\right)dt\right] \\
&= E_t\left[d(e^{-\phi t}J_t)\right] + \phi e^{-\phi t}\frac{c_t^{1-\gamma}}{1-\gamma}dt \\
&= E_t\left[\int_{s=t}^{\infty} d(e^{-\phi s}J_s)\right] + \phi E_t\int_{s=t}^{\infty} e^{-\phi s}\frac{c_s^{1-\gamma}}{1-\gamma}ds
\end{aligned}
$$

[second step is introduce factor of integration, third step is integrate and use iterated expectations.]

- Assume transversality conditions holds.

$$\implies J(W_t, \mathbf{X}_t) = \phi E_t\left[\int_{s=t}^{\infty} e^{\phi(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right]$$

Example V

Setting $\psi \to 1$ with SDU:

$$
\begin{aligned}
J(W_t, y_t) &= \max_{\{\alpha_s, c_s\}_{t\le s}} E_t\left[\int_{s=t}^{\infty} f(c_s, J(W_s, y_s))ds\right] \\
\text{Felicity: } f(c, J) &= \phi(1-\gamma)\left(\log(c) - \frac{1}{1-\gamma}\log((1-\gamma)J)\right)J \\
\text{Wealth: } dW_t &= W_t(\alpha_t(\mu - r) + r)dt + W_t\alpha_t\sqrt{\frac{1}{y_t}}dz_{1t} - c_t dt, \forall t \\
\text{Precision: } dy_t &= \kappa(\bar{y} - y_t)dt + \sigma\sqrt{y_t}dz_{2t}
\end{aligned}
$$

where $W_t \geq 0$, W_0 given, and $dz_{1t}dz_{2t} = \rho dt$.

- HJB:

$$
0 = \max_{\alpha_t, c_t} J_W(W(\alpha_t(\mu - r) + r) - c_t) + h(c_t, J) + \frac{1}{2}J_{WW}W_t^2\alpha_t^2\frac{1}{y_t} + J_y\kappa(\theta - y_t) + \frac{1}{2}J_{yy}\sigma^2 y_t + J_{Wy}W_t\alpha_t\rho\sigma.
$$

- Guess: $J(W, y) = \frac{W^{1-\gamma}}{1-\gamma}h(y)$
- Solution:

$$
\begin{aligned}
c_t &= \phi W_t \\
\alpha_t &= \frac{1}{\gamma}(\mu - r)y_t + (1 - 1/\gamma)(-\rho)\sigma\frac{A}{1-\gamma}y_t \\
h(y) &= \exp(Ay + B)
\end{aligned}
$$

where A and B solve a system of quadratic equations.

Challenge

- In Ai (JF, 2010), an investor chooses consumption and savings when capital (wealth) follows CRS process.
- The productivity rate of capital x_t is unobservable and the investor continuously updates a posterior distribution that has mean m_t.
- The investor solves

$$
\begin{aligned}
J(K, m_t) &= \max_{\{c_s\}_{t\le s}} E_t\left[\int_{s=t}^{\infty} f(c_t, J(K_t, m_t))ds\right] \\
dK_t &= K_t m_t dt - c_t dt + K_t\sigma dz_t \\
dm_t &= \kappa(\mu - m_t)dt + \sigma_m dz_t^m
\end{aligned}
$$

where $K_t \geq 0$, K_0 and m_0 given, $dz_t dz_t^m = \rho dt$, $\sigma > 0, \kappa > 0, \mu > 0, \sigma_m > 0, \rho > 0$ constants.

- To do:

1. Write out the HJB equation.
2. Show that the standard result for J_t holds, where H is the wealth consumption ratio.
3. Show that the HJB (given the optimal consumption policy) is a partial differential equation in H as a function of m. (The HJB has no known general solution.)
4. Assume a special case holds: $m_t = \mu$, a constant. Solve for H as a constant, and show that it satisfies the Constant Dividend Growth Model of Gordon.

4 Part IV

Setup

- We assume:

  1. A probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.
  2. A d-dimensional Brownian motion $\mathbf{z} : \Omega \times [0, \infty) \to \mathbb{R}^d$.
  3. An investor's information (filtration $\mathcal{F}_t$) is defined by the history of $\mathbf{z}$.

- All adapted processes are Ito process with increments written:

$$dS_t = \mu(\omega, t)dt + \sigma(\omega, t)d\mathbf{z}_t$$

- If $S_t > 0$ assumed for all t, an alternative expression is $dS_t./S_t = \mu(\omega, t)dt + \sigma(\omega, t)d\mathbf{z}_t$ with $\mu(\omega, t)$ and $\sigma(\omega, t)$ appropriately redefined.

- Any scalar martingale satisfies $dM_t = \gamma(\omega, t)d{\cdot}\mathbf{z}_t$ and any positive martingale satisfies $dM_t = M_t\gamma(\omega, t)\cdot d\mathbf{z}_t$ for some adapted $\gamma(\omega, t)$.

Deflated Gains and SDF

- A deflator is a process Y such that $P(Y_t > 0) = 1$ for all t. A deflator changes the numeraire.

- For example if $Y_t = \exp(-\int_0^t r_s ds)$, which is the inverse of gains rolling over at the risk-free rate r_s, $S^Y := YS$ is the (vector) of units of an account with value $\exp(\int_0^t r_s ds$ required to buy one unit of each asset at the prices in S_t.

- A deflated gains process if $G_t^Y := S_t Y_t + \int_{s=0}^t D_s Y_s ds$.

- Another example: A stochastic discount factor is a deflator π such that G_t^π is a martingale: $E_t(G_s^\pi) = G_t^\pi, t < s$.

- If $dS_t./S_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma} d\mathbf{z}_t$ and if G_t^Y is a martingale with $Y_t = \exp(-\int_0^t r_s ds)$, then

$$0 = E_t dG_t^Y = S_t Y_t(-r_t \mathbb{1} dt + E_t[dS_t./S_t + D_t./S_t dt]) \implies r_t \mathbb{1} = \boldsymbol{\mu}_t + D_t./S_t$$

- Let a pure discount bond have payoff of 1 at time $T > 0$ and price P_t. If the deflated gains are a martingale, and if the risk-free rate is Markov, say $dr_t = \mu(r_t)dt + \sigma(r_t) \cdot d\mathbf{z}_t$,

$$
\begin{aligned}
P_t \exp\Bigg(- \int_0^t r_s ds \Bigg) &= E_t\Bigg(\exp\Bigg(- \int_0^t r_s ds \Bigg)\Bigg) \\
P_t &= E_t\Bigg(\exp\Bigg(- \int_t^T r_s ds \Bigg)\Bigg) \\
&:= p(r_t, t) \\
p(r_t, t) r_t &= p_r \mu(r_t) + \frac{1}{2} p_{rr} \sigma(r_t) \cdot \sigma(r_t) + p_t
\end{aligned}
$$

Arbitrage

- A self-financing portfolio $\theta_t, 0 \geq t \geq T$ (trading strategy) is an arbitrage if for any $T > 0$ either:
  1. $W_0 \leq 0, W_t \geq 0$ with probability one, and $W_t > 0$ with positive probability, or
  2. $W_0 < 0$ and $W_t \geq 0$ with probability one.
- A trading strategy is self-financing if $\theta_t \cdot S_t = \theta_0 \cdot S_0 + \int_{v=0}^t \theta_v \cdot dS_v + \int_{v=0}^t \theta_v \cdot D_v dv$, which is alternatively written $d(\theta_t \cdot S_t) = \theta_t \cdot dS_t + \theta_t \cdot D_t dt$.
- Numeraire Invariance Theorem: A trading strategy θ is self-financing with respect to the gains process G iff it is self-financing with respect to the deflated process G^Y for any positive deflator.
- Corollary. A trading strategy is an arbitrage with respect to G iff it is an arbitrage with respect to G^Y.

No Arbitrage

- Consider two spaces of trading strategies, $\mathcal{H}^2(S^\pi)$ and $\underline{\Theta}(S^\pi)$, where

$$
\mathcal{H}^2(S^\pi) := \begin{cases} \theta \text{ adapted;} \\ P(\int_0^\tau |\theta_t \cdot \mu_t \pi_t| dt < \infty) = 1; \\ E[(\int_0^\tau \theta_t \cdot \mu_t \pi_t dt)^2] < \infty; \\ P(\int_0^\tau (\theta_t \cdot \mu_t \pi_t)^2 dt < \infty) = 1; \\ E[(\int_0^\tau (\theta_t \cdot \mu_t \pi_t)^2 dt] < \infty, \\ \text{for } \tau > 0 \end{cases}
$$

$$
\underline{\Theta}(S^\pi) := \begin{cases} \theta \text{ adapted;} \\ P(\theta_t \cdot S_t \pi_t \geq k) = 1, t > 0, \text{ for constant } k \end{cases}
$$

- Proposition: Assume dividends are zero. If a sdf π exists such that $E_t(S_s \pi_s) = S_t \pi_t, s > t$, there is no arbitrage in either $\mathcal{H}^2(S^\pi)$ or $\underline{\Theta}(S^\pi)$.

Doubling Strategy

- For a stock with driftless price S with $dS_t = S_t dz_t$, and bond with price $\beta_t = 1$ (and zero risk-free rate), consider the trading strategy $\theta_t(a_t, b_t)$ where

$$a_t = \begin{cases} \frac{1}{S_t\sqrt{T-t}}, & t \le \tau \\ 0, & \tau < t \end{cases}$$

$$b_t = -a_t S_t + \int_{v=0}^{t} a_v S_v$$

$$\tau = \inf\{t : \int_{v=0}^{\tau} \frac{1}{\sqrt{T-v}} dz_v = \alpha > 0\}$$

- For this strategy, $W_0 = b_0 + a_0 S_0 = 0$, while $W_\tau = b_\tau + a_\tau S_\tau = \int_{v=0}^{\tau} a_v dS_v = \alpha$.
- Also, $P(0 < \tau < T) = 1$.

Equivalent Martingale Measure

- We say the deflated price and dividend processes admit an equivalent martingale measure if there is a measure Q equivalent to P such that $E_t^Q(G_T^Y) = G_t^Y$ for all dates $T > t$.
- Theorem: Assume dividends are zero. If the deflated gains process admits an equivalent martingale measures, there is no arbitrage in either $\mathcal{H}^2(S^\pi)$ or $\underline{\Theta}(S^\pi)$.

SDF and Measure Q

- Assume zero dividends. The value $\xi_t = \exp(\int_0^t r_s ds)\frac{\pi_t}{\pi_0}$ is the density of an equivalent martingale measure.
- We have $E_t(\xi_T) = \xi_t$ and

$$S_t = E_t[\frac{\xi_T}{\xi_t}\exp(-\int_t^T r_s ds)S_T] = E_t^Q(\exp(-\int_t^T r_s ds)S_T)$$

- Alternatively, if ξ_t is the density for an equivalent martingale measure, meaning $E_t(\xi_T) = \xi_t$ and $S_t = E_t^Q(\exp(-\int_t^T r_s ds)S_T), \pi_r = \exp(-\int_0^t r_s ds)\xi_t$ is a stochastic discount factor and $\pi_t S_t = E_t(\pi_T S_T)$.

Girsanov's Theorem

- Because ξ_t is adapted, positive almost surely, and is a martingale, $d\xi_t = \gamma_t \cdot d\mathbf{z}_t = -\xi_t\eta_t \cdot d\mathbf{z}_t$, where $\eta_t := -\frac{\gamma_t}{\xi_t}$ for some (vector) adapted process γ_t, and $\xi_t = \exp(-\int_0^t \eta_s \cdot d\mathbf{z}_s - \frac{1}{2}\int_0^t \eta_s \cdot \eta_s ds)$.
- Also, $\frac{d\pi_t}{\pi_t} = -r_t dt - \eta_t \cdot d\mathbf{z}_t$ where $\pi_t = \exp(-\int_0^t r_s ds)\xi_t$.
- Girsanov's Theorem: Under measure Q, $\mathbf{z}_t^Q = \mathbf{z}_t + \int_0^t \eta_s ds$ is standard Brownian motion.

Market Prices of Risk

- Under $P : dS_t./S_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{z}_t$, and

$$\frac{1}{\exp(-\int_0^t r_s ds)S_t} d(\exp(-\int_0^t r_s ds)S_t) = (\boldsymbol{\mu}_t - r_t\mathbb{1})dt + \boldsymbol{\sigma}(d\mathbf{z}_t + \eta_s^T dt - \eta_s^T dt) = (\boldsymbol{\mu}_t - r_t - \boldsymbol{\sigma}_t\eta_s^T)dt + \boldsymbol{\sigma}_t d\mathbf{z}_t^Q$$

- Under Q: $dS_t./S_t = \boldsymbol{\mu}_t^Q dt + \boldsymbol{\sigma}_t d\mathbf{z}_t^Q$, where $\boldsymbol{\mu}_t^Q = \boldsymbol{\mu}_t - \boldsymbol{\sigma}_t\eta_s^T = r_t\mathbb{1}$ and

$$\frac{1}{\exp(-\int_0^t r_s ds)S_t} d(\exp(-\int_0^t r_s ds)S_t) = \boldsymbol{\sigma} d\mathbf{z}_t^Q$$

The Risk-Free Rate is a State Variable

- Suppose a representative investor with time-additive logarithmic utility lives in an economy with one productive asset.
  1. Marginal utility is $u'(c_t, t) = \exp(-\beta t)c_t^{-1} = \exp(-\beta t - \ln c_t)$.
  2. Wealth in units of consumptions follows $dW_t = W_t x_t dt + W_t \sigma_w \sqrt{x_t} dz_{1t} - c_t dt$.
  3. The productivity rate x follows $dx_t = \kappa(\theta_x - x_t)dt + \sigma_x \sqrt{x_t}(\rho dz_{1t} + \sqrt{1-\rho^2} dz_{2t})$.
- In the equilibrium of CIR, consumption is $c_t = \beta W_t$ and it follows the subordinated process:

$$d \ln c_t = -\beta dt + (1 - \frac{1}{2}\sigma_w^2)x_t dt + \sigma_w \sqrt{x_t} dz_{1t}$$

- As a consequence,

$$\begin{aligned}
\frac{d\pi_t}{\pi_t} &= \beta dt - d\ln c_t + \frac{1}{2}(d \ln c_t)^2 \\
&= -(1-\sigma_w^2)x_t dt - \sigma_w \sqrt{x_t} dz_{1t} \\
&:= -\delta x_t dt - \eta_t dz_{1t} \\
r_t &= -E_t \frac{d\exp(-\beta t - \ln c_t)}{\exp(-\beta t - \ln c_t)dt} \\
&= \delta x_t, \\
dr_t &= \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}(\rho dz_{1t} + \sqrt{1-\rho^2} dz_{2t}) \\
&= \kappa(\theta - r_t)dt - \sigma\sqrt{r_t}\rho\eta_t dt + \sigma\sqrt{r_t}(\rho(dz_{1t} + \eta_t dt) + \sqrt{1-\rho^2} dz_{2t}) \\
&= \kappa(\theta - r_t)dt - \lambda r_t dt + \sigma\sqrt{r_t} dz^Q
\end{aligned}$$

where $\theta = \delta\theta_x$ and $\sigma = \sqrt{\delta}\sigma_x$.

Cox, Ingersoll, and Ross

- Consider a short rate that follows the process:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t} dz,$$

where z is scalar Brownian motion under the statistical measure P.

- Suppose also that under the risk-neutral measure Q, the short rate follows

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t} dz^Q - \lambda r_t dt,$$

where λ is a constant and z^Q is scalar Brownian motion. Then the price of a discount bond maturing at date T is:

$$P(t,T) = \exp(-YTM(t,T)(T-t) = E_t^Q(\exp(-\int_t^T r_s ds)) = A(t,T)\exp(-B(t,T)r_t)$$

$$\begin{aligned}
\text{where } A(t,T) &\equiv \left[\frac{2\gamma\exp([(\kappa+\lambda+\gamma)(T-t)]/2)}{(\gamma+\kappa+\lambda)(\exp(\gamma(T-t))-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2} \\
B(t,T) &\equiv \frac{2(\exp(\gamma(T-t)-1)}{(\gamma+\kappa+\lambda)(\exp(\gamma(T-t))-1)+2\gamma} \\
\gamma &\equiv ((\kappa+\lambda)^2 + 2\sigma^2)^{1/2}
\end{aligned}$$

Solution

- Three alternative methods of solution:
  1. Calculate the expectation
  2. Discretize the Ito process for r_t, and use Monte Carlo to approximate the expectation
  3. Solve a p.d.e.

- Suppose the bond price is a function $P(t,T) = p(r_t, t)$ that satisfies the conditions required for Ito's lemma.

  Then $p(r_t, t) = E_t^Q(\exp(-\int_t^T r_s ds))$, with $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz^Q - \lambda r_t dt$.

  By Feynman-Kac,

$$0 = p_r(\kappa\theta - (\kappa + \lambda)r) + p_{rr}\frac{1}{2}\sigma^2 r + p_t - pr$$

  with $p(r_T, T) = 1$ for all r_T.

- Discretize: $r_{i,j\Delta} = \kappa(\theta - r_{i,(j-1)\Delta})\Delta - \lambda r_{i,(j-1)\Delta}\Delta + \sigma\sqrt{r_{i,(j-1)\Delta}}\sqrt{\Delta}z_{i,j}$, $r_{i,0} = r_t$, where $j = 1, ..., n$ with $\Delta = (T-t)/n; i = 1, ..., m$, and $z_{i,j} \sim N(0,1)$ for all i, j.

- Approximate: $E_t^Q(\exp(-\int_t^T r_s ds)) \approx \frac{1}{m}\sum_{i=1}^m \exp(-\frac{1}{2}\sum_{j=1}^n (r_{i,j\Delta} + r_{i,(j-1)\Delta})\Delta)$