ECON 703 - PS 6

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(1) Alices lives on the farm "Happy Cow", which is located in the forest 5 kilometers from the main road. Bob lives in the house, located on the main road and 13 kilometers from Alice's farm. Bob wants to visit his friend Alice. He walks with the speed 5km/hour on the road and 3km/hour in the forest. What is the smallest time Bob needs to reach "Happy Cow" from home?

Define x as the distance Bob travels on the road, y as the distance he travels through the forest, and $T = x/5 + y/3$ as the total time it takes him to walk to "Happy Cow". Since the path through the forest that is perpendicular to the road starts at $\sqrt{13^2 - 5^2} = 12$, we can write y as:

$$y = \sqrt{5^2 + (12 - x)^2} = \sqrt{169 - 24x + x^2}$$

So $T = x/5 + (1/3)\sqrt{169 - 24x + x^2}$. Setting the first order condition to zero:

$$\begin{aligned}
\frac{dT}{dx} &= 1/5 + (1/3)(1/2)(169 - 24x + x^2)^{-1/2}(2x - 24) \\
0 &= 1/5 + (1/3)(1/2)(169 - 24x + x^2)^{-1/2}(2x - 24) \\
-5(2x - 24) &= 6\sqrt{169 - 24x + x^2} \\
\frac{120 - 10x}{6} &= \sqrt{169 - 24x + x^2} \\
\frac{14400 - 2400x + 100x^2}{36} &= 169 - 24x + x^2 \\
14400 - 2400x + 100x^2 &= 6084 - 864x + 36x^2 \\
8316 - 1536x + 64x^2 &= 0 \\
4(4x - 33)(4x - 63) &= 0
\end{aligned}$$

The roots are $(33/4, 63/4)$, but the second root is larger than 12. Thus, $x = 33/4$, so $y = \sqrt{169 - 24x + x^2} = 25/4$ and $T = (33/4)/5 + (25/4)/3 = 224/60 \approx 3.733$ hours.

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(2) Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $B_\varepsilon(x_0)$ for some $x_0 \in \mathbb{R}, \varepsilon > 0$. Suppose also that $f'(x) < 0$ for any $x \in B_\varepsilon(x_0) \setminus \{x_0\}$. Can the point x_0 be a local maximum or minimum of f?

No, the point x_0 cannot be a local maximum or minimum of f.

Proof: Fix $\varepsilon > 0$. Assume for the sake a contradiction that x_0 is a local maximum. Then, choose $x_1 \in (x_0 - \varepsilon, x_0)$. By the mean value theorem, there exists a x_2 such that

$$f(x_1) - f(x_0) = f'(x_2)(x_1 - x_0)f(x_1) - f(x_0) = f'(x_2)(x_0 - \varepsilon - x_0)f(x_1) - f(x_0) = f'(x_2)(-\varepsilon)$$

Since $x_2 \in B_\varepsilon(x_0)$, $f'(x_2) < 0$, so the $f(x_1) - f(x_0) > 0 \implies f(x_1) > f(x_0)$. This is a contradiction because x_0 is assumed to be a local maximum.

Similarly, assume for the sake a contradiction that x_0 is a local minimum. Then, choose $x_1 \in (x_0, x_0 + \varepsilon)$. By the mean value theorem, there exists a x_2 such that

$$f(x_0) - f(x_1) = f'(x_2)(x_0 - x_1)f(x_0) - f(x_1) = f'(x_2)(x_0 - \varepsilon - x_0)f(x_0) - f(x_1) = f'(x_2)(-\varepsilon)$$

Since $x_2 \in B_\varepsilon(x_0)$, $f'(x_2) < 0$, so the $f(x_0) - f(x_1) > 0 \implies f(x_1) < f(x_0)$. This is a contradiction because x_0 is assumed to be a local minimum.

Therefore, x_0 cannot be a local minimum or maximum. $\square$

(3) Let $w = f(x, y, z) = xy^2z$, with $x = r + 2s + t, y = 2r + 3s + t, z = 3r + s + t$. Use the chain rule to calculate $\partial w/\partial r, \partial w/\partial s, \partial w/\partial t$.

$$\frac{\partial w}{\partial x} = y^2 z = (2r + 3s + t)^2(3r + s + t)$$
$$\frac{\partial w}{\partial y} = 2xyz = 2(r + 2s + t)(2r + 3s + t)(3r + s + t)$$
$$\frac{\partial w}{\partial z} = xy^2 = (r + 2s + t)(2r + 3s + t)^2$$

$$\frac{\partial x}{\partial r} = 1$$
$$\frac{\partial x}{\partial s} = 2$$
$$\frac{\partial x}{\partial t} = 1$$
$$\frac{\partial y}{\partial r} = 2$$
$$\frac{\partial y}{\partial s} = 3$$
$$\frac{\partial y}{\partial t} = 1$$
$$\frac{\partial z}{\partial r} = 3$$
$$\frac{\partial z}{\partial s} = 1$$
$$\frac{\partial z}{\partial t} = 1$$

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (1)(2r + 3s + t)^2(3r + s + t) + (2)(2)(r + 2s + t)(2r + 3s + t)(3r + s + t) + (3)(r + 2s + t)(2r + 3s + t)^2 \\
&= (2r + 3s + t)^2(3r + s + t) + 4(r + 2s + t)(2r + 3s + t)(3r + s + t) + 3(r + 2s + t)(2r + 3s + t)^2 \\
&= 48r^3 + 192r^2s + 84r^2t + 238rs^2 + 212rst + 46rt^2 + 87s^3 + 122s^2t + 55st^2 + 8t^3 \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (2)(2r + 3s + t)^2(3r + s + t) + (3)(2)(r + 2s + t)(2r + 3s + t)(3r + s + t) + (1)(r + 2s + t)(2r + 3s + t)^2 \\
&= 2(2r + 3s + t)^2(3r + s + t) + 6(r + 2s + t)(2r + 3s + t)(3r + s + t) + (r + 2s + t)(2r + 3s + t)^2 \\
&= 64r^3 + 238r^2s + 106r^2t + 261rs^2 + 244rst + 55rt^2 + 72s^3 + 117s^2t + 58st^2 + 9t^3 \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&= (1)(2r + 3s + t)^2(3r + s + t) + (1)(2)(r + 2s + t)(2r + 3s + t)(3r + s + t) + (1)(r + 2s + t)(2r + 3s + t)^2 \\
&= (2r + 3s + t)^2(3r + s + t) + 2(r + 2s + t)(2r + 3s + t)(3r + s + t) + (r + 2s + t)(2r + 3s + t)^2 \\
&= 28r^3 + 106r^2s + 46r^2t + 122rs^2 + 110rst + 24rt^2 + 39s^3 + 58s^2t + 27st^2 + 4t^3
\end{aligned}$$

(4) Let $f : X \to \mathbb{R}^n$ be a continuously differentiable function on the open set $X \subset \mathbb{R}^n$. Show that f is locally Lipschitz on X (use Euclidean distance).

Proof: Let $f : X \to \mathbb{R}^n$ be a continuously differentiable function on the open set $X \subset \mathbb{R}^n$. Choose $x \in X$ and $\varepsilon > 0$ such that the closed ball $B_\varepsilon[x] \subset X$.

Since f is continuously differentiable on X, it is continuously differentiable on $B_\varepsilon[x]$. Since Df is continuous on $B_\varepsilon[x]$, it is bounded. Define $a^{i,j}$ as the upper bound of Df on $B_\varepsilon[x_0]$ in the ith input dimension and jth output dimension for $i, j \in \{1, ..., n\}$. Similarly, define $b^{i,j}$ as the lower bound of Df on $B_\varepsilon[x]$ in the ith input dimension and jth output dimension for $i, j \in \{1, ..., n\}$. Let $M := \max_{i,j\in\{1,...,n\}}\{|a_{i,j}|, |b_{i,j}|\}$ and let $\vec{M}$ be an n-element vector with M in each dimension. Notice that $\forall v \in B_\varepsilon[x]$, $||Df(v)|| \leq ||\vec{M}||$.

Choose $w, y \in B_\varepsilon[x]$. By the mean value theorem, there exists $z_1, ..., z_n \in \{\alpha w + (1-\alpha)y : \alpha \in [0,1]\}$ such that $f_i(w) - f_i(y) = Df^i(z_i)(w_i - y_i)$. By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|f_i(w) - f_i(y)| &\leq ||Df_i(z_i)|| \cdot |w_i - y_i| \\
|f_i(w) - f_i(y)| &\leq ||\vec{M}|| \cdot |w_i - y_i| \\
\sqrt{\sum_{i=1}^{n}(f_i(w) - f_i(y))^2} &\leq \sqrt{\sum_{i=1}^{n} ||\vec{M}||^2 \cdot (w_i - y_i)^2} \\
||f(w) - f(y)|| &\leq ||\vec{M}|| \cdot ||w - y||
\end{aligned}
$$

Thus, f is locally Lipschitz on X.

(5) Let $f(x,y) = x^5 - x^2 + x - y^3 - 2y + 2$ and let $x(y)$ satisfy $x(1) = 1$ and $f(x(y), y) = 0$. Calculate $\left.\frac{\partial x(y)}{\partial y}\right|_{y=1}$.

I apply the implicit function theorem. Note that f is continuously differentiable:

$$\begin{aligned}\frac{\partial f}{\partial x} =& 5x^4 - 2x + 1 \\ \frac{\partial f}{\partial y} =& -3y^2 - 2\end{aligned}$$

If $x_0 = x(y)$ and $a_0 = y$, then $f(x_0, a_0) = f(x(y), y) = 0$. Furthermore, $\det(D_X f(x,y)) = 5x^4 - 2x + 1 > 0$ $\forall x(y) \in \mathbb{R}$. By the implicit function theorem:

$$\begin{aligned}Dx(y) &= -(5x(y)^4 - 2x(y) + 1)^{-1}(-3y - 2) \\ Dx(1) &= -(5(1)^4 - 2(1) + 1)^{-1}(-3(1)^2 - 2) \\ &= 5/4\end{aligned}$$

(6) Find all local minima/maxima of $f(x,y) = 2x^4 + y^2 - xy + 1$. Does it have a global maximum/minimum?

The Jacobian is:

$$Df(x,y) = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \end{pmatrix} = \begin{pmatrix} 8x^3 - y & 2y - x \end{pmatrix} = \vec{0}$$

and the Hessian is:

$$D^2 f(x,y) = \begin{pmatrix} \partial^2 f/\partial^2 x & \partial^2 f/(\partial y \partial x) \\ \partial^2 f/(\partial x \partial y) & \partial^2 f/\partial^2 y \end{pmatrix} = \begin{pmatrix} 24x^2 & -1 \\ -1 & 2 \end{pmatrix}$$

Setting the Jacobian to zero implies

$$0 = 2y - x \implies x = 2y \implies 0 = 8(2y)^3 - y \implies 0 = y(8y-1)(8y+1)$$

Thus, there are potential minima/maxima at $(0,0)$, $(1/4, 1/8)$, $(-1/4, -1/8)$.

For $(0,0)$,

$$D^2 f(1/4, 1/8) = \begin{pmatrix} 24(0)^2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

The characteristic polynomial is $-\lambda(2 - \lambda) - (-1)(-1) = 0 \implies \lambda^2 - 2\lambda - 1 \implies \lambda_1 = 1 - \sqrt{2}, \lambda_2 = 1 + \sqrt{2}$. Since one eigenvalue is positive and one eigenvalue is negative, f has a saddle point at $(0,0)$.

For $(1/4, 1/8)$ and $(-1/4, -1/8)$,

$$D^2 f(-1/4, 1/8) = D^2 f(1/4, 1/8) = \begin{pmatrix} 24(1/4)^2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3/2 & -1 \\ -1 & 2 \end{pmatrix}$$

The characteristic polynormal is $(3/2 - \lambda)(2 - \lambda) - 1 = 0 \implies \lambda^2 - 7/2\lambda + 2 \implies \lambda_1 = 7/4 + \sqrt{17}/4, \lambda_2 = 7/4 - \sqrt{17}/4$. Since both eigenvalues are positive, f has local minima at $(1/4, 1/8)$ and $(-1/4, -1/8)$. Since $f(1/4, 1/8) = f(-1/4, -1/8)$ and $\lim_{x \to \infty} f(x,y) = \lim_{x \to -\infty} f(x,y) = \lim_{y \to \infty} f(x,y) = \lim_{y \to -\infty} f(x,y) = \infty$, $(1/4, 1/8)$ and $(-1/4, -1/8)$ are also global minima. Since f has no local maxima, it has no global maxima.