ECON 703 Final Cheatsheet

Let X be a vector space and $T \in L(X,X)$. If $T(v) = \lambda v$, λ is an **eigenvalue** of T and $v \neq \bar{0}$ is an **eigenvector** corresp. to λ.

Let W be a basis of X. λ is an eigenvalue of T iff λ is an eigenvalue of $\text{mtx}_W(T)$. v is an eigenvector of T corresp. to λ iff $\text{crd}_W(v)$ is an eigenvector of $\text{mtx}_W(T)$ corresp. to λ.

If $\dim X = n$, $\text{mtx}_W(T)$ is **diagonalizable** if $\exists$ basis U s.t. $\text{mtx}_U(T) = diag(\lambda_1, ..., \lambda_n)$. Thus, $\lambda_1, ..., \lambda_n$ are eigenvalues of T and $U = \{u_1, ..., u_n\}$ are eigenvectors of T.

$\text{mtx}_W(T)$ is diagonalizable $\iff$ eigenvectors of T form a basis of X $\iff$ eigenvectors of $\text{mtx}_W(T)$ form a basis of $\mathbb{R}^n$.

If $\lambda_1, ..., \lambda_m$ are distinct eigenvalues of T with corresponding eigenvectors $v_1, ..., v_m$, then $v_1, ..., v_m$ are linearly independent.

If $\dim X = n$ and T has n distinct eigenvalues, then X has a basis consisting of T's eigenvectors. Thus, if W is a basis of X, $\text{mtx}_W(T)$ is diagonalizable.

$A \in M_{n\times n}$ is **symmetric** if $a_{ij} = a_{ji}$ for all $i,j = 1, ..., n$,

$A \in M_{n\times n}$ is **orthogonal** if $A^{-1} = A'$.

A basis $V = \{v_1, ..., v_n\}$ of $\mathbb{R}^n$ is **orthonormal** if $v_i \cdot v_j = 1$ when $i = j$ and $v_i \cdot v_j = 0$ when $i \neq j$.

A real $n \times n$ matrix A is orthogonal iff A's columns are orthonormal. (Thus, A's columns = basis of $\mathbb{R}^n$.)

Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and W be the standard basis of $\mathbb{R}^n$. If $\text{mtx}_W(T)$ is symmetric, then

- T has n eigenvalues.
- T's eigenvectors $\{v_1, ..., v_n\}$ are orthonormal basis of $\mathbb{R}^n$
- $\text{mtx}_W(T)$ is diagonaliable: $\text{mtx}_W(T) = \text{mtx}_{W,V}(id) \cdot \text{mtx}_V(T) \cdot \text{mtx}_{V,W}(id)$ ($\text{mtx}_V(T)$ is diagonal and $\text{mtx}_{W,V}(id), \text{mtx}_{V,W}(id)$ are orthogonal)

Quadratic Form

$f(x_1, ..., x_n) = \sum_{i=1}^n \alpha_{ii} x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j \equiv x'Ax$,

$$A = \begin{pmatrix} \alpha_{11} & ... & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & ... & \alpha_{nn} \end{pmatrix}, \alpha_{ij} = \begin{cases} \beta_{ij}/2, i < j \\ \beta_{ji}/2, i > j \end{cases}$$

A is symmetric $\implies$ A is diagonalizable, $A = U'DU$.

A's eigenvectors $= V = \{v_1, ..., v_n\}$ are an orthonormal basis of $\mathbb{R}^n$.

$U = (v_1 ... v_n) = \text{mtx}_{V,W}(id), W =$ standard basis of $\mathbb{R}^n$.

$\forall x \in \mathbb{R}^n$: $x = \sum_{i=1}^n \beta_i v_i, \beta_i = x \cdot v_i$

$f(x) = x'Ax = (\beta_1, ..., \beta_n) D (\beta_1, ..., \beta_n)^T = \sum_{i=1}^n \lambda_i \beta_i^2$, where $D = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Let X be a vector space. A **norm** on X is a function $||\cdot|| : X \to \mathbb{R}_+$ s.t.

- $||x|| \geq 0 \forall x \in X$
- $||x|| = 0 \iff x = \bar{0}$
- $||x + y|| \leq ||x|| + ||y|| \; \forall x, y \in X$
- $||\alpha x|| = |\alpha| \cdot ||x|| \; \forall \alpha \in \mathbb{R}, x \in X$

A **normed vector space** is a vector space equipped with a norm.

Let $(X, ||\cdot||)$ be a normed vector space. Define $d : X \times X \to \mathbb{R}_+$ s.t. $d(x,y) = ||x - y||$. Then (X, d) is a **metric space**.

For $X = \mathbb{R}^n$, $||x||_2 = \sqrt{\sum_{i=1}^n x_i^2}$, $||x||_1 = \sum_{i=1}^n |x_i|$, $||x||_\infty = \max\{|x_1|, ..., |x_n|\}$.

For $X = C([0,1])$, $||f||_2 = \sqrt{\int_0^1 f^2(t)dt}$, $||f||_1 = \int_0^1 |f(t)|dt$, $||f||_\infty = \sup_{t\in[0,1]} |f(t)|$.

Suppose X, Y are normed vector spaces and $T \in L(X,Y)$. We say that T is **bounded** if $\exists \beta \in \mathbb{R}$ s.t. $||T(x)||_Y \leq \beta ||x||_X$ $\forall x \in X$. T is **bounded** is equivalent to:

- T is **continuous at** $x_0 \in X$.
- T is **continuous** $\forall x \in X$.
- T is **uniformly continuous**.
- T is **Lipschitz**.

Let X, Y be normed vector spaces, $\dim X = n$. Then **every** $T \in L(X,Y)$ is **bounded**.

$B(X,Y) = \{T \in L(X,Y) | T \text{ is bounded }\}$. It $\dim X = n$, then $B(X,Y) \equiv L(X,Y)$.

$||T||_{B(X,Y)} = \sup_{x\in X, x\neq\bar{0}} \left\{ \frac{||T(x)||_Y}{||x||_X} \right\} = \sup_{||x||_X = 1} \{||T(x)||_Y\}$

Working with $B(X,Y)$ instead of $L(X,Y)$ guarantees that sup exists.

$(B(X,Y), ||\cdot||_{B(X,Y)})$ is **normed vector space**.

Let $f : I \to \mathbb{R}, I \subset \mathbb{R}$ is an open interval. f is **differentiable at** $x \in I$ if $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = a$ for some $a \in \mathbb{R}$.

Let $f : X \to \mathbb{R}, X \subset \mathbb{R}^n$ is an open set. f is **differentiable at** $x \in X$ if $\lim_{h\to 0, h\in\mathbb{R}^n} \frac{|f(x+h) - (f(x) + a_1^x h_1 + ... + a_n^x h_n)|}{||h||} = 0$ for some $(a_1^x, ..., a_n^x) \in \mathbb{R}^n$.

f is differentiable if it is differentiable at all $x \in X$.

Let $f : X \to \mathbb{R}^m, X \subset \mathbb{R}^n$ is an open set. f is **differentiable at** $x \in X$ if $\lim_{h\to 0, h\in\mathbb{R}^n} \frac{||f(x+h) - (f(x) + A_x h)||}{||h||} = 0$ for some $A_x \in M_{m\times n}$.

$\to f(x + h) \approx f(x) + A_x h$. Matrix $A_x =$ **Jacobian matrix**, denoted $Df(x)$.

Linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ represented by $A_x =$ **differential**, denoted df_x.

If f is differentiable at x, then its differential df_x is **unique**.

If f is differentiable at x, then f is **continuous** at x.

$$Df(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x) & ... & \frac{\partial f^1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x_1}(x) & ... & \frac{\partial f^m}{\partial x_n}(x) \end{pmatrix}$$

The partial derivative of f is $\frac{\partial f^i}{\partial x_j}(x) := \lim_{\varepsilon\to 0} \frac{f^i(x + \varepsilon e_j) - f^i(x)}{\varepsilon}, i = 1, ..., m, j = 1, ..., n$.

Chain Rule: Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be open, $f : X \to Y, g : Y \to \mathbb{R}^k$. Let $x_0 \in X$ and $F := g \circ f$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then F is differentiable at x_0 and $dF_{x_0} = dg_{f(x_0)} \circ df_{x_0}$ and $DF(x_0) = Fg(f(x_0))Df(x_0)$.

MVT: Let $f : [a,b] \to \mathbb{R}$ be continuous $[a,b]$ and differential on (a,b). Then there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

MVT: Let $f : \mathbb{R}^n \to \mathbb{R}$ be differential on an open set $X \in \mathbb{R}^n, x, y \in X$, and $l(x,y) := \{\alpha x + (1-\alpha)y | \alpha \in [0,1]\} \subset X$. Then there exists $z \in l(x,y)$ such that $f(y) - f(x) = Df(z)(y-x)$.

Rolle's Thm: Let $f : [a,b] \to \mathbb{R}$ be continuous $[a,b]$ and differential on (a,b). Assume that $f(a) = f(b) = 0$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Taylor's Thm: Let $f : I \to \mathbb{R}$ is n times differentiable with $I \subset \mathbb{R}$ is open and $[x, x+h] \subset I$. Then $f(x+h) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)h^k}{k!} + \frac{f^{(n)}(x+\lambda h)h^n}{n!}, \lambda \in (0,1)$ and $f(x+h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} + o(h^n)$ as $h \to 0$.

If f is $(n+1)$ times differentiable, then $f(x+h) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)h^k}{k!} + O(h^{n+1})$ as $h \to 0$.

Taylor's Thm: Let $f : X \to \mathbb{R}^m$ is differentiable with $X \subset \mathbb{R}^n$ is open and $x \in X$. Then $f(x+h) = f(x) + Df(x)h + o(||h||)$ as $h \to 0$. If additionally, $f \in C^2$, then $f(x+h) = f(x) + Df(x)h + o(||h||^2)$ as $h \to 0$.

For $f : X \to \mathbb{R}, x \subset \mathbb{R}^n$, the **Hessian matrix** is

$$D^2 f(x) := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \ldots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

If $f \in C^2$, then $D^2 f(x)$ is symmetric.

Taylor Thm: Let $f : X \to \mathbb{R}$ is C^2 with $X \subset \mathbb{R}^n$ is open and $x \in X$. Then $f(x+h) = f(x) + Df(x)h + \frac{1}{2}h' D^2(x)h + o(||h||^2)$ as $h \to 0$. If additionally $f \in C^3$, then $f(x+h) = f(x) + Df(x)h + \frac{1}{2}h' D^2 f(x)h + O(||h||^3)$ as $h \to 0$.

Let $f : X \to \mathbb{R}, X \in \mathbb{R}^n, f \in C^2$, then $D^2 f(x)$ has eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{R}$. If f has a local max/min at x, then $Df(x) = 0$. If $Df(x) = 0$, then

- $\lambda_1, ..., \lambda_n > 0 \implies f$ has a local minimum at x.
- $\lambda_1, ..., \lambda_n < 0 \implies f$ has a local maximum at x.
- $\exists i, j$ s.t. $\lambda_i > 0, \lambda_j < 0 \implies f$ has a saddle point at x.
- $\lambda_1, ..., \lambda_n \geq 0, \lambda_i > 0$ for some $i \implies f$ has a local minimum or saddle at x.
- $\lambda_1, ..., \lambda_n \leq 0, \lambda_i < 0$ for some $i \implies f$ has a local maximum or saddle at x.
- $\lambda_1 = ... = \lambda_n = 0$ gives no information.

Inverve Fn Thm: Let $f : X \to \mathbb{R}^n$ be a continuously differentiable function, $X \subset \mathbb{R}^n$ be open, $x_0 \in X$. If $\det(Df(x_0)) \neq 0$, then there exists an open neighborhood U of x_0 such that

- f is one-to-one in U
- $V = f(U)$ is an open set, $y_0 := f(x_0) \in V$
- f^{-1} is continuously differentiable and $Df^{-1}(y_0) = (Df(x_0))^{-1}$.

Implicit Fn Thm: Suppose $X \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^p$ are open, $f : X \times A \to \mathbb{R}^n$ is continuously differentiable, $f(x_0, a_0) = 0$ and $\det(D_x f(x_0, a_0)) \neq 0$. Then there exist open neighborhoods U of x_0 and W of a_0 such that

- $\forall a \in W\ \exists! \equiv g(a) \in U$ s.t. $f(x,a) = f(g(a), a) = 0$
- g is continuously differentiable
- $Dg(a_0) = -(D_x f(x_0, a_0))^{-1} D_a f(x_0, a_0)$

A set $X \subset \mathbb{R}^n$ is **convex** if $\forall \lambda \in [0,1], x', x'' \in X$, the point $x_\lambda := (1-\lambda)x' + \lambda x'' \in X$

Any intersection of convex sets is convex.

If X, Y are convex sets in $\mathbb{R}^n$, then for any $\alpha, \beta \in \mathbb{R}$, the set $z = \alpha X + \beta Y := \{z \in \mathbb{R}^n | z = \alpha x + \beta y \text{ for some } x \in X, y \in Y\}$ is also convex.

A vector $p \neq \bar{0}$ in $\mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$ define the **hyperplane** $H(p, \alpha)$ given by $H(p, \alpha) = \{x \in \mathbb{R}^n | p \cdot x := \sum_{i=1}^n p_i x_i = \alpha\}$.

Vector p is called the **normal** to the hyperplane $H(p, \alpha)$.

If $x', x'' \in H(p, \alpha), \lambda \in \mathbb{R}$, then $(1-\lambda)x' + \lambda x'' \in H(p, \alpha)$.

Sets X and Y are **separated** by a hyperplane $H(p, \alpha)$ if $p \cdot x \leq \alpha p \leq \cdot y\ \forall x \in X, y \in Y$.

Sets X and Y are **strictly separated** by a hyperplane $H(p, \alpha)$ if $p \cdot x < \alpha < p \cdot y\ \forall x \in X, y \in Y$.

A hyperplane $H(p, \alpha)$ **supports** a set X if either $\alpha = \inf_{x \in X}(p \cdot x)$ and $\alpha = \sup_{x \in X}(p \cdot x)$

Let X be a nonempty, closed, convex set in $\mathbb{R}^n, z \notin X$. Then

- There exists $x^0 \in X$ and $H(p, \alpha)$ s.t. $x^0 \in H(p, \alpha), H(p, \alpha)$ supports X, and separates X and $\{z\}$.
- There exists a hyperplane $H(p, \beta)$ that strictly separates X and $\{z\}$.

Let X be a nonempty convex set in $\mathbb{R}^n, z \notin X$. Then there exists $H(p, \alpha)$ s.t., $z \in H(p, \alpha)$ and $H(p, \alpha)$ separates X and $\{z\}$.

SHT: Let X and Y be disjoint and nonempty convex sets in $\mathbb{R}^n$. Then there exists a hyperplane $H(p, \alpha)$ that separates X and Y.

Let $f : X \to X$. A point $x^* \in X$ is a **fixed point** of f if $f(x^*) = x^*$.

Let $f : [a,b] \to [a,b]$ be continuous. Then f has a **fixed point**.

Brouwer's Fixed Point Theorem: Let $X \subset \mathbb{R}^n$ be nonempty, compact, and convex, and let $f : X \to X$ be continuous. Then f has a fixed point.