Notes on Farhi and Werning (2016): Section 5.1

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1 Environment

1.1 Timing

- Three periods $t \in \{0, 1, 2\}$
- Two goods: consumption and labor

1.2 Agents

1.2.1 Households

- Two types of agents $i \in \{1, 2\}$ with fractions $\phi^1 + \phi^2 = 1$
- Type 1 agents are savers and type 2 agents are borrowers
- Type i agents have preferences over consumption C_t^i and labor N_t^i

$$\sum_{t=0}^{2} \beta_i^t [u(C_t^i) - v(N_t^i)]$$

- $\beta_1 \geq \beta_2 \implies$ type 1 agents (savers) are more patient than type 2 agents (borrowers)
- Budget constraint of type i agents is

$$P_t C_t^i + B_t^i + \tau_t^i \leq W_t N_t^i + \frac{1}{1 + i_t} B_{t+1}^i$$

  where B_t^i is nominal debt holdings, i_t is nominal interest rate, τ_t^i is lump-sum taxes (nonzero only in $t = 0$), and W_t is nominal wage

- Initial bond holdings are zero, $B_1^i = 0$, and final period bond holdings are zero, $B_3^i = 0$
- Exogenous borrowing constraint on type-2 agents (borrowers) in period 1:

$$B_2^2 \leq P_2 \bar{B}_2$$

1.2.2 Government

- Government can implement borrowing constraint on type-2 agents (borrowers) in $t = 0$:

$$B_1^2 \leq P_1 \bar{B}_1$$

- Government chooses nonnegative nominal interest rates $i_t \geq 0$ in all t
- Government levies lump-sum taxes in period 0 subject to government budget constraint

$$\phi^1 \tau_0^1 + \phi^2 \tau_0^2 = 0$$

1.3 Technology

1.3.1 Production

- Final good is produced by competitive firms with production function

$$Y_t = AN_t$$

1.3.2 Sticky wages and rationing

- Wages are sticky: $W_t = W$ for all t
- Work is equally rationed across types: $N_t^1 = N_t^2 = N_t$ for all t

1.4 Info Structure

- All objects are public information

2 Equilibrium

- An allocation x is output $\{Y_t\}$ consumption $\{C_t^i\}$, labor supply, $\{N_t^i\}$, and debt holding $\{B_t^i\}$ for all i and t
- A policy π is nominal interest rates $\{i_t\}$ for all t, borrowing constraint on type-2 agents in period 0 $\bar{B}_1$, period-0 lump-sum taxes $\{\tau_0^i\}$
- A price system q is prices $\{P_t\}$ and wages $\{W_t\}$ for all t
- An equilibrium is an allocation x, policy π, and price system q such that

– Given π and q, type-i agents choose $\{C_t^i, N_t^i, B_t^i\}$ for all t to solve their problem

$$\max_{\{C_t^i, N_t^i, B_t^i\}} \sum_{t=0}^{2} \beta_i^t [u(C_t^i) - v(N_t^i)]$$
$$\text{s.t. } P_t C_t^i + B_t^i + \tau_t^i \leq W_t N_t^i + \frac{1}{1+i_t} B_{t+1}^i$$
$$B_1^2 \leq P_1 \bar{B}_1$$
$$B_2^2 \leq P_2 \bar{B}_2$$

where $B_3^i = 0$

– Given π and q, firms choose N_t^i to solve their problem

$$\max_{N_t} P_t A N_t - W N_t$$

– GBC holds

$$\phi^1 \tau_0^1 + \phi^2 \tau_0^2 = 0$$

– Markets clear for all t:

$$\sum_{i=1}^{2} \phi^i C_t^i = \sum_{i=1}^{2} \phi^i Y_t \qquad \text{[Goods]}$$
$$\phi^1 B_t^1 + \phi^2 B_t^2 = 0 \qquad \text{[Bonds]}$$

– Wages are sticky $W_t = W$ for all t

– Equal rationing of labor $N_t^1 = N_t^2 = N_t$

3 Solution

3.1 Implementability

- Notice that government choosing lump-sum taxes is equivalent to choosing (possibly nonzero) initial levels of debt holding subject to bond market clearing

- Given nominal wage W, firm solves

$$\max_{N_t} P_t A N_t - W N_t \implies P_t = \frac{W}{A}$$

- Type 1 agents (savers) problem implies Euler equation for $t = 0, 1$:

$$\frac{1}{1+i_t} \frac{P_{t+1}}{P_t} = \frac{\beta_1 u'(C_{t+1}^1)}{u'(C_t^1)}$$
$$\implies u'(C_t^1) = \beta_1 (1+i_t) u'(C_{t+1}^1)$$

- Type 2 agents (borrowers) problem implies Euler equation for $t = 0, 1$:

$$\begin{aligned} \frac{1}{1+i_t}\frac{P_{t+1}}{P_t} &\geq \frac{\beta_2 u'(C_{t+1}^2)}{u'(C_t^2)} \\ \implies u'(C_t^2) &\geq \beta_2(1+i_t)u'(C_{t+1}^2) \end{aligned}$$

  with equality if borrowing constraint in t is slack

- Combining $t = 3$ budget constraint of type-2 agent and the exogenous borrowing constraint

$$\begin{aligned} P_2C_2^2 + B_2^2 = (AP_2)N_2 &\implies B_2^2 = WN_2 - P_2C_2^2 \\ B_2^2 &\leq P_2\bar{B}_2 \\ \implies AP_2N_2 - P_2C_2^2 &\leq P_2\bar{B}_2 \\ \implies C_2^2 &\geq AN_2 - \bar{B}_2 \end{aligned}$$

- Combine borrower Euler equation and borrowing constraint to get complimentary slackness condition
$$(C_2^2 - AN_2 + \bar{B}_2)[u'(C_t^1) - \beta_2(1+i_1)u'(C_2^2)] = 0$$

- These conditions are necessary and sufficient for implementation

4 Ramsey Problem

- Given Pareto weights λ^i, Ramsey planner maximizes weight average of utilities:

$$\max_{\{C_t^i, N_t, i_t\}} \sum_{i=1}^{2}\sum_{t=0}^{2} \lambda^i \phi^i \beta_i^t [u(C_t^i) - v(N_t)]$$

$$\begin{aligned} \text{s.t. } \sum_{i=1}^{2}\phi^i C_t^i &= \sum_{i=1}^{2}\phi^i AN_t, \forall t && \text{(RC)} \\ C_2^2 &\geq AN_2 - \bar{B}_2 && \text{(BC)} \\ u'(C_t^1) &= \beta_1(1+i_t)u'(C_{t+1}^1), \forall t && \text{(EE 1)} \\ u'(C_t^2) &\geq \beta_2(1+i_t)u'(C_{t+1}^2), \forall t && \text{(EE 2)} \\ (C_2^2 - AN_2 + \bar{B}_2)[u'(C_1^2) - \beta_2(1+i_1)u'(C_2^2)] &= 0 && \text{(CS 2)} \\ i_t &\geq 0, \forall t && \text{(ZLB)} \end{aligned}$$

- Case of interest:
  - Borrower is borrowing constrained in period 1 $\implies$ (BC) holds with equality and (EE 2) in period 1 holds with strict inequality
  - Assume ZLB is slack in period 0

- Thus, the Ramsey problem for this case of interest is

$$\max_{\{C_t^i, N_t, i_t\}} \sum_{i=1}^{2} \sum_{t=0}^{2} \lambda_t^i \phi^i [u(C_t^i) - v(N_t)]$$

$$\text{s.t.} \quad \sum_{i=1}^{2} \phi^i C_t^i = \sum_{i=1}^{2} \phi^i A N_t, \forall t \tag{RC}$$

$$C_2^2 = AN_2 - \bar{B}_2 \tag{BC}$$

$$u'(C_0^1) = \beta_1 (1 + i_0) u'(C_1^1) \tag{EE 1, 0}$$

$$u'(C_1^1) = \beta_1 (1 + i_1) u'(C_2^1) \tag{EE 1, 1}$$

$$u'(C_0^2) \geq \beta_2 (1 + i_0) u'(C_1^2) \tag{EE 2, 0}$$

$$u'(C_1^2) > \beta_2 (1 + i_1) u'(C_2^2) \tag{EE 2, 1}$$

$$i_1 \geq 0 \tag{ZLB}$$

where $\lambda_t^i \equiv \lambda^i \beta_i^t$