ECON 714A - Problem Set 5

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A representative household maximizes lifetime expected utility

$$E_0 \sum_{t=0}^{\infty} \beta^t (\log C_t - L_t)$$

subject to the budget constraint

$$P_t C_t + B_t = W_t L_t + \Pi_t + (1 + i_{t-1}) B_{t-1} + T_t$$

where consumption bundle C_t is a standard CES aggregator of individual varieties with the elasticity of substitution θ. Firms are monopolistic competitors and use a linear technology $Y_{it} = A_t L_{it}$ to produce a continuum of unique varieties $i \in [0, 1]$. Each firm has to hire additional $\frac{\varphi}{2}(\frac{Pit - Pit-1}{Pit-1})^2$ units of labor to adjust the price from P_{it-1} to P_{it} in period t.

1. Consider a flexible-price version of the model with $\varphi = 0$. Describe the deterministic steady state of the economy and the linearized dynamics around it.

The household problem is to choose consumption of each product, supply labor, and purchase bonds:

$$\max_{\{(C_{it}, L_t, B_t)\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t (\log C_t - L_t)$$

$$\text{s.t. } P_t C_t + B_t = W_t L_t + \Pi_t + (1 + i_{t-1}) B_{t-1} + T_t$$

$$\text{and } C_t = \left(\int C_{it}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$$

The resulting optimality conditions are the same as in the RBC and Dixit-Stiglitz models.

Demand for products:

$$C_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\theta} C_t$$

Labor supply condition:

$$C_t = \frac{W_t}{P_t} \implies L_t = 1$$

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The Euler equation:

$$\beta E_t \left(\frac{C_{t+1}}{C_t} \right)^{-1} \frac{P_t}{P_{t+1}} (1 + i_t) = 1$$

With $\varphi = 0$, the problem of each firm to maximize profits becomes static:

$$\max P_{it} C_{it} - W_t L_{it}$$
$$\text{s.t. } C_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\theta} C_t$$
$$\text{and } C_{it} = A_t L_{it}$$

From the Dixit-Stiglitz model, the optimal price is:

$$P_{it} = \frac{\theta}{\theta - 1} \frac{W_t}{A_t}$$

$$P_t = \left(\int P_{it}^{1-\theta} di \right)^{\frac{1}{1-\theta}}$$

In the steady state, the labor supply condition implies:

$$\bar{C} = \bar{W} / \bar{P}$$

From the price aggregation condition:

$$\bar{P} = \left(\int \bar{P}_i^{1-\theta} di \right)^{\frac{1}{1-\theta}}$$

From the optimal markup condition:

$$\bar{P} = \frac{\theta}{\theta - 1} \frac{\bar{W}}{\bar{A}}$$

From the Euler equation (and symmetry of $\bar{P}_i = \bar{P}$):

$$\beta E_t \left(\frac{\bar{C}}{\bar{C}} \right)^{-1} \frac{\bar{P}}{\bar{P}} (1 + \bar{i}) = 1 \implies \bar{i} = 1/\beta - 1$$

From the output equation:

$$\bar{C} = \bar{A} \bar{L}$$

Thus,

$$\bar{L} = \frac{\bar{C}}{\bar{A}} = \frac{\bar{W}}{\bar{P} \bar{A}} = \frac{\bar{W}}{\bar{A}} \frac{\theta - 1}{\theta} \frac{\bar{A}}{\bar{W}} = \frac{\theta - 1}{\theta}$$

Now let us consider the linearized dynamics around the steady state. Inelastic labor supply means that $l_t = 0$. The labor supply condition implies:

$$c_t = w_t - p_t$$

The Euler equation implies:

$$E_t[c_t - c_{t+1} + p_t - p_{t+1} + i_t] = 0$$

The optimal price implies:

$$p_{it} = w_t - a_t$$

Finally the price aggregation condition implies:

$$p_t = p_{it}$$

The linearized labor supply condition, optimal price condition, and price aggregation condition imply:

$$c_t = w_t - (w_t - a_t) \implies c_t = a_t$$

2. Derive the NKPC following these steps:

(a) write the FOC of an individual firm,

From the Euler equation, we can define the stochastic discount factor: $\Theta_{t,t+j} = \beta^j \frac{C_t}{C_{t+j}} \frac{P_t}{P_{t+j}}$. The firm's problem with sticky prices is to maximize discounted profits:

$$\max_{\{P_{it}\}_{t=0}^{\infty}} E_t \sum_{j=0}^{\infty} \Theta_{t,t+j} \left[P_{i,t+j} C_{i,t+j} - \frac{W_{t+j}}{A_{t+j}} C_{i,t+j} - W_{t+j} \frac{\varphi}{2} \left(\frac{P_{i,t+j} - P_{i,t+j-1}}{P_{i,t+j-1}} \right)^2 \right]$$

$$\text{s.t. } C_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\theta} C_t$$

$$\implies \max_{\{P_{it}\}_{t=0}^{\infty}} E_t \sum_{j=0}^{\infty} \Theta_{t,t+j} \left[P_{i,t+j}^{1-\theta} P_{t+j}^{\theta} C_{t+j} - \frac{W_{t+j}}{A_{t+j}} P_{i,t+j}^{-\theta} P_{t+j}^{\theta} C_{t+j} - W_{t+j} \frac{\varphi}{2} \left(\frac{P_{i,t+j} - P_{i,t+j-1}}{P_{i,t+j-1}} \right)^2 \right]$$

$$\begin{aligned}
\implies \max_{\{P_{it}\}_{t=0}^{\infty}} E_t & \left[P_{i,t}^{1-\theta} P_t^{\theta} C_t - \frac{W_t}{A_t} P_{i,t}^{-\theta} P_t^{\theta} C_t - W_t \frac{\varphi}{2} \left(\frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \right)^2 \right] \\
& + \Theta_{t,t+1} \left[P_{i,t+1}^{1-\theta} P_{t+1}^{\theta} C_{t+1} - \frac{W_{t+1}}{A_{t+1}} P_{i,t+1}^{-\theta} P_{t+1}^{\theta} C_{t+1} - W_{t+1} \frac{\varphi}{2} \left(\frac{P_{i,t+1} - P_{i,t}}{P_{i,t}} \right)^2 \right] \\
& + \sum_{j=2}^{\infty} \Theta_{t,t+j} \left[P_{i,t+j}^{1-\theta} P_{t+j}^{\theta} C_{t+j} - \frac{W_{t+j}}{A_{t+j}} P_{i,t+j}^{-\theta} P_{t+j}^{\theta} C_{t+j} - W_{t+j} \frac{\varphi}{2} \left(\frac{P_{i,t+j} - P_{i,t+j-1}}{P_{i,t+j-1}} \right)^2 \right]
\end{aligned}$$

FOC $[P_{i,t}]$:

$$E_t\left[(1-\theta)P_{i,t}^{-\theta}P_t^{\theta}C_t-(-\theta)\frac{W_t}{A_t}P_{i,t}^{-\theta-1}P_t^{\theta}C_t-\frac{W_t\varphi}{P_{i,t-1}}\left(\frac{P_{i,t}-P_{i,t-1}}{P_{i,t-1}}\right)+\Theta_{t,t+1}\frac{W_{t+1}\varphi P_{i,t+1}}{P_{i,t}^2}\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right)\right]=0$$

$$\implies (1-\theta)C_{i,t}+\theta\frac{W_t}{A_t}P_{i,t}^{-1}C_{i,t}-\frac{W_t\varphi}{P_{i,t-1}}\left(\frac{P_{i,t}-P_{i,t-1}}{P_{i,t-1}}\right)=-E_t\left[\beta\frac{C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\frac{W_{t+1}\varphi P_{i,t+1}}{P_{i,t}^2}\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right)\right]$$

$$\implies (1-\theta)C_{i,t}P_{i,t}+\theta\frac{W_t}{A_t}C_{i,t}-\frac{W_t\varphi P_{i,t}}{P_{i,t-1}}\left(\frac{P_{i,t}-P_{i,t-1}}{P_{i,t-1}}\right)=-E_t\left[\beta\frac{C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\frac{W_{t+1}\varphi P_{i,t+1}}{P_{i,t}}\left(\frac{P_{i,t+1}-P_{i,t}}{P_{i,t}}\right)\right]$$

(b) impose symmetry across producers and define inflation rate as $\pi_t=\frac{P_t}{P_{t-1}}-1$,

Symmetry across producers implies that $P_{i,t}=P_t$ and $C_{t,t}=C_t$:

$$(1-\theta)C_tP_t+\theta\frac{W_t}{A_t}C_t-\frac{W_t\varphi P_t}{P_{t-1}}\left(\frac{P_t-P_{t-1}}{P_{t-1}}\right)=-E_t\left[\beta\frac{C_t}{C_{t+1}}\frac{P_t}{P_{t+1}}\frac{W_{t+1}\varphi P_{t+1}}{P_t}\left(\frac{P_{t+1}-P_t}{P_t}\right)\right]$$

$$(1-\theta)C_tP_t+\theta\frac{W_t}{A_t}C_t-\frac{W_t\varphi P_t}{P_{t-1}}\left(\frac{P_t-P_{t-1}}{P_{t-1}}\right)=-E_t\left[\beta\frac{C_t}{C_{t+1}}W_{t+1}\varphi\left(\frac{P_{t+1}-P_t}{P_t}\right)\right]$$

Define inflation rate as $\pi_t=\frac{P_t}{P_{t-1}}-1=\frac{P_t-P_{t-1}}{P_{t-1}}$:

$$(1-\theta)P_t+\theta\frac{W_t}{A_t}=\varphi\left(\frac{W_t}{C_t}\pi_t(\pi_t+1)-\beta E_t\left[\frac{W_{t+1}}{C_{t+1}}\pi_{t+1}\right]\right)$$

(c) take the first-order approximation,

Define

$$\begin{aligned}U_t &:= (1-\theta)P_t+\theta\frac{W_t}{A_t}\\ V_t &:= \varphi\left(\frac{W_t}{C_t}\pi_t(\pi_t+1)-\beta E_t\left[\frac{W_{t+1}}{C_{t+1}}\pi_{t+1}\right]\right)\end{aligned}$$

Notice that $\bar{\pi}=0\implies\bar{V}=0\implies\bar{U}=0$:

$$\begin{aligned}u_t &= (1-\theta)\bar{P}(1+p_t)+\theta\frac{\bar{W}}{\bar{A}}(1+w_t-a_t)\\ &= (1-\theta)\bar{P}p_t+\theta\frac{\bar{W}}{\bar{A}}(w_t-a_t)+(1-\theta)\bar{P}+\theta\frac{\bar{W}}{\bar{A}}\\ &= (1-\theta)\bar{P}p_t+\theta\frac{\bar{W}}{\bar{A}}(w_t-a_t)+\bar{U}\\ &= (1-\theta)\bar{P}p_t+\theta\frac{\bar{W}}{\bar{A}}(w_t-a_t)\end{aligned}$$

Turning to V_t, define

$$M_t := \frac{W_t}{C_t}\pi_t$$

$$N_t := \frac{W_t}{C_t}\pi_t(\pi_t + 1)$$

Notice that $\bar{M} = \bar{N} = 0$:

$$m_t = \frac{\bar{W}}{\bar{C}}(1 + w_t - c_t)\pi_t = \frac{\bar{W}}{\bar{C}}[\pi_t + \pi_t(w_t - c_t)] \approx \frac{\bar{W}}{\bar{C}}\pi_t$$

Notice that the first order approximation of $\pi_t(\pi_t + 1)$ is π_t, so $n_t = m_t$. Thus, $u_t = v_t$ implies:

$$(1-\theta)\bar{P}p_t + \theta\frac{\bar{W}}{\bar{A}}(w_t - a_t) = \varphi\frac{\bar{W}}{\bar{C}}\Big(\pi_t - \beta E_t[\pi_{t+1}]\Big)$$

(d) write the NKPC in terms of inflation rate and output gap,

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(e) compare the NKPC to the one under the Calvo pricing.

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3. What is the source of inflation costs in this model? Is it different from the one in the Calvo model?

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4. Let the monetary policy be described by the Taylor rule $i_t = \varphi x_t + u_t$. What restrictionson the coefficient ϕ_x ensure uniqueness of the equilibrium?

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