# ECON 711 - PS 1

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## Question 1: The Law of Supply

Suppose $k = 3$, and a firm uses goods one and two as inputs and produces good three as output. (Formally, $y \in Y$ requires $y_1, y_2 \leq 0$.) For each of the following, either give an example showing it's possible or prove that it's impossible. (Feel free to use examples where $Y$ contains only a few points.)

(a) If $p_3$ falls and $p_1$ and $p_2$ stay the same, can the firm's output $y_3$ go up?

No.

Proof: Let $p = (p_1, p_2, p_3)$ and $p' = (p_1, p_2, p_3 + \delta)$, so $\Delta p = (p_1 - p_1, p_2 - p_2, p_3 - \delta - p_3) = (0, 0, -\delta)$ where $\delta > 0$. Furthermore, let $y = (y_1, y_2, y_3)$, $y' = (y_1', y_2', y_3')$, and $\Delta y = (y_1' - y_1, y_2' - y_2, y_3' - y_3) = (\Delta y_1, \Delta y_2, \Delta y_3)$. The general producer framework implies that the law of supply holds:

$$\begin{aligned}
\Delta p \cdot \Delta y &\geq 0 \\
\implies (0, 0, -\delta) \cdot (\Delta y_1, \Delta y_2, \Delta y_3) &\geq 0 \\
\implies (0)\Delta y_1 + (0)\Delta y_2 + (-\delta)\Delta y_3 &\geq 0 \\
\implies (-\delta)\Delta y_3 &\geq 0 \\
\implies \Delta y_3 &\leq 0
\end{aligned}$$

Therefore, the firm's output $y_3$ cannot increase. $\square$

(b) If $p_1$ rises and $p_2$ and $p_3$ stay the same, can the firm's output $y_3$ go up?

Yes.

Consider price vectors $p = (1, 1, 1)$ and $p' = (2, 1, 1)$ as well as production set $Y = \{y, z\}$ where $y = (-5, -1, 10)$ and $z = (-1, -7, 11)$. Notice that $\Delta p_1 = 1$, $\Delta p_2 = \Delta p_3 = 0$, and $\Delta y_3 = 1$. When the firm faces prices $p$, it chooses production plan $y$ because $p \cdot y > p \cdot z$ with $p \cdot y = -5 - 1 + 10 = 4$ and $p \cdot z = -1 - 7 + 11 = 3$. When the firm faces prices $p'$, it chooses production plan $z$ because $p' \cdot y < p' \cdot z$ with $p' \cdot y = -10 - 1 + 10 = -1$ and $p' \cdot z = -2 - 7 + 11 = 2$.

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, Tyler Welch, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

(c) If $p_1$ and $p_2$ both increase and $p_3$ stays the same, can the firm's output $y_3$ go up? What if $p_1$ and $p_2$ both increase by 10%?

Yes.

Consider price vectors $p = (1, 1, 1)$ and $p' = (2, 1.1, 1)$ as well as production set $Y = \{y, z\}$ where $y = (-5, -1, 10)$ and $z = (-1, -7, 11)$. Notice that $\Delta p_1 = 1, \Delta p_2 = 0.1, \Delta p_3 = 0$, and $\Delta y_3 = 1$. When the firm faces prices $p$, it chooses production plan $y$ because $p \cdot y > p \cdot z$ (with $p \cdot y = -5 - 1 + 10 = 4$ and $p \cdot z = -1 - 7 + 11 = 3$). When the firm faces prices $p'$, it chooses production plan $z$ because $p' \cdot y < p' \cdot z$ (with $p' \cdot y = -10 - 1.1 + 10 = -1.1$ and $p' \cdot z = -2 - 7.7 + 11 = 1.3$).

If $p_1, p_2$ both increase by 10% and $p_3 > 0$, $y_3$ cannot increase.[1] Below I prove this if $p_1, p_2$ increase to $\alpha p_1, \alpha p_2$ for any $\alpha > 1$.

Proof: Define $p = (p_1, p_2, p_3)$ and $p' = (\alpha p_1, \alpha p_2, p_3)$ where $\alpha > 1$ and $p_3 > 0$. Based on the optimal supply correspondences being homogeneous of order 0,

$$\begin{aligned} Y^*(p') &= Y^*((\alpha p_1, \alpha p_2, p_3)) \\ &= Y^*(\alpha(p_1, p_2, (1/\alpha)p_3)) \\ &= Y^*(p_1, p_2, (1/\alpha)p_3) \\ &= Y^*(\tilde{p}) \end{aligned}$$

where $\tilde{p} = (p_1, p_2, (1/\alpha)p_3)$. Thus, the set of optimal production plans is the same under $p'$ and $\tilde{p}$. Define $\Delta p = \tilde{p} - p = (0, 0, \frac{1-\alpha}{\alpha} p_3)$. Because $\alpha > 1$, $\Delta p_3 < 0$. By the law of supply, $\Delta p \cdot \Delta y \geq 0 \implies \Delta y_3 \Delta p_3 \geq 0 \implies \Delta y_3 \leq 0$. $\square$

[1] If $p_3 = 0$, the optimal supply correspondences are homogeneous of order 0 implies that

$$\begin{aligned} Y^*((p_1, p_2, 0)) &= Y^*((1/\alpha)(\alpha p_1, \alpha p_2, 0)) \\ &= Y^*((\alpha p_1, \alpha p_2, 0)) \end{aligned}$$

So, if $Y^*((p_1, p_2, 0))$ contains multiple elements, the firm may switch to a production plan with higher $y_3$ when faced by $(\alpha p_1, \alpha p_2, 0)$.

## Question 2: Rationalizability

Consider the following two "datasets":

Dataset 1

| $p$ | $y(p)$ |
|---|---|
| (7, 4) | (-20, 40) |
| (5, 5) | (-50, 60) |
| (4, 8) | (-70, 90) |

Dataset 2

| $p$ | $y(p)$ |
|---|---|
| (7, 4) | (-20, 40) |
| (5, 5) | (-40, 70) |
| (4, 8) | (-70, 90) |

For each one, determine whether the three observations are consistent with a profit-maximizing firm. If not, explain why not. If so, draw or describe:

(a) the smallest production set that can rationalize the data
(b) the smallest convex production set with free disposal and the shutdown property that can rationalize the data
(c) the largest production set that can rationalize the data

Data are consistent with a profit-maximizing firm iff the weak axiom of profit maximization holds.

The weak axiom of profit maximization fails to hold for Dataset 1. When $p = (5, 5)$, the firm chose production plan $(-50, 60)$. The profit associated with these prices and this production plan is 50 whereas the profit associated with the other two production plans in the data, $(-20, 40)$ and $(-70, 90)$, would have resulted in a profit of 100. Dot products of prices and production plans in the production set of Dataset 1 are below:

| $y \cdot p =$ | $y = (-20, 40)$ | $y = (-50, 60)$ | $y = (-70, 90)$ |
|---|---|---|---|
| $p = (7, 4)$ | 20 | -110 | -130 |
| $p = (5, 5)$ | 100 | 50 | 100 |
| $p = (4, 8)$ | 240 | 280 | 440 |

Proof: Define

$$
\begin{aligned}
Y^I &= \{(-20, 40), (-50, 60), (-70, 90)\} \\
Y^O &= \{y \in \mathbb{R}^k : (7, 4) \cdot y \leq 20, (5, 5) \cdot y \leq 50, \text{ and } (4, 8) \cdot y \leq 440\}.
\end{aligned}
$$

Consider $y' = (-20, 40) \in Y^I$, but $y' \notin Y^O$ because $y' \cdot (5, 5) = -20(5) + 40(4) = 100 > 50$. The weak axiom of profit maximization does not hold. Thus, Dataset 1 cannot be rationalized. $\square$

The weak axiom of profit maximization holds for Dataset 2. In Dataset 2, for each set of prices faced by the firm, there was no feasible production plan in the inner bound that would have been more profitable than the one the firm chose.

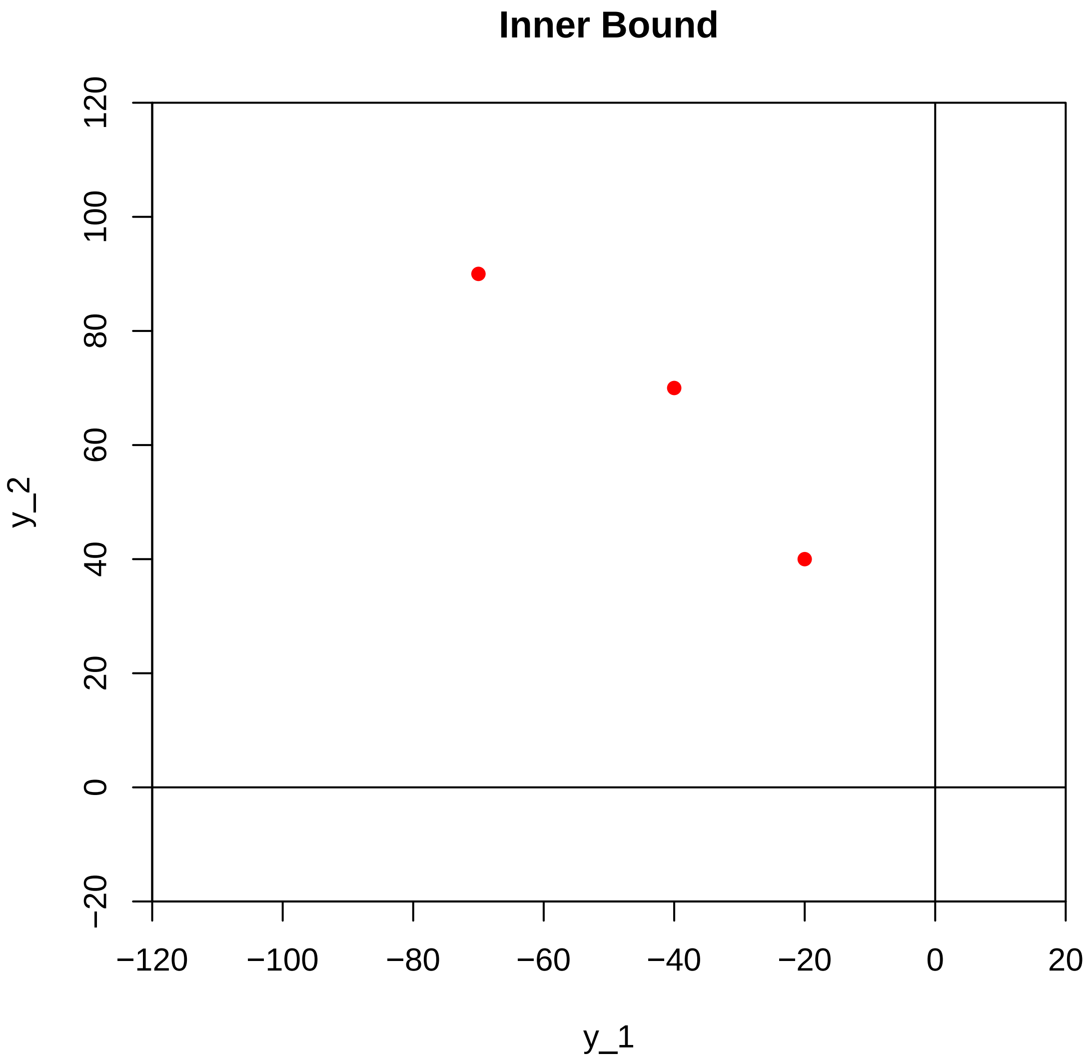
| $y \cdot p =$ | $y = (-20, 40)$ | $y = (-50, 70)$ | $y = (-70, 90)$ |
|---|---|---|---|
| $p = (7, 4)$ | 20 | -110 | -130 |
| $p = (5, 5)$ | 100 | 150 | 100 |
| $p = (4, 8)$ | 240 | 280 | 440 |

Proof: Define

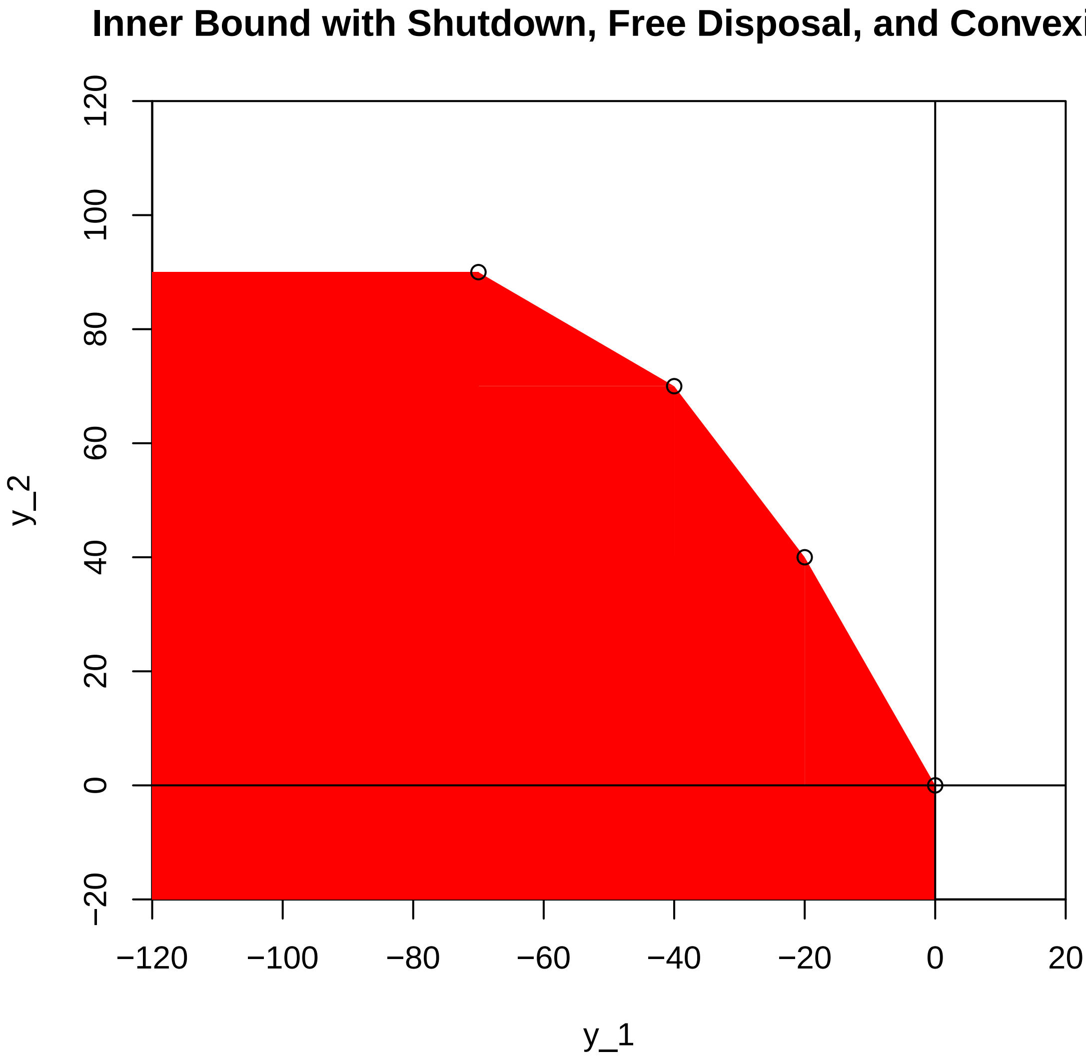
$$
\begin{aligned}
Y^I &= \{(-20, 40), (-50, 70), (-70, 90)\} \\
Y^O &= \{y \in \mathbb{R}^k : (7, 4) \cdot y \leq 20, (5, 5) \cdot y \leq 150, \text{ and } (4, 8) \cdot y \leq 440\}.
\end{aligned}
$$

For $(-20, 40) \in Y^I$, $(-20, 40) \in Y^O$ because $(7, 4) \cdot (-20, 40) = 20, (5, 5) \cdot (-20, 40) = 100 < 150$, and $(4, 8) \cdot (-20, 40) = 240 < 440$. For $(-50, 70) \in Y^I$, $(-50, 70) \in Y^O$ because $(7, 4) \cdot (-50, 70) = -110 < 20, (5, 5) \cdot (-50, 70) = 150$, and $(4, 8) \cdot (-50, 70) = 280 < 440$. For $(-70, 90) \in Y^I$, $(-70, 90) \in Y^O$ because $(7, 4) \cdot (-70, 90) = -130 < 20, (5, 5) \cdot (-70, 90) = 100 < 150$, and $(4, 8) \cdot (-70, 90) = 440$. Thus, for all $y \in Y^I$, $y \in Y^O$. The weak axiom of profit-maximization holds, so the data is rationalizable. $\square$

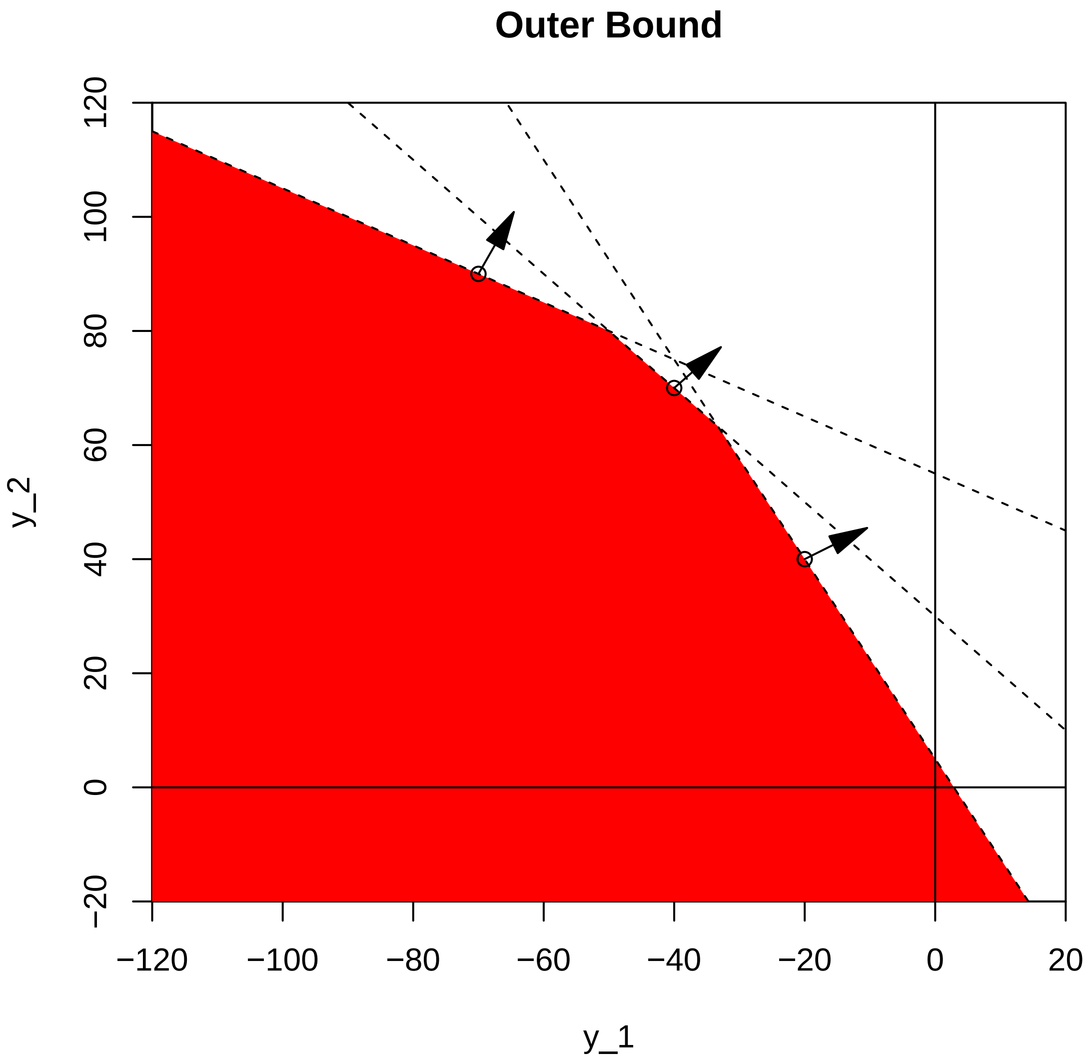
(a) The smallest production set that can rationalize the data is $Y^I$.



(b) The smallest convex production set with free disposal and the shutdown property that can rationalize the data is $\{z \in \mathbb{R}^k : z = yt + (1-t)y, t \in (0,1), y \in \{x \in \mathbb{R}^k : x \leq w \text{ for some } w \in Y^I \cup \{(0,0)\}\}\}$.



(c) The largest production set that can rationalize the data is $Y^O$.



# Question 3. Aggregate Production

Suppose an industry consists of $n$ profit-maximizing, price-taking firms, each with its own production set $Y_1, Y_2, ..., Y_n$. You observe industry-level data at several price vectors: instead of observing individual firm production $(y_1(p), y_2(p), ..., y_n(p))$, you observe only the sum $y_1(p) + ... + y_n(p)$. Will this aggregate data satisfy the Weak Axiom? Can industry be rationalized as if it were the choice of a single profit-maximizing firm? Explain. (You may find it helpful to use an example.)

Proof: For $p, p' \in P$, let $y_i \in Y_i^*(p)$ and $y_i' \in Y_i^*(p')$ be the production of firm $i$ under $p$ and $p'$, respectively. Since each firm $i$ is profit-maximizing, the weak axiom of profit maximization holds at the firm level, $p \cdot y_i \geq p \cdot y_i'$. Let $z = y_1 + ... + y_n$ and $z' = y_1' + ... + y_n'$ be aggregate production under $p$ and $p'$, respectively. To prove that aggregate data satisfy the weak axiom, we need to show that $p \cdot z \geq p \cdot z'$:

$$\begin{aligned} p \cdot z' &= p \cdot (y_1' + ... + y_n) \\ &= p \cdot y_1' + ... + p \cdot y_n' \\ &\leq p \cdot y_1 + ... + p \cdot y_n \\ &= p \cdot z \end{aligned}$$

Thus, the aggregate data satisfies the Weak Axiom and is rationalizable. □

So, yes, since the industry data is rationalizable, the industry can be treated as a single price-taking profit-maximizing firm.