ECON 711 - PS 6

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Question 1. Rationalizing Demand

Suppose you observe the following data on prices, wealth, and chosen consumption bundles for a certain consumer at four points in time:

w	p	x
100	(5, 5, 5)	(12, 4, 4)
100	(7, 4, 5)	(9, 3, 5)
100	(2, 4, 1)	(27, 9, 10)
150	(7, 4, 5)	(15, 5, 5)

(a) Are the data consistent with Walras Law?

Yes.

$$\begin{aligned}(5,5,5)\cdot(12,4,4) &= 100\\ (7,4,5)\cdot(9,3,5) &= 100\\ (2,4,1)\cdot(27,9,10) &= 100\\ (7,4,5)\cdot(15,5,5) &= 150\end{aligned}$$

(b) Can these data be rationalized by a continuous, monotonic and concave utility function?[1]

By Afriat's Theorem, we know that if data satisfy GARP, then there exists a LNS, continuous, concave, monotonic utility function that rationalizes the data.

- Consider x^1 and x^2. $x^1 \cdot p^1 = (12,4,4)\cdot(5,5,5) = 100$ and $x^2 \cdot p^1 = (9,3,5)\cdot(5,5,5) = 85$. So $x^1 \succ^D x^2$.
- Consider x^1 and x^3. Notice $x^3 > x^1$. By the hint, $p \cdot x^3 > p \cdot x^1$ for any $p >> 0$. So $x^3 \succ^D x^1$.
- Consider x^1 and x^4. Notice $x^4 > x^1$. By the hint, $x^4 \succ^D x^1$.
- Consider x^2 and x^3. Notice $x^3 > x^2$. By the hint, $x^3 \succ^D x^2$.
- Consider x^2 and x^4. Notice $x^4 > x^2$. By the hint, $x^4 \succ^D x^2$.
- Consider x^3 and x^4. Notice $x^3 > x^4$. By the hint, $x^3 \succ^D x^4$.

Thus, $x^3 \succ^D x^4 \succ^D x^1 \succ^D x^2$. The data satisfy GARP.

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[1]Hint: you don't need to calculate the cost of every bundle at every price; if $x^i > x^j$, then $p \cdot x^i > p \cdot x^j$ for any $p >> 0$.

Question 2. Aggregating Demand

Suppose there are n consumers, and consumer $i \in \{1, 2, ..., n\}$ has indirect utility function $v^i = a_i(p) + b(p)w_i$ where $\{a_i\}_{i=1}^n$ and b are differentiable functions from $\mathbb{R}_+^k$ to $\mathbb{R}$.

(a) Use Roy's Identity to calculate each consumer's Marshallian demand $x^i(p, w_i)$.

Since $\{a_i\}_{i=1}^n$ and b are differentiable, v^i is differentiable.

$$\frac{\partial v^i}{\partial w_i} = b(p)$$

$$\frac{\partial v^i}{\partial p} = \frac{\partial a_i}{\partial p} + \frac{\partial b}{\partial p} w_i$$

By Roy's Identity,

$$x^i(p, w_i) = -\frac{\partial v^i / \partial p}{\partial v / \partial w_i} = -\frac{\frac{\partial a_i}{\partial p} + \frac{\partial b}{\partial p} w_i}{b(p)}$$

(b) Calculate the Marshallian demand $X(p, W)$ of a "representative consumer" with wealth W and indirect utility function $V(p, W) = \sum_{i=1}^n a_i(p) + b(p)W$ show that $X(p, \sum_{i=1}^n w_i) = \sum_{i=1}^n x^i(p, w_i)$.

$$\frac{\partial V}{\partial W} = b(p)$$

$$\frac{\partial V}{\partial p} = \sum_{i=1}^n \frac{\partial a_i}{\partial p} + \frac{\partial b}{\partial p} W$$

By Roy's identity,

$$X(p, W) = -\frac{\partial V / \partial p}{\partial V / \partial W} = -\frac{\sum_{i=1}^n \frac{\partial a_i}{\partial p} + \frac{\partial b}{\partial p} W}{b(p)}$$

$$X(p, \sum_{i=1}^n w_i) = \frac{\sum_{i=1}^n a_i'(p) + b'(p) \sum_{i=1}^n w_i}{b(p)} = \frac{\sum_{i=1}^n [a_i'(p) + b'(p) w_i]}{b(p)} = \sum_{i=1}^n \frac{a_i'(p) + b'(p) w_i}{b(p)} = \sum_{i=1}^n x^i(p, w_i)$$

Question 3. Homothetic Preferences

Complete, transitive preferences $\succsim$ on $\mathbb{R}^k_+$ are called homothetic if for all $x, y \in \mathbb{R}^k_+$ and all $t > 0$, $x \succsim y \iff tx \succsim ty$.

(a) Show that if preferences are homothetic, Marshallian demand is homogeneous of degree 1 in wealth: for any $t > 0, x(p, tw) = tx(p, w)$.

Fix $w \in \mathbb{R}_+$ and $p \in \mathbb{R}^k_+$. Let $\succsim_H$ be a homothetic preference relation. Let $x^* := x(p, w) = \{x \in B(p, w) : x \succsim_H y \; \forall y \in B(p, w)\}$ and $x' \in B(p, w)$. By definition, $x^* \succsim_H x'$. Since $\succsim_H$ is homothetic, $tx^* \succsim_H tx'$ for $t > 0$. Since the budget set is linear, $tx^*, tx' \in B(p, tw)$. Since tx' represents an arbitrary element of $B(p, tw)$, $tx^* = \{x \in B(p, tw) : x \succsim_H y \; \forall y \in B(p, tw)\} = x(p, tw)$. Therefore, Marshallian demand is homogeneous of degree 1 in wealth.

(b) Show that if preferences are homothetic, monotone, and continuous, they can be represented by a utility function which is homogeneous of degree 1. (Hint try the utility function we used to prove existence of a utility function in class.)

Let $\succsim_H$ be a homothetic, monotone, and continuous preference relation. Since $\succsim_H$ is homothetic, it is also complete and transitive. Thus, $\succsim_H$ can be represented by utility function u such that, for $x \in \mathbb{R}^k_+$, $u(x) = \alpha$ where $x \succsim_H \alpha e$ and $\alpha e \succsim_H x$ and $e = (1, ..., 1)$. Thus, because $\succsim_H$ is homothetic, $tx \succsim_H t\alpha e$ and $t\alpha e \succsim_H tx$ for $t > 0$, so $u(tx) = t\alpha$.

(c) Show that given (a) and (b), the indirect utility function takes the form $v(p, w) = b(p)w$ for some function b.

Fix $w \in \mathbb{R}_+$ and $p \in \mathbb{R}^k_+$. Since Marshallian demand and utility functions associated with homothetic preferences are homogeneous of degree 1:

$$v(p, w) = u(x(p, w)) = u(wx(p, 1)) = wu(x(p, 1)) = wb(p)$$

where $b(p) = u(x(p, 1))$.

Question 4. Quasilinear Utility

Let $X = \mathbb{R} \times \mathbb{R}_+^{k-1}$ (allow positive or negative consumption of the first good), suppose utility $u(x) = x_1 + U(x_2, ..., x_k)$ is quasilinear, and fix the price of the first good $p_1 = 1$.

(a) Show that Marshallian demand for goods 2 through k does not depend on wealth.

Notice that u represents LNS preferences; for any $x = (x_1, x_2, ..., x_k) \in X$ and $\varepsilon > 0$, let $x' = (x_1 + \varepsilon, x_2, ..., x_k) \in X$.

$$||x' - x|| = ||(x_1 + \varepsilon, x_2, ..., x_k) - (x_1, x_2, ..., x_k)|| = ||(\varepsilon, 0, ..., 0)|| = \varepsilon$$

$$u(x_1 + \varepsilon, x_2, ..., x_k) = x_1 + \varepsilon + U(x_2, ..., x_k) > x_1 + U(x_2, ..., x_k) = u(x_1, x_2, ..., x_k) \implies x' \succ x$$

Because u represents LNS preferences, we know Walras' Law hold and the budget constraint holds with equality. Thus, we can substitute in $x_1 = w - p_2x_2 - ... - p_kx_k$ into Marshallian demand:

$$\begin{aligned}
x(p,w) &= \arg\max_{x \in B(p,w)} u(x) \\
&= \arg\max_{x \in B(p,w)} \{x_1 + U(x_2, ..., x_k)\} \\
&= \arg\max\{w - p_2x_2 - ... - p_kx_k + U(x_2, ..., x_k)\} \\
&= \arg\max\{U(x_2, ..., x_k) - p_2x_2 - ... - p_kx_k\}
\end{aligned}$$

So Marshallian demand does not depend on w.

(b) Show that indirect utility can be written as $v(p,w) = w + \tilde{v}(p)$ for some function $\tilde{v}$.

As we showed in (a), u represents LNS preferences, so Walras' Law holds:

$$\begin{aligned}
v(p,w) &= \max_{x \in B(p,w)} u(x) \\
&= \max_{x \in B(p,w)} \{x_1 + U(x_2, ..., x_k)\} \\
&= \max\{w - p_2x_2 - ... - p_kx_k + U(x_2, ..., x_k)\} \\
&= w + \max\{U(x_2, ..., x_k) - p_2x_2 - ... - p_kx_k\} \\
&= w + \tilde{v}(p)
\end{aligned}$$

where $\tilde{v}(p) = \max\{U(x_2, ..., x_k) - p_2x_2 - ... - p_kx_k\}$.

(c) Show the expenditure function can be written as $e(p, u) = u - f(p)$ for some function f.

Assume that U is continuous, so u is continuous. Thus, the "no excess utility" condition holds, so if $u \geq u(0)$, then for any $x \in h(p, u), u(x) = u$. Thus, $x_1 = u - U(x_2, ..., x_k)$:

$$\begin{aligned}
e(p, u) &= \min_x \{x_1 + p_2x_2 + ... + p_kx_k\} \\
&= \min\{u - U(x_2, ..., x_k) + p_2x_2 + ... + p_kx_k\} \\
&= u + \min\{p_2x_2 + ... + p_kx_k - U(x_2, ..., x_k)\} \\
&= u + f(p)
\end{aligned}$$

where $f(p) = \min_{\{x_2,...,x_k\}}\{p_2x_2 + ... + p_kx_k - U(x_2, ..., x_k)\}$.

(d) Show that the Hicksian demand for goods 2 through k does not depend on target utility.

Because the "no excess utility" condition holds,

$$\begin{aligned}
h(p, u) &= \arg\min_x \{u - U(x_2, ..., x_k) + p_2x_2 + ... + p_kx_k\} \\
&= \arg\min\{p_2x_2 + ... + p_kx_k - U(x_2, ..., x_k)\}
\end{aligned}$$

So Hicksian demand does not depend on w.

(e) Show that Compensating Variation and Equivalent Variation are the same when the price of good $i \neq 1$ changes, and also equal to Consumer Surplus.

Let $p^0 \in \mathbb{R}^k_+$, $u^0, u^1 \in \mathbb{R}$. Construct Δp such that $\Delta p_i > 0$ and $\Delta p_j = 0\ \forall j \neq i$. Let $p^1 := p^0 + \Delta p$. With homothetic preferences, Compensating Variation and Equivalent Variation both equal $u^1 - u^0$:

$$EV = e(p^0, u^1) - e(p^1, u^1) = e(p^0, u^1) - e(p^0, u^0) = (u^1 + f(p^0)) - (u^0 + f(p^0)) = u^1 - u^0$$

$$CV = e(p^0, u^0) - e(p^1, u^0) = e(p^1, u^1) - e(p^1, u^0) = (u^1 + f(p^1)) - (u^0 + f(p^1)) = u^1 - u^0$$

As we saw in (a) and (d), Marshallian demand is the arg max of a function and Hicksian demand is the arg min of the negative of that function:

$$x(p, w) = \arg\max\{U(x_2, ..., x_k) - p_2x_2 - ... - p_kx_k\} = \arg\min\{p_2x_2 + ... + p_kx_k - U(x_2, ..., x_k)\} = h(p, u)$$

Thus, consumer surplus equals Compensating Variation and Equivalent Variation because there is no wealth effects.