ECON 714A - Problem Set 2

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Consider a growth model with preferences $\sum_{t=0}^{\infty} \beta^t \log C_t$, production function $Y_t = AK_t^\alpha$, the capital law of motion $K_{t+1} = K_t^{1-\delta} I_t^\delta$, and the resource constraint $Y_t = C_t + I_t$.

1. Write down the social planner's problem and derive the Euler equation. Provide the intuition to this optimality condition using the perturbation argument.

The social planner's problem is

$$
\begin{aligned}
&\max_{\{C_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log C_t \\
&\text{s.t. } AK_t^\alpha = C_t + I_t \\
&\quad K_{t+1} = K_t^{1-\delta} I_t^\delta \\
\implies &\max_{\{C_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log C_t \\
&\text{s.t. } 0 = K_t^{1-\delta}(AK_t^\alpha - C_t)^\delta - K_{t+1}
\end{aligned}
$$

The legrangian is

$$
\sum_{t=0}^{\infty} \beta^t \log C_t + \lambda_t [K_t^{1-\delta}(AK_t^\alpha - C_t)^\delta - K_{t+1}]
$$

FOC (C_t):

$$
\begin{aligned}
0 &= \frac{\beta^t}{C_t} - \lambda_t \delta K_t^{1-\delta}(AK_t^\alpha - C_t)^{\delta - 1} \\
\lambda_t &= \frac{\beta^t}{C_t \delta K_t^{1-\delta} I_t^{\delta-1}}
\end{aligned}
$$

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FOC (K_{t+1}):

$$\begin{aligned}0 &= -\lambda_t + \lambda_{t+1}[(1-\delta)K_{t+1}^{-\delta}(AK_{t+1}^{\alpha} - C_{t+1})^{\delta} + \delta K_{t+1}^{1-\delta}(AK_{t+1}^{\alpha} - C_{t+1})^{\delta-1}A\alpha K_{t+1}^{\alpha-1}] \\ \frac{\lambda_t}{\lambda_{t+1}} &= (1-\delta)(\frac{I_{t+1}}{K_{t+1}})^{\delta} + \delta(\frac{I_{t+1}}{K_{t+1}})^{\delta-1}A\alpha K_{t+1}^{\alpha-1}\end{aligned}$$

FOCs imply the consumption Euler equation:

$$\begin{aligned}\frac{(\frac{\beta^t}{C_t\delta K_t^{1-\delta}I_t^{\delta-1}})}{(\frac{\beta^{t+1}}{C_{t+1}\delta K_{t+1}^{1-\delta}I_{t+1}^{\delta-1}})} &= (1-\delta)(\frac{I_{t+1}}{K_{t+1}})^{\delta} + \delta(\frac{I_{t+1}}{K_{t+1}})^{\delta-1}A\alpha K_{t+1}^{\alpha-1} \\ \frac{C_{t+1}K_{t+1}^{1-\delta}I_{t+1}^{\delta-1}}{\beta C_t K_t^{1-\delta}I_t^{\delta-1}} &= (1-\delta)(\frac{I_{t+1}}{K_{t+1}})^{\delta} + \delta(\frac{I_{t+1}}{K_{t+1}})^{\delta-1}A\alpha K_{t+1}^{\alpha-1} \\ C_{t+1} &= C_t\beta(\frac{K_t}{K_{t+1}})^{1-\delta}(\frac{I_t}{I_{t+1}})^{\delta-1}[(1-\delta)(\frac{I_{t+1}}{K_{t+1}})^{\delta} + \delta(\frac{I_{t+1}}{K_{t+1}})^{\delta-1}A\alpha K_{t+1}^{\alpha-1}] \\ C_{t+1} &= C_t\beta\left(\frac{K_t}{I_t}\right)^{1-\delta}\left[(1-\delta)\frac{I_{t+1}}{K_{t+1}} + \delta A\alpha K_{t+1}^{\alpha-1}\right]\end{aligned}$$

Perturbation argument:

To argue that the EE represents the optimal path, assume we're on the equilibrium path for capital $\{K_t\}_{t=0}^{\infty}$ and a deviation at some t such that $\tilde{K}_{t+1} < K_{t+1}$. To achieve such a deviation, $\tilde{C}_t = C_t + \Delta$ for some $\Delta > 0$. The feasibility constraint implies that $\tilde{I}_t = I_t - \Delta$. Thus, capital in period $t+1$ can be approximated as:

$$\begin{aligned}\tilde{K}_{t+1} &= \tilde{K}_t^{1-\delta}\tilde{I}_t^{\delta} \\ &= K_t^{1-\delta}(I_t - \Delta)^{\delta} \\ &\approx K_t^{1-\delta}I_t^{\delta} - \delta K_t^{1-\delta}I_t^{\delta-1}\Delta \\ &= K_{t+1} - K_{t+1}\frac{\delta\Delta}{I_t}\end{aligned}$$

With lower capital, production in $t+1$ is less:

$$\begin{aligned}\tilde{Y}_{t+1} &= A\tilde{K}_{t+1}^{\alpha} \\ &= A\left(K_{t+1} - \frac{\delta K_{t+1}\Delta}{I_t}\right)^{\alpha} \\ &\approx AK_{t+1}^{\alpha} - A\alpha K_{t+1}^{\alpha-1}\frac{\delta K_{t+1}\Delta}{I_t} \\ &= Y_{t+1} - Y_{t+1}\frac{\alpha\delta\Delta}{I_t}\end{aligned}$$

2. Derive the system of equations that pins down the steady state of the model.

In a steady state, $C_t = C_{t+1} = \bar{C}, K_t = K_{t+1} = \bar{K}$, and $I_t = I_{t+1} = \bar{I}$.

From the law of motion of capital:

$$\bar{K} = \bar{K}^{1-\delta}\bar{I}^{\delta} \implies \bar{K} = \bar{I}$$

From the consumption Euler equation and $\bar{K} = \bar{I}$:

$$\bar{C} = \bar{C}\beta(\frac{\bar{K}}{\bar{K}})^{1-\delta}[(1-\delta)\frac{\bar{K}}{\bar{K}} + \delta A\alpha\bar{K}^{\alpha-1}] \implies \bar{K} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\delta A\alpha}\right)^{\frac{1}{\alpha-1}}$$

From the resource constraint and $\bar{K} = \bar{I}$:

$$A\bar{K}^{\alpha} = \bar{C} + \bar{K} \implies \bar{C} = A\bar{K}^{\alpha} - \bar{K}$$

3. Log-linearize the equilibrium conditions around the steady state.

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4. Write down a dynamic system with one state variable and one control variable. Use the Blanchard-Kahn method to solve this system for a saddle path.

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5. Show that the obtained solution is not just locally accurate, but is in fact the exact solution to the planner's problem.

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6. Generalize the (global) solution to the case of stochastic productivity shocks A_t.

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7. The analytical tractability of the model is due to special functional form assumptions, which however, have strong economic implications. What is special about consumption behavior in this model? Provide economic intuition.

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8. Bonus task: can you introduce labor into preferences and production function without compromising the analytical tractability of the model?

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