ECON 709 - PS 1

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(1) For two events $A, B \in S$, prove that $A \cup B = (A \cap B) \cup ((A \cap B^C) \cup (B \cap A^C))$.

Proof: Applying the partition rule and the properties of set operators,

$$\begin{aligned} A \cup B &= ((A \cap B) \cup (A \cap B^C)) \cup B \\ &= ((A \cap B) \cup (A \cap B^C)) \cup ((B \cap A) \cup (B \cap A^C)) \\ &= ((A \cap B) \cup (B \cap A)) \cup ((A \cap B^C) \cup (B \cap A^C)) \\ &= (A \cap B) \cup ((A \cap B^C) \cup (B \cap A^C)) \end{aligned}$$

□

(2) Prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof: Applying the partition rule and axoims of the probability measure,

$$\begin{aligned} P(A \cup B) &= P(((A \cap B) \cup (A \cap B^C)) \cup ((B \cap A) \cup (B \cap A^C))) \\ &= P((A \cap B) \cup (A \cap B^C) \cup (B \cap A^C)) \\ &= P(A \cap B) + P(A \cap B^C) + P(B \cap A^C) \\ &= P(A \cap B) + P(A \cap B^C) + P(B \cap A^C) + P(A \cap B) - P(A \cap B) \\ &= P((A \cap B) \cup (A \cap B^C)) + P(B \cap A^C) \cup (A \cap B)) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

□

(3) Suppose that the unconditional probability of a disease is 0.0025. A screening test for this disease has a detection rate of 0.9, and has a false positive rate of 0.01. Given that the screening test returns positive, what is the conditional probability of having the disease?

(4) Suppose that a pair of events A and B are mutually exclusive, i.e., $A \cap B = \emptyset$, and that $P(A) > 0$ and $P(B) > 0$. Prove that A and B are not independent.

(5) (Conditional Independence) Sometimes, we may also use the concept of conditional independence. The definition is as follows: let A, B, C be three events with positive probabilities. Then A and B are independent given C if $P(A \cap B|C) = P(A|C)P(B|C)$. Consider the experiment of tossing two dice. Let $A = \{\text{First die is 6}\}$, $B = \{\text{Second die is 6}\}$, and $C = \{\text{Both dice are the same}\}$.

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(a) Show that A and B are independent (unconditionally), but A and B are dependent given C.

(b) Consider the following experiment: let there be two urns, one with 9 black balls and 1 white balls and the other with 1 black ball and 9 white balls. First randomly (with equal probability) select one urn. Then take two draws with replacement from the selected urn. Let A and B be drawing a black ball in the first and the second draw, respectively, and let C be the event urn 1 is selected. Show that A and B are not independent, but are conditionally independent given C.

(6) A CDF F_X is stochastically greater than a CDF F_Y if $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Prove that if $X \sim F_X$ and $Y \sim F_Y$, then

$$P(X > t) \geq P(Y > t) \text{ for every } t,$$

and

$$P(X > t) > P(Y > t) \text{ for some } t,$$

that is, X tends to be bigger than Y.

(7) Show that the function $F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - \exp(-x), & x \geq 0 \end{cases}$ is a CDF, and $f_X(x)$ and $F_X^{-1}(y)$.