ECON 709 - PS 1

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9/14/2020

1. Suppose that $Y = X^3$ and $f_X(x) = 42x^5(1-x), x \in (0,1)$. Find the PDF of Y, and show that the PDF integrates to 1.

Notice that $Y = X^3$ is a monotone transformation, so we can use the following theorem from the lecture notes:

$$
\begin{aligned}
f_Y(y) &= \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|, y \in Y \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} 42(y^{1/3})^5(1-y^{1/3})|(1/3)y^{-2/3}|, y \in (0,1) \\ 0, \text{ otherwise} \end{cases} \\
&= \begin{cases} 14y(1-y^{1/3}), y \in (0,1) \\ 0, \text{ otherwise} \end{cases}
\end{aligned}
$$

where $g^{-1}(y) = y^{1/3}$ and $Y = \{0^3, 1^3\} = \{0, 1\}$.

$f_Y(y)$ integrates to 1:

$$
\begin{aligned}
\int_0^1 14t(1-t^{1/3})dt &= 14\left[y^2/2 - \frac{y^{7/3}}{7/3}\right]_0^1 \\
&= 14\left[\frac{1}{2} - \frac{3}{7}\right] \\
&= 1
\end{aligned}
$$

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

2. For the following CDF and PDF, show that f_X is the density function of F_X as long as $a \geq 0$. That is, show that for all $x \in [0,1], F_X(x) = \int_0^x f_X(t)dt$.

$$F_X(x) = \begin{cases} 1.2x, x \in [0, 0.5) \\ 0.2 + 0.8x, x \in [0.5, 1] \end{cases}$$

$$f_X(x) = \begin{cases} 1.2, x \in [0, 0.5) \\ a, x = 0.5 \\ 0.8, x \in (0.5, 1] \end{cases}$$

Case 1: $x < 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^x 1.2dt \\ &= 1.2x \\ &= F_X(x) \end{aligned}$$

Case 2: $x = 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} adt \\ &= 1.2(0.5) + 0 \\ &= 0.6 \\ &= 0.2 + 0.8(0.5) \\ &= F_X(0.5) \end{aligned}$$

Case 3: $x > 0.5$

$$\begin{aligned} \int_0^x f_X(t)dt &= \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} adt + \int_{0.5}^{x} 0.8dt \\ &= 1.2(0.5) + 0 + 0.8x - 0.8(0.5) \\ &= 0.6 + 0.8x - 0.4 \\ &= 0.2 + 0.8x \\ &= F_X(x) \end{aligned}$$

3. Let X have PDF $f_X(x) = \frac{2}{9}(x+1), x \in [-1, 2]$. Find the PDF of $Y = X^2$.

For $x \in [-1, 2]$

$$\begin{aligned} F_X(x) &= \int_{-1}^{x} \frac{2}{9}(t+1)dt \\ &= \frac{2}{9}\left[\frac{t^2}{2} + t\right]_{-1}^{x} \\ &= \frac{2}{9}\left[\frac{x^2}{2} + x - \left(\frac{1}{2} - 1\right)\right] \\ &= \frac{x^2}{9} + \frac{2x}{9} + \frac{1}{9} \end{aligned}$$

Thus,

$$F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{x^2}{9} + \frac{2x}{9} + \frac{1}{9}, & x \in [-1, 2] \\ 1, & x > 2 \end{cases}$$

Consider $Y = X^2$. First, notice that $y \in [0, 4]$. I consider two cases $y \in [0, 1]$ and $y \in (1, 4]$

Case 1: $y \in [0, 1]$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\ &= \left[\frac{y}{9} + \frac{2\sqrt{y}}{9} + \frac{1}{9}\right] - \left[\frac{y}{9} - \frac{2\sqrt{y}}{9} + \frac{1}{9}\right] \\ &= \frac{4\sqrt{y}}{9} \end{aligned}$$

Case 2: $y \in (1, 4]$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) \\ &= \frac{y}{9} + \frac{2\sqrt{y}}{9} + \frac{1}{9} \end{aligned}$$

Thus, the CDF and PDF of Y is:

$$F_X(x) = \begin{cases} 0, & y < 0 \\ \frac{4\sqrt{y}}{9}, & y \in [0, 1] \\ \frac{y}{9} + \frac{2\sqrt{y}}{9} + \frac{1}{9}, & y \in (1, 4] \\ 1, & y > 4 \end{cases}$$

$$f_X(x) = \begin{cases} \frac{2}{0\sqrt{y}}, & y \in [0, 1] \\ \frac{1}{9} + \frac{1}{9\sqrt{y}}, & y \in (1, 4] \\ 0, & \text{otherwise.} \end{cases}$$

4. A median of a distribution is a value m such that $P(X \leq m) \geq 1/2$ and $P(X \geq m) \geq 1/2$. Find the median of the distribution $f(x) = \frac{1}{\pi(1+x^2)}, x \in \mathbb{R}$.

The CDF of X is

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+t^2} dt \\ &= \frac{1}{\pi} \Big[\tan^{-1}(t) \Big]_{-\infty}^{x} \\ &= \frac{1}{\pi} \left[\tan^{-1}(x) - \lim_{t \to -\infty} \tan^{-1}(t) \right] \\ &= \frac{1}{\pi} \left[\tan^{-1}(x) - \frac{\pi}{2} \right] \end{aligned}$$

Now, notice that the distribution is symmetric around 0, so we will consider $m = 0$

$$\begin{aligned} P(X \leq 0) &= F(0) \\ &= \frac{1}{\pi} \left[\tan^{-1}(0) - \frac{\pi}{2} \right] \\ &= \frac{1}{\pi} \left[0 - \frac{\pi}{2} \right] \\ &= \frac{1}{2} \\ P(X \geq 0) &= 1 - P(X \leq 0) \\ &= 1 - F(0) \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

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