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ECON 709 Midterm

① $\log(X) \sim N(0,1) \Rightarrow \log(X) = Z \quad Z \sim N(0,1)$
$\Rightarrow X = \exp(Z)$

(a) Find $P(X \leq x \mid X > 1)$ for $x > 1$

$$P(X \leq x \mid X > 1) = \frac{P(X \leq x \wedge X > 1)}{P(X > 1)}$$

$$= \frac{P(1 < X \leq x)}{P(X > 1)}$$

$\log(X) = Z$
$\Rightarrow X = \exp(Z)$
$\Rightarrow g(x) = \exp(x)$
$\Rightarrow g^{-1}(y) = \log(y)$

$$= \frac{F_x(x) - F_x(1)}{F_x(1)}$$

$$= \frac{\Phi(\log(x)) - \Phi(\log(1))}{\Phi(\log(1))}$$

$$= \frac{\Phi(\log(x)) - \Phi(0)}{\Phi(0)}$$

$$= \boxed{\frac{\Phi(\log(x)) - 0.5}{0.5}}$$

① b. $\log(X) \sim N(\mu, 1)$ propose unbiased estimator and prove the unbiasness

Notice $E[\log(x)] = \mu$.

So let $\hat{\mu}_n = n^{-1} \sum_{i=1}^{n} \log(X_i)$

$\hat{\mu}_n$ is unbiased:

$$E(\hat{\mu}_n) = E\left(n^{-1} \sum_{i=1}^{n} \log(X_i)\right)$$

$$= n^{-1} \sum_{i=1}^{n} E(\log(X_i))$$

$$= n^{-1} \sum_{i=1}^{n} \mu$$

$$= n^{-1} n \mu$$

$$= \mu.$$

2.

		Y		
		2	1	0
X	4	.08	.02	0
	2	.07	.10	.03
	0	.05	.48	.17

(a) Find marginal distribution of Y.

$$\Pr\{Y=2\} = \Pr\{Y=2 \mid X=4\} + \Pr\{Y=2 \mid X=2\} + \Pr\{Y=2 \mid X=0\}$$

$$= .08 + .07 + .05$$
$$= .2$$

$$\Pr\{Y=1\} = \Pr\{Y=1 \mid X=4\} + \Pr\{Y=1 \mid X=2\} + \Pr\{Y=1 \mid X=0\}$$

$$= .02 + .10 + .48$$

$$= .6$$

$$\Pr\{Y=0\} = \Pr\{Y=0 \mid X=4\} + \Pr\{Y=0 \mid X=2\} + \Pr\{Y=0 \mid X=0\}$$

$$= 0 + .03 + .17$$

$$= .2$$

$$\boxed{f(y) = \begin{cases} .2 & \text{if } y=2 \\ .6 & \text{if } y=1 \\ .2 & \text{if } y=0 \\ 0 & \text{otherwise} \end{cases}}$$

② (a) cont

$E(Y) = 2\cdot Pr\{Y=2\} + 1\cdot Pr\{Y=1\} + 0\cdot Pr\{Y=0\}$

$= 2\cdot(.2) + 1\cdot(.6)$

$= .4 + .6$

$= \boxed{1}$

② b. $E[Y|X]$ PMF

$E[Y|X=4] = (.08(2) + .02(1) + 0\cdot 0)/(.08+.02+0)$

$\boxed{P(X=4) = .08 + .02 = .1}$

$= (.16 + .02)/(.1)$

$= (.18) = 1.8$

$E[Y|X=2] = (.07(2) + .10(1) + 0\cdot .03)/(.07+.1+.2)$

$\boxed{P(X=2) = .07 + .1 + .03 = .2}$

$= (.14 + .10)/(.2)$

$= (.28) = 1.4$

$E[Y|X=0] = (.05(2) + .48(1) + 0(.17))/(.05+.48+.17)$

$\boxed{P(X=0) = .05 + .48 + .17 = .7}$

$= (.1 + .48)/(.7)$

$= .58/(.7) \approx 0.83$

PMF of $z = E[Y|X]$ is

$$f(z) = \begin{cases} .1 & \text{if } z = 1.8 \\ .2 & \text{if } z = 1.4 \\ .7 & \text{if } z \approx 0.83 \\ 0 & \text{otherwise} \end{cases}$$

~~2) c. $E[E[Y|X]] = E[Z]$~~

~~$= .1 \cdot .18 + .2 \cdot .28 + .7 \cdot .58$~~

~~≈ 0.04~~

~~I think $E[Y|X] = E[Y]$, but I must have made an ~~math~~ algebra error~~

2) c. $E[E(Y|X)] = E[Z]$

$= .1 \cdot 1.8 + .2 \cdot 1.4 + .7 \cdot .83$

$\approx 1.$

$E(E(Y|X)) = E(Y).$ ~~The [illegible]~~

The arithmetic is slightly off due to round for $E(Y|X=0)$. This makes sense the overall average equal the average of the group averages.

3. $f_X(x) = \begin{cases} cx^2 & x \in [-2, 1] \\ 0 & \text{otherwise} \end{cases}$

(a) ~~If~~ The PDF should integrate to 1

$$\int_{-2}^{1} f_X(x)\,dx = 1$$

$$\Rightarrow \int_{-2}^{1} cx^2\,dx = 1$$

$$\Rightarrow \left[\frac{cx^3}{3}\right]_{-2}^{1} = 1$$

$$\Rightarrow c\left[\frac{1^3}{3} - \frac{(-2)^3}{3}\right] = 1$$

$$\Rightarrow c\left[\frac{1}{3} - \frac{-8}{3}\right] = 1$$

$$3c = 1$$

$$\boxed{c = 1/3}$$

③ b. $Y = X^2$. Find PDF of Y

CDF of X is

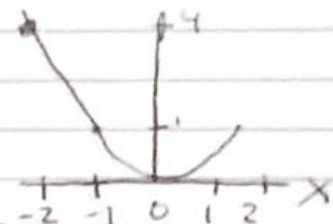
$$F_X(x) = \int_{-2}^{x} \frac{t^2}{3}\, dt = \left[\frac{t^3}{9}\right]_{-2}^{x}$$

$$= \left[\frac{x^3}{9} - \frac{(-8)}{9}\right] = \frac{x^3}{9} + 8/9$$

[Check $F_X(-2) = \frac{(-2)^3}{9} + \frac{8}{9} = \frac{-8}{9} + \frac{8}{9} = 0$

$F_X(1) = \frac{1^3}{9} + \frac{8}{9} = 1.$]

$Supp(X) = [-2, 1] \Rightarrow Supp(Y) = [0, 4]$



For $Y \in [0, 1]$

$F_Y(y) = Pr\{Y \le y\}$

$= Pr\{X^2 \le y\} = Pr\{-\sqrt{y} \le X^2 \le \sqrt{y}\}$

$= Pr\{X \le \sqrt{y}\} - Pr\{X \le -\sqrt{y}\} = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$= \left[\frac{\sqrt{y}^3}{9} + \frac{8}{9}\right] - \left[\frac{(-\sqrt{y})^3}{9} + \frac{8}{9}\right]$

$= \frac{y^{3/2}}{9} + \frac{y^{3/2}}{9} = \frac{2y^{3/2}}{9}$

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{\cancel{2} y^{1/2}}{3\cancel{9}}\left(\frac{\cancel{3}}{\cancel{2}}\right) = \frac{\sqrt{y}}{3}$

3 b. cont For $Y \in \cancel{\ }$ $[1,4]$

$$F_Y(y) = Pr\{Y \le y\} = Pr\{X^2 \le y\}$$

$$= Pr\{-\sqrt{y} \le X \le \sqrt{y}\} = Pr\{-\sqrt{y} \le X \le 1\}$$

$$= F_X(1) - F_X(-\sqrt{y})$$

$$= \left[\frac{1}{9} + \frac{8}{9}\right] - \left[\frac{(-\sqrt{y})^3}{9} + \frac{8}{9}\right] = \frac{1}{9} + \frac{y^{3/2}}{9}$$

$$f_Y(y) = \frac{d}{dy}\left(\frac{1}{9} + \frac{y^{3/2}}{9}\right) = \left(\frac{3}{2}\right)\frac{\sqrt{y}}{9_3} = \frac{\sqrt{y}}{6}$$

$$\boxed{f_Y(y) = \begin{cases} \sqrt{y}/3 & \text{if } y \in [0,1] \\ \sqrt{y}/6 & \text{if } y \in (1,4] \\ 0 & \text{, otherwise} \end{cases}}$$

independent

4. $X_i \overset{iid}{\sim} N(\mu_x, \sigma_x^2)$ $\quad Y_i \overset{iid}{\sim} N(\mu_Y, \sigma_Y^2)$

a. What is the distribution of ~~$\hat{\theta} = \bar{X}_X - \bar{X}_Y$~~

~~By CLT~~ $\hat{\theta} = \bar{X}_{n_x} - \bar{Y}_{n_y}$

By CLT, we know $\dfrac{\sqrt{n_x}(\bar{X}_{n_x} - \mu_x)}{\sigma_x} \sim N(0,1)$.

So $\bar{X}_{n_x} \sim N(\mu_x, \frac{\sigma_x^2}{n_x})$. Similarly, $\bar{Y}_{n_y} \sim N(\mu_Y, \frac{\sigma_Y^2}{n_Y})$.

So $\hat{\theta} = \bar{X}_{n_x} - \bar{Y}_{n_y} \sim N(\mu_x - \mu_Y; \frac{\sigma_x^2}{n_x} + \frac{\sigma_Y^2}{n_Y})$.

b. $\sigma_x^2 = \sigma_Y^2 = 1$. Propose a hypothesis test $H_0: \mu_x = \mu_Y$

$H_1: \mu_x \neq \mu_Y$.

Propose test statistic $\left| \dfrac{(\bar{X}_{n_x} - \bar{Y}_{n_Y})}{\sqrt{\frac{1}{n_x} + \frac{1}{n_Y}}} \right|$ with critical

value $(1-\frac{\alpha}{2})$th percentile of standard normal distribution

Based on (a) and assuming H_0 holds the

test statistic is distributed standard normal.

$\dfrac{\hat{\theta}}{\sqrt{1/n_x + 1/n_Y}} \sim N(0,1)$. because $\hat{\theta} \sim N(\mu_x - \mu_Y, \frac{\sigma_x^2}{n_x} + \frac{\sigma_x^2}{n_y})$

if H_0 holds $\mu_x = \mu_Y$ then $\hat{\theta} \sim N(0, \frac{1}{n_x} + \frac{1}{n_Y})$ and

if $\sigma_x^2 = \sigma_Y^2 = 1$. Thus, the proposed critical value results in a rejection probability under H_0 of α.

4(c) $n_x = n_y = 100$ $\bar{X}_{n_x} = 13$ & $\bar{Y}_{n_y} = 13.9$

$$T = \left| \frac{13 - 13.9}{\sqrt{\frac{1}{100} + \frac{1}{100}}} \right| = \frac{.9}{\sqrt{\frac{2}{100}}} = \frac{9}{\sqrt{2}} = 6.364$$

6.364 exceed the $1 - \frac{\alpha}{2}$ percentile for more standard α-levels.

The p-value is basically zero. So there is enough evidence to reject the null hypothesis at the 1%, level.

5. $P(X=x) = p_1^{1(x=1)} \, p_2^{1(x=2)} \, (1-p_1-p_2)^{1(x=3)}$

or

$$f(x) = \begin{cases} p_1 & \text{if } X=1 \\ p_2 & \text{if } X=2 \\ (1-p_1-p_2) & \text{if } X=3 \end{cases}$$

X_i is iid.

(a) Find l_n

$$\mathcal{L}_n = \prod_{i=1}^{n} p_1^{1(x_i=1)} \, p_2^{1(x_i=2)} \, (1-p_1-p_2)^{1(x_i=3)}$$

$$l_n = \sum_{i=1}^{n} \ln \left[p_1^{1(x_i=1)} \, p_2^{1(x_i=2)} \, (1-p_1-p_2)^{1(x_i=3)} \right]$$

$$= \sum_{i=1}^{n} \left[1(X_i=1) \ln p_1 + 1(X_i=2) \ln p_2 + 1(X_i=3) \ln (1-p_1-p_2) \right]$$

$$\boxed{= \ln p_1 \sum_{i=1}^{n} 1(X_i=1) + \ln p_2 \sum_{i=1}^{n} 1(X_i=2) + \ln(1-p_1-p_2) * \sum_{i=1}^{n} 1(X_i=3)}$$

(b) Find MLE for $\theta = (p_1, p_2)'$

$$\frac{\partial l_n}{\partial p_1} = 0 \Rightarrow \frac{\sum_{i=1}^{n} 1(X_i=1)}{p_1} - \frac{\sum_{i=1}^{n} 1(X_i=3)}{1-p_1-p_2} = 0$$

$$\Rightarrow \sum_{i=1}^{n} 1(X_i=1) \, (1-p_1-p_2) = \sum_{i=1}^{n} 1(X_i=3) \, p_1$$

$$\Rightarrow \sum_{i=1}^{n} 1(X_i=1) \, (1-p_2) = \left[\sum_{i=1}^{n} 1(X_i=3) + \sum_{i=1}^{n} 1(X_i=1) \right] p_1$$

$$\Rightarrow p_1 = \frac{\sum 1(X_i=1)}{\sum 1(X_i=3) + \sum 1(X_i=1)} \, (1-p_2)$$

5.(b) con't

Same algebra as for p_1 swaps "2" & "2"

$$\frac{\partial l_n}{\partial p_2} = 0 \Rightarrow \frac{\sum 1(X_i=2)}{p_2} - \frac{\sum 1(X_i=3)}{(1-p_1-p_2)} = 0$$

$$\Rightarrow p_2 = \frac{\sum 1(X_i=2)}{\sum 1(X_i=2) + \sum 1(X_i=3)}(1-p_1)$$

$$\Rightarrow p_1 = \frac{\sum 1(X_i=1)}{\sum 1(X_i=1) + \sum 1(X_i=3)}\left(1 - \frac{\sum 1(X_i=2)}{\sum 1(X_i=2) + \sum 1(X_i=3)}\right)$$

$$= \frac{\sum 1(X_i=1)}{\sum 1(X_i=1) + \sum 1(X_i=3)} \cdot \frac{\sum 1(X_i=3)}{\sum 1(X_i=2) + \sum 1(X_i=3)}$$

$$\Rightarrow \hat{p}_1 = \frac{\sum 1(X_i=1)\sum 1(X_i=3)}{[\sum 1(X_i=1) + \sum 1(X_i=3)][\sum 1(X_i=2) + \sum 1(X_i=3)]}$$

$$\Rightarrow \hat{p}_2 = \frac{\sum 1(X_i=2)\sum 1(X_i=3)}{[\sum 1(X_i=2) + \sum 1(X_i=3)][\sum 1(X_i=1) + \sum 1(X_i=3)]}$$

I'm running out of time, but my intuition is that

$$\hat{p}_1 = \frac{\sum 1(X_i=1)}{\sum 1(X_i=1) + \sum 1(X_i=2) + \sum 1(X_i=3)}$$

$$\hat{p}_2 = \frac{\sum 1(X_i=2)}{\sum 1(X_i=1) + \sum 1(X_i=2) + \sum 1(X_i=3)}$$

is a better estimator.

5. (c)