ECON 711 - PS 7

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A Risky Investment

You have wealth $w > 0$ and preferences over lotteries represented by a von Neumann-Morgenstern expected utility function with Bernoulli utility u which is strictly increasing, twice differentiable, and weakly concave. Your friend wants you to invest in his startup; you can choose any amount $a \leq w$ to invest, and your investment will either triple in value (with probability p) or become worthless (with probability $1 - p$). Your expected utility if you invest a is therefore

$$U(a) = pu(w - a + 3a) + (1 - p)u(w - a) = pu(w + 2a) + (1 - p)u(w - a)$$

(a) Show that if u is linear, then you invest all your wealth if $p > \frac{1}{3}$ and nothing if $p < \frac{1}{3}$.

If u is linear and strictly increasing, u can be represented as $u(x) = mx + b$ for some $m \in \mathbb{R}_{++}, b \in \mathbb{R}$:

$$\begin{aligned}
U(a) &= pu(w + 2a) + (1 - p)u(w - a) \\
&= p(m(w + 2a) + b) + (1 - p)(m(w - a) + b) \\
&= pwm + 2pam + pb + wm - pwm - am + pam + b - pb \\
&= (3p - 1)ma + mw + b
\end{aligned}$$

If $p > \frac{1}{3} \implies 3p - 1 > 0$, so the coefficent on a in utility function is positive. Thus, to maximize U, you want to invest as much as possible, which is all your wealth. If $p < \frac{1}{3} \implies 3p - 1 < 0$, so the coefficent on a in utility function is negative. Thus, to maximize U, you want to invest as little as possible, which is nothing.

From here on, assume $p > \frac{1}{3}$, so the expected value of the investment is positive; and assume that you are strictly risk-averse ($u'' < 0$).

(b) Show that it's optimal to invest a strictly positive amount.[1]

$$U'(a) = pu'(w + 2a)(2) + (1 - p)u'(w - a)(-1) = 2pu'(w + 2a) - (1 - p)u'(w - a)$$

$$U'(0) = 2pu'(w + 2(0)) - (1 - p)u'(w - (0)) = 2pu'(w) - (1 - p)u'(w) = (3p - 1)u'(w)$$

$U'(0) > 0$ because $3p - 1 > 0$ and $u'(w) > 0$.

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[1] You can do this by showing that $U'(0) > 0$ - the marginal expected utility of increasing a is positive when $a = 0$.

(c) Show that $U(a)$ is strictly concave in a, so that except at a corner solution, the first-order condition is necessary and sufficient to find a^*.

$U(a)$ is strictly concave in a iff $U(ta + (1-t)b) < tU(a) + (1-t)U(b)$ for $a, b \in [0, w]$ and $t \in [0, 1]$. Because $u'' < 0$,

$$\begin{aligned}
U(ta + (1-t)b) &= pu(w + 2(ta + (1-t)b)) + (1-p)u(w - (ta + (1-t)b)) \\
&= pu(t(w + 2a) + (1-t)(w + 2b)) + (1-p)u(t(w-a) + (1-t)(w-b)) \\
&< p(tu(w + 2a) + (1-t)u(w + 2b)) + (1-p)(tu(w-a) + (1-t)u(w-b)) \\
&= t(pu(w + 2a) + (1-p)u(w-a)) + (1-t)(pu(w + 2b) + (1-p)u(w-b)) \\
&= tU(a) + (1-t)U(b)
\end{aligned}$$

(d) Show that if $u'(0)$ is infinite, it's not optimal to invest all your wealth; and that if $u'(0)$ is finite, then there's a cutoff $\bar{p}$ such that it's optimal to invest all of your wealth if $p \geq \bar{p}$.

From (c), we know that the first-order condition is necessary and sufficient to find a^*. The derivative of the utility function at $a = w$ is

$$U'(w) = 2pu'(w + 2(w)) - (1-p)u'(w - (w)) = 2pu'(3w) - (1-p)u'(0)$$

Thus, if $u'(0)$ is infinite, $U'(w) = -\infty$, so you're infinitely better off investing $w - \varepsilon$ instead of w for a small positive ε.

If $u'(0)$ is finite, the first order condition is:

$$U'(w) = 0 \implies 2pu'(3w) - (1-p)u'(0) = 0 \implies \bar{p} = \frac{u'(0)}{2u'(3w) + u'(0)}$$

Thus, if $p \geq \bar{p}$ investing all of your wealth is optimal.

From here on, assume that either $u'(0)$ is infinite or $p \in (\frac{1}{3}, \bar{p})$, so the optimal level of investment a^* is strictly positive but below w.

(e) Show that if $u(x) = 1 - e^{-cx}$ (the Constant Absolute Risk Aversion or CARA utility function), your optimal investment a^* does not depend on w.

$$U(a) = p(1 - e^{-c(w+2a)}) + (1-p)(1 - e^{-c(w-a)}) = p(1 - e^{-cw}e^{-2ac}) + (1-p)(1 - e^{-cw}e^{ac})$$

The first order condition implies

$$\begin{aligned}
U'(a) &= 0 \\
\implies p(-e^{-cw}e^{-2ac}(-2c)) + (1-p)(-e^{-cw}e^{ac}(c)) &= 0 \\
\implies 2pe^{-2ac} &= (1-p)e^{ac} \\
\implies a^* &= \frac{3c\ln(1-p)}{\ln(2p)}
\end{aligned}$$

Thus, a^* does not depend on w.

(f) For general u, show that if your Coefficient of Absolute Risk Aversion $A(x) = -\frac{u''(x)}{u'(x)}$ is decreasing, you invest more as w increases.

Notice that if $U'(a)$ is strictly increasing in w at $a = a^*(w)$, then a^* is strictly increasing in w because $a^*(w) = \arg\max U(a)$, U is differentiable and strictly concave in a, and $U'(a)$ is strictly increasing in w when $U'(a) = 0$.

From (b), we found $U'(a)$, so

$$\begin{aligned}
\frac{\partial}{\partial w}(U'(a)) &= 2pu''(w+2a) - (1-p)u''(w-a) \\
&= -2pu'(w+2a)\left(-\frac{u''(w+2a)}{u'(w+2a)}\right) + (1-p)u'(w-a)\left(-\frac{u''(w-a)}{u'(w-a)}\right) \\
&= -2pu'(w+2a)A(w+2a) + (1-p)u'(w-a)A(w-a)
\end{aligned}$$

At the optimum, $U'(a) = 0 \implies 2pu'(w+2a) = (1-p)u'(w-a)$. Thus, because A is decreasing $\implies A(w+2a^*) < A(w-a^*)$,

$$\frac{\partial}{\partial w}(U'(a))\bigg|_{a=a^*(w)} = (1-p)u'(w-a^*)(A(w-a^*) - A(w+2a^*)) > 0$$

Thus, you invest more as w increases.

Now reframe the question as deciding what fraction t of your wealth to invest; writing $a = tw$,

$$U(t) = pu(w(1+2t)) + (1-p)u(w(1-t))$$

(g) Show that if $u(x) = \frac{1}{1-\rho}x^{1-\rho}$, with $\rho \leq 1$ and $\rho \neq 0$ (the Constant Relative Risk Aversion or CRRA utility function), you invest the same fraction of your wealth regardless of w.

$$U(t) = p\frac{1}{1-\rho}\Big(w(1+2t)\Big)^{1-\rho} + (1-p)\frac{1}{1-\rho}\Big(w(1-t)\Big)^{1-\rho}$$

First order conditions imply:

$$\begin{aligned}
U'(t) &= 0 \\
\implies p\frac{1-\rho}{1-\rho}\Big(w(1+2t)\Big)^{-\rho}(2w) + (1-p)\frac{1-\rho}{1-\rho}\Big(w(1-t)\Big)^{-\rho}(-w) &= 0 \\
2wp\Big(w(1+2t)\Big)^{-\rho} - w(1-p)\Big(w(1-t)\Big)^{-\rho} &= 0 \\
\implies 2p(1+2t)^{-\rho} - (1-p)(1-t)^{-\rho} &= 0
\end{aligned}$$

Since the above equation does not depend upon w, t^* does not depend upon w, so you invest the same fraction of your wealth regardless of w.

(h) For general u, show that if your Coefficient of Relative Risk Aversion $R(x) = -\frac{xu''(x)}{u'(x)}$ is increasing, you invest a smaller fraction of your wealth as w increases.

$$
\begin{aligned}
U'(t) &= 2wpu'(w(1+2t)) + (1-p)u'(w(1-t))(-w) \\
&= 2wpu'(w(1+2t)) - w(1-p)u'(w(1-t)) \\
\frac{\partial}{\partial w}(U'(t)) &= 2pu'(w(1+2t)) + 2wpu''(w(1+2t))(1+2t) \\
&\quad - (1-p)u'(w(1-t)) - w(1-p)u''(w(1-t))(1-t)
\end{aligned}
$$

At the optimum, $U'(t) = 0 \implies 2pu'(w(1+2t)) = (1-p)u'(w(1-t))$

$$
\begin{aligned}
\frac{\partial}{\partial w}(U'(t))\bigg|_{t=t^*(w)} &= 2wpu''(w(1+2t))(1+2t) - w(1-p)u''(w(1-t))(1-t) \\
&= -2pu'(w(1+2t))\left(-w(1+2t)\frac{u''(w(1+2t))}{u'(w(1+2t))}\right) \\
&\quad + (1-p)u'(w(1-t))\left(-w(1-t)\frac{u''(w(1-t))(1-t)}{u'(w(1-t))(1-t)}\right) \\
&= -2pu'(w(1+2t))R(w(1+2t)) + (1-p)u'(w(1-t))R(w(1-t)) \\
&= 2pu'(w(1+2t))(R(w(1-t)) - R(w(1+2t)))
\end{aligned}
$$

Thus, since R is increasing $\implies R(w(1-t)) < R(w(1+2t))$, so $\frac{\partial}{\partial w}(U'(t))|_{t=t^*(w)} < 0$. Therefore, you invest a smaller fraction of your wealth as w increases.