Lecture 1

Lecture 2

Lecture 3

Lecture 4

Let (x, d) and (Y, ρ) be two metric spaces. A function $f : X \to Y$ is **continuous** at a point x^0 if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $d(x, x^0) < \delta$, then $\rho(f(x), f(x^0)) < \varepsilon$.

Continuity at x^0 requires $f(x^0)$ is defined and either x^0 is an isolated point of X ($\exists x^0$ s.t. $B_\varepsilon(x^0) = \{x^0\}$) or $\lim_{x\to x^0} f(x)$ exists and equals $f(x^0)$.

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y$. Then f is continuous at x^0 iff either (1) $f(x)$ is defined and either x^0 is an isolated point or x^0 is a limit point of X and $\lim_{x\to x^0} = f(X^0)$ or (2) for any sequence $\{x_n\}$ s.t. $x_n \to x^0$, the sequence $\{f(x_n)\}$ converges to $f(x^0)$.

A function f is continuous if it is continuous at every point of its domain.

$f^{-1}(A) = \{x \in X | f(x) \to A\}$

Let (X, d) and (Y, ρ) be two metric spaces, $f : x \to Y$. Then f is continuous iff for any closed set C in (Y, ρ), the set $f^{-1}(C)$ is closed in (X, d).

Let (X, d) and (Y, ρ) be two metric spaces, $f : x \to Y$. Then f is continuous iff for any open set C in (Y, ρ), the set $f^{-1}(C)$ is open in (X, d).

Let (X, d) and (Y, ρ) be two metric spaces. A function $f : X \to Y$ is uniformly continuous if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $d(x, x^0) < \delta$, then $\rho(f(x), f(x^0)) < \varepsilon$ (δ depends only on ε not on x^0).

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y, E \subset X$. Then f is Lipschitz on E if $\exists K > 0$ s.t. $\rho(f(x), f(y)) \leq K d(x, y) \forall x, y \in E$.

Let (X, d) and (Y, ρ) be two metric spaces, $f : X \to Y, E \subset X$. Then f is locally Lipschitz on E if $\forall x \in E \exists \varepsilon > 0$ s.t. f is Lipschitz on $B_\varepsilon(x) \cap E$.

Lipschitz continuity $\implies$ uniform continuity $\implies$ continuity

Lecture 5

Let $X \subset \mathbb{R}$. Then $u \in \mathbb{R}$ is an upper bound for X if $x \leq u$ for all $x \in X$.

Let $X \subset \mathbb{R}$. Then $l \in \mathbb{R}$ is an lower bound for X if $x \geq l$ for all $x \in X$.

Suppose X is bounded above. The **supremum** of X, $\sup X$, is the smallest upper bound for X. That is, $\sup X$ satisfies $\sup X \geq x \forall x \in X$ and $\forall y < \sup X \exists x \in X$ s.t. $x > y$.

Suppose X is bounded below. The **infimum** of X, $\inf X$, is the largest lower bound for X. That is, $\inf X$ satisfies $\inf X \leq x \forall x \in X$ and $\forall y > \inf X \exists x \in X$ s.t. $x < y$.

Supremum Property: Every nonempty set of real numbers that is bounded above has a supremum. This supremum is a real number.

EVT: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains its maximum and minimum on $[a, b]$: $f(x_M) = \sup_{x\in[a,b]} f(x), f(x_m) = inf_{x\in[a,b]} f(x), x_M, x_m \in [a, b]$.

IVT: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any $\gamma \in [f(a), f(b)]$ there exists $c \in [a, b]$ s.t. $f(c) = \gamma$.

$f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing if $\forall x, y, x < y$ implies $f(x) < f(y)$.

Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Then one-sided limits $f(x^+) := \lim_{x\to x^+} f(y)$ and $f(x^-) := \lim_{x\to x^-} f(y)$ exist $\forall x \in (a, b)$. Moreover, $\sup\{f(s)|a < s < x\} = f(x^-) \leq f(x) \leq f(x^+) = \inf\{f(s)|x < s < b\}$.

Lecture 6

A sequence $\{x_n\}$ in a metric space (X, d) is **Cauchy** if $\forall \varepsilon > 0 \exists N > 0$ s.t. if $m, n > N$, then $d(x_n, x_m) < \varepsilon$.

Every **convergent** sequence in a metric space is **Cauchy**.

A metric space (X, d) is **complete** if every Cauchy sequence contained in X converges to some point in X.

Euclidean space $(\mathbb{R}^m, d_E)$ is complete for any m.

If (X, d) is a complete metric space and $Y \subset X$, then (Y, d) is complete iff Y is closed.

A function $T : X \to X$ from a metric space to itself is called and **operator**.

An operator $T : X \to X$ is a **contraction of modulus** β if $\beta < 1$ and $d(T(x), T(y)) \leq \beta d(x, y) \forall x, y \in X$.

Every contraction is uniformly continuous.

A fixed point of an operator T is an element $x^* \in X$ s.t. $T(x^*) = x^*$.

Contraction Mapping Theorem: Let (X, d) be a nonempty complete metric space and $T : X \to X$ a contraction with modulus $\beta < 1$. Then T has a unique fixed point x^* and $\forall x_0 \in X$ the sequence $\{x_n\}$, where $x_n = T^n(x_0) = T(T(...T(x_0)))$ converges to x^*.

Continuous Dependence of the Fixed Point on Parameters: Let (X, d) and (Ω, ρ) be two metric spaces and $T : X \times \Omega \to X$. For each $\omega \in \Omega$, let $T_\omega : X \to X$ be defined by $T_\omega(x) = T(x, \omega)$. Suppose (X, d) is complete, T is continuous in ω, and $\exists \beta < 1$ s.t. T_ω is a contraction of modulus β for all $\omega \in \Omega$. Then the fixed point function $x^* : \Omega \to X$ defined by $x^*(\omega) = T_\omega(x^*(\omega))$ is continuous.

Blackwell's Sufficient Conditions: Let $B(X)$ be the set of all bounded functions from X to $\mathbb{R}$ with metric $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$. Let $T : B(X) \to B(X)$ satisfy monotonicity (if $f(x) \leq g(x) \forall x \in X$, then $(T(f))(x) \leq (T(g))(x)$ for all $x \in X$) and discounting ($\exists \beta \in (0, 1)$ s.t. for every $\alpha \geq 0$ and $x \in X$, $(T(f + a))(x) \leq (T(f))(x) + \beta\alpha$), then T is a contraction with modulus β.

Lecture 7

A collection of sets $\mathcal{U} = \{U_\lambda | \lambda \in \Lambda\}$ in a metric space (X, d) is an **open cover** of the set A if U_λ is open for all $\lambda \in \Lambda$ and $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$.

A set A in a metric space is **compact** if every open cover of A contains a **finite subcover** of A. That is, if $\{U_\lambda | \lambda \in \Lambda\}$ is an open cover of A, then $\exists n \in \mathbb{N}$ and $\lambda_1, ..., \lambda_n \in \Lambda$ such that $A \subset U_{\lambda_1} \cup ... \cup U_{\lambda_n}$.

Any closed subset of a compact space is compact.

If A is a compact subset of a metric space, then A is closed and bounded.

Heine-Borel Theorem: If $A \subset \mathbb{R}^m$, then A is compact iff A is closed and bounded.

Closed interval $[a, b] = \{a \in \mathbb{R}^m | a_i \leq x_i \leq b_i, i = 1, ..., m\}$ is compact in $(\mathbb{R}^m, d_E)$ for any $a, b, \in \mathbb{R}^m$.

Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and C is a compact set in (X, d), then $f(C)$ is compact in (Y, ρ).

EVT: If C is a compact set in a metric space (X, d) and $f : C \to \mathbb{R}$ is continuous, then f is bounded on C and attains its maximum and minimum.

Let (X, d) and (Y, ρ) be metric spaces, $C \subset X$ compact, $f : C \to Y$ continuous. Then f is uniformly continuous on C.

Lecture 8

A **vector space** V is a collection of obejcts called vectors, which may be added together and multiplied by real numbers, called scalars, satisfying $\forall x, y, z \in V$, $\forall \alpha, \beta \in \mathbb{R}$: (1) $(x + y) + z = x + (y + z)$, (2) $x + y = y + x$, (3) $\exists \bar{0} \in V$ s.t. $x + \bar{0} = \bar{0} + x = x$, (4) $\exists (-x) \in V$ s.t. $x + (-x) = \bar{0}$, (5) $\alpha(x + y) = \alpha x + \alpha y$, (6) $(\alpha + \beta)x = \alpha x + \beta x$, (7) $(\alpha \cdot \beta)x = \alpha(\beta \cdot x)$, (8) $1 \cdot x = x$.

Lecture 9