ECON 709B - Problem Set 1

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1. 2.1 - 2.2[1]

2.1 Find $E[E[E[Y|X_1, X_2, X_3]|X_1, X_2]|X_1]$.

Using the law of iterated expectations,

$$E[E[E[Y|X_1, X_2, X_3]|X_1, X_2]|X_1] = E[E[Y|X_1, X_2]|X_1] = E[Y|X_1]$$

2.2 If $E[Y|X] = a + bX$, find $E[YX]$ as a function of moments of X.

Using the law of iterated expectations,

$$E[YX] = E[E[YX|X]] = E[XE[Y|X]] = E[X(a + bX)] = E[aX + bX^2] = aE[X] + bE[X^2]$$

2. 2.3 Prove conclusion (4) of Theorem 2.4.

If $E|Y| < \infty$ then for any function $h(x)$ such that $E|h(X)e| < \infty$ then $E[h(X)e] = 0$.

Proof: Using the law of iterated expectations, Theorem 2.3, and conclusion (1) (i.e., $E[e|X] = 0$),

$$E[h(X)e] = E[E[h(X)e|X]] = E[h(X)E[e|X]] = E[h(X)(0)] = E[0] = 0$$

□

*I worked on this problem set with a study group of Michael Nattinger, Andrew Smith, and Ryan Mather. I also discussed problems with Emily Case, Sarah Bass, and Danny Edgel.

[1]These problems come from *Econometrics* by Bruce Hansen, revised on October 23, 2020.

3. 2.4 Suppose that the random variables Y and X only take the values 0 and 1, and have the following joint probability distribution

	$X=0$	$X=1$
$Y=0$	.1	.2
$Y=1$	.4	.3

Find $E[Y|X]$, $E[Y^2|X]$ and $\text{var}[Y|X]$ for $X=0$, $X=1$.

$$\begin{aligned}
E[Y|X=0] &= (1)P[Y=1|X=0]+(0)P[Y=0|X=0] = (1)(.4)/(.5) = .8 \\
E[Y|X=1] &= (1)P[Y=1|X=1]+(0)P[Y=0|X=1] = (1)(.3)/(.5) = .6 \\
E[Y^2|X=0] &= (1)^2P[Y=1|X=0]+(0)^2P[Y=0|X=0] = (1)^2(.4)/(.5) = .8 \\
E[Y^2|X=1] &= (1)^2P[Y=1|X=1]+(0)^2P[Y=0|X=1] = (1)^2(.3)/(.5) = .6 \\
\text{var}[Y|X=0] &= E[Y^2|X=0]-(E[Y|X=0])^2 = (.8)-(.8)^2 = 0.16 \\
\text{var}[Y|X=1] &= E[Y^2|X=1]-(E[Y|X=1])^2 = (.6)-(.6)^2 = 0.24
\end{aligned}$$

4. 2.5 (c) Show that $\sigma^2(X)$ is the best predictor of e^2 given X. Show that $\sigma^2(X)$ minimizes the mean-squared error and is thus the best predictor.

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5. 2.8 Suppose that Y is discrete-valued, taking values only on the non-negative integers, and the conditional distribution of Y given $X=x$ is Poisson:

$$P[Y=j|X=x] = \frac{\exp(-x'\beta)(x'\beta)^j}{j!}, j=0,1,2,...$$

Compute $E[Y|X]$ and $\text{var}[Y|X]$. Does this justify a linear regression model of the form $Y = X'\beta + e$?[2]

Using the hint, we know that $E[Y|X] = x'\beta$ and $\text{var}[Y|X] = x'\beta$.

Yes, this justifies a linear regression model because $E[e|X] = E[Y - X'\beta|X] = E[Y|X] - E[X'\beta|X] = x'\beta - x'\beta = 0$.

6. 2.10 - 2.14 Explain your answers.

2.10 If $Y = X\beta + e, X \in \mathbb{R}$, and $E[e|X] = 0$, then $E[X^2e] = 0$.

True, based on the law of iterated expectation:

$$E[X^2e] = E[E[X^2e|X]] = E[X^2E[e|X]] = E[X^2(0)] = E[0] = 0$$

2.11 If $Y = X\beta + e, X \in \mathbb{R}$, and $E[Xe] = 0$, then $E[X^2e] = 0$.

False, for a counter example, assume $X \sim N(0,1)$ and e is a degenerate random variable equal to 1. Notice that $E[Xe] = E[X] = 0$ and $E[X^2e] = E[X^2] = 1$.

2.12 If $Y = X'\beta + e$, and $E[e|X] = 0$, then e is independent of X.

False, for a counter example...

2.13 If $Y = X'\beta + e$, and $E[Xe] = 0$, then $E[e|X] = 0$.

[2]Hint: $P[Y=j] = \frac{\exp(-\lambda)(\lambda)^j}{j!}$, then $E[Y] = \lambda$ and $\text{var}[Y] = \lambda$.

False, for a counter example, assume $X \sim N(0,1)$ and e is a degenerate random variable equal to 1. Notice that $E[Xe] = E[X] = 0$ and $E[e|X] = E[e] = 1$.

2.14 If $Y = X'\beta + e$, and $E[e|X] = 0$, and $E[e^2|X] = \sigma^2$, then e is independent of X.

False,...

7. 2.16 Let X and Y have the joint density $f(x,y) = \frac{3}{2}(x^2 + y^2)$ on $0 \leq x \leq 1, 0 \leq y \leq 1$. Compute the coefficients of the best linear predictor $Y = \alpha + \beta X + e$. Compute the conditional expectation $m(x) = E[Y|X = x]$. Are the best linear predictor and conditional expectation different?

8. 4.1 - 4.6

4.1 For some integer k, set $\mu_k = E[Y^k]$.

(a) Construct an estimator $\hat{\mu}_k$ for μ_k.

...

(b) Show that $\hat{\mu}_k$ is unbaised for μ_k.

...

(c) Calculate the variance of $\hat{\mu}_k$, say $\text{var}[\hat{\mu}_k]$. What assumption is needed for $\text{var}[\hat{\mu}_k]$ to be finite?

...

(d) Propose an estimator of $\text{var}[\hat{\mu}_k]$.

...

4.2 Calculate $E[(\bar{Y} - \mu)^3]$, the skewness of $\bar{y}$. Under what conditions is it zero?

...

4.3 Explain the difference between $\bar{Y}$ and μ. Explain the difference between $n^{-1}\sum_{i=1}^{n} X_i X_i'$ and $EX_i X_i'$.

...

4.4 True or False. If $Y_i = X_i\beta + e_i, X_i \in \mathbb{R}, E[x_i|X_i] = 0$, and $\hat{e}_i$ is the OLS residual from the regression of Y_i on X_i, then $\sum_{i=1}^{n} X_i^2 \hat{e}_i = 0$.

...

4.5 Prove (4.15) and (4.16).

(4.15) $E[\hat{\beta}|X] = \beta$

...

(4.16) $\text{var}[\hat{\beta}|X] = (X'X)^{-1}(X'\Omega X)(X'X)^{-1}$

...

4.6 Prove Theorem 4.5.

Theorem 4.5 Generalized Guass-Markov

In the linear regression model (Assumption 4.2) and $\Omega > 0$, if $\tilde{\beta}$ is a linear unbiased estimator of β then $\text{var}[\tilde{\beta}|X] \geq (X'\Omega^{-1}X)^{-1}$

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