ECON 710A - Problem Set 4

Alex von Hafften*

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1. Let X be generated by the following random coefficients discrete choice model $X = 1\{-U_0 + ZU_1 > 0\}$ where $U = (U_0, U_1)'$ is independent of Z and $Z \in \{0, 1\}$. Provide conditions on U such that $Pr(Defying) = 0$ and $Pr(Complying) > 0$.

$Pr(Defying) = 0$ iff $X_U(1) = 0 \implies X_U(0) = 0$ and $X_U(0) = 1 \implies X_U(1) = 1$. $X_U(1) = 0 \implies X_U(0) = 0$ iff $-U_0 + (1)U_1 = -U_0 + U_1 < 0 \implies -U_0 + (0)U_1 = -U_0 < 0$. $X_U(0) = 1 \implies X_U(1) = 1$ iff $-U_0 + (0)U_1 = -U_0 > 0 \implies -U_0 + (1)U_1 = -U_0 + U_1 > 0$. Thus, $U_1 \geq 0$.

$Pr(Complying) > 0 \iff Pr(X_U(1) = 1 \text{ and } X_U(0) = 0) > 0$. Since $U_1 \geq 0$, this implies that $U_1 > U_0 \geq 0$.

2. Let $\{Y_t\}_{t=1}^T$ be generated by the following MA(q) model, i.e., $Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$ where $\{\varepsilon_t\}_{t=0}^T$ are i.i.d. random variables with mean zero and variance σ^2.

(i) Find the autocovariance function $\gamma(k)$.

For $k = 0$:

$$\begin{aligned}
\gamma(0) &= Cov(Y_t, Y_t) \\
&= Var(Y_t) \\
&= Var(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}) \\
&= Var(\varepsilon_t) + \theta_1^2 Var(\varepsilon_{t-1} + ... + \theta_q^2 Var(\varepsilon_{t-q}) \\
&= \sigma^2 + \theta_1^2 \sigma^2 + ... + \theta_q^2 \sigma^2 \\
&= \sigma^2 (1 + \theta_1^2 + ... + \theta_q^2)
\end{aligned}$$

For $k = 1$:

$$\begin{aligned}
\gamma(1) &= Cov(Y_t, Y_{t+1}) \\
&= Cov(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}, \mu + \varepsilon_{t+1} + \theta_1 \varepsilon_{t+1-1} + ... + \theta_q \varepsilon_{t+1-q}) \\
&= Cov(\varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}, \varepsilon_{t+1} + \theta_1 \varepsilon_t + ... + \theta_q \varepsilon_{t+1-q}) \\
&= Cov(\varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_{q-1} \varepsilon_{t+1-q}, \theta_1 \varepsilon_t + ... + \theta_q \varepsilon_{t+1-q}) \\
&= \theta_1 Var(\varepsilon_t) + \theta_1 \theta_2 Var(\varepsilon_{t-1}) + ... + \theta_{q-1} \theta_q Var(\varepsilon_{t+1-q}) \\
&= \sigma^2 (\theta_1 + \theta_1 \theta_2 + ... + \theta_{q-1} \theta_q)
\end{aligned}$$

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For general k:

$$\gamma(k) = \begin{cases} \sigma^2(1+\theta_1^2+...+\theta_q^2) & \text{if } k=0 \\ \sigma^2(\theta_k+\theta_1\theta_{k+1}+\theta_2\theta_{k+1}+...+\theta_{q-k}\theta_q) & \text{if } 0<k\leq q \\ 0, & \text{if } k>q \end{cases}$$

(ii) Suppose that $q=1$ and find the autocorrelation function, $\rho(k)=\frac{\gamma(k)}{\gamma(0)}$.

If $q=1$:

$$Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1}$$

$\gamma(k)$ simplies to

$$\gamma(k) = \begin{cases} \sigma^2(1+\theta_1^2), & \text{if } k=0 \\ \sigma^2\theta_1, & \text{if } k=1 \\ 0, & \text{if } k>1 \end{cases}$$

The autocorrelation function:

$$\begin{aligned} \rho(k) &= \frac{\gamma(k)}{\gamma(0)} \\ &= \begin{cases} \frac{\sigma^2(1+\theta_1^2)}{\sigma^2(1+\theta_1^2)}, & \text{if } k=0 \\ \frac{\sigma^2\theta_1}{\sigma^2(1+\theta_1^2)}, & \text{if } k=1 \\ \frac{0}{\sigma^2(1+\theta_1)}, & \text{if } k>1 \end{cases} \\ &= \begin{cases} 1, & \text{if } k=0 \\ \frac{\theta_1}{1+\theta_1^2}, & \text{if } k=1 \\ 0, & \text{if } k>1 \end{cases} \end{aligned}$$

(iii) Is θ_1 identified from the autocorrelation function?

No. First, notice that θ_1 only appears in autocorrelation function when $k=1$. If $\theta_1=x$, $\theta_1=1/x$ yields the same value from the autocorrelation function:

$$\begin{aligned} \rho(1|\theta_1=x) &= \frac{x}{1+x^2} \\ \rho(1|\theta_1=x^{-1}) &= \frac{x^{-1}}{1+(x^{-1})^2} \\ &= \frac{x^{-1}}{1+x^{-2}}\frac{x^2}{x^2} \\ &= \frac{x}{1+x^2} \end{aligned}$$

(iv) Suppose $\theta_1\in[-1,1]$. Does you answer to (iii) change?

Yes, θ_1 is identified by the autocorrelation function because if $\theta_1=x\in[-1,1] \implies 1/x\notin[-1,1]$.

3. Consider an ARMA(1,1) model: $Y_t = \alpha_0 + Y_{t-1}\rho + U_t$ and $U_t = \varepsilon_t + \theta\varepsilon_{t-1}$ for all $t = 1, ..., T$; $Y_0 = \mu + \varepsilon_0 + \nu$ where $|\rho| < 1$, $|\theta| \leq 1$, $\varepsilon_0, ..., \varepsilon_T$ are idd $N(0, \sigma^2)$ and independent of $\nu \sim N(0, \tau)$.

(i) Find μ and τ (as functions of α_0, ρ, θ, and/or σ^2) such that $E[Y_t]$ and $Var(Y_t)$ does not depend on t.

If $E[Y_t]$ does not depend on $t \implies E[Y_0] = E[Y_1]$:

$$\begin{aligned} E[Y_0] &= E[\mu + \varepsilon_0 + \nu] \\ &= \mu \end{aligned}$$

$$\begin{aligned} E[Y_1] &= E[\alpha_0 + Y_0\rho + U_1] \\ &= E[\alpha_0 + Y_0\rho + \varepsilon_1 + \theta\varepsilon_0] \\ &= \alpha_0 + E[Y_0]\rho + E[\varepsilon_1] + \theta E[\varepsilon_0] \\ &= \alpha_0 + \mu\rho \end{aligned}$$

$$\begin{aligned} E[Y_0] &= E[Y_1] \\ \implies \mu &= \alpha_0 + \mu\rho \\ \implies \mu &= \frac{\alpha_0}{1-\rho} \end{aligned}$$

If $Var[Y_t]$ does not depend on $t \implies Var[Y_0] = Var[Y_1]$:

$$\begin{aligned} Var[Y_0] &= Var[\mu + \varepsilon_0 + \nu] \\ &= Var[\varepsilon_0] + Var[\nu] \\ &= \sigma^2 + \tau \end{aligned}$$

$$\begin{aligned} Var[Y_1] &= Var[\alpha_0 + Y_0\rho + U_1] \\ &= Var[Y_0\rho + \varepsilon_1 + \theta\varepsilon_0] \\ &= \rho^2 Var[Y_0] + Var[\varepsilon_1] + \theta^2 Var[\varepsilon_0] + 2\rho\theta Cov[Y_0, \varepsilon_0] \\ &= \rho^2(\sigma^2 + \tau) + \sigma^2 + \theta^2\sigma^2 + 2\rho\theta\sigma^2 \end{aligned}$$

$$\begin{aligned} Var[Y_0] &= Var[Y_1] \\ \implies \sigma^2 + \tau &= \rho^2\sigma^2 + \rho^2\tau + \sigma^2 + \theta^2\sigma^2 + 2\rho\theta\sigma^2 \\ \tau - \rho^2\tau &= \rho^2\sigma^2 + \theta^2\sigma^2 + 2\rho\theta\sigma^2 \\ \tau &= \frac{\sigma^2(\theta+\rho)^2}{1-\rho^2} \end{aligned}$$

(ii) For the μ and τ found above, you may use without proof that $\{Y_t\}_{t=1}^{T}$ is covariance stationary. Under what conditions on α_0, ρ, θ, and/or σ^2 is $(1, Y_{t-2})$ a valid instrument for $(1, Y_{t-1})$.

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