Econ712 - Handout 4b

1 Measure space and Measurable function

Definition 1. For a set S and a set of its subsets $\mathscr{S}$, $\mathscr{S}$ is a **σ-algebra** if

1. $\emptyset, S \in \mathscr{S}$
2. If $A \in \mathscr{S}$ then $A^c \in \mathscr{S}$
3. If $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$ then $\cup_{n=1}^{\infty} A_n \in \mathscr{S}$

The pair $(S, \mathscr{S})$ is called a measurable space, and any $A \in \mathscr{S}$ is called an $\mathscr{S}$-measurable set.

Definition 2. For any set S and any collection $\mathscr{A}$ of subsets of S, the smallest σ-algebra that contains $\mathscr{A}$ is called the σ-algebra generated by $\mathscr{A}$

Definition 3. The Borel algebra $\mathscr{B}^n$ of R^n is the σ-algebra generated by the open sets of R^n

Definition 4. Let $(S, \mathscr{S})$ be a measurable space. A measure is an extended real valued function $\mu : \mathscr{S} \to \bar{R}$ s.t.

1. $\mu(\emptyset) = 0$
2. $\mu(A) \geq 0$, $\forall A \in \mathscr{S}$
3. If $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence of subsets in $\mathscr{S}$, then $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

Definition 5. A measure space is a triple $(S, \mathscr{S}, \mu)$, where S is a set, $\mathscr{S}$ is a σ-algebra of its subsets, and μ is a measure defined on $\mathscr{S}$

Definition 6. Given a measure space $(S, \mathscr{S}, \mu)$, if $\mu(S) = 1$ then μ is a probability measure and $(S, \mathscr{S}, \mu)$ is a probability space

Definition 7. Given a measurable space $(S, \mathscr{S})$, a real-valued function $f : S \to R$ is measurable wrt $\mathscr{S}$ if

$$\{s \in S : f(s) \leq a\} \in \mathscr{S}, \forall a \in R$$

Exercise. For some $S \in \mathscr{B}^n$, define $\mathscr{B}_s = \{A \in \mathscr{B}^n; A \subseteq S\}$. Show that $\mathscr{B}_s$ is a σ-algebra

Exercise. Let $(S, \mathscr{S})$ be a measurable space; let μ_1, μ_2 be measures on it. Show that $\lambda : \mathscr{S} \to \bar{R}$ defined by $\lambda(A) = \mu_1(A) + \mu_2(A)$ is a measure on $(S, \mathscr{S})$

Exercise. Show that any monotone or continuous function $f : R \to R$ is measurable wrt to $\mathscr{B}$

2 Transition functions

Definition 8. Let $(Z, \mathscr{L})$ be a measurable space. A transition function is a function $Q : Z \times \mathscr{L} \to [0, 1]$ s.t.

1. for each $z \in Z$, $Q(z, \cdot)$ is a probability measure on $(Z, \mathscr{L})$ and
2. for each $A \in \mathscr{L}$, $Q(\cdot, A)$ is a $\mathscr{L}$-measurable function

The interpretation is that $Q(a, A) = Pr\left\{z_{t+1} \in A | z_t = a\right\}$, where z_t is the random stat in period t

For any $\mathscr{L}$-measurable function f, define Tf by

$$Tf(z) = \int f(z')Q(z, dz'), \quad \forall z \in Z$$

The interpretation is $Tf(z) = E\left[f(z')|z\right]$

For any probability measure λ on $(Z, \mathscr{L})$, define $T^*\lambda$ by

$$T^*\lambda(A) = \int Q(z, A)\lambda(dz), \quad \forall A \in \mathscr{L}$$

The interpretation is $T^*\lambda(A) = Pr_\lambda\left\{z_{t+1} \in A\right\}$