Econ712 - Problem set 1

1 Variations on the baseline OG

Consider an overlapping generations economy of 2-period-lived agents. There is a measure N of agents in each generation. New young agents enter the economy at each date $t \geq 1$. Half of the young agents are endowed with w_1 when young and 0 when old. The other half are endowed with 0 when young and w_2 when old. There is no savings technology. Agents order their consumption stream by $U(c_t^t, c_{t+1}^t) = \ln c_t^t + \ln c_{t+1}^t$. There is a measure N of inital old agents. Half of them are endowed with w_2 and the other half endowed with 0. Each old agent order their consumption by c_1^0.

1. In a given period t, set up and solve the problem of a planner that weighs everyone alive equally.

   (a) Given t, the problem is

$$\max_{c_t^{t-1}, c_t^t} N \log c_t^{t-1} + N \log c_t^t \quad s.t.$$

$$\frac{N}{2} w_1 + \frac{N}{2} w_2 = N c_t^t + N c_t^{t-1}$$

   FOCs give

$$\frac{1}{c_t^{t-1}} = \frac{1}{c_t^t} = \lambda$$

$$\frac{w_1 + w_2}{2} = c_t^{t-1} + c_t^t$$

$$\Rightarrow c_t^{t-1} = c_t^t = \frac{w_1 + w_2}{4}$$

   Since everyone is weighted equally, and agents have concave prefs, the planner gives everyone the same amount of consumption.

2. Suppose that agents who are alive can make fully enforcible contracts among themselves, and there exists a market for these contracts. What kind of contracts (ie who trades with whom) would be traded?

   (a) Since there is a single good, and multiple time periods, the only trades that would happen are intertemporal ones. At period t, the old will not trade with the young, since they cannot honour their contract next period. So the only possible trades are between agents of their respective generations.

   (b) Would these trades happen? Generally yes, since agents within a generation have different endowments.

3. Set up and solve the problem of young agents born at time t, assuming they have access to the above market for contracts. Agents take prices as given.

   (a) We are assuming a market for these contracts to resolve issues of bargaining. A young agent can buy/sell 1 unit of tomorrow's good at price p_t today. That is, if they spend $s_t p_t$ at time t, they get s_t at time $t+1$ (an alternative setup is to spend s_t today and get $s_t(1 + r_{t+1})$ tomorrow).

(b) A young agent, with endowments (y, o), solves the following problem:

$$\max_{c_t^t, c_{t+1}^t} \log c_t^t + \log c_{t+1}^t \quad s.t.$$

$$c_t^t + p_t s_t = y$$
$$c_{t+1}^t = s_t + o$$

Consolodate the BC into $c_t^t + p_t c_t^{t+1} = y + p_t o$. FOCs give

$$c_t = p_t c_{t+1}^t = \frac{y + p_t o}{2} \quad s_t = \frac{y - p_t o}{2p_t}$$

4. Impose market clearing to solve for the prices and allocations. Compare the allocations to that in (1), and give intuition as to why they are similar/different.

(a) Denote agents with endowments $(w_1, 0), (0, w_2)$ by $1, 2$. Market clearing:

$$\frac{1}{2}s_{1t} + \frac{1}{2}s_{2t} = 0$$
$$\Rightarrow \frac{w_1}{2} - p_t \frac{w_2}{2} = 0$$
$$p_t = \frac{w_1}{w_2}$$

(b) Sub back to get consumptions:

$$c_{1t}^t = c_{2t}^t = \frac{w_1}{2}; c_{1t+1}^t = c_{2t+1}^t = \frac{w_2}{2}$$

(c) Here, agents within a generation equalize consumption, but across generations within a period, consumptions differ (except for when $w_1 = w_2$). This is different to the planner's allocations. The reason is that there is a "missing market" for trade across generations - which is a source of inefficiency, so that the optimal planner's outcome cannot be achieved.

5. Now suppose there is a government that has access to generation specific lumpsum taxes and transfers (they can only impose different tax/transfer across generations, whereas individuals of the same generation face the same tax/transfer). Shutting down the above contracting market, can the government implement the planner's optimal allocations? Can they implement the planner's allocation with the above contracting market?

(a) In a given period t, the government impose taxes T_t^t, T_t^{t-1} on the young and old, taking into account the young and old's responses. Their budget constraint is

$$T_t^t + T_t^{t-1} = 0$$

(b) Without the above contracting market, since the government can only reallocate across generations, there would remain consumption differences within generations. Hence they cannot implement the planner's allocation.

(c) The contracting market allows for consumption equalization within generations, and with the lumpsum taxes/transfers to reallocate across generations, the planner's allocation can be obtained.

(d) Assume the government wants to implement the planner's allocation for each period t. Young agents solve

$$\max_{c_t^t, c_{t+1}^t} \log c_t^t + \log c_{t+1}^t \quad s.t.$$

$$c_t^t + p_t c_t^{t+1} = y + p_t o + T_t^t + p_t T_{t+1}^t$$

which gives $c_t = p_t c_{t+1}^t = \frac{y + p_t o + T_t^t + p_t T_{t+1}^t}{2} \quad s_t = \frac{y + T_t^t - p_t o - p_t T_{t+1}^t}{2p_t}$. Market clearing gives

$$w_1 + T_t^t - p_t T_{t+1}^t + T_t^t - p_t w_2 - p_t T_{t+1}^t = 0$$

$$p_t = \frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t}$$

so that

$$c_{1t} = \frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} c_{1t+1} = \frac{w_1 + T_t^t + \frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} T_{t+1}^t}{2}$$

$$c_{2t} = \frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} c_{2t+1} = \frac{\frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} w_2 + T_t^t + \frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} T_{t+1}^t}{2}$$

(e) We want to implement $c_t^{t-1} = c_t^t = \frac{w_1 + w_2}{4}$. This requires

$$\frac{w_1 + T_t^t + \frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} T_{t+1}^t}{2} = \frac{\frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} w_2 + T_t^t + \frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} T_{t+1}^t}{2}$$

and

$$\frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t} = 1$$

We also have to satisfy $T_t^t = -T_t^{t-1}$. Subbing out $\frac{w_1 + 2T_t^t}{w_2 + 2T_{t+1}^t}$ in the first equation implies

$$\frac{w_1}{2} = \frac{w_2}{2}$$

which is not true generally.

(f) With taxes and transfers working to redistribute across generations, and the contracting market working to equalize marginal consumption within generations, one might expect the government to be able to implement the planner's solution. However, compared to (4.), taxes/transfers create a wealth effect so that agents within a generation may no longer equalize consumption.

6. Briefly discuss how incorporating population growth/shrinkage might change the above answers.

   (a) Now assume the measure of agents in each generation is N_t, with $N_{t+1} = nN_t$, $n > 0$. Growth is captured by $n > 1$, and shrinkage captured by $n < 1$.

   (b) Changing population will affect the resource constraint, given a period t. The planner still weighs everyone equally so that all consumptions are equalized.

   (c) Within generations, agents will also equalize consumption if they have access to the contracting market.

   (d) The government's problem will then follow through as before.

2 Setting up a problem

For the following, please set up (just set up, solve at your own leisure) the optimizing problem of the associated agent(s). Feel free to impose (and explicitly state) additional assumptions to help with modeling.

1. Time is discrete and runs forever. There are 2 goods in the economy, a capital good k and an all purpose good y. y can either be used for consumption c or for investment i to create more capital for the next period. There is a production technology that generates all purpose goods according to the function $y = k^\alpha$. Capital depreciates at rate δ, such that capital tomorrow is $k_{t+1} = (1-\delta)k_t + i_t$. The economy is endowed with k_0 at the start of period 0. The planner maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t)$, choosing how much to consume and invest each period.

   (a) $\max_{\{c_t, k_{t+1}, i_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$ s.t. $k_{t+1} = (1-\delta)k_t + i_t$; $i_t + c_t = k_t^\alpha$; k_0 given; $c_t, k_{t+1} \geq 0$

2. Consider an agent that lives for T periods with time seperable utility. She ranks consumption each period according to $u(C)$, and discounts future consumption geometrically at rate β. Each period she is endowed with w units of consumption good. She has access to a perfect storage technology, whereby 1 unit of good saved today will give her 1 unit of good tomorrow. Her problem is to optimize over savings and consumption for every period.

   (a) $\max_{\{c_t, s_{t+1}\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$ s.t. $c_t + s_{t+1} = w + s_t$; $s_0 = 0$; $c_t, s_{t+1} \geq 0$

3. Consider an individual just entering the workforce, who will retire after N periods. Each period, his earnings y_t is the product of the number of hours worked h_t, the level of his human capital k_t, and the wage w_t: $y_t = h_t k_t w_t$. Assume that the wage is constant overtime. In each period, the worker can accumulate human capital by reducing the amount of hours worked. Specifically, $h_t = \phi(k_{t+1}/k_t)$. The worker seeks to maximize the present value of his earnings, at the discount rate $(1+r)^{-1}$, with a given k_0.

   (a) $\max_{\{h_t, k_{t+1}\}_{t=0}^{N-1}} \sum_{t=0}^{N-1} (1+r)^{-t} h_t k_t w$ s.t. $h_t = \phi(k_{t+1}/k_t)$; k_0 given.

4. Consider a tree whose growth is given by the function h. That is, if k_t is the size of the tree in period t, then $k_{t+1} = h(k_t)$. Assume that once cut down, all of the wood must be sold at price p_t. Assume that the interest rate r is constant over time. Assume that it is costless to cut down the tree, and that the tree must be cut down by time T. The problem is to find the optimal time to cut down the tree to maximize the present discounted value, given an initial size k_0.

   (a) $\max_{t \in \{0,1,\ldots,T\}} \{(1+r)^{-t} k_t p_t\}$ s.t. $k_{t+1} = h(k_t)$; k_0 given.

5. For a firm producing a new product, suppose that the marginal cost of production is constant in each period at c_t, but this marginal cost falls over time as a function of cumulative experience. Denote production and cumulative experience at time t by q_t and Q_t. Then $Q_0 = 0$ and $Q_{t+1} = Q_t + q_t$. Let marginal cost be related to experience by $c_t = \gamma(Q_t)$. The inverse demand is $p(q_t)$. The firm chooses production each period to maximize its present value of profits, at discount $(1 + r)^{-1}$.

   (a) $\max_{\{q_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (1 + r)^{-t} q_t \left[p(q_t) - \gamma(Q_t)\right]$ s.t. $Q_{t+1} = Q_t + q_t$; $Q_0 = 0$; $q_t \geq 0$.