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MINIMUM RANK AND MINIMUM TRACE OF COVARIANCE MATRICES

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This paper considers some mathematical aspects of minimum trace factor analysis (MTFA). The uniqueness of an optimal point of MTFA is proved and necessary and sufficient conditions for a point x to be optimal are established. Finally, some results about the connection between MTFA and the classical minimum rank factor analysis will be presented.

Key words: factor analysis, covariance matrices, minimum trace, constrained minimum trace, minimum rank.

Introduction

Consider the well-known factor analytic decomposition of an $n \times n$ population covariance matrix Σ in a common (true-score) and a unique (error) part:

$$\Sigma = (\Sigma - D) + D \tag{1}$$

where matrix $\Sigma - D$ is Gramian and matrix D is diagonal. For given Σ the specific decomposition (1) is not unique and in the so-called minimum-trace factor analysis (MTFA) one selects D in such a way that the trace of the reduced matrix $\Sigma - D$ is minimal. The MTFA solution D^* corresponds to a lower bound to reliability defined by Bentler [1972] as

$$\rho = 1 - \frac{l^t D^* l}{l^t \Sigma l}, \tag{2}$$

where l is the vector of unit weights. However the MTFA could lead to negative entries of D (Heywood case). Jackson and Agunwamba [1977], and independently Bentler and Woodward [1980], considered the MTFA with the further constraint that D be Gramian. This approach provides the greatest lower bound to reliability and will be referred to as the constrained minimum trace factor analysis (CMTFA).

Della Riccia [1973, 1980] and Della Riccia, de Santis and Sessa [1978] independently developed the MTFA method but for specific purposes of pattern recognition problems. Ledermann [1939] has discussed the relation between MTFA and minimal rank of a reduced matrix, which is the essential concept in minimum-rank factor analysis (MRFA). He showed that minimum-trace and minimum-rank solutions do not necessarily coincide even in the case of minimal rank equal to one. However, it was noted [Ledermann, 1939; Bentler & Woodward, 1980] that MTFA and MFRA solutions usually coincide when the minimal rank is small relative to n. At the end of this paper we shall prove a theorem that gives some additional insight on the relation between minimum-trace and minimum-rank. In practice one substitutes a sample covariance matrix S for Σ. However, complications of the sampling issues need not be a concern of this paper, since by continuity we expect that close covariance matrices will lead to close numerical results.

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Bentler [1972] proposed necessary conditions for the MTFA solution. We shall show that similar conditions are necessary and sufficient for MTFA as well as for CMTFA. A uniqueness proof for the CMTFA (or MTFA) solution is also offered. Independently ten Berge, Snijders, and Zegers [1981] arrived at results closely related to ours with completely different methods. We utilize convex analysis techniques recently developed in nonlinear programming for purposes of optimization problems. The next section shall summarize the basic facts in convex analysis to be used in this paper.

The following notation and definitions shall be used. $D = \operatorname{diag}(d_i)$, $W = \operatorname{diag}(w_i)$, etc., denote diagonal matrices. Vectors are denoted by small letters: d, w, x, etc., and are given in column form; for instance $d = (d_1, \ldots, d_n)^t$ where t indicates the transpose operation. For any vector u, we denote $u^2 = (u_1^2, \ldots u_n^2)^t$. The null space of a given $n \times m$ matrix A is denoted by $N(A) = \{x \mid x^t A = 0, x \in R^n\}$. $x > 0$ $(x \geq 0)$ indicates that vector x has positive (non-negative) coordinates. For a differentiable function $f\colon R^n \to R$ we denote $\nabla f(x) = (\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n)^t$ the gradient of f at x. A convex combination of $x, y \in R^n$ is $\alpha x + (1 - \alpha)y$ with $0 \leq \alpha \leq 1$. The convex hull conv $\{\Delta\}$ of a subset Δ of R^n is obtained from all convex combinations of $x, y \in \Delta$. $\overline{\text{conv}}$ $\{\Delta\}$ denotes the topological closure of the convex hull of the subset $\Delta \subset R^n$.

Convex Analysis

A function $f\colon R^n \to R$ is said to be convex if the inequality

$$f[\alpha x + (1 - \alpha)y] \leq \alpha f(x) + (1 - \alpha) f(y) \tag{3}$$

holds for all $x, y \in R^n, 0 \leq \alpha \leq 1$.

A function f is concave if $-f$ is convex.

The subdifferential $\partial f(x)$ of a convex function f at a point x is defined as the following set:

$$\partial f(x) = \{y : f((z) - f(x) \geq y^t(z - x), \qquad \forall z \in R^n\}. \tag{4}$$

It is known that the subdifferential is a nonempty, convex, compact set. Furthermore, f is differentiable at x if and only if $\partial f(x)$ is a singleton (i.e., the set $\partial f(x)$ contains only one vector). In the last case $\partial f(x) = \{\nabla f(x)\}$ [Rockafellar, 1970].

Let $f_s(x)$, $s \in S$, be a family of convex functions. Then the max-function $f(x)$ is

$$f(x) = \sup_{s \in S} f_s(x)$$

where sup stands for supremum. The max-function is convex whenever defined. The differential properties of max-function were studied by various authors. The following basic result and appropriate references can be found in Ioffe and Tihomirov [1979].

Theorem 1. Let S be a compact topological space, and let $f(x, s) = f_s(x)$ be a function on $R^n \times S$, convex in x for every $s \in S$ and continuous in s for every $x \in R^n$. Then

$$\partial f(x) = \overline{\text{conv}} \left\{ \bigcup_{s \in S_0(x)} \partial f_s(x) \right\} \tag{5}$$

where $S_0(x) = \{s \in S : f(x, s) = f(x)\}$.

Let $f_0, \ldots, f_m$ be functions defined on R^n. We consider the problem:

$$\inf f_0(x) \quad \text{subject to} \quad f_1(x) \leq 0, \ldots, f_m(x) \leq 0. \tag{6}$$

Problem (6) is said to be a convex programming problem, if all functions $f_0, \ldots, f_m$ are convex.

Theorem 2 (*necessary and sufficient conditions*). If x^* is a solution of the convex programming problem (6), then there exist Lagrange multipliers $\alpha_0 \geq 0, \ldots, \alpha_m \geq 0$ not all zero, such that

$$0 \in \alpha_0 \, \partial f_0(x^*) + \ldots + \alpha_m \, \partial f_m(x^*) \tag{7}$$

and

$$\alpha_i f_i(x^*) = 0, \qquad i = 1, \ldots, m. \tag{8}$$

Moreover, if there exists a vector x such that $f_i(x) < 0$ for all $i = 1, \ldots, m$ (the Slater condition), then $\alpha_0 \neq 0$ and one can set $\alpha_0 = 1$. In this case, the relations (7) and (8) are sufficient for the point x^*, satisfying $f_i(x^*) \leq 0$, $i = 1, \ldots m$, to be a solution of the problem (6).

We emphasize that (7) refers to multiple sets of vectors.

Necessary and Sufficient Conditions in CMTFA

Let us consider the CMTFA problem with weight vector w:

$$\max_D w^t D w \qquad \text{subject to } \Sigma - D \text{ Gramian} \quad \text{and} \quad D \text{ diagonal and Gramian} \tag{9}$$

Denote $\lambda(d)$ the minimal eigenvalue of $\Sigma - D$. It is well-known that $\Sigma - D$ is Gramian if and only if $\lambda(d) \geq 0$. Then we can formulate problem (9) in the following way

$$\min_d l(d) \qquad \text{subject to } -\lambda(d) \leq 0 \quad \text{and} \quad -d_i \leq 0, \qquad i = 1, \ldots, n \tag{10}$$

where $l(d) = -\sum_{i=1}^n w_i^2 \, d_i$.

Lemma 1. The function $-\lambda(d)$ is convex and

$$\partial[-\lambda(d)] = \overline{\text{conv}} \, \{y \colon y = x^2, \, x \text{ is a unit eigenvector of } \Sigma - D \text{ corresponding to } \lambda(d)\}. \tag{11}$$

Proof. It is known [e.g., Bellman, 1960] that the minimal eigenvalue $\lambda(d)$ can be represented as

$$\lambda(d) = \min_x x^t(\Sigma - D)x, \qquad x^t x = 1. \tag{12}$$

Then we have

$$-\lambda(d) = \max_x f_x(d), \qquad x^t x = 1, \tag{13}$$

where $f_x(d) = x^t D x - x^t \sum x$. For fixed x the function $f_x(d)$ is linear and therefore convex in d. Thus we obtain that $-\lambda(d)$ is convex as a maximum of convex functions. Furthermore $\nabla f_x(d) = x^2$ and a minimum in (12) is attained at a unit eigenvector of $\Sigma - D$ corresponding to the minimal eigenvalue. Then (11) follows at once from Theorem 1. □

We have shown that problem (10) is a convex programming problem. It can be easily seen that the Slater condition is satisfied and $\nabla l(d) = -w^2$, $\nabla(-d_i) = -\xi_i$, where ξ_i is the vector $(0, \ldots, 1, \ldots, 0)^t$ with all components equal to zero except the ith component which is equal to one. Furthermore, the minimum in (10) is attained on the boundary $\{d \colon \lambda(d) = 0\}$. Then from the necessary and sufficient conditions of Theorem 2 we obtain that d^* is a solution of problem (10) if and only if $\lambda(d^*) = 0$, $d^* \geq 0$, with

$$0 \in -w^2 + \alpha \partial[-\lambda(d^*)] - \sum \alpha_i \xi_i, \tag{14}$$

$\alpha_i d_i = 0$, $\alpha \geq 0$ and $\alpha_i \geq 0$, $i = 1, \ldots, n$. This with (11) implies:

Theorem 3 (*necessary and sufficient conditions in CMTFA*). The point d^* is a solution of the CMTFA problem if and only if $\lambda(d^*) = 0$, $d^* \geq 0$ and the weight vector w can be represented in the form:

$$w^2 = \sum_{i=1}^{k} x_i^2 - \sum_{j \in I(d^*)} \alpha_j \xi_j, \tag{15}$$

where $x_i \in N(\Sigma - D^*)$, $i = 1, \ldots, k$, $I(d) = \{i: d_i = 0, i \leq n\}$ and α_j, $j \in I(d^*)$, are non-negative numbers.

Remark 1. For the MTFA case (or for the CMTFA case when $d^* > 0$, i.e., when the set $I(d^*)$ is empty) the sum $\sum \alpha_j \xi_j$ in representation (15) has to be omitted.

Corollary 1. If d^* is a solution of CMTFA (of MTFA) then there is at least one vector $x \in N(\Sigma - D^*)$ with nonzero coordinates corresponding to nonzero coordinates of w. In particular if all coordinates of w are different from zero then there exists $x \in N(\Sigma - D^*)$ without zero coordinates.

We can now prove the uniqueness of a solution [cf. Hakim, Lochard, Olivier & Térouanne, 1976].

Theorem 4. If the weight vector w has no zero components then the CMTFA (the MTFA) problem has a unique solution d^*.

Proof. Let us assume that there are two solutions d^* and d_0. Then we have that $\lambda(d^*) = \lambda(d_0) = 0$ and from convexity it follows that each point of the interval $\mu d^* + (1 - \mu)d_0$, $0 \leq \mu \leq 1$, is a solution. Set $\mu = \frac{1}{2}$ and denote $d_1 = (d^* + d_0)/2$. By Corollary 1 the space $N(\Sigma - D_1)$ contains a vector x without zero coordinates. Since $(\Sigma - D_1)x = 0$ it follows that $x'(\Sigma - D^*)x + x'(\Sigma - D_0)x = 0$ and because $x'(\Sigma - D^*)x \geq 0$, $x'(\Sigma - D_0)x \geq 0$ we obtain that $x'(\Sigma - D^*)x = x'(\Sigma - D_0)x = 0$. Then $(\Sigma - D^*)x = (\Sigma - D_0)x = 0$ and hence $D^*x = D_0 x$. All coordinates of x are different from zero and thus $D^* = D_0$. The proof is complete. □

Relation Between MTFA and MRFA

In the classical MRFA one is interested in lowest rank solutions. The following theorem gives some insight into the question of the relation between the minimum-rank and minimum-trace approaches to factor analysis.

Theorem 5. Let r_1, r_2, be the rank of the Gramian matrices $\Sigma - D^*$, $\Sigma - D$, respectively. If:

(a) $r_1 + r_2 < n$

and

(b) all principal minors of $\Sigma - D^*$ of size r_1 are different from zero, then $r_1 \leq r_2$ and $d^* - d \geq 0$.

(A principal minor is a minor obtained by deleting a certain number of rows and columns having the same indices).

Proof. Condition (a) implies that

$$\dim N(\Sigma - D^*) + \dim N(\Sigma - D) = (n - r_1) + (n - r_2) = 2n - (r_1 + r_2) > n.$$

Therefore there exists a vector $x \in N(\Sigma - D^*) \cap N(\Sigma - D)$, $x \neq 0$, and we have $[(\Sigma - D^*) - (\Sigma - D)]x = (D - D^*)x = 0$. Condition (b) on $\Sigma - D^*$ implies that every

$x \in N(\Sigma - D^*)$ has at least $r_1 + 1$ coordinates different from zero. Without loss of generality we can assume that the first r coordinates of x are different from zero. Since $(D - D^*)x = 0$ we have that $d_i^* = d_i$, $i = 1, \ldots, r$. Therefore we can write

$$\Sigma - D^* = \begin{array}{cc} \begin{bmatrix} \Sigma_{11} - D_1^* & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} - D_2^* \end{bmatrix} & \begin{array}{c} r_1 \\ n - r_1 \end{array} \\ \begin{array}{cc} r_1 & \quad n - r_1 \end{array} & \end{array}$$

$$\underline{\Sigma} - D = \begin{bmatrix} \Sigma_{11} - D_1^* & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} - D_2 \end{bmatrix}.$$

By condition (b) matrix $\Sigma_{11} - D_1^*$ is nonsingular. Because $\Sigma - D^*$ has rank r_1 it yields

$$(\Sigma_{22} - D_2^*) - \Sigma_{21}(\Sigma_{11} - D_1^*)^{-1}\Sigma_{12} = 0. \tag{16}$$

Furthermore matrix

$$(\Sigma_{22} - D_2) - \Sigma_{21}(\Sigma_{11} - D_1^*)^{-1}\Sigma_{12}$$

is Gramian and then using (16) we have that $(\Sigma_{22} - D_2) - (\Sigma_{22} - D_2^*) = D_2^* - D_2$ is Gramian. We obtain that $d_2^* - d_2 \geq 0$ and the proof is complete. □

Corollary 2. Let the conditions of Theorem 5 be satisfied and $r < n/2$. Then d^* is the unique minimal rank solution.

Conclusion

After the first paper of Bentler [1972] the MTFA approach was developed by some authors, mainly for the purpose of reliability estimation. A survey of the existing literature on the subject can be found in the paper of Bentler and Woodward [1980]. This paper presented some mathematical results that are connected with MTFA. The necessary optimality conditions of Theorem 3 are equivalent to those stated by Bentler [1972] and Bentler and Woodward [1980]; our approach based on convex analysis enabled us to prove their sufficiency. In applied problems it is important to know the relation between MTFA and MRFA results; in that sense Theorem 4 provides a theoretical explanation to the empirical observation [e.g. Bentler & Woodward, 1980] that MTFA and MRFA approaches usually lead to close numerical results.

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