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LOWER BOUNDS FOR THE RELIABILITY OF THE TOTAL SCORE ON A TEST COMPOSED OF NON-HOMOGENEOUS ITEMS: I: ALGEBRAIC LOWER BOUNDS

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Let Σ_X be the (population) dispersion matrix, assumed well-estimated, of a set of non-homogeneous item scores. Finding the greatest lower bound for the reliability of the total of these scores is shown to be equivalent to minimizing the trace of Σ_X by reducing the diagonal elements while keeping the matrix non-negative definite. Using this approach, Guttman's bounds are reviewed, a method is established to determine whether his λ_4 (maximum split-half coefficient alpha) is the greatest lower bound in any instance, and three new bounds are discussed. A geometric representation, which sheds light on many of the bounds, is described.

Key words: reliability bounds, coefficient alpha, non-homogeneous composites.

1. Introduction

Guttman [1945], Gulliksen [1950], Cronbach [1951], Novick and Lewis [1967], Bentler [1972], and others have investigated the problem of finding and interpreting lower bounds for the reliability of the total score on a test whose items (or subtests) are not parallel, using data from a single test administration. The most familiar such bound is coefficient alpha.

The present paper gives a mathematical formulation of the problem of finding the *greatest* lower bound which can be obtained in the same circumstances, and investigates algebraic methods of solution. A companion paper will describe a computer search which finds the greatest lower bound.

The following notation is used. The subtest observed, true, and error scores are denoted by X_i, T_i, and E_i respectively ($i = 1, 2, \cdots, n$), and their totals by X, T, and E. The population dispersion (variance-covariance) matrices of the vectors $(X_1, X_2, \cdots, X_n)$, $(T_1, T_2, \cdots, T_n)$ and $(E_1, E_2, \cdots, E_n)$ are written Σ_X, Σ_T, Σ_E, and the variances of the totals X, T and E are written σ_X^2, σ_T^2 and σ_E^2. The symbol 1 denotes an n-dimensional column vector with elements unity.

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The usual axioms (or definitions) of classical test theory include the uncorrelatedness of E_i with T_j, and of E_i with E_j for $i \neq j$. It follows that

$$\Sigma_X = \Sigma_T + \Sigma_E, \tag{1.1}$$

where Σ_E is a diagonal matrix. If we denote the elements of Σ_X by σ_{ij}, and those of Σ_E by θ_i, then the off-diagonal elements of Σ_T are σ_{ij}, and the diagonal elements are $t_i = \sigma_{ii} - \theta_i$. The reliability ρ of X (in the population under discussion) can be variously written

$$\rho = \frac{\sigma_T^2}{\sigma_X^2} = \frac{l'\Sigma_T l}{\sigma_X^2} \tag{1.2}$$

$$= 1 - \frac{\sigma_E^2}{\sigma_X^2} = 1 - \frac{\Sigma\theta_i}{\sigma_X^2} \tag{1.3}$$

The positive definite matrix Σ_X is supposed known (that is, well-estimated), and so $\sigma_X^2 = l'\Sigma_X l$ is also known. Equation (1.3) shows that the smallest possible value of ρ is found by maximizing the sum $\sum \theta_i$ of the (diagonal) elements of Σ_E, subject to the constraints that Σ_T and Σ_E are non-negative definite matrices. Note that these are the only constraints: any set of θ_i satisfying them could be the actual error variances of the items.

Alternatively, the problem may be viewed as that of minimizing the trace of a matrix Σ_T whose off-diagonal elements are given, subject to the conditions

(A) Σ_T must be non-negative definite, and
(B) $t_i \leq \sigma_{ii}$ for all i.

The existence of two constraints gives rise to a mildly paradoxical situation. In practical applications it appears that (B) is rarely an active constraint in determining the *greatest* lower bound (though artificial examples are easily constructed), and Guttman's first five bounds make no use of it. However, his λ_6, which is frequently the best of his bounds, does require the use of (B).

This approach shows, by a continuity argument, that the greatest lower bound will correspond to a positive semi-definite (and hence singular) matrix Σ_T, and will be attainable, that is, will be equal to the smallest possible value of ρ. It has also a natural interpretation. The evidence for the reliability of X resides in the known covariances σ_{ij} between the item true scores. The greatest lower bound for ρ corresponds to the smallest average true-score variance which is consistent with these covariances, provided that no individual true-score variance is thereby required to be larger than the corresponding observed-score variance.

Since in this paper we shall sometimes approach the greatest lower bound from below, seeking the best in a class of lower bounds, and sometimes from above, seeking the smallest possible value of ρ, some mental confusion may be prevented by the remark that there can be no such thing as an upper bound for ρ (other than the trivial one of unity), since from a single test-administration we have no way of knowing that Σ_T is not identical with Σ_X, implying that the item scores are error free, and the total score perfectly reliable.

2. *Guttman's Lower Bounds*

A number of lower bounds for ρ, due to Guttman [1945], can be derived from necessary conditions for Σ_T to be non-negative definite. An almost trivial condition is that $t_i \geq 0$, for all i, and so $\theta_i \leq \sigma_{ii}$. Thus, from (1.3),

$$\lambda_1 = 1 - \sum \frac{\sigma_{ii}}{\sigma_X^2} \tag{2.1}$$

is a lower bound for ρ.

If u is an n-vector with i-th and j-th elements $+1$ and -1, and remaining elements zero, then

$$0 \leq u'\Sigma_T u = t_i + t_j - 2\sigma_{ij}. \tag{2.2}$$

Summing inequalities (2.2) over the $n(n-1)$ pairs $i \neq j$, and dividing by $2(n-1)$, we obtain

$$\sum t_i \geq (n-1)^{-1} \sum_{i \neq j}\sum \sigma_{ij} \tag{2.3}$$

Hence

$$\begin{aligned} l'\Sigma_T l &= \sum t_i + \sum_{i \neq j}\sum \sigma_{ij} \\ &\geq \frac{n}{n-1} \sum_{i \neq j}\sum \sigma_{ij} = \frac{n}{n-1}\left(\sigma_X^2 - \sum \sigma_{ii}\right), \end{aligned}$$

and so, from (1.2),

$$\lambda_3 = \frac{n}{n-1}\left(1 - \frac{\sum \sigma_{ii}}{\sigma_X^2}\right) = \frac{n\lambda_1}{(n-1)} \tag{2.4}$$

is a lower bound for ρ, commonly known as coefficient alpha, following Cronbach [1951]. Clearly λ_3 is a better (higher) lower bound than the trivial bound λ_1 whenever $\lambda_1 > 0$, but only marginally so when the number of items (subtests) is large. If $\lambda_1 \leq 0$, neither bound is useful since the reliability is always non-negative.

The bound λ_3 exploits information in the 2×2 principal submatrices of Σ_T, with diagonal elements t_i, t_j and off-diagonal elements $\sigma_{ij} = \sigma_{ji}$. (Similarly λ_1 uses the 1×1 principal submatrices σ_{ii}.) However, λ_3 does not, in general, fully exploit the non-negative definite property of these submatrices. This can be done using the fact that the determinant must be non-negative, that is,

$$t_i t_j \geq \sigma_{ij}^2, \qquad i \neq j. \tag{2.5}$$

To relate the inequalities (2.5) to the trace of Σ_T we use the fact that, for *any* set of t_i.

$$2(n-1)\left(\sum t_i\right)^2 = 2(n-1)\sum t_i^2 + 2(n-1)\sum_{i \neq j}\sum t_i t_j$$

$$= \sum_{i \neq j}\sum (t_i - t_j)^2 + 2n \sum_{i \neq j}\sum t_i t_j. \tag{2.6}$$

Since $t_i \geq 0$ for all i, $\sum t_i \geq 0$. Hence by (2.6) and then (2.5),

$$\sum \theta_i = \sum \sigma_{ii} - \sum t_i$$

$$\leq \sum \sigma_{ii} - \left[n(n-1)^{-1} \sum_{i \neq j}\sum t_i t_j\right]^{1/2}$$

$$\leq \sum \sigma_{ii} - \left[n(n-1)^{-1} \sum_{i \neq j}\sum \sigma_{ij}^2\right]^{1/2},$$

and from (1.3),

$$\lambda_2 = 1 - \sum \frac{\sigma_{ii}}{\sigma_X^2} + \frac{\left[n(n-1)^{-1} \sum_{i \neq j}\sum \sigma_{ij}^2\right]^{1/2}}{\sigma_X^2} \tag{2.7}$$

is a lower bound for ρ. For the lower bound to be attained, from (2.6) we require that $t_i = t_j = t$ (say) for all i, j; and from (2.5) we require that $\sigma_{ij} = \pm t$ all $i \neq j$. Typically Σ_X will not possess this property, and we shall know that λ_2 is not the *greatest* lower bound which the data can provide.

If $\sigma_{ij} = t \geq 0$ for all $i \neq j$, then $\lambda_2 = \lambda_3$. In every other case it is readily verified, using the fact that the sum of squares of the $n(n-1)$ values σ_{ij} ($i \neq j$) about their own mean is non-negative, that $\lambda_2 > \lambda_3$. In the days of hand calculation, λ_3 had the advantage that it required the calculation of only the item variances, not the covariances. This is no longer of practical importance. Clearly we have also $\lambda_2 > \lambda_1$ unless Σ_X is a diagonal matrix.

In Lord and Novick [1968] it is erroneously asserted (p. 95) that Guttman's λ_2 (incorrectly referred to as λ_3) is always positive. To see that this is not so we have only to take $n > 2$, $\sigma_{ij} = -t$ all $i \neq j$, where $t > 0$, and any set of σ_{ii} large enough to make Σ_X positive definite. Then $\lambda_1 = -n(n-1)t/\sigma_X^2$, $\lambda_3 = -n^2 t/\sigma_X^2$, and $\lambda_2 = -n(n-2)t/\sigma_X^2$, so all three bounds derived so far are useless. In practice, of course, it is most unlikely that one would use as a test score the sum of scores on items negatively correlated among themselves.

We next inquire whether the methods used to derive λ_2 and λ_3 can be extended to make use of information in the $k \times k$ principal submatrices, $3 \leq k \leq n$. We shall study the determinantal form of the non-negative definite condition for the case $k = 3$ in a companion paper. It does not appear to yield any simple inequality for $\sum t_i$. The same is true a fortiori for $k > 3$.

The natural generalization of (2.2) is to consider n-vectors u with k elements either $+1$ or -1 and the remaining elements zero, making use of the inequality

$$u'\Sigma_T u \geq 0. \tag{2.8}$$

If we proceed directly to the case $k = n$, there are 2^n such vectors u, of which only 2^{n-1} need be considered since replacing u by $-u$ in (2.8) yields the same inequality. For any one of these, by (1.1),

$$u'\Sigma_X u \geq u'\Sigma_E u = \sum \theta_i. \tag{2.9}$$

Hence, by (1.3),

$$\lambda_4 = \max_u \, (1 - u'\Sigma_X u/\sigma_X^2) \tag{2.10}$$

is a lower bound for ρ.

Now if we write $X = (X_1, X_2, \cdots, X_n)'$, $Y_1 = \frac{1}{2}(1 + u)'X$, and $Y_2 = \frac{1}{2}(1 - u)'X$, then Y_1 and Y_2 are the sums of the scores on the items corresponding respectively to the positive and negative elements of u. Thus to each u corresponds a split of the test into two "halves" (since no assumption of parallelism is made for items, the number of items in each "half" is irrelevant). The covariance of Y_1 and Y_2 is

$$\sigma(Y_1, Y_2) = \tfrac{1}{4}(1 + u)'\Sigma_X(1 - u) = \tfrac{1}{4}(\sigma_X^2 - u'\Sigma_X u) \tag{2.11}$$

and the value of coefficient alpha based on the split-half scores Y_1, Y_2, is

$$\frac{4\sigma(Y_1, Y_2)}{\sigma_X^2} = 1 - \frac{u'\Sigma_X u}{\sigma_X^2}. \tag{2.12}$$

Thus (2.10) is the same as Guttman's λ_4 (or, strictly speaking, the maximum of λ_4 over all splits, which is the bound he actually proposed). For moderate values of n, it is now a much more practical proposition than when Guttman wrote to compute all 2^{n-1} possible values of $u'\Sigma_X u$ and select the smallest for use in calculating λ_4.

Since the evidence for the reliability of the test resides in the correlations between its components, and the computation of λ_4 involves a thorough examination of these correlations, one might speculate that λ_4 is the *greatest* lower bound the data provide. In the companion paper we shall see that this is not necessarily true, although for a somewhat unexpected reason. However, the present method of derivation enables us to determine, in any given instance, *whether* λ_4 is the greatest lower bound.

Let $\nu = (\nu_1, \nu_2, \cdots, \nu_n)'$, $\nu_i = \pm 1$, denote the u-vector which yields the greatest split-half bound, λ_4. In order for it to be possible to have $\rho = \lambda_4$ we require equality in (2.8) and (2.9). Since Σ_T is non-negative definite there exists an $n \times n$ matrix B such that $\Sigma_T = B'B$. Thus we require

$$0 = \nu'\Sigma_T\nu = \nu'B'B\nu = (B\nu)'(B\nu).$$

This is possible only if $B\nu = 0$, in which case $\Sigma_T \nu = B'B\nu = 0$, and so

$$\Sigma_X \nu = \Sigma_E \nu = (\nu_1\theta_1, \nu_2\theta_2, \cdots, \nu_n\theta_n)'. \tag{2.13}$$

Thus if λ_4 is the greatest lower bound, we must have

$$\theta_i = \nu_i(\Sigma_X \nu)_i \tag{2.14}$$

where the second factor denotes the i-th component of the vector $\Sigma_X \nu$. If this set of θ_i satisfies the conditions that $\Sigma_T = \Sigma_X - \mathrm{diag}\,(\theta_1, \theta_2, \cdots, \theta_n)$ is non-negative definite, and $\theta_i \geq 0$ for all i, then λ_4 is a possible value of ρ; but λ_4 is certainly *a* lower bound for ρ; hence it is the greatest lower bound. If either of the conditions is violated, we know that the greatest lower bound is greater than λ_4.

A major weakness of bounds λ_1, λ_2 and λ_3 is that they are symmetric in the covariances σ_{ij}, whereas a little experimentation with artificial matrices Σ_X (with $n = 3$ or 4) shows that the main constraint on $\sum t_i$ comes from those covariances which are large in absolute value. If there is a whole column of Σ_X, say the j-th, in which the covariances are relatively large, we can exploit this fact as follows. Let $\sum'$ denote a summation for fixed j over all values of i except j. Using (2.5) we have

$$\begin{aligned} \sum t_i = t_j + \sum' t_i &\geq t_j + t_j^{-1} \sum' \sigma_{ij}^2 \\ &\geq 2\left(\sum' \sigma_{ij}^2\right)^{1/2}, \end{aligned} \tag{2.15}$$

the last inequality following from the fact that the arithmetic mean of two positive numbers is not less than their geometric mean. Thus

$$\sum \theta_i \leq \sum \sigma_{ii} - 2\left(\sum' \sigma_{ij}^2\right)^{1/2}, \tag{2.16}$$

and by substitution in (1.3) we obtain a lower bound for ρ. By selecting the column for which the right hand side of (2.15) is greatest, we obtain the lower bound

$$\lambda_5 = 1 - \sum \frac{\sigma_{ii}}{\sigma_X^2} + \max_j \frac{2\left(\sum' \sigma_{ij}^2\right)^{1/2}}{\sigma_X^2}. \tag{2.17}$$

On comparing (2.17) with (2.7) it follows that $\lambda_5 > \lambda_2$ if the average squared covariance in the maximizing column is greater than $n/2$ times the average of the remaining squared covariances. It is possible to have $\lambda_5 > \lambda_4$ if λ_4 is not the greatest lower bound; an example is given in the companion paper.

Guttman's sixth bound can be established as follows. For any vector $\beta = (\beta_1, \beta_2, \cdots, \beta_n)'$ we have

$$\begin{aligned} \beta'\Sigma_X\beta &= \beta'\Sigma_T\beta + \beta'\Sigma_E\beta \\ &\geq \beta'\Sigma_E\beta = \sum \beta_i^2\theta_1 \\ &\geq \beta_i^2\theta_1. \end{aligned} \tag{2.18}$$

Without loss of generality take $\beta_1 = 1$, partition β as $(1, \gamma)$ and Σ_X as

$$\Sigma_X = \begin{pmatrix} \sigma_{11} & \sigma_1' \\ \sigma_1 & \Sigma_{11} \end{pmatrix}.$$

Then

$$(2.19) \quad \beta'\Sigma_X\beta = \sigma_{11} - \sigma_1'\Sigma_{11}^{-1}\sigma_1 + (\gamma - \Sigma_{11}^{-1}\sigma_1)'\Sigma_{11}(\gamma - \Sigma_{11}^{-1}\sigma_1).$$

Since Σ_{11} is positive definite it follows that

$$(2.20) \qquad \begin{aligned} \theta_1 &\le \sigma_{11} - \sigma_1'\Sigma_{11}^{-1}\sigma_1 \\ &= (\Sigma_X^{-1})_{11}^{-1}, \end{aligned}$$

the reciprocal of the first diagonal element of Σ_X^{-1}. Now (2.19) is exactly the argument by which the linear regression equation of X_1 on $X_2, X_3, \cdots, X_n$ is determined, and $(\Sigma_X^{-1})_{11}^{-1}$ is the residual variance, ϵ_1^2 say. Replacing $b_1^2\theta_1$ by $b_i^2\theta_i$ in (2.18) we obtain in the same way $\theta_i \le \epsilon_i^2$, where ϵ_i^2 is the residual variance in the regression of X_i on the remaining subtest scores. Substituting in (1.3) we obtain as a lower bound for ρ

$$(2.21) \qquad \lambda_6 = 1 - \sum \frac{\epsilon_i^2}{\sigma_X^2}.$$

Note that this is the only one of Guttman's bounds which uses—in obtaining (2.18)—condition (B) that $\theta_i \ge 0$ for all i.

Rozeboom [1966, pp. 334–341] discusses Guttman's bounds in a factor analytic context, using the concepts of saturation and homogeneity; he also considers the situation where the assumption of uncorrelated subtest errors is unrealistic.

3. Other Algebraic Lower Bounds

Two of Guttman's bounds, λ_1 and λ_6, are derived by bounding the individual error variances θ_i. We have seen that

$$(3.1) \qquad \theta_i \le b_{1i} = \sigma_{ii},$$

and

$$(3.2) \qquad \theta_i \le b_{2i} = (\Sigma_X^{-1})_{ii}^{-1}.$$

For any such set of bounds b_{hi}, $i = 1, 2, \cdots, n$, if we write $b_h = \sum_i b_{hi}$ we obtain as a lower bound for the reliability

$$(3.3) \qquad \rho_h = 1 - \frac{b_h}{\sigma_X^2}.$$

From (2.20), $b_{2i} \le b_{1i}$; however, computation of b_{2i} requires matrix inversion which, while no longer a major obstacle, involves an expenditure of computing time of an order considerably higher than that required for some of the other bounds. It may therefore be useful to consider alternative bounds on the θ_i. From (2.5),

$$t_i \geq \frac{\sigma_{ij}^2}{t_j} \geq \frac{\sigma_{ij}^2}{\sigma_{jj}} \quad \text{for any } j \neq i. \tag{3.4}$$

Hence

$$\theta_i \leq b_{3i} = \min_{j \neq i} \left(\sigma_{ii} - \frac{\sigma_{ij}^2}{\sigma_{jj}} \right) = \sigma_{ii}(1 - \max_{j \neq i} r_{ij}^2), \tag{3.5}$$

where r_{ij} is the correlation between X_i and X_j. This bound is likely to be useful when the actual error variances are small, in which case the correlations between subtest scores are large (and the composite score is highly reliable). Of course, since

$$b_{2i} = \sigma_{ii}(1 - R_i^2), \tag{3.6}$$

where R_i is the coefficient of multiple correlation between X_i and the remaining subtest scores, $b_{2i} \leq b_{3i}$, and b_{2i} is an even better bound in the same circumstances.

Inequality (3.4) can also be used to improve Guttman's λ_2. Let g denote the value of j ($\neq i$) for which $\sigma_{ij}^2/\sigma_{jj}$ is largest, and similarly let $\sigma_{kj}^2/\sigma_{kk}$ be the largest value of $\sigma_{ij}^2/\sigma_{ii}$ for $i \neq j$. Then

$$t_i t_j \geq \frac{\sigma_{ig}^2 \sigma_{kj}^2}{\sigma_{gg}\sigma_{kk}} = \sigma_{ii}\sigma_{jj} r_{ig}^2 r_{kj}^2, \tag{3.7}$$

and we can substitute this inequality in the derivation of λ_2 whenever it gives a better bound than (2.5), which we can write as

$$t_i t_j \geq \sigma_{ii}\sigma_{jj} r_{ij}^2. \tag{3.8}$$

Thus if

$$d_{ij}^2 = \sigma_{ii}\sigma_{jj} \max (r_{ig}^2 r_{kj}^2, r_{ij}^2), \tag{3.9}$$

we obtain the lower bound

$$\lambda_7 = 1 - \sum \frac{\sigma_{ii}}{\sigma_X^2} + \frac{\left[n(n-1)^{-1} \sum_{i \neq j} \sum d_{ij}^2 \right]^{1/2}}{\sigma_X^2}. \tag{3.10}$$

This bound will be substantially better than λ_2 when there is considerable variation among the squared correlations.

Another upper bound for θ_i can be obtained as follows. Let α_p, $p = 1, 2, \cdots, n$ be the eigen values of Σ_X, and let a_p be corresponding normalized eigen vectors with elements a_{pi}. Then $\Sigma_X a_p = \alpha_p a_p$, and so

$$\alpha_p = \alpha_p a_p' a_p = a_p' \Sigma_X a_p \geq a_p' \Sigma_E a_p = \sum_j a_{pj}^2 \theta_j \geq a_{pi}^2 \theta_i \tag{3.11}$$

for any i. Hence

$$\theta_i \leq b_{4i} = \min_p \left(\frac{\alpha_p}{a_{pi}^2}\right). \tag{3.12}$$

This bound is of academic interest, and readily computable with modern packages. However, the amount of computation required is typically greater than for b_{2i}, and since we shall show in the next section that $b_{4i} \geq b_{2i}$, there seems to be little practical value in computing it. (In computing b_{2i}, $i = 1, 2, \cdots, n$, we require only the diagonal elements of Σ_X^{-1}. If Σ_X is poorly conditioned, it may be better to compute these directly as det Σ_{ij}/det Σ_X than to attempt to use an inversion package.)

4. *A Geometrical Representation of the Problem*

There is a geometrical representation which sheds useful light on many of the bounds discussed in this paper, and on the possibility of improving upon them. If Y is a vector of coordinates in n-dimensional Cartesian space, the equation

$$Y'\Sigma_X Y = 1 \tag{4.1}$$

represents an n-dimensional ellipsoid, C_X say, which is fixed since we are supposing Σ_X known. (The easily visualised case $n = 3$ is of sufficient generality to illustrate all the points to be discussed.) In the notation of preceding sections, the intercepts on the coordinate axes are $\pm(\sigma_{ii})^{-1/2}$; the principal axes of the ellipsoid are of length $2\alpha_p^{-1/2}$ and have direction cosines $(a_{p1}, a_{p2}, \cdots, a_{pn})$; the tangent hyperplane at Y_0 has equation

$$Y'\Sigma_X Y_0 = 1; \tag{4.2}$$

and the equation of the tangent hyperplane whose normal has direction cosines $l = (l_1, l_2, \cdots, l_n)'$ is

$$l'Y = (l'\Sigma_X^{-1}l)^{1/2}, \tag{4.3}$$

where the right member of the equation is the distance of the plane from the centre of the ellipsoid (the origin).

Similarly the equation

$$\underline{Y'\Sigma_E Y} = \Sigma\theta_i Y_i^2 = 1 \tag{4.4}$$

represents an ellipsoid C_E whose principal axes are the co-ordinate axes, and are of length $2\theta^{-1/2}$. Condition (B) requires that (4.4) should *be* an ellipsoid (degenerate if $\theta_i = 0$ for some i), not some other variety of conicoid; in particular it must be convex. Since $Y'\Sigma Y < 1$ at interior points of the ellipsoid $Y'\Sigma Y = 1$, Condition (A), which can be written

$$Y'\Sigma_X Y \geq Y'\Sigma_E Y \quad \text{for all} \quad Y, \tag{4.5}$$

implies that no point of C_E lies inside C_X.

The inequalities $\theta_i \leq b_{hi}$ of the previous section ($h = 1,2,3,4$) express the fact that the following points on the i-th co-ordinate axis, whose distances from the origin 0 are $(b_{hi})^{-1/2}$, cannot lie outside C_E, and so $(b_{hi})^{-1/2} \leq \theta_i^{-1/2}$:

(i) The point at which C_X meets the axis, at a distance $(\sigma_{ii})^{-1/2}$ from 0.

(ii) The point at which the axis is met by the tangent hyperplane of C_X normal to it. If l_i denotes the vector with i-th element l and remaining elements zero, by (4.3) the distance of this point from 0 is $(l_i'\Sigma_X^{-1}l)^{1/2} = (\Sigma_X^{-1})_{ii}^{1/2}$. Note that the squared reciprocal of this distance is the residual variance in the multiple linear regression of X_i on the remaining subtest scores.

(iii) For any j, the point at which the axis is met by the tangent drawn perpendicular to it to the elliptical section of C_X by the plane of the i-th and j-th co-ordinate axes; hence the furthest of these points from the origin. As in (ii), the squared reciprocal of the distance of each of these points from the origin is the residual variance in the linear regression of X_i on X_j, leading to equation (3.5).

(iv) For any p, the projection on the axis of the end of the p-th principal axis of C_X, at a distance $a_{pi}\alpha_p^{-1/2}$ from 0; hence the furthest of these points from the origin.

From this representation it is clear that $b_{1i} \geq b_{3i} \geq b_{2i}$ and $b_{4i} \geq b_{2i}$ but no general statement can be made about the relative sizes of b_{3i} and b_{4i}. It also seems to follow that no direct algebraic bound for the reliability obtained by bounding the individual θ_i is likely to give a general improvement on Guttman's λ_6. Note that all the bounds except b_{1i} make use of Condition (B).

Geometrically, the problem of finding the greatest lower bound for ρ is that of finding the "smallest" ellipsoid C_E which circumscribes C_X, where the criterion of size is the harmonic mean of the principal semi-axes, $n/\sum \theta_i$. If l is a vector of direction cosines each $\pm n^{-1/2}$, the point kl lies on C_E if $k^2 l'\Sigma_E l = 1$, that is, $k^2 = n/\sum \theta_i$, and on C_X if $k^2 = 1/l'\Sigma_X l = n/u\Sigma_X u$, where $u_i = n^{1/2}l_i = \pm 1$. Thus Condition (A) yields the inequality $\sum \theta_i \geq u'\Sigma_X u$ on which λ_4 is based.

If it is possible to arrange that C_E and C_X touch at the points $\pm kl$ with $k^2 = \max(1/l'\Sigma_X l)$, then we have found the "smallest" ellipsoid C_E and λ_4 is the *greatest* lower bound for ρ. From (4.3) this requires $k\Sigma_X^{-1}l = k\Sigma_E^{-1}l$, or $\Sigma_X^{-1}u = (u_1\theta_1, u_2\theta_2, \cdots, u_n\theta_n)'$ as in (2.13). However, since the orientations of both C_X and C_E are fixed, this may not be possible: C_E "cannot get round the corner".

5. Conclusions

Given modern computational facilities, only bounds λ_4, λ_5, λ_6 and λ_7 are worth considering. Of these, λ_4 has the advantage that, unless Σ_X has very special properties, unlikely to be encountered in practice, it is the only one which may actually be the *greatest* lower bound: in Section 2 a method was given for determining in any instance whether this is the case. For moderate values of n, computation of λ_4 presents no problems, but if the test comprises a large number of items the direct computer search for the greatest lower bound described in the companion paper is likely to be more efficient. Research could usefully be directed to the question of whether the optimizing split of the test can be found by examining a subset of the 2^{n-1} possible splits, by analogy with linear programming, where a subset of the vertices determined by the constraints is examined to find the optimizing vertex.

Of the remaining bounds, λ_6, which takes simultaneous account of the information in all the elements of Σ_X, should be particularly advantageous in the fairly typical situation where the inter-item correlations are positive, modest in size, and rather similar. Its computation does, however, require matrix inversion. If n is large and facilities for inversion of large matrices are not available, one might consider the bound (Section 3)

$$\rho_3 = 1 - \sigma_X^{-2} \sum_i \sigma_{ii}(1 - \max_{j \neq i} r_{ij}^2).$$

The comparison between λ_6 and ρ_3 least favourable to the latter is obtained by supposing that all the observed-score correlations are equal to $r > 0$. Then

$$\begin{aligned} b_{2i} &= \sigma_{ii}(1 - r)\{1 + (n - 1)r\}/\{1 + (n - 2)r\} \\ &= \sigma_{ii}(1 - r), \end{aligned}$$

approximately, if n is large, while

$$b_{3i} = \sigma_{ii}(1 - r^2).$$

In absolute terms, the difference between b_{3i} and b_{2i} is greatest when $r = .5$. The effect on the corresponding lower bounds for ρ, however, is expressed by the equation

$$(1 - \rho_3) = (1 + r)(1 - \lambda_6),$$

showing that, in terms of the ratio of the distances of the two bounds from unity, the advantage of λ_6 increases as r increases.

It does not seem possible to make simple general statements about the sizes of λ_5 and λ_7 relative to λ_4 and λ_6. However, since they involve relatively little computational effort, it will be worth examining whether, in any particular application, one of them provides a better bound than any other which has been computed.

Examples of all the bounds, using both real and artificial matrices Σ_X, will be given in the companion paper, where the greatest lower bound will also be available for comparison.

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