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TEST THEORY WITH MINIMAL ASSUMPTIONS

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Using the concepts of conditional probability, conditional expectation, and conditional independence, the main results of the classical test theory model can be derived in a very few steps with minimal assumptions. It is well known that the variance of a random variable is the sum of the variance of its conditional expectation and the expectation of its conditional variance. These concepts lead to another fundamental relation: the reliability coefficient, regarded as a correlation between two conditionally independent, conditionally identically distributed random variables (exchangeable random variables, parallel measurements) is the ratio of the variance of their common conditional expectation to their common unconditional variance. Formulas describing the reliability of lengthened tests follow easily, and the same result also applies to stochastic processes involving joint distributions of many dependent random variables.

The theory of mental tests has a long history, going back to the first decade of this century. From the beginning it had many applications in Psychology, Education, and other fields, and it was developed extensively even before its mathematical basis was completely clarified. At first, the structure of the theory appeared to be quite simple. It was said that an observed score is the sum of a true score and an error score,

$$X = T + E,$$

that the mean of error scores is zero, and that true scores and error scores are uncorrelated. Then, observed variance, true variance, and error variance were related to one another, and test reliability was expressed as a correlation between observed scores on "parallel" tests, or as the ratio of true variance to observed variance (see, for example, Gulliksen, 1950).

In time, however, these ideas were examined more critically, and the question arose as to how the theory could be formulated with a minimum of assumptions. It became apparent that the simplicity of the model was somewhat deceptive and that these basic components of scores needed to be defined more explicitly. Guttman (1945) pointed out that the earlier theory that had been proposed by Yule contained more assumptions than necessary and suggested a formulation which he believed to be more consistent with the original intention of Spearman. For example, the statements that true scores and error scores are uncorrelated, that error scores on parallel tests are uncorrelated, and that the mean of error scores is zero were shown to be consequences of other definitions. Later, Novick (1966) and Lord and Novick (1968) also emphasized this fact. All of these models have been based on the idea that an observed score is a sum of true and error components, and certain other assumptions have been made about random sampling of individuals and of measurements, "experimental independence," and so on.

The present paper explores the possibility that these theories can be further condensed. It is guided by the following considerations. We want a bare minimum of assumptions, a minimum of special terminology and notation, and derivations requiring as few steps as possible. We want to show the natural relationship of the main concepts to probability theory generally and to the models employed in statistical sampling theory, especially the familiar independent subexperiments model that describes random sampling by sequences of independent random variables. We will assume as known standard results in probability, including definitions of random variables, expectation, variance, correlation, and so on. However, some concepts that are not so well known, such as conditional independence and conditional expectation regarded as a random variable, will be defined more explicitly and treated in some detail. With this preparation it is found that the desired results can be obtained quite rapidly. The basic ideas that are familiar in test theory are contained in theorem 1, which requires only a few sentences for its statement and proof.

Conditional Independence and Conditional Expectation

The concepts of conditional independence and conditional expectation are particularly important for our present purposes. After these are defined and some of their properties listed, the main results of interest for test theory, contained in theorem 1, can be proved with very little effort.

We assume familiarity with the concepts of probability spaces, random points and random variables, and independence. Let $(\Omega, \mathfrak{A}, P)$

be a probability space, X_1: $\Omega \rightarrow R$ and X_2: $\Omega \rightarrow R$ two random variables with finite non-zero variance, and f: $\Omega \rightarrow \Psi$ a random point taking values in an arbitrary set Ψ with range $A = f(\Omega)$. We emphasize that X_1 and X_2 take their values in the set of real numbers, while f may take values in a set that does not necessarily have the algebraic and topological structure of the real numbers. For example, in finite cases $A = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$.

Definition 1

Random variables X_1 and X_2 are ***conditionally independent*** given a random point f, if, for every $\alpha \in A$, the random variables $X_1| f = \alpha$ and $X_2| f=\alpha$ are independent.

Conditional independence does not imply independence, and independence does not imply conditional independence.

Example 1

Select a point α at random on the unit interval [0,1]. Then select two points at random on the interval [0, α] and let X_1 and X_2 represent the real numbers obtained. Although the random variables X_1 and X_2 are dependent, they are conditionally independent, since for all $\alpha \in A =$ [0,1] the random variables $X_1 \mid f = \alpha$ and $X_2 \mid f = \alpha$ are independent.

Definition 2

Random variables X_1 and X_2 are ***conditionally identically distributed*** given a random point f, if, for every $\alpha \in A$, the random variables $X_1 \mid f = \alpha$ and $X_2 \mid f = \alpha$ are identically distributed, that is, if their induced probability measures are equal.

If random variables are conditionally identically distributed, they are identically distributed, but the converse is not generally true. If random variables $\{X_i\}$, $i = 1,2, \cdots, n$ are both conditionally independent and conditionally identically distributed, they are *exchangeable*, that is, for every positive integer $m \leq n$, all ordered subsets $(X_1, X_2, \cdots, X_m)$ containing m elements induce the same joint distribution on R^m (see, for example, Feller, 1966, Loève, 1963 for discussion of the concept of exchangeable random variables).

Example 2

In example 1 the dependent random variables X_1 and X_2 are conditionally independent and conditionally identically distributed, and the joint distribution of (X_1,X_2) is the same as that of (X_2,X_1). Let $X_3 = X_2 + 1$. Then X_1 and X_3 are conditionally independent, but not conditionally identically distributed, and the joint distribution of (X_1,X_3) differs from that of (X_3,X_1).

Definition 3

The *conditional expectation* of a random variable X given a random point f is the random variable M_X: $\Omega \rightarrow R$ defined by

$$M_X(\omega) = m_X(f(\omega)), \text{ for each } \omega \in \Omega,$$

where $m_X: A \to R$ is the function defined by

$$m_X(\alpha) = E\,[X \mid f = \alpha], \text{ for each } \alpha \in A.$$

For the subsequent derivations, an important fact is that if two random variables X_1 and X_2 are conditionally independent, the conditional expectation of their product is the product of their conditional expectations—that is, $M_{X_1X_2} = M_{X_1}M_{X_2}$. Also, if two random variables X_1 and X_2 are conditionally identically distributed, their conditional expectations are equal—that is, M_{X_1} and M_{X_2} are the same function from Ω to R.

In the same way we regard the *conditional variance* of a random variable X given a random point f as the random variable $V_X: \Omega \to R$ defined by $V_X(\omega) = v_X(f(\omega))$, for each $\omega \in \Omega$, where $v_X: A \to R$ is the function defined by $v_X(\alpha) = \text{Var}\,[X \mid f = \alpha]$, for each $\alpha \in A$. Two relations that can now be expressed easily in this notation are that $EX = EM_X$ and that $\text{Var}\,X = \text{Var}\,M_X + EV_X$, that is, the expectation of a random variable equals the expectation of its conditional expectation, and the variance of a random variable equals the variance of its conditional expectation plus the expectation of its conditional variance. Note that the random variables X, M_X, and V_X and the random point f are all defined on the same fundamental probability set Ω, while m_X and v_X are functions defined on the range of f.

Example 3

In example 1, f induces a uniform density on the unit interval, and $X_1|f = \alpha$ induces a uniform density on the interval $[0,\alpha]$, so we have $m_{X_1}(\alpha) = E[X_1 \mid f = \alpha] = \alpha/2$, for each $\alpha \in A = [0,1]$. Since $\Omega = \{(\alpha, \beta_1, \beta_2) \in R^3\colon 0 \leq \alpha \leq 1, 0 \leq \beta_1 \leq \alpha, 0 \leq \beta_2 \leq \alpha\}$, it follows that $M_{X_1}(\omega) = M_{X_1}(\alpha, \beta_1, \beta_2) = \alpha/2$, for each $\omega \in \Omega$.

Reliability of Measurements

We continue to consider a random point f, random variables X_1 and X_2, and conditional expectations M_{X_1} and M_{X_2}, all with the same associated probability space $(\Omega, \mathfrak{A}, P)$. The correlation between X_1 and X_2 will be denoted by $\rho(X_1, X_2)$ and the standard deviation of X_1 by σX_1. Here is the main theorem:

Theorem 1

Let X_1 and X_2 be two conditionally independent, conditionally identically distributed random variables with finite non-zero variance. Then,

$$\rho(X_1,\ X_2) = \frac{\text{Var}\ M_{X_1}}{\text{Var}\ X_1},$$

and, if Var $M_{X_1} > 0$,

$$\rho(X_1, M_{X_1}) = \frac{\sigma M_{X_1}}{\sigma X_1}.$$

Proof: Denoting by X_1X_2 the random variable defined by $(X_1X_2)(\omega) = X_1(\omega)X_2(\omega)$, for each $\omega \in \Omega$, and by $M_{X_1X_2}$ the conditional expectation of X_1X_2, we have

$$\begin{aligned}\mathrm{Cov}(X_1,X_2) &= E(X_1X_2) - (EX_1)(EX_2)\\ &= EM_{X_1X_2} - (EM_{X_1})(EM_{X_2})\\ &= E(M_{X_1}M_{X_2}) - (EM_{X_1}(EM_{X_2})\\ &= \mathrm{Cov}(M_{X_1},M_{X_2}) = \mathrm{Var}\ M_{X_1}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(X_1,M_{X_1}) &= E(X_1M_{X_1}) - (EX_1)(EM_{X_1})\\ &= EM_{X_1M_{X_1}} - (EM_{X_1})(EM_{X_1})\\ &= E(M_{X_1}M_{X_1}) - (EM_{X_1})(EM_{X_1})\\ &= \mathrm{Var}\ M_{X_1}.\end{aligned}$$

Substitution in $\rho(X_1,X_2) = \mathrm{Cov}(X_1,X_2)/\sigma X_1 \sigma X_2$ and $\rho(X_1, M_{X_1}) = \mathrm{Cov}(X_1, M_{X_1})/\sigma X_1 \sigma M_{X_1}$ completes the proof.

Since Var X_1 = Var $M_{X_1} + EV_{X_1}$, it follows that $\rho(X_1,X_2) = 1 - M_{X_1}/EV_{X_1}$, X_1. Various other relations follow easily from substitutions. To summarize, if Var $M_{X_1} > 0$, then $0 \leq \rho^2(X_1, M_{X_1}) = \rho(X_1,X_2)$ = Var M_{X_1}/Var $X_1 = 1 - EV_{X_1}$/Var $X_1 \leq 0$, and if Var $M_{X_1} = 0$, all the above relations hold, except that $\rho(X_1,M_{X_1})$ is undefined.

Example 4

Table 1 shows a random point f, two random variables X_1 and X_2, the conditional expectation M_{X_1} and the conditional variance V_{X_1}, all defined on $\Omega = \{\omega_1, \cdots, \omega_8\}$, with $A = \{\alpha_1, \alpha_2\}$. It is found that Var $M_{X_1} = 1$, $EV_{X_1} = 5$, Var X_1 = Var $M_{X_1} + EV_{X_1} = 6$, $\rho(X_1, X_2)$ = Var M_{X_1}/Var $X_1 = 1/6$, and $\rho(X_1, M_{X_1}) = \sigma M_{X_1}/\sigma X_1 = 1/\sqrt{6}$.

Example 5

We will show now that the hypothesis of theorem 1 can be weakened and the same conclusion obtained. The random variables X_1 and X_2 in Table 2 are not conditionally independent, not conditionally identically distributed, and their conditional expectations are not equal. However, they are *conditionally uncorrelated,* they have the same variance, and their conditional expectations have the same variance and covariance, that is, Var M_{X_1} = Var M_{X_2} = Cov (M_{X_1}, M_{X_2}). Also, Var X_1 = Var $X_2 = 3/4$, and $\rho(X_1, X_2)$ = Var M_{X_1}/Var $X_1 = 1/3$. The proof of theorem 1 uses only the following facts: $EX_1 = EM_{X_1}$ (which is true for any random variable with expectation), $M_{X_1X_2} = M_{X_1}M_{X_2}$, Var X_1 = Var X_2, and Var M_{X_1} = Var M_{X_2} = Cov (M_{X_1}, M_{X_2}), and all of these follow from the weaker hypothesis.

TABLE 1
Random Variables in Example 4

ω	$p(\omega)$	$f(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$M_{X_1}(\omega)$	$V_{X_1}(\omega)$
ω_1	1/8	α_1	0	0	1	1
ω_2	1/8	α_1	0	2	1	1
ω_3	1/8	α_1	2	0	1	1
ω_4	1/8	α_1	2	2	1	1
ω_5	1/8	α_2	0	0	3	9
ω_6	1/8	α_2	0	6	3	9
ω_7	1/8	α_2	6	0	3	9
ω_8	1/8	α_2	6	6	3	9

Example 6

A point α is selected at random on the unit interval. If $\alpha > \frac{1}{2}$, the number 1 is assigned as the outcome of the experiment. If $0 \leq \alpha \leq \frac{1}{2}$, a second point β is selected at random on the unit interval, and then if $\beta > \alpha$, the number 0 is assigned, and if $0 \leq \beta \leq \alpha$, the number 1 is assigned. We have

$$\omega \in \Omega = \{(\alpha, \beta) \in R^2\colon 0 \leq \alpha \leq \tfrac{1}{2}, 0 \leq \beta \leq 1\}$$

$$\cup \{(\alpha, \beta) \in R^2\colon \tfrac{1}{2} < \alpha \leq 1, \beta = 0\}$$

$$f(\alpha,\beta) = \alpha$$

$$X(\alpha, \beta) = \begin{cases} 1 & \text{if } \beta \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

$$M_X(\alpha, \beta) = \begin{cases} \alpha & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ 1 & \text{if } \alpha > \frac{1}{2} \end{cases}$$

$$V_X(\alpha, \beta) = \begin{cases} \alpha(1 - \alpha) & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ 0 & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

TABLE 2
Random Variables in Example 5

ω	$p(\omega)$	$f(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$M_{X_1}(\omega)$	$M_{X_2}(\omega)$
ω_1	1/8	α_1	0	2	1	2
ω_2	1/8	α_1	1	1	1	2
ω_3	1/8	α_1	2	2	1	2
ω_4	1/8	α_1	1	3	1	2
ω_5	1/8	α_2	1	3	2	3
ω_6	1/8	α_2	2	2	2	3
ω_7	1/8	α_2	3	3	2	3
ω_8	1/8	α_2	2	4	2	3

Hence, Var $M_X = 29/192$, $EV_X = 1/12$, Var $X = 15/64$, and Var M_X/Var $X = 29/45$. The number Var M_X/Var X can be interpreted as the correlation between two random variables X_1 and X_2 defined as follows: A point α is selected at random on the unit interval, and if $\alpha > \frac{1}{2}$, both X_1 and X_2 take the value 1. If $0 \leq \alpha \leq \frac{1}{2}$, two points β_1 and β_2 are selected at random on the unit interval. If $\beta_i > \alpha$, X_i takes the value 0, and if $0 \leq \beta_i \leq \alpha$, X_i takes the value 1, $i = 1,2$.

In order to interpret these results in the context of test theory all that is required are the following coordinating definitions: a score is a random variable X, and a collection of individuals or experimental objects is the range of a random point f. This implies, of course, that the outcomes of the experiment comprise a fundamental probability set Ω and that a probability measure is defined.

We have proved that the correlation between conditionally independent, conditionally identically distributed random variables X_1 and X_2 (scores) equals the ratio of the variance of their common conditional expectation to their common unconditional variance. It is natural to regard X_1 and X_2 as "parallel measurements," although as shown in example 5, a weaker concept of parallelism can be adopted if desired and the same result obtained.

The real number $\rho(X_1, X_2) = \text{Var } M_{X_1}/\text{Var } X_1$ can be designated a "reliability coefficient." If one wishes to call M_X the "true score" instead of the conditional expectation, then in this terminology theorem 1 states that the reliability coefficient is the proportion of observed variance that is true variance. It is also possible to introduce an "error score." However, note that the function m_X is not the same as the random variable M_X, although if a suitable probability measure is defined on a collection of subsets of A, then m_X and M_X are identically distributed random variables with different associated probability spaces. Also, the random variable $[X \mid f = \alpha] - m_X(\alpha)$ is not the same as the random variable $X - M_X$. Actually, neither of these "error" random variables is needed in the above proof of theorem 1.

To continue the interpretation given above, the random point f can be regarded as "selection of an individual at random" and the random variable X as taking a measurement that also is subject to random variability. Hence, the distribution of the conditional random variable $X \mid f = \alpha$ represents the measurements that might be obtained on a particular individual α. Now, in test procedures a somewhat different scheme at first appears more natural. Typically, scores are obtained on every member of a fixed set of k individuals, so that a particular outcome is represented by a collection of random variables $X_1, X_2, \cdots, X_k$. However, we will see in the next section that the first point of view actually is more fundamental and that the second can be derived from it.

Much of classical test theory has been concerned with the reliability of lengthened tests. Along these lines, we have the following theorem (Spearman-Brown formula). It is stated so as to reveal the analogy which prevails with the usual sampling model and with certain probability limit theorems. Recall that if $X_1, X_2, \cdots, X_n$ are independent, identically distributed random variables and if $S_n = X_1 + X_2 + \cdots + X_n$, then $\text{Var}\,(S_n/n) = \text{Var}\,X_1/n$.

Theorem 2

Let $X_1, \cdots, X_n, X_{n+1}, \cdots, X_{2n}$ be conditionally independent, conditionally identically distributed random variables with finite non-zero variance, and let $S_n = X_1 + X_2 + \cdots + X_n$ and $S_n' = X_{n+1} + X_{n+2} + \cdots + X_{2n}$. Then,

$$\rho(S_n, S_n') = \frac{n\rho(X_1, X_2)}{1 + (n-1)\rho(X_1, X_2)}.$$

Proof: Since $\text{Var}\,S_n = n\text{Var}\,X_1 + n(n-1)\text{Cov}(X_1, X_2)$ and since $\text{Var}\,M_{S_n} = n^2\text{Var}\,M_{X_1}$, substitution in $\rho(S_n,S_n') = \text{Var}\,M_{S_n}/\text{Var}\,S_n$ yields the above result.

Example 7

Select a point α at random on the interval $[0,a]$. Then select a point at random on the interval $[\alpha, \alpha + b]$ and let X_1 represent the real number obtained. By theorem 1, $\text{Var}\,M_{X_1} = a^2/12$, $EV_{X_1} = b^2/12$, $\text{Var}\,X_1 = (a_2 + b_2)/12$, and $\rho(X_1,X_2) = a^2/(a^2 + b^2)$. A larger experiment can be described as follows: select a point α at random on the interval $[0,a]$, and then select n points at random on the interval $[\alpha, \alpha + b]$. Let $X_1, X_2, \cdots, X_n$ represent the real numbers obtained, and let $S_n = X_1 + X_2 + \cdots + X_n$. By theorem 2, $\rho(S_n,S_n') = na_2/(na_2 + b^2)$.

Random Points and Random Variables Associated with Stochastic Processes

Consider an ordered pair (f,X), with associated $(\Omega,\mathfrak{A},P)$ where f takes values in an arbitrary set Ψ and X takes values in the set of real numbers. If X has finite non-zero variance, we can find the number $\text{Var}\,M_X/\text{Var}\,X$, where $0 \leq \text{Var}\,M_x/\text{Var}\,X \leq 1$. In a sense, this number characterizes the joint distribution induced by the ordered pair (f,X), just as the mean and variance characterize the distribution induced by a random variable X. If $(f_1\, X_1)$ and (f_2,X_2) are identically distributed, that is, if the induced probability measures are equal $P_{f_1,X_1} = P_{f_2,X_2}$, then Var M_{X_1}/Var X_1 = Var M_{X_2}/Var X_2. Hence, from this point of view the reliability coefficient can be regarded naturally as a parameter characterizing joint distributions where conditional random variables are of interest, much as the mean and variance characterize the distributions of random variables.

Given one random variable X with associated $(\Omega,\mathfrak{A},P)$, it is always possible to construct a probability space $(\Omega',\mathfrak{A}',P')$ and independent, identically distributed random variables X_1 and X_2 having the same distribution as X. This is accomplished by taking $\Omega' = \Omega \times \Omega$, by assigning probabilities in Ω' according to a product rule, and by making the value of X_i at each point in Ω' agree with the value of X at the ith coordinate projection on Ω , $i = 1,2$. This construction sometimes is called an independent subexperiments model.

Similarly, given an ordered pair (f,X), it is always possible to construct a probability space $(\Omega',\mathfrak{A}',P')$, a random point f', and random variables X_1 and X_2, such that X_1 and X_2 are conditionally independent and such that (f,X), (f',X_1), and (f',X_2) are identically distributed. This is accomplished by taking $\Omega' = \bigcup_{\alpha\epsilon A} \{f^{-1}(\alpha) \times f^{-1}(\alpha)\}$, by assigning conditional probabilities in $\{f^{-1}(\alpha) \times f^{-1}(\alpha)\}$ according to a product rule, by letting the value of f' on $\{f^{-1}(\alpha)\, x\, f^{-1}(\alpha)\}$ be α, and by making the value of $X|f = \alpha$ at each point in $f'^{-1}(\alpha)$ agree with the value of $X \mid f = \alpha$ at the ith coordinate projection on $f^{-1}(\alpha)$, $i = 1,2$. This construction produces conditionally independent, conditionally identically distributed random variables given (f,X), just as the usual independent subexperiments model produces independent, identically distributed random variables given X. According to theorem 1, we can be assured that the number Var M_X/Var X equals $\rho(X_1,X_2)$, where X_1 and X_2 are conditionally independent and conditionally identically distributed. As described above, it is natural to interpret the random point f as selection of an individual or experimental object, the random variable X as assignment of a number according to a test procedure, and the random variables X_1 and X_2 as "parallel" measurements having the same distribution as X.

In some procedures scores are obtained on every member of a collection of individuals or experimental objects, and the outcome can be represented by a random vector $(X_{\alpha_1}, X_{\alpha_2}, \cdots, X_{\alpha_m})$. For example, a test is administered to m subjects, resulting in m scores. Here, the total variability in scores over all subjects is of interest, and we will see that the same concepts presented above apply to this situation. To generalize further, we can just as well consider an A-indexed collection of random variables $\{X_\alpha, \alpha \in A\}$, where A is a non-denumerable parameter set. The total variability then can be represented by a random variable defined on the product set $A \times \Omega$, taking the value $X_\alpha(\omega)$ at each $(\alpha,\omega) \in A \times \Omega$.

Therefore, to continue along these lines we consider a stochastic process $\{X_\alpha, \alpha \in A\}$, consisting of a collection of random variables with finite variances with the same associated probability space $(\Omega,\mathfrak{A},P)$, indexed by an arbitrary parameter set A. A probability measure can be introduced in the product set $A \times \Omega$, and the function X, defined by $X(\alpha,\omega) = X_\alpha(\omega)$, can be regarded as a random variable defined on this product space. If A is a finite set, so that $(X_{\alpha_1}, X_{\alpha_2}, \cdots, X_{\alpha_m})$ is a random vector, it is natural to take $p'(\alpha,\omega) = 1/m\, p(\omega)$, for each $(\alpha,\omega) \in A \times \Omega$.

Definition 4

The *random variable X associated with a stochastic process* $\{X_\alpha, \alpha \in A\}$ is the function X: $A \times \Omega \rightarrow R$ defined by

$$X(\alpha,\omega) = X_\alpha(\omega), \text{ for each } (\alpha,\omega) \in A \times \Omega\,,$$

regarded as a random variable defined on $A \times \Omega$.

Definition 5

Two stochastic processes $\{X_\alpha^{(1)}, \alpha \in A\}$ and $\{X_\alpha^{(2)}, \alpha \in A\}$ are *independent and identically distributed* if, for every finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ of A the random vectors $(X_{\alpha_1}^{(1)}, X_{\alpha_2}^{(21)}, \cdots, X_{\alpha_m}^{(1)})$ and $X_{\alpha_2}^{(2)}, \cdots ; X_{\alpha_m}^{(2)})$ are independent and identically distributed.

Definition 6

The *random point f associated with a stochastic process* $\{X_\alpha, \alpha \in A\}$ is the projection f: $A \times \Omega \rightarrow A$ defined by

$$f(\alpha,\omega) = \alpha, \text{ for each } (\alpha,\omega) \in A \times \Omega,$$

regarded as a random point defined on $A \times \Omega$.

Definition 7

The *mean-value function* of a stochastic process $\{X_\alpha, \alpha \in A\}$ is the function M_X: $A \times \Omega \rightarrow R$ defined by

$$M_X(\alpha,\omega) = m_X(f(\omega)), \text{ for each } (\alpha,\omega) \in A \times \Omega\,,$$

regarded as a random variable defined on $A \times \Omega$, where m_X: $A \rightarrow R$ is the function defined by

$$m_X(\alpha) = EX_\alpha, \text{ for each } \alpha \in A.$$

In standard usage m_X, not M_X, is called the mean-value function. The above definition is introduced so that X, f, and M_X will all have the same associated probability space, as before.

Theorem 3

Let $X^{(1)}$ and $X^{(2)}$ be random variables associated with independent, identically distributed stochastic processes $\{X_\alpha^{(1)}, \alpha \in A\}$ and $\{X_\alpha^{(2)}, \alpha$

$\in A\}$, and let $M_{X^{(1)}}$ be the conditional expectation of $X^{(1)}$, given the random point f associated with these processes. Then,

$$\rho(X^{(1)}, X^{(2)}) = \frac{\text{Var } M_{X^{(1)}}}{\text{Var } X^{(1)}},$$

and, if Var $M_{X^{(1)}} > 0$,

$$\rho(X^{(1)}, M_{X^{(1)}}) = \frac{\sigma M_{X^{(1)}}}{\sigma_{X^{(1)}}}.$$

Proof: We need only observe that $X^{(1)}$, $X^{(2)}$, f, and $M_{X^{(1)}}$regarded as random variables and a random point defined on $A \times \Omega$, satisfy the hypothesis of theorem 1. Note that $E[X^{(1)}|f = \alpha] = EX_\alpha^{(1)}$, for all $\alpha \in A$.

Example 8

A point ω is selected at random on the unit interval. Also, let A be the unit interval, and define a collection of indicators $\{I_\alpha, \alpha \in A\}$ by

$$\mathrm{I}_\alpha(\omega) = \begin{cases} 1 & \text{if} \quad \alpha > \frac{1}{2} \quad \text{and} \quad \omega > \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\mathrm{I}(\alpha, \omega) = \mathrm{I}_\alpha(\omega),$$

$$f(\alpha, \omega) = \alpha,$$

$$M_I(\alpha, \omega) = \begin{cases} \frac{1}{2} & \text{if} \quad \alpha > \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$V_I(\alpha, \omega) = \begin{cases} \frac{1}{4} & \text{if} \quad \alpha > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

(where V_I is defined in the obvious way), $EI = \frac{1}{4}$, Var $M_I = 1/16$, $EV_I = 1/8$, Var $I = 3/16$, and Var M_I/Var $I = \rho(I_1,I_2) = 1/3$.

Example 9

Let $A = \{\alpha \in R\colon 0 \leq \alpha \leq a\}$ and let $\{X_\alpha, \alpha \in A\}$ be a Poisson process restricted to A with intensity λ. Then, for all $\alpha \in A$, $EX_\alpha = \lambda\alpha$ and Var $X_\alpha = \lambda\alpha$, so that Var $M_X = \lambda^2a^2/12$, $EV_X = \lambda a/2$, Var $X = (\lambda^2a^2 + 6\lambda a)/12$, and Var M_X/Var $X = \rho(X_1,X_2) = \lambda a/(\lambda a + 6)$.

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