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NOTES AND COMMENTS

A NOTE ON THE ASYMPTOTIC DISTRIBUTION OF THE GREATEST LOWER BOUND TO RELIABILITY

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In a recent article Bentler and Woodward (1983) discussed computational and statistical issues related to the greatest lower bound ρ_+ to reliability. Although my work (Shapiro, 1982) was cited frequently some results presented were misunderstood. A sample estimate $\hat{\rho}_+$ of ρ_+ was considered and it was claimed (Bentler & Woodward) that: "Since $\hat{\rho}_+$ is not a closed form expression ... an exact analytic expression for **h** has not been found" (p. 247). (**h** is a vector of partial derivatives of ρ_+ as a function of the covariance matrix.) Therefore Bentler and Woodward proposed to use numerical derivatives in order to evaluate the asymptotic variance avar $(\hat{\rho}_+)$ of $\hat{\rho}_+$.

However, an *exact analytic* expression for **h** was given in Shapiro (1982) in the form of differentials. Under normality assumptions the resulting analytic expression for avar $(\hat{\rho}_+)$ was also presented (Shapiro, 1982, Theorem 4.2) and can be written as follows:

$$\text{avar}\,(\hat{\rho}_+) = 2N^{-1}(\mathbf{1}'\Sigma\mathbf{1})^{-2}[\text{tr}\,(T'_+\Psi T_+)^2 - 2(1-\rho_+)(\mathbf{1}'\Psi T_+ T'_+\Psi\mathbf{1}) + (\text{tr}\,\Psi)^2]. \qquad (1)$$

Here N is the sample size, Σ is the population covariance matrix, T_+ is a matrix that minimizes tr $T'\Sigma T$ subject to diag $(TT') \geq I$ and Ψ is the corresponding diagonal matrix defined by $(\Sigma - \Psi)T = O$. For a given Σ both matrices T_+ and Ψ are calculated by Bentler's algorithm (Bentler & Woodward, pp. 244–246).

Of course, in reality the population covariance matrix Σ is unknown and is replaced by a sample estimate S. Replacing Σ, T_+ and Ψ in (1) by their sample counterparts S, $\hat{T}_+$ and $\hat{\Psi}$ one obtains a consistent estimate of avar $(\hat{\rho}_+)$.

Formula (1) was applied to two practical examples reported by Bentler and Woodward (Tables 14.1, 14.2). Using this formula the estimated asymptotic standard error of $\hat{\rho}_+$ was found to be 0.0037 and 0.0100 respectively. The corresponding standard errors given in Bentler and Woodward are 0.0037 and 0.0098. Thus in these two examples both approaches lead to essentially the same numerical results. The difference is that formula (1) is easily computed and only requires a pocket calculator, whereas the numerical differentiation required by Bentler and Woodward involves a substantial amount of calculation and the use of a computer is essential. Moreover, formula (1) is exact whereas the Bentler and Woodward method, which employs numerical rather than analytical derivatives, is approximate (although the approximation may be excellent).

We remark that in order to invoke formula (1) as well as the Bentler and Woodward approach, the greatest lower bound ρ_+ must be a differentiable function of the covariance matrix S at $S = \Sigma$. Regularity conditions ensuring such differentiability are given in Shapiro (1982). If those conditions do not hold, then the asymptotic distribution of $N^{1/2}(\hat{\rho}_+ - \rho_+)$ is *not normal* and $\hat{\rho}_+$ has a bias of order $O(N^{-1/2})$. It is interesting to note that ρ_+ is differentiable at *almost every* value of S. Therefore dealing with a *sample* covariance

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matrix S, having a continuous distribution, it is difficult to detect the effect of nondifferentiability.

References

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