If you selected incorrectly, when the game show host opens a door to show you that the prize is not behind it, the prize is behind the other door. You will always win if your initial choice was incorrect and you change doors. So, by changing doors, the probability you win is $2/3$. In other words, you should always change doors when given the chance to do so by the game show host. This doubles the probability that you will win. (A more rigorous treatment of this puzzle can be found in Exercise 15 of Section 7.3. For much more on this notorious puzzle and its variations, see [Ro09].) ◀

Exercises

1. What is the probability that a card selected at random from a standard deck of 52 cards is an ace?

2. What is the probability that a fair die comes up six when it is rolled?

3. What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?

4. What is the probability that a randomly selected day of a leap year (with 366 possible days) is in April?

5. What is the probability that the sum of the numbers on two dice is even when they are rolled?

6. What is the probability that a card selected at random from a standard deck of 52 cards is an ace or a heart?

7. What is the probability that when a coin is flipped six times in a row, it lands heads up every time?

8. What is the probability that a five-card poker hand contains the ace of hearts?

9. What is the probability that a five-card poker hand does not contain the queen of hearts?

10. What is the probability that a five-card poker hand contains the two of diamonds and the three of spades?

11. What is the probability that a five-card poker hand contains the two of diamonds, the three of spades, the six of hearts, the ten of clubs, and the king of hearts?

12. What is the probability that a five-card poker hand contains exactly one ace?

13. What is the probability that a five-card poker hand contains at least one ace?

14. What is the probability that a five-card poker hand contains cards of five different kinds?

15. What is the probability that a five-card poker hand contains two pairs (that is, two of each of two different kinds and a fifth card of a third kind)?

16. What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit?

17. What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive kinds? (Note that an ace can be considered either the lowest card of an A-2-3-4-5 straight or the highest card of a 10-J-Q-K-A straight.)

18. What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds?

***19.** What is the probability that a five-card poker hand contains cards of five different kinds and does not contain a flush or a straight?

20. What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?

21. What is the probability that a fair die never comes up an even number when it is rolled six times?

22. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

23. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

24. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 30. **b)** 36. **c)** 42. **d)** 48.

25. Find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 50. **b)** 52. **c)** 56. **d)** 60.

26. Find the probability of selecting none of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 40. **b)** 48. **c)** 56. **d)** 64.

27. Find the probability of selecting exactly one of the correct six integers in a lottery, where the order in which these integers are selected does not matter, from the positive integers not exceeding

a) 40. **b)** 48. **c)** 56. **d)** 64.

28. In a superlottery, a player selects 7 numbers out of the first 80 positive integers. What is the probability that a person wins the grand prize by picking 7 numbers that are among the 11 numbers selected at random by a computer.

29. In a superlottery, players win a fortune if they choose the eight numbers selected by a computer from the positive integers not exceeding 100. What is the probability that a player wins this superlottery?

30. What is the probability that a player of a lottery wins the prize offered for correctly choosing five (but not six) numbers out of six integers chosen at random from the integers between 1 and 40, inclusive?

31. Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Michelle wins one of these prizes if she is one of the contestants?

32. Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Kumar, Janice, and Pedro each win a prize if each has entered the contest?

33. What is the probability that Abby, Barry, and Sylvia win the first, second, and third prizes, respectively, in a drawing if 200 people enter a contest and

a) no one can win more than one prize.

b) winning more than one prize is allowed.

34. What is the probability that Bo, Colleen, Jeff, and Rohini win the first, second, third, and fourth prizes, respectively, in a drawing if 50 people enter a contest and

a) no one can win more than one prize.

b) winning more than one prize is allowed.

35. In roulette, a wheel with 38 numbers is spun. Of these, 18 are red, and 18 are black. The other two numbers, which are neither black nor red, are 0 and 00. The probability that when the wheel is spun it lands on any particular number is $1/38$.

a) What is the probability that the wheel lands on a red number?

b) What is the probability that the wheel lands on a black number twice in a row?

c) What is the probability that the wheel lands on 0 or 00?

d) What is the probability that in five spins the wheel never lands on either 0 or 00?

e) What is the probability that the wheel lands on one of the first six integers on one spin, but does not land on any of them on the next spin?

36. Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?

37. Which is more likely: rolling a total of 9 when two dice are rolled or rolling a total of 9 when three dice are rolled?

38. A player in the Mega Millions lottery picks five different integers between 1 and 70, inclusive, and a sixth integer between 1 and 25, inclusive, which may duplicate one of the earlier five integers. The player wins the jackpot if all six numbers match the numbers drawn.

a) What is the probability that a player wins the jackpot?

b) What is the probability that a player wins \$1,000,000, the prize for matching the first five numbers, but not the sixth number, drawn?

c) What is the probability that a player wins \$500, the prize for matching exactly four of the first five numbers, but not the sixth number, drawn?

d) What is the probability that a player wins \$10, the prize for matching exactly three of the first five numbers but not the sixth number drawn, or for matching exactly two of the first five numbers and the sixth number drawn?

39. When a player buys a Mega Millions ticket in many states (see Exercise 38), the player can also buy the Megaplier, which multiplies the size of a prize other than a jackpot by a multiplier ranging from two to five. The Megaplier is drawn using a pool of 15 balls, with five marked 2X, six marked 3X, three marked 4X, and one marked 5X, where each ball has the same likelihood of being drawn. Find the probability that a player who buys a Mega Millions ticket and the Megaplier wins

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a) \$5,000,000? (The only way to do this is to match the first five numbers drawn but not the sixth number drawn, with Megaplier 5X.)

b) \$30,000? (The only way to do this is to match exactly four of the first five numbers drawn and the sixth number drawn, with Megaplier 3X.)

c) \$20? (The three ways to do this are to match exactly three of the first five numbers drawn, but not the sixth number drawn, or exactly two of the first five numbers and the sixth number, with Megaplier 2X, or to match exactly one of the first five numbers and the sixth number, with Megaplier 5X.)

d) \$8? (The two ways to do this are to match exactly one of the first five numbers and the sixth number drawn, with a multiplier of 2X, or to match the sixth number but none of the first five numbers, with Megaplier 4X.)

40. A player in the Powerball lottery picks five different integers between 1 and 69, inclusive, and a sixth integer between 1 and 26, which may duplicate one of the earlier five integers. The player wins the jackpot if all six numbers match the numbers drawn.

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a) What is the probability that a player wins the jackpot?

b) What is the probability that a player wins \$1,000,000, which is the prize for matching the first five numbers, but not the sixth number, drawn?

c) What is the probability that a player wins \$100 by matching exactly three of the first five and the sixth numbers drawn, or four of the first five numbers, but not the sixth number, drawn?

d) What is the probability that a player wins a prize of \$4, which is the prize when the player matches the sixth number, and either one or none of the first five numbers drawn?

41. A player in the Powerball lottery (see Exercise 40) can purchase the Power Play option. When this option has been purchased, prizes other than the jackpot are multiplied by a multiplier, chosen using a random number generator with weighted values for the different multipliers. When the jackpot is more than \$150,000,000, the weighted values are 24 for 2X, 13 for 3X, 3 for 4X, and 2 for 5X. When the jackpot does not exceed \$150,000,000, the weighted values are 24 for 2X, 13 for 3X, 3 for 4X, 2 for 5X, and 1 for 10X. All non-jackpot prizes are multiplied by the multiplier chosen, except for the \$1,000,000 prize, which is doubled when the Power Play option is in effect regardless of the multiplier chosen. What is the probability that a play who has purchased a Powerball ticket and Power Play wins

a) \$2,000,000, if the jackpot is more than \$150,000,000?

b) \$2,000,000, if the jackpot does not exceed \$150,000,000?

c) \$1000, if the jackpot does not exceed \$150,000,000? (The two ways to do this are for the Power Play multiplier to be 10X, and to match either exactly four of the first five numbers but not the sixth number drawn, or exactly three of the first five numbers and the sixth number drawn.)

d) \$12, if the jackpot is more than \$150,000,000? (The two ways to do this are for the Power Play multiplier to be 3X and to match the sixth number and either one or none of the first five numbers drawn.)

42. Two events E_1 and E_2 are called **independent** if $p(E_1 \cap E_2) = p(E_1)p(E_2)$. For each of the following pairs of events, which are subsets of the set of all possible outcomes when a coin is tossed three times, determine whether or not they are independent.

a) E_1: tails comes up with the coin is tossed the first time; E_2: heads comes up when the coin is tossed the second time.

b) E_1: the first coin comes up tails; E_2: two, and not three, heads come up in a row.

c) E_1: the second coin comes up tails; E_2: two, and not three, heads come up in a row.

(We will study independence of events in more depth in Section 7.2.)

43. Explain what is wrong with the statement that in the Monty Hall Three-Door Puzzle the probability that the prize is behind the first door you select and the probability that the prize is behind the other of the two doors that Monty does not open are both $1/2$, because there are two doors left.

44. Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?

45. This problem was posed by the Chevalier de Méré and was solved by Blaise Pascal and Pierre de Fermat.

a) Find the probability of rolling at least one six when a fair die is rolled four times.

b) Find the probability that a double six comes up at least once when a pair of dice is rolled 24 times. Answer the query the Chevalier de Méré made to Pascal asking whether this probability was greater than $1/2$.

c) Is it more likely that a six comes up at least once when a fair die is rolled four times or that a double six comes up at least once when a pair of dice is rolled 24 times?

7.2 Probability Theory

7.2.1 Introduction

Links

In Section 7.1 we introduced the notion of the probability of an event. (Recall that an event is a subset of the possible outcomes of an experiment.) We defined the probability of an event E as Laplace did, that is,

$$p(E) = \frac{|E|}{|S|},$$

the number of outcomes in E divided by the total number of outcomes. This definition assumes that all outcomes are equally likely. However, many experiments have outcomes that are not equally likely. For instance, a coin may be biased so that it comes up heads twice as often as tails. Similarly, the likelihood that the input of a linear search is a particular element in a list, or is not in the list, depends on how the input is generated. How can we model the likelihood of events in such situations? In this section we will show how to define probabilities of outcomes to study probabilities of experiments where outcomes may not be equally likely.

Suppose that a fair coin is flipped four times, and the first time it comes up heads. Given this information, what is the probability that heads comes up three times? To answer this and similar questions, we will introduce the concept of *conditional probability*. Does knowing that the first flip comes up heads change the probability that heads comes up three times? If not, these two events are called *independent*, a concept studied later in this section.