**COROLLARY 1**   If an m-ary tree of height h has l leaves, then $h \geq \lceil \log_m l \rceil$. If the m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$. (We are using the ceiling function here. Recall that $\lceil x \rceil$ is the smallest integer greater than or equal to x.)

Proof: We know that $l \leq m^h$ from Theorem 5. Taking logarithms to the base m shows that $\log_m l \leq h$. Because h is an integer, we have $h \geq \lceil \log_m l \rceil$. Now suppose that the tree is balanced. Then each leaf is at level h or $h - 1$, and because the height is h, there is at least one leaf at level h. It follows that there must be more than m^{h-1} leaves (see Exercise 30). Because $l \leq m^h$, we have $m^{h-1} < l \leq m^h$. Taking logarithms to the base m in this inequality gives $h - 1 < \log_m l \leq h$. Hence, $h = \lceil \log_m l \rceil$.   ◄
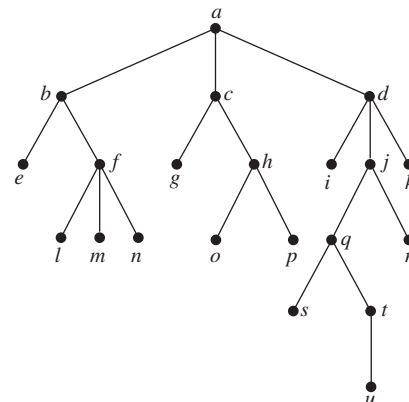
Exercises

1. Which of these graphs are trees?

a)

b)

c)

d)

e)

f)
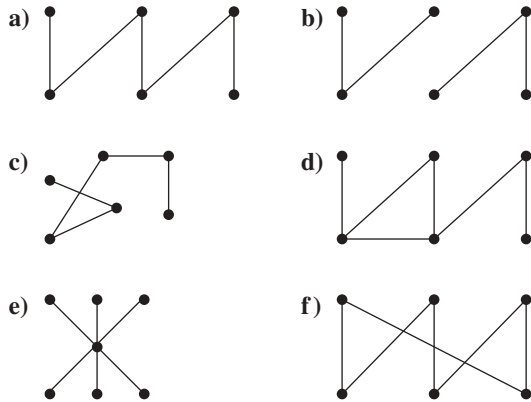
2. Which of these graphs are trees?

**a)**

b)

c)

d)

e)

f)
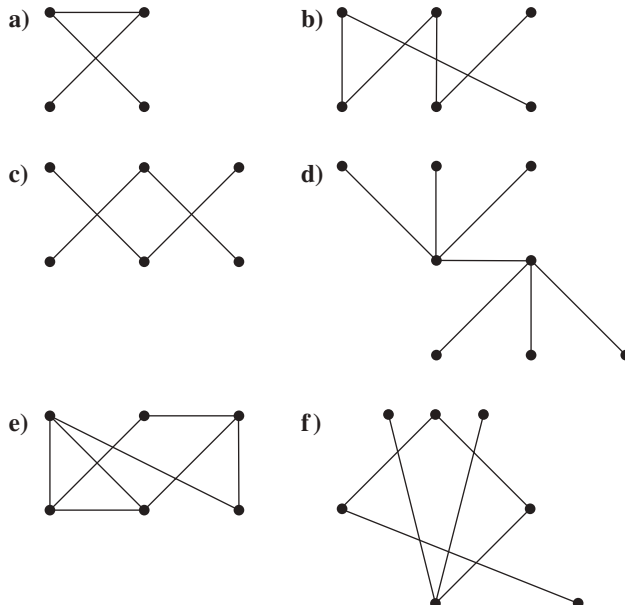
3. Answer these questions about the rooted tree illustrated.

a) Which vertex is the root?

b) Which vertices are internal?

c) Which vertices are leaves?

d) Which vertices are children of j?

e) Which vertex is the parent of h?

f) Which vertices are siblings of o?

g) Which vertices are ancestors of m?

h) Which vertices are descendants of b?

4. Answer the same questions as listed in Exercise 3 for the rooted tree illustrated.
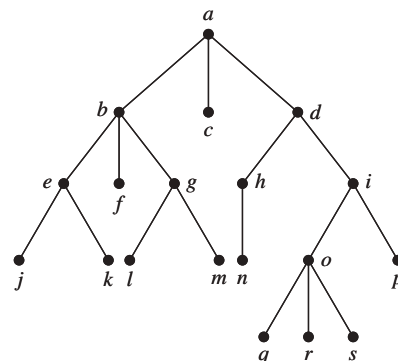
5. Is the rooted tree in Exercise 3 a full *m*-ary tree for some positive integer *m*?

6. Is the rooted tree in Exercise 4 a full *m*-ary tree for some positive integer *m*?

7. What is the level of each vertex of the rooted tree in Exercise 3?

8. What is the level of each vertex of the rooted tree in Exercise 4?

9. Draw the subtree of the tree in Exercise 3 that is rooted at

a) *a.*   **b)** *c.*   **c)** *e.*

10. Draw the subtree of the tree in Exercise 4 that is rooted at

a) *a.*   **b)** *c.*   **c)** *e.*

11. a) How many nonisomorphic unrooted trees are there with three vertices?

   b) How many nonisomorphic rooted trees are there with three vertices (using isomorphism for directed graphs)?

*12. a) How many nonisomorphic unrooted trees are there with four vertices?

   b) How many nonisomorphic rooted trees are there with four vertices (using isomorphism for directed graphs)?

*13. a) How many nonisomorphic unrooted trees are there with five vertices?

   b) How many nonisomorphic rooted trees are there with five vertices (using isomorphism for directed graphs)?

*14. Show that a simple graph is a tree if and only if it is connected but the deletion of any of its edges produces a graph that is not connected.

☞***15.** Let *G* be a simple graph with *n* vertices. Show that

   a) *G* is a tree if and only if it is connected and has $n - 1$ edges.

   b) *G* is a tree if and only if *G* has no simple circuits and has $n - 1$ edges. [*Hint:* To show that *G* is connected if it has no simple circuits and $n - 1$ edges, show that *G* cannot have more than one connected component.]

16. Which complete bipartite graphs $K_{m,n}$, where *m* and *n* are positive integers, are trees?

17. How many edges does a tree with 10,000 vertices have?

18. How many vertices does a full 5-ary tree with 100 internal vertices have?

19. How many edges does a full binary tree with 1000 internal vertices have?

20. How many leaves does a full 3-ary tree with 100 vertices have?

21. Suppose 1000 people enter a chess tournament. Use a rooted tree model of the tournament to determine how many games must be played to determine a champion, if a player is eliminated after one loss and games are played until only one entrant has not lost. (Assume there are no ties.)

22. A chain letter starts when a person sends a letter to five others. Each person who receives the letter either sends it to five other people who have never received it or does not send it to anyone. Suppose that 10,000 people send out the letter before the chain ends and that no one receives more than one letter. How many people receive the letter, and how many do not send it out?

23. A chain letter starts with a person sending a letter out to 10 others. Each person is asked to send the letter out to 10 others, and each letter contains a list of the previous six people in the chain. Unless there are fewer than six names in the list, each person sends one dollar to the first person in this list, removes the name of this person from the list, moves up each of the other five names one position, and inserts his or her name at the end of this list. If no person breaks the chain and no one receives more than one letter, how much money will a person in the chain ultimately receive?

*24. Either draw a full *m*-ary tree with 76 leaves and height 3, where *m* is a positive integer, or show that no such tree exists.

*25. Either draw a full *m*-ary tree with 84 leaves and height 3, where *m* is a positive integer, or show that no such tree exists.

*26. A full *m*-ary tree *T* has 81 leaves and height 4.

   a) Give the upper and lower bounds for *m*.

   b) What is *m* if *T* is also balanced?

A **complete *m*-ary tree** is a full *m*-ary tree in which every leaf is at the same level.

27. Construct a complete binary tree of height 4 and a complete 3-ary tree of height 3.

28. How many vertices and how many leaves does a complete *m*-ary tree of height *h* have?

29. Prove

   a) part (*ii*) of Theorem 4.

   b) part (*iii*) of Theorem 4.

☞ **30.** Show that a full *m*-ary balanced tree of height *h* has more than m^{h-1} leaves.

31. How many edges are there in a forest of *t* trees containing a total of *n* vertices?

32. Explain how a tree can be used to represent the table of contents of a book organized into chapters, where each chapter is organized into sections, and each section is organized into subsections.

33. How many different isomers do these saturated hydrocarbons have?

   a) C_3H_8 **b)** C_5H_{12} **c)** C_6H_{14}

34. What does each of these represent in a tree that represents the structure of an organization?

   a) the parent of a vertex
   b) a child of a vertex
   c) a sibling of a vertex
   d) the ancestors of a vertex
   e) the descendants of a vertex
   f) the level of a vertex
   **g)** the height of the tree

35. Answer the same questions as those given in Exercise 34 for a rooted tree representing a computer file system.

36. a) Draw the complete binary tree with 15 vertices that represents a tree-connected network of 15 processors.

  b) Show how 16 numbers can be added using the 15 processors in part (a) using four steps.

37. Let n be a power of 2. Show that n numbers can be added in $\log n$ steps using a tree-connected network of $n - 1$ processors.

*38. A **labeled tree** is a tree where each vertex is assigned a label. Two labeled trees are considered isomorphic when there is an isomorphism between them that preserves the labels of vertices. How many nonisomorphic trees are there with three vertices labeled with different integers from the set $\{1, 2, 3\}$? How many nonisomorphic trees are there with four vertices labeled with different integers from the set $\{1, 2, 3, 4\}$?

The **eccentricity** of a vertex in an unrooted tree is the length of the longest simple path beginning at this vertex. A vertex is called a **center** if no vertex in the tree has smaller eccentricity than this vertex. In Exercises 39–41 find every vertex that is a center in the given tree.

39.

40.

41.
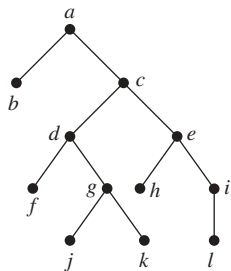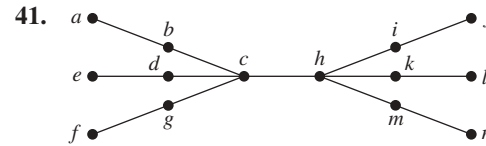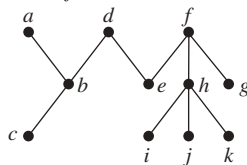
42. Show that a center should be chosen as the root to produce a rooted tree of minimal height from an unrooted tree.

*43. Show that a tree has either one center or two centers that are adjacent.

44. Show that every tree can be colored using two colors.

The **rooted Fibonacci trees** T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n = 3, 4, \ldots$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree.

45. Draw the first seven rooted Fibonacci trees.

*46. How many vertices, leaves, and internal vertices does the rooted Fibonacci tree T_n have, where n is a positive integer? What is its height?

47. What is wrong with the following "proof" using mathematical induction of the statement that every tree with n vertices has a path of length $n - 1$. *Basis step:* Every tree with one vertex clearly has a path of length 0. *Inductive step:* Assume that a tree with n vertices has a path of length $n - 1$, which has u as its terminal vertex. Add a vertex v and the edge from u to v. The resulting tree has $n + 1$ vertices and has a path of length n. This completes the inductive step.

*48. Show that the average depth of a leaf in a binary tree with n vertices is $\Omega(\log n)$.

11.2   Applications of Trees

11.2.1   Introduction

We will discuss three problems that can be studied using trees. The first problem is: How should items in a list be stored so that an item can be easily located? The second problem is: What series of decisions should be made to find an object with a certain property in a collection of objects of a certain type? The third problem is: How should a set of characters be efficiently coded by bit strings?

11.2.2   Binary Search Trees

Links ▶

Searching for items in a list is one of the most important tasks that arises in computer science. Our primary goal is to implement a searching algorithm that finds items efficiently when the items are totally ordered. This can be accomplished through the use of a **binary search tree**,