Chapter 5
Functions

Discrete Structures for Computing

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Contents

Course outcomes

	Course learning outcomes
L.O.1	Understanding of logic and discrete structures
	L.O.1.1 – Describe definition of propositional and predicate logic L.O.1.2 – Define basic discrete structures: set, mapping, graphs
L.O.2	Represent and model practical problems with discrete structures
	L.O.2.1 – Logically describe some problems arising in Computing L.O.2.2 – Use proving methods: direct, contrapositive, induction L.O.2.3 – Explain problem modeling using discrete structures
L.O.3	Understanding of basic probability and random variables
	L.O.3.1 – Define basic probability theory L.O.3.2 – Explain discrete random variables
L.O.4	Compute quantities of discrete structures and probabilities
	L.O.4.1 – Operate (compute/ optimize) on discrete structures L.O.4.2 – Compute probabilities of various events, conditional ones, Bayes theorem

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Introduction

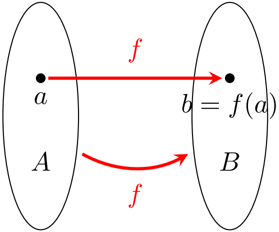
- Each student is assigned a grade from set $\{0, 0.1, 0.2, 0.3, \ldots, 9.9, 10.0\}$ at the end of semester
- Function is extremely important in mathematics and computer science
  - linear, polynomial, exponential, logarithmic,...
- Don't worry! For discrete mathematics, we need to understand functions at a basic set theoretic level

Function

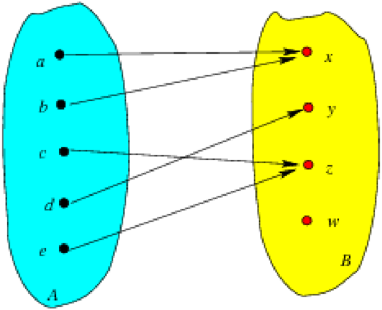
Definition

Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A.

- $f : A \to B$
- A: domain (*miền xác định*) of f
- B: codomain (*miền giá trị*) of f
- For each $a \in A$, if $f(a) = b$
  - b is an image (*ảnh*) of a
  - a is pre-image (*nghịch ảnh*) of $f(a)$
- Range of f is the set of all images of elements of A
- f maps (*ánh xạ*) A to B



Example



Example:

- y is an image of d
- c is a pre-image of z

Example

Example

What are domain, codomain, and range of the function that assigns grades to students includes: student A: 5, B: 3.5, C: 9, D: 5.2, E: 4.9?

Example

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ assign the the square of an integer to this integer. What is $f(x)$? Domain, codomain, range of f?

- $f(x) = x^2$
- Domain: set of all integers
- Codomain: Set of all integers
- Range of f : $\{0, 1, 4, 9, \ldots\}$

Add and multiply real-valued functions

Definition

Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

Example

Let $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

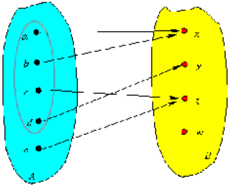
$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + x - x^2 = x$$
$$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2(x - x^2) = x^3 - x^4$$

Image of a subset

Definition

Let $f : A \to B$ and $S \subseteq A$. The image of S:

$$f(S) = \{f(s) \mid s \in S\}$$



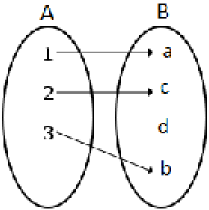
$f(\{a, b, c, d\}) = \{x, y, z\}$

One-to-one

Definition

A function f is one-to-one or injective (*đơn ánh*) if and only if

$$\forall a \forall b \, (f(a) = f(b) \rightarrow a = b)$$



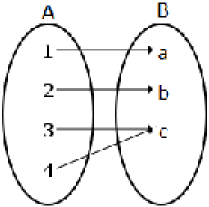
- Is $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x + 1$ one-to-one?
- Is $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x^2$ one-to-one?

Onto

Definition

$f : A \to B$ is onto or surjective (*toàn ánh*) if and only if

$$\forall b \in B, \exists a \in A : \ f(a) = b$$



- Is $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x + 1$ onto?
- Is $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2$ onto?

One-to-one and onto (bijection)

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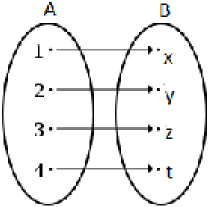
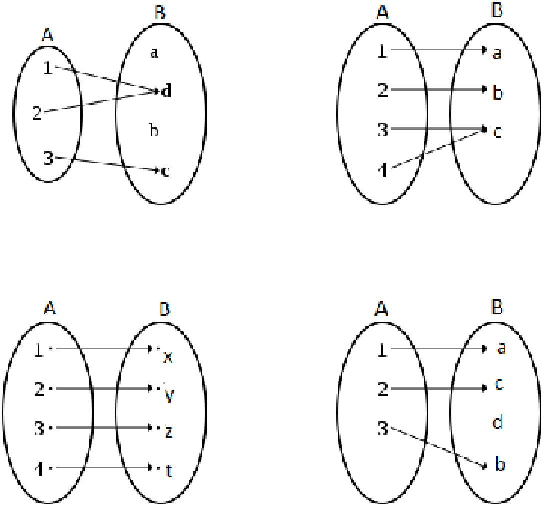
Definition

$f : A \rightarrow B$ is bijective (one-to-one correspondence) (*song ánh*) if and only if f is injective and surjective



- Let f be the function from $\{a, bc, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$. Is f a bijection?

Example



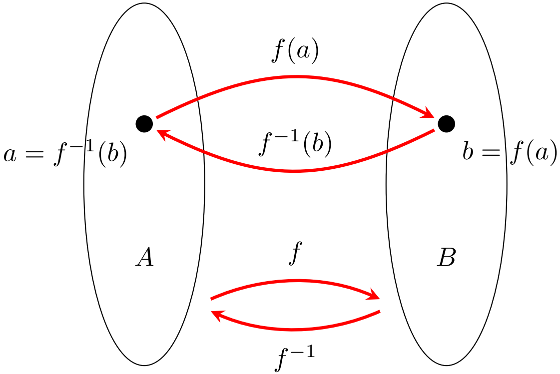
Inverse function (Hàm ngược)

Definition

Let $f : A \to B$ be a bijection then the inverse of f is the function $f^{-1} : B \to A$ defined by

$$if\ f(a) = b\ then\ f^{-1}(b) = a$$

A one-to-one correspondence is call invertible (*khả nghịch*) because we can define the inverse of this function.



Example

Example

$A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ with

$$f(a) = 2 \qquad f(b) = 3 \qquad f(c) = 1$$

f is invertible and its inverse is

$$f^{-1}(1) = c \qquad f^{-1}(2) = a \qquad f^{-1}(3) = b$$

Example

Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. If f invertible?

Example

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = 2x + 1$$

$$f^{-1} : \mathbb{R} \to \mathbb{R}$$

$$f^{-1}(x) = \frac{x-1}{2}$$

Function Composition

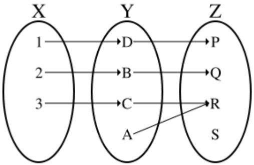
Definition

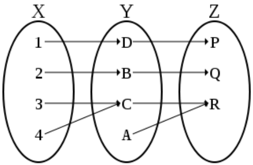
Given a pair of functions $g : A \to B$ and $f : B \to C$. Then the composition (*hợp thành*) of f and g, denoted $f \circ g$ is defined by

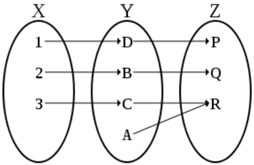
$$f \circ g : A \to C$$

$$f \circ g(a) = f(g(a))$$

Example



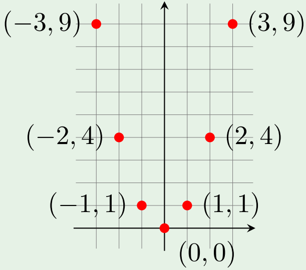




Graphs of Functions

Example

The graph of $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$.



Definition

Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

Important Functions

Definition

Floor function (*hàm sàn*) of x ($\lfloor x \rfloor$): the largest integer $\leq x$

$\lfloor \frac{1}{2} \rfloor = 0, \lfloor 3.1 \rfloor = 3, \lfloor 7 \rfloor = 7$

Ceiling function (*hàm trần*) of x ($\lceil x \rceil$): the smallest integer $\geq x$

$\lceil \frac{1}{2} \rceil = 1, \lceil 3.1 \rceil = 4, \lceil 7 \rceil = 7$

Bảng: Properties (n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ iff $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ iff $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ iff $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ iff $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Sequences

What are the rule of these sequences (*dãy*)?

Example

$1, 3, 5, 7, 9, \dots \qquad a_n = 2n - 1$

Arithmetic sequence (*cấp số cộng*)

Example

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \qquad a_n = \frac{1}{2^{n-1}}$

Geometric sequence (*cấp số nhân*)

Example

$\{a_n\}$ 5, 11, 17, 23, 29, 35, 41, 47, ... $a_n = 6n - 1$

$\{b_n\}$ 1, 7, 25, 79, 241, 727, 2185, ... $b_n = 3^n - 2$

Recurrence Relations

Example

$\{a_n\}$ 5, 11, 17, 23, 29, 35, 41, 47, ...
$a_n = a_{n-1} + 6$ for $n = 2, 3, 4, \ldots$ and $a_1 = 5$
Recurrence relations: *công thức truy hồi*

Definition (Fibonacci Sequence)

Initial condition: $f_0 = 0$ and $f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$

Example

Find the Fibonacci numbers f_2, f_3, f_4, f_5 and f_6

$f_2 = f_1 + f_0 = 1 + 0 = 1$
$f_3 = f_2 + f_1 = 1 + 1 = 2$
$f_4 = f_3 + f_2 = 2 + 1 = 3$
$f_5 = f_4 + f_3 = 3 + 2 = 5$
$f_6 = f_5 + f_4 = 5 + 3 = 8$

Exercise (1)

Initial deposit: \$10,000
Interest: 11%/year, compounded annually (*lãi suất kép*)

After 30 years, how much do you have in your account?

Solution:
Let P_n be the amount in the account after n years. The sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10,000$

Step 1. Solve the recurrence relation (iteration technique)

$P_1 = (1.11)P_0$
$P_2 = (1.11)P_1 = (1.11)^2P_0$
$P_3 = (1.11)P_2 = (1.11)^3P_0$
$\vdots$
$P_n = (1.11)P_{n-1} = (1.11)^nP_0.$

Step 2. Calculate
$P_{30} = (1.11)^{30}10,000 = \$228,922.97.$

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Exercise (2)

What is the 2012th number in the sequence $\{x_n\}$: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6,...

Solution:

In this sequence, integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and so on. Therefore integer n appears n times in the sequence.

We can prove that (try it!)

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

and can easily calculate that

$$\sum_{i=1}^{62} i = 1953$$

so the next 63 numbers (until 2016) is 63.

Therefore, 2012th number in the sequence is 63.

Theorem

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \textit{if } r \neq 1 \\ (n+1)a & \textit{if } r = 1. \end{cases}$$

Chứng minh.

Let $S_n = \sum_{j=0}^{n} ar^j$.

$$\begin{aligned} rS_n &= r\sum_{j=0}^{n} ar^j \\ &= \sum_{j=0}^{n} ar^{j+1} \\ &= \sum_{k=1}^{n+1} ar^k \\ &= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a) \\ &= S_n + (ar^{n+1} - a) \end{aligned}$$

Solving for S_n shows that if $r \neq 1$, then $S_n = \frac{ar^{n+1}-a}{r-1}$

If $r = 1$, then $S_n = \sum_{j=0}^{n} a = (n+1)a$ $\square$

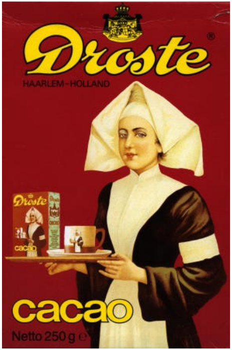
Recursion

Definition (Recurrence Relation)

An equation that recursively defines a sequence.

Definition (Recursion (đệ quy))

The act of defining an object (usually a function) in terms of that object itself.



Recursive Algorithms

Definition

An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

Example

Give a recursive algorithm for computing $n!$, where n is a nonnegative integer.

Solution. We base on the recursive definition of $n!$:
$n! = n \cdot (n-1)!$ and $0! = 1$.

procedure *factorial*(n: nonnegative integer)
if $n = 0$ **then return** 1
else return $n \cdot$ *factorial*(n - 1)
{output is $n!$}

Algorithms for Fibonacci Numbers

Recursive Algorithm

procedure *fibonacci*(n: nonnegative integer)
if $n = 0$ **then return** 0
else if $n = 1$ **then return** 1
else return *fibonacci*(n-1) + *fibonacci*(n-2)
{output is *fibonacci*(n)}

Iterative Algorithm

procedure *iterative fibonacci*(n: nonnegative integer)
if $n = 0$ **then return** 0
else
    $x := 0$
    $y := 1$
    for $i := 1$ to $n - 1$
        $z := x + y$
        $x := y$
        $y := z$
    return y
{output is the *nth* Fibonacci number}

Tower of Hanoi

There is a tower in Hanoi that has three pegs mounted on a board together with 64 gold disks of different sizes.

Initially, these disks are placed on the first peg in order of size, with the largest on the borrom.

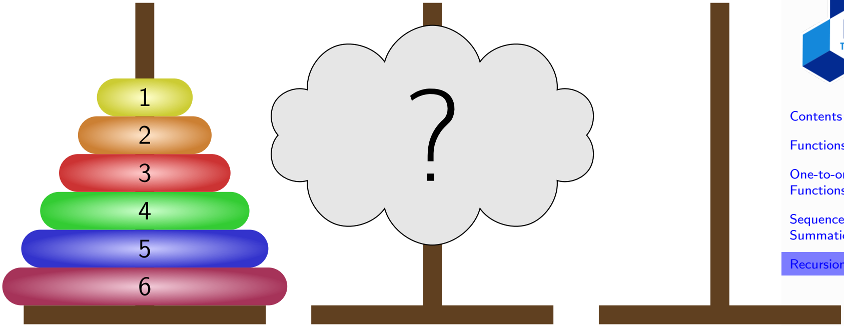
The rules:

1. Move one at a time from one peg to another
2. A disk is never placed on top of a smaller disk

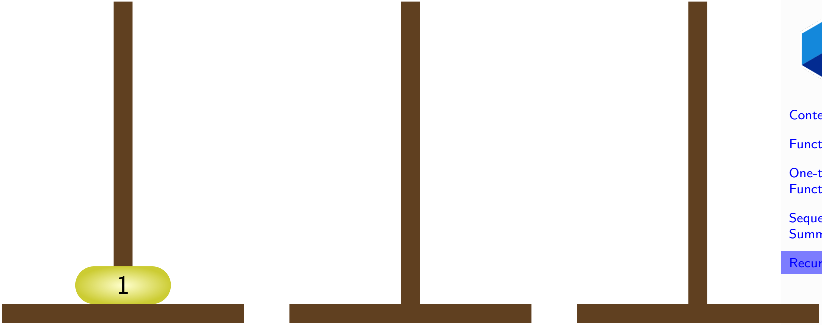
Goals: all the disks on the third peg in order of size.

The myth says that **the world will end** when they finish the puzzle.

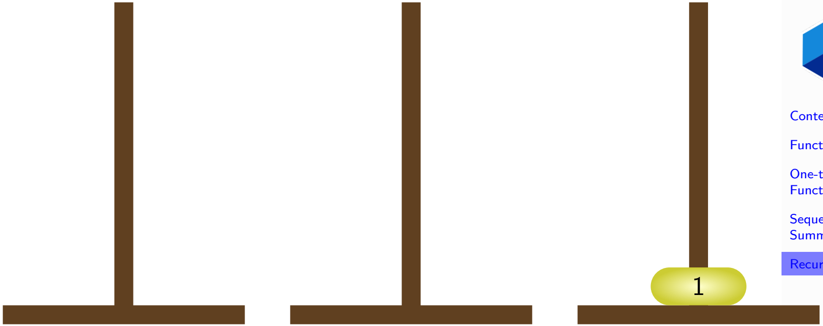
Tower of Hanoi – 64 Discs



Tower of Hanoi – 1 Disc



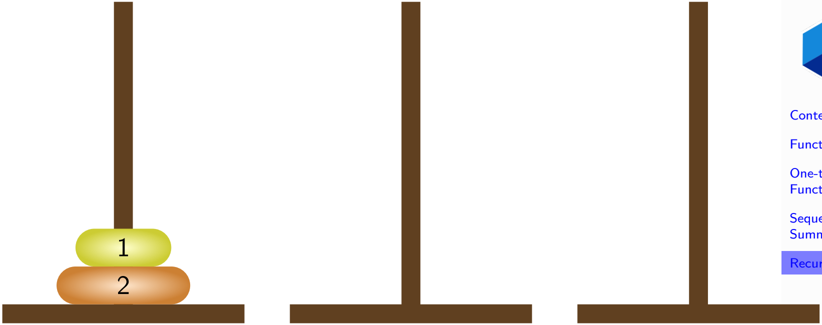
Tower of Hanoi – 1 Disc



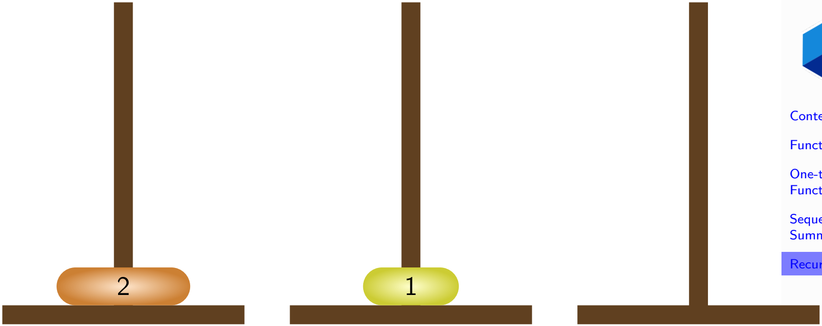
Moved disc from peg 1 to peg 3.

Tower of Hanoi – 1 Disc

Tower of Hanoi – 2 Discs

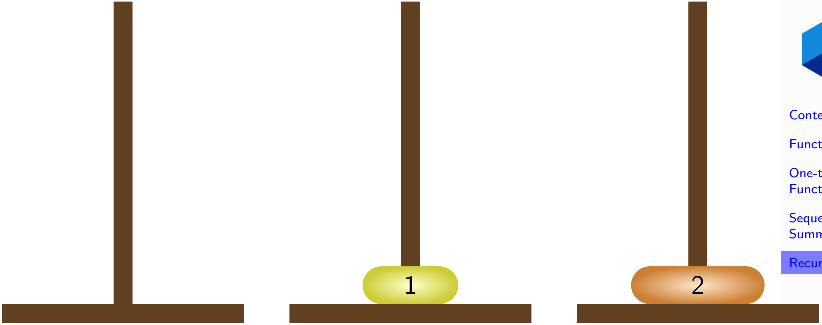


Tower of Hanoi – 2 Discs



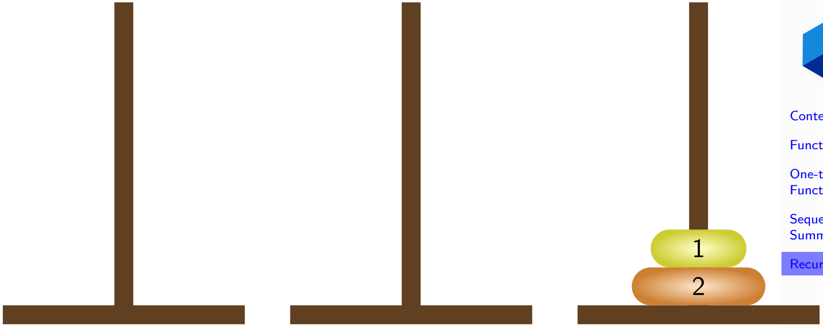
Moved disc from peg 1 to peg 2.

Tower of Hanoi – 2 Discs



Moved disc from peg 1 to peg 3.

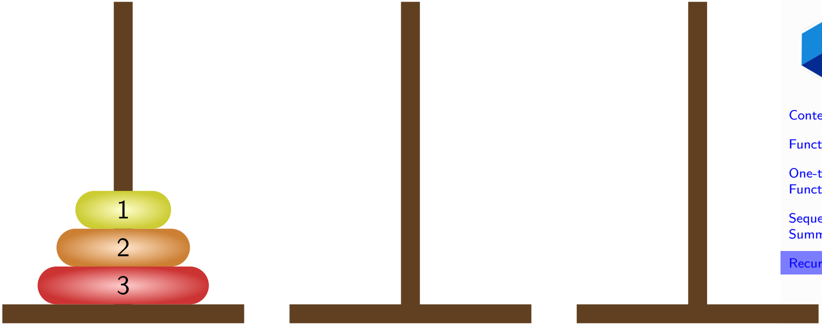
Tower of Hanoi – 2 Discs



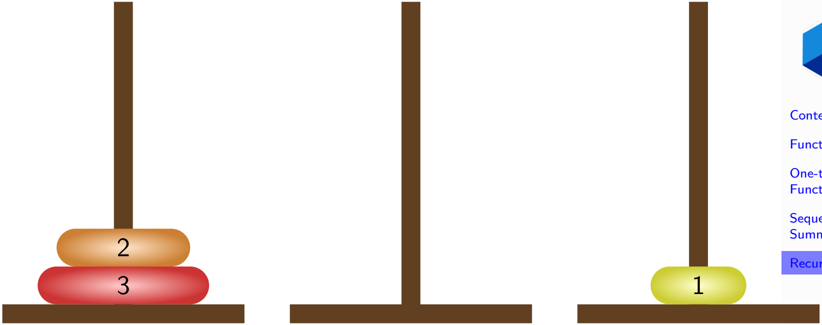
Moved disc from peg 2 to peg 3.

Tower of Hanoi – 2 Discs

Tower of Hanoi – 3 Discs

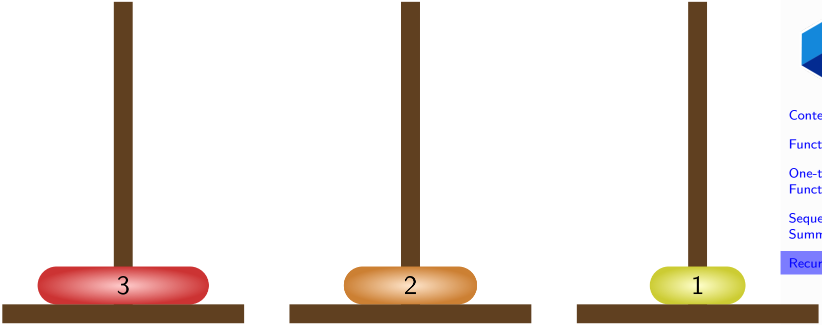


Tower of Hanoi – 3 Discs



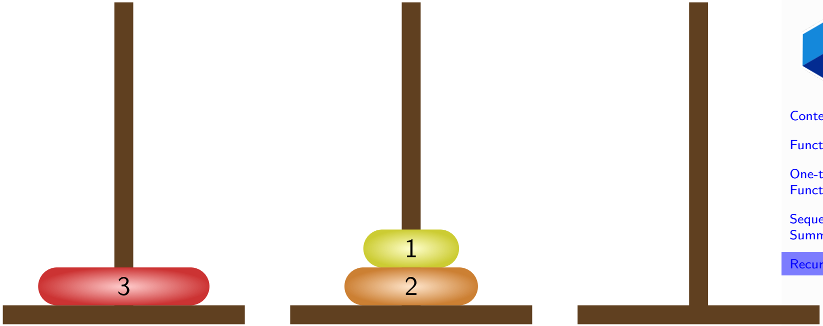
Moved disc from peg 1 to peg 3.

Tower of Hanoi – 3 Discs



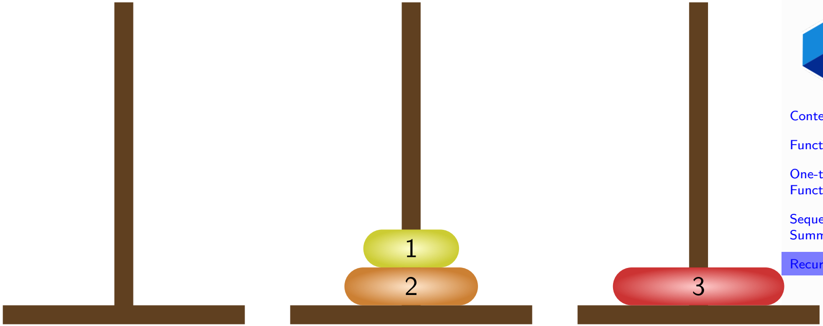
Moved disc from peg 1 to peg 2.

Tower of Hanoi – 3 Discs



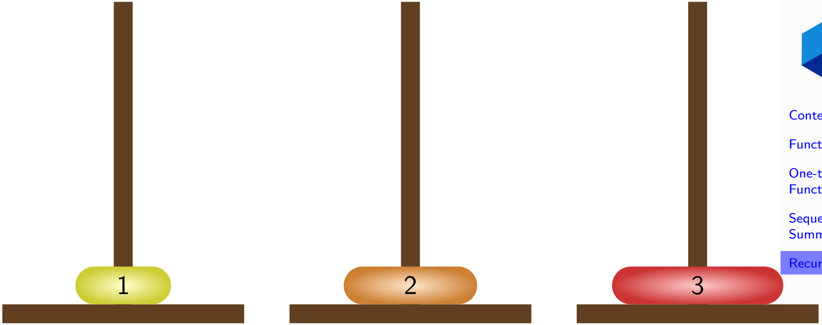
Moved disc from peg 3 to peg 2.

Tower of Hanoi – 3 Discs



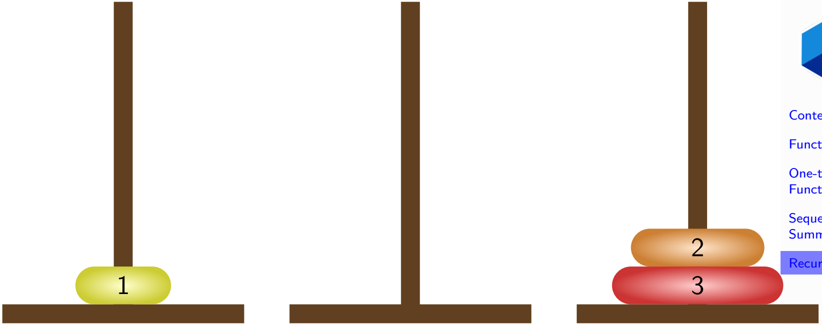
Moved disc from peg 1 to peg 3.

Tower of Hanoi – 3 Discs



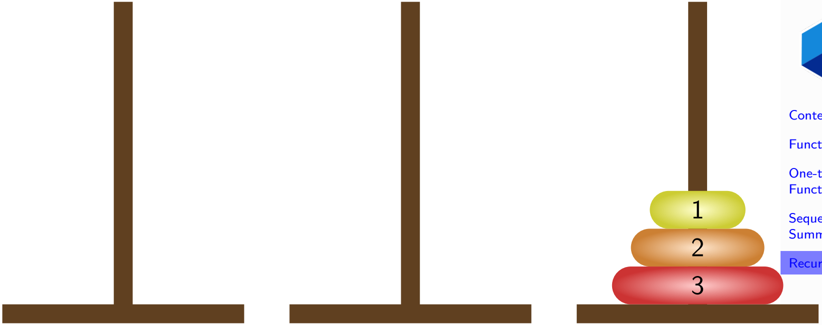
Moved disc from peg 2 to peg 1.

Tower of Hanoi – 3 Discs



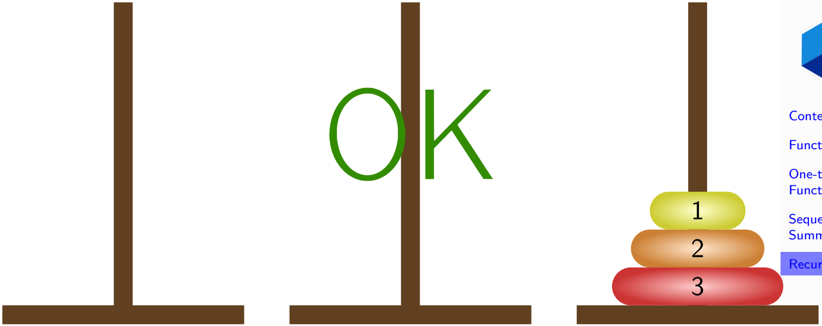
Moved disc from peg 2 to peg 3.

Tower of Hanoi – 3 Discs

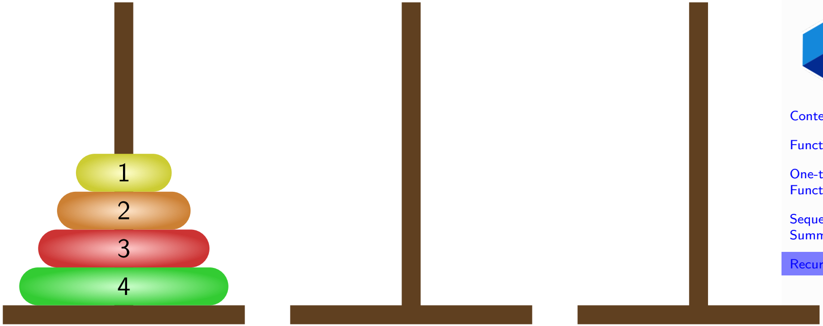


Moved disc from peg 1 to peg 3.

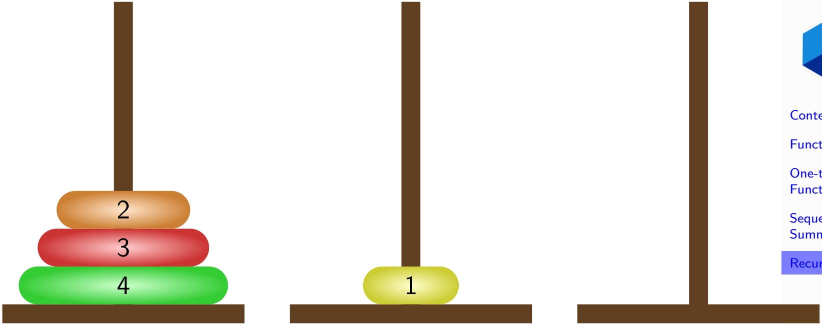
Tower of Hanoi – 3 Discs



Tower of Hanoi – 4 Discs

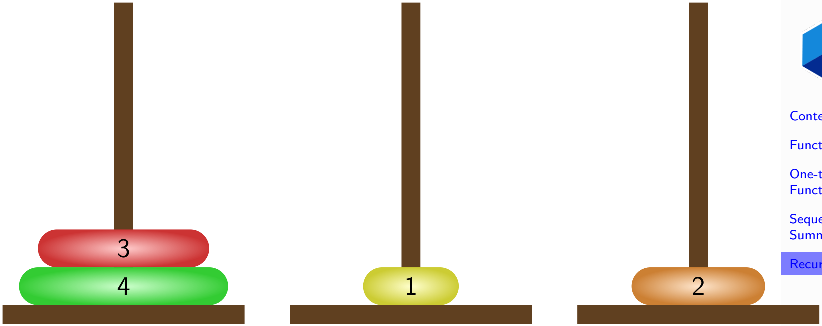


Tower of Hanoi – 4 Discs



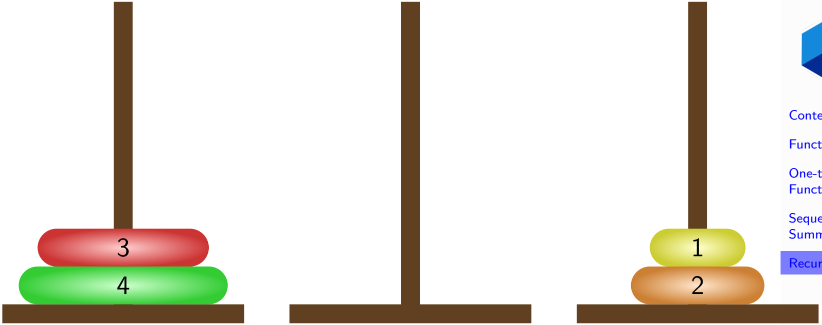
Moved disc from peg 1 to peg 2.

Tower of Hanoi – 4 Discs



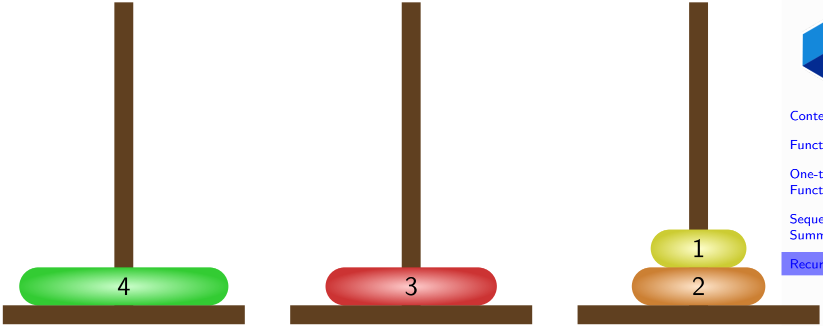
Moved disc from peg 1 to peg 3.

Tower of Hanoi – 4 Discs



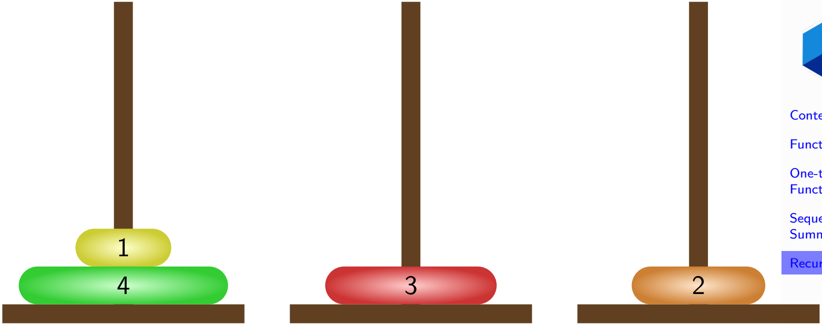
Moved disc from peg 2 to peg 3.

Tower of Hanoi – 4 Discs



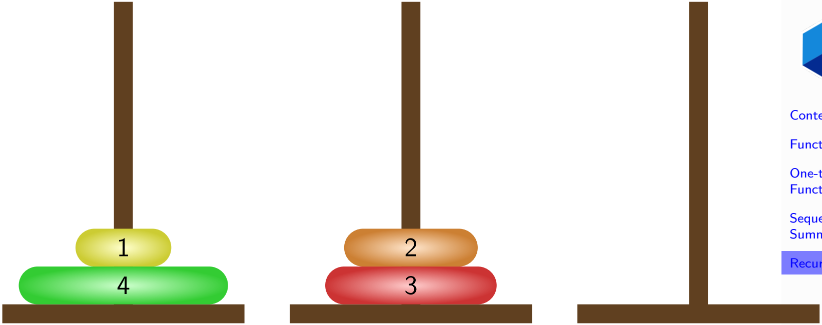
Moved disc from peg 1 to peg 2.

Tower of Hanoi – 4 Discs



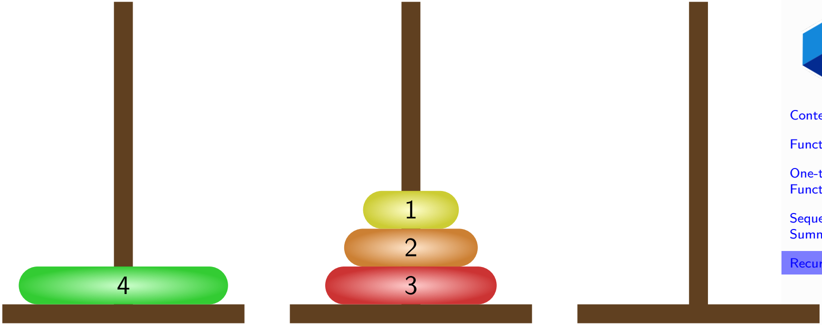
Moved disc from peg 3 to peg 1.

Tower of Hanoi – 4 Discs



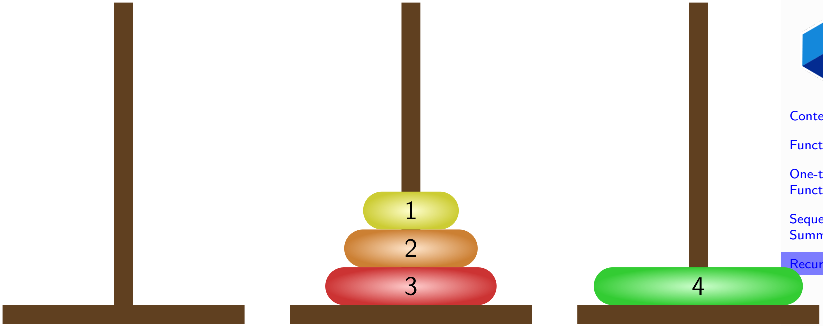
Moved disc from peg 3 to peg 2.

Tower of Hanoi – 4 Discs



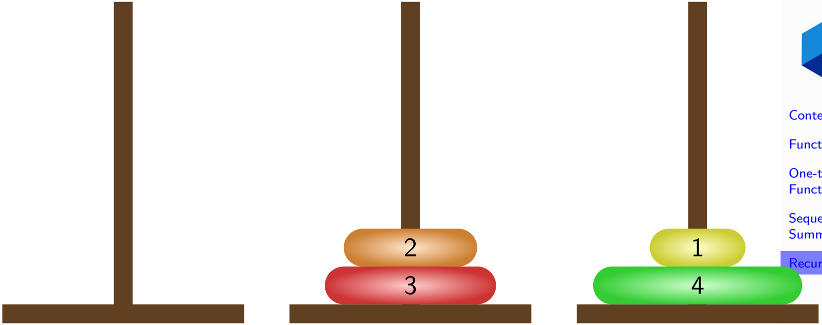
Moved disc from peg 1 to peg 2.

Tower of Hanoi – 4 Discs



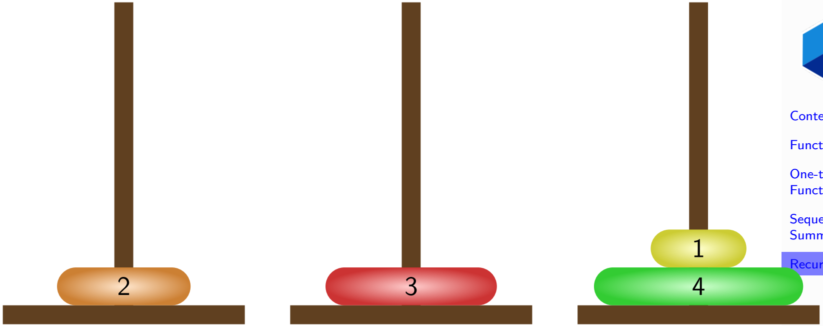
Moved disc from peg 1 to peg 3.

Tower of Hanoi – 4 Discs



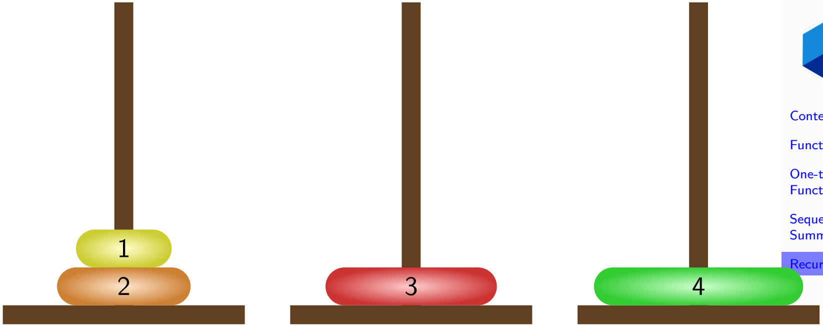
Moved disc from peg 2 to peg 3.

Tower of Hanoi – 4 Discs



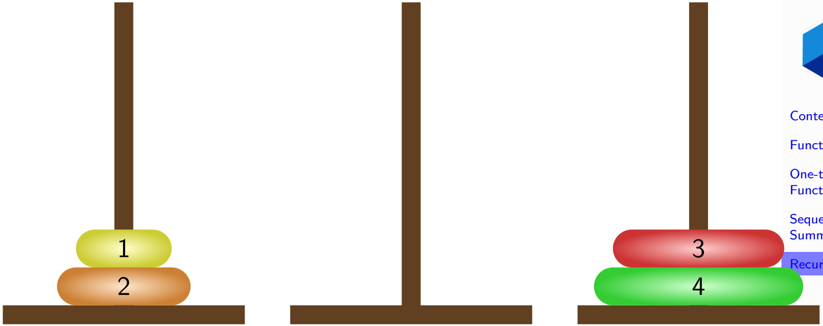
Moved disc from peg 2 to peg 1.

Tower of Hanoi – 4 Discs



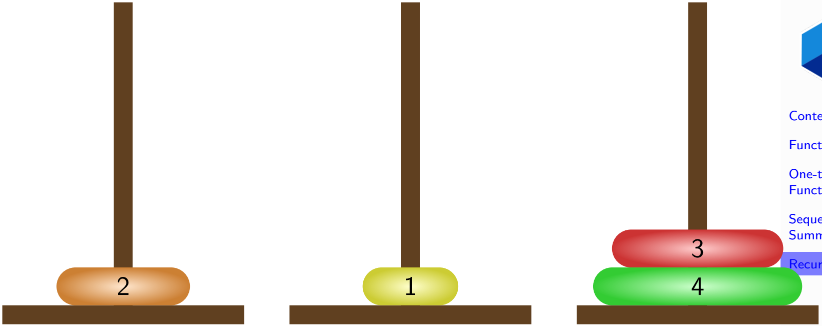
Moved disc from peg 3 to peg 1.

Tower of Hanoi – 4 Discs



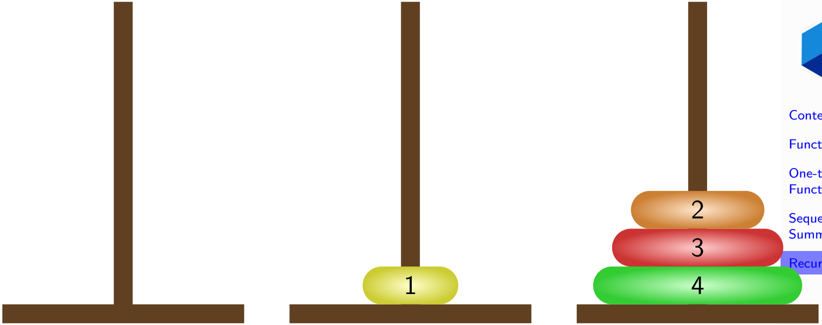
Moved disc from peg 2 to peg 3.

Tower of Hanoi – 4 Discs



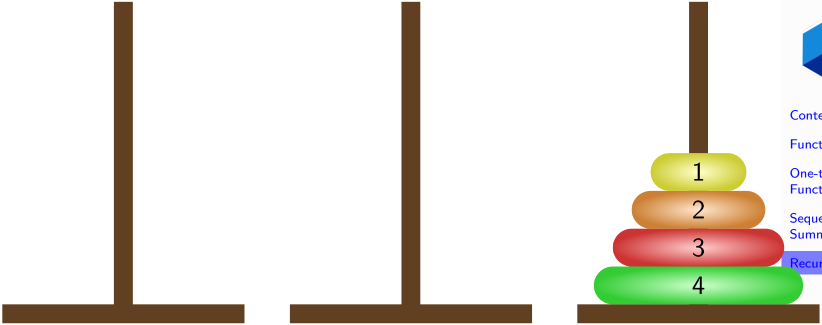
Moved disc from peg 1 to peg 2.

Tower of Hanoi – 4 Discs



Moved disc from peg 1 to peg 3.

Tower of Hanoi – 4 Discs



Moved disc from peg 2 to peg 3.

Tower of Hanoi – 4 Discs

Tower of Hanoi

Algorithm

procedure *hanoi*(n, A, B, C)
if $n = 1$ **then** move the disk from A to C
else
  call *hanoi*($n-1$, A, C, B)
  move disk n from A to C
  call *hanoi*($n-1$, B, A, C)

Recurrence Relation

$$H(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2H(n-1) + 1 & \text{if } n > 1. \end{cases}$$

Recurrence Solving

$H(n) = 2^n - 1$
If one move takes 1 second, for $n = 64$

$$\begin{aligned} 2^{64} - 1 &\approx 2 \times 10^{19} \text{ sec} \\ &\approx 500 \text{ billion years!}. \end{aligned}$$