The continuum hypothesis was stated by Cantor in 1877. He labored unsuccessfully to prove it, becoming extremely dismayed that he could not. By 1900, settling the continuum hypothesis was considered to be among the most important unsolved problems in mathematics. It was the first problem posed by David Hilbert in his 1900 list of open problems in mathematics.

The continuum hypothesis is still an open question and remains an area for active research. However, it has been shown that it can be neither proved nor disproved under the standard set theory axioms in modern mathematics, the Zermelo-Fraenkel axioms. The Zermelo-Fraenkel axioms were formulated to avoid the paradoxes of naive set theory, such as Russell's paradox, but there is much controversy whether they should be replaced by some other set of axioms for set theory.

Exercises

1. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the negative integers
b) the even integers
c) the integers less than 100
d) the real numbers between 0 and $\frac{1}{2}$
e) the positive integers less than 1,000,000,000
f) the integers that are multiples of 7

2. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the integers greater than 10
b) the odd negative integers
c) the integers with absolute value less than 1,000,000
d) the real numbers between 0 and 2
e) the set $A \times \mathbf{Z}^+$ where $A = \{2, 3\}$
f) the integers that are multiples of 10

3. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) all bit strings not containing the bit 0
b) all positive rational numbers that cannot be written with denominators less than 4
c) the real numbers not containing 0 in their decimal representation
d) the real numbers containing only a finite number of 1s in their decimal representation

4. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) integers not divisible by 3
b) integers divisible by 5 but not by 7
c) the real numbers with decimal representations consisting of all 1s
d) the real numbers with decimal representations of all 1s or 9s

5. Show that a finite group of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

6. Suppose that Hilbert's Grand Hotel is fully occupied, but the hotel closes all the even numbered rooms for maintenance. Show that all guests can remain in the hotel.

7. Suppose that Hilbert's Grand Hotel is fully occupied on the day the hotel expands to a second building which also contains a countably infinite number of rooms. Show that the current guests can be spread out to fill every room of the two buildings of the hotel.

8. Show that a countably infinite number of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

***9.** Suppose that a countably infinite number of buses, each containing a countably infinite number of guests, arrive at Hilbert's fully occupied Grand Hotel. Show that all the arriving guests can be accommodated without evicting any current guest.

10. Give an example of two uncountable sets A and B such that $A - B$ is

a) finite.
b) countably infinite.
c) uncountable.

11. Give an example of two uncountable sets A and B such that $A \cap B$ is

a) finite.
b) countably infinite.
c) uncountable.

12. Show that if A and B are sets and $A \subset B$ then $|A| \leq |B|$.

13. Explain why the set A is countable if and only if $|A| \leq |\mathbf{Z}^+|$.

14. Show that if A and B are sets with the same cardinality, then $|A| \leq |B|$ and $|B| \leq |A|$.

☞ **15.** Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

☞ **16.** Show that a subset of a countable set is also countable.

17. If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

18. Show that if A and B are sets $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

19. Show that if A, B, C, and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

20. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

21. Show that if A, B, and C are sets such that $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

22. Suppose that A is a countable set. Show that the set B is also countable if there is an onto function f from A to B.

23. Show that if A is an infinite set, then it contains a countably infinite subset.

24. Show that there is no infinite set A such that $|A| < |\mathbf{Z}^+| = \aleph_0$.

25. Prove that if it is possible to label each element of an infinite set S with a finite string of keyboard characters, from a finite list characters, where no two elements of S have the same label, then S is a countably infinite set.

26. Use Exercise 25 to provide a proof different from that in the text that the set of rational numbers is countable. [*Hint:* Show that you can express a rational number as a string of digits with a slash and possibly a minus sign.]

***27.** Show that the union of a countable number of countable sets is countable.

28. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable.

***29.** Show that the set of all finite bit strings is countable.

***30.** Show that the set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, and c are integers, is countable.

***31.** Show that $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m, n) = (m + n - 2)(m + n - 1)/2 + m$ is one-to-one and onto.

***32.** Show that when you substitute $(3n + 1)^2$ for each occurrence of n and $(3m + 1)^2$ for each occurrence of m in the right-hand side of the formula for the function $f(m, n)$ in Exercise 31, you obtain a one-to-one polynomial function $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$. It is an open question whether there is a one-to-one polynomial function $\mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$.

33. Use the Schröder-Bernstein theorem to show that $(0, 1)$ and $[0, 1]$ have the same cardinality.

34. Show that $(0, 1)$ and $\mathbf{R}$ have the same cardinality by

a) showing that $f(x) = \frac{2x-1}{2x(1-x)}$ is a bijection from $(0, 1)$ to $\mathbf{R}$.

b) using the Schröder-Bernstein theorem.

35. Show that there is no one-to-one correspondence from the set of positive integers to the power set of the set of positive integers. [*Hint:* Assume that there is such a one-to-one correspondence. Represent a subset of the set of positive integers as an infinite bit string with ith bit 1 if i belongs to the subset and 0 otherwise. Suppose that you can list these infinite strings in a sequence indexed by the positive integers. Construct a new bit string with its ith bit equal to the complement of the ith bit of the ith string in the list. Show that this new bit string cannot appear in the list.]

***36.** Show that there is a one-to-one correspondence from the set of subsets of the positive integers to the set real numbers between 0 and 1. Use this result and Exercises 34 and 35 to conclude that $\aleph_0 < |\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}|$. [*Hint:* Look at the first part of the hint for Exercise 35.]

***37.** Show that the set of all computer programs in a particular programming language is countable. [*Hint:* A computer program written in a programming language can be thought of as a string of symbols from a finite alphabet.]

***38.** Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. [*Hint:* First set up a one-to-one correspondence between the set of real numbers between 0 and 1 and a subset of these functions. Do this by associating to the real number $0.d_1d_2 \ldots d_n \ldots$ the function f with $f(n) = d_n$.]

***39.** We say that a function is **computable** if there is a computer program that finds the values of this function. Use Exercises 37 and 38 to show that there are functions that are not computable.

***40.** Show that if S is a set, then there does not exist an onto function f from S to $\mathcal{P}(S)$, the power set of S. Conclude that $|S| < |\mathcal{P}(S)|$. This result is known as **Cantor's theorem**. [*Hint:* Suppose such a function f existed. Let $T = \{s \in S \mid s \notin f(s)\}$ and show that no element s can exist for which $f(s) = T$.]

***41.** In this exercise, we prove the Schröder-Bernstein theorem. Suppose that A and B are sets where $|A| \leq |B|$ and $|B| \leq |A|$. This means that there are injections $f : A \to B$ and $g : B \to A$. To prove the theorem, we must show that there is a bijection $h : A \to B$, implying that $|A| = |B|$.

To build h, we construct the *chain* of an element $a \in A$. This chain contains the elements $a, f(a), g(f(a)), f(g(f(a))), g(f(g(f(a)))), \ldots$. It also may contain more elements that precede a, extending the chain backwards. So, if there is a $b \in B$ with $g(b) = a$, then b will be the term of the chain just before a. Because g may not be a surjection, there may not be any such b, so that a is the first element of the chain. If such a b exists, because g is an injection, it is the unique element of B mapped by g to a; we denote it by $g^{-1}(a)$. (Note that this defines g^{-1} as a partial function from B to A.) We extend the chain backwards as long as possible in the same way, adding $f^{-1}(g^{-1}(a))$, $g^{-1}(f^{-1}(g^{-1}(a)))$, ... To construct the proof, complete these five parts.

a) Show that every element in A or in B belongs to exactly one chain.

b) Show that there are four types of chains: chains that form a loop, that is, carrying them forward from every element in the chain will eventually return to this element (type 1), chains that go backwards without stopping (type 2), chains that go backwards and end in the set A (type 3), and chains that go backwards and end in the set B (type 4).

c) We now define a function $h : A \to B$. We set $h(a) = f(a)$ when a belongs to a chain of type 1, 2, or 3. Show that we can define $h(a)$ when a is in a chain of type 4, by taking $h(a) = g^{-1}(a)$. In parts (d) and (e), we show that this function is a bijection from A to B, proving the theorem.

d) Show that h is one-to-one. (You can consider chains of types 1, 2, and 3 together, but chains of type 4 separately.)

e) Show that h is onto. (You need to consider chains of types 1, 2, and 3 together, but chains of type 4 separately.)