Exercises

1. Use truth tables to verify these equivalences.
   a) $p \wedge \mathbf{T} \equiv p$
   b) $p \vee \mathbf{F} \equiv p$
   c) $p \wedge \mathbf{F} \equiv \mathbf{F}$
   d) $p \vee \mathbf{T} \equiv \mathbf{T}$
   e) $p \vee p \equiv p$
   f) $p \wedge p \equiv p$
2. Show that $\neg(\neg p)$ and p are logically equivalent.
3. Use truth tables to verify the commutative laws
   a) $p \vee q \equiv q \vee p$.
   b) $p \wedge q \equiv q \wedge p$.
4. Use truth tables to verify the associative laws
   a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
   b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.
5. Use a truth table to verify the distributive law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.
6. Use a truth table to verify the first De Morgan law $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
7. Use De Morgan's laws to find the negation of each of the following statements.
   a) Jan is rich and happy.
   b) Carlos will bicycle or run tomorrow.
   c) Mei walks or takes the bus to class.
   d) Ibrahim is smart and hard working.
8. Use De Morgan's laws to find the negation of each of the following statements.
   a) Kwame will take a job in industry or go to graduate school.
   b) Yoshiko knows Java and calculus.
   c) James is young and strong.
   d) Rita will move to Oregon or Washington.
9. For each of these compound propositions, use the conditional-disjunction equivalence (Example 3) to find an equivalent compound proposition that does not involve conditionals.
   a) $p \to \neg q$
   b) $(p \to q) \to r$
   c) $(\neg q \to p) \to (p \to \neg q)$
10. For each of these compound propositions, use the conditional-disjunction equivalence (Example 3) to find an equivalent compound proposition that does not involve conditionals.
    a) $\neg p \to \neg q$
    b) $(p \vee q) \to \neg p$
    c) $(p \to \neg q) \to (\neg p \to q)$
11. Show that each of these conditional statements is a tautology by using truth tables.
    a) $(p \wedge q) \to p$
    b) $p \to (p \vee q)$
    c) $\neg p \to (p \to q)$
    d) $(p \wedge q) \to (p \to q)$
    e) $\neg(p \to q) \to p$
    f) $\neg(p \to q) \to \neg q$
12. Show that each of these conditional statements is a tautology by using truth tables.
    a) $[\neg p \wedge (p \vee q)] \to q$
    b) $[(p \to q) \wedge (q \to r)] \to (p \to r)$
    c) $[p \wedge (p \to q)] \to q$
    d) $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$
13. Show that each conditional statement in Exercise 11 is a tautology using the fact that a conditional statement is false exactly when the hypothesis is true and the conclusion is false. (Do not use truth tables.)
14. Show that each conditional statement in Exercise 12 is a tautology using the fact that a conditional statement is false exactly when the hypothesis is true and the conclusion is false. (Do not use truth tables.)
15. Show that each conditional statement in Exercise 11 is a tautology by applying a chain of logical identities as in Example 8. (Do not use truth tables.)
16. Show that each conditional statement in Exercise 12 is a tautology by applying a chain of logical identities as in Example 8. (Do not use truth tables.)
17. Use truth tables to verify the absorption laws.
    a) $p \vee (p \wedge q) \equiv p$
    b) $p \wedge (p \vee q) \equiv p$
18. Determine whether $(\neg p \wedge (p \to q)) \to \neg q$ is a tautology.
19. Determine whether $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

Each of Exercises 20–32 asks you to show that two compound propositions are logically equivalent. To do this, either show that both sides are true, or that both sides are false, for exactly the same combinations of truth values of the propositional variables in these expressions (whichever is easier).

20. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.
21. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.
22. Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.
23. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.
24. Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.
25. Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.
26. Show that $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.
27. Show that $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$ are logically equivalent.
28. Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent.
29. Show that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent.
30. Show that $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent.
31. Show that $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are logically equivalent.
32. Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent.
33. Show that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.
34. Show that $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$ is a tautology.
35. Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.
36. Show that $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ are not logically equivalent.

37. Show that $(p \to q) \to (r \to s)$ and $(p \to r) \to (q \to s)$ are not logically equivalent.

The **dual** of a compound proposition that contains only the logical operators $\vee$, $\wedge$, and $\neg$ is the compound proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each **T** by **F**, and each **F** by **T**. The dual of s is denoted by s^*.

38. Find the dual of each of these compound propositions.

a) $p \vee \neg q$ **b)** $p \wedge (q \vee (r \wedge \mathbf{T}))$
c) $(p \wedge \neg q) \vee (q \wedge \mathbf{F})$

39. Find the dual of each of these compound propositions.

a) $p \wedge \neg q \wedge \neg r$ **b)** $(p \wedge q \wedge r) \vee s$
c) $(p \vee \mathbf{F}) \wedge (q \vee \mathbf{T})$

40. When does $s^* = s$, where s is a compound proposition?

41. Show that $(s^*)^* = s$ when s is a compound proposition.

42. Show that the logical equivalences in Table 6, except for the double negation law, come in pairs, where each pair contains compound propositions that are duals of each other.

****43.** Why are the duals of two equivalent compound propositions also equivalent, where these compound propositions contain only the operators $\wedge$, $\vee$, and $\neg$?

44. Find a compound proposition involving the propositional variables p, q, and r that is true when p and q are true and r is false, but is false otherwise. [*Hint:* Use a conjunction of each propositional variable or its negation.]

45. Find a compound proposition involving the propositional variables p, q, and r that is true when exactly two of p, q, and r are true and is false otherwise. [*Hint:* Form a disjunction of conjunctions. Include a conjunction for each combination of values for which the compound proposition is true. Each conjunction should include each of the three propositional variables or its negations.]

☞ **46.** Suppose that a truth table in n propositional variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations, with one conjunction included for each combination of values for which the compound proposition is true. The resulting compound proposition is said to be in **disjunctive normal form**.

A collection of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

47. Show that $\neg$, $\wedge$, and $\vee$ form a functionally complete collection of logical operators. [*Hint:* Use the fact that every compound proposition is logically equivalent to one in disjunctive normal form, as shown in Exercise 46.]

***48.** Show that $\neg$ and $\wedge$ form a functionally complete collection of logical operators. [*Hint:* First use a De Morgan law to show that $p \vee q$ is logically equivalent to $\neg(\neg p \wedge \neg q)$.]

***49.** Show that $\neg$ and $\vee$ form a functionally complete collection of logical operators.

We now present a group of exercises that involve the logical operators *NAND* and *NOR*. The proposition p *NAND* q is true when either p or q, or both, are false; and it is false when both p and q are true. The proposition p *NOR* q is true when both p and q are false, and it is false otherwise. The propositions p *NAND* q and p *NOR* q are denoted by $p \mid q$ and $p \downarrow q$, respectively. (The operators $\mid$ and $\downarrow$ are called the **Sheffer stroke** and the **Peirce arrow** after H. M. Sheffer and C. S. Peirce, respectively.)

50. Construct a truth table for the logical operator *NAND*.

51. Show that $p \mid q$ is logically equivalent to $\neg(p \wedge q)$.

52. Construct a truth table for the logical operator *NOR*.

53. Show that $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$.

54. In this exercise we will show that $\{\downarrow\}$ is a functionally complete collection of logical operators.

a) Show that $p \downarrow p$ is logically equivalent to $\neg p$.
b) Show that $(p \downarrow q) \downarrow (p \downarrow q)$ is logically equivalent to $p \vee q$.
c) Conclude from parts (a) and (b), and Exercise 49, that $\{\downarrow\}$ is a functionally complete collection of logical operators.

***55.** Find a compound proposition logically equivalent to $p \to q$ using only the logical operator $\downarrow$.

56. Show that $\{\mid\}$ is a functionally complete collection of logical operators.

57. Show that $p \mid q$ and $q \mid p$ are equivalent.

58. Show that $p \mid (q \mid r)$ and $(p \mid q) \mid r$ are not equivalent, so that the logical operator $\mid$ is not associative.

***59.** How many different truth tables of compound propositions are there that involve the propositional variables p and q?

60. Show that if p, q, and r are compound propositions such that p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent.

61. The following sentence is taken from the specification of a telephone system: "If the directory database is opened, then the monitor is put in a closed state, if the system is not in its initial state." This specification is hard to understand because it involves two conditional statements. Find an equivalent, easier-to-understand specification that involves disjunctions and negations but not conditional statements.

62. How many of the disjunctions $p \vee \neg q$, $\neg p \vee q$, $q \vee r$, $q \vee \neg r$, and $\neg q \vee \neg r$ can be made simultaneously true by an assignment of truth values to p, q, and r?

63. How many of the disjunctions $p \vee \neg q \vee s$, $\neg p \vee \neg r \vee s$, $\neg p \vee \neg r \vee \neg s$, $\neg p \vee q \vee \neg s$, $q \vee r \vee \neg s$, $q \vee \neg r \vee \neg s$, $\neg p \vee \neg q \vee \neg s$, $p \vee r \vee s$, and $p \vee r \vee \neg s$ can be made simultaneously true by an assignment of truth values to p, q, r, and s?

64. Show that the negation of an unsatisfiable compound proposition is a tautology and the negation of a compound proposition that is a tautology is unsatisfiable.

65. Determine whether each of these compound propositions is satisfiable.

a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
b) $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$
c) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$

66. Determine whether each of these compound propositions is satisfiable.

a) $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg s) \wedge (p \vee \neg r \vee \neg s) \wedge (\neg p \vee \neg q \vee \neg s) \wedge (p \vee q \vee \neg s)$
b) $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg s) \wedge (\neg p \vee \neg r \vee \neg s) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg r \vee \neg s)$
c) $(p \vee q \vee r) \wedge (p \vee \neg q \vee \neg s) \wedge (q \vee \neg r \vee s) \wedge (\neg p \vee r \vee s) \wedge (\neg p \vee q \vee \neg s) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee s) \wedge (\neg p \vee \neg r \vee \neg s)$

67. Find the compound proposition Q constructed in Example 10 for the n-queens problem, and use it to find all the ways that n queens can be placed on an $n \times n$ chessboard, so that no queen can attack another when n is

a) 2. **b)** 3. **c)** 4.

68. Starting with the compound proposition Q found in Example 10, construct a compound proposition that can be used to find all solutions of the n-queens problem where the queen in the first column is in an odd-numbered row.

69. Show how the solution of a given 4×4 Sudoku puzzle can be found by solving a satisfiability problem.

70. Construct a compound proposition that asserts that every cell of a 9×9 Sudoku puzzle contains at least one number.

71. Explain the steps in the construction of the compound proposition given in the text that asserts that every column of a 9×9 Sudoku puzzle contains every number.

***72.** Explain the steps in the construction of the compound proposition given in the text that asserts that each of the nine 3×3 blocks of a 9×9 Sudoku puzzle contains every number.

1.4 Predicates and Quantifiers

1.4.1 Introduction

Propositional logic, studied in Sections 1.1–1.3, cannot adequately express the meaning of all statements in mathematics and in natural language. For example, suppose that we know that

"Every computer connected to the university network is functioning properly."

No rules of propositional logic allow us to conclude the truth of the statement

"MATH3 is functioning properly,"

where MATH3 is one of the computers connected to the university network. Likewise, we cannot use the rules of propositional logic to conclude from the statement

"CS2 is under attack by an intruder,"

where CS2 is a computer on the university network, to conclude the truth of

"There is a computer on the university network that is under attack by an intruder."

In this section we will introduce a more powerful type of logic called **predicate logic**. We will see how predicate logic can be used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects. To understand predicate logic, we first need to introduce the concept of a predicate. Afterward, we will introduce the notion of quantifiers, which enable us to reason with statements that assert that a certain property holds for all objects of a certain type and with statements that assert the existence of an object with a particular property.

1.4.2 Predicates

Statements involving variables, such as

"$x > 3$," "$x = y + 3$," "$x + y = z$,"