A path of length $r + 1$ from v_i to v_j is made up of a path of length r from v_i to some intermediate vertex v_k, and an edge from v_k to v_j. By the product rule for counting, the number of such paths is the product of the number of paths of length r from v_i to v_k, namely, b_{ik}, and the number of edges from v_k to v_j, namely, a_{kj}. When these products are added for all possible intermediate vertices v_k, the desired result follows by the sum rule for counting. ◁

EXAMPLE 15 How many paths of length four are there from a to d in the simple graph G in Figure 8?

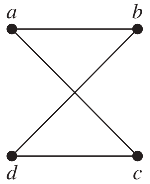


FIGURE 8 The graph *G*.

Solution: The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Hence, the number of paths of length four from a to d is the (1, 4)th entry of $\mathbf{A}^4$. Because

$$\mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix},$$

Extra Examples

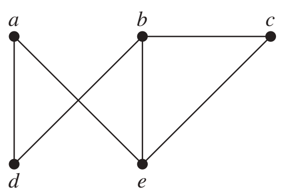
there are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths of length four from a to d. ◀

Theorem 2 can be used to find the length of the shortest path between two vertices of a graph (see Exercise 56), and it can also be used to determine whether a graph is connected (see Exercises 61 and 62).

Exercises

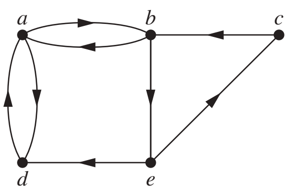
1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, e, b, c, b **b)** a, e, a, d, b, c, a
c) e, b, a, d, b, e **d)** c, b, d, a, e, c



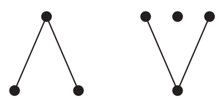
2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, b, e, c, b **b)** a, d, a, d, a
c) a, d, b, e, a **d)** a, b, e, c, b, d, a

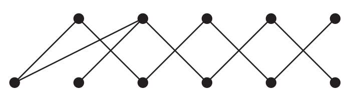


In Exercises 3–5 determine whether the given graph is connected.

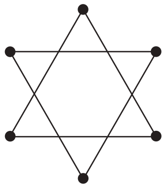
3.



4.



5.



6. How many connected components does each of the graphs in Exercises 3–5 have? For each graph find each of its connected components.

7. What do the connected components of acquaintanceship graphs represent?

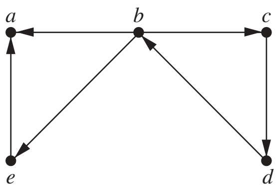
8. What do the connected components of a collaboration graph represent?

9. Explain why in the collaboration graph of mathematicians (see Example 3 in Section 10.1) a vertex representing a mathematician is in the same connected component as the vertex representing Paul Erdős if and only if that mathematician has a finite Erdős number.

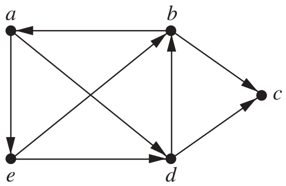
10. In the Hollywood graph (see Example 3 in Section 10.1), when is the vertex representing an actor in the same connected component as the vertex representing Kevin Bacon?

11. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

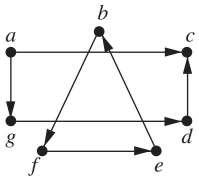
a)



b)

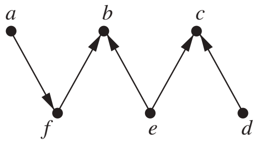


c)

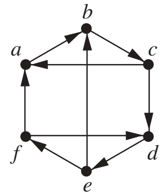


12. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

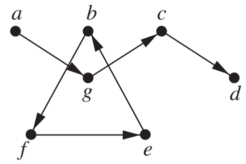
a)



b)



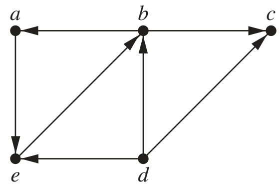
c)



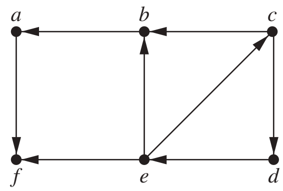
13. What do the strongly connected components of a telephone call graph represent?

14. Find the strongly connected components of each of these graphs.

a)



b)



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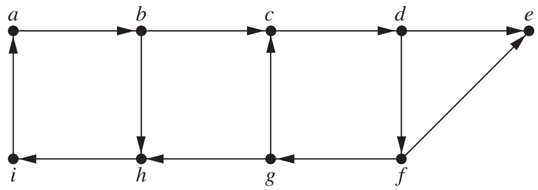
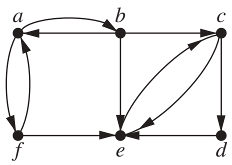


15. Find the strongly connected components of each of these graphs.

a)



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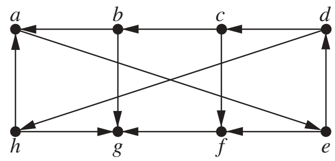
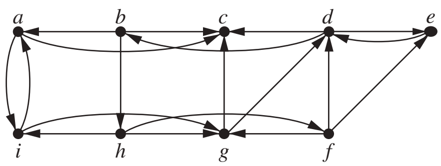


c)



Suppose that $G = (V, E)$ is a directed graph. A vertex $w \in V$ is **reachable** from a vertex $v \in V$ if there is a directed path from v to w. The vertices v and w are **mutually reachable** if there are both a directed path from v to w and a directed path from w to v in G.

16. Show that if $G = (V, E)$ is a directed graph and u, v, and w are vertices in V for which u and v are mutually reachable and v and w are mutually reachable, then u and w are mutually reachable.

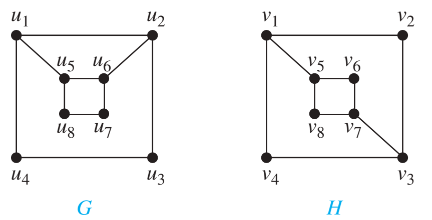
17. Show that if $G = (V, E)$ is a directed graph, then the strong components of two vertices u and v of V are either the same or disjoint. [*Hint:* Use Exercise 16.]

18. Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in this strongly connected component.

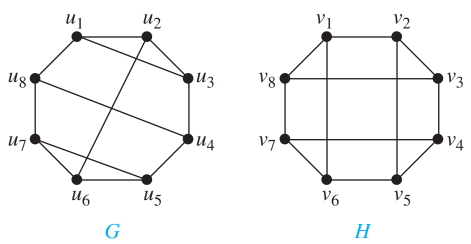
19. Find the number of paths of length n between two different vertices in K_4 if n is

a) 2. **b)** 3. **c)** 4. **d)** 5.

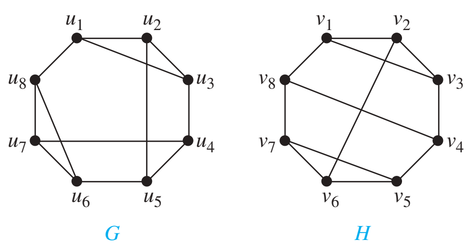
20. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.



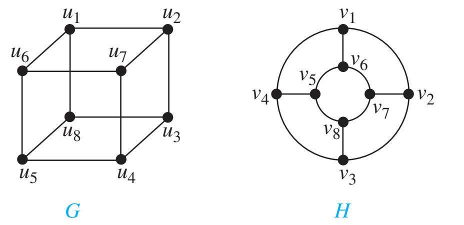
21. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



22. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



23. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



24. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ for the values of n in Exercise 19.

25. Find the number of paths of length n between any two nonadjacent vertices in $K_{3,3}$ for the values of n in Exercise 19.

26. Find the number of paths between c and d in the graph in Figure 1 of length

a) 2. **b)** 3. **c)** 4. **d)** 5. **e)** 6. **f)** 7.

27. Find the number of paths from a to e in the directed graph in Exercise 2 of length

a) 2. **b)** 3. **c)** 4. **d)** 5. **e)** 6. **f)** 7.

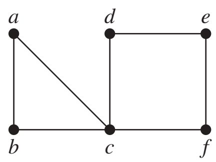
***28.** Show that every connected graph with n vertices has at least $n - 1$ edges.

29. Let $G = (V, E)$ be a simple graph. Let R be the relation on V consisting of pairs of vertices (u, v) such that there is a path from u to v or such that $u = v$. Show that R is an equivalence relation.

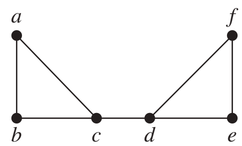
***30.** Show that in every simple graph there is a path from every vertex of odd degree to some other vertex of odd degree.

In Exercises 31–33 find all the cut vertices of the given graph.

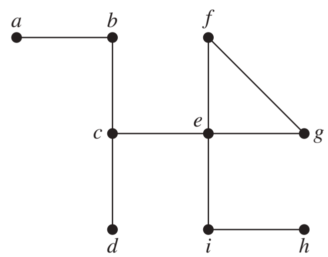
31.



32.



33.



34. Find all the cut edges in the graphs in Exercises 31–33.

***35.** Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

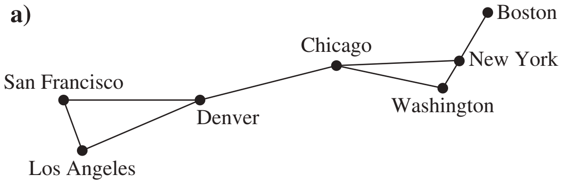
***36.** Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.

***37.** Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices.

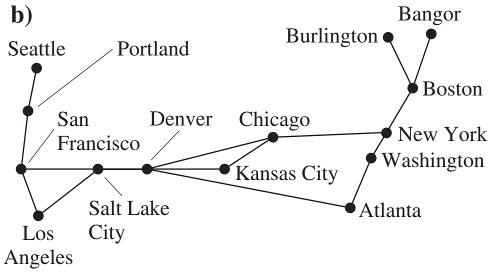
* **38.** Show that an edge in a simple graph is a cut edge if and only if this edge is not part of any simple circuit in the graph.

39. A communications link in a network should be provided with a backup link if its failure makes it impossible for some message to be sent. For each of the communications networks shown here in (a) and (b), determine those links that should be backed up.

a)



b)



A **vertex basis** in a directed graph G is a minimal set B of vertices of G such that for each vertex v of G not in B there is a path to v from some vertex B.

40. Find a vertex basis for each of the directed graphs in Exercises 7–9 of Section 10.2.

41. What is the significance of a vertex basis in an influence graph (described in Example 2 of Section 10.1)? Find a vertex basis in the influence graph in that example.

42. Show that if a connected simple graph G is the union of the graphs G_1 and G_2, then G_1 and G_2 have at least one common vertex.

* **43.** Show that if a simple graph G has k connected components and these components have $n_1, n_2, \ldots, n_k$ vertices, respectively, then the number of edges of G does not exceed

$$\sum_{i=1}^{k} C(n_i, 2).$$

* **44.** Use Exercise 43 to show that a simple graph with n vertices and k connected components has at most $(n-k)(n-k+1)/2$ edges. [*Hint:* First show that

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k),$$

where n_i is the number of vertices in the ith connected component.]

* **45.** Show that a simple graph G with n vertices is connected if it has more than $(n-1)(n-2)/2$ edges.

46. Describe the adjacency matrix of a graph with n connected components when the vertices of the graph are listed so that vertices in each connected component are listed successively.

47. How many nonisomorphic connected simple graphs are there with n vertices when n is

a) 2? **b)** 3? **c)** 4? **d)** 5?

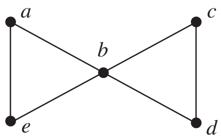
48. Show that each of the following graphs has no cut vertices.

a) C_n where $n \geq 3$
b) W_n where $n \geq 3$
c) $K_{m,n}$ where $m \geq 2$ and $n \geq 2$
d) Q_n where $n \geq 2$

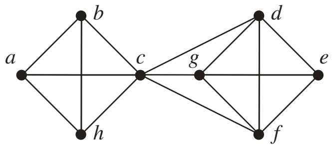
49. Show that each of the graphs in Exercise 48 has no cut edges.

50. For each of these graphs, find $\kappa(G)$, $\lambda(G)$, and $\min_{v\in V} \deg(v)$, and determine which of the two inequalities in $\kappa(G) \leq \lambda(G) \leq \min_{v\in V} \deg(v)$ are strict.

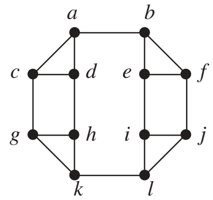
a)



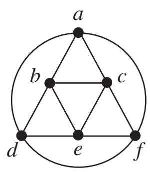
b)



c)



d)



51. Show that if G is a connected graph, then it is possible to remove vertices to disconnect G if and only if G is not a complete graph.

52. Show that if G is a connected graph with n vertices then

a) $\kappa(G) = n - 1$ if and only if $G = K_n$.
b) $\lambda(G) = n - 1$ if and only if $G = K_n$.

53. Find $\kappa(K_{m,n})$ and $\lambda(K_{m,n})$, where m and n are positive integers.

54. Construct a graph G with $\kappa(G) = 1$, $\lambda(G) = 2$, and $\min_{v\in V} \deg(v) = 3$.

* **55.** Show that if G is a graph, then $\kappa(G) \leq \lambda(G)$.

56. Explain how Theorem 2 can be used to find the length of the shortest path from a vertex v to a vertex w in a graph.

57. Use Theorem 2 to find the length of the shortest path between a and f in the graph in Figure 1.

58. Use Theorem 2 to find the length of the shortest path from a to c in the directed graph in Exercise 2.

☞ **59.** Let P_1 and P_2 be two simple paths between the vertices u and v in the simple graph G that do not contain the same set of edges. Show that there is a simple circuit in G.

60. Show that the existence of a simple circuit of length k, where k is an integer greater than 2, is an invariant for graph isomorphism.

61. Explain how Theorem 2 can be used to determine whether a graph is connected.

62. Use Exercise 61 to show that the graph G_1 in Figure 2 is connected whereas the graph G_2 in that figure is not connected.

63. Show that a simple graph G is bipartite if and only if it has no circuits with an odd number of edges.

64. In an old puzzle attributed to Alcuin of York (735–804), a farmer needs to carry a wolf, a goat, and a cabbage across a river. The farmer only has a small boat, which can carry the farmer and only one object (an animal or a vegetable). He can cross the river repeatedly. However, if the farmer is on the other shore, the wolf will eat the goat, and, similarly, the goat will eat the cabbage. We can describe each state by listing what is on each shore. For example, we can use the pair (FG, WC) for the state where the farmer and goat are on the first shore and the wolf and cabbage are on the other shore. [The symbol $\emptyset$ is used when nothing is on a shore, so that $(FWGC, \emptyset)$ is the initial state.]

a) Find all allowable states of the puzzle, where neither the wolf and the goat nor the goat and the cabbage are left on the same shore without the farmer.

b) Construct a graph such that each vertex of this graph represents an allowable state and the vertices representing two allowable states are connected by an edge if it is possible to move from one state to the other using one trip of the boat.

c) Explain why finding a path from the vertex representing $(FWGC, \emptyset)$ to the vertex representing $(\emptyset, FWGC)$ solves the puzzle.

d) Find two different solutions of the puzzle, each using seven crossings.

e) Suppose that the farmer must pay a toll of one dollar whenever he crosses the river with an animal. Which solution of the puzzle should the farmer use to pay the least total toll?

***65.** Use a graph model and a path in your graph, as in Exercise 64, to solve the **jealous husbands problem**. Two married couples, each a husband and a wife, want to cross a river. They can only use a boat that can carry one or two people from one shore to the other shore. Each husband is extremely jealous and is not willing to leave his wife with the other husband, either in the boat or on shore. How can these four people reach the opposite shore?

66. Suppose that you have a three-gallon jug and a five-gallon jug. You may fill either jug with water, you may empty either jug, and you may transfer water from either jug into the other jug. Use a path in a directed graph to show that you can end up with a jug containing exactly one gallon. [*Hint:* Use an ordered pair (a, b) to indicate how much water is in each jug. Represent these ordered pairs by vertices. Add an edge for each allowable operation with the jugs.]

10.5 Euler and Hamilton Paths

10.5.1 Introduction

Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge of the graph exactly once? Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly once? Although these questions seem to be similar, the first question, which asks whether a graph has an *Euler circuit*, can be easily answered simply by examining the degrees of the vertices of the graph, while the second question, which asks whether a graph has a *Hamilton circuit*, is quite difficult to solve for most graphs. In this section we will study these questions and discuss the difficulty of solving them. Although both questions have many practical applications in many different areas, both arose in old puzzles. We will learn about these old puzzles as well as modern practical applications.

10.5.2 Euler Paths and Circuits

Links

The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic), was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions. Figure 1 depicts these regions and bridges.

Only five bridges connect Kaliningrad today. Of these, just two remain from Euler's day.

The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.