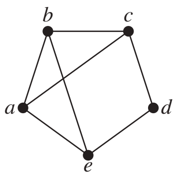
Gray codes are named after Frank Gray, who invented them in the 1940s at AT&T Bell Laboratories to minimize the effect of errors in transmitting digital signals. ◀

Exercises

In Exercises 1–8 determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

1.



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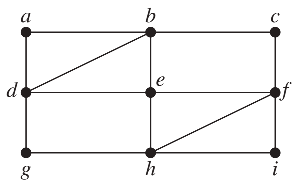

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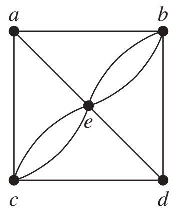

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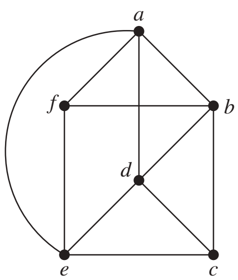

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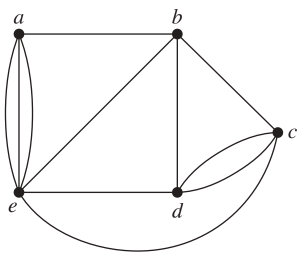

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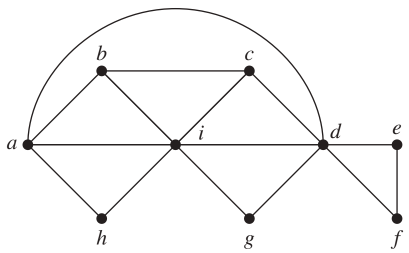

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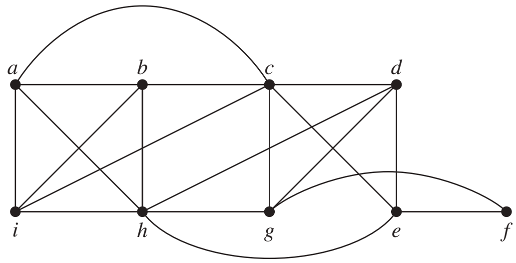

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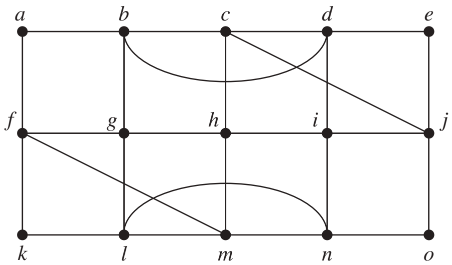
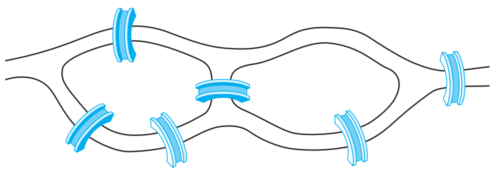


9. Suppose that in addition to the seven bridges of Königsberg (shown in Figure 1) there were two additional bridges, connecting regions B and C and regions B and D, respectively. Could someone cross all nine of these bridges exactly once and return to the starting point?

10. Can someone cross all the bridges shown in this map exactly once and return to the starting point?

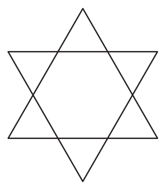


11. When can the centerlines of the streets in a city be painted without traveling a street more than once? (Assume that all the streets are two-way streets.)

12. Devise a procedure, similar to Algorithm 1, for constructing Euler paths in multigraphs.

In Exercises 13–15 determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.

13.



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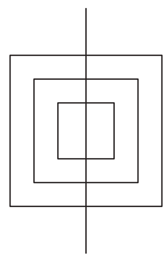

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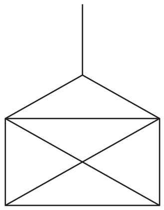
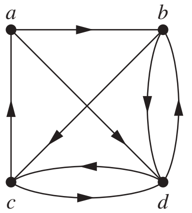


***16.** Show that a directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

***17.** Show that a directed multigraph having no isolated vertices has an Euler path but not an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree one larger than its out-degree and the other that has out-degree one larger than its in-degree.

In Exercises 18–23 determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path. Construct an Euler path if one exists.

18.



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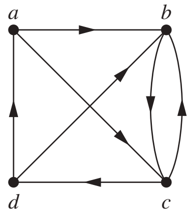

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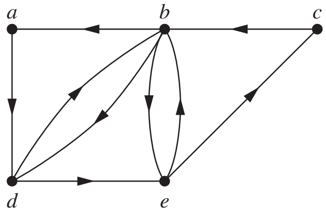

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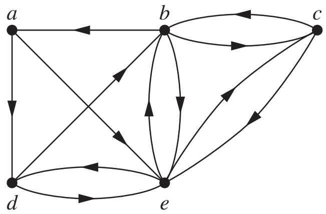

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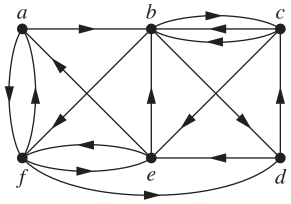

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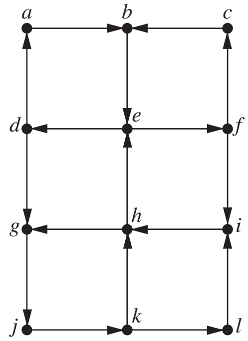


***24.** Devise an algorithm for constructing Euler circuits in directed graphs.

25. Devise an algorithm for constructing Euler paths in directed graphs.

26. For which values of n do these graphs have an Euler circuit?

a) K_n **b)** C_n **c)** W_n **d)** Q_n

27. For which values of n do the graphs in Exercise 26 have an Euler path but no Euler circuit?

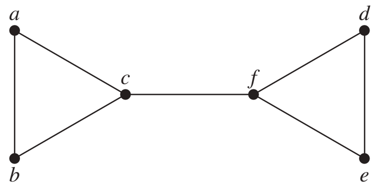
28. For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

a) Euler circuit?
b) Euler path?

29. Find the least number of times it is necessary to lift a pencil from the paper when drawing each of the graphs in Exercises 1–7 without retracing any part of the graph.

In Exercises 30–36 determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

30.



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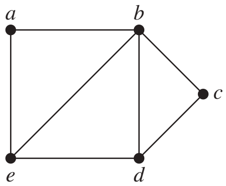

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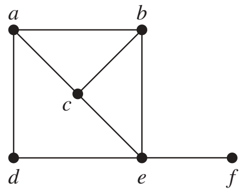

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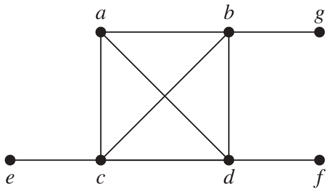

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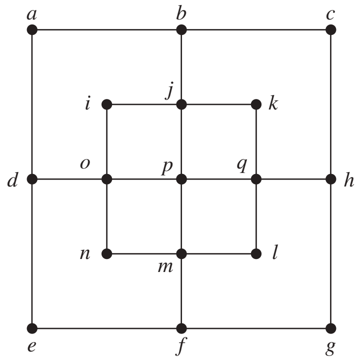

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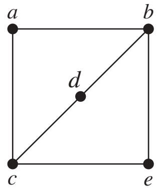

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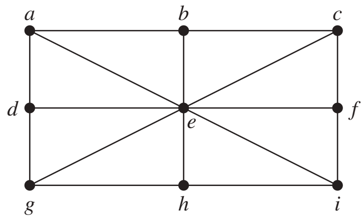


37. Does the graph in Exercise 30 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

38. Does the graph in Exercise 31 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

39. Does the graph in Exercise 32 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

40. Does the graph in Exercise 33 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

*__41.__ Does the graph in Exercise 34 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

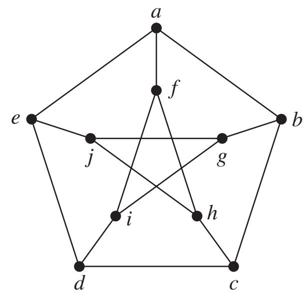
42. Does the graph in Exercise 35 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

43. Does the graph in Exercise 36 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

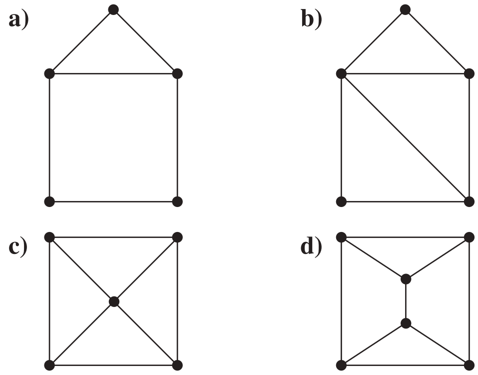
44. For which values of n do the graphs in Exercise 26 have a Hamilton circuit?

45. For which values of m and n does the complete bipartite graph $K_{m,n}$ have a Hamilton circuit?

*__46.__ Show that the **Petersen graph**, shown here, does not have a Hamilton circuit, but that the subgraph obtained by deleting a vertex v, and all edges incident with v, does have a Hamilton circuit.



47. For each of these graphs, determine (i) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit, (ii) whether Ore's theorem can be used to show that the graph has a Hamilton circuit, and (iii) whether the graph has a Hamilton circuit.



48. Can you find a simple graph with n vertices with $n \geq 3$ that does not have a Hamilton circuit, yet the degree of every vertex in the graph is at least $(n-1)/2$?

*__49.__ Show that there is a Gray code of order n whenever n is a positive integer, or equivalently, show that the n-cube Q_n, $n > 1$, always has a Hamilton circuit. [*Hint:* Use mathematical induction. Show how to produce a Gray code of order n from one of order $n-1$.]

Fleury's algorithm, published in 1883, constructs Euler circuits by first choosing an arbitrary vertex of a connected multigraph, and then forming a circuit by choosing edges successively. Once an edge is chosen, it is removed. Edges are chosen successively so that each edge begins where the last edge ends, and so that this edge is not a cut edge unless there is no alternative.

50. Use Fleury's algorithm to find an Euler circuit in the graph G in Figure 5.

*__51.__ Express Fleury's algorithm in pseudocode.

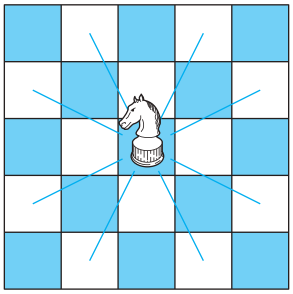
**__52.__ Prove that Fleury's algorithm always produces an Euler circuit.

*__53.__ Give a variant of Fleury's algorithm to produce Euler paths.

54. A diagnostic message can be sent out over a computer network to perform tests over all links and in all devices. What sort of paths should be used to test all links? To test all devices?

55. Show that a bipartite graph with an odd number of vertices does not have a Hamilton circuit.

A **knight** is a chess piece that can move either two spaces horizontally and one space vertically or one space horizontally and two spaces vertically. That is, a knight on square (x, y) can move to any of the eight squares $(x \pm 2, y \pm 1)$, $(x \pm 1, y \pm 2)$, if these squares are on the chessboard, as illustrated here.



A **knight's tour** is a sequence of legal moves by a knight starting at some square and visiting each square exactly once. A knight's tour is called **reentrant** if there is a legal move that takes the knight from the last square of the tour back to where the tour began. We can model knight's tours using the graph that has a vertex for each square on the board, with an edge connecting two vertices if a knight can legally move between the squares represented by these vertices.

56. Draw the graph that represents the legal moves of a knight on a 3×3 chessboard.

57. Draw the graph that represents the legal moves of a knight on a 3×4 chessboard.

58. **a)** Show that finding a knight's tour on an $m \times n$ chessboard is equivalent to finding a Hamilton path on the graph representing the legal moves of a knight on that board.

b) Show that finding a reentrant knight's tour on an $m \times n$ chessboard is equivalent to finding a Hamilton circuit on the corresponding graph.

***59.** Show that there is a knight's tour on a 3×4 chessboard.

***60.** Show that there is no knight's tour on a 3×3 chessboard.

***61.** Show that there is no knight's tour on a 4×4 chessboard.

62. Show that the graph representing the legal moves of a knight on an $m \times n$ chessboard, whenever m and n are positive integers, is bipartite.

63. Show that there is no reentrant knight's tour on an $m \times n$ chessboard when m and n are both odd. [*Hint:* Use Exercises 55, 58b, and 62.]

***64.** Show that there is a knight's tour on an 8×8 chessboard. [*Hint:* You can construct a knight's tour using a method invented by H. C. Warnsdorff in 1823: Start in any square, and then always move to a square connected to the fewest number of unused squares. Although this method may not always produce a knight's tour, it often does.]

65. The parts of this exercise outline a proof of Ore's theorem. Suppose that G is a simple graph with n vertices, $n \geq 3$, and $\deg(x) + \deg(y) \geq n$ whenever x and y are nonadjacent vertices in G. Ore's theorem states that under these conditions, G has a Hamilton circuit.

a) Show that if G does not have a Hamilton circuit, then there exists another graph H with the same vertices as G, which can be constructed by adding edges to G, such that the addition of a single edge would produce a Hamilton circuit in H. [*Hint:* Add as many edges as possible at each successive vertex of G without producing a Hamilton circuit.]

b) Show that there is a Hamilton path in H.

c) Let $v_1, v_2, \ldots, v_n$ be a Hamilton path in H. Show that $\deg(v_1) + \deg(v_n) \geq n$ and that there are at most $\deg(v_1)$ vertices not adjacent to v_n (including v_n itself).

Links

©*Paul Fearn/Alamy Stock Photo*

JULIUS PETER CHRISTIAN PETERSEN (1839–1910) Julius Petersen was born in the Danish town of Sorø. His father was a dyer. In 1854 his parents were no longer able to pay for his schooling, so he became an apprentice in an uncle's grocery store. When this uncle died, he left Petersen enough money to return to school. After graduating, he began studying engineering at the Polytechnical School in Copenhagen, later deciding to concentrate on mathematics. He published his first textbook, a book on logarithms, in 1858. When his inheritance ran out, he had to teach to make a living. From 1859 until 1871 Petersen taught at a prestigious private high school in Copenhagen. While teaching high school he continued his studies, entering Copenhagen University in 1862. He married Laura Bertelsen in 1862; they had three children, two sons and a daughter.

Petersen obtained a mathematics degree from Copenhagen University in 1866 and finally obtained his doctorate in 1871 from that school. After receiving his doctorate, he taught at a polytechnic and military academy. In 1887 he was appointed to a professorship at the University of Copenhagen. Petersen was well known in Denmark as the author of a large series of textbooks for high schools and universities. One of his books, *Methods and Theories for the Solution of Problems of Geometrical Construction*, was translated into eight languages, with the English language version last reprinted in 1960 and the French version reprinted as recently as 1990, more than a century after the original publication date.

Petersen worked in a wide range of areas, including algebra, analysis, cryptography, geometry, mechanics, mathematical economics, and number theory. His contributions to graph theory, including results on regular graphs, are his best-known work. He was noted for his clarity of exposition, problem-solving skills, originality, sense of humor, vigor, and teaching. One interesting fact about Petersen was that he preferred not to read the writings of other mathematicians. This led him often to rediscover results already proved by others, often with embarrassing consequences. However, he was often angry when other mathematicians did not read his writings!

Petersen's death was front-page news in Copenhagen. A newspaper of the time described him as the Hans Christian Andersen of science—a child of the people who made good in the academic world.

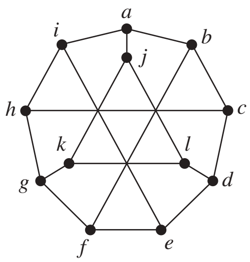
d) Let S be the set of vertices preceding each vertex adjacent to v_1 in the Hamilton path. Show that S contains $\deg(v_1)$ vertices and $v_n \notin S$.

e) Show that S contains a vertex v_k that is adjacent to v_n, implying that there are edges connecting v_1 and v_{k+1} and v_k and v_n.

f) Show that part (e) implies that $v_1, v_2, \ldots, v_{k-1}, v_k, v_n, v_{n-1}, \ldots, v_{k+1}, v_1$ is a Hamilton circuit in G. Conclude from this contradiction that Ore's theorem holds.

66. Show that if u and v are nonadjacent vertices in a graph G with n vertices and $\deg(u) + \deg(v) \geq n$, then G has a Hamilton circuit if and only if $G + \{u, v\}$ has a Hamilton circuit.

***67.** Show that this graph does *not* have a Hamilton circuit.



***68.** Show that the worst case computational complexity of Algorithm 1 for finding Euler circuits in a connected graph with all vertices of even degree is $O(m)$, where m is the number of edges of G.

10.6 Shortest-Path Problems

10.6.1 Introduction

Many problems can be modeled using graphs with weights assigned to their edges. As an illustration, consider how an airline system can be modeled. We set up the basic graph model by representing cities by vertices and flights by edges. Problems involving distances can be modeled by assigning distances between cities to the edges. Problems involving flight time can be modeled by assigning flight times to edges. Problems involving fares can be modeled by assigning fares to the edges. Figure 1 displays three different assignments of weights to the edges of a graph representing distances, flight times, and fares, respectively.

Graphs that have a number assigned to each edge are called **weighted graphs**. Weighted graphs are used to model computer networks. Communications costs (such as the monthly cost of leasing a telephone line), the response times of the computers over these lines, or the distance between computers, can all be studied using weighted graphs. Figure 2 displays weighted graphs that represent three ways to assign weights to the edges of a graph of a computer network, corresponding to distance, response time, and cost.

Several types of problems involving weighted graphs arise frequently. Determining a path of least length between two vertices in a network is one such problem. To be more specific, let the **length** of a path in a weighted graph be the sum of the weights of the edges of this path. (The reader should note that this use of the term *length* is different from the use of *length* to denote the number of edges in a path in a graph without weights.) The question is: What is a shortest path, that is, a path of least length, between two given vertices? For instance, in the airline system represented by the weighted graph shown in Figure 1, what is a shortest path in air distance between Boston and Los Angeles? What combinations of flights has the smallest total flight time (that is, total time in the air, not including time between flights) between Boston and Los Angeles? What is the cheapest fare between these two cities? In the computer network shown in Figure 2, what is a least expensive set of telephone lines needed to connect the computers in San Francisco with those in New York? Which set of telephone lines gives a fastest response time for communications between San Francisco and New York? Which set of lines has a shortest overall distance?

Another important problem involving weighted graphs asks for a circuit of shortest total length that visits every vertex of a complete graph exactly once. This is the famous *traveling salesperson problem*, which asks for an order in which a salesperson should visit each of the cities on his route exactly once so that he travels the minimum total distance. We will discuss the traveling salesperson problem later in this section.