Key Terms and Results

TERMS

undirected edge: an edge associated to a set $\{u, v\}$, where u and v are vertices

directed edge: an edge associated to an ordered pair (u, v), where u and v are vertices

multiple edges: distinct edges connecting the same vertices

multiple directed edges: distinct directed edges associated with the same ordered pair (u,v), where u and v are vertices

loop: an edge connecting a vertex with itself

undirected graph: a set of vertices and a set of undirected edges each of which is associated with a set of one or two of these vertices

simple graph: an undirected graph with no multiple edges or loops

multigraph: an undirected graph that may contain multiple edges but no loops

pseudograph: an undirected graph that may contain multiple edges and loops

directed graph: a set of vertices together with a set of directed edges each of which is associated with an ordered pair of vertices

directed multigraph: a graph with directed edges that may contain multiple directed edges

simple directed graph: a directed graph without loops or multiple directed edges

adjacent: two vertices are adjacent if there is an edge between them

incident: an edge is incident with a vertex if the vertex is an endpoint of that edge

deg v (degree of the vertex v in an undirected graph): the number of edges incident with v with loops counted twice

$\deg^-(v)$ (the in-degree of the vertex v in a graph with directed edges): the number of edges with v as their terminal vertex

$\deg^+(v)$ (the out-degree of the vertex v in a graph with directed edges): the number of edges with v as their initial vertex

underlying undirected graph of a graph with directed edges: the undirected graph obtained by ignoring the directions of the edges

K_n (complete graph on n vertices): the undirected graph with n vertices where each pair of vertices is connected by an edge

bipartite graph: a graph with vertex set that can be partitioned into subsets V_1 and V_2 so that each edge connects a vertex in V_1 and a vertex in V_2. The pair (V_1, V_2) is called a **bipartition** of V.

$K_{m,n}$ (complete bipartite graph): the graph with vertex set partitioned into a subset of m elements and a subset of n elements with two vertices connected by an edge if and only if one is in the first subset and the other is in the second subset

C_n (cycle of size n), $n \geq 3$: the graph with n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\ldots$, $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$

W_n (wheel of size n), $n \geq 3$: the graph obtained from C_n by adding a vertex and edges from this vertex to the original vertices in C_n

Q_n (n-cube), $n \geq 1$: the graph that has the 2^n bit strings of length n as its vertices and edges connecting every pair of bit strings that differ by exactly one bit

matching in a graph G: a set of edges such that no two edges have a common endpoint

complete matching M from V_1 to V_2: a matching such that every vertex in V_1 is an endpoint of an edge in M

maximum matching: a matching containing the most edges among all matchings in a graph

isolated vertex: a vertex of degree zero

pendant vertex: a vertex of degree one

regular graph: a graph where all vertices have the same degree

subgraph of a graph $G = (V, E)$: a graph (W, F), where W is a subset of V and F is a subset of E

$G_1 \cup G_2$ (union of G_1 and G_2): the graph $(V_1 \cup V_2, E_1 \cup E_2)$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

adjacency matrix: a matrix representing a graph using the adjacency of vertices

incidence matrix: a matrix representing a graph using the incidence of edges and vertices

isomorphic simple graphs: the simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one correspondence f from V_1 to V_2 such that $\{f(v_1), f(v_2)\} \in E_2$ if and only if $\{v_1, v_2\} \in E_1$ for all v_1 and v_2 in V_1

invariant for graph isomorphism: a property that isomorphic graphs either both have or both do not have

path from u to v in an undirected graph: a sequence of edges $e_1, e_2, \ldots, e_n$, where e_i is associated to $\{x_i, x_{i+1}\}$ for $i = 0, 1, \ldots, n$, where $x_0 = u$ and $x_{n+1} = v$

path from u to v in a graph with directed edges: a sequence of edges $e_1, e_2, \ldots, e_n$, where e_i is associated to (x_i, x_{i+1}) for $i = 0, 1, \ldots, n$, where $x_0 = u$ and $x_{n+1} = v$

simple path: a path that does not contain an edge more than once

circuit: a path of length $n \geq 1$ that begins and ends at the same vertex

connected graph: an undirected graph with the property that there is a path between every pair of vertices

cut vertex of G: a vertex v such that $G - v$ is disconnected

cut edge of G: an edge e such that $G - e$ is disconnected

nonseparable graph: a graph without a cut vertex

vertex cut of G: a subset V' of the set of vertices of G such that $G - V'$ is disconnected

$\kappa(G)$ (the vertex connectivity of G): the size of a smallest vertex cut of G

k-connected graph: a graph that has a vertex connectivity no smaller than k

edge cut of G: a set of edges E' of G such that $G - E'$ is disconnected

$\lambda(G)$ (the edge connectivity of G): the size of a smallest edge cut of G

connected component of a graph G: a maximal connected subgraph of G

strongly connected directed graph: a directed graph with the property that there is a directed path from every vertex to every vertex

strongly connected component of a directed graph G: a maximal strongly connected subgraph of G

Euler path: a path that contains every edge of a graph exactly once

Euler circuit: a circuit that contains every edge of a graph exactly once

Hamilton path: a path in a graph that passes through each vertex exactly once

Hamilton circuit: a circuit in a graph that passes through each vertex exactly once

weighted graph: a graph with numbers assigned to its edges

shortest-path problem: the problem of determining the path in a weighted graph such that the sum of the weights of the edges in this path is a minimum over all paths between specified vertices

traveling salesperson problem: the problem that asks for the circuit of shortest total length that visits every vertex of a weghted graph exactly once

planar graph: a graph that can be drawn in the plane with no crossings

regions of a representation of a planar graph: the regions the plane is divided into by the planar representation of the graph

elementary subdivision: the removal of an edge $\{u, v\}$ of an undirected graph and the addition of a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$

homeomorphic: two undirected graphs are homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions

graph coloring: an assignment of colors to the vertices of a graph so that no two adjacent vertices have the same color

chromatic number: the minimum number of colors needed in a coloring of a graph

RESULTS

The handshaking theorem: If $G = (V, E)$ be an undirected graph with m edges, then $2m = \sum_{v \in V} \deg(v)$.

Hall's marriage theorem: The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

There is an Euler circuit in a connected multigraph if and only if every vertex has even degree.

There is an Euler path in a connected multigraph if and only if at most two vertices have odd degree.

Dijkstra's algorithm: a procedure for finding a shortest path between two vertices in a weighted graph (see Section 10.6)

Euler's formula: $r = e - v + 2$ where r, e, and v are the number of regions of a planar representation, the number of edges, and the number of vertices, respectively, of a connected planar graph

Kuratowski's theorem: A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5. (Proof beyond the scope of this book.)

The four color theorem: Every planar graph can be colored using no more than four colors. (Proof far beyond the scope of this book!)

Review Questions

1. a) Define a simple graph, a multigraph, a pseudograph, a directed graph, and a directed multigraph.

b) Use an example to show how each of the types of graph in part (a) can be used in modeling. For example, explain how to model different aspects of a computer network or airline routes.

2. Give at least four examples of how graphs are used in modeling.

3. What is the relationship between the sum of the degrees of the vertices in an undirected graph and the number of edges in this graph? Explain why this relationship holds.

4. Why must there be an even number of vertices of odd degree in an undirected graph?

5. What is the relationship between the sum of the in-degrees and the sum of the out-degrees of the vertices in a directed graph? Explain why this relationship holds.

6. Describe the following families of graphs.

a) K_n, the complete graph on n vertices

b) $K_{m,n}$, the complete bipartite graph on m and n vertices

c) C_n, the cycle with n vertices

d) W_n, the wheel of size n

e) Q_n, the n-cube

7. How many vertices and how many edges are there in each of the graphs in the families in Question 6?

8. a) What is a bipartite graph?

b) Which of the graphs K_n, C_n, and W_n are bipartite?

c) How can you determine whether an undirected graph is bipartite?

9. a) Describe three different methods that can be used to represent a graph.

b) Draw a simple graph with at least five vertices and eight edges. Illustrate how it can be represented using the methods you described in part (a).

10. a) What does it mean for two simple graphs to be isomorphic?

b) What is meant by an invariant with respect to isomorphism for simple graphs? Give at least five examples of such invariants.

c) Give an example of two graphs that have the same numbers of vertices, edges, and degrees of vertices, but that are not isomorphic.

d) Is a set of invariants known that can be used to efficiently determine whether two simple graphs are isomorphic?

11. a) What does it mean for a graph to be connected?

b) What are the connected components of a graph?

12. a) Explain how an adjacency matrix can be used to represent a graph.

b) How can adjacency matrices be used to determine whether a function from the vertex set of a graph G to the vertex set of a graph H is an isomorphism?

c) How can the adjacency matrix of a graph be used to determine the number of paths of length r, where r is a positive integer, between two vertices of a graph?

13. a) Define an Euler circuit and an Euler path in an undirected graph.

b) Describe the famous Königsberg bridge problem and explain how to rephrase it in terms of an Euler circuit.

c) How can it be determined whether an undirected graph has an Euler path?

d) How can it be determined whether an undirected graph has an Euler circuit?

14. a) Define a Hamilton circuit in a simple graph.

b) Give some properties of a simple graph that imply that it does not have a Hamilton circuit.

15. Give examples of at least two problems that can be solved by finding the shortest path in a weighted graph.

16. a) Describe Dijkstra's algorithm for finding the shortest path in a weighted graph between two vertices.

b) Draw a weighted graph with at least 10 vertices and 20 edges. Use Dijkstra's algorithm to find the shortest path between two vertices of your choice in the graph.

17. a) What does it mean for a graph to be planar?

b) Give an example of a nonplanar graph.

18. a) What is Euler's formula for connected planar graphs?

b) How can Euler's formula for planar graphs be used to show that a simple graph is nonplanar?

19. State Kuratowski's theorem on the planarity of graphs and explain how it characterizes which graphs are planar.

20. a) Define the chromatic number of a graph.

b) What is the chromatic number of the graph K_n when n is a positive integer?

c) What is the chromatic number of the graph C_n when n is an integer greater than 2?

d) What is the chromatic number of the graph $K_{m,n}$ when m and n are positive integers?

21. State the four color theorem. Are there graphs that cannot be colored with four colors?

22. Explain how graph coloring can be used in modeling. Use at least two different examples.

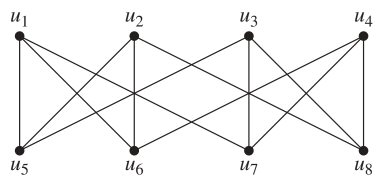
Supplementary Exercises

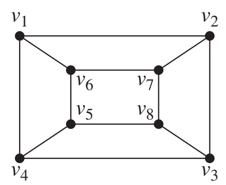
1. How many edges does a 50-regular graph with 100 vertices have?

2. How many nonisomorphic subgraphs does K_3 have?

In Exercises 3–5 determine whether two given graphs are isomorphic.

3.





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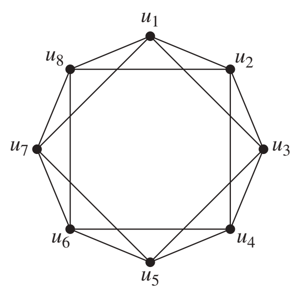

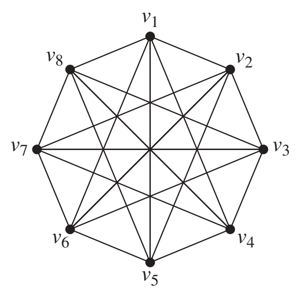

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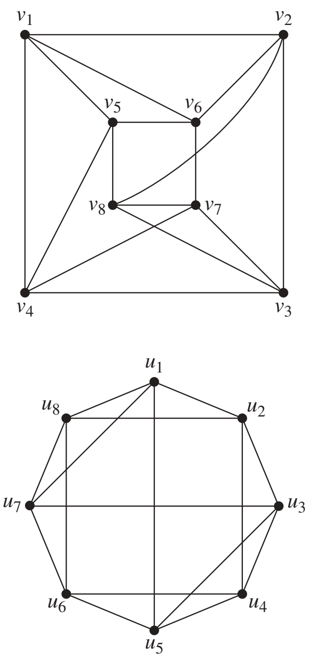


The **complete m-partite graph** $K_{n_1,n_2,\ldots,n_m}$ has vertices partitioned into m subsets of $n_1, n_2, \ldots, n_m$ elements each, and vertices are adjacent if and only if they are in different subsets in the partition.

6. Draw these graphs.

   a) $K_{1,2,3}$ b) $K_{2,2,2}$ c) $K_{1,2,2,3}$

*7. How many vertices and how many edges does the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$ have?

8. Prove or disprove that there are always two vertices of the same degree in a finite multigraph having at least two vertices.

9. Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that

   a) $N(A \cup B) = N(A) \cup N(B)$.

   b) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

10. Let $G = (V, E)$ be an undirected graph. Show that

   a) $|N(v)| \leq \deg(v)$ for all $v \in V$.

   b) $|N(v)| = \deg v$ for all $v \in V$ if and only if G is a simple graph.

Suppose that $S_1, S_2, \ldots, S_n$ is a collection of subsets of a set S where n is a positive integer. A **system of distinct representatives (SDR)** for this family is an ordered n-tuple $(a_1, a_2, \ldots, a_n)$ with the property that $a_i \in S_i$ for $i = 1, 2, \ldots, n$ and $a_i \neq a_j$ for all $i \neq j$.

11. Find a SDR for the sets $S_1 = \{a, c, m, e\}$, $S_2 = \{m, a, c, e\}$, $S_3 = \{a, p, e, x\}$, $S_4 = \{x, e, n, a\}$, $S_5 = \{n, a, m, e\}$, and $S_6 = \{e, x, a, m\}$.

12. Use Hall's marriage theorem to show that a collection of finite subsets $S_1, S_2, \ldots, S_n$ of a set S has a SDR $(a_1, a_2, \ldots, a_n)$ if and only if $|\bigcup_{i \in I} S_i| \geq |I|$ for all subsets I of $\{1, 2, \ldots, n\}$.

13. a) Use Exercise 12 to show that the collection of sets $S_1 = \{a, b, c\}$, $S_2 = \{b, c, d\}$, $S_3 = \{a, b, d\}$, $S_4 = \{b, c, d\}$ has a SDR without finding one explicitly.

   b) Find a SDR for the family of four sets in part (a).

14. Use Exercise 12 to show that collection of sets $S_1 = \{a, b, c\}$, $S_2 = \{a, c\}$, $S_3 = \{c, d, e\}$, $S_4 = \{b, c\}$, $S_5 = \{d, e, f\}$, $S_6 = \{a, c, e\}$, and $S_7 = \{a, b\}$ does not have a SDR.

The **clustering coefficient** $C(G)$ of a simple graph G is the probability that if u and v are neighbors and v and w are neighbors, then u and w are neighbors, where u, v, and w are distinct vertices of G.

15. We say that three vertices u, v, and w of a simple graph G form a triangle if there are edges connecting all three pairs of these vertices. Find a formula for $C(G)$ in terms of the number of triangles in G and the number of paths of length two in the graph. [*Hint:* Count each triangle in the graph once for each order of three vertices that form it.]

16. Find the clustering coefficient of each of the graphs in Exercise 20 of Section 10.2.

17. Explain what the clustering coefficient measures in each of these graphs.

   a) the Hollywood graph

   b) the graph of Facebook friends

   c) the academic collaboration graph for researchers in graph theory

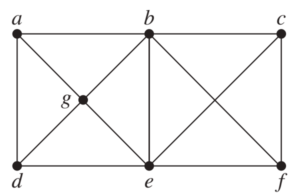
   d) the protein interaction graph for a human cell

   e) the graph representing the routers and communications links that make up the worldwide Internet

18. For each of the graphs in Exercise 17, explain whether you would expect its clustering coefficient to be closer to 0.01 or to 0.10 and why you expect this.

A **clique** in a simple undirected graph is a complete subgraph that is not contained in any larger complete subgraph. In Exercises 19–21 find all cliques in the graph shown.

19.



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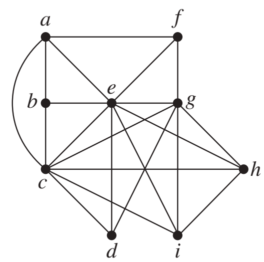

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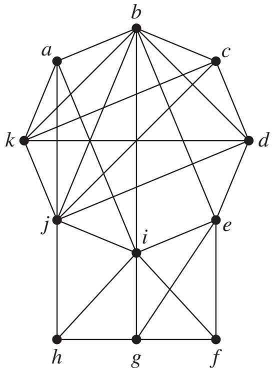
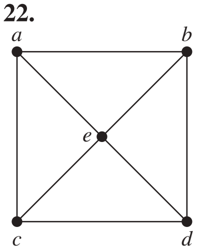


A **dominating set** of vertices in a simple graph is a set of vertices such that every other vertex is adjacent to at least one vertex of this set. A dominating set with the least number of vertices is called a **minimum dominating set**. In Exercises 22–24 find a minimum dominating set for the given graph.

22.



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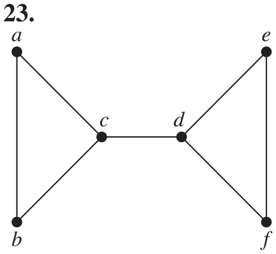

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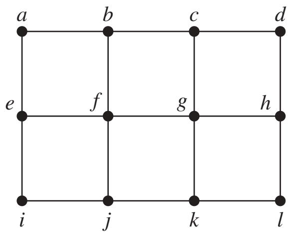
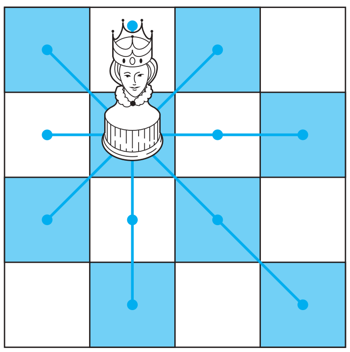


A simple graph can be used to determine the minimum number of queens on a chessboard that control the entire chessboard. An $n \times n$ chessboard has n^2 squares in an $n \times n$ configuration. A queen in a given position controls all squares in the same row, the same column, and on the two diagonals containing this square, as illustrated. The appropriate simple graph has n^2 vertices, one for each square, and two vertices are adjacent if a queen in the square represented by one of the vertices controls the square represented by the other vertex.



The squares controlled by a queen.

25. Construct the simple graph representing the $n \times n$ chessboard with edges representing the control of squares by queens for

a) $n = 3$. **b)** $n = 4$.

26. Explain how the concept of a minimum dominating set applies to the problem of determining the minimum number of queens controlling an $n \times n$ chessboard.

****27.** Find the minimum number of queens controlling an $n \times n$ chessboard for

a) $n = 3$. **b)** $n = 4$. **c)** $n = 5$.

28. Suppose that G_1 and H_1 are isomorphic and that G_2 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.

29. Show that each of these properties is an invariant that isomorphic simple graphs either both have or both do not have.

a) connectedness

b) the existence of a Hamilton circuit

c) the existence of an Euler circuit

d) having crossing number C

e) having n isolated vertices

f) being bipartite

30. How can the adjacency matrix of $\overline{G}$ be found from the adjacency matrix of G, where G is a simple graph?

31. How many nonisomorphic connected bipartite simple graphs are there with four vertices?

***32.** How many nonisomorphic simple connected graphs with five vertices are there

a) with no vertex of degree more than two?

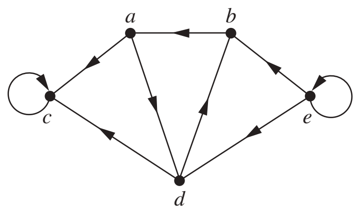
b) with chromatic number equal to four?

c) that are nonplanar?

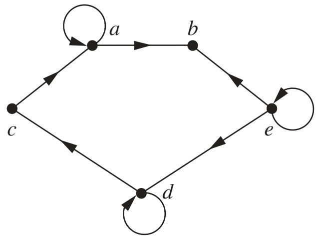
A directed graph is **self-converse** if it is isomorphic to its converse.

33. Determine whether the following graphs are self-converse.

a)



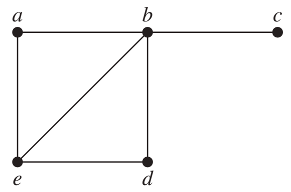
b)



34. Show that if the directed graph G is self-converse and H is a directed graph isomorphic to G, then H is also self-converse.

An **orientation** of an undirected simple graph is an assignment of directions to its edges such that the resulting directed graph is strongly connected. When an orientation of an undirected graph exists, this graph is called **orientable**. In Exercises 35–37 determine whether the given simple graph is orientable.

35.



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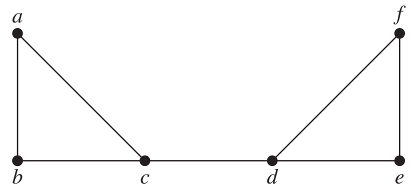

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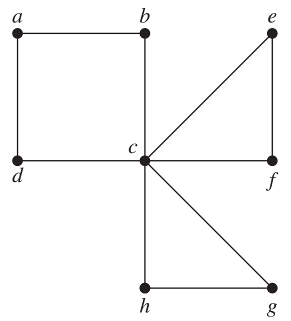

38. Because traffic is growing heavy in the central part of a city, traffic engineers are planning to change all the streets, which are currently two-way, into one-way streets. Explain how to model this problem.

***39.** Show that a graph is not orientable if it has a cut edge.

A **tournament** is a simple directed graph such that if u and v are distinct vertices in the graph, exactly one of (u, v) and (v, u) is an edge of the graph.

40. How many different tournaments are there with n vertices?

41. What is the sum of the in-degree and out-degree of a vertex in a tournament?

***42.** Show that every tournament has a Hamilton path.

43. Given two chickens in a flock, one of them is dominant. This defines the **pecking order** of the flock. How can a tournament be used to model pecking order?

44. Suppose that a connected graph G has n vertices and vertex connectivity $\kappa(G) = k$. Show that G must have at least $\lceil kn/2 \rceil$ edges.

A connected graph $G = (V, E)$ with n vertices and m edges is said to have **optimal connectivity** if $\kappa(G) = \lambda(G) = \min_{v \in V} \deg v = 2m/n$.

45. Show that a connected graph with optimal connectivity must be regular.

46. Show these graphs have optimal connectivity.

a) C_n for $n \geq 3$
b) K_n for $n \geq 3$
c) $K_{r,r}$ for $r \geq 2$

***47.** Find the two nonisomorphic simple graphs with six vertices and nine edges that have optimal connectivity.

48. Suppose that G is a connected multigraph with $2k$ vertices of odd degree. Show that there exist k subgraphs that have G as their union, where each of these subgraphs has an Euler path and where no two of these subgraphs have an edge in common. [*Hint:* Add k edges to the graph connecting pairs of vertices of odd degree and use an Euler circuit in this larger graph.]

In Exercises 49 and 50 we consider a puzzle posed by Petković in [Pe09] (based on a problem in [AvCh80]). Suppose that King Arthur has gathered his $2n$ knights of the Round Table for an important council. Every two knights are either friends or enemies, and each knight has no more than $n - 1$ enemies among the other $2n - 1$ knights. The puzzle asks whether King Arthur can seat his knights around the Round Table so that each knight has two friends for his neighbors.

49. a) Show that the puzzle can be reduced to determining whether there is a Hamilton circuit in the graph in which each knight is represented by a vertex and two knights are connected in the graph if they are friends.

b) Answer the question posed in the puzzle. [*Hint:* Use Dirac's theorem.]

50. Suppose that are eight knights Alynore, Bedivere, Degore, Gareth, Kay, Lancelot, Perceval, and Tristan.

Their lists of enemies are A (D, G, P), B (K, P, T), D (A, G, L), G (A, D, T), K (B, L, P), L (D, K, T), P (A, B, K), T (B, G, L), where we have represented each knight by the first letter of his name and shown the list of enemies of that knight following this first letter. Draw the graph representing these eight knights and their friends and find a seating arrangement where each knight sits next to two friends.

***51.** Let G be a simple graph with n vertices. The **bandwidth** of G, denoted by $B(G)$, is the minimum, over all permutations $a_1, a_2, \ldots, a_n$ of the vertices of G, of $\max\{|i - j| \mid a_i$ and a_j are adjacent$\}$. That is, the bandwidth is the minimum over all listings of the vertices of the maximum difference in the indices assigned to adjacent vertices. Find the bandwidths of these graphs.

a) K_5
b) $K_{1,3}$
c) $K_{2,3}$
d) $K_{3,3}$
e) Q_3
f) C_5

***52.** The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The **radius** of a graph is the minimum over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

a) K_6. **b)** $K_{4,5}$. **c)** Q_3. **d)** C_6.

***53.** **a)** Show that if the diameter of the simple graph G is at least four, then the diameter of its complement $\overline{G}$ is no more than two.

b) Show that if the diameter of the simple graph G is at least three, then the diameter of its complement $\overline{G}$ is no more than three.

***54.** Suppose that a multigraph has $2m$ vertices of odd degree. Show that any circuit that contains every edge of the graph must contain at least m edges more than once.

55. Find the second shortest path between the vertices a and z in Figure 3 of Section 10.6.

56. Devise an algorithm for finding the second shortest path between two vertices in a simple connected weighted graph.

57. Find the shortest path between the vertices a and z that passes through the vertex f in the weighted graph in Exercise 3 in Section 10.6.

58. Devise an algorithm for finding the shortest path between two vertices in a simple connected weighted graph that passes through a specified third vertex.

***59.** Show that if G is a simple graph with at least 11 vertices, then either G or $\overline{G}$, the complement of G, is nonplanar.

A set of vertices in a graph is called **independent** if no two vertices in the set are adjacent. The **independence number** of a graph is the maximum number of vertices in an independent set of vertices for the graph.

***60.** What is the independence number of

a) K_n? **b)** C_n? **c)** Q_n? **d)** $K_{m,n}$?

61. Show that the number of vertices in a simple graph is less than or equal to the product of the independence number and the chromatic number of the graph.

62. Show that the chromatic number of a graph is less than or equal to $n - i + 1$, where n is the number of vertices in the graph and i is the independence number of this graph.

63. Suppose that to generate a random simple graph with n vertices we first choose a real number p with $0 \leq p \leq 1$. For each of the $C(n, 2)$ pairs of distinct vertices we generate a random number x between 0 and 1. If $0 \leq x \leq p$, we connect these two vertices with an edge; otherwise these vertices are not connected.

a) What is the probability that a graph with m edges where $0 \leq m \leq C(n, 2)$ is generated?

b) What is the expected number of edges in a randomly generated graph with n vertices if each edge is included with probability p?

c) Show that if $p = 1/2$ then every simple graph with n vertices is equally likely to be generated.

A property retained whenever additional edges are added to a simple graph (without adding vertices) is called **monotone increasing**, and a property that is retained whenever edges are removed from a simple graph (without removing vertices) is called **monotone decreasing**.

64. For each of these properties, determine whether it is monotone increasing and determine whether it is monotone decreasing.

a) The graph G is connected.

b) The graph G is not connected.

c) The graph G has an Euler circuit.

d) The graph G has a Hamilton circuit.

e) The graph G is planar.

f) The graph G has chromatic number four.

g) The graph G has radius three.

h) The graph G has diameter three.

65. Show that the graph property P is monotone increasing if and only if the graph property Q is monotone decreasing where Q is the property of not having property P.

****66.** Suppose that P is a monotone increasing property of simple graphs. Show that the probability a random graph with n vertices has property P is a monotonic nondecreasing function of p, the probability an edge is chosen to be in the graph.

Computer Projects

Write programs with these input and output.

1. Given the vertex pairs associated to the edges of an undirected graph, find the degree of each vertex.

2. Given the ordered pairs of vertices associated to the edges of a directed graph, determine the in-degree and out-degree of each vertex.

3. Given the list of edges of a simple graph, determine whether the graph is bipartite.

4. Given the vertex pairs associated to the edges of a graph, construct an adjacency matrix for the graph. (Produce a version that works when loops, multiple edges, or directed edges are present.)

5. Given an adjacency matrix of a graph, list the edges of this graph and give the number of times each edge appears.

6. Given the vertex pairs associated to the edges of an undirected graph and the number of times each edge appears, construct an incidence matrix for the graph.

7. Given an incidence matrix of an undirected graph, list its edges and give the number of times each edge appears.

8. Given a positive integer n, generate a simple graph with n vertices by producing an adjacency matrix for the graph so that all simple graphs with n vertices are equally likely to be generated.

9. Given a positive integer n, generate a simple directed graph with n vertices by producing an adjacency matrix for the graph so that all simple directed graphs with n vertices are equally likely to be generated.

10. Given the lists of edges of two simple graphs with no more than six vertices, determine whether the graphs are isomorphic.

11. Given an adjacency matrix of a graph and a positive integer n, find the number of paths of length n between two vertices. (Produce a version that works for directed and undirected graphs.)

*12. Given the list of edges of a simple graph, determine whether it is connected and find the number of connected components if it is not connected.

13. Given the vertex pairs associated to the edges of a multigraph, determine whether it has an Euler circuit and, if not, whether it has an Euler path. Construct an Euler path or circuit if it exists.

*14. Given the ordered pairs of vertices associated to the edges of a directed multigraph, construct an Euler path or Euler circuit, if such a path or circuit exists.

**15. Given the list of edges of a simple graph, produce a Hamilton circuit, or determine that the graph does not have such a circuit.

**16. Given the list of edges of a simple graph, produce a Hamilton path, or determine that the graph does not have such a path.

17. Given the list of edges and weights of these edges of a weighted connected simple graph and two vertices in this graph, find the length of a shortest path between them using Dijkstra's algorithm. Also, find a shortest path.

18. Given the list of edges of an undirected graph, find a coloring of this graph using the algorithm given in the exercise set of Section 10.8.

19. Given a list of students and the courses that they are enrolled in, construct a schedule of final exams.

20. Given the distances between pairs of television stations and the minimum allowable distance between stations, assign frequencies to these stations.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Display all simple graphs with four vertices.

2. Display a full set of nonisomorphic simple graphs with six vertices.

3. Display a full set of nonisomorphic directed graphs with four vertices.

4. Generate at random 10 different simple graphs each with 20 vertices so that each such graph is equally likely to be generated.

5. Construct a Gray code where the code words are bit strings of length six.

6. Construct knight's tours on chessboards of various sizes.

7. Determine whether each of the graphs you generated in Exercise 4 of this set is planar. If you can, determine the thickness of each of the graphs that are not planar.

8. Determine whether each of the graphs you generated in Exercise 4 of this set is connected. If a graph is not connected, determine the number of connected components of the graph.

9. Generate at random simple graphs with 10 vertices. Stop when you have constructed one with an Euler circuit. Display an Euler circuit in this graph.

10. Generate at random simple graphs with 10 vertices. Stop when you have constructed one with a Hamilton circuit. Display a Hamilton circuit in this graph.

11. Find the chromatic number of each of the graphs you generated in Exercise 4 of this set.

**12. Find the shortest path a traveling salesperson can take to visit each of the capitals of the 50 states in the United States, traveling by air between cities in a straight line.

*13. Estimate the probability that a randomly generated simple graph with n vertices is connected for each positive integer n not exceeding ten by generating a set of random simple graphs and determining whether each is connected.

**14. Work on the problem of determining whether the crossing number of $K_{7,7}$ is 77, 79, or 81. It is known that it equals one of these three values.

Writing Projects

Respond to these with essays using outside sources.

1. Describe the origins and development of graph theory prior to the year 1900.
2. Discuss the applications of graph theory to the study of ecosystems.
3. Discuss the applications of graph theory to sociology and psychology.
4. Discuss what can be learned by investigating the properties of the web graph.
5. Explain what community structure is in a graph representing a network, such as a social network, a computer network, an information network, or a biological network. Define what a community in such a graph is, and explain what communities represent in graphs representing the types of networks listed.
6. Describe some of the algorithms used to detect communities in graphs representing networks of the types listed in Question 5.
7. Describe algorithms for drawing a graph on paper or on a display given the vertices and edges of the graph. What considerations arise in drawing a graph so that it has the best appearance for understanding its properties?
8. Explain how graph theory can help uncover networks of criminals or terrorists by studying relevant social and communication networks.
9. What are some of the capabilities that a software tool for inputting, displaying, and manipulating graphs should have? Which of these capabilities do available tools have?
10. Describe some of the algorithms available for determining whether two graphs are isomorphic and the computational complexity of these algorithms. What is the most efficient such algorithm currently known?
11. What is the subgraph isomorphism problem and what are some of its important applications, including those to chemistry, bioinformatics, electronic circuit design, and computer vision?
12. Explain what the area of graph mining, an important area of data mining, is and describe some of the basic techniques used in graph mining.
13. Describe how Euler paths can be used to help determine DNA sequences.
14. Define *de Bruijn sequences* and discuss how they arise in applications. Explain how de Bruijn sequences can be constructed using Euler circuits.
15. Describe the *Chinese postman problem* and explain how to solve this problem.
16. Describe some of the different conditions that imply that a graph has a Hamilton circuit.
17. Describe some of the strategies and algorithms used to solve the traveling salesperson problem.
18. Describe several different algorithms for determining whether a graph is planar. What is the computational complexity of each of these algorithms?
19. In modeling, very large scale integration (VLSI) graphs are sometimes embedded in a book, with the vertices on the spine and the edges on pages. Define the *book number* of a graph and find the book number of various graphs including K_n for $n = 3, 4, 5$, and 6.
20. Discuss the history of the four color theorem.
21. Describe the role computers played in the proof of the four color theorem. How can we be sure that a proof that relies on a computer is correct?
22. Describe and compare several different algorithms for coloring a graph, in terms of whether they produce a coloring with the least number of colors possible and in terms of their complexity.
23. Explain how graph multicolorings can be used in a variety of different models.
24. Describe some of the applications of edge colorings.
25. Explain how the theory of random graphs can be used in nonconstructive existence proofs of graphs with certain properties.