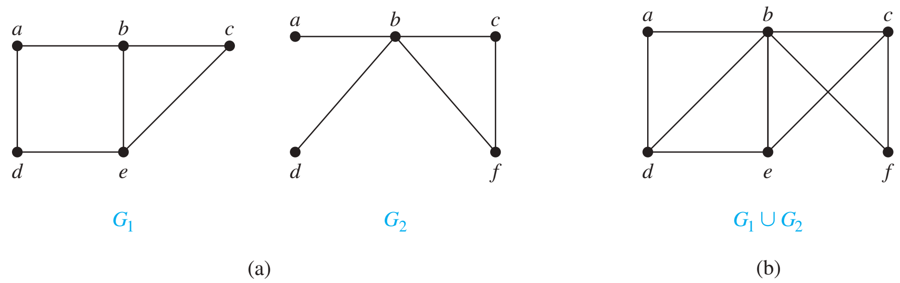


FIGURE 17 **(a) The simple graphs G_1 and G_2. (b) Their union $G_1 \cup G_2$.**

Definition 9

The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

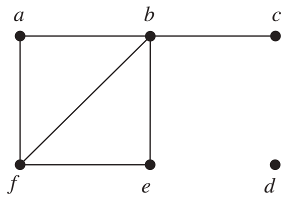
EXAMPLE 20 Find the union of the graphs G_1 and G_2 shown in Figure 17(a). ◀

Solution: The vertex set of the union $G_1 \cup G_2$ is the union of the two vertex sets, namely, $\{a, b, c, d, e, f\}$. The edge set of the union is the union of the two edge sets. The union is displayed in Figure 17(b).

Exercises

In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

1.



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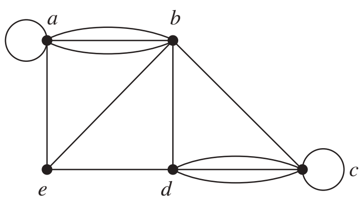

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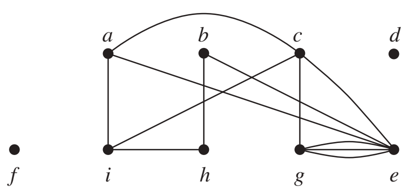


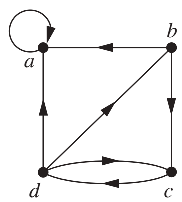
4. Find the sum of the degrees of the vertices of each graph in Exercises 1–3 and verify that it equals twice the number of edges in the graph.

5. Can a simple graph exist with 15 vertices each of degree five?

6. Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

In Exercises 7–9 determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

7.



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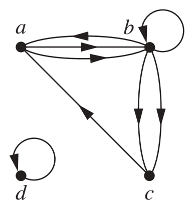

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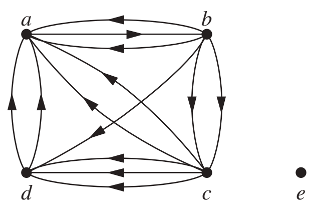


10. For each of the graphs in Exercises 7–9 determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of edges in the graph.

11. Construct the underlying undirected graph for the graph with directed edges in Figure 2.

12. What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the world? What does the neighborhood of a vertex in this graph represent? What do isolated and pendant vertices in this graph represent? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model?

13. What does the degree of a vertex represent in an academic collaboration graph? What does the neighborhood of a vertex represent? What do isolated and pendant vertices represent?

14. What does the degree of a vertex in the Hollywood graph represent? What does the neighborhood of a vertex represent? What do the isolated and pendant vertices represent?

15. What do the in-degree and the out-degree of a vertex in a telephone call graph, as described in Example 4 of Section 10.1, represent? What does the degree of a vertex in the undirected version of this graph represent?

16. What do the in-degree and the out-degree of a vertex in the web graph, as described in Example 5 of Section 10.1, represent?

17. What do the in-degree and the out-degree of a vertex in a directed graph modeling a round-robin tournament represent?

18. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

19. Use Exercise 18 to show that in a group of people, there must be two people who are friends with the same number of other people in the group.

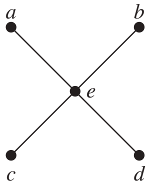
20. Draw these graphs.

a) K_7 **b)** $K_{1,8}$ **c)** $K_{4,4}$

d) C_7 **e)** W_7 **f)** Q_4

In Exercises 21–25 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

21.



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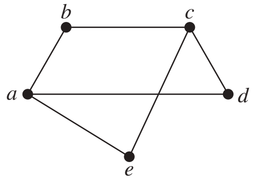

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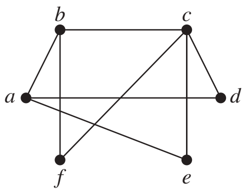

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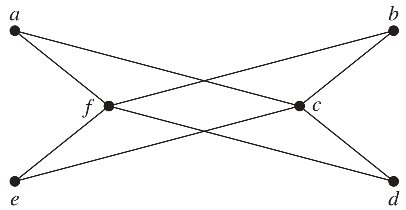

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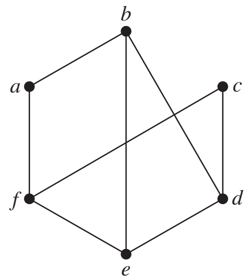


26. For which values of n are these graphs bipartite?

a) K_n **b)** C_n **c)** W_n **d)** Q_n

27. Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

a) Use a bipartite graph to model the four employees and their qualifications.

b) Use Hall's theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support.

c) If an assignment of employees to support areas so that each employee is assigned to one support area exists, find one.

28. Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of six responsiblities: planning, publicity, sales, marketing, development, and industry relations. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning or development; Smith could do publicity, sales, or industry relations; Chou could do planning, sales, or industry relations; and Macintyre could do planning, publicity, sales, or industry relations.

a) Model the capabilities of these employees using a bipartite graph.

b) Find an assignment of responsibilites such that each employee is assigned one responsibility.

c) Is the matching of responsibilities you found in part (b) a complete matching? Is it a maximum matching?

29. Suppose that there are five young women and five young men on an island. Each man is willing to marry some of the women on the island and each woman is willing to marry any man who is willing to marry her. Suppose that Sandeep is willing to marry Tina and Vandana; Barry is willing to marry Tina, Xia, and Uma; Teja is willing to marry Tina and Zelda; Anil is willing to marry Vandana and Zelda; and Emilio is willing to marry Tina and Zelda. Use Hall's theorem to show there is no matching of the young men and young women on the island such that each young man is matched with a young woman he is willing to marry.

30. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

a) Model the possible marriages on the island using a bipartite graph.

b) Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.

c) Is the matching you found in part (b) a complete matching? Is it a maximum matching?

Each of Exercises 31–33 can be solved using Hall's theorem.

***31.** Suppose there is an integer k such that every man on a desert island is willing to marry exactly k of the women on the island and every woman on the island is willing to marry exactly k of the men. Also, suppose that a man is willing to marry a woman if and only if she is willing to marry him. Show that it is possible to match the men and women on the island so that everyone is matched with someone that they are willing to marry.

***32.** Suppose that $2n$ tennis players compete in a round-robin tournament. Every player has exactly one match with every other player during $2n - 1$ consecutive days. Every match has a winner and a loser. Show that it is possible to select a winning player each day without selecting the same player twice.

***33.** Suppose that m people are selected as prize winners in a lottery, where each winner can select two prizes from a collection of different prizes. Show if there are $2m$ prizes that every winner wants, then every winner is able to select two prizes that they want.

***34.** In this exercise we prove a theorem of Øystein Ore. Suppose that $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2) and that $A \subseteq V_1$. Show that the maximum number of vertices of V_1 that are the endpoints of a matching of G equals $|V_1| - \max_{A \subseteq V_1} \operatorname{def}(A)$, where $\operatorname{def}(A) = |A| - |N(A)|$. (Here, def$(A)$ is called the **deficiency** of A.) [*Hint:* Form a larger graph by adding $\max_{A \subseteq V_1} \operatorname{def}(A)$ new vertices to V_2 and connect all of them to the vertices of V_1.]

35. For the graph G in Exercise 1 find

a) the subgraph induced by the vertices a, b, c, and f.

b) the new graph G_1 obtained from G by contracting the edge connecting b and f.

36. Let n be a positive integer. Show that a subgraph induced by a nonempty subset of the vertex set of K_n is a complete graph.

37. How many vertices and how many edges do these graphs have?

a) K_n **b)** C_n **c)** W_n
d) $K_{m,n}$ **e)** Q_n

The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. For example, the degree sequence of the graph G in Example 1 is 4, 4, 4, 3, 2, 1, 0.

38. Find the degree sequences for each of the graphs in Exercises 21–25.

39. Find the degree sequence of each of the following graphs.

a) K_4 **b)** C_4 **c)** W_4
d) $K_{2,3}$ **e)** Q_3

40. What is the degree sequence of the bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.

41. What is the degree sequence of K_n, where n is a positive integer? Explain your answer.

42. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

43. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

A sequence $d_1, d_2, \ldots, d_n$ is called **graphic** if it is the degree sequence of a simple graph.

44. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0 **b)** 6, 5, 4, 3, 2, 1 **c)** 2, 2, 2, 2, 2, 2
d) 3, 3, 3, 2, 2, 2 **e)** 3, 3, 2, 2, 2, 2 **f)** 1, 1, 1, 1, 1, 1
g) 5, 3, 3, 3, 3, 3 **h)** 5, 5, 4, 3, 2, 1

45. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 3, 3, 3, 3, 2 **b)** 5, 4, 3, 2, 1 **c)** 4, 4, 3, 2, 1
d) 4, 4, 3, 3, 3 **e)** 3, 2, 2, 1, 0 **f)** 1, 1, 1, 1, 1

***46.** Suppose that $d_1, d_2, \ldots, d_n$ is a graphic sequence. Show that there is a simple graph with vertices $v_1, v_2, \ldots, v_n$ such that $\deg(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$.

***47.** Show that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ so that the terms are in nonincreasing order is a graphic sequence.

***48.** Use Exercise 47 to construct a recursive algorithm for determining whether a nonincreasing sequence of positive integers is graphic.

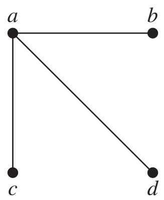
49. Show that every nonincreasing sequence of nonnegative integers with an even sum of its terms is the degree sequence of a pseudograph, that is, an undirected graph where loops are allowed. [*Hint:* Construct such a graph by first adding as many loops as possible at each vertex. Then add additional edges connecting vertices of odd degree. Explain why this construction works.]

50. How many subgraphs with at least one vertex does K_2 have?

51. How many subgraphs with at least one vertex does K_3 have?

52. How many subgraphs with at least one vertex does W_3 have?

53. Draw all subgraphs of this graph.



54. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that

a) $2e/v \geq m$. **b)** $2e/v \leq M$.

A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called n-**regular** if every vertex in this graph has degree n.

55. For which values of n are these graphs regular?

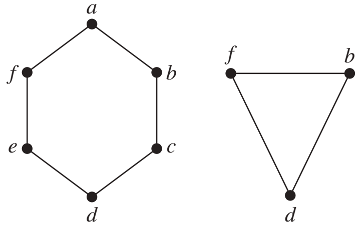
a) K_n **b)** C_n **c)** W_n **d)** Q_n

56. For which values of m and n is $K_{m,n}$ regular?

57. How many vertices does a regular graph of degree four with 10 edges have?

In Exercises 58–60 find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

58.



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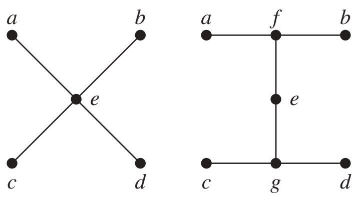

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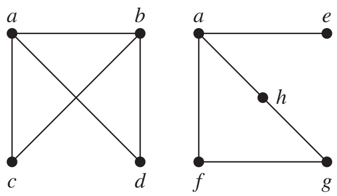

61. The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of these graphs.

a) $\overline{K_n}$ **b)** $\overline{K_{m,n}}$ **c)** $\overline{C_n}$ **d)** $\overline{Q_n}$

62. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?

63. If the simple graph G has v vertices and e edges, how many edges does $\overline{G}$ have?

64. If the degree sequence of the simple graph G is 4, 3, 3, 2, 2, what is the degree sequence of $\overline{G}$?

65. If the degree sequence of the simple graph G is $d_1, d_2, \ldots, d_n$, what is the degree sequence of $\overline{G}$?

***66.** Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

67. Show that if G is a simple graph with n vertices, then the union of G and $\overline{G}$ is K_n.

***68.** Describe an algorithm to decide whether a graph is bipartite based on the fact that a graph is bipartite if and only if it is possible to color its vertices two different colors so that no two vertices of the same color are adjacent.

The **converse** of a directed graph $G = (V, E)$, denoted by G^{conv}, is the directed graph (V, F), where the set F of edges of G^{conv} is obtained by reversing the direction of each edge in E.

69. Draw the converse of each of the graphs in Exercises 7–9 in Section 10.1.

70. Show that $(G^{conv})^{conv} = G$ whenever G is a directed graph.

71. Show that the graph G is its own converse if and only if the relation associated with G (see Section 9.3) is symmetric.

72. Show that if a bipartite graph $G = (V, E)$ is n-regular for some positive integer n (see the preamble to Exercise 55) and (V_1, V_2) is a bipartition of V, then $|V_1| = |V_2|$. That is, show that the two sets in a bipartition of the vertex set of an n-regular graph must contain the same number of vertices.

73. Draw the mesh network for interconnecting nine parallel processors.

74. In a variant of a mesh network for interconnecting $n = m^2$ processors, processor $P(i, j)$ is connected to the four processors $P((i \pm 1) \textbf{ mod } m, j)$ and $P(i, (j \pm 1) \textbf{ mod } m)$, so that connections wrap around the edges of the mesh. Draw this variant of the mesh network for 16 processors.

75. Show that every pair of processors in a mesh network of $n = m^2$ processors can communicate using $O(\sqrt{n}) = O(m)$ hops between directly connected processors.