$$\begin{aligned} P - Q &= \{\max(4-3,0)\cdot a, \max(1-4,0)\cdot b, \max(3-0,0)\cdot c, \max(0-2,0)\cdot d\} \\ &= \{1\cdot a, 0\cdot b, 3\cdot c, 0\cdot d\} = \{1\cdot a, 3\cdot c\}, \text{ and} \end{aligned}$$

$$\begin{aligned} P + Q &= \{(4+3)\cdot a, (1+4)\cdot b, (3+0)\cdot c, (0+2)\cdot d\} \\ &= \{7\cdot a, 5\cdot b, 3\cdot c, 2\cdot d\}. \end{aligned}$$

◀

Exercises

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$ **b)** $A \cup B$
c) $A - B$ **d)** $B - A$

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school
b) the set of sophomores at your school who are not taking discrete mathematics
c) the set of students at your school who either are sophomores or are taking discrete mathematics
d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. **b)** $A \cap B$.
c) $A - B$. **d)** $B - A$.

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$. **b)** $A \cap B$.
c) $A - B$. **d)** $B - A$.

In Exercises 5–10 assume that A is a subset of some underlying universal set U.

5. Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.

6. Prove the identity laws in Table 1 by showing that

a) $A \cup \emptyset = A$. **b)** $A \cap U = A$.

7. Prove the domination laws in Table 1 by showing that

a) $A \cup U = U$. **b)** $A \cap \emptyset = \emptyset$.

8. Prove the idempotent laws in Table 1 by showing that

a) $A \cup A = A$. **b)** $A \cap A = A$.

9. Prove the complement laws in Table 1 by showing that

a) $A \cup \overline{A} = U$. **b)** $A \cap \overline{A} = \emptyset$.

10. Show that

a) $A - \emptyset = A$. **b)** $\emptyset - A = \emptyset$.

11. Let A and B be sets. Prove the commutative laws from Table 1 by showing that

a) $A \cup B = B \cup A$.
b) $A \cap B = B \cap A$.

12. Prove the first absorption law from Table 1 by showing that if A and B are sets, then $A \cup (A \cap B) = A$.

13. Prove the second absorption law from Table 1 by showing that if A and B are sets, then $A \cap (A \cup B) = A$.

14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

a) by showing each side is a subset of the other side.
b) using a membership table.

16. Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$. **b)** $A \subseteq (A \cup B)$.
c) $A - B \subseteq A$. **d)** $A \cap (B - A) = \emptyset$.
e) $A \cup (B - A) = A \cup B$.

17. Show that if A and B are sets in a universe U then $A \subseteq B$ if and only if $\overline{A} \cup B = U$.

18. Given sets A and B in a universe U, draw the Venn diagrams of each of these sets.

a) $A \rightarrow B = \{x \in U \mid x \in A \rightarrow x \in B\}$
b) $A \leftrightarrow B = \{x \in U \mid x \in A \leftrightarrow x \in B\}$

19. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

a) by showing each side is a subset of the other side.
b) using a membership table.

20. Let A, B, and C be sets. Show that

a) $(A \cup B) \subseteq (A \cup B \cup C)$.
b) $(A \cap B \cap C) \subseteq (A \cap B)$.
c) $(A - B) - C \subseteq A - C$.
d) $(A - C) \cap (C - B) = \emptyset$.
e) $(B - A) \cup (C - A) = (B \cup C) - A$.

21. Show that if A and B are sets, then

a) $A - B = A \cap \overline{B}$.
b) $(A \cap B) \cup (A \cap \overline{B}) = A$.

22. Show that if A and B are sets with $A \subseteq B$, then

a) $A \cup B = B$.
b) $A \cap B = A$.

23. Prove the first associative law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.

24. Prove the second associative law from Table 1 by showing that if A, B, and C are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.

25. Prove the first distributive law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

26. Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

27. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find

a) $A \cap B \cap C$. **b)** $A \cup B \cup C$.
c) $(A \cup B) \cap C$. **d)** $(A \cap B) \cup C$.

28. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B \cup C)$ **b)** $\overline{A} \cap \overline{B} \cap \overline{C}$
c) $(A - B) \cup (A - C) \cup (B - C)$

29. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$ **b)** $(A \cap B) \cup (A \cap C)$
c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

30. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.

a) $(A \cap B) \cup (C \cap D)$ **b)** $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$
c) $A - (B \cap C \cap D)$

31. What can you say about the sets A and B if we know that

a) $A \cup B = A$? **b)** $A \cap B = A$?
c) $A - B = A$? **d)** $A \cap B = B \cap A$?
e) $A - B = B - A$?

32. Can you conclude that $A = B$ if A, B, and C are sets such that

a) $A \cup C = B \cup C$? **b)** $A \cap C = B \cap C$?
c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

33. Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

34. Let A, B, and C be sets. Use the the identity $A - B = A \cap \overline{B}$, which holds for any sets A and B, and the identities from Table 1 to show that $(A - B) \cap (B - C) \cap (A - C) = \emptyset$.

35. Let A, B, and C be sets. Use the identities in Table 1 to show that $\overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$.

36. Prove or disprove that for all sets A, B, and C, we have

a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

37. Prove or disprove that for all sets A, B, and C, we have

a) $A \times (B - C) = (A \times B) - (A \times C)$.
b) $\overline{A} \times \overline{(B \cup C)} = \overline{A \times (B \cup C)}$.

The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

38. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

39. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.

40. Draw a Venn diagram for the symmetric difference of the sets A and B.

41. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

42. Show that $A \oplus B = (A - B) \cup (B - A)$.

43. Show that if A is a subset of a universal set U, then

a) $A \oplus A = \emptyset$. **b)** $A \oplus \emptyset = A$.
c) $A \oplus U = \overline{A}$. **d)** $A \oplus \overline{A} = U$.

44. Show that if A and B are sets, then

a) $A \oplus B = B \oplus A$. **b)** $(A \oplus B) \oplus B = A$.

45. What can you say about the sets A and B if $A \oplus B = A$?

***46.** Determine whether the symmetric difference is associative; that is, if A, B, and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

***47.** Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

48. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?

49. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

50. Show that if A and B are finite sets, then $A \cup B$ is a finite set.

51. Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set.

***52.** Show that if A, B, and C are sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)

53. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find

a) $\displaystyle\bigcup_{i=1}^{n} A_i$. **b)** $\displaystyle\bigcap_{i=1}^{n} A_i$.

54. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

a) $\displaystyle\bigcup_{i=1}^{n} A_i$. **b)** $\displaystyle\bigcap_{i=1}^{n} A_i$.

55. Let A_i be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding i. Find

a) $\displaystyle\bigcup_{i=1}^{n} A_i$. **b)** $\displaystyle\bigcap_{i=1}^{n} A_i$.

56. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{i, i + 1, i + 2, \ldots\}$.
b) $A_i = \{0, i\}$.
c) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.
d) $A_i = (i, \infty)$, that is, the set of real numbers x with $x > i$.

57. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{-i, -i + 1, \ldots, -1, 0, 1, \ldots, i - 1, i\}$.
b) $A_i = \{-i, i\}$.
c) $A_i = [-i, i]$, that is, the set of real numbers x with $-i \le x \le i$.
d) $A_i = [i, \infty)$, that is, the set of real numbers x with $x \ge i$.

58. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$
b) $\{1, 3, 6, 10\}$
c) $\{2, 3, 4, 7, 8, 9\}$

59. Using the same universal set as in the last exercise, find the set specified by each of these bit strings.

a) 11 1100 1111
b) 01 0111 1000
c) 10 0000 0001

60. What subsets of a finite universal set do these bit strings represent?

a) the string with all zeros
b) the string with all ones

61. What is the bit string corresponding to the difference of two sets?

62. What is the bit string corresponding to the symmetric difference of two sets?

63. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$ **b)** $A \cap B$
c) $(A \cup D) \cap (B \cup C)$ **d)** $A \cup B \cup C \cup D$

64. How can the union and intersection of n sets that all are subsets of the universal set U be found using bit strings?

The **successor** of the set A is the set $A \cup \{A\}$.

65. Find the successors of the following sets.

a) $\{1, 2, 3\}$ **b)** $\emptyset$
c) $\{\emptyset\}$ **d)** $\{\emptyset, \{\emptyset\}\}$

66. How many elements does the successor of a set with n elements have?

67. Let A and B be the multisets $\{3 \cdot a, 2 \cdot b, 1 \cdot c\}$ and $\{2 \cdot a, 3 \cdot b, 4 \cdot d\}$, respectively. Find

a) $A \cup B$. **b)** $A \cap B$. **c)** $A - B$.
d) $B - A$. **e)** $A + B$.

68. Assume that $a \in A$, where A is a set. Which of these statements are true and which are false, where all sets shown are ordinary sets, and not multisets. Explain each answer.

a) $\{a, a\} \cup \{a, a, a\} = \{a, a, a, a, a\}$
b) $\{a, a\} \cup \{a, a, a\} = \{a\}$
c) $\{a, a\} \cap \{a, a, a\} = \{a, a\}$
d) $\{a, a\} \cap \{a, a, a\} = \{a\}$
e) $\{a, a, a\} - \{a, a\} = \{a\}$

69. Answer the same questions as posed in Exercise 68 where all sets are multisets, and not ordinary sets.

70. Suppose that A is the multiset that has as its elements the types of computer equipment needed by one department of a university and the multiplicities are the number of pieces of each type needed, and B is the analogous multiset for a second department of the university. For instance, A could be the multiset $\{107 \cdot$ personal computers, $44 \cdot$ routers, $6 \cdot$ servers$\}$ and B could be the multiset $\{14 \cdot$ personal computers, $6 \cdot$ routers, $2 \cdot$ mainframes$\}$.

a) What combination of A and B represents the equipment the university should buy assuming both departments use the same equipment?
b) What combination of A and B represents the equipment that will be used by both departments if both departments use the same equipment?
c) What combination of A and B represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?
d) What combination of A and B represents the equipment that the university should purchase if the departments do not share equipment?

The **Jaccard similarity** $J(A, B)$ of the finite sets A and B is $J(A, B) = |A \cap B|/|A \cup B|$, with $J(\emptyset, \emptyset) = 1$. The **Jaccard distance** $d_J(A, B)$ between A and B equals $d_J(A, B) = 1 - J(A, B)$.

71. Find $J(A, B)$ and $d_J(A, B)$ for these pairs of sets.

a) $A = \{1, 3, 5\}, B = \{2, 4, 6\}$
b) $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6\}$
c) $A = \{1, 2, 3, 4, 5, 6\}, B = \{1, 2, 3, 4, 5, 6\}$
d) $A = \{1\}, B = \{1, 2, 3, 4, 5, 6\}$

72. Prove that each of the properties in parts (a)–(d) holds whenever A and B are finite sets.

a) $J(A, A) = 1$ and $d_J(A, A) = 0$
b) $J(A, B) = J(B, A)$ and $d_J(A, B) = d_J(B, A)$
c) $J(A, B) = 1$ and $d_J(A, B) = 0$ if and only if $A = B$
d) $0 \le J(A, B) \le 1$ and $0 \le d_J(A, B) \le 1$
****e)** Show that if A, B, and C are sets, then $d_J(A, C) \le d_J(A, B) + d_J(B, C)$. (This inequality is known as the **triangle inequality** and together with parts (a), (b), and (c) implies that d_J is a **metric**.)

Fuzzy sets are used in artificial intelligence. Each element in the universal set U has a **degree of membership**, which is a real number between 0 and 1 (including 0 and 1), in a fuzzy set S. The fuzzy set S is denoted by listing the elements with their degrees of membership (elements with 0 degree of membership are not listed). For instance, we write {0.6 Alice, 0.9 Brian, 0.4 Fred, 0.1 Oscar, 0.5 Rita} for the set F (of famous people) to indicate that Alice has a 0.6 degree of membership in F, Brian has a 0.9 degree of membership in F, Fred has a 0.4 degree of membership in F, Oscar has a 0.1 degree of membership in F, and Rita has a 0.5 degree of membership in F (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that R is the set of rich people with R = {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.9 Oscar, 0.7 Rita}.

73. The **complement** of a fuzzy set S is the set $\overline{S}$, with the degree of the membership of an element in $\overline{S}$ equal to 1 minus the degree of membership of this element in S. Find $\overline{F}$ (the fuzzy set of people who are not famous) and $\overline{R}$ (the fuzzy set of people who are not rich).

74. The **union** of two fuzzy sets S and T is the fuzzy set $S \cup T$, where the degree of membership of an element in $S \cup T$ is the maximum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cup R$ of rich or famous people.

75. The **intersection** of two fuzzy sets S and T is the fuzzy set $S \cap T$, where the degree of membership of an element in $S \cap T$ is the minimum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cap R$ of rich and famous people.