

Winning team
shown in color

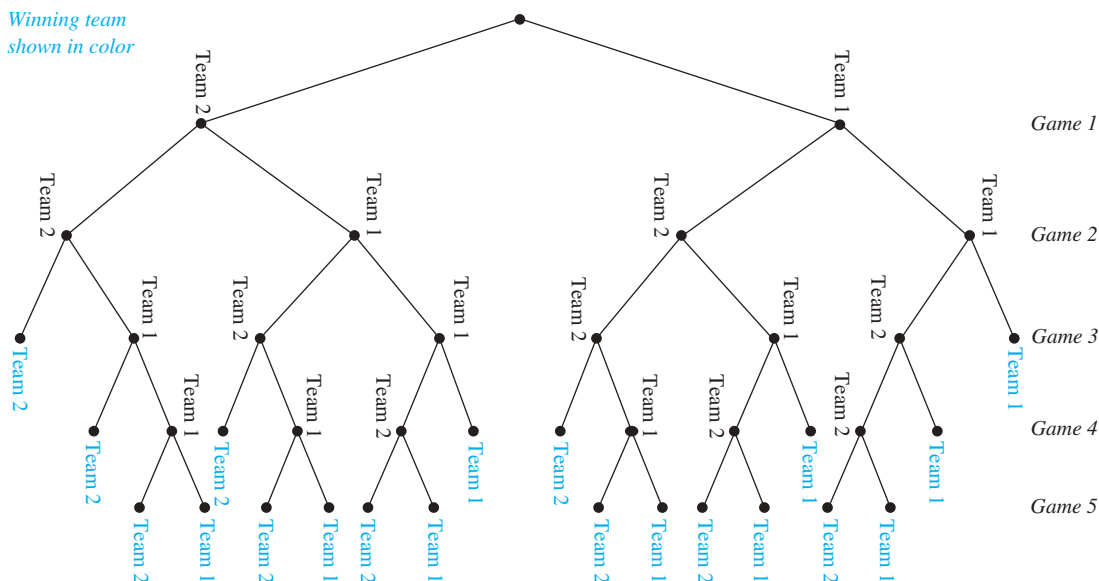


FIGURE 5 Best three games out of five playoffs.

EXAMPLE 24 Suppose that “I Love New Jersey” T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

Solution: The tree diagram in Figure 6 displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different T-shirts. ◀

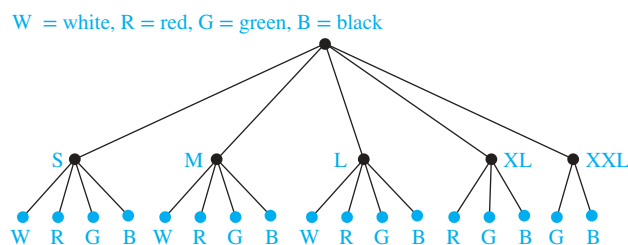


FIGURE 6 Counting varieties of T-shirts.

Exercises

- There are 18 mathematics majors and 325 computer science majors at a college.
 - In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?
 - In how many ways can one representative be picked who is either a mathematics major or a computer science major?
- An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?
- A multiple-choice test contains 10 questions. There are four possible answers for each question.
 - In how many ways can a student answer the questions on the test if the student answers every question?
 - In how many ways can a student answer the questions on the test if the student can leave answers blank?

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?
5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco?
6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?
7. How many different three-letter initials can people have?
8. How many different three-letter initials with none of the letters repeated can people have?
9. How many different three-letter initials are there that begin with an A?
10. How many bit strings are there of length eight?
11. How many bit strings of length ten both begin and end with a 1?
12. How many bit strings are there of length six or less, not counting the empty string?
13. How many bit strings with length not exceeding n , where n is a positive integer, consist entirely of 1s, not counting the empty string?
14. How many bit strings of length n , where n is a positive integer, start and end with 1s?
15. How many strings are there of lowercase letters of length four or less, not counting the empty string?
16. How many strings are there of four lowercase letters that have the letter x in them?
17. How many strings of five ASCII characters contain the character @ ("at" sign) at least once? [Note: There are 128 different ASCII characters.]
18. How many 5-element DNA sequences
 - a) end with A?
 - b) start with T and end with G?
 - c) contain only A and T?
 - d) do not contain C?
19. How many 6-element RNA sequences
 - a) do not contain U?
 - b) end with GU?
 - c) start with C?
 - d) contain only A or U?
20. How many positive integers between 5 and 31
 - a) are divisible by 3? Which integers are these?
 - b) are divisible by 4? Which integers are these?
 - c) are divisible by 3 and by 4? Which integers are these?
21. How many positive integers between 50 and 100
 - a) are divisible by 7? Which integers are these?
 - b) are divisible by 11? Which integers are these?
 - c) are divisible by both 7 and 11? Which integers are these?
22. How many positive integers less than 1000
 - a) are divisible by 7?
 - b) are divisible by 7 but not by 11?
 - c) are divisible by both 7 and 11?
 - d) are divisible by either 7 or 11?
 - e) are divisible by exactly one of 7 and 11?
 - f) are divisible by neither 7 nor 11?
 - g) have distinct digits?
 - h) have distinct digits and are even?
23. How many positive integers between 100 and 999 inclusive
 - a) are divisible by 7?
 - b) are odd?
 - c) have the same three decimal digits?
 - d) are not divisible by 4?
 - e) are divisible by 3 or 4?
 - f) are not divisible by either 3 or 4?
 - g) are divisible by 3 but not by 4?
 - h) are divisible by 3 and 4?
24. How many positive integers between 1000 and 9999 inclusive
 - a) are divisible by 9?
 - b) are even?
 - c) have distinct digits?
 - d) are not divisible by 3?
 - e) are divisible by 5 or 7?
 - f) are not divisible by either 5 or 7?
 - g) are divisible by 5 but not by 7?
 - h) are divisible by 5 and 7?
25. How many strings of three decimal digits
 - a) do not contain the same digit three times?
 - b) begin with an odd digit?
 - c) have exactly two digits that are 4s?
26. How many strings of four decimal digits
 - a) do not contain the same digit twice?
 - b) end with an even digit?
 - c) have exactly three digits that are 9s?
27. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?
28. How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?
29. How many license plates can be made using either two uppercase English letters followed by three digits or two digits followed by four uppercase English letters?
30. How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?
31. How many license plates can be made using either two or three uppercase English letters followed by either two or three digits?

32. How many strings of eight uppercase English letters are there
- if letters can be repeated?
 - if no letter can be repeated?
 - that start with X, if letters can be repeated?
 - that start with X, if no letter can be repeated?
 - that start and end with X, if letters can be repeated?
 - that start with the letters BO (in that order), if letters can be repeated?
 - that start and end with the letters BO (in that order), if letters can be repeated?
 - that start or end with the letters BO (in that order), if letters can be repeated?
33. How many strings of eight English letters are there
- that contain no vowels, if letters can be repeated?
 - that contain no vowels, if letters cannot be repeated?
 - that start with a vowel, if letters can be repeated?
 - that start with a vowel, if letters cannot be repeated?
 - that contain at least one vowel, if letters can be repeated?
 - that contain exactly one vowel, if letters can be repeated?
 - that start with X and contain at least one vowel, if letters can be repeated?
 - that start and end with X and contain at least one vowel, if letters can be repeated?
34. How many different functions are there from a set with 10 elements to sets with the following numbers of elements?
- 2
 - 3
 - 4
 - 5
35. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?
- 4
 - 5
 - 6
 - 7
36. How many functions are there from the set $\{1, 2, \dots, n\}$, where n is a positive integer, to the set $\{0, 1\}$?
37. How many functions are there from the set $\{1, 2, \dots, n\}$, where n is a positive integer, to the set $\{0, 1\}$
- that are one-to-one?
 - that assign 0 to both 1 and n ?
 - that assign 1 to exactly one of the positive integers less than n ?
38. How many partial functions (see Section 2.3) are there from a set with five elements to sets with each of these number of elements?
- 1
 - 2
 - 5
 - 9
39. How many partial functions (see Definition 13 of Section 2.3) are there from a set with m elements to a set with n elements, where m and n are positive integers?
40. How many subsets of a set with 100 elements have more than one element?
41. A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?
42. How many 4-element DNA sequences
- do not contain the base T?
 - contain the sequence ACG?
 - contain all four bases A, T, C, and G?
 - contain exactly three of the four bases A, T, C, and G?
43. How many 4-element RNA sequences
- contain the base U?
 - do not contain the sequence CUG?
 - do not contain all four bases A, U, C, and G?
 - contain exactly two of the four bases A, U, C, and G?
44. On each of the 22 work days in a particular month, every employee of a start-up venture was sent a company communication. If a total of 4642 total company communications were sent, how many employees does the company have, assuming that no staffing changes were made that month?
45. At a large university, 434 freshmen, 883 sophomores, and 43 juniors are enrolled in an introductory algorithms course. How many sections of this course need to be scheduled to accommodate all these students if each section contains 34 students?
46. How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?
47. How many ways are there to seat six people around a circular table where two seatings are considered the same when everyone has the same two neighbors without regard to whether they are right or left neighbors?
48. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if
- the bride must be in the picture?
 - both the bride and groom must be in the picture?
 - exactly one of the bride and the groom is in the picture?
49. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if
- the bride must be next to the groom?
 - the bride is not next to the groom?
 - the bride is positioned somewhere to the left of the groom?
50. How many bit strings of length seven either begin with two 0s or end with three 1s?
51. How many bit strings of length 10 either begin with three 0s or end with two 0s?
- * 52. How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?
- ** 53. How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?
54. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

55. How many positive integers not exceeding 100 are divisible either by 4 or by 6?
56. How many different initials can someone have if a person has at least two, but no more than five, different initials? Assume that each initial is one of the 26 uppercase letters of the English language.
57. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters *, >, <, !, +, and =.
 - a) How many different passwords are available for this computer system?
 - b) How many of these passwords contain at least one occurrence of at least one of the six special characters?
 - c) Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that it takes one nanosecond for a hacker to check each possible password.
58. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)
59. The name of a variable in the JAVA programming language is a string of between 1 and 65,535 characters, inclusive, where each character can be an uppercase or a lowercase letter, a dollar sign, an underscore, or a digit, except that the first character must not be a digit. Determine the number of different variable names in JAVA.
60. The International Telecommunications Union (ITU) specifies that a telephone number must consist of a country code with between 1 and 3 digits, except that the code 0 is not available for use as a country code, followed by a number with at most 15 digits. How many available possible telephone numbers are there that satisfy these restrictions?
61. Suppose that at some future time every telephone in the world is assigned a number that contains a country code 1 to 3 digits long, that is, of the form X , XX , or XXX , followed by a 10-digit telephone number of the form $NXX-NXX-XXXX$ (as described in Example 8). How many different telephone numbers would be available worldwide under this numbering plan?
62. A key in the Vigenère cryptosystem is a string of English letters, where the case of the letters does not matter. How many different keys for this cryptosystem are there with three, four, five, or six letters?
63. A wired equivalent privacy (WEP) key for a wireless fidelity (WiFi) network is a string of either 10, 26, or 58 hexadecimal digits. How many different WEP keys are there?
64. Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion–exclusion to find the number of positive integers not exceeding n that are relatively prime to n .
65. Use the principle of inclusion–exclusion to find the number of positive integers less than 1,000,000 that are not divisible by either 4 or by 6.
66. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.
67. How many ways are there to arrange the letters a , b , c , and d such that a is not followed immediately by b ?
68. Use a tree diagram to find the number of ways that the World Series can occur, where the first team that wins four games out of seven wins the series.
69. Use a tree diagram to determine the number of subsets of $\{3, 7, 9, 11, 24\}$ with the property that the sum of the elements in the subset is less than 28.
70. a) Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64-ounce bottles.
 - b) Answer the question in part (a) using counting rules.
71. a) Suppose that a popular style of running shoe is available for both men and women. The woman's shoe comes in sizes 6, 7, 8, and 9, and the man's shoe comes in sizes 8, 9, 10, 11, and 12. The man's shoe comes in white and black, while the woman's shoe comes in white, red, and black. Use a tree diagram to determine the number of different shoes that a store has to stock to have at least one pair of this type of running shoe for all available sizes and colors for both men and women.
 - b) Answer the question in part (a) using counting rules.
72. Determine the number of matches played in a single-elimination tournament with n players, where for each game between two players the winner goes on, but the loser is eliminated.
73. Determine the minimum and the maximum number of matches that can be played in a double-elimination tournament with n players, where after each game between two players, the winner goes on and the loser goes on if and only if this is not a second loss.
- *74. Use the product rule to show that there are 2^{2^n} different truth tables for propositions in n variables.
75. Use mathematical induction to prove the sum rule for m tasks from the sum rule for two tasks.
76. Use mathematical induction to prove the product rule for m tasks from the product rule for two tasks.

77. How many diagonals does a convex polygon with n sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)
78. Data are transmitted over the Internet in **datagrams**, which are structured blocks of bits. Each datagram contains header information organized into a maximum of 14 different fields (specifying many things, including the source and destination addresses) and a data area that contains the actual data that are transmitted. One of the 14 header fields is the **header length field** (denoted by HLEN), which is specified by the protocol to be 4 bits long and that specifies the header length in terms of 32-bit blocks of bits. For example, if $\text{HLEN} = 0110$, the header is made up of six 32-bit blocks. Another of the 14 header fields is the 16-bit-long **total length field** (denoted

by TOTAL LENGTH), which specifies the length in bits of the entire datagram, including both the header fields and the data area. The length of the data area is the total length of the datagram minus the length of the header.

- a) The largest possible value of TOTAL LENGTH (which is 16 bits long) determines the maximum total length in octets (blocks of 8 bits) of an Internet datagram. What is this value?
- b) The largest possible value of HLEN (which is 4 bits long) determines the maximum total header length in 32-bit blocks. What is this value? What is the maximum total header length in octets?
- c) The minimum (and most common) header length is 20 octets. What is the maximum total length in octets of the data area of an Internet datagram?
- d) How many different strings of octets in the data area can be transmitted if the header length is 20 octets and the total length is as long as possible?

6.2 The Pigeonhole Principle

6.2.1 Introduction



Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the **pigeonhole principle**, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it (see Figure 1). This principle is extremely useful; it applies to much more than pigeons and pigeonholes.

THEOREM 1 THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

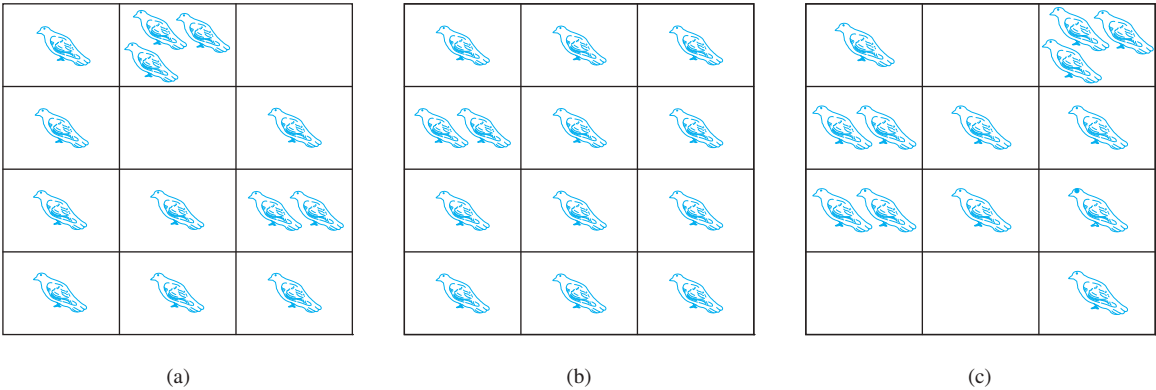


FIGURE 1 There are more pigeons than pigeonholes.