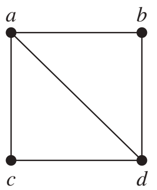
structural isomers, molecules with identical molecular formulas but with atoms bonded differently, have nonisomorphic molecular graphs. When a potentially new chemical compound is synthesized, a database of molecular graphs is checked to see whether the molecular graph of the compound is the same as one already known.

Electronic circuits are modeled using graphs in which vertices represent components and edges represent connections between them. Modern integrated circuits, known as chips, are miniaturized electronic circuits, often with millions of transistors and connections between them. Because of the complexity of modern chips, automation tools are used to design them. Graph isomorphism is the basis for the verification that a particular layout of a circuit produced by an automated tool corresponds to the original schematic of the design. Graph isomorphism can also be used to determine whether a chip from one vendor includes intellectual property from a different vendor. This can be done by looking for large isomorphic subgraphs in the graphs modeling these chips.

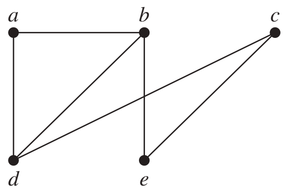
Exercises

In Exercises 1–4 use an adjacency list to represent the given graph.

1.



2.



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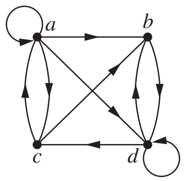
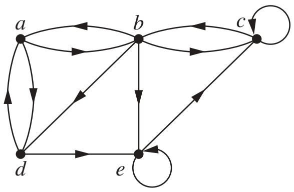


4.



5. Represent the graph in Exercise 1 with an adjacency matrix.

6. Represent the graph in Exercise 2 with an adjacency matrix.

7. Represent the graph in Exercise 3 with an adjacency matrix.

8. Represent the graph in Exercise 4 with an adjacency matrix.

9. Represent each of these graphs with an adjacency matrix.

a) K_4 **b)** $K_{1,4}$ **c)** $K_{2,3}$

d) C_4 **e)** W_4 **f)** Q_3

In Exercises 10–12 draw a graph with the given adjacency matrix.

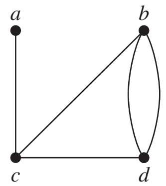
10. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

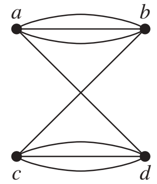
12. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

In Exercises 13–15 represent the given graph using an adjacency matrix.

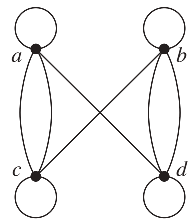
13.



14.



15.



In Exercises 16–18 draw an undirected graph represented by the given adjacency matrix.

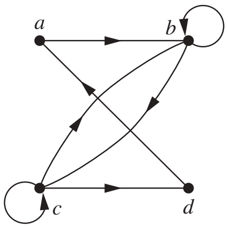
16. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

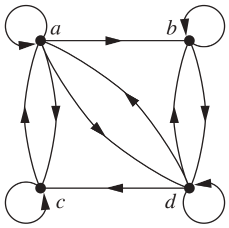
18. $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$

In Exercises 19–21 find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

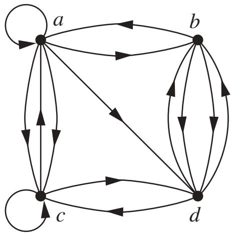
19.



20.



21.



In Exercises 22–24 draw the graph represented by the given adjacency matrix.

22. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **23.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ **24.** $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

The **density** of an undirected graph G is the number of edges of G divided by the number of possible edges in an undirected graph with $|G|$ vertices. Consequently, the density of $G(V, E)$ is

$$\frac{2|E|}{|V|(|V|-1)}.$$

A family of graphs G_n, $n = 1, 2, \ldots$ is **sparse** if the limit of the density of G_n is zero as n grows without bound, while it is **dense** if this proportion approaches a positive real number. As mentioned in the text, an individual graph is called sparse when it contains relatively few edges and dense if it contains many edges. These terms can be defined precisely depending on the context, but different definitions generally will not agree.

25. Find the density of the graph in
- **a)** Figure 1 of Section 10.1.
- **b)** Figure 6 of Section 10.1.
- **c)** Figure 12 of Section 10.1.

26. Find the density of the graph in
- **a)** Figure 12 of Section 10.2.
- **b)** Figure 13 of Section 10.2.
- **c)** Figure 14 of Section 10.2.

27. (*Requires calculus*) For each of these families of graphs, determine whether the graph family is sparse, dense, or neither. (Refer to the results of Exercise 37 in Section 10.2.)
- **a)** K_n **b)** C_n **c)** $K_{n,n}$
- **d)** Q_n **e)** W_n **f)** $K_{3,n}$

28. Determine whether an undirected graph is sparse, dense, or neither, and explain your answer, if it is used to model
- **a)** the street network in a city (where the vertices are street intersections).
- **b)** whether buildings in a city are within two miles of one another.
- **c)** whether two people in the world are siblings.
- **d)** whether two people work for the same company.

29. Is every zero–one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?

30. Use an incidence matrix to represent the graphs in Exercises 1 and 2.

31. Use an incidence matrix to represent the graphs in Exercises 13–15.

***32.** What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

***33.** What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?

34. What is the sum of the entries in a row of the incidence matrix for an undirected graph?

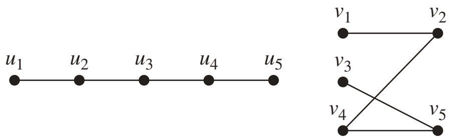
35. What is the sum of the entries in a column of the incidence matrix for an undirected graph?

***36.** Find an adjacency matrix for each of these graphs.
a) K_n **b)** C_n **c)** W_n **d)** $K_{m,n}$ **e)** Q_n

***37.** Find incidence matrices for the graphs in parts (a)–(d) of Exercise 36.

In Exercises 38–48 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists. For additional exercises of this kind, see Exercises 3–5 in the Supplementary Exercises.

38.



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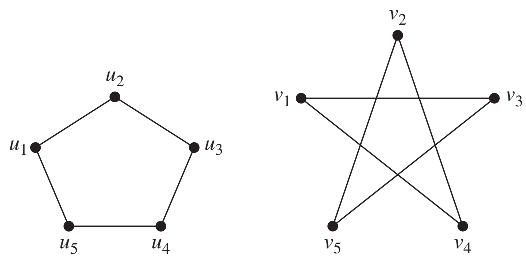

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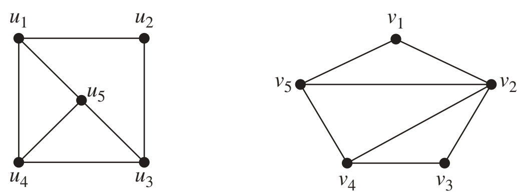

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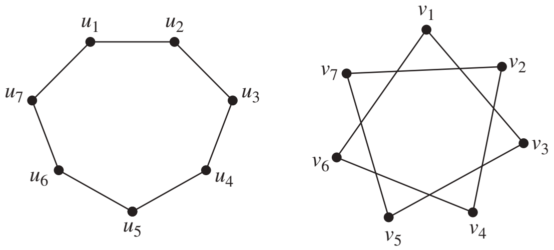

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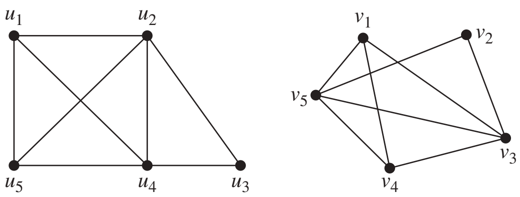

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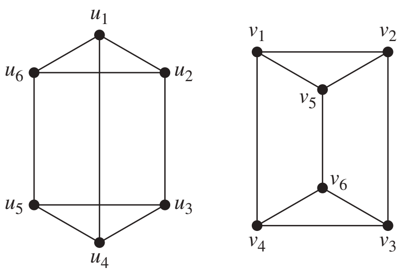

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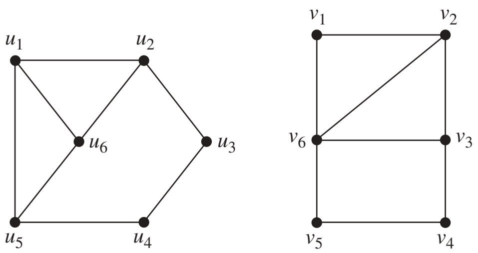

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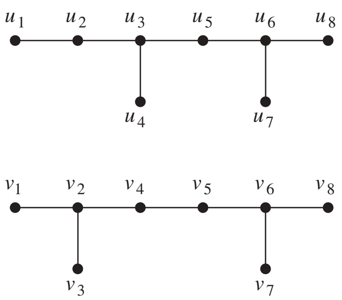

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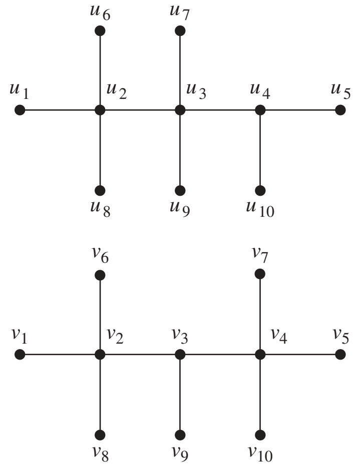

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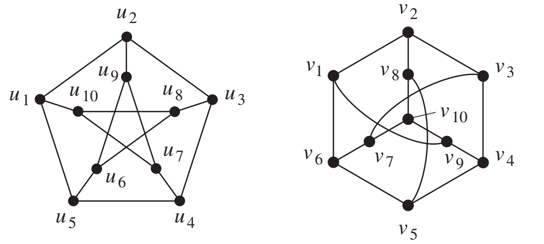

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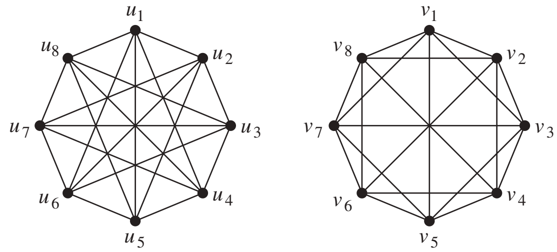


49. Show that isomorphism of simple graphs is an equivalence relation.

50. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs $\overline{G}$ and $\overline{H}$ are also isomorphic.

51. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertex.

52. Describe the row of an incidence matrix of a graph corresponding to an isolated vertex.

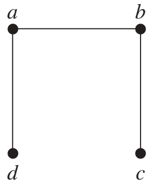
53. Show that the vertices of a bipartite graph with two or more vertices can be ordered so that its adjacency matrix has the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix},$$

where the four entries shown are rectangular blocks.

A simple graph G is called **self-complementary** if G and $\overline{G}$ are isomorphic.

54. Show that this graph is self-complementary.



55. Find a self-complementary simple graph with five vertices.

***56.** Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

57. For which integers n is C_n self-complementary?

58. How many nonisomorphic simple graphs are there with n vertices, when n is

a) 2? **b)** 3? **c)** 4?

59. How many nonisomorphic simple graphs are there with five vertices and three edges?

60. How many nonisomorphic simple graphs are there with six vertices and four edges?

***61.** Find the number of nonisomorphic simple graphs with six vertices in which each vertex has degree three.

62. Find the number of nonisomorphic simple graphs with seven vertices in which each vertex has degree two.

63. Are the simple graphs with the following adjacency matrices isomorphic?

a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

64. Determine whether the graphs without loops with these incidence matrices are isomorphic.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

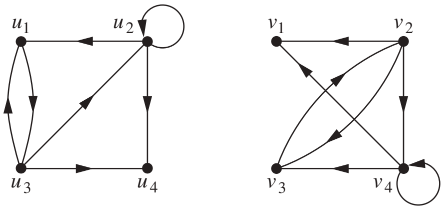
b) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

65. Extend the definition of isomorphism of simple graphs to undirected graphs containing loops and multiple edges.

66. Define isomorphism of directed graphs.

In Exercises 67–70 determine whether the given pair of directed graphs are isomorphic. (See Exercise 66.)

67.



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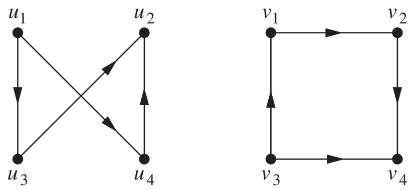

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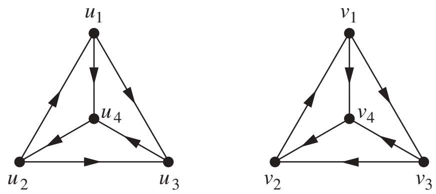

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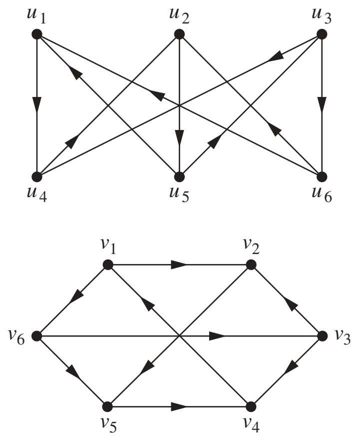

71. Show that if G and H are isomorphic directed graphs, then the converses of G and H (defined in the preamble of Exercise 69 of Section 10.2) are also isomorphic.

72. Show that the property that a graph is bipartite is an isomorphic invariant.

73. Find a pair of nonisomorphic graphs with the same degree sequence (defined in the preamble to Exercise 38 in Section 10.2) such that one graph is bipartite, but the other graph is not bipartite.

***74.** How many nonisomorphic directed simple graphs are there with n vertices, when n is

a) 2? **b)** 3? **c)** 4?

***75.** What is the product of the incidence matrix and its transpose for an undirected graph?

***76.** How much storage is needed to represent a simple graph with n vertices and m edges using

a) adjacency lists?
b) an adjacency matrix?
c) an incidence matrix?

A **devil's pair** for a purported isomorphism test is a pair of nonisomorphic graphs that the test fails to show that they are not isomorphic.

77. Find a devil's pair for the test that checks the degree sequence (defined in the preamble to Exercise 38 in Section 10.2) in two graphs to make sure they agree.

78. Suppose that the function f from V_1 to V_2 is an isomorphism of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Show that it is possible to verify this fact in time polynomial in terms of the number of vertices of the graph, in terms of the number of comparisons needed.

10.4 Connectivity

10.4.1 Introduction

Many problems can be modeled with paths formed by traveling along the edges of graphs. For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model. Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks, and so on can be solved using models that involve paths in graphs.

10.4.2 Paths

Informally, a **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these edges.

A formal definition of paths and related terminology is given in Definition 1.

Definition 1

Let n be a nonnegative integer and G an undirected graph. A *path* of *length* n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has, for $i = 1, \ldots, n$, the endpoints x_{i-1} and x_i. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$ (because listing these vertices uniquely determines the path). The path is a *circuit* if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to *pass through* the vertices $x_1, x_2, \ldots, x_{n-1}$ or *traverse* the edges $e_1, e_2, \ldots, e_n$. A path or circuit is *simple* if it does not contain the same edge more than once.

When it is not necessary to distinguish between multiple edges, we will denote a path $e_1, e_2, \ldots, e_n$, where e_i is associated with $\{x_{i-1}, x_i\}$ for $i = 1, 2, \ldots, n$ by its vertex sequence $x_0, x_1, \ldots, x_n$. This notation identifies a path only as far as which vertices it passes through. Consequently, it does not specify a unique path when there is more than one path that passes through this sequence of vertices, which will happen if and only if there are multiple edges between some successive vertices in the list. Note that a path of length zero consists of a single vertex.

Remark: There is considerable variation of terminology concerning the concepts defined in Definition 1. For instance, in some books, the term **walk** is used instead of *path*, where a walk is defined to be an alternating sequence of vertices and edges of a graph, $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$, where v_{i-1} and v_i are the endpoints of e_i for $i = 1, 2, \ldots, n$. When this terminology is used, **closed walk** is used instead of *circuit* to indicate a walk that begins and ends at the same vertex, and **trail** is used to denote a walk that has no repeated edge (replacing the term *simple path*). When this terminology is used, the terminology **path** is often used for a trail with no repeated vertices, conflicting with the terminology in Definition 1. Because of this variation in terminology, you will need to make sure which set of definitions are used in a