Proof: We note that the theorem holds for $k = 2$ and $k = 3$ because $R(2, 2) = 2$ and $R(3, 3) = 6$, as was shown in Section 6.2. Now suppose that $k \geq 4$. We will use the probabilistic method to show that if there are fewer than $2^{k/2}$ people at a party, it is possible that no k of them are mutual friends or mutual enemies. This will show that $R(k, k)$ is at least $2^{k/2}$.

To use the probabilistic method, we assume that it is equally likely for two people to be friends or enemies. (Note that this assumption does not have to be realistic.) Suppose there are n people at the party. It follows that there are $\binom{n}{k}$ different sets of k people at this party, which we list as $S_1, S_2, \ldots, S_{\binom{n}{k}}$. Let E_i be the event that all k people in S_i are either mutual friends or mutual enemies. The probability that there are either k mutual friends or k mutual enemies among the n people equals $p(\bigcup_{i=1}^{\binom{n}{k}} E_i)$.

According to our assumption it is equally likely for two people to be friends or enemies. The probability that two people are friends equals the probability that they are enemies; both probabilities equal $1/2$. Furthermore, there are $\binom{k}{2} = k(k-1)/2$ pairs of people in S_i because there are k people in S_i. Hence, the probability that all k people in S_i are mutual friends and the probability that all k people in S_i are mutual enemies both equal $(1/2)^{k(k-1)/2}$. It follows that $p(E_i) = 2(1/2)^{k(k-1)/2}$.

The probability that there are either k mutual friends or k mutual enemies in the group of n people equals $p(\bigcup_{i=1}^{\binom{n}{k}} E_i)$. Using Boole's inequality (Exercise 15), it follows that

$$p\left(\bigcup_{i=1}^{\binom{n}{k}} E_i\right) \leq \sum_{i=1}^{\binom{n}{k}} p(E_i) = \binom{n}{k} \cdot 2\left(\frac{1}{2}\right)^{k(k-1)/2}.$$

By Exercise 21 in Section 6.4, we have $\binom{n}{k} \leq n^k/2^{k-1}$. Hence,

$$\binom{n}{k} 2\left(\frac{1}{2}\right)^{k(k-1)/2} \leq \frac{n^k}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2}.$$

Now if $n < 2^{k/2}$, we have

$$\frac{n^k}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2} < \frac{2^{k(k/2)}}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2} = 2^{2-(k/2)} \leq 1,$$

where the last step follows because $k \geq 4$.

We can now conclude that $p(\bigcup_{i=1}^{\binom{n}{k}} E_i) < 1$ when $k \geq 4$. Hence, the probability of the complementary event, that there is no set of either k mutual friends or mutual enemies at the party, is greater than 0. It follows that if $n < 2^{k/2}$, there is at least one set such that no subset of k people are mutual friends or mutual enemies. ◁

Exercises

1. What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?
2. Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.
3. Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.
4. Show that conditions (i) and (ii) are met under Laplace's definition of probability, when outcomes are equally likely.
5. A pair of dice is loaded. The probability that a 4 appears on the first die is $2/7$, and the probability that a 3 appears on the second die is $2/7$. Other outcomes for each die

appear with probability $1/7$. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

6. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?
a) 1 precedes 3.
b) 3 precedes 1.
c) 3 precedes 1 and 3 precedes 2.

7. What is the probability of these events when we randomly select a permutation of $\{1, 2, 3, 4\}$?
a) 1 precedes 4.
b) 4 precedes 1.
c) 4 precedes 1 and 4 precedes 2.
d) 4 precedes 1, 4 precedes 2, and 4 precedes 3.
e) 4 precedes 3 and 2 precedes 1.

8. What is the probability of these events when we randomly select a permutation of $\{1, 2, \ldots, n\}$ where $n \geq 4$?
a) 1 precedes 2.
b) 2 precedes 1.
c) 1 immediately precedes 2.
d) n precedes 1 and $n-1$ precedes 2.
e) n precedes 1 and n precedes 2.

9. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?
a) The permutation consists of the letters in reverse alphabetic order.
b) z is the first letter of the permutation.
c) z precedes a in the permutation.
d) a immediately precedes z in the permutation.
e) a immediately precedes m, which immediately precedes z in the permutation.
f) m, n, and o are in their original places in the permutation.

10. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?
a) The first 13 letters of the permutation are in alphabetical order.
b) a is the first letter of the permutation and z is the last letter.
c) a and z are next to each other in the permutation.
d) a and b are not next to each other in the permutation.
e) a and z are separated by at least 23 letters in the permutation.
f) z precedes both a and b in the permutation.

11. Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

12. Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$.

13. Show that if E and F are events, then $p(E \cap F) \geq p(E) + p(F) - 1$. This is known as **Bonferroni's inequality**.

14. Use mathematical induction to prove the following generalization of Bonferroni's inequality:

$$p(E_1 \cap E_2 \cap \cdots \cap E_n) \geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n-1),$$

where $E_1, E_2, \ldots, E_n$ are n events.

☞ **15.** Show that if $E_1, E_2, \ldots, E_n$ are events from a finite sample space, then

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) \leq p(E_1) + p(E_2) + \cdots + p(E_n).$$

This is known as **Boole's inequality**.

16. Show that if E and F are independent events, then $\overline{E}$ and $\overline{F}$ are also independent events.

17. If E and F are independent events, prove or disprove that $\overline{E}$ and F are necessarily independent events.

In Exercises 18, 20, and 21 assume that the year has 366 days and all birthdays are equally likely. In Exercise 19 assume it is equally likely that a person is born in any given month of the year.

18. a) What is the probability that two people chosen at random were born on the same day of the week?
b) What is the probability that in a group of n people chosen at random, there are at least two born on the same day of the week?
c) How many people chosen at random are needed to make the probability greater than $1/2$ that there are at least two people born on the same day of the week?

19. a) What is the probability that two people chosen at random were born during the same month of the year?
b) What is the probability that in a group of n people chosen at random, there are at least two born in the same month of the year?
c) How many people chosen at random are needed to make the probability greater than $1/2$ that there are at least two people born in the same month of the year?

20. Find the smallest number of people you need to choose at random so that the probability that at least one of them has a birthday today exceeds $1/2$.

21. Find the smallest number of people you need to choose at random so that the probability that at least two of them were both born on April 1 exceeds $1/2$.

*__22.__ February 29 occurs only in leap years. Years divisible by 4, but not by 100, are always leap years. Years divisible by 100, but not by 400, are not leap years, but years divisible by 400 are leap years.
a) What probability distribution for birthdays should be used to reflect how often February 29 occurs?
b) Using the probability distribution from part (a), what is the probability that in a group of n people at least two have the same birthday?

23. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

24. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

25. What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)

26. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

27. Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. Are E and F independent if

a) $n = 2$? **b)** $n = 4$? **c)** $n = 5$?

28. Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has

a) exactly three boys?
b) at least one boy?
c) at least one girl?
d) all children of the same sex?

29. A group of six people play the game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments. What is the probability that there is an odd person out after the coins are flipped once?

30. Find the probability that a randomly generated bit string of length 10 does not contain a 0 if bits are independent and if

a) a 0 bit and a 1 bit are equally likely.
b) the probability that a bit is a 1 is 0.6.
c) the probability that the ith bit is a 1 is $1/2^i$ for $i = 1, 2, 3, \ldots, 10$.

31. Find the probability that a family with five children does not have a boy, if the sexes of children are independent and if

a) a boy and a girl are equally likely.
b) the probability of a boy is 0.51.
c) the probability that the ith child is a boy is $0.51 - (i/100)$.

32. Find the probability that a randomly generated bit string of length 10 begins with a 1 or ends with a 00 for the same conditions as in parts (a), (b), and (c) of Exercise 30, if bits are generated independently.

33. Find the probability that the first child of a family with five children is a boy or that the last two children of the family are girls, for the same conditions as in parts (a), (b), and (c) of Exercise 31.

34. Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

a) the probability of no successes
b) the probability of at least one success
c) the probability of at most one success
d) the probability of at least two successes

35. Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p.

a) the probability of no failures
b) the probability of at least one failure
c) the probability of at most one failure
d) the probability of at least two failures

36. Use mathematical induction to prove that if $E_1, E_2, \ldots, E_n$ is a sequence of n pairwise disjoint events in a sample space S, where n is a positive integer, then $p(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} p(E_i)$.

***37.** (*Requires calculus*) Show that if $E_1, E_2, \ldots$ is an infinite sequence of pairwise disjoint events in a sample space S, then $p(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} p(E_i)$. [*Hint:* Use Exercise 36 and take limits.]

38. A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers in the affirmative.

a) What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

b) Suppose that the honest observer tells us that at least one die came up five. What is the probability the sum of the numbers that came up on the dice is seven, given this information?

****39.** This exercise employs the probabilistic method to prove a result about round-robin tournaments. In a **round-robin tournament** with m players, every two players play one game in which one player wins and the other loses.

We want to find conditions on positive integers m and k with $k < m$ such that it is possible for the outcomes of the tournament to have the property that for every set of k players, there is a player who beats every member in this set. So that we can use probabilistic reasoning to draw conclusions about round-robin tournaments, we assume that when two players compete it is equally likely that either player wins the game and we assume that the outcomes of different games are independent. Let E be the event that for every set S with k players, where k is a positive integer less than m, there is a player who has beaten all k players in S.

a) Show that $p(\overline{E}) \leq \sum_{j=1}^{\binom{m}{k}} p(F_j)$, where F_j is the event that there is no player who beats all k players from the jth set in a list of the $\binom{m}{k}$ sets of k players.

b) Show that the probability of F_j is $(1-2^{-k})^{m-k}$.

c) Conclude from parts (a) and (b) that $p(\overline{E}) \leq \binom{m}{k}(1 - 2^{-k})^{m-k}$ and, therefore, that there must be a tournament with the described property if $\binom{m}{k}(1-2^{-k})^{m-k} < 1$.

d) Use part (c) to find values of m such that there is a tournament with m players such that for every set S of two players, there is a player who has beaten both players in S. Repeat for sets of three players.

***40.** Devise a Monte Carlo algorithm that determines whether a permutation of the integers 1 through n has already been sorted (that is, it is in increasing order), or instead, is a random permutation. A step of the algorithm should answer "true" if it determines the list is not sorted and "unknown" otherwise. After k steps, the algorithm decides that the integers are sorted if the answer is "unknown" in each step. Show that as the number of steps increases, the probability that the algorithm produces an incorrect answer is extremely small. [*Hint:* For each step, test whether certain elements are in the correct order. Make sure these tests are independent.]

41. Use pseudocode to write out the probabilistic primality test described in Example 16.

7.3 Bayes' Theorem

7.3.1 Introduction

There are many times when we want to assess the probability that a particular event occurs on the basis of partial evidence. For example, suppose we know the percentage of people who have a particular disease for which there is a very accurate diagnostic test. People who test positive for this disease would like to know the likelihood that they actually have the disease. In this section we introduce a result that can be used to determine this probability, namely, the probability that a person has the disease given that this person tests positive for it. To use this result, we will need to know the percentage of people who do not have the disease but test positive for it and the percentage of people who have the disease but test negative for it.

Similarly, suppose we know the percentage of incoming e-mail messages that are spam. We will see that we can determine the likelihood that an incoming e-mail message is spam using the occurrence of words in the message. To determine this likelihood, we need to know the percentage of incoming messages that are spam, the percentage of spam messages in which each of these words occurs, and the percentage of messages that are not spam in which each of these words occurs.

The result that we can use to answer questions such as these is called Bayes' theorem and dates back to the eighteenth century. In the past two decades, Bayes' theorem has been extensively applied to estimate probabilities based on partial evidence in areas as diverse as medicine, law, machine learning, engineering, and software development.

7.3.2 Bayes' Theorem

We illustrate the idea behind Bayes' theorem with an example that shows that when extra information is available, we can derive a more realistic estimate that a particular event occurs. That is, suppose we know $p(F)$, the probability that an event F occurs, but we have knowledge that an event E occurs. Then the conditional probability that F occurs given that E occurs, $p(F \mid E)$, is a more realistic estimate than $p(F)$ that F occurs. In Example 1 we will see that we can find $p(F \mid E)$ when we know $p(F)$, $p(E \mid F)$, and $p(E \mid \overline{F})$.

EXAMPLE 1

Extra Examples

We have two boxes. The first contains two green balls and seven red balls; the second contains four green balls and three red balls. Bob selects a ball by first choosing one of the two boxes at random. He then selects one of the balls in this box at random. If Bob has selected a red ball, what is the probability that he selected a ball from the first box?

Solution: Let E be the event that Bob has chosen a red ball; $\overline{E}$ is the event that Bob has chosen a green ball. Let F be the event that Bob has chosen a ball from the first box; $\overline{F}$ is the event that Bob has chosen a ball from the second box. We want to find $p(F \mid E)$, the probability that the ball Bob selected came from the first box, given that it is red. By the definition of conditional probability, we have $p(F \mid E) = p(F \cap E)/p(E)$. Can we use the information provided to determine both $p(F \cap E)$ and $p(E)$ so that we can find $p(F \mid E)$?