FIGURE 3   The circuit for $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$.

EXAMPLE 11   Build a digital circuit that produces the output $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$ when given input bits p, q, and r.

Solution: To construct the desired circuit, we build separate circuits for $p \vee \neg r$ and for $\neg p \vee (q \vee \neg r)$ and combine them using an AND gate. To construct a circuit for $p \vee \neg r$, we use an inverter to produce $\neg r$ from the input r. Then, we use an OR gate to combine p and $\neg r$. To build a circuit for $\neg p \vee (q \vee \neg r)$, we first use an inverter to obtain $\neg r$. Then we use an OR gate with inputs q and $\neg r$ to obtain $q \vee \neg r$. Finally, we use another inverter and an OR gate to get $\neg p \vee (q \vee \neg r)$ from the inputs p and $q \vee \neg r$.

To complete the construction, we employ a final AND gate, with inputs $p \vee \neg r$ and $\neg p \vee (q \vee \neg r)$. The resulting circuit is displayed in Figure 3.   ◀

We will study logic circuits in great detail in Chapter 12 in the context of Boolean algebra, and with different notation.

Exercises

In Exercises 1–6, translate the given statement into propositional logic using the propositions provided.

1. You cannot edit a protected Wikipedia entry unless you are an administrator. Express your answer in terms of e: "You can edit a protected Wikipedia entry" and a: "You are an administrator."

2. You can see the movie only if you are over 18 years old or you have the permission of a parent. Express your answer in terms of m: "You can see the movie," e: "You are over 18 years old," and p: "You have the permission of a parent."

3. You can graduate only if you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book. Express your answer in terms of g: "You can graduate," m: "You owe money to the university," r: "You have completed the requirements of your major," and b: "You have an overdue library book."

4. To use the wireless network in the airport you must pay the daily fee unless you are a subscriber to the service. Express your answer in terms of w: "You can use the wireless network in the airport," d: "You pay the daily fee," and s: "You are a subscriber to the service."

5. You are eligible to be President of the U.S.A. only if you are at least 35 years old, were born in the U.S.A., or at the time of your birth both of your parents were citizens, and you have lived at least 14 years in the country. Express your answer in terms of e: "You are eligible to be President of the U.S.A.," a: "You are at least 35 years old," b: "You were born in the U.S.A.," p: "At the time of your birth, both of your parents were citizens," and r: "You have lived at least 14 years in the U.S.A."

6. You can upgrade your operating system only if you have a 32-bit processor running at 1 GHz or faster, at least 1 GB RAM, and 16 GB free hard disk space, or a 64-bit processor running at 2 GHz or faster, at least 2 GB RAM, and at least 32 GB free hard disk space. Express your answer in terms of u: "You can upgrade your operating system," b_{32}: "You have a 32-bit processor," b_{64}: "You have a 64-bit processor," g_1: "Your processor runs at 1 GHz or faster," g_2: "Your processor runs at 2 GHz or faster," r_1: "Your processor has at least 1 GB RAM," r_2: "Your processor has at least 2 GB RAM," h_{16}: "You have at least 16 GB free hard disk space," and h_{32}: "You have at least 32 GB free hard disk space."

7. Express these system specifications using the propositions p: "The message is scanned for viruses" and q: "The message was sent from an unknown system" together with logical connectives (including negations).

   a) "The message is scanned for viruses whenever the message was sent from an unknown system."

b) "The message was sent from an unknown system but it was not scanned for viruses."

c) "It is necessary to scan the message for viruses whenever it was sent from an unknown system."

d) "When a message is not sent from an unknown system it is not scanned for viruses."

8. Express these system specifications using the propositions p: "The user enters a valid password," q: "Access is granted," and r: "The user has paid the subscription fee" and logical connectives (including negations).

a) "The user has paid the subscription fee, but does not enter a valid password."

b) "Access is granted whenever the user has paid the subscription fee and enters a valid password."

c) "Access is denied if the user has not paid the subscription fee."

d) "If the user has not entered a valid password but has paid the subscription fee, then access is granted."

9. Are these system specifications consistent? "The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode."

10. Are these system specifications consistent? "Whenever the system software is being upgraded, users cannot access the file system. If users can access the file system, then they can save new files. If users cannot save new files, then the system software is not being upgraded."

11. Are these system specifications consistent? "The router can send packets to the edge system only if it supports the new address space. For the router to support the new address space it is necessary that the latest software release be installed. The router can send packets to the edge system if the latest software release is installed. The router does not support the new address space."

12. Are these system specifications consistent? "If the file system is not locked, then new messages will be queued. If the file system is not locked, then the system is functioning normally, and conversely. If new messages are not queued, then they will be sent to the message buffer. If the file system is not locked, then new messages will be sent to the message buffer. New messages will not be sent to the message buffer."

13. What Boolean search would you use to look for Web pages about beaches in New Jersey? What if you wanted to find Web pages about beaches on the isle of Jersey (in the English Channel)?

14. What Boolean search would you use to look for Web pages about hiking in West Virginia? What if you wanted to find Web pages about hiking in Virginia, but not in West Virginia?

15. What Google search would you use to look for Web pages relating to Ethiopian restaurants in New York or New Jersey?

16. What Google search would you use to look for men's shoes or boots not designed for work?

17. Suppose that in Example 7, the inscriptions on Trunks 1, 2, and 3 are "The treasure is in Trunk 3," "The treasure is in Trunk 1," and "This trunk is empty." For each of these statements, determine whether the Queen who never lies could state this, and if so, which trunk the treasure is in.

a) "All the inscriptions are false."

b) "Exactly one of the inscriptions is true."

c) "Exactly two of the inscriptions are true."

d) "All three inscriptions are true."

18. Suppose that in Example 7 there are treasures in two of the three trunks. The inscriptions on Trunks 1, 2, and 3 are "This trunk is empty," "There is a treasure in Trunk 1," and "There is a treasure in Trunk 2." For each of these statements, determine whether the Queen who never lies could state this, and if so, which two trunks the treasures are in.

a) "All the inscriptions are false."

b) "Exactly one of the inscriptions is true."

c) "Exactly two of the inscriptions are true."

d) "All three inscriptions are true."

*19. Each inhabitant of a remote village always tells the truth or always lies. A villager will give only a "Yes" or a "No" response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit; the other branch leads deep into the jungle. A villager is standing at the fork in the road. What one question can you ask the villager to determine which branch to take?

20. An explorer is captured by a group of cannibals. There are two types of cannibals—those who always tell the truth and those who always lie. The cannibals will barbecue the explorer unless he can determine whether a particular cannibal always lies or always tells the truth. He is allowed to ask the cannibal exactly one question.

a) Explain why the question "Are you a liar?" does not work.

b) Find a question that the explorer can use to determine whether the cannibal always lies or always tells the truth.

21. When three professors are seated in a restaurant, the hostess asks them: "Does everyone want coffee?" The first professor says "I do not know." The second professor then says "I do not know." Finally, the third professor says "No, not everyone wants coffee." The hostess comes back and gives coffee to the professors who want it. How did she figure out who wanted coffee?

22. When planning a party you want to know whom to invite. Among the people you would like to invite are three touchy friends. You know that if Jasmine attends, she will become unhappy if Samir is there, Samir will attend only if Kanti will be there, and Kanti will not attend unless Jasmine also does. Which combinations of these three friends can you invite so as not to make someone unhappy?

Exercises 23–27 relate to inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, *A* and *B*. Determine, if possible, what *A* and *B* are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusions?

23. *A* says "At least one of us is a knave" and *B* says nothing.

24. *A* says "The two of us are both knights" and *B* says "*A* is a knave."

25. *A* says "I am a knave or *B* is a knight" and *B* says nothing.

26. Both *A* and *B* say "I am a knight."

27. *A* says "We are both knaves" and *B* says nothing.

Exercises 28–35 relate to inhabitants of an island on which there are three kinds of people: knights who always tell the truth, knaves who always lie, and spies (called normals by Smullyan [Sm78]) who can either lie or tell the truth. You encounter three people, *A*, *B*, and *C*. You know one of these people is a knight, one is a knave, and one is a spy. Each of the three people knows the type of person each of other two is. For each of these situations, if possible, determine whether there is a unique solution and determine who the knave, knight, and spy are. When there is no unique solution, list all possible solutions or state that there are no solutions.

28. *A* says "*C* is the knave," *B* says "*A* is the knight," and *C* says "I am the spy."

29. *A* says "I am the knight," *B* says "I am the knave," and *C* says "*B* is the knight."

30. *A* says "I am the knave," *B* says "I am the knave," and *C* says "I am the knave."

31. *A* says "I am the knight," *B* says "*A* is telling the truth," and *C* says "I am the spy."

32. *A* says "I am the knight," *B* says "*A* is not the knave," and *C* says "*B* is not the knave."

33. *A* says "I am the knight," *B* says "I am the knight," and *C* says "I am the knight."

34. *A* says "I am not the spy," *B* says "I am not the spy," and *C* says "*A* is the spy."

35. *A* says "I am not the spy," *B* says "I am not the spy," and *C* says "I am not the spy."

Exercises 36–42 are puzzles that can be solved by translating statements into logical expressions and reasoning from these expressions using truth tables.

36. The police have three suspects for the murder of Mr. Cooper: Mr. Smith, Mr. Jones, and Mr. Williams. Smith, Jones, and Williams each declare that they did not kill Cooper. Smith also states that Cooper was a friend of Jones and that Williams disliked him. Jones also states that he did not know Cooper and that he was out of town the day Cooper was killed. Williams also states that he saw both Smith and Jones with Cooper the day of the killing and that either Smith or Jones must have killed him. Can you determine who the murderer was if

 a) one of the three men is guilty, the two innocent men are telling the truth, but the statements of the guilty man may or may not be true?

 b) innocent men do not lie?

37. Steve would like to determine the relative salaries of three coworkers using two facts. First, he knows that if Fred is not the highest paid of the three, then Janice is. Second, he knows that if Janice is not the lowest paid, then Maggie is paid the most. Is it possible to determine the relative salaries of Fred, Maggie, and Janice from what Steve knows? If so, who is paid the most and who the least? Explain your reasoning.

38. Five friends have access to a chat room. Is it possible to determine who is chatting if the following information is known? Either Kevin or Heather, or both, are chatting. Either Randy or Vijay, but not both, are chatting. If Abby is chatting, so is Randy. Vijay and Kevin are either both chatting or neither is. If Heather is chatting, then so are Abby and Kevin. Explain your reasoning.

39. A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook; the cook and the gardener cannot both be telling the truth; the gardener and the handyman are not both lying; and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning.

40. Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said, "Carlos did it." John said, "I did not do it." Carlos said, "Diana did it." Diana said, "Carlos lied when he said that I did it."

 a) If the authorities also know that exactly one of the four suspects is telling the truth, who did it? Explain your reasoning.

 b) If the authorities also know that exactly one is lying, who did it? Explain your reasoning.

41. Suppose there are signs on the doors to two rooms. The sign on the first door reads "In this room there is a lady, and in the other one there is a tiger"; and the sign on the second door reads "In one of these rooms, there is a lady, and in one of them there is a tiger." Suppose that you know that one of these signs is true and the other is false. Behind which door is the lady?

Links

*42. Solve this famous logic puzzle, attributed to Albert Einstein, and known as the **zebra puzzle**. Five men with different nationalities and with different jobs live in consecutive houses on a street. These houses are painted different colors. The men have different pets and have different favorite drinks. Determine who owns a zebra and whose favorite drink is mineral water (which is one of the favorite drinks) given these clues: The Englishman lives in the red house. The Spaniard owns a dog. The Japanese man is a painter. The Italian drinks tea. The Norwegian lives in the first house on the left. The green house is immediately to the right of the white one. The photographer breeds snails. The diplomat lives in the yellow house. Milk is drunk in the middle house. The owner of the green house drinks coffee. The Norwegian's house is next to the

blue one. The violinist drinks orange juice. The fox is in a house next to that of the physician. The horse is in a house next to that of the diplomat. [*Hint:* Make a table where the rows represent the men and columns represent the color of their houses, their jobs, their pets, and their favorite drinks and use logical reasoning to determine the correct entries in the table.]

43. Freedonia has 50 senators. Each senator is either honest or corrupt. Suppose you know that at least one of the Freedonian senators is honest and that, given any two Freedonian senators, at least one is corrupt. Based on these facts, can you determine how many Freedonian senators are honest and how many are corrupt? If so, what is the answer?

44. Find the output of each of these combinatorial circuits.

a)

b)
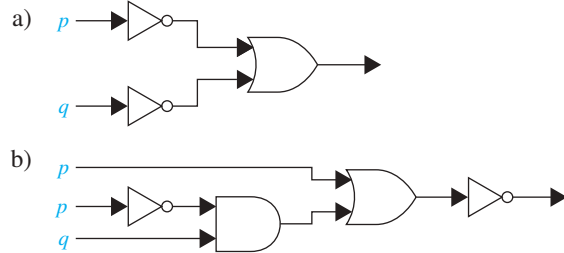
45. Find the output of each of these combinatorial circuits.

a)

b)
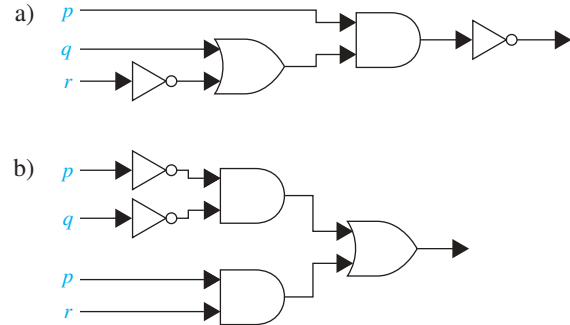
46. Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $(p \wedge \neg r) \vee (\neg q \wedge r)$ from input bits p, q, and r.

47. Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $((\neg p \vee \neg r) \wedge \neg q) \vee (\neg p \wedge (q \vee r))$ from input bits p, q, and r.

1.3   Propositional Equivalences

1.3.1   Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term "compound proposition" to refer to an expression formed from propositional variables using logical operators, such as $p \wedge q$.

We begin our discussion with a classification of compound propositions according to their possible truth values.

Definition 1

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Tautologies and contradictions are often important in mathematical reasoning. Example 1 illustrates these types of compound propositions.

EXAMPLE 1   We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction.   ◄