Exercises

1. What is the expected number of heads that come up when a fair coin is flipped five times?
2. What is the expected number of heads that come up when a fair coin is flipped 10 times?
3. What is the expected number of times a 6 appears when a fair die is rolled 10 times?
4. A coin is biased so that the probability a head comes up when it is flipped is 0.6. What is the expected number of heads that come up when it is flipped 10 times?
5. What is the expected sum of the numbers that appear on two dice, each biased so that a 3 comes up twice as often as each other number?
6. What is the expected value when a $1 lottery ticket is bought in which the purchaser wins exactly $10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, \ldots, 50\}$ and the purchaser wins nothing otherwise?
7. The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?
8. What is the expected sum of the numbers that appear when three fair dice are rolled?
9. Suppose that the probability that x is in a list of n distinct integers is $2/3$ and that it is equally likely that x equals any element in the list. Find the average number of comparisons used by the linear search algorithm to find x or to determine that it is not in the list.
10. Suppose that we flip a fair coin until either it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?
11. Suppose that we roll a fair die until a 6 comes up or we have rolled it 10 times. What is the expected number of times we roll the die?
12. Suppose that we roll a fair die until a 6 comes up.
    a) What is the probability that we roll the die n times?
    b) What is the expected number of times we roll the die?
13. Suppose that we roll a pair of fair dice until the sum of the numbers on the dice is seven. What is the expected number of times we roll the dice?
14. Show that the sum of the probabilities of a random variable with geometric distribution with parameter p, where $0 < p \leq 1$, equals 1.
15. Show that if the random variable X has the geometric distribution with parameter p, and j is a positive integer, then $p(X \geq j) = (1 - p)^{j-1}$.
16. Let X and Y be the random variables that count the number of heads and the number of tails that come up when two fair coins are flipped. Show that X and Y are not independent.
17. Estimate the expected number of integers with 1000 digits that need to be selected at random to find a prime, if the probability a number with 1000 digits is prime is approximately $1/2302$.
18. Suppose that X and Y are random variables and that X and Y are nonnegative for all points in a sample space S. Let Z be the random variable defined by $Z(s) = \max(X(s), Y(s))$ for all elements $s \in S$. Show that $E(Z) \leq E(X) + E(Y)$.
19. Let X be the number appearing on the first die when two fair dice are rolled and let Y be the sum of the numbers appearing on the two dice. Show that $E(X)E(Y) \neq E(XY)$.
*20. Show that if $X_1, X_2, \ldots, X_n$ are mutually independent random variables, then $E(\prod_{i=1}^{n} X_i) = \prod_{i=1}^{n} E(X_i)$.

The **conditional expectation** of the random variable X given the event A from the sample space S is $E(X|A) = \sum_{r \in X(S)} r \cdot P(X = r|A)$.

21. What is the expected value of the sum of the numbers appearing on two fair dice when they are rolled given that the sum of these numbers is at least nine. That is, what is $E(X|A)$ where X is the sum of the numbers appearing on the two dice and A is the event that $X \geq 9$?

The **law of total expectation** states that if the sample space S is the disjoint union of the events $S_1, S_2, \ldots, S_n$ and X is a random variable, then $E(X) = \sum_{j=1}^{n} E(X|S_j)P(S_j)$.

22. Prove the law of total expectations.
23. Use the law of total expectation to find the average weight of a breeding elephant seal, given that 12% of the breeding elephant seals are male and the rest are female, and the expected weights of a breeding elephant seal is 4200 pounds for a male and 1100 pounds for a female.
24. Let A be an event. Then I_A, the **indicator random variable** of A, equals 1 if A occurs and equals 0 otherwise. Show that the expectation of the indicator random variable of A equals the probability of A, that is, $E(I_A) = p(A)$.
25. A **run** is a maximal sequence of successes in a sequence of Bernoulli trials. For example, in the sequence $S, S, S, F, S, S, F, F, S$, where S represents success and F represents failure, there are three runs consisting of three successes, two successes, and one success, respectively. Let R denote the random variable on the set of sequences of n independent Bernoulli trials that counts the number of runs in this sequence. Find $E(R)$. [*Hint:* Show that $R = \sum_{j=1}^{n} I_j$, where $I_j = 1$ if a run begins at the jth Bernoulli trial and $I_j = 0$ otherwise. Find $E(I_1)$ and then find $E(I_j)$, where $1 < j \leq n$.]
26. Let $X(s)$ be a random variable, where $X(s)$ is a nonnegative integer for all $s \in S$, and let A_k be the event that $X(s) \geq k$. Show that $E(X) = \sum_{k=1}^{\infty} p(A_k)$.
27. What is the variance of the number of heads that come up when a fair coin is flipped 10 times?
28. What is the variance of the number of times a 6 appears when a fair die is rolled 10 times?

29. Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.

a) What is the expected value of X_n?
b) What is the variance of X_n?

30. Show that if X and Y are independent random variables, then $V(XY) = E(X)^2V(Y) + E(Y)^2V(X) + V(X)V(Y)$.

31. Let $A(X) = E(|X - E(X)|)$, the expected value of the absolute value of the deviation of X, where X is a random variable. Prove or disprove that $A(X + Y) = A(X) + A(Y)$ for all random variables X and Y.

32. Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances when the random variables are not independent.

33. Suppose that X_1 and X_2 are independent Bernoulli trials each with probability $1/2$, and let $X_3 = (X_1 + X_2) \mathbf{mod}\ 2$.

a) Show that X_1, X_2, and X_3 are pairwise independent, but X_3 and $X_1 + X_2$ are not independent.
b) Show that $V(X_1 + X_2 + X_3) = V(X_1) + V(X_2) + V(X_3)$.
c) Explain why a proof by mathematical induction of Theorem 7 does not work by considering the random variables X_1, X_2, and X_3.

***34.** Prove the general case of Theorem 7. That is, show that if $X_1, X_2, \ldots, X_n$ are pairwise independent random variables on a sample space S, where n is a positive integer, then $V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$. [*Hint:* Generalize the proof given in Theorem 7 for two random variables. Note that a proof using mathematical induction does not work; see Exercise 33.]

35. Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a fair coin is tossed n times deviates from the mean by more than $5\sqrt{n}$.

36. Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a biased coin with probability of heads equal to 0.6 is tossed n times deviates from the mean by more than $\sqrt{n}$.

37. Let X be a random variable on a sample space S such that $X(s) \geq 0$ for all $s \in S$. Show that $p(X(s) \geq a) \leq E(X)/a$ for every positive real number a. This inequality is called **Markov's inequality**.

38. Suppose that the number of cans of soda pop filled in a day at a bottling plant is a random variable with an expected value of 10,000 and a variance of 1000.

a) Use Markov's inequality (Exercise 37) to obtain an upper bound on the probability that the plant will fill more than 11,000 cans on a particular day.
b) Use Chebyshev's inequality to obtain a lower bound on the probability that the plant will fill between 9000 and 11,000 cans on a particular day.

39. Suppose that the number of aluminum cans recycled in a day at a recycling center is a random variable with an expected value of 50,000 and a variance of 10,000.

a) Use Markov's inequality (Exercise 37) to find an upper bound on the probability that the center will recycle more than 55,000 cans on a particular day.
b) Use Chebyshev's inequality to provide a lower bound on the probability that the center will recycle 40,000 to 60,000 cans on a certain day.

***40.** Suppose the probability that x is the ith element in a list of n distinct integers is $i/[n(n + 1)]$. Find the average number of comparisons used by the linear search algorithm to find x or to determine that it is not in the list.

***41.** In this exercise we derive an estimate of the average-case complexity of the variant of the bubble sort algorithm that terminates once a pass has been made with no interchanges. Let X be the random variable on the set of permutations of a set of n distinct integers $\{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ such that $X(P)$ equals the number of comparisons used by the bubble sort to put these integers into increasing order.

a) Show that, under the assumption that the input is equally likely to be any of the $n!$ permutations of these integers, the average number of comparisons used by the bubble sort equals $E(X)$.
b) Use Example 5 in Section 3.3 to show that $E(X) \leq n(n - 1)/2$.
c) Show that the sort makes at least one comparison for every inversion of two integers in the input.
d) Let $I(P)$ be the random variable that equals the number of inversions in the permutation P. Show that $E(X) \geq E(I)$.
e) Let $I_{j,k}$ be the random variable with $I_{j,k}(P) = 1$ if a_k precedes a_j in P and $I_{j,k} = 0$ otherwise. Show that $I(P) = \sum_k \sum_{j<k} I_{j,k}(P)$.
f) Show that $E(I) = \sum_k \sum_{j<k} E(I_{j,k})$.
g) Show that $E(I_{j,k}) = 1/2$. [*Hint:* Show that $E(I_{j,k})$ = probability that a_k precedes a_j in a permutation P. Then show it is equally likely for a_k to precede a_j as it is for a_j to precede a_k in a permutation.]
h) Use parts (f) and (g) to show that $E(I) = n(n - 1)/4$.
i) Conclude from parts (b), (d), and (h) that the average number of comparisons used to sort n integers is $\Theta(n^2)$.

***42.** In this exercise we find the average-case complexity of the quick sort algorithm, described in the preamble to Exercise 50 in Section 5.4, assuming a uniform distribution on the set of permutations.

a) Let X be the number of comparisons used by the quick sort algorithm to sort a list of n distinct integers. Show that the average number of comparisons used by the quick sort algorithm is $E(X)$ (where the sample space is the set of all $n!$ permutations of n integers).
b) Let $I_{j,k}$ denote the random variable that equals 1 if the jth smallest element and the kth smallest element of the initial list are ever compared as the quick sort algorithm sorts the list and equals 0 otherwise. Show that $X = \sum_{k=2}^{n} \sum_{j=1}^{k-1} I_{j,k}$.

c) Show that $E(X) = \sum_{k=2}^{n} \sum_{j=1}^{k-1} p$(the jth smallest element and the kth smallest element are compared).

d) Show that p(the jth smallest element and the kth smallest element are compared), where $k > j$, equals $2/(k - j + 1)$.

e) Use parts (c) and (d) to show that $E(X) = 2(n + 1)(\sum_{i=2}^{n} 1/i) - 2(n - 1)$.

f) Conclude from part (e) and the fact that $\sum_{j=1}^{n} 1/j \approx \ln n + \gamma$, where $\gamma = 0.57721\ldots$ is Euler's constant, that the average number of comparisons used by the quick sort algorithm is $\Theta(n \log n)$.

***43.** What is the variance of the number of **fixed elements**, that is, elements left in the same position, of a randomly selected permutation of n elements? [*Hint:* Let X denote the number of fixed points of a random permutation. Write $X = X_1 + X_2 + \cdots + X_n$, where $X_i = 1$ if the permutation fixes the ith element and $X_i = 0$ otherwise.]

The **covariance** of two random variables X and Y on a sample space S, denoted by $\mathrm{Cov}(X, Y)$, is defined to be the expected value of the random variable $(X - E(X))(Y - E(Y))$. That is, $\mathrm{Cov}(X, Y) = E((X - E(X))(Y - E(Y)))$.

44. Show that $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)$, and use this result to conclude that $\mathrm{Cov}(X, Y) = 0$ if X and Y are independent random variables.

45. Show that $V(X + Y) = V(X) + V(Y) + 2\,\mathrm{Cov}(X, Y)$.

46. Find $\mathrm{Cov}(X, Y)$ if X and Y are the random variables with $X((i, j)) = 2i$ and $Y((i, j)) = i + j$, where i and j are the numbers that appear on the first and second of two dice when they are rolled.

47. When m balls are distributed into n bins uniformly at random, what is the probability that the first bin remains empty?

48. What is the expected number of balls that fall into the first bin when m balls are distributed into n bins uniformly at random?

49. What is the expected number of bins that remain empty when m balls are distributed into n bins uniformly at random?

Key Terms and Results

TERMS

sample space: the set of possible outcomes of an experiment

event: a subset of the sample space of an experiment

probability of an event (Laplace's definition): the number of successful outcomes of this event divided by the number of possible outcomes

probability distribution: a function p from the set of all outcomes of a sample space S for which $0 \le p(x_i) \le 1$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p(x_i) = 1$, where $x_1, \ldots, x_n$ are the possible outcomes

probability of an event *E*: the sum of the probabilities of the outcomes in E

***p*(*E*|*F*) (conditional probability of *E* given *F*):** the ratio $p(E \cap F)/p(F)$

independent events: events E and F such that $p(E \cap F) = p(E)p(F)$

pairwise independent events: events $E_1, E_2, \ldots, E_n$ such that $p(E_i \cap E_j) = p(E_i)p(E_j)$ for all pairs of integers i and j with $1 \le j < k \le n$

mutually independent events: events $E_1, E_2, \ldots, E_n$ such that $p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_m}) = p(E_{i_1})p(E_{i_2}) \cdots p(E_{i_m})$ whenever $i_j, j = 1, 2, \ldots, m$, are integers with $1 \le i_1 < i_2 < \cdots < i_m \le n$ and $m \ge 2$

random variable: a function that assigns a real number to each possible outcome of an experiment

distribution of a random variable *X*: the set of pairs $(r, p(X = r))$ for $r \in X(S)$

uniform distribution: the assignment of equal probabilities to the elements of a finite set

expected value of a random variable: the weighted average of a random variable, with values of the random variable weighted by the probability of outcomes, that is, $E(X) = \sum_{s \in S} p(s)X(s)$

geometric distribution: the distribution of a random variable X such that $p(X = k) = (1 - p)^{k-1}p$ for $k = 1, 2, \ldots$ for some real number p with $0 \le p \le 1$

independent random variables: random variables X and Y such that $p(X = r_1 \text{ and } Y = r_2) = p(X = r_1)p(Y = r_2)$ for all real numbers r_1 and r_2

variance of a random variable *X*: the weighted average of the square of the difference between the value of X and its expected value $E(X)$, with weights given by the probability of outcomes, that is, $V(X) = \sum_{s \in S}(X(s) - E(X))^2 p(s)$

standard deviation of a random variable *X*: the square root of the variance of X, that is, $\sigma(X) = \sqrt{V(X)}$

Bernoulli trial: an experiment with two possible outcomes

probabilistic (or Monte Carlo) algorithm: an algorithm in which random choices are made at one or more steps

probabilistic method: a technique for proving the existence of objects in a set with certain properties that proceeds by assigning probabilities to objects and showing that the probability that an object has these properties is positive

RESULTS

The probability of exactly k successes when n independent Bernoulli trials are carried out equals $C(n, k)p^k q^{n-k}$, where p is the probability of success and $q = 1 - p$ is the probability of failure.