CHEATSHEET - MATH MODELING - FINSEM241

Chapter 3: Integer linear programming

Standard form:

$$\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \ \text{for } i = \overline{1, m} \\
& x_1, \ldots, x_n \geq 0
\end{aligned}$$

or

$$\max \left\{ \mathbf{c}^\top \mathbf{x} \,\middle|\, \mathbf{A}\mathbf{x} \leq \mathbf{b}, \, \mathbf{x} \geq \mathbf{0} \right\}$$

LO-model conversion rules:

- A minimizing model is transformed into a maximizing model by using the fact that minimizing a function is equivalent to maximizing minus that function.
- A '$\geq$' constraint is transformed into a '$\leq$' constraint by multiplying both sides of the inequality by -1 and reversing the inequality sign.
- A '$=$' constraint of the form '$\mathbf{a}^\top \mathbf{x} = b$' can be written as '$\mathbf{a}^\top \mathbf{x} \leq b$' and '$\mathbf{a}^\top \mathbf{x} \geq b$'. The second inequality in this expression is then transformed into a '$\leq$' constraint.
- A nonpositivity constraint is transformed into a nonnegativity constraint by replacing the corresponding variable by its negative.
- A free variable is replaced by the difference of two new nonnegative variables.

Recognition of special cases

- A basic feasible solution is called *degenerate* if one of its right hand side coefficients (excluding the objective value) is 0.
- If at any step in the execution of the simplex algorithm, there is at least one entering variable but no exiting variables or vice versa, the model is unbounded.

Cycling & anti-cycling methods

- If a sequence of pivots starting from some basic feasible solution ends up at the exact same basic feasible solution, then we refer to this as *cycling*. If the simplex method cycles, it can cycle forever.
- Anti-cycling methods:
  - Pertubation method: Choosing a small positive value for ε (e.g., $\varepsilon = 10^{-5}$), and take $\varepsilon_i = \varepsilon^i$ ($i = \overline{1, m}$). The original LO-model then changes into the *perturbed model*:

$$\begin{aligned}
\max \ & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
\text{subject to } & a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1 + \varepsilon^1 \\
& a_{21} x_1 + \cdots + a_{2n} x_n \leq b_2 + \varepsilon^2 \\
& \vdots \\
& a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m + \varepsilon^m \\
& x_i \geq 0, \ i = \overline{1, m}
\end{aligned}$$

  - Bland's rule:
    1. The entering variable is the nonbasic variable with a positive current objective coeffcient that has the smallest index (i.e., which is leftmost in the simplex tableau).
    2. If there is a tie in the minimum-ratio test, then the 'minimum-ratio' row corresponding to the basic variable with the smallest index is chosen.

Chapter 4: Dynamical systems

Solutions of basic differential equations

Linear dynamical systems	**Solutions**
$a_{n+1} = ra_n \ (r = \text{const})$	$a_k = r^k a_0$
$a_{n+1} = ra_n + b \ (r, b \in \mathbb{R})$	$a_k = r^k c + \dfrac{b}{1-r}$

Model 1: Unlimited population growth

$$\begin{cases} \dfrac{\mathrm{d}P}{P} = k\,\mathrm{d}t \\ P(t_0) = P_0 \end{cases} \Leftrightarrow \boxed{P(t) = P_0 e^{k(t-t_0)}}$$

Model 2: Limited growth model

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = r(M-P)P \\ M : \text{maximum population} \\ r = \text{const} \\ P(t_0) = P_0 \end{cases} \Leftrightarrow \boxed{P(t) = \dfrac{MP_0}{P_0 + (M - P_0)\, e^{-rM(t-t_0)}}}$$

The population P in the logistic equation reaches $\dfrac{M}{2}$ at time

$$t^* = t_0 - \frac{1}{rM} \ln \frac{P_0}{M - P_0}$$

Model 3: Highly communicable disease model

$$\begin{cases} N : \text{population size} \\ X : \text{the number of people have been infected} \\ k \in \mathbb{R}^+ \end{cases} \Leftrightarrow \boxed{X(t) = \dfrac{N}{1 + e^{-kN(t-t_0)}}}$$

Model 4: Bank interest rate - basic model

$$Q = \left(1 + \frac{i}{n}\right)^n Q_0 \quad \text{which} \begin{cases} i : & \text{annual interest rate} \\ n : & \text{number of periods in a year} \end{cases}$$