



# Chapter 1c

## Advanced Predicate Logic

*Discrete Mathematics II*

(Materials drawn from **Chapter 2** in:

"Michael Huth and Mark Ryan. *Logic in Computer Science: Modelling and Reasoning about Systems*, 2nd Ed., Cambridge University Press, 2006.")

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- Propositional logic can easily handle simple declarative statements such as:

## Example

Student Hung enrolled in DMII.



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- Propositional logic can easily handle simple declarative statements such as:

## Example

Student Hung enrolled in DMII.

- Propositional logic can also handle combinations of such statements such as:

## Example

Student Hung enrolled in Tutorial 1, *and* student Cuong is enrolled in Tutorial 2.



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- Propositional logic can easily handle simple declarative statements such as:

## Example

Student Hung enrolled in DMII.

- Propositional logic can also handle combinations of such statements such as:

## Example

Student Hung enrolled in Tutorial 1, *and* student Cuong is enrolled in Tutorial 2.

- *But:* How about statements with “*there exists...*” or “*every...*” or “*among...*”?



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# What is needed?

## Example

*Every student is younger than some instructor.*



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# What is needed?

## Example

*Every student is younger than some instructor.*

What is this statement about?



## Example

Every student is younger than *some* instructor.

What is this statement about?

- Being a student
- Being an instructor
- Being younger than somebody else

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# What is needed?

## Example

Every student is younger than *some* instructor.

What is this statement about?

- Being a student
- Being an instructor
- Being younger than somebody else

These are *properties* of elements of a *set* of objects.



## Example

Every student is younger than *some* instructor.

What is this statement about?

- Being a student
- Being an instructor
- Being younger than somebody else

These are *properties* of elements of a *set* of objects.

We express them in predicate logic using *predicates*.

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## Example

Every student is younger than *some* instructor.

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## Example

Every student is younger than *some* instructor.

- $S(An)$  could denote that An is a student.
- $I(Binh)$  could denote that Binh is an instructor.
- $Y(An, Binh)$  could denote that An is younger than Binh.

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## Example

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## Example

*Every student is younger than some instructor.*

We use the predicate  $S$  to denote student-hood.

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## Example

*Every student is younger than some instructor.*

We use the predicate  $S$  to denote student-hood.  
How do we express “*every student*”?

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## Example

Every student is younger than *some* instructor.

We use the predicate  $S$  to denote student-hood.

How do we express “*every student*”?

We need *variables* that can stand for constant values, and a *quantifier* symbol that denotes “*every*”.

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## Example

Every student is younger than *some* instructor.

Using variables and quantifiers, we can write:

$$\forall x(S(x) \rightarrow (\exists y(I(y) \wedge Y(x, y)))).$$

Literally: For every  $x$ , if  $x$  is a student, then there is some  $y$  such that  $y$  is an instructor and  $x$  is younger than  $y$ .

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## English

Not all birds can fly.

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## English

Not all birds can fly.

## Predicates

$B(x)$ :  $x$  is a bird

$F(x)$ :  $x$  can fly

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## Another Example

### English

Not all birds can fly.

### Predicates

$B(x)$ :  $x$  is a bird

$F(x)$ :  $x$  can fly

### The sentence in predicate logic

$$\neg(\forall x(B(x) \rightarrow F(x)))$$

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# A Third Example

## English

Every girl is younger than her mother.



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## A Third Example

### English

Every girl is younger than her mother.

### Predicates

$G(x)$ :  $x$  is a girl

$M(x, y)$ :  $y$  is  $x$ 's mother

$Y(x, y)$ :  $x$  is younger than  $y$

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## English

Every girl is younger than her mother.

## Predicates

$G(x)$ :  $x$  is a girl

$M(x, y)$ :  $y$  is  $x$ 's mother

$Y(x, y)$ :  $x$  is younger than  $y$

## The sentence in predicate logic

$$\forall x \forall y (G(x) \wedge M(x, y) \rightarrow Y(x, y))$$

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# A “Mother” Function

## The sentence in predicate logic

$$\forall x \forall y (G(x) \wedge M(x, y) \rightarrow Y(x, y))$$

Note that  $y$  is only introduced to denote the mother of  $x$ .



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## The sentence in predicate logic

$$\forall x \forall y (G(x) \wedge M(x, y) \rightarrow Y(x, y))$$

Note that  $y$  is only introduced to denote the mother of  $x$ .

If everyone has exactly one mother, the predicate  $M(x, y)$  is a function, when read from right to left.

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## The sentence in predicate logic

$$\forall x \forall y (G(x) \wedge M(x, y) \rightarrow Y(x, y))$$

Note that  $y$  is only introduced to denote the mother of  $x$ .

If everyone has exactly one mother, the predicate  $M(x, y)$  is a function, when read from right to left.

We introduce a function symbol  $m$  that can be applied to variables and constants as in

$$\forall x (G(x) \rightarrow Y(x, m(x)))$$

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# A Drastic Example

## English

An and Binh have the same maternal grandmother.



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# A Drastic Example

## English

An and Binh have the same maternal grandmother.

## The sentence in predicate logic without functions

$$\forall x \forall y \forall u \forall v (M(y, x) \wedge M(An, y) \wedge M(v, u) \wedge M(Binh, v) \rightarrow x = u)$$



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# A Drastic Example

## English

An and Binh have the same maternal grandmother.

## The sentence in predicate logic without functions

$$\forall x \forall y \forall u \forall v (M(y, x) \wedge M(An, y) \wedge M(v, u) \wedge M(Binh, v) \rightarrow x = u)$$

## The same sentence in predicate logic with functions

$$m(m(An)) = m(m(Binh))$$



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**Syntax:** We formalize the language of predicate logic,  
including scoping and substitution.



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**Syntax:** We formalize the language of predicate logic, including scoping and substitution.

**Proof theory:** We extend natural deduction from propositional to predicate logic

**Semantics:** We describe models in which predicates, functions, and formulas have meaning.

**Further topics:** Soundness/completeness (beyond scope of module), undecidability, incompleteness results, compactness results, extensions



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# ① Predicate Logic: Motivation, Syntax, Proof Theory

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# ② Semantics of Predicate Logic

# ③ Soundness and Completeness of Predicate Logic

# ④ Undecidability of Predicate Logic

# ⑤ Compactness of Predicate Calculus



At any point in time, we want to describe the features of a particular “world”, using predicates, functions, and constants. Thus, we introduce for this world:

- a set of predicate symbols  $\mathcal{P}$

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At any point in time, we want to describe the features of a particular “world”, using predicates, functions, and constants. Thus, we introduce for this world:

- a set of predicate symbols  $\mathcal{P}$
- a set of function symbols  $\mathcal{F}$

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# Predicate Vocabulary

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Huynh Tuong Nguyen



At any point in time, we want to describe the features of a particular “world”, using predicates, functions, and constants. Thus, we introduce for this world:

- a set of predicate symbols  $\mathcal{P}$
- a set of function symbols  $\mathcal{F}$
- a set of constant symbols  $\mathcal{C}$

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Every function symbol in  $\mathcal{F}$  and predicate symbol in  $\mathcal{P}$  comes with a fixed arity, denoting the number of arguments the symbol can take.

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Every function symbol in  $\mathcal{F}$  and predicate symbol in  $\mathcal{P}$  comes with a fixed arity, denoting the number of arguments the symbol can take.

## Special case

Function symbols with arity 0 are called *constants*.

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# Terms



$$t ::= x \mid c \mid f(t, \dots, t)$$

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$$t ::= x \mid c \mid f(t, \dots, t)$$

where

- $x$  ranges over a given set of variables **var**,

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$$t ::= x \mid c \mid f(t, \dots, t)$$

where

- $x$  ranges over a given set of variables **var**,
- $c$  ranges over nullary function symbols in  $\mathcal{F}$ , and

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$$t ::= x \mid c \mid f(t, \dots, t)$$

where

- $x$  ranges over a given set of variables **var**,
- $c$  ranges over nullary function symbols in  $\mathcal{F}$ , and
- $f$  ranges over function symbols in  $\mathcal{F}$  with arity  $n > 0$ .

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If  $n$  is nullary,  $f$  is unary, and  $g$  is binary, then examples of terms are:

- $g(f(n), n)$
- $f(g(n, f(n)))$

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## More Examples of Terms

If  $0, 1, \dots$  are nullary,  $s$  is unary, and  $+, -$  and  $*$  are binary, then

$$*(-(2, +(s(x), y)), x)$$

is a term.

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## More Examples of Terms

If  $0, 1, \dots$  are nullary,  $s$  is unary, and  $+, -$  and  $*$  are binary, then

$$*(-(2, +(s(x), y)), x)$$

is a term.

Occasionally, we allow ourselves to use infix notation for function symbols as in

$$(2 - (s(x) + y)) * x$$



$\phi ::= P(t_1, t_2, \dots, t_n) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid$   
 $(\phi \rightarrow \phi) \mid (\forall x\phi) \mid (\exists x\phi)$

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$$\begin{aligned}\phi ::= & P(t_1, t_2, \dots, t_n) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \\ & (\phi \rightarrow \phi) \mid (\forall x\phi) \mid (\exists x\phi)\end{aligned}$$

where

- $P \in \mathcal{P}$  is a predicate symbol of arity  $n \geq 1$ ,

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where

- $P \in \mathcal{P}$  is a predicate symbol of arity  $n \geq 1$ ,
- $t_i$  are terms over  $\mathcal{F}$  and

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where

- $P \in \mathcal{P}$  is a predicate symbol of arity  $n \geq 1$ ,
- $t_i$  are terms over  $\mathcal{F}$  and
- $x$  is a variable.

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Just like for propositional logic, we introduce convenient conventions to reduce the number of parentheses:

- $\neg$ ,  $\forall x$  and  $\exists x$  bind most tightly;
- then  $\wedge$  and  $\vee$ ;
- then  $\rightarrow$ , which is right-associative.

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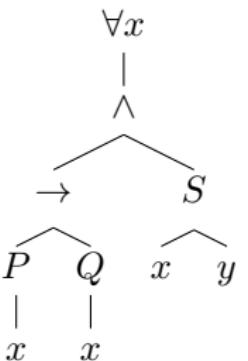
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$$\forall x((P(x) \rightarrow Q(x)) \wedge S(x, y))$$

has parse tree



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## Another Example

Every son of my father is my brother.

### Predicates

$S(x, y)$ :  $x$  is a son of  $y$

$B(x, y)$ :  $x$  is a brother of  $y$

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## Functions

$m$ : constant for “me”

$f(x)$ : father of  $x$



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## The sentence in predicate logic

$$\forall x(S(x, f(m)) \rightarrow B(x, m))$$



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## The sentence in predicate logic

$$\forall x(S(x, f(m)) \rightarrow B(x, m))$$

Does this formula hold?

# Equality as Predicate

Equality is a common predicate, usually used in infix notation.

$$= \in \mathcal{P}$$



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# Equality as Predicate

Equality is a common predicate, usually used in infix notation.

$$= \in \mathcal{P}$$

## Example

Instead of the formula

$$= (f(x), g(x))$$

we usually write the formula

$$f(x) = g(x)$$

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Consider the formula

$$\forall x((P(x) \rightarrow Q(x)) \wedge S(x, y))$$



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Consider the formula

$$\forall x((P(x) \rightarrow Q(x)) \wedge S(x, y))$$

What is the relationship between variable “binder”  $x$  and occurrences of  $x$ ?



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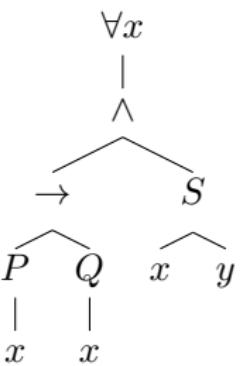
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Consider the formula

$$(\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y))$$



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Which variable *occurrences* are free; which are bound?



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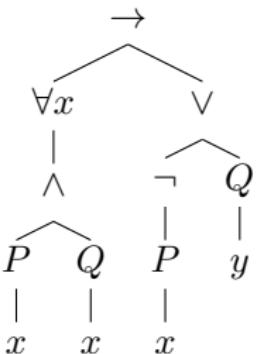
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# Substitution

Variables are *placeholders*. Replacing them by terms is called *substitution*.



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# Substitution

Variables are *placeholders*. Replacing them by terms is called *substitution*.

## Definition

Given a variable  $x$ , a term  $t$  and a formula  $\phi$ , we define  $[x \Rightarrow t]\phi$  to be the formula obtained by replacing each free occurrence of variable  $x$  in  $\phi$  with  $t$ .

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## Example

$$\begin{aligned}[x \Rightarrow f(x, y)](\forall x(P(x) \wedge Q(x))) &\rightarrow (\neg P(x) \vee Q(y)) \\ &= \forall x(P(x) \wedge Q(x)) \rightarrow (\neg P(f(x, y)) \vee Q(y))\end{aligned}$$

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Instead of

$$[x \Rightarrow t]\phi$$

the textbook uses the notation

$$\phi[t/x]$$

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Instead of

$$[x \Rightarrow t]\phi$$

the textbook uses the notation

$$\phi[t/x]$$

(we find the order of arguments in the latter notation hard to remember)

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# Example as Parse Tree

$$[x \Rightarrow f(x, y)]((\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y)))$$
$$= (\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(f(x, y)) \vee Q(y))$$


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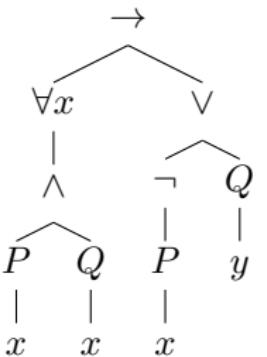
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$$[x \Rightarrow f(x, y)]((\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y)))$$

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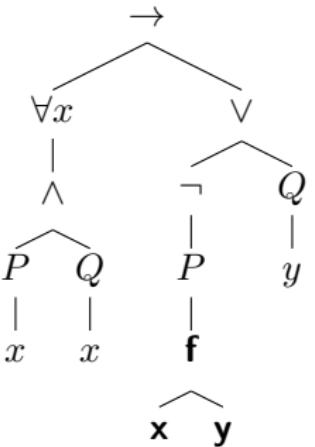
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# Capturing in $[x \Rightarrow t]\phi$

## Problem

$t$  contains variable  $y$  and  $x$  occurs under the scope of  $\forall y$  in  $\phi$



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# Capturing in $[x \Rightarrow t]\phi$

## Problem

$t$  contains variable  $y$  and  $x$  occurs under the scope of  $\forall y$  in  $\phi$

## Example

$$[x \Rightarrow f(y, y)](S(x) \wedge \forall y(P(x) \rightarrow Q(y)))$$



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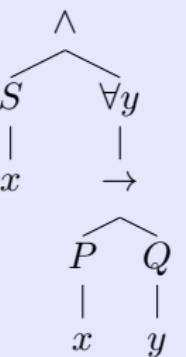
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# Avoiding Capturing

## Definition

Given a term  $t$ , a variable  $x$  and a formula  $\phi$ , we say that  $t$  is free for  $x$  in  $\phi$  if no free  $x$  leaf in  $\phi$  occurs in the scope of  $\forall y$  or  $\exists y$  for any variable  $y$  occurring in  $t$ .



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## Free-ness as precondition

In order to compute  $[x \Rightarrow t]\phi$ , we demand that  $t$  is free for  $x$  in  $\phi$ .



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## Free-ness as precondition

In order to compute  $[x \Rightarrow t]\phi$ , we demand that  $t$  is free for  $x$  in  $\phi$ .

## What if not?



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## Free-ness as precondition

In order to compute  $[x \Rightarrow t]\phi$ , we demand that  $t$  is free for  $x$  in  $\phi$ .

## What if not?

Rename the bound variable!

# Example of Renaming

Advanced Predicate Logic

Nguyen An Khuong,  
Huynh Tuong Nguyen

$$[x \Rightarrow f(y, y)](S(x) \wedge \forall y(P(x) \rightarrow Q(y)))$$



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# Example of Renaming

$$[x \Rightarrow f(y, y)](S(x) \wedge \forall y(P(x) \rightarrow Q(y)))$$



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# Example of Renaming

$$[x \Rightarrow f(y, y)](S(x) \wedge \forall y(P(x) \rightarrow Q(y)))$$

$$[x \Rightarrow f(y, y)](S(x) \wedge \forall z(P(x) \rightarrow Q(z)))$$

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# Example of Renaming


$$[x \Rightarrow f(y, y)](S(x) \wedge \forall y(P(x) \rightarrow Q(y)))$$

$$[x \Rightarrow f(y, y)](S(x) \wedge \forall z(P(x) \rightarrow Q(z)))$$


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# Example of Renaming

$$[x \Rightarrow f(y, y)](S(x) \wedge \forall y(P(x) \rightarrow Q(y)))$$


$$[x \Rightarrow f(y, y)](S(x) \wedge \forall z(P(x) \rightarrow Q(z)))$$


$$S(f(y, y)) \wedge \forall z(P(f(y, y)) \rightarrow Q(z))$$

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## Relationship between propositional and predicate logic

If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic.



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## Relationship between propositional and predicate logic

If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic.

## Inheriting natural deduction

We can translate the rules for natural deduction in propositional logic directly to predicate logic.



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## Relationship between propositional and predicate logic

If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic.



## Inheriting natural deduction

We can translate the rules for natural deduction in propositional logic directly to predicate logic.

## Example

$$\frac{\phi \quad \psi}{\phi \wedge \psi} [\wedge i]$$

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# Built-in Rules for Equality



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$$\frac{}{t = t} [= i] \qquad \frac{t_1 = t_2 \quad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi} [= e]$$



# Properties of Equality

We show:

$$f(x) = g(x) \vdash h(g(x)) = h(f(x))$$

using

$$\frac{}{t = t} [= i] \qquad \frac{t_1 = t_2 \quad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi} [= e]$$

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# Properties of Equality

We show:

$$f(x) = g(x) \vdash h(g(x)) = h(f(x))$$

using

$$\frac{}{t = t} [= i] \qquad \frac{t_1 = t_2 \quad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi} [= e]$$

- |   |                     |             |
|---|---------------------|-------------|
| 1 | $f(x) = g(x)$       | premise     |
| 2 | $h(f(x)) = h(f(x))$ | $= i$       |
| 3 | $h(g(x)) = h(f(x))$ | $= e \ 1,2$ |

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# Rules for Universal Quantification

Advanced Predicate Logic

Nguyen An Khuong,  
Huynh Tuong Nguyen



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$$\frac{\forall x \phi}{[x \Rightarrow t] \phi}$$

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$$\frac{\forall x \phi}{[ \forall x \ e ]} \\ [x \Rightarrow t] \phi$$

We prove:  $F(m(Duong)), \forall x(F(x) \rightarrow \neg M(x)) \vdash \neg M(m(Duong))$

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## Example



$$\frac{\forall x \phi}{[ \forall x \ e ]} \\ [x \Rightarrow t] \phi$$

We prove:  $F(m(Duong)), \forall x(F(x) \rightarrow \neg M(x)) \vdash \neg M(m(Duong))$

- |   |  |                       |
|---|--|-----------------------|
| 1 | $F(m(Duong))$                              | premise               |
| 2 | $\forall x(F(x) \rightarrow \neg M(x))$    | premise               |
| 3 | $F(m(Duong)) \rightarrow \neg M(m(Duong))$ | $\forall x \ e \ 2$   |
| 4 | $\neg M(m(Duong))$                         | $\rightarrow e \ 3,1$ |

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# Rules for Universal Quantification

If we manage to establish a formula  $\phi$  about a fresh variable  $x_0$ , we can assume  $\forall x\phi$ .



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If we manage to establish a formula  $\phi$  about a fresh variable  $x_0$ , we can assume  $\forall x \phi$ .

$$\frac{\boxed{\begin{array}{c} x_0 \\ \vdots \\ [x \Rightarrow x_0] \phi \end{array}}}{\forall x \ i}$$

 $\forall x \phi$ 

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$x_0$   
⋮  
 $[x \Rightarrow x_0]\phi$

## Example

$\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall xQ(x)$  via

$\forall x\phi$



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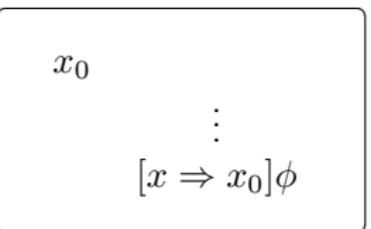
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## Example



$\forall x(P(x) \rightarrow Q(x)), \forall xP(x) \vdash \forall xQ(x)$  via

$\forall x\phi$

|   |                                       |                       |
|---|---------------------------------------|-----------------------|
| 1 | $\forall x(P(x) \rightarrow Q(x))$    | premise               |
| 2 | $\forall xP(x)$                       | premise               |
| 3 | $x_0 \quad P(x_0) \rightarrow Q(x_0)$ | $\forall x \ e \ 1$   |
| 4 | $P(x_0)$                              | $\forall x \ e \ 2$   |
| 5 | $Q(x_0)$                              | $\rightarrow e \ 3,4$ |
| 6 | $\forall xQ(x)$                       | $\forall x \ i \ 3-5$ |



$$\frac{[x \Rightarrow t]\phi}{\exists x \ i} \quad \exists x \phi$$

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**Rules for Existential Quantification**

# Rules for Existential Quantification

Advanced Predicate Logic

Nguyen An Khuong,  
Huynh Tuong Nguyen



$$\frac{\frac{[x \Rightarrow t]\phi}{\exists x \ i} \quad \exists x \phi}{\chi}$$
$$\frac{\exists x \phi \quad \boxed{x_0 \quad [x \Rightarrow x_0]\phi \quad \vdots \quad \chi}}{[\exists e]}$$

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## Example

$$\forall x(P(x) \rightarrow Q(x)), \exists xP(x) \vdash \exists xQ(x)$$

|   |                                    |                         |
|---|------------------------------------|-------------------------|
| 1 | $\forall x(P(x) \rightarrow Q(x))$ | premise                 |
| 2 | $\exists xP(x)$                    | premise                 |
| 3 | $x_0 \quad P(x_0)$                 | assumption              |
| 4 | $P(x_0) \rightarrow Q(x_0)$        | $\forall x \ e \ 1$     |
| 5 | $Q(x_0)$                           | $\rightarrow e \ 4,3$   |
| 6 | $\exists xQ(x)$                    | $\exists x \ i \ 5$     |
| 7 | $\exists xQ(x)$                    | $\exists x \ e \ 2,3-6$ |

# Examples of Quantifier Equivalences

Nguyen An Khuong,  
Huynh Tuong Nguyen



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Assume  $x$  is not free in  $\psi$ :

$$\neg \forall x \phi \dashv \vdash \exists x \neg \phi$$

$$\neg \exists x \phi \dashv \vdash \forall x \neg \phi$$

$$\exists x \exists y \phi \dashv \vdash \exists y \exists x \phi$$

$$\begin{array}{lcl} \forall x \phi \wedge \psi & \dashv \vdash & \forall x (\phi \wedge \psi) \\ \exists x (\psi \rightarrow \phi) & \dashv \vdash & \psi \rightarrow \exists x \phi \end{array}$$



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## Definition

Let  $\mathcal{F}$  contain function symbols and  $\mathcal{P}$  contain predicate symbols.  
A model  $\mathcal{M}$  for  $(\mathcal{F}, \mathcal{P})$  consists of:

- ① A non-empty set  $A$ , the *universe*;
- ② for each nullary function symbol  $f \in \mathcal{F}$  a concrete element  $f^{\mathcal{M}} \in A$ ;
- ③ for each  $f \in \mathcal{F}$  with arity  $n > 0$ , a concrete function  $f^{\mathcal{M}} : A^n \rightarrow A$ ;
- ④ for each  $P \in \mathcal{P}$  with arity  $n > 0$ , a set  $P^{\mathcal{M}} \subseteq A^n$ .

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## Example

Let  $\mathcal{F} = \{e, \cdot\}$  and  $\mathcal{P} = \{\leq\}$ .

Let model  $\mathcal{M}$  for  $(\mathcal{F}, \mathcal{P})$  be defined as follows:

- ① Let  $A$  be the set of binary strings over the alphabet  $\{0, 1\}$ ;
- ② let  $e^{\mathcal{M}} = \epsilon$ , the empty string;
- ③ let  $\cdot^{\mathcal{M}}$  be defined such that  $s_1 \cdot^{\mathcal{M}} s_2$  is the concatenation of the strings  $s_1$  and  $s_2$ ; and
- ④ let  $\leq^{\mathcal{M}}$  be defined such that  $s_1 \leq^{\mathcal{M}} s_2$  iff  $s_1$  is a prefix of  $s_2$ .



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## Example (continued)

- ① Let  $A$  be the set of binary strings over the alphabet  $\{0, 1\}$ ;
- ② let  $e^M = \epsilon$ , the empty string;
- ③ let  $\cdot^M$  be defined such that  $s_1 \cdot^M s_2$  is the concatenation of the strings  $s_1$  and  $s_2$ ; and
- ④ let  $\leq^M$  be defined such that  $s_1 \leq^M s_2$  iff  $s_1$  is a prefix of  $s_2$ .

Some Elements of  $A$ 

- 10001
- $\epsilon$
- $1010 \cdot^M 1100$



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- ① Let  $A$  be the set of binary strings over the alphabet  $\{0, 1\}$ ;
- ② let  $e^M = \epsilon$ , the empty string;
- ③ let  $\cdot^M$  be defined such that  $s_1 \cdot^M s_2$  is the concatenation of the strings  $s_1$  and  $s_2$ ; and
- ④ let  $\leq^M$  be defined such that  $s_1 \leq^M s_2$  iff  $s_1$  is a prefix of  $s_2$ .

Some Elements of  $A$ 

- 10001
- $\epsilon$
- $1010 \cdot^M 1100 = 10101100$
- $\epsilon$
- $000 \cdot^M \epsilon$



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- ① Let  $A$  be the set of binary strings over the alphabet  $\{0, 1\}$ ;
- ② let  $e^M = \epsilon$ , the empty string;
- ③ let  $\cdot^M$  be defined such that  $s_1 \cdot^M s_2$  is the concatenation of the strings  $s_1$  and  $s_2$ ; and
- ④ let  $\leq^M$  be defined such that  $s_1 \leq^M s_2$  iff  $s_1$  is a prefix of  $s_2$ .

Some Elements of  $A$ 

- 10001
- $\epsilon$
- $1010 \cdot^M 1100 = 10101100$
- $\epsilon$
- $000 \cdot^M \epsilon = 000$



## Interpretation of equality

Usually, we require that the equality predicate  $=$  is interpreted as same-ness.

## Extensionality restriction

This means that allowable models are restricted to those in which  $a =^M b$  holds if and only if  $a$  and  $b$  are the same elements of the model's universe.

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- ① Let  $A$  be the set of binary strings over the alphabet  $\{0, 1\}$ ;
- ② let  $e^M = \epsilon$ , the empty string;
- ③ let  $\cdot^M$  be defined such that  $s_1 \cdot^M s_2$  is the concatenation of the strings  $s_1$  and  $s_2$ ; and
- ④ let  $\leq^M$  be defined such that  $s_1 \leq^M s_2$  iff  $s_1$  is a prefix of  $s_2$ .

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## Example (continued)

- ① Let  $A$  be the set of binary strings over the alphabet  $\{0, 1\}$ ;
- ② let  $e^{\mathcal{M}} = \epsilon$ , the empty string;
- ③ let  $\cdot^{\mathcal{M}}$  be defined such that  $s_1 \cdot^{\mathcal{M}} s_2$  is the concatenation of the strings  $s_1$  and  $s_2$ ; and
- ④ let  $\leq^{\mathcal{M}}$  be defined such that  $s_1 \leq^{\mathcal{M}} s_2$  iff  $s_1$  is a prefix of  $s_2$ .

Equality in  $\mathcal{M}$ 

- $000 =^{\mathcal{M}} 000$
- $001 \neq^{\mathcal{M}} 100$



Let  $\mathcal{F} = \{z, s\}$  and  $\mathcal{P} = \{\leq\}$ .

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## Another Example

Let  $\mathcal{F} = \{z, s\}$  and  $\mathcal{P} = \{\leq\}$ .

Let model  $\mathcal{M}$  for  $(\mathcal{F}, \mathcal{P})$  be defined as follows:

- ① Let  $A$  be the set of natural numbers;



Let  $\mathcal{F} = \{z, s\}$  and  $\mathcal{P} = \{\leq\}$ .

Let model  $\mathcal{M}$  for  $(\mathcal{F}, \mathcal{P})$  be defined as follows:

- ① Let  $A$  be the set of natural numbers;
- ② let  $z^{\mathcal{M}} = 0$ ;

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Let  $\mathcal{F} = \{z, s\}$  and  $\mathcal{P} = \{\leq\}$ .

Let model  $\mathcal{M}$  for  $(\mathcal{F}, \mathcal{P})$  be defined as follows:

- ① Let  $A$  be the set of natural numbers;
- ② let  $z^{\mathcal{M}} = 0$ ;
- ③ let  $s^{\mathcal{M}}$  be defined such that  $s(n) = n + 1$ ; and

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## Another Example

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Let model  $\mathcal{M}$  for  $(\mathcal{F}, \mathcal{P})$  be defined as follows:

- ① Let  $A$  be the set of natural numbers;
- ② let  $z^{\mathcal{M}} = 0$ ;
- ③ let  $s^{\mathcal{M}}$  be defined such that  $s(n) = n + 1$ ; and
- ④ let  $\leq^{\mathcal{M}}$  be defined such that  $n_1 \leq^{\mathcal{M}} n_2$  iff the natural number  $n_1$  is less than or equal to  $n_2$ .

# How To Handle Free Variables?

Nguyen An Khuong,  
Huynh Tuong Nguyen



## Idea

We can give meaning to formulas with free variables by providing an environment (lookup table) that assigns variables to elements of our universe:

$$l : \mathbf{var} \rightarrow A.$$

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# How To Handle Free Variables?

## Idea

We can give meaning to formulas with free variables by providing an environment (lookup table) that assigns variables to elements of our universe:

$$l : \mathbf{var} \rightarrow A.$$

## Environment extension

We define environment extension such that  $l[x \mapsto a]$  is the environment that maps  $x$  to  $a$  and any other variable  $y$  to  $l(y)$ .



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The model  $\mathcal{M}$  satisfies  $\phi$  with respect to environment  $l$ , written  
 $\mathcal{M} \models_l \phi$ :

- in case  $\phi$  is of the form  $P(t_1, t_2, \dots, t_n)$ , if the result  $(a_1, a_2, \dots, a_n)$  of evaluating  $t_1, t_2, \dots, t_n$  with respect to  $l$  is in  $P^{\mathcal{M}}$ ;
- in case  $\phi$  has the form  $\forall x\psi$ , if the  $\mathcal{M} \models_{l[x \mapsto a]} \psi$  holds for all  $a \in A$ ;

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## Satisfaction Relation

The model  $\mathcal{M}$  satisfies  $\phi$  with respect to environment  $l$ , written  $\mathcal{M} \models_l \phi$ :

- in case  $\phi$  is of the form  $P(t_1, t_2, \dots, t_n)$ , if the result  $(a_1, a_2, \dots, a_n)$  of evaluating  $t_1, t_2, \dots, t_n$  with respect to  $l$  is in  $P^{\mathcal{M}}$ ;
- in case  $\phi$  has the form  $\forall x\psi$ , if the  $\mathcal{M} \models_{l[x \mapsto a]} \psi$  holds for all  $a \in A$ ;
- in case  $\phi$  has the form  $\exists x\psi$ , if the  $\mathcal{M} \models_{l[x \mapsto a]} \psi$  holds for some  $a \in A$ ;

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- in case  $\phi$  has the form  $\neg\psi$ , if  $\mathcal{M} \models_l \psi$  does not hold;

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# Satisfaction Relation (continued)

- in case  $\phi$  has the form  $\neg\psi$ , if  $\mathcal{M} \models_l \psi$  does not hold;
- in case  $\phi$  has the form  $\psi_1 \vee \psi_2$ , if  $\mathcal{M} \models_l \psi_1$  holds or  $\mathcal{M} \models_l \psi_2$  holds;



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# Satisfaction Relation (continued)

- in case  $\phi$  has the form  $\neg\psi$ , if  $\mathcal{M} \models_l \psi$  does not hold;
- in case  $\phi$  has the form  $\psi_1 \vee \psi_2$ , if  $\mathcal{M} \models_l \psi_1$  holds or  $\mathcal{M} \models_l \psi_2$  holds;
- in case  $\phi$  has the form  $\psi_1 \wedge \psi_2$ , if  $\mathcal{M} \models_l \psi_1$  holds and  $\mathcal{M} \models_l \psi_2$  holds; and



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- in case  $\phi$  has the form  $\psi_1 \wedge \psi_2$ , if  $\mathcal{M} \models_l \psi_1$  holds and  $\mathcal{M} \models_l \psi_2$  holds; and
- in case  $\phi$  has the form  $\psi_1 \rightarrow \psi_2$ , if  $\mathcal{M} \models_l \psi_1$  holds whenever  $\mathcal{M} \models_l \psi_2$  holds.

# Satisfaction of Closed Formulas

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If a formula  $\phi$  has no free variables, we call  $\phi$  a *sentence*.



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If a formula  $\phi$  has no free variables, we call  $\phi$  a *sentence*.

$\mathcal{M} \models_l \phi$  holds or does not hold regardless of the choice of  $l$ . Thus we write  $\mathcal{M} \models \phi$  or  $\mathcal{M} \not\models \phi$ .

# Semantic Entailment and Satisfiability

Let  $\Gamma$  be a possibly infinite set of formulas in predicate logic and  $\psi$  a formula.



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# Semantic Entailment and Satisfiability

Let  $\Gamma$  be a possibly infinite set of formulas in predicate logic and  $\psi$  a formula.

## Entailment

$\Gamma \models \psi$  iff for all models  $\mathcal{M}$  and environments  $l$ , whenever  $\mathcal{M} \models_l \phi$  holds for all  $\phi \in \Gamma$ , then  $\mathcal{M} \models_l \psi$ .



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## Satisfiability of Formulas

$\psi$  is satisfiable iff there is some model  $\mathcal{M}$  and some environment  $l$  such that  $\mathcal{M} \models_l \psi$  holds.



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# Semantic Entailment and Satisfiability

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$\Gamma \models \psi$  iff for all models  $\mathcal{M}$  and environments  $l$ , whenever  $\mathcal{M} \models_l \phi$  holds for all  $\phi \in \Gamma$ , then  $\mathcal{M} \models_l \psi$ .

## Satisfiability of Formulas

$\psi$  is satisfiable iff there is some model  $\mathcal{M}$  and some environment  $l$  such that  $\mathcal{M} \models_l \psi$  holds.

## Satisfiability of Formula Sets

$\Gamma$  is satisfiable iff there is some model  $\mathcal{M}$  and some environment  $l$  such that  $\mathcal{M} \models_l \phi$ , for all  $\phi \in \Gamma$ .



Let  $\Gamma$  be a possibly infinite set of formulas in predicate logic and  $\psi$  a formula.

## Validity

$\psi$  is valid iff for all models  $\mathcal{M}$  and environments  $l$ , we have  
 $\mathcal{M} \models_l \psi$ .

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# The Problem with Predicate Logic

## Entailment ranges over models

Semantic entailment between sentences:  $\phi_1, \phi_2, \dots, \phi_n \models \psi$  requires that in *all* models that satisfy  $\phi_1, \phi_2, \dots, \phi_n$ , the sentence  $\psi$  is satisfied.



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## How to effectively argue about all possible models?

Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.



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# The Problem with Predicate Logic

Advanced Predicate Logic

Nguyen An Khuong,  
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## Entailment ranges over models

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## How to effectively argue about all possible models?

Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.

## Idea from propositional logic

Can we use natural deduction for showing entailment?

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# Central Result of Natural Deduction



$$\phi_1, \dots, \phi_n \models \psi$$

iff

$$\phi_1, \dots, \phi_n \vdash \psi$$

proven by Kurt Gödel, in 1929 in his doctoral dissertation

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# Recall: Decidability

## Decision problems

A *decision problem* is a question in some formal system with a yes-or-no answer.



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# Recall: Decidability

## Decision problems

A *decision problem* is a question in some formal system with a yes-or-no answer.

## Decidability

Decision problems for which there is an algorithm that returns “yes” whenever the answer to the problem is “yes”, and that returns “no” whenever the answer to the problem is “no”, are called *decidable*.



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# Recall: Decidability

## Decision problems

A *decision problem* is a question in some formal system with a yes-or-no answer.

## Decidability

Decision problems for which there is an algorithm that returns “yes” whenever the answer to the problem is “yes”, and that returns “no” whenever the answer to the problem is “no”, are called *decidable*.

## Decidability of satisfiability

The question, whether a given propositional formula is satisfiable, is decidable.

# Undecidability of Predicate Logic

## Theorem

The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula  $\phi$  in that language, decides whether  $\models \phi$ .



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## Proof

- Establish that the Post Correspondence Problem (PCP) is undecidable (here only as sketch).



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# Undecidability of Predicate Logic

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## Proof

- Establish that the Post Correspondence Problem (PCP) is undecidable (here only as sketch).
- Translate an arbitrary PCP, say  $C$ , to a formula  $\phi$ .

## Theorem

The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula  $\phi$  in that language, decides whether  $\models \phi$ .

## Proof

- Establish that the Post Correspondence Problem (PCP) is undecidable (here only as sketch).
- Translate an arbitrary PCP, say  $C$ , to a formula  $\phi$ .
- Establish that  $\models \phi$  holds if and only if  $C$  has a solution.

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The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula  $\phi$  in that language, decides whether  $\models \phi$ .

## Proof

- Establish that the Post Correspondence Problem (PCP) is undecidable (here only as sketch).
- Translate an arbitrary PCP, say  $C$ , to a formula  $\phi$ .
- Establish that  $\models \phi$  holds if and only if  $C$  has a solution.
- Conclude that validity of pred. logic formulas is undecidable.



# Post Correspondence Problem

## Informally

Can we line up copies of the cards such that the top row spells out the same sequence as the bottom row?

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## Informally

Can we line up copies of the cards such that the top row spells out the same sequence as the bottom row?

## Formally

Given a finite sequence of pairs  $(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$  such that all  $s_i$  and  $t_i$  are binary strings of positive length, is there a sequence of indices  $i_1, i_2, \dots, i_n$  with  $n \geq 1$  such that the concatenations  $s_{i_1} s_{i_2} \dots s_{i_n}$  and  $t_{i_1} t_{i_2} \dots t_{i_n}$  are equal?

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# Undecidability of Post Correspondence Problem

## Turing machines

Basic abstract symbol-manipulating devices that can simulate in principle any computer algorithm. The input is a string of symbols on a *tape*, and the machine “accepts” the input string, if it reaches one of a number of *accepting states*.

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# Undecidability of Post Correspondence Problem

## Turing machines

Basic abstract symbol-manipulating devices that can simulate in principle any computer algorithm. The input is a string of symbols on a *tape*, and the machine “accepts” the input string, if it reaches one of a number of *accepting states*.

## Termination of Programs is Undecidable

It is undecidable, whether program with input terminates.



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# Undecidability of Post Correspondence Problem

## Turing machines

Basic abstract symbol-manipulating devices that can simulate in principle any computer algorithm. The input is a string of symbols on a *tape*, and the machine “accepts” the input string, if it reaches one of a number of *accepting states*.

## Termination of Programs is Undecidable

It is undecidable, whether program with input terminates.

## Proof idea

For a Turing machine with a given input, construct a PCP such that a solution of the PCP exists if and only if the Turing machine accepts the solution.



## Bits as Functions

Represent bits 0 and 1 by functions  $f_0$  and  $f_1$ .

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# Translate Post Correspondence Problem to Formula

Advanced Predicate Logic

Nguyen An Khuong,  
Huynh Tuong Nguyen



## Bits as Functions

Represent bits 0 and 1 by functions  $f_0$  and  $f_1$ .

## Strings as Terms

Represent the empty string by a constant  $e$ .

The string  $b_1 b_2 \dots b_l$  corresponds to the term

$$f_{b_l}(f_{b_{l-1}} \dots (f_{b_2}(f_{b_1}(e))) \dots)$$

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# Towards a Formula for a PCP



Let C be the problem

$$\begin{array}{cccc} s_1 & s_2 & \dots & s_k \\ t_1 & t_2 & \dots & t_k \end{array}$$

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Let C be the problem

$$\begin{array}{cccc} s_1 & s_2 & \dots & s_k \\ t_1 & t_2 & \dots & t_k \end{array}$$

### Idea

$P(s, t)$  holds iff there is a sequence of indices  $(i_1, i_2, \dots, i_m)$  such that  $s$  is  $s_{i_1} s_{i_2} \dots s_{i_m}$  and  $t$  is  $t_{i_1} t_{i_2} \dots t_{i_m}$ .

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# The Formula $\phi$



$\phi = \phi_1 \wedge \phi_2 \rightarrow \phi_3$ , where

$$\phi_1 = \bigwedge_{i=1}^k P(f_{s_i}(e), f_{t_i}(e))$$

$$\phi_2 = \forall v \forall w (P(v, w) \rightarrow \bigwedge_{i=1}^k P(f_{s_i}(v), f_{t_i}(w)))$$

$$\phi_3 = \exists z P(z, z)$$

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# Undecidability of Predicate Logic

## So Far

Post correspondence problem is undecidable.

Constructed  $\phi_C$  for Post correspondence problem  $C$ .

## To Show

$\models \phi_C$  holds if and only if  $C$  has a solution.

## Proof

Proof via construction of  $\phi_C$ . Formally construct an interpretation of strings and show that whenever there is a solution, the formula  $\phi_C$  holds and vice versa.



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# Summary of Undecidability Proof

## Theorem

The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula  $\phi$  in that language, decides whether  $\models \phi$ .

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## Proof

- Establish that the Post Correspondence Problem (PCP) is undecidable



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## Theorem

The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula  $\phi$  in that language, decides whether  $\models \phi$ .

## Proof

- Establish that the Post Correspondence Problem (PCP) is undecidable
- Translate an arbitrary PCP, say  $C$ , to a formula  $\phi$ .



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The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula  $\phi$  in that language, decides whether  $\models \phi$ .

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- Translate an arbitrary PCP, say  $C$ , to a formula  $\phi$ .
- Establish that  $\models \phi$  holds if and only if  $C$  has a solution.

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# Summary of Undecidability Proof

## Theorem

The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula  $\phi$  in that language, decides whether  $\models \phi$ .

## Proof

- Establish that the Post Correspondence Problem (PCP) is undecidable
- Translate an arbitrary PCP, say  $C$ , to a formula  $\phi$ .
- Establish that  $\models \phi$  holds if and only if  $C$  has a solution.
- Conclude that validity of pred. logic formulas is undecidable.

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# Compactness Theorem

Let  $\Gamma$  be a set of sentences of predicate logic. If all finite subsets of  $\Gamma$  are satisfiable, then  $\Gamma$  is satisfiable.

# Proof of Compactness Theorem

Advanced Predicate Logic

Nguyen An Khuong,  
Huynh Tuong Nguyen



Assume  $\Gamma$  is not satisfiable.

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# Proof of Compactness Theorem

Advanced Predicate Logic

Nguyen An Khuong,  
Huynh Tuong Nguyen



Assume  $\Gamma$  is not satisfiable.  
We thus have  $\Gamma \models \perp$ .

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# Proof of Compactness Theorem

Advanced Predicate Logic

Nguyen An Khuong,  
Huynh Tuong Nguyen



Assume  $\Gamma$  is not satisfiable.

We thus have  $\Gamma \models \perp$ .

Via completeness, we have  $\Gamma \vdash \perp$ .

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# Proof of Compactness Theorem



Assume  $\Gamma$  is not satisfiable.

We thus have  $\Gamma \models \perp$ .

Via completeness, we have  $\Gamma \vdash \perp$ .

The proof is finite, thus only uses a finite subset  $\Delta \subset \Gamma$  of premises.

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# Proof of Compactness Theorem



Assume  $\Gamma$  is not satisfiable.

We thus have  $\Gamma \models \perp$ .

Via completeness, we have  $\Gamma \vdash \perp$ .

The proof is finite, thus only uses a finite subset  $\Delta \subset \Gamma$  of premises.

Thus,  $\Delta \vdash \perp$ , and  $\Delta \models \perp$  via soundness.

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# Reachability not Expressible in Predicate Logic

Nguyen An Khuong,  
Huynh Tuong Nguyen



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There is no predicate logic formula  $\phi_{G,u,v}$  with  $u$  and  $v$  as its only free variables and  $R$  as its only predicate symbol, such that  $\phi_{G,u,v}$  holds iff there is a path from  $u$  to  $v$  in  $G$ .



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Let  $\psi$  be a sentence of predicate logic such that for any natural number  $n \geq 1$  there is a model of  $\psi$  with at least  $n$  elements.  
Then  $\psi$  has a model with infinitely many elements.

# Homeworks and Next Week Plan?

## Homeworks

It is recommended that you should do as much as you can ALL marked exercises in [2, Sect. 2.8] (notice that sample solutions for these exercises are available in [3]). For this lecture, the following are recommended exercises [2]:

- 2.1: 1a); 2a)
- 2.2: 6
- 2.3: 1a); 1b); 6a); 6b); 6c); 7b); 9b); 9c); 13d)
- 2.4: 2); 3); 11a); 11c); 12e); 12f); 12h); 12k)
- 2.5: 1c); 1e).



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## Next Weeks?

- Exercises Session;

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## Next Weeks?

- Exercises Session;
- Applications of FoL.