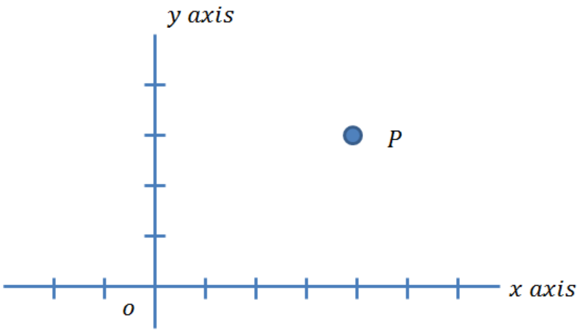
MATH1350H-A: Linear Algebra 1

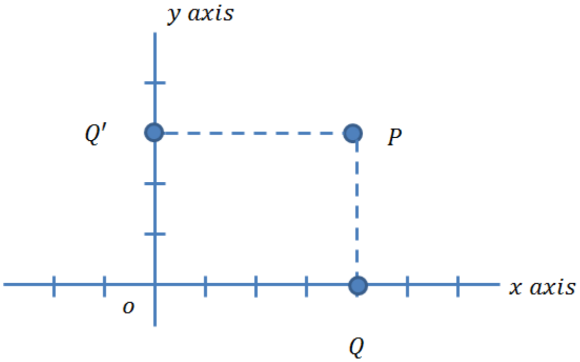
Trent University

Lesson 4.1: Introduction to Vectors



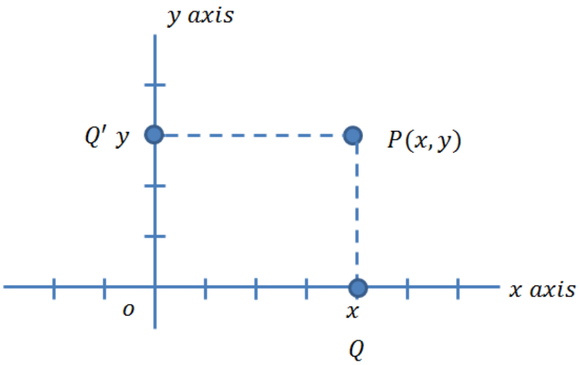
Consider a point P plotted in the xy-plane.

Chapter 4: Vector Spaces
4.1 Introduction to Vectors

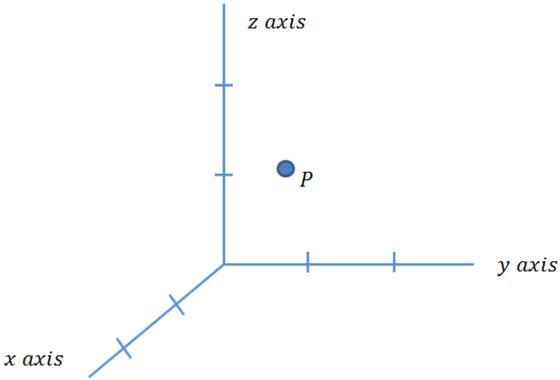


Let Q be the projection of P onto the x-axis, and Q' be the projection of P onto the y-axis.

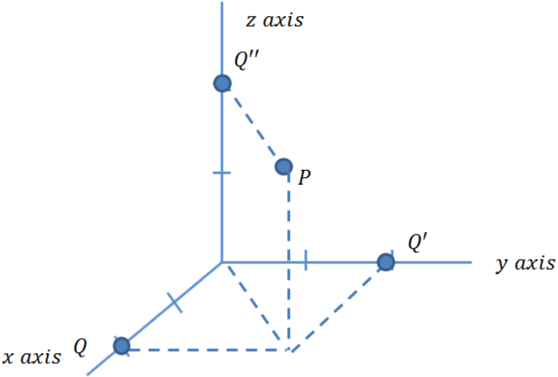
Chapter 4: Vector Spaces
4.1 Introduction to Vectors



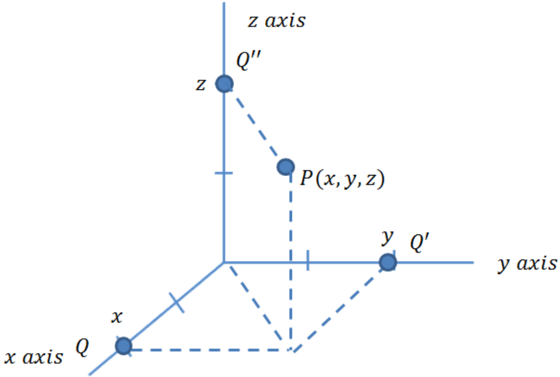
We associate to each point Q and Q' a real number value representing their distance from the origin. Call these numbers x and y respectively. Assign to point P the ordered pair (x, y).



Consider now a point P in three dimensional space.



Let Q, Q' and Q'' be the projections of P onto the x-axis, y-axis, and z-axis respectively.



Again, we associate to each point Q, Q' and Q'' a real numbers, x, y and z representing their respective distances from the origin. We may now assign to point P the ordered triple (x, y, z).

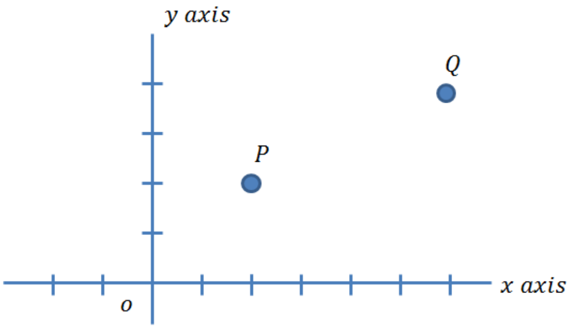
Coordinate systems in $\mathbb{R}^2$ and $\mathbb{R}^3$

- We define $\mathbb{R}^2$ to be the set of all ordered pairs (x, y) where x and y are in $\mathbb{R}$, or

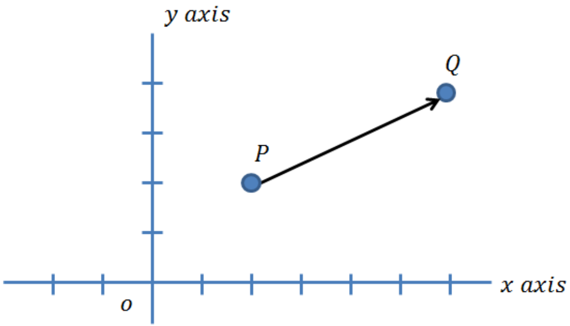
$$\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$$

- Similarly define $\mathbb{R}^3$ as

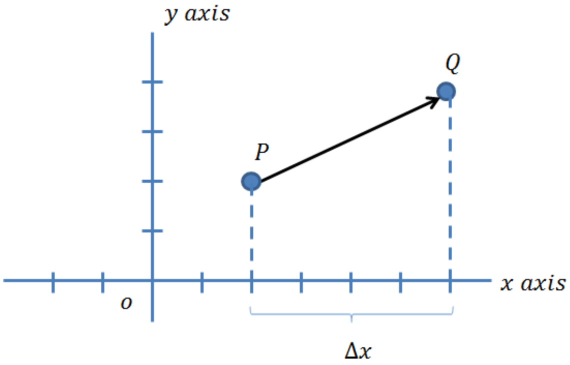
$$\mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$$



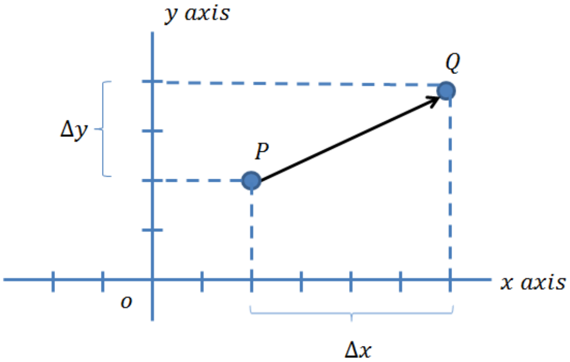
Let P and Q be two points in the xy-plane.



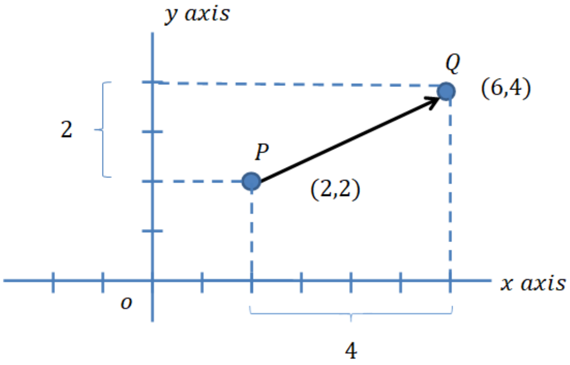
The directed line segment from P to Q is called a vector and can be written as $\mathbf{v} = \overrightarrow{PQ}$. The point P is called the initial point and Q is called the terminal point.



We may consider the vector's distance in the x direction by projecting P and Q to the x-axis. This quantity represents the change in x direction, from P to Q, and we can call it the vector's x component.



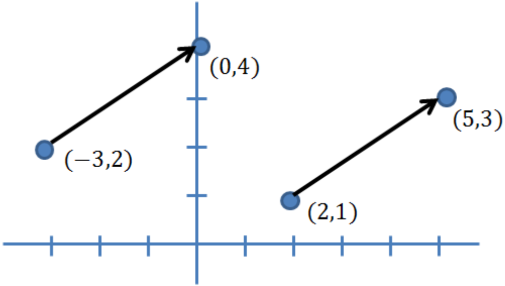
Similarly we consider the vector's distance in the y direction. This is called the vector's y component, and it represents the change in y direction from P to Q.



In this case P is point $(2, 2)$ and Q is point $(6, 4)$, so $\Delta x = 4$ and $\Delta y = 2$. We therefore assign the following 2×1 column matrix to represent this vector,

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Example



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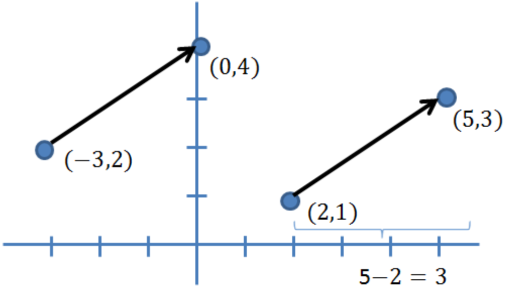

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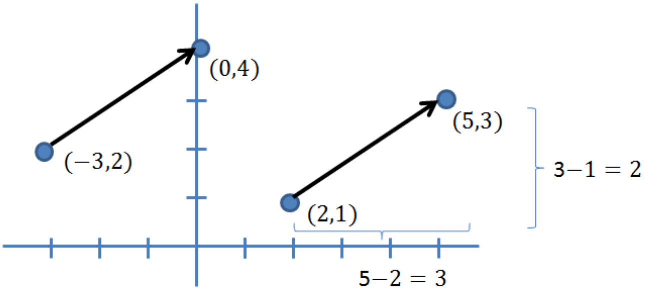

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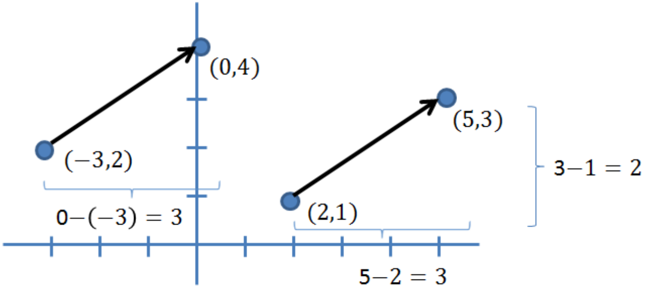
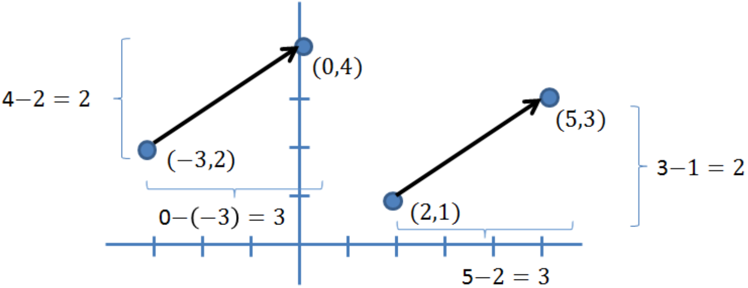
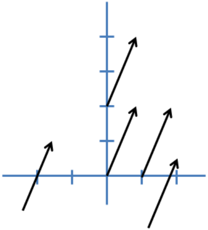


Example



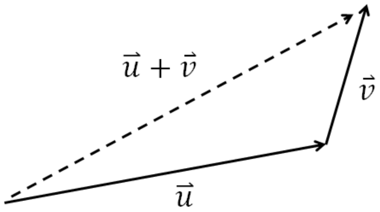
- The previous example illustrates the fact that vectors do not have a fixed position in space.
- Two vectors are equal if they have the same length and direction, but do not need to have the same initial point.
- The following vectors are all equal



Addition of Vectors in the Plane

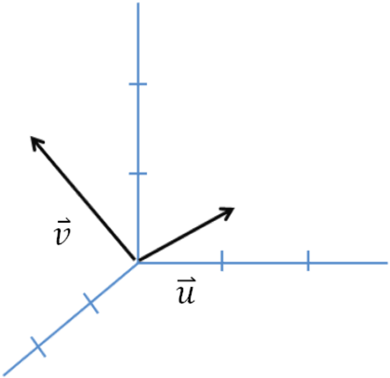
Placing the initial point of $\overrightarrow{v}$ at the terminal point $\overrightarrow{u}$, the sum $\overrightarrow{u} + \overrightarrow{v}$ is the vector whose initial point is that of $\overrightarrow{u}$ and whose terminal point is that of $\overrightarrow{v}$.

Example



Addition in 3D space

Example



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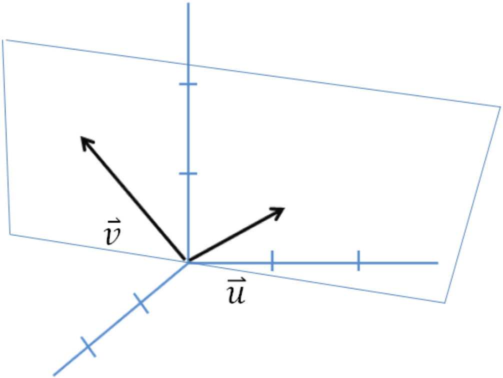

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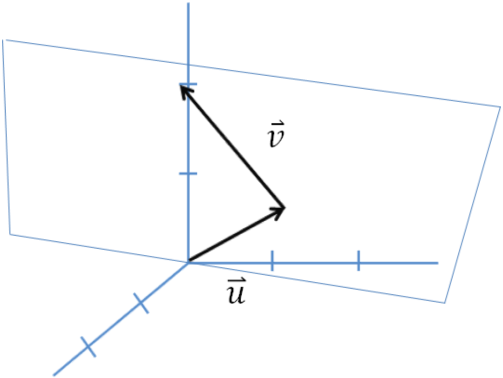

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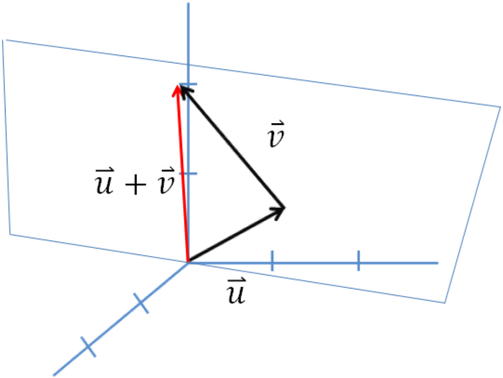
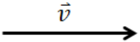


Scalar multiple of a vector

- We have already seen how to compute the scalar multiple of a column matrix.
- How does the vector look geometrically?

Example



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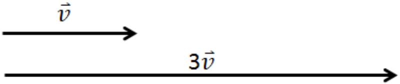

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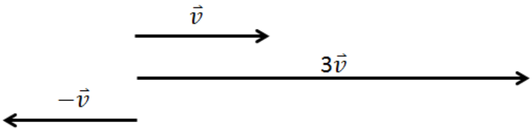
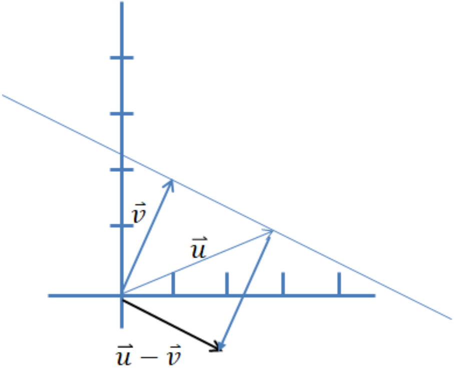
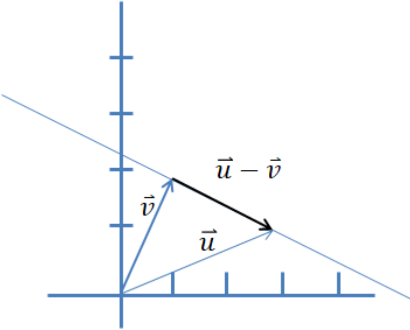


Subtraction of vectors

Subtraction, $\overrightarrow{u} - \overrightarrow{v}$ is performed by adding $-\overrightarrow{v}$ to $\overrightarrow{u}$. This can be seen geometrically in two ways.

Example



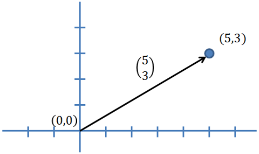


Sets of vectors

- The set of all points in the xy plane as $\mathbb{R}^2$, and the set of all vectors in the plane is also denoted by $\mathbb{R}^2$:

$$\mathbb{R}^2 = \left\{ \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) \middle| x_1, x_2 \in \mathbb{R} \right\}.$$

- To each point, or each ordered pair, we may assign a vector; namely, the vector that begins at the origin and ends at that point. For example, to the point (5,3) we assign the vector which starts at (0,0) and ends at (5,3).



This vector is written with column matrix $\left(\begin{array}{c} 5 \\ 3 \end{array} \right)$.

Sets of vectors

- The same can be said of points in three dimensional space. Denote the set of all vectors in $\mathbb{R}^3$ by

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| x_1, x_2, x_3 \in \mathbb{R} \right\}$$

Length of a vector

Pythagorean Theorem

Length of a vector

Definition

- For vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$, define the <u>length</u>, or <u>norm</u>, of $\mathbf{v}$ to be
$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$$
- For vector $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$, define the length, or norm, of $\mathbf{w}$ to be
$$||\mathbf{w}|| = \sqrt{w_1^2 + w_2^2 + w_3^2}$$
- We also refer to this as the <u>magnitude</u> of a vector.

Example

Find the lengths of $\mathbf{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Unit vector

Definition

- A vector which has length equal to 1 is called a <u>unit vector</u>.
- For any nonzero vector $\mathbf{v}$, the vector $\mathbf{u} = \frac{1}{||\mathbf{v}||}\mathbf{v}$ is unit vector in the same direction as $\mathbf{v}$ (prove this as an exercise).
- Finding this unit vector in the same direction of $\mathbf{v}$ is called <u>normalizing</u> $\mathbf{v}$.

Example

Find unit vectors in the same direction as $\mathbf{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

n-Vectors, and $\mathbb{R}^n$

- So far we have dealt with vectors in the familiar spaces of $\mathbb{R}^2$ and $\mathbb{R}^3$; i.e. things we can draw.
- We described a vector as being an object which has a magnitude and a direction, but no fixed position in space.
- We are going to learn the proper definition of a vector space (and hence a vector) but first we generalize what we have learned about $\mathbb{R}^2$ and $\mathbb{R}^3$ to n-space.
- Note that here we are unable to draw diagrams so we rely on the algebra to paint the picture for us.

n-Vectors, and $\mathbb{R}^n$

Definition

Let $n \in \mathbb{N}$. Then $\mathbb{R}^n$ can be described in two ways; the set of n-tuples

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) | x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

or as the set of $n \times 1$ column matrices

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}$$

Recall that these column matrices are also denoted $\mathcal{M}_{n\times 1}(\mathbb{R})$.

Adding vectors in $\mathbb{R}^n$

Example

Adding vectors and scalar multiplication in $\mathbb{R}^n$. (we have already seen this more generally with any size matrices)

Norm of a vector

Definition

The <u>norm</u> of the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, is the number

$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

which describes its magnitude.

Norm of a vector

Example

Compute the norm of the vector $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 3 \end{pmatrix} \in \mathbb{R}^4$.

Norm of a vector

Theorem

Let $\mathbf{v}$ *be a vector in* $\mathbb{R}^n$ *and* $c \in \mathbb{R}$ *a scalar. Then*

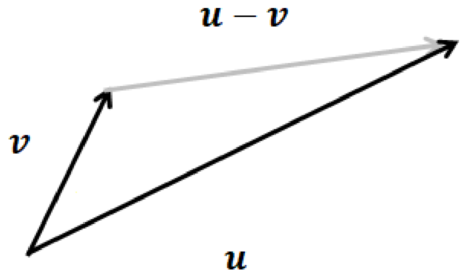
$$||c\mathbf{v}|| = |c|||\mathbf{v}||$$

where $|c|$ *is the absolute value of* c*.*

Example

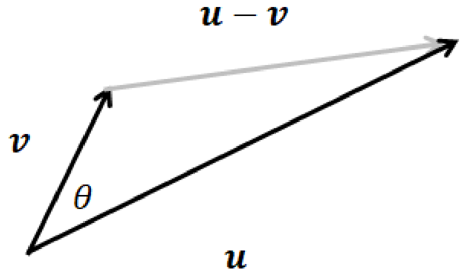
- $$||12\mathbf{v}|| = 12||\mathbf{v}||$$
- $$||-5\mathbf{v}|| = 5||\mathbf{v}||$$
- $$||-\mathbf{v}|| = ||\mathbf{v}||.$$

Distance between vectors



Define the distance between vectors $\mathbf{u}$ and $\mathbf{v}$ to be $||\mathbf{u} - \mathbf{v}||$.

Cosine Law



For a triangle in the plane, with sides labeled above, recall the cosine law:

$$||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||||\mathbf{v}||\cos\theta.$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

Dot product in $\mathbb{R}^n$

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ with

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

The <u>dot product</u> $\mathbf{u} \cdot \mathbf{v}$, of vectors $\mathbf{u}$ and $\mathbf{v}$ is the real number given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Properties of the dot product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$, and $c \in \mathbb{R}$.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$.
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$.

- An operation between vectors which satisfies these properties is called an <u>inner product</u>.
- Note that $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$, so $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.

The Triangle Inequality

For any $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$

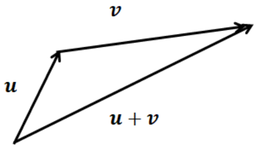
Theorem (The Cauchy-Schwarz Inequality)

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \; ||\mathbf{v}||.$$

(on the left side is the absolute value of $\mathbf{u} \cdot \mathbf{v}$*.)*

Theorem (The Triangle Inequality)

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$



Angle between two vectors

Definition

Let $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is defined to be

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}\right)$$

(note that $-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||} \leq 1$ by Cauchy-Schwarz inequality)

Theorem

Vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ are orthogonal (at right angles to each other) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

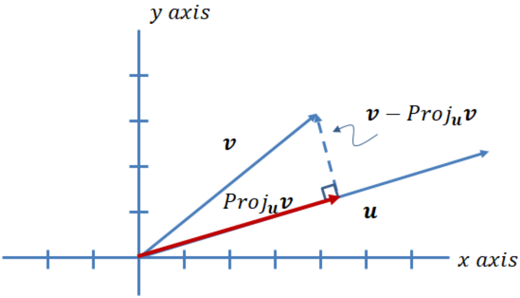
Angle between two vectors

Example

Find the angle between

$$\mathbf{u} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Projections



(vector $\mathbf{u}$, in blue, begins at the origin. Vector $\text{proj}_{\mathbf{u}}(\mathbf{v})$ is in red.)

Definition

Let $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ with $\mathbf{u} \neq 0$. The projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||^2}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}.$$

Projections

Example

Find the projection of $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, onto $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

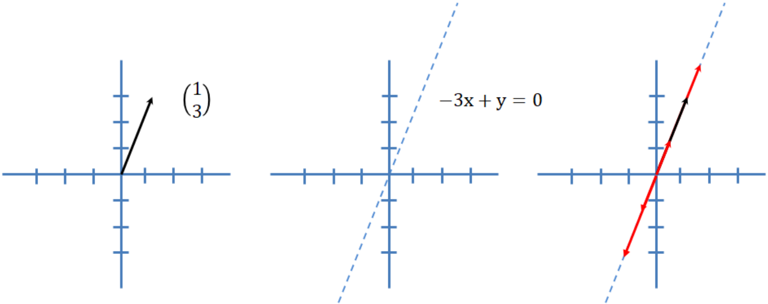
Lines in $\mathbb{R}^2$

A line in $\mathbb{R}^2$ is the set of points $(x, y) \in \mathbb{R}^2$ which satisfy an equation of the form:

- General form: $ax + by = c$, where $a, b, c \in \mathbb{R}$.
- Slope-intercept form: $y = mx + b_0$, where m is its slope, b_0 is its y-intercept
- Point-slope form: $(y - y_1) = m(x - x_1)$, where m is its slope and (x_1, y_1) is any point on the line.

Lines in $\mathbb{R}^2$

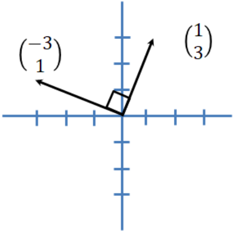
- The vector $\mathbf{d} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ plotted at the origin lies on the line $\ell : -3x + y = 0$.

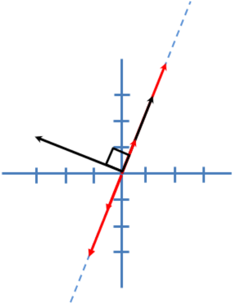


- Line ℓ is all endpoints of the multiples of $\mathbf{d}$ (from the origin).
- Describe this line with vectors as $\ell : \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ t \in \mathbb{R}$.
- We say $\mathbf{d}$ is a <u>direction vector</u> for ℓ.

Lines in $\mathbb{R}^2$

- Vector $\mathbf{n} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is orthogonal to $\mathbf{d} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, since $\mathbf{n} \cdot \mathbf{d} = 0$.

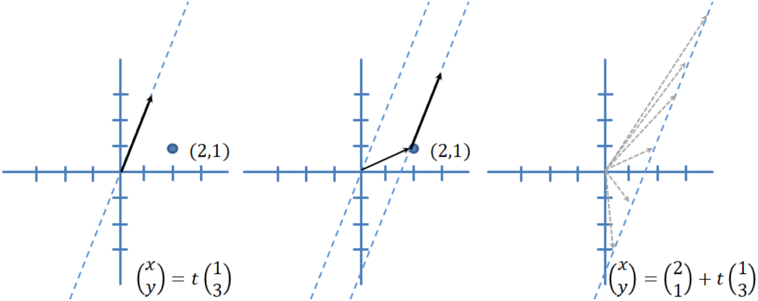




- Also $\mathbf{n} \cdot (t\mathbf{d}) = 0$ for any $t \in \mathbb{R}$.
- $\ell : -3x + y = 0$ described as vectors $\mathbf{x} \in \mathbb{R}^2$ orthogonal to $\mathbf{n}$.
- The equation $\mathbf{n} \cdot \mathbf{x} = 0$ defines all $\mathbf{x} \in \mathbb{R}^2$ that make up ℓ.
- We call $\mathbf{n}$ a <u>normal vector</u> to this line .

Lines in $\mathbb{R}^2$

- Consider the line with direction $\mathbf{d} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ through $P = (2, 1)$.

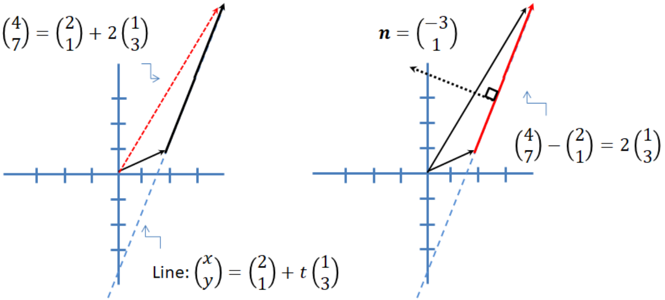


- Let $\mathbf{p} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
- This line described with vectors is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ or simply $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, for $t \in \mathbb{R}$.

Lines in $\mathbb{R}^2$

- Rearrange the equation $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ to get $\mathbf{x} - \mathbf{p} = t\mathbf{d}$.
- If $\mathbf{n} \cdot (t\mathbf{d}) = 0$ then $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$.
- Line $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ described as all $\mathbf{x} \in \mathbb{R}^2$ such that $(\mathbf{x} - \mathbf{p})$ orthogonal to $\mathbf{n}$.
- Equivalently all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$.

Example



Vector form of a line in $\mathbb{R}^2$ or $\mathbb{R}^3$

Definition

The vector form of the equation of a line ℓ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + t\mathbf{d}$$

where $t \in \mathbb{R}$, $\mathbf{p}$ is a specific point on ℓ, and $\mathbf{d} \neq 0$ is the direction vector for ℓ. This may be written

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + t \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + t \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

The components are called the parametric equations of ℓ:

$$\begin{aligned} x &= p_1 + td_1 \\ y &= p_2 + td_2 \end{aligned} \qquad\qquad \begin{aligned} x &= p_1 + td_1 \\ y &= p_2 + td_2 \\ z &= p_3 + td_3 \end{aligned}$$

Vector form of a line in $\mathbb{R}^2$ or $\mathbb{R}^3$

Example

Find the vector and parametric equations of the line in $\mathbb{R}^3$ through the point $P = (0, 3, -2)$ and parallel to the vector $\mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Normal form of a line in $\mathbb{R}^2$

Definition

The normal form of the equation of a line ℓ in $\mathbb{R}^2$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where $\mathbf{p}$ is a specific point on ℓ, and $\mathbf{n} \neq 0$ is the normal vector for ℓ. If $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ we can write

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

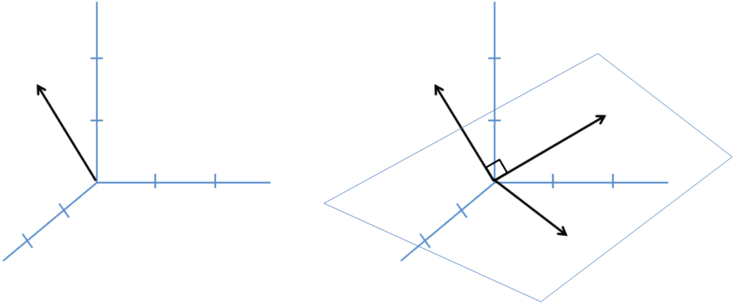
Computing dot products yields the general form for ℓ:

$$n_1 x + n_2 y = (n_1 p_1 + n_2 p_2).$$

So given a line $\ell : ax + by = c$, the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is normal to ℓ.

Planes in $\mathbb{R}^3$

- Let $\mathbf{n}$ be a nonzero vector in $\mathbb{R}^3$. The set of all vectors $\mathbf{x} \in \mathbb{R}^3$ which are orthogonal to $\mathbf{n}$ forms a plane through the origin.



Definition

- In general $\mathcal{P} : \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ (or $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$) defines a plane, where $\mathbf{p}$ is a point on $\mathcal{P}$, and $\mathbf{n} \neq 0$ is a normal vector for $\mathcal{P}$.
- This is called the normal form of the equation of the plane in $\mathbb{R}^3$.

Example

The normal form of the equation of the plane in $\mathbb{R}^3$ through the point $(6, 0, 1)$ with normal vector $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$$

Computing the dot product gives the general form of the equation for this plane:

$$x + 2y + 3z = 9.$$

Notice that this equation has solution set

$$\left\{ \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$$

Planes in $\mathbb{R}^3$

Definition

The vector form of the equation of a plane $\mathcal{P}$ in $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v} \quad \text{or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + s \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

where $s, t \in \mathbb{R}$, $\mathbf{p}$ is a point on $\mathcal{P}$, and $\mathbf{u}, \mathbf{v}$ are nonzero vectors parallel to $\mathcal{P}$, but not parallel to each other. The components are the parametric equations for $\mathcal{P}$:

$$\begin{aligned} x &= p_1 + su_1 + tv_1 \\ y &= p_2 + su_2 + tv_2 \\ z &= p_3 + su_3 + tv_3 \end{aligned}$$

Example

Find the

(a) Vector form

(b) Normal form

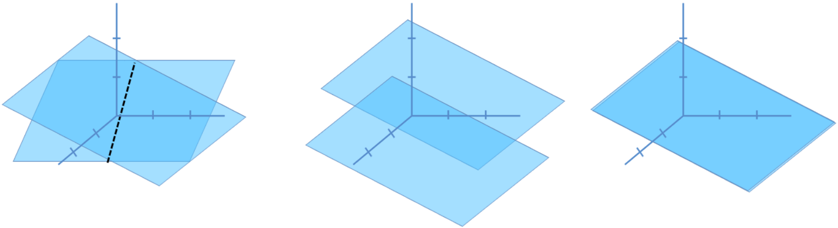
(c) General form

(d) Parametric form

which describe the following planes:

1. The plane through the point $P = (1, -1, 0)$ that has normal vector $\mathbf{n} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$.
2. The plane through the point $P = (4, 0, 9)$ which is parallel to both $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$
3. The plane through points $A = (1, 0, 1)$, $B = (1, -2, 0)$ and $C = (0, 4, 4)$.

Intersection of planes in $\mathbb{R}^3$



- Two planes in $\mathbb{R}^3$ can either intersect in a line, not intersect at all (they are parallel), or coincide (they are the same).
- The line of intersection in first case lies on both planes, and so it is orthogonal to normal vectors for both planes.
- Thus a line in $\mathbb{R}^3$ may be represented as a system of equations of two non parallel planes.

Normal form of a line in $\mathbb{R}^3$

Definition

The normal form of the equation of a line ℓ in $\mathbb{R}^3$ is the system

$$\mathbf{n}_1 \cdot \mathbf{x} = \mathbf{n}_1 \cdot \mathbf{p}$$
$$\mathbf{n}_2 \cdot \mathbf{x} = \mathbf{n}_2 \cdot \mathbf{q}$$

where each equation is the normal form of a plane in $\mathbb{R}^3$ and the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel. If

$$\mathbf{n}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

then computing dot products yields the general form for ℓ:

$$a_1 x + b_1 y + c_1 z = (a_1 p_1 + b_1 p_2 + c_1 p_3)$$
$$a_2 x + b_2 y + c_2 z = (a_2 q_1 + b_2 q_2 + c_3 q_3)$$

The cross product in $\mathbb{R}^3$

Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Definition

The <u>cross product</u> of vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, denoted $\mathbf{u} \times \mathbf{v}$, is the vector in $\mathbb{R}^3$ defined by

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{e}_1 + (u_3 v_1 - u_1 v_3)\mathbf{e}_2 + (u_1 v_2 - u_2 v_1)\mathbf{e}_3$$

or

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

Remember $\mathbf{u} \times \mathbf{v}$ by $\det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$.

The cross product in $\mathbb{R}^3$

- The cross product $\mathbf{u} \times \mathbf{v}$ (by construction) returns a vector in $\mathbb{R}^3$ which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
- In other words it gives a normal to the plane parallel to both $\mathbf{u}$ and $\mathbf{v}$.
- It can be shown that $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \, ||\mathbf{v}|| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ (see Exercise 8 section 1.3).

The cross product in $\mathbb{R}^3$

Example

Find the cross product of vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$.

Example

Find the cross product of vectors $\mathbf{u} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix}$.

Distance between a point and line/plane

Theorem (Pythagorean Theorem in $\mathbb{R}^n$)

For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *with* $\mathbf{u} \cdot \mathbf{v} = 0$,

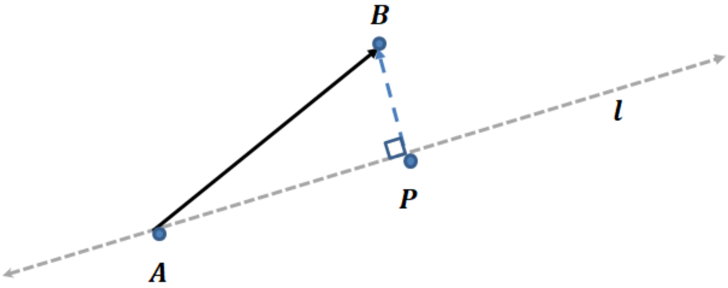
$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Proof.

$$||\mathbf{u}+\mathbf{v}||^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = ||\mathbf{u}||^2+2(\mathbf{u}\cdot\mathbf{v})+||\mathbf{v}||^2 = ||\mathbf{u}||^2+||\mathbf{v}||^2.$$

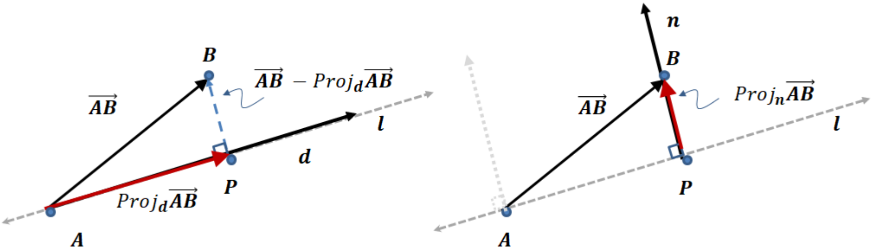
$\square$

Distance between a point and line/plane



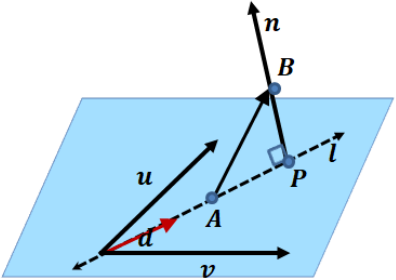
- The Pythagorean Theorem shows that the shortest distance from point B to the line ℓ is the length of the normal vector connecting B and ℓ at point P.
- This also gives the shortest distance from B to a plane, where ℓ is the line on the plane that passes through A and P.

Distance between a point and line/plane



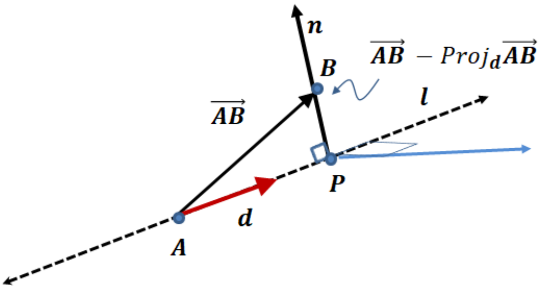
- We want the distance from point B to line ℓ.
- Point P is not known, but we can find a point A which is known to lie on the line/plane.
- The diagram above shows two ways to find vector $\overrightarrow{PB}$ depending on whether the direction vector $\mathbf{d}$ of the line ℓ is known, or if the normal vector $\mathbf{n}$ is known.
- Once $\overrightarrow{PB}$ is found we compute its length.

Distance between a point and line/plane



- The plane $\mathcal{P}$ may be given in vector form $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$.
- The direction vectors $\mathbf{u}$ and $\mathbf{v}$ are not (necessarily) the same as the direction vector $\mathbf{d}$ along ℓ.
- In this case find the general form of the plane $ax + by + cz = d$, and use the normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when finding the distance from B to plane.

Distance between a point and line/plane



- A line in $\mathbb{R}^3$ has more than one non parallel normal vector.
- In this case use $\overrightarrow{AB} - \text{proj}_{\mathbf{d}}\overrightarrow{AB}$ to find the distance from B to the line in $\mathbb{R}^3$.

Distance between a point and line/plane

Example

Find the distance between the point $B(1, 0, 2)$ to the line through $A = (3, 1, 1)$ with direction vector $\mathbf{d} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

Example

Find the distance between the point $B(4, -4, 3)$ to the plane $2x - 2y + 5z + 8 = 0$.

Distance between a point and line/plane

- If a line ℓ in $\mathbb{R}^2$ has general form $ax + by = c$, the distance from point $B = (x_0, y_0)$ to ℓ is

$$\frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$

- If a plane $\mathcal{P}$ in $\mathbb{R}^3$ has general form $ax + by + cz = d$, the distance from point $B = (x_0, y_0.z_0)$ to $\mathcal{P}$ is

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$