Intersection Number of Two Curves on a Torus

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1 Mapping $\mathbb{R}^2$ to the Quotient Space $\mathbb{R}^2/\sim$

Consider the universal cover of the torus, $\mathbb{R}^2$. The quotient space $\mathbb{R}^2/\sim$ is formed by identifying points whose difference is an integer vector. The equivalence relation $\sim$ is defined as:
Two points (x_1, y_1) and (x_2, y_2) are equivalent under $\sim$ if:

$$(x_1, y_1) \sim (x_2, y_2) \iff (x_2 - x_1, y_2 - y_1) \in \mathbb{Z}^2.$$

The quotient map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ is then defined as:

$$\pi(x, y) = (x \mod 1, y \mod 1),$$

which maps each point in $\mathbb{R}^2$ to its equivalence class on the torus. Geometrically, this means every point $(x, y) \in \mathbb{R}^2$ is mapped to a point inside the unit square $[0, 1) \times [0, 1)$, representing a point on the torus.

2 Defining the Map A on $\mathbb{R}^2$

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map represented by a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For any point $(x, y) \in \mathbb{R}^2$, the action of A is given by:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

3 Inducing a Well-Defined Map on the Quotient Space

To induce a well-defined map $\bar{A} : \mathbb{R}^2/\sim \to \mathbb{R}^2/\sim$,we must show if two points (x, y) and $(x + m, y + n)$ are equivalent under the quotient relation, their images under A must also be equivalent.

Applying A to the equivalent point $(x + m, y + n)$:

$$A \begin{bmatrix} x+m \\ y+n \end{bmatrix} = \begin{bmatrix} a(x+m) + b(y+n) \\ c(x+m) + d(y+n) \end{bmatrix}.$$

$$= \begin{bmatrix} (ax+by) + (am+bn) \\ (cx+dy) + (cm+dn) \end{bmatrix}.$$

Since m and n are integers, the terms $am+bn$ and $cm+dn$ are also integers. Therefore, the transformed point differs from $A(x, y)$ by an integer vector, meaning it lies in the same equivalence class in the quotient space. Thus, the map $\bar{A}$ is well-defined on the torus.

4 Commutative Diagram

The following commutative diagram illustrates the relationship between the maps:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{R}^2/\sim & \xrightarrow{\bar{A}} & \mathbb{R}^2/\sim \end{array}$$

This diagram commutes if:

$$\bar{A}(\pi(p)) = \pi(A(p)) \quad \text{for all } p \in \mathbb{R}^2.$$

In other words, applying A on the plane and then projecting to the torus gives the same result as projecting to the torus first and then applying the induced map $\bar{A}$.