Jones Polynomial

victoria

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The Jones polynomial associates a Laurent polynomial with integer coefficients to every knot and link. This polynomial is invariant under Reidemeister moves, making it a useful tool for distinguishing knots.

In order to define the Jones polynomial, we first need to define the Kauffman Bracket.

1 The Kauffman Bracket

The Kauffman bracket is a function from unoriented link diagrams in the oriented plane (or S^2) to Laurent polynomials with integer coefficients in an indeterminate $A$$. It maps a diagram D to $\langle D \rangle \in \mathbb{Z}[A^{-1}, A]$ and is characterized by:

,where $\bigcirc$ is the unknot and $D \sqcup \bigcirc$ consists of the diagram D along with an additional closed curve that does not cross itself or D. The skein relation (iii) expresses how the polynomial changes when resolving a crossing.

The bracket polynomial of a diagram with n crossings can be computed by resolving crossings recursively, reducing it to a sum of 2^n crossing-free diagrams. If a diagram has c components and no crossings, its polynomial follows as $(-A^{-2} - A^2)^{c-1}$.

The ordering of crossing resolutions does not affect the final result, ensuring the well-defined nature of the Kauffman bracket. If an orientation-preserving homeomorphism of the plane is applied to the diagram, the polynomial remains unchanged.

We will now analyze the effect of Reidemeister moves on the bracket polynomial.

2 Effect of Reidemeister Moves

If a diagram is changed by a **Type I Reidemeister** move, its bracket polynomial changes as follows:

$$\langle \boxed{\texttt{reidemeister\_I.pdf}} \rangle = (A(-A^{-2} - A^{2}) + A^{-1})\langle D \rangle. \tag{1}$$

This follows from an application of the skein relation. If the crossing were reversed, the formula remains the same except for an interchange of A and A^{-1}. This symmetry results from rotating the skein relation by $\pi/2$. If D' is the reflection of D (with all crossings flipped), then $\langle D' \rangle$ is obtained by exchanging A and A^{-1} in $\langle D \rangle$.

This lemma is used in subsequent calculations, including the bracket polynomial of a two-component link and a trefoil knot. For example:

$$\langle \boxed{\texttt{two\_component\_link.pdf}} \rangle = A\langle \boxed{\texttt{resolved\_1.pdf}} \rangle + A^{-1}\langle \boxed{\texttt{resolved\_2.pdf}} \rangle (-A^{4} - A^{-4}). \tag{2}$$

For the trefoil knot:

$$\langle \boxed{\texttt{trefoil.pdf}} \rangle = A\langle \boxed{\texttt{resolved\_1.pdf}} \rangle + A^{-1}\langle \boxed{\texttt{resolved\_2.pdf}} \rangle A(-A^{4} - A^{-4}) + A^{-7} = (A^{-7} - A^{-3} - A^{5}). \tag{3}$$