Discrete Mathematics 46 (1983) 107–108
North-Holland

NOTE

ON (+1, −1)-MATRICES WITH VANISHING PERMANENT

Rodica SIMION and Frank W. SCHMIDT

Department of Mathematics, Southern Illinois University, Carbondale, IL 62901, USA

Received 28 May 1982
Revised 29 September 1982

In this paper we show that if A is an $n\times n$ $(+1,-1)$-matrix and if $n=2^m-1$ for some positive integer m, then $\mathrm{per}(A)\neq 0$. This answers partially a question raised by E.T.H. Wang in [3]

For an $n\times n$ matrix $A=(a_{ij})$ the permanent of A is $\sum_\sigma \prod_{i=1}^n a_{i\sigma(i)}$ where the summation is taken over the symmetric group of degree n. (For a thorough treatment of the permanent function, see [2]). In [3], E.T.H. Wang investigated various problems involving permanents of $(+1,-1)$-matrices. One of these problems was to determine those values of n for which there exists an $n\times n$ $(+1,-1)$-matrix A such that $\mathrm{per}(A)=0$. He showed that such matrices exist if n is even or if $n>1$ and $n\equiv 1 \pmod 4$. The case when $n\equiv 3 \pmod 4$ was left open though it was shown that when $n=3$ no such matrices exist In this note we show that, in fact, such matrices do not exist if $n=2^m-1$ for some positive integer m. As a result, the first unsettled case is $n=11$.

Theorem. *If $n=2^m-1$ for some positive integer m, then there does not exist an $n\times n$ $(+1,-1)$-matrix A with* $\mathrm{per}(A)=0$.

Proof. Let A be an $n\times n$ $(+1,-1)$-matrix and write A as $A=J-2B$ where $J=$ the $n\times n$ matrix all of whose entries are $+1$ and B is a uniquely determined $(0,1)$-matrix.

Then by the expansion formula for the permanent of the sum of two matrices (see [2, p. 18, Theorem 1.4]) we have

$$(*)\qquad \begin{aligned}\mathrm{per}(A)&=\mathrm{per}(J-2B)\\ &=n!-2(n-1)!\,p_1(B)+2^2(n-2)!\,p_2(B)+\cdots+(-1)^n2^np_n(B)\end{aligned}$$

where $p_i(B)$ denotes the sum of the permanents of all the $i\times i$ submatrices of B. In particular, $p_1(B)=$ sum of all entries in B, and $p_n(B)=\mathrm{per}(B)$.

We now examine the exact power of two which divides the RHS of $(*)$. In order to do this we recall (see [1]) that $\mathrm{ord}_2(k!)=k-s_k$, where $s_k=$ sum of the digits of k written in base two; e.g. $\mathrm{ord}_2(4!) = \mathrm{ord}_2(24) = 4 - 1 = 3$. Then for each term $\pm 2^{n-k}k!\ p_{n-k}(B)$ of $(*)$ we see

$$\mathrm{ord}_2(\pm 2^{n-k}k!\ p_{n-k}(B)) \geq (n-k) + (k - s_k) = n - s_k \quad (k < n),$$
$$\mathrm{ord}_2(n!) = n - s_n \quad (k = n).$$

Now the form of $n = 2^m - 1$ in binary notation implies that $s_k < s_n$, and so $n - s_n < n - s_k$, for any $k < n$. Therefore, the RHS of $(*)$ cannot vanish since it is divisible by precisely 2^{n-s_n}. □

References

[1] N. Koblitz, p-Adic Numbers, p-Adic Analysis and Zeta Functions (Springer-Verlag, New York, 1977), p. 7, exercise 13.

[2] H. Minc, Permanents, in: G.-C. Rota, ed., Encyclopedia of Mathematics and its Applications, Vol. 6 (Addison-Wesley, Reading, MA, 1978).

[3] E.T.H. Wang, On permanents of (+1, −1)-matrices, Israel J. of Math. 18 (1974) 353–361.