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Permanents of matrices of signed ones

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By calculating the permanents for all Hadamard matrices of orders up to and including 28 we answer a problem posed by E.T.H. Wang and a similar question asked by H. Perfect. Both questions are answered by the existence of Hadamard matrices of order 20 which do not seem to be simply related but nevertheless have the same permanent.

For orders up to and including 20 we also settle several other existence questions involving permanents of $(+1, -1)$-matrices. Specifically, we establish the lowest positive value taken by the permanent in these cases and find matrices which have equal permanent and determinant when such a matrix exists.

Our results address Conjectures 19 and 36 and Problems 5 and 7 in Minc's well known catalogue of unsolved problems on permanents. We also include a little-known proof that there exists a $(+1, -1)$-matrix A of order n such that $\mathrm{per}(A) = 0$ if and only if $n+1$ is not a power of 2.

Keywords: Permanent; Hadamard matrix; $(1, -1)$-matrix; Toeplitz matrix

Mathematics Subject Classifications: 15A15; 05B20; 15A36

1. Introduction

In this article we report on the calculation of permanents of a number of $(+1, -1)$-matrices, including all Hadamard matrices of orders ≤ 28. These calculations help to answer a number of existence questions. In all cases, permanents were calculated using the Nijenhuis–Wilf [1] adaptation of Ryser's method, which is the fastest known general algorithm for computing permanents.

Our results address several of the problems in Minc's catalogue [2–4] of open problems. For a recent survey of progress on all of Minc's problems, see [5].

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2. Hadamard matrices

A *Hadamard matrix* H of order n is a $(+1,-1)$-matrix of order n which satisfies $HH^T = nI_n = H^TH$, where I_n is the identity matrix of order n and a superscript T denotes transposition. The following operations on H clearly preserve the Hadamard property:

(1) Permutations of the rows or columns,
(2) Multiplication of a row or column by -1,
(3) Transposition of the matrix.

In the study of Hadamard matrices, it is traditional to call two matrices *equivalent* if one can be converted into the other by a sequence of operations (1) and (2). We will write $H_1 \sim H_2$ to denote that H_1 is equivalent to H_2. This relation partitions the set of Hadamard matrices into *equivalence classes*. We will say that H_1 *resembles* H_2 if $H_1 \sim H_2$ or $H_1 \sim H_2^T$. This relation partitions the set of Hadamard matrices into *resemblance classes*. Clearly, each resemblance class consists of either one or two entire equivalence classes.

For any $(+1,-1)$-matrix M let $\mathcal{P} = \mathcal{P}(M)$ denote the absolute value of the permanent of M. It is clear that $\mathcal{P}$ is an invariant of a resemblance class. Wang [6] asked whether there exist two Hadamard matrices H_1 and H_2 which do not resemble each other but nonetheless satisfy $\mathcal{P}(H_1) = \mathcal{P}(H_2)$. This question was rephrased as Conjecture 19 in [2], which asserts that no such pair of matrices exists (although it is not clear that Wang believed this conjecture). One purpose of this first section is to answer Wang's question by reporting that Conjecture 19 in [2] is false. In doing so we will also answer a question posed by Perfect [7].

Hadamard matrices of order n exist only if $n \le 2$ or n is divisible by 4. The number of equivalence classes of Hadamard matrices of order $4n$ is sequence A007299 in [8]. Currently the sequence is known to begin 1, 1, 1, 5, 3, 60, 487. Representatives of the equivalence classes up to order 28 can be downloaded from [9]. We refer to the representative of the Yth equivalence class of order X in Sloane's catalogue as $\mathcal{H}_{X,Y}$. The unique (up to equivalence) Hadamard matrices of orders 1, 2, 4, 8 and 12 have respective $\mathcal{P}$ values of 1, 0, 8, 384, 46080.

The smallest order for which there are inequivalent Hadamard matrices is 16. Here there are known to be five equivalence classes. The last two of these (using Sloane's numbering) are known to resemble each other. They have $\mathcal{P} = 8028160 = 2^{15} \cdot 5 \cdot 7^2$. The first three equivalence classes have $\mathcal{P}$ equal to, respectively, $50692096 = 2^{15} \cdot 7 \cdot 13 \cdot 17$, $17137664 = 2^{15} \cdot 523$ and $360448 = 2^{15} \cdot 11$. We conclude that $\mathcal{H}_{16,1}$ (which is the equivalence class containing the direct product of Hadamard matrices of order 2 and order 8) achieves the maximum value of $\mathcal{P}$ for Hadamard matrices of order 16.

Next we found that $\mathcal{P}(H) = 219414528 = 2^{18} \cdot 3^3 \cdot 31$ for every Hadamard matrix H of order 20. Since there are three resemblance classes of this order (each consisting of a single equivalence class), we see immediately that Wang's question is answered and that order 20 is the smallest order for which the answer becomes clear. At the same time we answer a question of Perfect [7, p. 235], who designed a test for non-resemblance of $(+1,-1)$-matrices and asked whether two matrices must resemble each other if they are indistinguishable by her test and have the same permanent.

Perfect's test works well in some situations but it cannot distinguish between Hadamard matrices. Hence her question reduces to Wang's in this case.

For order 24 there are 36 resemblance classes of Hadamard matrix comprised of a total of 60 equivalence classes. On these 36 resemblance classes, $\mathcal{P}$ takes only 27 different values. The gcd of these values is $g = 2^{22} \cdot 3^2$. The values of $\mathcal{P}/g$ are

$$\{73, 567, \mathbf{585}, \mathbf{969}, \mathbf{1207}, 1353, 1609, 2167, 2441, 2761, \mathbf{2889}, 3145, 3273, \\ 3401, 3785, \mathbf{4041}, \mathbf{4279}, 4425, \mathbf{5449}, \mathbf{5961}, \mathbf{6327}, 6985, 14007, 18249, \\ 18615, 43191, 116919\}. \tag{1}$$

In particular, it is striking that in all cases $\mathcal{P}/g$ is odd. The minimum value of $\mathcal{P}$ (namely $2^{22} \cdot 3^2 \cdot 73$) is achieved uniquely by $\mathcal{H}_{24,20}$. The maximum value of $\mathcal{P}$ (namely $2^{22} \cdot 3^4 \cdot 11 \cdot 1181$) is achieved uniquely by $\mathcal{H}_{24,1}$ (which contains the direct product of Hadamard matrices of orders 2 and 12). The values shown in **bold** in (1) are achieved by two distinct resemblance classes and hence yield further counterexamples to Conjecture 19. As a concrete example, we note that $\mathcal{H}_{24,43}$ (which resembles $\mathcal{H}_{24,44}$) has $\mathcal{P} = 2^{22}3^4449$, as does $\mathcal{H}_{24,45}$ (which is equivalent to its own transpose).

For order 28 there are 294 resemblance classes comprised of a total of 487 equivalence classes. On these 294 resemblance classes $\mathcal{P}(H)$ takes 288 different values. The gcd of these values is $g = 2^{25}$. The lowest and highest values of $\mathcal{P}/g$ are

$$\{3445,\ 5387,\ 12203,\ 15851,\ 16021,\ 19211,\ 19477,\ 54155,\ 56917,\ 59797, \ldots \\ \ldots,\ 1821781,\ 1881109,\ 2308117,\ 3127787,\ 3146773,\ 9272341,\ 19926517\}.$$

Again, $\mathcal{P}/g$ is invariably odd. The minimum $\mathcal{P}$ value (namely $2^{25}.5.13.53$) is achieved uniquely by $\mathcal{H}_{28,185}$. The maximum $\mathcal{P}$ value (namely $2^{25}.13.743.2063$) is achieved uniquely by $\mathcal{H}_{28,487}$, which is the equivalence class of the Paley Hadamard matrix based on the Galois field of order 27.

The five values of $\mathcal{P}/g$ shared by more than one resemblance class, together with the representatives which achieve them, are as follows:

$$354517: \mathcal{H}_{28,198}, \mathcal{H}_{28,361} \sim \mathcal{H}^T_{28,440}.$$

$$364213: \mathcal{H}_{28,93} \sim \mathcal{H}^T_{28,175}, \mathcal{H}_{28,203} \sim \mathcal{H}^T_{28,278}, \mathcal{H}_{28,359} \sim \mathcal{H}^T_{28,389}.$$

$$400117: \mathcal{H}_{28,207} \sim \mathcal{H}^T_{28,262}, \mathcal{H}_{28,265} \sim \mathcal{H}^T_{28,433}.$$

$$469141: \mathcal{H}_{28,119} \sim \mathcal{H}^T_{28,162}, \mathcal{H}_{28,299} \sim \mathcal{H}^T_{28,345}.$$

$$599701: \mathcal{H}_{28,43} \sim \mathcal{H}^T_{28,110}, \mathcal{H}_{28,374}.$$

The above computations raise an interesting point. Let

$$f(n) = \sum_{k \geq 1} \lfloor 2^{-k} n \rfloor$$

be the number of factors of 2 which divide $n!$. Then we see that every Hadamard matrix H of order $n < 32$ has the property that $\mathcal{P}(H)$ is divisible by $2^{f(n)}$. Except in the trivial case where $n = 2$, we also see that $\mathcal{P}(H)$ is not divisible by $2^{f(n)+1}$. It would be very interesting to know if these properties generalise. Proving that they did would be one way to resolve problem #5 in Minc's catalogue of open problems [2], which asks whether the permanent of an $n \times n$ Hadamard matrix can vanish for $n > 2$. This problem, which is originally due to Wang [6] has in any case been answered in the negative for $n < 32$.

We will see below in (3) that a high power of two does always divide $\mathcal{P}(H)$, however the above question will remain open. The question of whether the permanent of a more general $(+1, -1)$-matrix can vanish will be answered in section 3.

3. Other (+1, −1)-matrices

In the remainder of this article we consider a number of questions regarding the permanent of more general $(+1, -1)$-matrices. To do so, we will need a number of examples of Toeplitz matrices. For any set S we denote by $T_n S$ the $(+1, -1)$-matrix of order n whose (i, j)th entry is -1 if $j - i \in S$ and $+1$ otherwise. For example, the following matrix can be written as $T_{11}\{-6, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$, which we abbreviate further to $T_{11}\{-6, -1..7\}$.

$$\begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 \\
+1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & +1 \\
+1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 & -1 \\
+1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 \\
+1 & +1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & -1 & +1 & +1 & +1 & +1 & -1 & -1
\end{pmatrix} \tag{2}$$

The first question we consider is due to Kräuter and is listed as Conjecture 36 by Minc [4]. This conjecture asserts that the minimum positive value taken by $\mathcal{P}$ on the set of $(+1, -1)$-matrices of order n is

$$2^{n - \lfloor \log_2(n+1) \rfloor}. \tag{3}$$

Since Kräuter and Seifter [10] have shown that (3) always divides $\mathcal{P}$, it follows that (3) is a lower bound on the minimum positive value taken by $\mathcal{P}$. Hence to prove Kräuter's conjecture it suffices to give a construction for each n which achieves the value in (3). We now do that for $n \leq 20$. For $n > 2$ we give two examples, the first of which is non-singular while the second is singular. This is merely to make the point that the minimum positive value of $\mathcal{P}$ seems to usually be achieved by both singular and non-singular matrices.

$T_1\{0\}$	$T_2\{0\}$
$T_3\{1, 2\}$	$T_3\{2\}$
$T_4\{1\}$	$T_4\{2, 3\}$
$T_5\{0\}$	$T_5\{0..2\}$
$T_6\{1, 2\}$	$T_6\{0..2\}$
$T_7\{1, 5, 6\}$	$T_7\{2..4\}$
$T_8\{1..3, 7\}$	$T_8\{2..4, 6\}$
$T_9\{0, 2..6\}$	$T_9\{2, 3, 5..7\}$
$T_{10}\{0..2, 5..9\}$	$T_{10}\{-5..1, 5..7\}$
$T_{11}\{1..3, 8, 9\}$	$T_{11}\{-1..2, 7\}$
$T_{12}\{0..3, 8\}$	$T_{12}\{-1..3, 6..10\}$
$T_{13}\{1..4, 7..12\}$	$T_{13}\{-11..-2, 4, 5, 12\}$
$T_{14}\{-6..-2, 2..11\}$	$T_{14}\{-1..2, 7, 9..13\}$
$T_{15}\{-1..1, 5, 7..13\}$	$T_{15}\{2..5, 9..11, 14\}$
$T_{16}\{-3..0, 4, 5, 11..13\}$	$T_{16}\{-1..1, 5, 7, 12..14\}$
$T_{17}\{-4..-2, 5, 10..16\}$	$T_{17}\{-5..1, 5..8, 10..12, 14\}$
$T_{18}\{-1..3, 6, 8, 9, 14..17\}$	$T_{18}\{-12..-4, 3..5, 9..14\}$
$T_{19}\{-2, -1, 2..5, 8, 9, 11..13\}$	$T_{19}\{-17, 4..9, 12, 13\}$
$T_{20}\{-4..1, 6, 7, 11..18\}$	$T_{20}\{-17, -8, -5..1, 8..11, 16..18\}$

We next consider another problem posed by Wang [6], which is to find non-singular solutions to

$$\mathcal{P}(A) = \left|\det(A)\right|$$

among the $(+1, -1)$-matrices of each given order n. It was shown by Kräuter and Seifter [10] that there are no solutions when $2 \le n \le 4$ or $n = 2^k - 1$ for an integer k, but that there are solutions when $n \in \{5, 6\}$. The following examples demonstrate existence of a non-singular solution for each order up to 20 (other than those ruled out above). In each case the common value of $\mathcal{P}$ and $|\det(H)|$ is given in parentheses after the matrix.

$T_5\{1..4\}$ (16)	$T_6\{1..4\}$ (32)
$T_8\{1, 4..6\}$ (128)	$T_9\{0..2, 6, 7\}$ (256)
$T_{10}\{1..5, 8, 9\}$ (512)	$T_1 1\{0..3, 7\}$ (1024)
$T_{12}\{1..6, 8\}$ (2048)	$T_{13}\{-10..2, 6, 7\}$ (4096)
$T_{14}\{-12..1, 8..10\}$ (8192)	$T_{16}\{-14..-6, -3..2, 13, 14\}$ (98304)
$T_{17}\{-9..-6, -1..7, 10..13, 16\}$ (2097152)	$T_{18}\{1..5, 7, 8, 12, 13\}$ (131072)
$T_{19}\{-18, -15..-12, 3..7, 13..15\}$ (2097152)	$T_{20}\{1..3, 5, 8..13, 16..18\}$ (524288)

It is worth noting that there are typically several options for obtaining the common value of $\mathcal{P}(A)$ and $|\det(A)|$. For order 9, for example, we could have chosen $T_9\{0,..,2,6,7\}$ (256), $T_9\{0,1,7,8\}$ (512), $T_9\{-4,..,0,4\}$ (768) or $T_9\{-2,..,1,5\}$ (1024). Having said that, it is well known that the determinant of a $(+1,-1)$-matrix of order n is divisible by 2^{n-1}, which does limit the options considerably.

4. Vanishing permanents

Wang [6] also asked for what orders there is a $(+1,-1)$-matrix with vanishing permanent. The example of order 11 given in (2) is such a matrix. Wang proved that an example exists if the order $n > 1$ and $n \not\equiv 3 \bmod 4$, but left open the case when $n \equiv 3 \bmod 4$. Kräuter and Seifter [10] and, independently, Simion and Schmidt [11] showed that there are no examples when $n = 2^k - 1$ for some integer $k \geq 1$. It was subsequently proved by Lothar Teichert that these are the only orders for which examples do not exist. It seems that Teichert left academia and never published his proof, although it is reproduced in [12]. As [12] is not widely available, we feel it worthwhile to prove Teichert's main result here.

Theorem 1 *There exists a square* $(+1,-1)$*-matrix A of order n such that* $\operatorname{per}(A) = 0$ *if and only if* $n+1$ *is not a power of* 2.

Proof The proof we give is essentially that in [12]. Given the preceding results, it suffices to construct A in the case when $n = 2^k r - 1$ for integers $k \geq 2$ and $r \geq 3$, where r is odd. Let $m = 2^k < n/2$ and consider the $n \times n$ matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} -1 & \text{if } i \leq m \text{ and } (j - i \bmod 2m) \in \{m, m+1, \ldots, 2m-1\}, \\ 1 & \text{otherwise.} \end{cases}$$

Let U denote the set of all $m \times m$ submatrices of the first m rows of A. It is clear that

$$\operatorname{per}(A) = (n-m)! \sum_{B \in U} \operatorname{per}(B).$$

If we can find a permutation ϕ of the elements of U such that $\operatorname{per}(B) = -\operatorname{per}(\phi(B))$ for all $B \in U$, then it will follow that $\operatorname{per}(A) = -\operatorname{per}(A)$ and hence $\operatorname{per}(A) = 0$, as required. For each $B \in U$ let $c(B)$ denote the set of indices of the m columns which were used to form B, and let $\mu = \mu(\mathrm{B})$ denote the minimum element of $\{1, 2, \ldots, n\} \backslash c(B)$. We define $\phi(B)$ by

$$c(\phi(B)) = \{1, 2, \ldots, \mu - 1\} \cup \{n + 1 + \mu - x : x \in c(B) \text{ and } x > \mu\}.$$

It is routine to check that ϕ is a permutation of U and that in each case $\phi(B)$ can be obtained from B by (1) reversing the order of the first μ rows, (2) reversing the order of, and negating the last $m - \mu$ rows, (3) reversing the order of, and negating the first $\mu - 1$ columns and (4) reversing the order of the last $m - \mu + 1$ columns. Now steps (1) and (4) leave the permanent unchanged while steps (2) and (3) multiply it by $(-1)^{m-\mu}$ and $(-1)^{\mu-1}$ respectively. Since m is even, the overall effect is to multiply by -1, so that $\text{per}(B) = -\text{per}(\phi(B))$ as required. ■

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