(1) that, $\cos^2\phi = \dfrac{\sin^2\theta}{1-\rho^{\frac{2}{3}}\sec^2\theta}$, and (2) that the base will be only just clear of the water when $\rho = (\cos 2\theta)^{\frac{3}{2}}$.

Solution by the PROPOSER.

1. Referring to the figure in my solution of Quest. 5133 (*Reprint*, Vol. XXVII., p. 57), let AN be the perpendicular on DE. Then we have

$$\cos^2\phi = \frac{\mathrm{AN}^2}{\mathrm{AF}^2} = \frac{\mathrm{AD}\,.\,\mathrm{AE}\sin^2\theta}{\mathrm{DF}\,.\,\mathrm{FE}} = \frac{\mathrm{AF}\,.\,\mathrm{AQ}\sin^2\theta}{\mathrm{AF}\,.\,\mathrm{FQ}},$$

or
$$\sin^2\theta = \frac{\mathrm{AF}\,.\,\mathrm{FQ}\cos^2\phi}{\mathrm{AF}\,.\,\mathrm{AQ}};$$

that is,
$$\sin^2\theta = \left(1-\frac{\mathrm{AF}^2}{\mathrm{AF}\,.\,\mathrm{FQ}}\right)\cos^2\phi = \left(1-\frac{\mathrm{AF}^2}{\mathrm{AD}\,.\,\mathrm{AE}}\right)\cos^2\phi$$
$$= \left(1-\frac{\mathrm{AD}\,.\,\mathrm{AE}}{\mathrm{AQ}^2}\right)\cos^2\phi = \left(1-\frac{\mathrm{AK}^2}{\mathrm{AQ}^2}\right)\cos^2\phi$$
$$= \left(1-\frac{\mathrm{AM}^2\sec^2\theta}{\mathrm{AQ}^2}\right)\cos^2\phi = (1-\rho^{\frac{2}{3}}\sec^2\theta)\cos^2\phi.$$

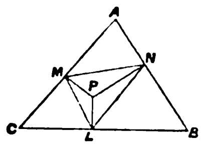
2. When C and E coincide, $\phi = \mathrm{QCD} = \mathrm{QAD} = \theta$, and the above relation becomes $\cos^2\theta-\rho^{\frac{2}{3}} = \sin^2\theta$, whence $\rho = (\cos 2\theta)^{\frac{3}{2}}$.

In order that ϕ should have a sensible magnitude, $1-\rho^{\frac{2}{3}}\sec^2\theta$ must be greater than $\sin^2\theta$, that is, than $1-\cos^2\theta$; or $\rho^{\frac{2}{3}}\sec^2\theta$ must be less than $\cos^2\theta$, or ρ less than $\cos^6\theta$. Thus, if $\theta = 30°$, ρ must be less than $\frac{27}{64}$ in order that an inclined position of floatation should be possible, but not less than $\dfrac{1}{2\sqrt{2}}$ in order that the base may be entirely unsubmerged.

5175. (By the Rev. H. G. DAY, M.A.)—From a random point within the area of an acute-angled triangle, perpendiculars are drawn on the sides; show that the average area of the triangle formed by joining the feet of these perpendiculars is, in parts of the area of the given triangle, $\frac{1}{12}(\sin^2\mathrm{A}+\sin^2\mathrm{B}+\sin^2\mathrm{C})$.

Solution by E. B. SEITZ; Rev. J. L. KITCHIN, M.A.; *and others.*

Let ABC be the given triangle, P the random point, PL, PM, PN the perpendiculars, and LMN the triangle formed. Put $\mathrm{PL}=x$, $\mathrm{PM}=y$, $\mathrm{PN}=z$, $\mathrm{BC}=a$, $\mathrm{CA}=b$, $\mathrm{AB}=c$, area $\mathrm{ABC}=\Delta$.



Then area $\mathrm{LMN} = \left(\dfrac{xy}{ab}+\dfrac{xz}{ac}+\dfrac{yz}{bc}\right)\Delta$,

and $ax+by+cz = 2\Delta$;

hence, substituting for z in terms of x and y, the average is

$$\Delta_1 = \int_0^{\frac{2\Delta}{a}}\int_0^{\frac{1}{b}(2\Delta-ax)}\left(\frac{xy}{ab}+\frac{2x\Delta}{ac^2}-\frac{x^2}{c^2}-\frac{bxy}{ac^2}+\frac{2y\Delta}{bc^2}-\frac{axy}{bc^2}-\frac{y^2}{c^2}\right)dx\,\mathrm{cosec}\,\mathrm{C}\,dy$$
$$= \int_0^{\frac{2\Delta}{a}}\left(\frac{4\Delta^3}{3b^3c^2}+\frac{4x\Delta^2\cos\mathrm{A}}{ab^2c}-\frac{4x^2\Delta\cos\mathrm{A}}{b^2c}-\frac{a^2x^2\Delta}{b^3c^2}+\frac{ax^3\cos\mathrm{A}}{b^2c}+\frac{a^3x^3}{3b^3c^2}\right)\mathrm{cosec}\,\mathrm{C}\,dx$$
$$= \tfrac{2}{3}\Delta^3\left(\frac{1}{b^2c^2}+\frac{\cos\mathrm{A}}{a^2bc}\right) = \tfrac{1}{12}\Delta(\sin^2\mathrm{A}+\sin^2\mathrm{B}+\sin^2\mathrm{C}).$$

[The same result has been otherwise obtained by the EDITOR, in the Solution of his Question 3960, on p. 33 of Vol. XIX. of the *Reprint*, where it is further shown that, if P range over the circle drawn round the triangle ABC, the average area of LMN is one-eighth of the triangle ABC.]

5204. (By H. M. DYER, B.A.)—If a semi-cubical parabola and a parabola have a common tangent at the vertices, and parallels to this tangent cut the semi-cubical in A′, B′, C′, and the parabola in A, B, C respectively; show that the ratio $\Delta\,\mathrm{A'B'C'} : \Delta\,\mathrm{ABC}$ varies as the sum of the products taken two at a time of the ordinates of A, B, C.

Solution by the PROPOSER; Prof. EVANS, M.A.; *and others.*

Let $y^2 = cx^3$ and $y^2 = px$ be the two curves; and (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_1, y_1'), (x_2, y_2'), (x_3, y_3') the coordinates of A, B, C, A′, B′, C′;

then
$$\frac{\Delta\,\mathrm{A'B'C'}}{\Delta\,\mathrm{ABC}} = \begin{vmatrix}1&1&1\\x_1&x_2&x_3\\y_1'&y_2'&y_3'\end{vmatrix} \div \begin{vmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{vmatrix}$$
$$= \left(\frac{c}{p}\right)^{\frac{1}{2}}\begin{vmatrix}1&1&1\\x_1&x_2&x_3\\x_1^{\frac{3}{2}}&x_2^{\frac{3}{2}}&x_3^{\frac{3}{2}}\end{vmatrix} \div \begin{vmatrix}1&1&1\\x_1&x_2&x_3\\x_1^{\frac{1}{2}}&x_2^{\frac{1}{2}}&x_3^{\frac{1}{2}}\end{vmatrix}$$
$$= \left(\frac{c}{p}\right)^{\frac{1}{2}}\frac{\Sigma\left(x_2x_3^{\frac{3}{2}}-x_3x_2^{\frac{3}{2}}\right)}{\Sigma\left(x_2x_3^{\frac{1}{2}}-x_3x_2^{\frac{1}{2}}\right)}$$
$$= \left(\frac{c}{p}\right)^{\frac{1}{2}}\left\{(x_2x_3)^{\frac{1}{2}}+(x_3x_1)^{\frac{1}{2}}+(x_1x_2)^{\frac{1}{2}}\right\}$$
$$= \left(\frac{c}{p^3}\right)^{\frac{1}{2}}(y_2y_3+y_3y_1+y_1y_2).$$

QUESTIONS FOR SOLUTION.

5420. (By Professor SYLVESTER, F.R.S.)—From the expansion of $\{\log(1+x)\}^n$, in a series according to powers of x, prove that $S_{i,j}$ [the coefficient of t^j is the developed product of $(1+t)(1+2t)\,\ldots\ldots\,(1+it)$] is divisible by every prime number greater than $j+1$ in the series

$$i+1,\ i,\ i-1,\ \ldots\ldots,\ i-j+1.$$

5421. (By Professor CAYLEY, F.R.S.)—Suppose $S_x = m_1(x-a_1)$, $m_2(x-a_2)$, $m_3(x-a_3)$, $m_4(x-a_4)$; where, for any given value of x, we write $+, -$, or 0, according as the linear function is positive, negative, or zero, and where the order of the terms is not attended to. If x is any one of the values a_1, a_2, a_3, a_4, the corresponding S is $0+++$, $0---$, $0++-$, or $0+--$; and if I denote indifferently the first or second form, and R denote indifferently the third or fourth form, then it is to be shown that the four S's are R, R, R, R, or else R, R, I, I.

5422. (By Professor TOWNSEND, F.R.S.)—Four points A, B, C, D being supposed given or taken arbitrarily on a straight line L; construct, by elementary geometry, the two X and Y, on the line, the distance of each of which from one of the four D shall be the harmonic mean of its distances from the remaining three A, B, C.

5423. (By Professor CLIFFORD, F.R.S.)—It is known that if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that if five lines be given, the five points so belonging to their five tetragrams lie on a circle [Miquel's Theorem, see *Diary* for 1861, p. 55]. Show that this series of propositions is interminable; so that, if $2n$ lines be given, they determine $2n$ circles that meet in a point; and if $2n+1$ lines be given, they determine $2n+1$ points that lie on a circle.

5424. (By Professor CROFTON, F.R.S.)—Prove that the mean value of the triangle formed by joining the centroid of a triangle and two points taken at random within it is

$$\frac{\Delta}{3^6}\left(34+\tfrac{1}{8}\log 2\right).$$

5425. (By Professor CHASE, LL.D.)—If f represent any force whatever that varies inversely as the square of the distance, and r be the radius of a perpetual circular oscillation produced by the force; find the mean velocity of a synchronous radial oscillation.

5426. (By Professor WOLSTENHOLME, M.A.)—Prove that (1) the two points whose distances from A, B, C, the angular points of a triangle, are as sin A, sin B, sin C, and the two whose distances are as cos A, cos B, cos C (one of which is the orthocentre), lie on the straight line joining the centre (O) of the circumscribed circle and the orthocentre (L); (2) the two former points Q, Q′ are real for any acute-angled triangle, and lie in LO produced, their positions being determined by

$$\frac{\mathrm{QL}}{\mathrm{OL}} = \frac{2k+2}{3k+1},\quad \frac{\mathrm{Q'L}}{\mathrm{OL}} = \frac{2-2k}{1-3k},$$

where $k^2 = \dfrac{\cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}}{1+\cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}}$; (3) P is always real, and lies in OL produced, so that OL . OP = square on the radius of the circumscribed circle,

and
$$\frac{\mathrm{AP}}{\mathrm{AL}} = \frac{\mathrm{BP}}{\mathrm{BL}} = \frac{\mathrm{CP}}{\mathrm{CL}} = \frac{\mathrm{OP}}{\mathrm{R}} = \frac{\mathrm{R}}{\mathrm{OL}} = \frac{1}{(1-8\cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C})^{\frac{1}{2}}}.$$

Hence the points will be fixed for all triangles inscribed in the same circle and having the same centroid.

5427. (Proposed by Professor MINCHIN, M.A.)—A body of any shape, with a plane base, rests with this base on a rough horizontal plane; a heavy beam moveable round a horizontal axis fixed in the plane rests at a single point against the body, the vertical plane through the beam containing the centre of gravity of the body. Show that limiting equilibrium of the system is impossible unless the normal to the surface of contact of the beam and body makes with the vertical an angle greater than the sum of the angles of friction between the body and the beam and the body and the ground.

5428. (By Professor ELLIOTT, M.A.)—Prove (1) that the highest point on the wheel of a carriage rolling on a horizontal plane moves twice as fast as each of two points in the rim whose distance from the ground is half the radius of the wheel; and (2) find the rate at which the carriage is travelling when the dirt thrown from the rim of the wheel to the greatest height attains a given level, explaining the two roots of the resulting equation.

5429. (By Professor LLOYD TANNER, M.A.)—If

$$\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2 = 0;$$

then, x being considered as a function of z, y,

$$\frac{d^2x}{dz^2}\cdot\frac{d^2x}{dy^2}-\left(\frac{d^2x}{dz\,dy}\right)^2 = 0.$$

5430. (By Professor NASH, M.A.)—Find the average length of the radius of curvature of a parabola for points situated between the vertex and the latus rectum.

5431. (By Professor GRIFFITHS, M.A.) — Supposing the Hooghly river to flow due south, prove that the pressure on the western bank at the depth d would be increased by the change of latitude of the running