Supplement to the solution of Exercise 18, (a)

For the given derivation of the delta method in (7), first we need to show:

$$r_g(X_n) \xrightarrow{P} 0 \quad \text{for} \quad n \to \infty$$

To that aim, let $\varepsilon > 0$ and $\gamma > 0$. Then, we have to show that there exists some $n_0 \in \mathbb{N}$ with:

(S1) $\quad P(\|r_\mu(X_n)\| > \varepsilon) < \gamma \quad \forall \; n \geq n_0$

(Here, we use the same symbol $\|.\|$ for the Euclidean norms in $\mathbb{R}^p$ and $\mathbb{R}^q$.)

By (2), for the given $\varepsilon > 0$, there exists some $\delta > 0$ with

(S2) $\quad \|r_\mu(x)\| \leq \varepsilon \quad \forall \; x \in \mathbb{R}^p$ with $\|x-\mu\| \leq \delta$

Further, according to (6), there exists some $n_0 \in \mathbb{N}$ with

(S3) $\quad P(\|X_n - \mu\| > \delta) < \gamma \quad \forall \; n \geq n_0$

Let $n \geq n_0$. Then, we obtain:

$$P(\|r_\mu(X_n)\| > \varepsilon)$$

$$= \underbrace{P(\{\|r_\mu(X_n)\| > \varepsilon\} \cap \{\|X_n-\mu\| \leq \delta\})}_{=0,} + P(\{\|r_\mu(X_n)\| > \varepsilon\} \cap \{\|X_n-\mu\| > \delta\})$$

since, by (S2) $\|X_n-\mu\| \leq \delta \Rightarrow \|r_\mu(X_n)\| \leq \varepsilon$
and thus, the two events
$\{\|r_\mu(X_n)\| > \varepsilon\}$ and $\{\|X_n-\mu\| \leq \delta\}$ are disjoint

$$= P(\{\|r_\mu(X_n)\| > \varepsilon\} \cap \{\|X_n-\mu\| > \delta\}) \leq P(\|X_n-\mu\| > \delta) < \gamma$$

by (S3) since $n \geq n_0$.

Thus, we have shown (S1).