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Applied Data Analysis

Exercise Sheet 1 - Solutions

Exercise 1

Let

$$A = V\Lambda V'$$

with $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_p) \in \mathbb{R}^{p\times p}$ and $V = [v_1|\dots|v_p]$ be the singular value decomposition (SVD) of A.

(a) According to Lemma I.1.4 (5) we get

$$\operatorname{rank}(A) \leq \min\{\operatorname{rank}(V), \operatorname{rank}(\Lambda), \operatorname{rank}(V')\}.$$

Since V is orthogonal it follows that $VV' = I_p$ and therefore $\det(VV') = (\det(V^2))^2 = 1$. In particular, $\det(V) \neq 0$ and V is of full rank (i.e. all orthogonal matrices are of full rank). Consequently,

$$\operatorname{rank}(A) \leq \min\{\underbrace{\operatorname{rank}(V)}_{=p}, \underbrace{\operatorname{rank}(\Lambda)}_{\leq p}, \underbrace{\operatorname{rank}(V')}_{=p}\} = \operatorname{rank}(\Lambda).$$

On the other hand, by substituting the SVD of A we get

$$V'AV = V'V\Lambda V'V = \Lambda.$$

Then, by analogous arguments it follows that

$$\operatorname{rank}(\Lambda) \leq \min\{\underbrace{\operatorname{rank}(V)}_{=p}, \underbrace{\operatorname{rank}(\Lambda)}_{\leq p}, \underbrace{\operatorname{rank}(V')}_{=p}\} = \operatorname{rank}(A).$$

Therefore,

$$\operatorname{rank}(A) = \operatorname{rank}(\Lambda) = \operatorname{rank}(\operatorname{diag}(\lambda_1, \dots, \lambda_p)) = |\{i \in \{1, \dots, p\} \mid \lambda_i \neq 0\}|.$$

(b),(c) V defines an orthonormal system of eigenvectors with corresponding eigenvalues $\lambda_1, \dots, \lambda_p$. Therefore, if $A \in \mathbb{R}^{p\times p}_{\geq 0}(\mathbb{R}^{p\times p}_{>0})$, then

$$\lambda_i = \lambda_i v_i' v_i = v_i' A v_i \geq (>) 0$$

for all $i \in \{1, \dots, p\}$.

On the other hand, if $\lambda_i \geq (> 0)$ for all $i \in \{1, \dots, p\}$, then, for any $x \in \mathbb{R}^p \backslash \{0_p\}$ represented as a linear combination $x = \sum_{i=1}^p \alpha_i v_i$, $\alpha_1, \dots, \alpha_p \in \mathbb{R}$, given by the orthonormal basis defined by V we get

$$\begin{aligned} x'Ax &= \left(\sum_{i=1}^p \alpha_i v_i'\right) A \left(\sum_{i=1}^p \alpha_i v_i\right) \\ &= \sum_{i=1}^p \sum_{j=1}^p \alpha_i \alpha_j v_j' \underbrace{A v_j}_{=\lambda_j v_j} \\ &= \sum_{i=1}^p \sum_{j=1}^p \alpha_i \alpha_j \lambda_j v_i' v_j \\ &= \sum_{i=1}^p \lambda_i \alpha_i^2 \geq (>) 0. \end{aligned}$$

Exercise 2

(a) Since $\Lambda^{1/2}$ is symmetric we get

$$(A^{1/2})' = (V\Lambda^{1/2}V')' = (V')'(\Lambda^{1/2})'V' = V\Lambda^{1/2}V' = A^{1/2}.$$

Therefore, the matrix $A^{1/2}$ is symmetric. Furthermore, for any $x \in \mathbb{R}^p$ with $y = (y_1, \ldots, y_p)' := V'x$ we get

$$x'A^{1/2}x = \underbrace{x'V}_{=y'} \Lambda^{1/2} \underbrace{xV'}_{=y} \Lambda^{1/2} y = \sum_{i=1}^p \lambda^{1/2} y_i^2 \geq 0.$$

It follows that $A \geq 0$ and finally

$$A^{1/2}A^{1/2} = V\Lambda^{1/2}V'V\Lambda^{1/2}V' = V\Lambda^{1/2}\Lambda^{1/2}V' = V\Lambda V' = A.$$

(b) The positive definiteness of A follows by noticing that $\lambda_i^{1/2} > 0$, $i \in \{1, \ldots, p\}$, acc. to Ex. 1. Furthermore, notice that according to Lemma I.1.4 (3) the following identities hold:

$$\Lambda^{-1} = \operatorname{diag}(\lambda_1^{-1}, \ldots, \lambda_p^{-1}), \quad A^{-1} = V\Lambda^{-1}V'.$$

(i)

$$A^{-1/2}A^{-1/2} = V\Lambda^{-1/2}V'V\Lambda^{-1/2}V' = V\Lambda^{-1}V' = A^{-1}.$$

(ii) We get

$$A^{1/2}A^{-1/2} = V\Lambda^{1/2}V'V\Lambda^{-1/2}V' = VV' = I_p$$

and

$$A^{-1/2}A^{1/2} = V\Lambda^{-1/2}V'V\Lambda^{1/2}V' = VV' = I_p.$$

(iii) We get

$$A^{1/2}A^{-1}A^{1/2} = A^{1/2}A^{-1/2}A^{-1/2}A^{1/2} = I_pI_p = I_p$$

and

$$A^{-1/2}AA^{-1/2} = A^{-1/2}A^{1/2}A^{1/2}A^{-1/2} = I_pI_p = I_p.$$

Exercise 3

(a) Let the SVD of A be given by $A = V\Lambda V'$.

To prove existence of B define $B = A^{1/2}$. Since $A \geq 0$ it follows that $B \geq 0$ and

$$BB' = B^2 = A^{1/2}A^{1/2} = A.$$

To prove uniqueness of B let $B_1, B_2 \in \mathbb{R}^{p\times p}_{\geq 0}$ with

$$A = B_1^2 = B_2^2.$$

Assuming $B_1 \neq B_2$ implies $B_1 - B_2 \neq 0$ as a symmetric matrix has at least one eigenvalue $\lambda \in \mathbb{R}\backslash\{0\}$ such that

$$(B_1 - B_2)x = \lambda x.$$

Applying the hint yields

$$\begin{aligned} 0 =& x'(B_1^2 - B_2^2)x \\ =& \frac{1}{2}(x'(B_1 + B_2)(B_1 - B_2)x + x'(B_1 - B_2)(B_1 + B_2)x) \\ =& \frac{1}{2}(x'(B_1 + B_2)\lambda x + \lambda x'(B_1 + B_2)x) \\ =& \lambda(\underbrace{x'B_1x}_{\geq 0} + \underbrace{x'B_2x}_{\geq 0}) \geq 0. \end{aligned}$$

Since $\lambda \neq 0$ this implies

$$0 = x'B_1x = x'B_2x = x'(B_1 - B_2)x = x'(B_1 - B_2)x = x'\lambda x = \lambda||x||^2 \neq 0$$

and therefore a contradiction.

(b) Let $A = BB'$ for some matrix $B \in \mathbb{R}^{p\times q}$. Then,

$$A' = (BB')' = BB' = A$$

and A is symmetric. Furthermore, for any $x \in \mathbb{R}^p$ we get

$$x'Ax = x'BB'x = (B'x)'(B'x) = ||B'x||^2 \geq 0$$

and A is positive semidefinite.

Exercise 4

(a) Following the hints some algebra yields

$$\begin{vmatrix} I_p & 0_{p\times q} \\ -CA^{-1} & I_q \end{vmatrix} \begin{vmatrix} A & B \\ C & D \end{vmatrix} \overset{I.1.4(8)}{=} \left| \begin{pmatrix} I_p & 0_{p\times q} \\ -CA^{-1} & I_q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right|$$

$$\overset{hint(i)}{=} \begin{vmatrix} A & B \\ O_{q\times p} & -CA^{-1}B + D \end{vmatrix} \overset{hint(ii)}{=} |A|\,|D - CA^{-1}B|.$$

Furthermore, since

$$\begin{vmatrix} I_p & 0_{p\times q} \\ -CA^{-1} & I_q \end{vmatrix} = \begin{vmatrix} I_p & (-CA^{-1})' \\ 0_{q\times p} & I_q \end{vmatrix} \overset{hint(ii)}{=} |I_p|\,|I_q| = 1$$

the proposition is true.

(b) Some algebra and application of hint (i) yields

$$\begin{aligned} &\begin{pmatrix} A & B \\ B' & D \end{pmatrix} \begin{pmatrix} A^{-1}+FE^{-1}F' & -FE^{-1} \\ -E^{-1}F' & E^{-1} \end{pmatrix} \\ =&\begin{pmatrix} I_p + AFE^{-1}F' - BE^{-1}F' & -AFE^{-1} + BE^{-1} \\ B'A^{-1} + B'FE^{-1}F' - DE^{-1}F' & -B'FE^{-1} + DE^{-1} \end{pmatrix} =: \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} M_1 &\overset{\text{def.}F}{=} I_p + AA^{-1}BE^{-1}F' - BE^{-1}F' = I_p \\ M_2 &= -AA^{-1}BE^{-1} + BE^{-1} = 0_{p\times q} \\ M_3 &= B'A^{-1} + B'A^{-1}BE^{-1}F' - DE^{-1}F' \\ &= B'A^{-1} - \underbrace{(D - B'A^{-1}B)}_{=E} E^{-1}\,(A^{-1}B)' \\ &= B'A^{-1} - B'(A')^{-1} = B'A^{-1} - B'A^{-1} = 0_{q\times p} \\ M_4 &= \underbrace{(D - B'A^{-1}B)}_{=E} E^{-1} = I_q. \end{aligned}$$

Exercise 5

(a) Assume $A = A^2$. First show the hint:

$$\mathbb{R}^n = Im(A) \oplus Ker(A).$$

Let $x \in \mathbb{R}^n$, $x_1 := Ax$, $x_0 := x - Ax \Longrightarrow x = Ax + x - Ax = x_1 + x_0$ with $x_1 \in Im(A)$ (by def.) and

$$Ax_0 = Ax - A^2x \underset{A=A^2}{=} Ax - Ax = 0,$$

and therefore $x_0 \in Ker(A)$.

Further assume that $\tilde{x}_1 \in Im(A)$, $\tilde{x}_0 \in Ker(A)$ with

$$\tilde{x}_1 + \tilde{x}_0 = x = x_1 + x_0.$$

Then, there exists $\tilde{y} \in \mathbb{R}^n$ such that $\tilde{x}_1 = A\tilde{y}$. Furthermore, $x_0 - \tilde{x}_0 \in Ker(A)$

$$\begin{aligned} &\Longrightarrow \quad 0 = A(x_0 - \tilde{x}_0) \underset{(2)}{=} A(\tilde{x}_1 - x_1) \overset{\tilde{x}_1 = A\tilde{y}}{\underset{x_1 = Ax}{=}} A^2\tilde{y} - A^2x \underset{A^2=A}{=} \tilde{x}_1 - x_1 \\ &\Longrightarrow \quad x_0 - \tilde{x}_0 = \tilde{x}_1 - x_1 = 0, \end{aligned}$$

Now, let $r := \operatorname{rank}(A) \underset{\text{Def.}}{=} \dim(Im(A))$.

<u>Case 1:</u> $r \in \{1, \ldots, n-1\}$.

Then, there exists an orthonormal basis $\{v_1, \ldots, v_r\}$ of $Im(A)$, and for $i \in \{1, \ldots, r\}$ there exists $u_i \in \mathbb{R}^n$ with $v_i = Au_i$.

$$\Longrightarrow Av_i = A^2 u_i \underset{A=A^2}{=} Au_i = v_i, \quad i \in \{1, \ldots, r\}.$$

Therefore: $v_1, \ldots, v_r$ are eigenvectors with eigenvalue $\lambda = 1$. By linear algebra we have $\dim(Ker(A)) = n - r$. Then, there exists an orthonormal basis $\{v_{r+1}, \ldots, v_n\}$ of $Ker(A)$ (by the Gram/Schmidt procedure, respectively, see numerical analysis or next exercise sheet).

$$\Longrightarrow Av_i = 0 = 0 \cdot v_i, \quad i \in \{r+1, \ldots, n\}.$$

Therefore: $v_{r+1}, \ldots, v_n$ are eigenvectors for the eigenvalue $\mu = 0$.

Then, $v_1, \ldots, v_n$ are a basis of $\mathbb{R}^n$, since by construction above each $x \in \mathbb{R}^n$ is representable as a suitable linear combination and, if

$0 = \sum_{i=1}^n \alpha_i v_i$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, then

$$\begin{aligned}
&\Longrightarrow 0 = A\Big(\sum_{i=1}^n \alpha_i v_i\Big) = \sum_{i=1}^n \alpha_i \underbrace{Av_i}_{=0 \text{ for } i \geq r+1} = \sum_{i=1}^r \alpha_i v_i, \\
&v_1, \ldots, v_r \text{ lin. ind. } \Longrightarrow \alpha_1 = \ldots = \alpha_r = 0 \\
&\Longrightarrow 0 = \sum_{i=r+1}^n \alpha_i v_i \Longrightarrow \alpha_{r+1} = \ldots = \alpha_n = 0, \text{ since } v_{r+1}, \ldots, v_n \text{ lin. ind.}
\end{aligned}$$

In summation: $v_1, \ldots, v_n$ are lin. ind.

Let $V := [v_1, \ldots, v_n]$.

<u>Case 2:</u> $r \in \{0, n\}$.

Then, either $Ker(A) = \mathbb{R}^n$ (for $r = 0$) or $Im(A) = \mathbb{R}^n$ (for $r = n$). Analogous to the above argumentation we have $\mu = 0$ or $\lambda = 1$ as the sole eigenvalue of A. Choose $V := [v_1, \ldots, v_n]$ for some basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$. (Note: here we have either $A = 0_{n \times n}$ for $r = 0$ or $A = I_n$ for $r = n$.)

In both cases (i.e. $r \in \{1, \ldots, n-1\}$, $r \in \{0, n\}$) is $V \in \mathbb{R}^{n \times n}$ regular, and $v_1, \ldots, v_n$ are eigenvectors for A (for the eigenvalue $\mu = 0$ or $\lambda = 1$, respectively). Let

$$\Lambda := \operatorname{diag}(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-r}).$$

Then,

$$\begin{aligned}
& AV = V\Lambda \quad \Longleftrightarrow \quad A = V\Lambda V^{-1} \\
\Longrightarrow \quad & \operatorname{tr}(A) = \operatorname{tr}(V\Lambda V^{-1}) = \operatorname{tr}(\underbrace{V^{-1}V}_{=I_n}\Lambda) \\
& = \operatorname{tr}(\Lambda) \underset{\text{def. } \Lambda}{=} r = \operatorname{rank}(A).
\end{aligned}$$

(b) Let $A = A'$ with eigenvalues in $\{0, 1\}$. Since A is symmetric we can find a SVD

$$A = V\Lambda V',$$

with $\Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of A and $V \in \mathbb{R}^{n\times n}$ orthogonal. By (a) it follows, that

$$\begin{aligned} r := \operatorname{rank}(A) &= |\{i \in \{1, \ldots, n\} | \lambda_i \neq 0\}| \\ &\underset{\text{ass.}}{=} |\{i \in \{1, \ldots, n\} | \lambda_i = 1\}| \end{aligned}$$

Since $\lambda_1 \geq \ldots \geq \lambda_n$ we get

$$\begin{aligned} && \Lambda &= \operatorname{diag}(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-r}) \\ \Longrightarrow && A^2 &= V\Lambda \underbrace{V'V}_{=I_n} \Lambda V' = V \underbrace{\Lambda\Lambda}_{=\Lambda} V' = V\Lambda V' = A \end{aligned}$$

If A is not symmetric, consider the following counterexample:

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow |A - \lambda I_2| = \lambda^2$$

That is, $\lambda = 0$ is the sole eigenvalue of A. But:

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq A.$$

and A is not idempotent.

(c) Let $A' = A = A^2$ and $x \in \mathbb{R}^n$.

$$\begin{aligned} \Longrightarrow \quad x'Ax &\underset{A^2=A}{=} x'AAx \underset{A=A'}{=} x'A'Ax \\ &= (Ax)'Ax = ||Ax||_2^2 \geq 0. \end{aligned}$$

Since $x \in \mathbb{R}^n$ was chosen arbitrarily we get $A \geq 0$

Furthermore,

$$\begin{aligned} (I_n - A)' &\underset{A=A'}{=} I_n - A \\ (I_n - A)^2 &= I_n^2 - I_nA - AI_n + A^2 \\ &\underset{A^2=A}{=} I_n - A - A + A = I_n - A. \end{aligned}$$

Therefore, $I_n - A$ is symmetric and idempotent. Analogous to the arguments for A we also get:

$$I_n - A \geq 0.$$

Exercise 6

(a) (i)

$$E_n' = I_n' - \frac{1}{n}\mathbb{1}_{n\times n}' = I_n - \frac{1}{n}\mathbb{1}_{n\times n} = E_n.$$

(ii)

$$\begin{aligned}
E_n^2 &\underset{\text{Def.}}{=} \Big(I_n - \frac{1}{n}\mathbb{1}_{n\times n}\Big)\Big(I_n - \frac{1}{n}\mathbb{1}_{n\times n}\Big) \\
&= I_n - \frac{2}{n}\mathbb{1}_{n\times n} + \frac{1}{n^2} \overbrace{\begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}\begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}}^{\mathbb{1}_{n\times n}^2} = \begin{pmatrix} n & \dots & n \\ \vdots & & \vdots \\ n & \dots & n \end{pmatrix} \\
&= I_n - \frac{2}{n}\mathbb{1}_{n\times n} + \frac{1}{n^2} n \mathbb{1}_{n\times n} \\
&= I_n - \frac{1}{n}\mathbb{1}_{n\times n} \underset{\text{def.}}{=} E_n.
\end{aligned}$$

(b) By (a) and Exercise 5 we have

$$\begin{aligned}
\operatorname{rank}(E_n) = \operatorname{tr}(E_n) &\underset{\text{def.}}{=} \operatorname{tr}\Big(I_n - \frac{1}{n}\mathbb{1}_{n\times n}\Big) \\
&= \sum_{i=1}^{n} \Big(1 - \frac{1}{n}\Big) = n\Big(1 - \frac{1}{n}\Big) = n - 1.
\end{aligned}$$

(c) We have

$$E_n \mathbb{1}_{n\times 1} \underset{\text{Def.}}{=} \mathbb{1}_{n\times 1} - \frac{1}{n} \underbrace{\mathbb{1}_{n\times n}\mathbb{1}_{n\times 1}}_{\underset{\text{s.o.}}{=} n\mathbb{1}_{n\times 1}} = 0_{n\times 1}$$

$$\Longrightarrow \{\lambda \mathbb{1}_{n\times 1} | \lambda \in \mathbb{R}\} \subseteq Ker(E_n).$$

Furthermore, by the dimension formula of linear algebra:

$$\dim(Ker(E_n)) = n - \operatorname{rank}(E_n) \underset{\text{(b)}}{=} 1.$$

Thus,

$$Ker(E_n) = \{\lambda \mathbb{1}_{n\times 1} | \lambda \in \mathbb{R}\}$$