Item 1 **4 points**

Let $\boldsymbol{X} \sim N_2(\boldsymbol{\mu}, \Sigma)$ be a normally distributed random vector with

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Sigma = \frac{1}{2} \cdot \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} = \begin{pmatrix} \frac{13}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{13}{2} \end{pmatrix}$$

.

eigenvectors   eigenvals, Diag

The singular value decomposition (SVD) is given by $\Sigma = V \Lambda V'$ where $V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 9^{\lambda_1} & 0 \\ 0 & 4^{\lambda_2} \end{pmatrix}$ (no proof required).

v_1 v_2

For the tasks below, please give the numerical results in the fill in blanks. **4 points**

Determine the eigenvalues λ_1, λ_2 of the covariance matrix Σ where $\lambda_1 \geq \lambda_2$. 0.5 points

$\lambda_1 =$

9 Number

$\lambda_2 =$ 0.5 points

4 Number

Th. I.2.11 "Distribution of linearly Transformed multinormal distribution

Let the matrix $B \in \mathbb{R}^{1\times 2}$ be given by $B = (1 \quad 2)$ and define $Y = 1 + B\boldsymbol{X}$. Then, the random variable $Y \sim N_1(\nu, \sigma^2)$ is also normally distributed with 1 point

$Y = 1 + \binom{1}{2} X$

$\sim N_2(\mu, \Sigma)$

expectation $\nu =$

$X \sim N_p(\mu, \Sigma)$ Then:
$Y = a + BX \sim N_k(a + B\mu, B\Sigma B')$

$1 + \underset{2\times1}{\binom{1}{2}} \underset{1\times2}{(1 \quad 0)} = 2$

Number

variance $\sigma^2 =$ 1 point

variance $\sigma^2 =$ $(1 \ \ 2)\begin{pmatrix} 13/2 & 5/2 \\ 5/2 & 13/2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} =$

$B\Sigma B'$

1 point

42.5

0.5 points

According to the assumptions, a matrix $A \in \mathbb{R}^{2\times 2}$ exists such that $A = \Sigma^{-1/2}$, $\Sigma = V\Lambda V' = V\Lambda^{1/2}V'$ $= U\Delta U'$

Th. I.2.11 + SVD

"Distribution of linearly transformed multiv. dist"

$$Z = A(X - \mu) \sim N_2(0, I_2),$$

where I_2 denotes the (2-dimensional) identity matrix in $\mathbb{R}^{2\times 2}$. Let $A = U\Delta U'$ be the SVD of A. Determine the diagonal entries δ_{11}, δ_{22} of the diagonal matrix Δ with $\delta_{11} \geq \delta_{22}$ with a precision of two decimals.

$\delta_{11} =$

① $Y = \Sigma^{-1/2}(X-\mu) \sim N_p(0, I_p)$ Th I.2.11

$= A(X-\mu) \sim N_p(0, I_p)$

0,5

② $\Sigma^{-1/2} = V\Lambda^{-1/2}V'$ (SVD)

$A = U\Delta U'$

0.5 points

$\delta_{22} =$ Using $\Sigma = \begin{pmatrix} 13/2 & 5/2 \\ 5/2 & 13/2 \end{pmatrix} \underset{SVD}{\Rightarrow} V\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}V' = \begin{pmatrix} 1/\sqrt{9} & 0 \\ 0 & 1/\sqrt{4} \end{pmatrix}$

0,33

Next

1 2 3 4 5 6 7

Item 2

3.5 points

Let $X = (X_1, X_2)' \sim N_2(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Sigma = \frac{1}{2}\begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} = \begin{array}{c|cc} & 1 & 2 \\ 1 & \frac{13}{2} & \frac{5}{2} \\ 2 & \frac{5}{2} & \frac{13}{2} \end{array}.$$

Parameters of conditionals of multivariate normal distributions.

For the tasks below, please give the numerical results in the fill in blanks.

3.5 points

Th I.2.8 "$X_K | X_L = x_L \sim N_K(\mu_K + \Sigma_{K,L}\Sigma_{L,L}^{-1}(x_L - \mu_L), \Sigma_{KK|L})$

1 point

Due to the properties of a multivariate normal distribution, the distribution of X_1 conditionally on $X_2 = 1$ can be expressed as $X_1|X_2 = 1 \sim N_1(\nu, \sigma^2)$. Determine the conditional expectation $\nu = E(X_1 \mid X_2 = 1)$ and the (conditional) variance $\sigma^2 = \mathrm{Var}(X_1 \mid X_2 = 1)$ with a precision of two decimals.

$\nu =$

$E(X_K | X_L = x_L) = \mu_K + \Sigma_{K,L}\Sigma_{L,L}^{-1}(x_L - \mu_L)$

$(X_1 | X_2 = 1) = \mu_1 + \Sigma_{1,2}\Sigma_{2,2}^{-1}(1-0)$

$= 1 + (5/2)(13/2)^{-1}(1) =$

1.38

1 point

$\sigma^2 =$

$\mathrm{COV}(X_K | X_L = x_L) = \Sigma_{KK|L}$

5.54

$\Sigma_{KK|L} = \Sigma_{K,K} - \Sigma_{K,L}\Sigma_{L,L}^{-1}\Sigma_{K,L}'$

1 2 3 4 5 6 7

$\mathrm{COV}(X_1 | X_2 = 1) = \Sigma_{11|2}$

$\Sigma_{11|2} = \Sigma_{1,1} - \Sigma_{1,2}\Sigma_{2,2}^{-1}\Sigma_{1,2}' = 13/2 - 5/2\,(13/2)^{-1}(5/2)$

Number

1 point

$\sigma^2 =$

Number

1 point

Check, whether an $x_2 \in \mathbb{R}$ exists with $X_1|X_2 = x_2 \sim N_1(1, \sigma^2)$. If such an $x_2 \in \mathbb{R}$ exists, then provide its value. If it does not exist, then type "NA" (without quotation marks) instead into the blank.

$\mu_K + \Sigma_{K,L} \Sigma_{L,L}^{-1} (x_L - \mu_L) = 1$

$\mu_1 + \Sigma_{1,2} \Sigma_{2,2}^{-1} (x_2 - \mu_2) = 1$

$1 + (5/2)(13/2)^{-1}(x_2 - 0) = 1$

0 Number

0.5 points

Check, whether an $x_1 \in \mathbb{R}$ exists with $X_2|X_1 = x_1 \sim N_1(0, 16)$. If such an $x_1 \in \mathbb{R}$ exists, then provide its value. If it does not exist, then type "NA" (without quotation marks) instead into the blank.

$\mu_K + \Sigma_{K,L} \Sigma_{L,L}^{-1} (x_L - \mu_L) = 0 \Rightarrow \mu_2 + \Sigma_{2,1} \Sigma_{1,1}^{-1} (x_1 - \mu_1) = 0$

"NA" Text

$\Sigma_{K,K} - \Sigma_{K,L} \Sigma_{L,L}^{-1} \Sigma'_{K,L} = 16 \Rightarrow \Sigma_{2,2} - \Sigma_{2,1} \Sigma_{1,1}^{-1} \Sigma'_{2,1}$

$13/2 - (5/2)(2/13)^{-1}(5/2) \neq 16$

← Previous Item

Next Item →

1 2 3 4 5 6 7

Item 3

Th. 2.14 "Independence of matrix vector multiplication"
$X \sim N_p(\mu, \Sigma)$ then AX and BX are independent if $A\Sigma B' = 0$

5.5 points

Let $X \sim N_2(\mu, I_2)$ for some $\mu \in \mathbb{R}^2$. Furthermore, define the matrices $A, B, C \in \mathbb{R}^{2\times 2}$ by

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_2 \\ -1 & 1 \end{pmatrix}.$$

Give your answers to the tasks below by filling in the blanks. Results that are numerical values should, if necessary, be rounded to two decimals. In case of multiple solutions, please order your solutions from smallest to largest and separate them by " & " (without quotations marks but with spaces, e.g.: 3 & 5). If such a value **does not exist**, then type "NA" (without quotation marks) instead into the blank. If the value can be **chosen arbitrarily**, then type "R" (without quotation marks) instead into the blank.

5.5 points

1 point

Determine the value for $b \in \mathbb{R}$ such that AX and BX are independent.

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

-1 Number

$\underbrace{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}}_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow 2+2b=0 \quad b=-1$

0.5 points

Assume that $c = c_1 = -c_2 \in \mathbb{R}$. Determine the value for $c \in \mathbb{R}$ such that AX and CX are independent.

1 2 3 4 5 6 7

should, if necessary, be rounded to two decimals. In case of multiple solutions, please order your solutions from smallest to largest and separate them by " & " (without quotations marks but with spaces, e.g.: 3 & 5). If such a value **does not exist**, then type "NA" (without quotation marks) instead into the blank. If the value can be **chosen arbitrarily**, then type "R" (without quotation marks) instead into the blank.

5.5 points

1 point

Determine the value for $b \in \mathbb{R}$ such that AX and BX are independent.

Number

0.5 points

Assume that $c = c_1 = -c_2 \in \mathbb{R}$. Determine the value for $c \in \mathbb{R}$ such that AX and CX are independent.

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 & -1 \\ c_2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} c & -1 \\ -c & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $2c_1 - 2c = 0$ $\cdot -2c_1 + 2 = 0$ $c = 0$

R

0.5 points

$A \varepsilon B'$

Assume that $c = c_1 = c_2 \in \mathbb{R}\backslash\{0\}$. Determine the value for $c \in \mathbb{R}\backslash\{0\}$ such that AX and CX are independent.

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & -1 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} c & -1 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $2c + 2c = 0$ $\cdot 4c = 0$ $c = 0$

$\cdot -2 + 2 = 0$

"NA"

1 point

Sheet 2 Ex. 9.a

Assume that $c_1 = c$ and $c_2 = c^2 - 2$ for some $c \in \mathbb{R}$. Determine the values for $c \in \mathbb{R}$ such that $X'AX$ and CX are independent.

using Conditions :- 1) A is Symmetric

2) $A, B \in \mathbb{R}^{p \times p}$

3) $B \cdot A = O_{p \times p}$

$C \cdot A = 0$

$\begin{pmatrix} c & c^2-2 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $2c + 2c^2 - 4$

ans : 1 , -2

1 point

Sheet 2 Ex. 9.b

Assume that $c_1 = c$ and $c_2 = c^2 - 2$ for some $c \in \mathbb{R}$. Determine the values for $c \in \mathbb{R}$ such that $X'AX$ and $X'CX$ are independent.

using Conditions :- 1) B is also Symmetric

$C = \begin{pmatrix} c & c^2-2 \\ -1 & 1 \end{pmatrix} \Rightarrow$ $c = 1$ $c^2 - 2 = -1 \Rightarrow$ $c = 1$

1

0.5 points

"Cochran's"

Assume that $\mu = 0 \in \mathbb{R}^2$. Find the value for $a \in \mathbb{R}\backslash\{0\}$ such that $a \cdot X'AX \sim \chi^2(2)$, where $\chi^2(d)$ denotes the χ^2-distribution with $d \in \mathbb{N}$ degrees of freedom.

using Th. I.3.6 "Cochran" The col. of A Should be independent thus $\sum_{j=1}^{n} A_j = I_p$ which is not the case.

"NA"

1 point

"Cochrans"

Assume that $\mu = 0 \in \mathbb{R}^2$. Find the value for $a \in \mathbb{R}\backslash\{0\}$ such that $a \cdot X'AX \sim \chi^2(1)$.

using Th. I.3.6

$\sum_{j=1}^{2} 1 = 2$

0.25

← Previous Item

Next Item →

All answers have been saved!

1 2 **3** 4 5 6 7

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Item1 E-test 2

Let the matrices $B, X \in R^{3\times 2}$ be

$$B = \begin{pmatrix} b_1 & b_2 \\ 3 & 3 \\ 1 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 2 & 2 \\ 0 & 1 \\ x_1 & x_2 \end{pmatrix}$$

Considere the 2 normal linear models

$$(1)\ Y = B\beta + \varepsilon \quad \text{and} \quad Y = X\beta + \varepsilon \ (2)$$

with $\beta = (\beta_1, \beta_2)'$ and $\varepsilon \sim N_3(0, \sigma^2\Sigma)$, $\sigma^2 > 0$, $\Sigma \in R^{3\times 3}_{\geq 0}$

a) Let $\Sigma = I_3$ asume $b_1 = b_2 = b \in R$ Find the largest set of values of b s.t. the LSE $\hat{\beta}$ for β in model = unique.

→ Not possible since B not full rank.

b) assume $x_1 = x_2 = x \in \mathbb{R}$ Find the largest set of values for x s.t. LSE unique for model (2).

$$X = \begin{pmatrix} 2 & 2 \\ 0 & 1 \\ x & x \end{pmatrix} \rightarrow X \text{ is full rank} \Rightarrow \text{estimator unique}$$

→ any x.

c) $b_1 = b$, $b_2 = 2b-1$ same... but NOT unique.

$$B = \begin{pmatrix} b & 2b-1 \\ 3 & 3 \\ 1 & 1 \end{pmatrix}$$ $\forall b \neq 1$ model unique → if $b = 1 \Rightarrow$ model not unique

d) Identifiability: Let $b_1 = b$ and $b_2 = -2b+1$. with $b \in R$, further more consider the funtion $g: R^2 \to R^2$ defined by.

$$g(\beta) = \begin{pmatrix} \beta_1 - \beta_2 \\ \beta_1 + \beta_2 \end{pmatrix}$$

Find the largest set of values for **b** $\in \mathbb{R}$ such that $g(\beta)$ not identifiable.

$g(\beta)$ identifiable means $B\beta$ identifiable →

Hence $\ker(B) = \{0\}$ and $\text{Rank}(B) = p$ → $\text{Rank}(B) \neq p$ and model not ident

$$B = \begin{pmatrix} b & -2b+1 \\ 3 & 3 \\ 1 & 1 \end{pmatrix} \quad b = -2b+1 \Rightarrow b = 1/3$$

e) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and model $AY = AX\beta + A\varepsilon$.

which results from model 2 by considering the first 2. observations Y_1 and Y_2., Assume that:

$$Cov(A\varepsilon) = Cov\left(\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}\right) = \frac{2}{3} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Determine the covariance matrix

$Y = -0.33$
$Y_1 = 4.33$
$Y_2 = 0.33$

$V = \begin{pmatrix} V_1 & V \\ V & V_2 \end{pmatrix}$ of $g(\hat{\beta}) = (\beta_1 - \beta_2, \beta_1 + \beta_2)'$ where $\hat{\beta} = (\beta_1, \beta_2)'$ is the MLE of β in model 3.

$\Rightarrow \hat{\Sigma}(Y) = A(Y)$ linear estimator wher $E(\hat{\Sigma}(Y)) = g(\beta)$

$$E\hat{\Sigma}(Y) = E(AY) = E(AB\beta + A\varepsilon) = \overset{I_p}{\overset{\shortparallel}{AB}}\beta = \beta$$

$\rightarrow Cov(\hat{\beta}) = \sigma^2(B'B)^{-1}$ $\quad B = AX$

$\sigma^2[(AX)'(AX)]^{-1}$ $\quad \sigma^2 = cov(A\varepsilon)$

$\rightarrow g(\hat{\beta}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = C\hat{\beta}$

$\rightarrow$ If $Cov(X) = \Sigma$ then $Cov(CX) = C\Sigma C'$

$$Cov(g(\hat{\beta})) = Cov(C\hat{\beta}) = C\,Cov(\hat{\beta})C' = C(\overbrace{\sigma^2((AX)'(AX))^{-1}}^{Cov(A\varepsilon)})C'$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{3}{2}\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3/4 & 1/4 \\ -3/4 & 1/4 \end{pmatrix} = \begin{pmatrix} 3/2 & 0 \\ 0 & 1/2 \end{pmatrix} \rightarrow \text{wrong}$$

$\sigma^{-1/2}$, Theorem 4.25 $\rightarrow \hat{\beta} \sim N_p(\beta, \underline{\sigma^2(B'B)^{-1}})$

Item 2: Consider the normal linear model $Y = B\beta + \varepsilon$ with

$$B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \varepsilon \sim N_3(0, \sigma^2 I_3)$$

$n = 3 \quad r = 2$

a) Denote $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)'$ the LSE of β. Find the missing numerical values.

An upper $(1-\alpha)$ confidence interval for β_1 is given by

$$I_{\beta_1} = [\hat{\beta}_1 - q(\alpha) \cdot \|Y - B\hat{\beta}\| \cdot d, \infty)$$

$\alpha = 0.05$

for $c(\beta) = \beta_0$

$$I_{\beta_0} = \left(\hat{\beta}_0 - t_{1-\alpha}(n-p) \cdot \frac{\sqrt{(B'B)^{-1}_{11}}\,\|Y - B\hat{\beta}\|}{\sqrt{(n-p)}}, \infty\right)$$

Theorem 4.32

$q(\alpha) = t_{(1-\alpha)}(n-p) = t^{\text{one tail}}_{(0.05)}(1) =$ $= t^{2\text{ tail}}_{(0.1)}(1) = 6.31$

$q(\alpha) = 6.31$

$$B'B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$d = \sqrt{c'(B'B^{-1})c} \,/\, \sqrt{n-p}$ $\quad c = (1, 0)$ $\quad (B'B)^{-1} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$

$$d = (1 \;\; 0)\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\tfrac{2}{3} \;\; \tfrac{1}{3}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2}{3} = 0.63$$

$d = c'(B'B^{-1})c \rightarrow$ or $\sqrt{c'(B'B)^{-1}c}$? $\quad d = \sqrt{0.63}$ ← something wrong

b) Similarly an upper $(1-\alpha)$-conf. interv. for $\gamma = \beta_1 - 2\beta_2$ is given by

$$I_\gamma = [\hat{\gamma} - \underbrace{q^*(\alpha)} \cdot \|Y - B\hat{\beta}\| \cdot d^*, \infty)$$

for $\alpha = 0,1$ determine $q^*(\alpha)$ and d^*

$q(\alpha) = t^{\text{one side}}_{(\alpha-1)}(n-p) = 3.078 = 3.08$ $\quad c = (1 \;\; -2)$

$$d^* = c'(B'B)c = (1 \;\; -2)\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = (1 \;\; -2)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \sqrt{2} = 1.41$$

$d^* = 1.41$

c) Consider the testing problem $\boxed{H_0: \beta_2 = 0 \leftrightarrow H_1 - \beta_2 \neq 0}$

Then, the exist an α- level statistic test H_0 whose decision rule is formulated as:

$$\text{Reject } H_0 \text{ if } \frac{Y' A_0 Y}{Y' A Y} > c(\alpha).$$

For some orthog. projectors A_0, A.

Find the matrix $B_0 = (X_1, X_2, X_3)'$ associated with the null hypothesis.

$$B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \rightarrow \quad B_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 = 1, x_2 = 1, x_2 = 0$$

d List the diagonal elements of $A_0 = (a^{\circ}_{ij})_{i,j}$ and $A = (a_{ij})_{i,j}$

$$A_0 = \frac{Q - Q_0}{r - r_0} = \frac{Q - Q_0}{2 - 1} \rightarrow$$

$$Q_0 = B_0 (B_0' B_0)^{-1} B_0'$$

$$(B'B)^{-1} = (2)^{-1} = 1/2$$

$$Q_0 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} (1 \; 1 \; 0) = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Q = B(B'B)^{-1} B'$$

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2/3 & -1/3 \\ 1 & -1 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & 1 & -1/3 \\ 1 & 2 & -1 \\ -1/3 & -1 & 2/3 \end{pmatrix} = Q$$

$$\Rightarrow Q - Q_0 = \begin{pmatrix} 1/6 & 1/2 & -1/3 \\ 1/2 & 3/2 & -1 \\ -1/3 & -1 & 2/3 \end{pmatrix}$$

$a^{\circ}_{11} = 1/6$ ✓ $\quad a^{\circ}_{22} = ?$

$a^{\circ}_{33} = 2/3$

$$\rightarrow A = (I_n - Q) = \begin{pmatrix} 1/3 & -1 & 1/3 \\ -1 & -1 & 1 \\ 1/3 & 1 & 1/3 \end{pmatrix}$$

$a_{11} = 1/3$ ✓ $\quad a_{22} = -1 \quad a_{33} = 1/3$ ✓

e) For $\alpha = 0{,}01$ find the critical value $c(\alpha)$.

$$c(\alpha) = F_{1-\alpha}(r - r_0, n - r) = F_{1\alpha}(1, 1) = 4052.181$$

Item 3

Consider measurments $x_1 = 1$, $x_2 = -1$, $x_3 = 2$ $x_4 = 0$ and the polynomial regression model

$$Y = \beta_0 + \beta_1 X^3$$

a) Consider the testing problem $H_0: \beta_0 = 0 \leftrightarrow H_1\ \beta_0 \neq 0$.
Determine the entries of the matrix $B = (b_1, b_2, b_3, b_4)$ associated with H_0.

$f(x) = \sum_{j=0}^{m} \beta_j x^j$ – intercept included →

$$B = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \longrightarrow B_0 \begin{pmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \\ x_4^3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 8 \\ 0 \end{pmatrix}$$

b) The decision rule in terms of quantiles of the t-distrib. can be formulated as.

$$\text{Reject } H_0 \text{ if } \left| \frac{\hat{\beta}_0}{c} \right| > t_{1-\alpha/2}(df). \qquad c \in \mathbb{R}$$

Determine the degrees of freedom of the t-distrib
See example 5.18

Rank $(B) = m+1 = 2$
$\rightarrow m = 1$
$n = 4$

$$T_k = \frac{\hat{\beta}_k}{\sqrt{(B'B)^{-1}_{kk} \, \|Y - B\hat{\beta}\| / (n-m-1)}} \sim t(n-m-1).$$

reject H_0 if $|T_k| > t_{1-\alpha/2}(n-m-1)$

$\rightarrow t_{1-\alpha/2}(n-m-1) = t_{1-\alpha/2}(4-1-1) = t_{1-\alpha/2}(2)$ ←

e-test 3

ITEM 1

Family of distributions given by pdfs with $\alpha>0, \gamma>0$:

$$f(x;\alpha,\gamma) = \sqrt{\frac{\gamma}{2\pi x^2}} \exp\left(-\frac{\gamma(x-\alpha)^2}{2\alpha^2 x}\right), \quad x>0$$

For $\gamma>0$ fixed, $f_\gamma(x;\alpha) = f(x;\alpha,\gamma)$ is a subfamily of the EDF with $c(x,\phi) = \frac{1}{2}(\ln(\gamma) - \ln(2\pi x^2) - \frac{\gamma}{x})$

a) Let $\gamma=2$ and $X \sim f_2(\cdot;\alpha)$

i) Determine θ and $a(\phi)$ for $\alpha=1$

EDF: $f(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$

$$f_2(x;\alpha) = \underbrace{\sqrt{\frac{\gamma}{2\pi x^2}}}_{\text{corresponds to first part of } c(x,\phi)} \exp\left(\underbrace{-\frac{\gamma(x-\alpha)^2}{2\alpha^2 x}}_{\text{look at this more closely}}\right)$$

$$\Rightarrow -\frac{\gamma}{2\alpha^2 x}(x^2 - 2x\alpha + \alpha^2) = -\frac{\gamma x}{2\alpha^2} + \frac{\gamma}{\alpha} \underbrace{-\frac{\gamma}{2x}}_{\text{covered by last part of } c(x,\phi)}$$

$$\Rightarrow \frac{x\cdot\theta - b(\theta)}{a(\phi)} \overset{!}{=} -\frac{\gamma x}{2\alpha^2} + \frac{\gamma}{\alpha}$$ $\to$ multiple parametrizations possible

One possible solution: $a(\phi) = \frac{2}{\gamma} = 1$, $\theta = -\frac{1}{\alpha^2} = -1$, $b(\theta) = -\frac{2}{\alpha} = -2$

ii) Calculate $E(X)$ for $\alpha=1$

$E(X) = b'(\theta)$ $\qquad \alpha = \pm\sqrt{-\theta}^{-1} \to$ Choose positive solution

$\Rightarrow b(\theta) = -2\sqrt{-\theta}$, $b'(\theta) = -2\cdot\frac{1}{2}\cdot(-1)\frac{1}{\sqrt{-\theta}} = \frac{1}{\sqrt{-\theta}}$

Insert $\theta = -1$: $E(X) = 1$

iii) Calculate $Var(X)$ for $\alpha=1$

$$Var(X) = b''(\theta)\,a(\phi) = -\frac{1}{2}\cdot(-1)\cdot\frac{1}{\sqrt[3]{-\theta}}\cdot\frac{2}{\gamma} = \frac{1}{\gamma\sqrt[3]{-\theta}} \overset{\gamma=2,\,\theta=-1}{=} \frac{1}{2}$$

b) Further assume that Y is a binary response variable and
$\pi(x) = P(Y=1 \mid X=x)$

Suppose $(X \mid Y=j) \sim f(\cdot \mid \alpha_j, \gamma_j)$, $\alpha_j, \gamma_j > 0$, $j \in \{0,1\}$

Model: $\text{logit}(\pi(x)) = \log\left(\frac{\pi(x)}{1-\pi(x)}\right) = \beta_0 + \beta_1 x^{-1} + \beta_2 x$

Assume $\alpha_0 = \gamma_0 = 2$, $\alpha_1 = \gamma_1 = 1$

Calculate β_1 and β_2

Use Bayes' Theorem: $f^{X_1 \mid X_2 = x_2}(x_1) = \frac{1}{f^{X_2}(x_2)} f^{X_2 \mid X_1 = x_1}(x_2) f^{X_1}(x_1) I_{supp}(X_2)(x_2)$, $x_1 \in \mathbb{R}$

Try to express $P(Y \mid X)$ through $P(X \mid Y)$:

$$P(Y \mid X) = \frac{P(X \mid Y)\, P(Y)}{P(X)}, \quad P(X) = P(X \mid Y=1) P(Y=1) + P(X \mid Y=0) P(Y=0)$$

$$P(X=x \mid Y=1) \sim f(x \mid 1,1) = \frac{1}{\sqrt{2\pi x^2}} \exp\left(-\frac{1}{2x}(x-1)^2\right)$$

$$P(X=x \mid Y=0) \sim f(x \mid 2,2) = \frac{1}{\sqrt{\pi x^2}} \exp\left(-\frac{1}{4x}(x-2)^2\right)$$

$$P(Y=1) = P(Y=2) = \frac{1}{2}$$

$$\Rightarrow P(Y=1 \mid X=x) = \frac{\frac{1}{\sqrt{2}} \exp\left(-\frac{1}{2x}(x-1)^2\right)}{\frac{1}{\sqrt{2}} \exp\left(-\frac{1}{2x}(x-1)^2\right) + \exp\left(-\frac{1}{4x}(x-2)^2\right)} = \pi(x)$$

$$1 - \pi(x) = \frac{\exp\left(-\frac{1}{4x}(x-2)^2\right)}{\frac{1}{\sqrt{2}} \exp\left(-\frac{1}{2x}(x-1)^2\right) + \exp\left(-\frac{1}{4x}(x-2)^2\right)}$$

$$\frac{\pi(x)}{1-\pi(x)} = \frac{1}{\sqrt{2}} \frac{\exp\left(-\frac{1}{2x}(x-1)^2\right)}{\exp\left(-\frac{1}{4x}(x-2)^2\right)} = \frac{1}{\sqrt{2}} \exp\left(-\frac{1}{2x}\left((x-1)^2 - \frac{1}{2}(x-2)^2\right)\right)$$

$$\text{logit}(\pi(x)) = \log\left(\frac{\pi(x)}{1-\pi(x)}\right) = \log\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2x}\left(\frac{1}{2}x^2 - 1\right) = \underbrace{\log\left(\frac{1}{\sqrt{2}}\right)}_{\beta_0} - \frac{x}{4} + \frac{1}{2x}$$

$$\beta_1 = \frac{1}{2} \qquad \beta_2 = -\frac{1}{4}$$

ITEM 2

GLM $E(\vec{Y}) = g(\vec{\mu}) = X\beta$ g is the canonical link

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad Y_i \sim P(\mu_i), \text{ Poisson distribution with } \mu_i = 2i^2$$

Determine the Fisher information matrix I_F

Canonical link : $g(\vec{\mu}) = \vec{\theta} \overset{11.2.15}{\Longrightarrow} g(\vec{\mu}) = \log(\vec{\mu})$ for $P(\mu)$

$\Rightarrow \mu_i = e^{\theta_i}$

11.2.24 : I_F for canonical link $\Rightarrow I_F = X' W_c X$

with $W_c = \operatorname{diag}\left(\frac{b''(\theta_i)}{a(\phi_i)}\right)$

11.2.5 : $b(\theta_i) = e^{\theta_i}$ $a(\phi) = 1$

$$I_F = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \mu_1 + \mu_3 & -\mu_3 \\ -\mu_3 & \mu_2 + \mu_3 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & -18 \\ -18 & 26 \end{pmatrix}$$

ITEM 3

$(X_i)_i \overset{iid}{\sim} P(\mu_1)$, $(Y_i)_i \overset{iid}{\sim} P(\mu_2)$

$\hat{\mu}_{1(n)} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \hat{\mu}_{2(n)} = \frac{1}{n} \sum_{i=1}^{n} Y_i \quad , n \in \mathbb{N}$

Derive asymptotic variance $\sigma^2 > 0$ of $Z_n = \sqrt{n}\left(\hat{\mu}_{1(n)} + \hat{\mu}_{2(n)}^2\right)$

as $n \to \infty$ for $\mu_1 = 1$, $\mu_2 = 2$

Use delta method from 11.2.31

We know: $\sqrt{n}\left(\vec{\hat{\mu}}_n - \vec{\mu}\right) \longrightarrow N_2(\vec{0}, \Sigma)$ with

$$\Sigma = \begin{pmatrix} Var(X) & 0 \\ 0 & Var(Y) \end{pmatrix}$$ since $Cov(X_i, Y) = 0$ bc $P(\mu_1)$ and $P(\mu_2)$ are independent

We also know that $Var(X) = \mu_1$ and $Var(Y) = \mu_2$ bc X, Y are poisson-distributed.

Delta Method: If $\sqrt{n}(\hat{\vec{\theta}}_n - \vec{\theta}_0) \to \mathcal{N}_d(\vec{0}, \Sigma)$

then $\sqrt{n}(g(\hat{\vec{\theta}}_n) - g(\vec{\theta}_0)) \to \mathcal{N}_d\left(\vec{0}, \frac{\partial}{\partial \vec{\theta}} g(\vec{\theta}_0) \Sigma \left(\frac{\partial}{\partial \vec{\theta}} g(\vec{\theta}_0)\right)'\right)$

Find $g: \mathbb{R}^2 \to \mathbb{R}$ s.th. $g(\hat{\vec{\mu}}_n) = \frac{Z_n}{\sqrt{n}}$

$$\Rightarrow \frac{Z_n}{\sqrt{n}} = \hat{\mu}_{1(n)} + \hat{\mu}_{2(n)}^2 = (1\ 0)\begin{pmatrix}\hat{\mu}_{1(n)} \\ \hat{\mu}_{2(n)}\end{pmatrix} + (\hat{\mu}_{1(n)}, \hat{\mu}_{2(n)})\begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\mu}_{1(n)} \\ \mu_{2(n)}\end{pmatrix}$$

$$= (1\ 0)\hat{\vec{\mu}}_n + \hat{\vec{\mu}}_n' \begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\hat{\vec{\mu}}_n = g(\hat{\vec{\mu}}_n)$$

$$\frac{\partial}{\partial \vec{\mu}} g(\vec{\mu}) = (1\ \ 0) + (0\ \ 2\mu_2) = (1\ \ 2\mu_2)$$

$$\sigma^2 = (1\ \ 2\mu_2)\begin{pmatrix}\mu_1 & 0 \\ 0 & \mu_2\end{pmatrix}\begin{pmatrix}1 \\ 2\mu_2\end{pmatrix} = (1\ \ 4)\begin{pmatrix}1 & 0 \\ 0 & 2\end{pmatrix}\begin{pmatrix}1 \\ 4\end{pmatrix} = 33 \,//$$

e - test 4

ITEM 1

$X_1 \dots X_n$ $n \in \mathbb{N}$ iid with pdf $f(x;\alpha,\gamma) = \begin{cases} \alpha\gamma^{\alpha}x^{-(\alpha+1)} & , \gamma \le x \\ 0 & , \text{else} \end{cases}$

cdf $F(x;\alpha,\beta) = \begin{cases} 1-\gamma^{\alpha}x^{-\alpha} & , \gamma \le x \\ 0 & , \text{else} \end{cases}$ $\quad \alpha > 1, \gamma > 0$

$L(\alpha,\gamma;\vec{x}) = \begin{cases} \alpha^{n}\gamma^{n\alpha}\prod_{i=1}^{n} x_i^{-(\alpha+1)} & , \gamma \le x_{(1)} \\ 0 & , \text{else} \end{cases}$ $\quad x_{(1)} = \min\{x_1, \dots x_n\}$

11.2.50 Consider profile likelihood approach where γ the nuisance parameter and α parameter of intereste.

$\gamma(\alpha)$: estimate of γ for fixed α

observed sample: $x_1 = 6$, $x_2 = 4$, $x_3 = 3$ $\quad n = 3$

Calculate $\gamma(\alpha)$ and give $\gamma(\alpha)$ for $\alpha = 2$

$l_{\alpha}(\gamma;\vec{x}) = \begin{cases} n\log(\alpha) + n\alpha\log(\gamma) - (\alpha+1)\sum_{i=1}^{n} x_i & \text{for } \gamma \le x_{(1)} \\ -\infty & \text{else} \end{cases}$

$\frac{\partial l_{\alpha}(\gamma,\vec{x})}{\partial\gamma} = \begin{cases} \frac{n\alpha}{\gamma} & , \gamma \le x_{(1)} \\ -\infty & \text{else} \end{cases} \overset{!}{=} 0$

$\frac{n\alpha}{\gamma} \to 0$ for $\gamma \to \infty$ but γ is limited by $x_{(1)} \Rightarrow \gamma = x_{(1)}$

$\Rightarrow \hat{\gamma}(\alpha) = 3$

Calculate $\hat{\alpha}_{\text{profile}}$

$\frac{\partial l_{\gamma}(\alpha,\vec{x})}{\partial\alpha} = \frac{n}{\alpha} + n\log(\gamma) - \sum_{i=1}^{n}\log(x_i)$ for $\gamma \le x_{(1)}$

$\frac{\partial l_{\gamma}(\alpha,\vec{x})}{\partial\alpha} \overset{!}{=} 0 \Rightarrow \alpha = -\frac{n}{n\log(\gamma) - \sum_{i=1}^{n}\log(x_i)} = 3.059$

$x_3 = n = \gamma = 3, x_1 = 6, x_2 = 4$

ITEM 2

Let $Y|X=k \sim \mathcal{P}(k)$, $P(X=k) = (1-p)^k p$, $k \in \mathbb{N}_0$
for $p \in (0,1)$, $E(X) = \frac{1-p}{p}$ $Var(X) = \frac{1-p}{p^2}$

a) Derive expectation and variance of Y for $p = \frac{1}{3}$
Use Solutions/Hint of theoretical exercise 25 of sheet 6:

$E(Y) = E(E(Y|X))$

$Var(Y) = Var(E(Y|X)) + E(Var(Y|X))$

$$\left.\begin{aligned} E(Y|X) &= X \\ Var(Y|X) &= X \end{aligned}\right\} \text{II.2.5}$$

$E(Y) = E(X) = \frac{1-p}{p} = 2$ $Var(Y) = Var(X) + E(X) = \frac{1-p}{p}\left(\frac{1}{p}+1\right) = 8$ //

b) p_{min} is the smallest $p \in (0,1)$ s.th. $P(Y=0) \geq \frac{1}{3}$. Find p_{min}
Hint: Law of total probability

$P(Y) = \sum_{k=0}^{\infty} P(Y|X=k)\, P(X=k) = \sum_{k=0}^{\infty} \frac{k^y}{y!} e^{-k} (1-p)^k p$

$\text{II.1.3} \quad P(y;k) = \frac{k^y}{y!} e^{-k}$

$P(Y=0) = \sum_{k=0}^{\infty} e^{-k}(1-p)^k p = \sum_{k=0}^{\infty} \left(\frac{1-p}{e}\right)^k p = \frac{p}{1-\frac{1-p}{e}} \stackrel{!}{=} \frac{1}{3}$

$\Rightarrow p = \frac{1-\frac{1}{e}}{3-\frac{1}{e}} \approx 0.24$ //

ITEM 3

$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{3}} \\ -1 & \frac{1}{\sqrt{3}} \end{pmatrix}$

Lasso and ridge regression: $\min_{\beta \in \mathbb{R}^2} \left\{ \frac{1}{2}\|\vec{y} - X\vec{\beta}\|^2 + \lambda \|\vec{\beta}\|_q^q \right\}$ $q \in \{1,2\}$

$\vec{y} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ← observed.

$X'X = I_2$ $\quad \hat{\vec{\beta}}^{LS} = (X'X)^{-1} X'\vec{Y}$ ← 1.4.12
$\quad = X'\vec{Y}$

$\frac{1}{2}\|\vec{y} - X\vec{\beta}\|^2 + \lambda(|\beta_1| + |\beta_2|) = \frac{1}{2}\vec{y}'\vec{y} - \vec{y}'X\vec{\beta} + \frac{1}{2}\vec{\beta}'X'X\vec{\beta} + \lambda(|\beta_1| + |\beta_2|)$
$= \frac{1}{2}\vec{y}'\vec{y} - (\hat{\vec{\beta}}^{LS})'\vec{\beta} + \frac{1}{2}\vec{\beta}^2 + \lambda(|\beta_1| + |\beta_2|)$

Find λ_{min} s.th. at least one of the lasso estimates $\hat{\beta}_i^{lasso}$ equals zero (with precision of two decimal places)

$X'X = I_2 \qquad \hat{\vec{\beta}}^{LS} = (X'X)^{-1} X \vec{y} \quad \leftarrow 1.4.12$

$= X\vec{y} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$

based on solution of theoretical ex. 27 sheet 6 $\rightarrow$

$\frac{1}{2}\|\vec{y} - X\vec{\beta}\|^2 + \lambda(|\beta_1| + |\beta_2|) = \frac{1}{2}\vec{y}'\vec{y} - \vec{y}'X\vec{\beta} + \lambda(|\beta_1| + |\beta_2|) + \frac{1}{2}\vec{\beta}'X'X\vec{\beta}$

$= \frac{1}{2}\vec{y}'\vec{y} \underbrace{- (\hat{\vec{\beta}}^{LS})'\vec{\beta} + \lambda(|\beta_1| + |\beta_2|) + \frac{1}{2}\vec{\beta}^2}_{:= l(\vec{\beta})}$

$\hat{\beta}_1^{LS} > 0$ so $\beta_1 \geq 0$ since it would not minimize $l(\vec{\beta})$ otherwise

$\Rightarrow \frac{\partial l(\vec{\beta})}{\partial \beta_1} = -\hat{\beta}_1^{LS} + \beta_1 + \lambda \stackrel{!}{=} 0$

$\Leftrightarrow \quad \beta_1 = \hat{\beta}_1^{LS} - \lambda = \frac{1}{2} - \lambda$

$\hat{\beta}_2^{LS} < 0$ so $\beta_2 \leq 0$ analogous to before

$\Rightarrow \frac{\partial l(\vec{\beta})}{\partial \beta_2} = -\hat{\beta}_2^{LS} + \beta_2 - \lambda \stackrel{!}{=} 0$

$\Leftrightarrow \quad \beta_2 = \hat{\beta}_2^{LS} + \lambda = -\frac{\sqrt{3}}{2} + \lambda$

$\Rightarrow \lambda = \frac{1}{2} \vee \lambda = \frac{\sqrt{3}}{2}$ with $\frac{1}{2} < \frac{\sqrt{3}}{2}$

$\Rightarrow \lambda_{min} = \frac{1}{2}$ //

Calculate the corresponding ridge regression estimates of λ_{min}

Use augmented model from ex. 27 sheet 6

$X_* = \begin{pmatrix} X \\ \sqrt{\lambda} I_p \end{pmatrix} \qquad \vec{Y}_* = \begin{pmatrix} \vec{Y} \\ 0_p \end{pmatrix} \qquad X'X = I_2$

$\hat{\vec{\beta}}^{ridge} = (X_*'X_*)^{-1} X_*' \vec{Y}_*$

$= (X'X + \lambda I_2)^{-1} X' \vec{Y}$

$= \frac{1}{1+\lambda} \hat{\vec{\beta}}^{LS}$

$= \frac{2}{3} \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$