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Applied Data Analysis

Exercise Sheet 2

Exercise 7

(a) Let $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$ with $\boldsymbol{\mu} \in \mathbb{R}^p$, $\Sigma \in \mathbb{R}^{p\times p}_{>0}$ and $\emptyset \neq K, L \subseteq \{1,\ldots,p\}, K \cap L = \emptyset, k = |K|, l = |L|$. Further, let $\Sigma_{K,L} = \mathrm{Cov}(\boldsymbol{X}_K, \boldsymbol{X}_L)$ and $\Sigma_{KK|L} = \Sigma_{K,K} - \Sigma_{K,L}\Sigma_{L,L}^{-1}\Sigma'_{K,L}$.

Show Theorem I 2.8, i.e., the conditional distribution of $\boldsymbol{X}_K$ given $\boldsymbol{X}_L = \boldsymbol{x}_L$ is given by

$$\boldsymbol{X}_K|\boldsymbol{X}_L = \boldsymbol{x}_L \sim N_k(\boldsymbol{\mu}_K + \Sigma_{K,L}\Sigma_{L,L}^{-1}(\boldsymbol{x}_L - \boldsymbol{\mu}_L), \Sigma_{KK|L})$$

(b) Show that $\boldsymbol{X}_K$ and $\boldsymbol{X}_L$ are independent if and only if $\mathrm{E}(\boldsymbol{X}_K|\boldsymbol{X}_L = \boldsymbol{x}_L) = \mathrm{E}(\boldsymbol{X}_K) = \boldsymbol{\mu}_K$ for all $\boldsymbol{x}_L \in \mathbb{R}^l$.

(c) Let $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$, $\rho \in (-1,1)$ and $\boldsymbol{X} = (X_1, X_2)' \sim N_2(\boldsymbol{\mu}, \Sigma)$ with

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \qquad \text{and} \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

For $t \in \mathbb{R}$ derive the conditional distribution of X_2 given $X_1 = t$.

Hint to (a): Argue that, without loss of generality, one may assume $K \cup L = \{1,\ldots,p\}$ and therefore

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_K \\ \boldsymbol{X}_L \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_K \\ \boldsymbol{\mu}_L \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \Sigma_{K,K} & \Sigma_{K,L} \\ \Sigma_{L,K} & \Sigma_{L,L} \end{pmatrix},$$

i.e $K = \{1,\ldots,k\}$, $L = \{k+1,\ldots,p\}$, and apply the results of Exercise 4 to factorize the density of $\boldsymbol{X} = \boldsymbol{X}_{K\cup L}$ into a marginal and conditional part.

Exercise 8

Show that if $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$, where Σ is an ortho-projection matrix and $\boldsymbol{\mu} \in Im(\Sigma)$, then $\boldsymbol{X}'\boldsymbol{X} \sim \chi^2(\mathrm{rank}(\Sigma), \frac{1}{2}\boldsymbol{\mu}'\boldsymbol{\mu})$.

Hint: Apply Theorem 1.3.5.

Exercise 9

Let $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \sigma^2 I_p)$ and $A, B \in \mathbb{R}^{p\times p}$, where A is symmetric and $BA = 0_{p\times p}$. Show the following statements.

(a) $\boldsymbol{X}'A\boldsymbol{X}$ and $B\boldsymbol{X}$ are independent.

(b) If, furthermore, B is also symmetric, then $\boldsymbol{X}'A\boldsymbol{X}$ and $\boldsymbol{X}'B\boldsymbol{X}$ are independent.

Exercise 10

Consider the linear model $\boldsymbol{Y} = B\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, with $\mathrm{Cov}(\boldsymbol{\varepsilon}) = \sigma^2 I_p$, $\sigma^2 > 0$. Furthermore, assume that $\det(B'B) > 0$. Then, the residuals are defined by

$$\hat{\boldsymbol{\varepsilon}} := (I_p - B(B'B)^{-1}B')\boldsymbol{Y}.$$

Calculate

(a) $\mathrm{E}(\hat{\boldsymbol{\varepsilon}})$,

(b) $\mathrm{Cov}(\hat{\boldsymbol{\varepsilon}})$.