Applied Data Analysis

Exercise Sheet 1 - Solutions

Exercise 1

According to the assumptions and Theorem I.2.5, there exists a representation

(1) $$A \;=\; V\,\Lambda\,V'$$

with $\Lambda := \operatorname{diag}(\lambda_1,\ldots,\lambda_p) \in \mathbb{R}^{p\times p}$ and $V := [\,\mathbf{v_1}|\ldots|\mathbf{v_p}\,] \in \mathbb{R}^{p\times p}$ orthogonal (SVD of A).

(a) First, since V is orthogonal, we obtain:

(2) $$\det(V)\,\det(V') \overset{\text{I.2.4,(8)}}{=} \det(VV') \overset{\text{I.2.3,(2)}}{=} \det(I_p) \;=\; 1$$

Thus, especially, $\det(V) \;=\; \det(V') \;\neq\; 0$, and thus V and V' are both regular.

Then, we obtain:

(3) $$\begin{aligned}\operatorname{rank}(A) &\overset{(1)}{=} \operatorname{rank}\big((V\,\Lambda)\,V'\big) \underset{V'\text{ regular}}{\overset{\text{I.2.4,(5)}}{=}} \operatorname{rank}(V\,\Lambda) \overset{\text{I.2.4,(6)}}{=} \operatorname{rank}(\Lambda\,V)\\ &\underset{V\text{ regular}}{\overset{\text{I.2.4,(5)}}{=}} \operatorname{rank}(\Lambda) \overset{\text{Def.}}{=} \operatorname{rank}\big(\operatorname{diag}(\lambda_1,\ldots,\lambda_p)\big)\\ &\underset{\text{Algebra}}{\overset{\text{Linear}}{=}} \big|\,\big\{i\in\{1,\ldots,p\}\,\big|\,\lambda_i\neq 0\big\}\,\big|\end{aligned}$$

(b),(c)

Since $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of A and $\{\mathbf{v_1},\ldots,\,\mathbf{v_p}\}$ an orthonormal system (i.e. an orthonormal basis) of corresponding eigenvectors, it holds:

(4) $$A\,\mathbf{v_i} \;=\; \lambda_i\,\mathbf{v_i}\;\;,\;\; i\in\{1,\ldots,p\}$$

(5) $$\mathbf{v_i}'\mathbf{v_j} \;=\; \underbrace{\delta_{ij}}_{\text{Kronecker symbol}} \;=\; \left\{\begin{array}{ccc} 1 & , & i=j\\ 0 & , & i\neq j\end{array}\right\}\;\;,\;\; i,j\in\{1,\ldots,p\}$$

$\Longrightarrow$: Let A be non-negative definite (positive definite).
Then, it holds for $i\in\{1,\ldots,p\}$:

$$\lambda_i \overset{(5)}{=} \lambda_i\,\mathbf{v_i}'\,\mathbf{v_i} \;=\; \mathbf{v_i}'\,\lambda_i\,\mathbf{v_i} \overset{(4)}{=} \mathbf{v_i}'\,A\,\mathbf{v_i} \left\{\begin{array}{l} \overset{\text{I.2.3}}{\geq}\;\; 0\;\;,\text{ if } A \text{ non-negative definite}\\ \overset{\text{I.2.3}}{>}\;\; 0\;\;,\text{ if } A \text{ positve definite}\end{array}\right.$$

Regarding the last step, notice that $\mathbf{v_i}\in\mathbb{R}^p\backslash\{\mathbf{0}\}$ for $i\in\{1,\ldots,p\}$ (as normal vector).

$\Longleftarrow$: Let $\lambda_i \geq 0$ for $i \in \{1, \ldots, p\}$ and $\mathbf{x} \in \mathbb{R}^p \backslash \{\mathbf{0}\}$.
Since $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a basis of $\mathbb{R}^p$, there exist $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$ with

$$(6) \qquad \mathbf{x} \;=\; \sum_{i=1}^{p} \alpha_i \, \mathbf{v_i}$$

Then, we get:

$$(7) \qquad \mathbf{x}' A \, \mathbf{x} \;\overset{(6)}{=}\; \left(\sum_{i=1}^{p} \alpha_i \, \mathbf{v_i}' \right) A \left(\sum_{j=1}^{p} \alpha_j \, \mathbf{v_j} \right) \;=\; \sum_{i=1}^{p} \sum_{j=1}^{p} \alpha_i \, \alpha_j \, \mathbf{v_i}' \, \underbrace{A \, \mathbf{v_j}}_{= \lambda_j \, \mathbf{v_j}}$$

$$\overset{(4)}{=}\; \sum_{i=1}^{p} \sum_{j=1}^{p} \alpha_i \, \alpha_j \, \lambda_j \, \underbrace{\mathbf{v_i}' \, \mathbf{v_j}}_{= \delta_{ij}} \;\overset{(5)}{=}\; \sum_{i=1}^{p} \lambda_i \, \alpha_i^2 \;\overset{\text{Ass.}}{\geq}\; 0$$

Now, let $\lambda_i > 0$ for $i \in \{1, \ldots, p\}$.
Since $\mathbf{x} \in \mathbb{R}^p \backslash \{\mathbf{0}\}$, in the representation (6) of $\mathbf{x}$, there must (at least) exist one $i_0 \in \{1, \ldots, p\}$ with $\alpha_{i_0} \neq 0$. Then, we obtain:

$$\mathbf{x}' A \, \mathbf{x} \;\overset{(7)}{=}\; \sum_{i=1}^{p} \lambda_i \, \alpha_i^2 \;\geq\; \lambda_{i_0} \, \alpha_{i_0}^2 \;>\; 0$$

Exercise 2

(a) By the assumptions and Exercise 1, (b), $\lambda_1 \geq \ldots \geq \lambda_p \geq 0$ and thus $\Lambda^{1/2}$ as well as $A^{1/2}$ both are well defined.

First, we have to show that $A^{1/2}$ is symmetric.

Since $\Lambda^{1/2}$ is symmetric (as a diagonal matrix), we get:

$$(1) \qquad (A^{1/2})' \;\overset{\text{Def.}}{=}\; \left(V \, \Lambda^{1/2} \, V' \right)' \;\overset{\text{I.2.4,(2)}}{=}\; (V')' \, (\Lambda^{1/2})' \, V' \;=\; V \, \Lambda^{1/2} \, V' \;\overset{\text{Def.}}{=}\; A^{1/2}$$

Further, let $\mathbf{x} \in \mathbb{R}^p$ and $\mathbf{y} \;=\; (y_1, \ldots, y_p)' \;:=\; V' \mathbf{x}$. Then, we obtain:

$$(2) \qquad \mathbf{x}' \, A^{1/2} \, \mathbf{x} \;\overset{\text{Def.}}{=}\; \underbrace{\mathbf{x}' \, V}_{= \mathbf{y}'} \, \Lambda^{1/2} \, \underbrace{V' \, \mathbf{x}}_{= \mathbf{y}} \;=\; \mathbf{y}' \, \Lambda^{1/2} \, \mathbf{y} \;=\; \sum_{i=1}^{p} \sqrt{\lambda_i} \, y_i^2 \;\geq\; 0$$

By (1) $A^{1/2}$ is symmetric and (then) by (2) $A^{1/2}$ is non-negative definite.

Finally, we obtain:

$$(3) \qquad A^{1/2} \, A^{1/2} \;\overset{\text{Def.}}{=}\; V \, \Lambda^{1/2} \, \underbrace{V' \, V}_{= I_p} \, \Lambda^{1/2} \, V' \;=\; V \, \underbrace{\Lambda^{1/2} \, \Lambda^{1/2}}_{= \Lambda} \, V' \;=\; V \, \Lambda \, V' \;\overset{\text{Ass.}}{=}\; A.$$

(b) Now, suppose that A is positive definite.

Then, by the assumptions and Exercise 1, (c), $\lambda_1 \geq \ldots \geq \lambda_p > 0$ and thus

$$\sqrt{\lambda_1} \geq \ldots \geq \sqrt{\lambda_p} > 0 \tag{4}$$

Especially, (4) yields that the following matrices (given as in Theorem I.2.5) are well-defined:

$$A^{-1/2} := V\,\Lambda^{-1/2}\,V' \quad \text{with} \tag{5}$$

$$\Lambda^{-1/2} := \operatorname{diag}\bigl(\lambda_1^{-1/2}, \ldots, \lambda_p^{-1/2}\bigr) = \operatorname{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \ldots, \frac{1}{\sqrt{\lambda_p}}\right) \tag{6}$$

Let $\mathbf{x} \in \mathbb{R}^p \backslash \{\mathbf{0}\}$ and (as before) $\mathbf{y} = (y_1, \ldots, y_p)' := V'\mathbf{x}$. Then, $\mathbf{y} \in \mathbb{R}^p \backslash \{\mathbf{0}\}$ since V' is regular. Thus, there exists (at least) one $i_0 \in \{1, \ldots, p\}$ with $y_{i_0} \neq 0$. It folows:

$$\mathbf{x}'\,A^{1/2}\,\mathbf{x} \overset{(2)}{=} \sum_{i=1}^{p} \sqrt{\lambda_i}\,y_i^2 \geq \sqrt{\lambda_{i_0}}\,y_{i_0}^2 \overset{(4)}{>} 0 \tag{7}$$

By (1) and (7), $A^{1/2}$ is positive definite.

Further, Λ and A both are regular with

$$\Lambda^{-1} = \operatorname{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_p}\right) \overset{\text{Def.}}{=} \Lambda^{-1/2}\,\Lambda^{-1/2} \tag{8}$$

$\bigl($since $\Lambda\,\operatorname{diag}(\lambda_1^{-1}, \ldots, \lambda_p^{-1}) = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)\,\operatorname{diag}(\lambda_1^{-1}, \ldots, \lambda_p^{-1}) = I_p\bigr)$

$$A^{-1} \overset{\text{Ass.}}{=} (V\,\Lambda\,V')^{-1} \overset{\text{I.2.4,(3)}}{=} \underbrace{(V')^{-1}}_{=V}\,\Lambda^{-1}\,\underbrace{V^{-1}}_{=V'} = V\,\Lambda^{-1}\,V' \tag{9}$$

Using these two equations, finally we obtain:

(i)
$$A^{-1/2}\,A^{-1/2} \underset{\text{(cf. (5))}}{\overset{\text{Def.}}{=}} V\,\Lambda^{-1/2}\,\underbrace{V'\,V}_{=I_p}\,\Lambda^{-1/2}\,V' = V\,\Lambda^{-1/2}\,\Lambda^{-1/2}\,V'$$
$$\overset{(8)}{=} V\,\Lambda^{-1}\,V' \overset{(9)}{=} A^{-1}$$

(ii)
$$A^{1/2}\,A^{-1/2} \underset{\text{(cf. (5))}}{\overset{\text{Def.}}{=}} V\,\Lambda^{1/2}\,\underbrace{V'\,V}_{=I_p}\,\Lambda^{-1/2}\,V' = V\,\underbrace{\Lambda^{1/2}\,\Lambda^{-1/2}}_{=I_p}\,V' = V\,V' = I_p$$

and analogously

$$A^{-1/2}\,A^{1/2} \underset{\text{(cf. (5))}}{\overset{\text{Def.}}{=}} V\,\Lambda^{-1/2}\,\underbrace{V'\,V}_{=I_p}\,\Lambda^{1/2}\,V' = V\,\underbrace{\Lambda^{-1/2}\,\Lambda^{1/2}}_{=I_p}\,V' = V\,V' = I_p$$

(iii)
$$A^{1/2}\,A^{-1}\,A^{1/2} \overset{\text{(i)}}{=} A^{1/2}\,A^{-1/2}\,A^{-1/2}\,A^{1/2} \overset{\text{(ii)}}{=} I_p\,I_p = I_p$$

and analogously

$$A^{-1/2}\,A\,A^{-1/2} \overset{\text{(a)}}{=} A^{-1/2}\,A^{1/2}\,A^{1/2}\,A^{-1/2} \overset{\text{(ii)}}{=} I_p\,I_p = I_p$$

Exercise 3

(a) Let A be non-negative definite. Then, especially, A is symmetric and by Theorem I.2.5, there exists a SVD of A given by

$$A \;=\; V\,\Lambda\,V' \tag{1}$$

Existence

Set $B := A^{1/2}$ with $A^{1/2}$ defined as in Exercise 2 (based on the SVD (1)).
Then, $B \in \mathbb{R}^{p\times p}$ is non-negative definite (and thus especially symmetric) by Ex. 2, (a).
Further, we obtain:

$$B\,B' \;\overset{\text{Symm.}}{=}\; B^2 \;\overset{\text{Def.}}{=}\; A^{1/2}\,A^{1/2} \;\overset{\text{Ex. 2, (a)}}{=}\; A \tag{2}$$

Uniqueness

To prove uniqueness of B, let $B_1, B_2 \in \mathbb{R}^{p\times p}$ both be non-negative definite with

$$A \;=\; B_1^2 \;=\; B_2^2 \tag{3}$$

Assumption: $B_1 \neq B_2$

Then, $B_1 - B_2 \neq 0_{p\times p}$ as a symmetric matrix has at least one eigenvalue $\lambda \in \mathbb{R}\backslash\{0\}$, i.e. for some $\mathbf{x} \in \mathbb{R}^p\backslash\{\mathbf{0}\}$, it holds:

$$(B_1 - B_2)\,\mathbf{x} \;=\; \lambda\,\mathbf{x} \tag{4}$$

Otherwise, in contradiction to $B_1 - B_2 \neq 0_{p\times p}$, we would obtain:

$$\operatorname{rank}(B_1 - B_2) \;\overset{\text{Ex. 1,(a)}}{=}\; \Big|\,\big\{i \in \{1,\ldots,p\}\,\big|\; \underbrace{\lambda_i(B_1 - B_2)}_{i\text{-th eigenvalue of } B_1 - B_2} \;\neq 0\big\}\,\Big| \;=\; 0$$

Then, applying the given hint yields:

$$0 \;\overset{(3)}{=}\; \mathbf{x}'\,\underbrace{(B_1^2 - B_2^2)}_{=\,0_{p\times p}}\,\mathbf{x} \;\overset{\text{Hint}}{=}\; \frac{1}{2}\Big(\mathbf{x}'\,(B_1 + B_2)\,(B_1 - B_2)\,\mathbf{x} + \mathbf{x}'\,(B_1 - B_2)\,(B_1 + B_2)\,\mathbf{x}\Big) \tag{5}$$

$$\overset{(4)}{=}\; \frac{1}{2}\big(\mathbf{x}'\,(B_1 + B_2)\,\lambda\,\mathbf{x} + \lambda\,\mathbf{x}'\,(B_1 + B_2)\,\mathbf{x}\big) \;=\; \lambda\,\big(\underbrace{\mathbf{x}'\,B_1\,\mathbf{x}}_{\geq 0} + \underbrace{\mathbf{x}'\,B_2\,\mathbf{x}}_{\geq 0}\big)$$

Here, in the third step, we used:

$$\mathbf{x}'\,(B_1 - B_2) \;\overset{\text{Symm.}}{=}\; \mathbf{x}'\,(B_1 - B_2)' \;=\; \big((B_1 - B_2)\,\mathbf{x}\big)' \;\overset{(4)}{=}\; \lambda\,\mathbf{x}'$$

Since $\lambda \neq 0$, (5) implies $\mathbf{x}'\,B_1\,\mathbf{x} \;=\; \mathbf{x}'\,B_2\,\mathbf{x} \;=\; 0$. Thus, we obtain:

$$0 \;=\; \mathbf{x}'\,(B_1 - B_2)\,\mathbf{x} \;\overset{(4)}{=}\; \mathbf{x}'\,\lambda\,\mathbf{x} \;=\; \lambda\,||\mathbf{x}||^2 \;\neq\; 0 \quad\text{since}\quad \lambda \in \mathbb{R}\backslash\{0\} \;\text{ and }\; \mathbf{x} \in \mathbb{R}^p\backslash\{\mathbf{0}\}$$

This is a contradiction and thus, we obtain $B_1 \;=\; B_2$.

(b) Let $A := B\,B'$ for some matrix $B \in \mathbb{R}^{p\times q}$. Then, it holds:

$$A' \;\overset{\text{Def.}}{=}\; (B\,B')' \;=\; (B')'\,B' \;=\; B\,B' \;\overset{\text{Def.}}{=}\; A \tag{6}$$

Furthermore, for any $\mathbf{x} \in \mathbb{R}^p$ we get:

$$\mathbf{x}'\,A\,\mathbf{x} \;\overset{\text{Def.}}{=}\; \mathbf{x}'\,B\,B'\,\mathbf{x} \;=\; (B'\,\mathbf{x})'\,B'\,\mathbf{x} \;=\; ||B'\,\mathbf{x}||^2 \;\geq\; 0 \tag{7}$$

By (6), A is symmetric and (then) by (7) non-negative definite.

Remark

Without demanding the matrix B to be non-negative definite in part (a), uniqueness gets lost. Then (for example) also $B := -A^{1/2}$ fulfills the equation $B\,B' = B^2 = A$.

Exercise 4

(a) Following the given hints, first we get:

$$(1)\quad \det\begin{pmatrix} I_p & 0_{p\times q} \\ -C\,A^{-1} & I_q \end{pmatrix} \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \overset{\text{I.1.4,(8)}}{=} \det\left(\begin{pmatrix} I_p & 0_{p\times q} \\ -C\,A^{-1} & I_q \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)$$

$$\overset{\text{Hint (i)}}{=} \det\begin{pmatrix} A & B \\ 0_{q\times p} & -C\,A^{-1}\,B + D \end{pmatrix} \overset{\text{Hint (ii)}}{=} \det(A)\,\det(D - C\,A^{-1}\,B)$$

Further, it holds:

$$(2)\quad \det\begin{pmatrix} I_p & 0_{p\times q} \\ -C\,A^{-1} & I_q \end{pmatrix} = \det\begin{pmatrix} I_p & (-C\,A^{-1})' \\ 0_{q\times p} & I_q \end{pmatrix} \overset{\text{Hint (ii)}}{=} \det(I_p)\,\det(I_q) = 1$$

(1) and (2) yield the statement of part (a).

(b) Applying Hint (i), we get:

$$(3)\qquad \begin{pmatrix} A & B \\ B' & D \end{pmatrix}\begin{pmatrix} A^{-1} + F\,E^{-1}\,F' & -F\,E^{-1} \\ -E^{-1}\,F' & E^{-1} \end{pmatrix}$$

$$\overset{\text{Hint (i)}}{=} \begin{pmatrix} I_p + A\,F\,E^{-1}\,F' - B\,E^{-1}\,F' & -A\,F\,E^{-1} + B\,E^{-1} \\ B'\,A^{-1} + B'\,F\,E^{-1}\,F' - D\,E^{-1}\,F' & -B'\,F\,E^{-1} + D\,E^{-1} \end{pmatrix} =: \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

Using $F \overset{\text{Def.}}{=} A^{-1}\,B$, the calculation of the four matrices $M_1, \ldots, M_4$ yields:

$$(4)\qquad M_1 = I_p + \underbrace{A\,A^{-1}}_{=I_p}\,B\,E^{-1}\,F' - B\,E^{-1}\,F' = I_p$$

$$(5)\qquad M_2 = -\underbrace{A\,A^{-1}}_{=I_p}\,B\,E^{-1} + B\,E^{-1} = 0_{p\times q}$$

$$(6)\qquad M_3 = B'\,A^{-1} + B'\,A^{-1}\,B\,E^{-1}\,F' - D\,E^{-1}\,F'$$

$$= B'\,A^{-1} - \underbrace{\big(D - B'\,A^{-1}\,B\big)}_{=E}\,E^{-1}\,F$$

$$\overset{\text{Def. } E,F}{=} B'\,A^{-1} - \underbrace{E\,E^{-1}}_{=I_q}(A^{-1}\,B)' = B'\,A^{-1} - B'\,\underbrace{(A^{-1})'}_{=(A')^{-1}} \overset{\text{I.2.4,(1)}}{=} 0_{q\times p}$$

$$(7)\qquad M_4 = (D - B'\,F)\,E^{-1} \overset{\text{Def. } F}{=} \underbrace{\big(D - B'\,A^{-1}\,B\big)}_{=E}\,E^{-1} \overset{\text{Def. } E}{=} E\,E^{-1} = I_q$$

By (3) – (7) using the definiton of Σ, finally we obtain:

$$(8)\qquad \Sigma\begin{pmatrix} A^{-1} + F\,E^{-1}\,F' & -F\,E^{-1} \\ -E^{-1}\,F' & E^{-1} \end{pmatrix} = \begin{pmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & I_q \end{pmatrix} = I_{p+q}$$

By (8), Σ is regular and the inverse Σ^{-1} of Σ has the given representation.

Remark: A proof of Hint (ii) is (for example) given in M. Koecher, *Lineare Algebra und analytische Geometrie*, 3. Aufl., Springer, 1992, Chapter 3, Section 1.4, p. 104.

Exercise 5

(a) Let $A = A^2$. Then first, we show the statement of the hint:

(1) $$\mathbb{R}^n \;=\; \mathrm{Im}(A) \oplus \mathrm{Ker}(A)$$

To that end, let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x_1} := A\,\mathbf{x}$, $\mathbf{x_0} := \mathbf{x} - A\,\mathbf{x}$. Then, we obtain:

(2) $$\mathbf{x} \;=\; A\,\mathbf{x} + \mathbf{x} - A\,\mathbf{x} \overset{\text{Def.}}{=} \mathbf{x_1} + \mathbf{x_0}$$

with $\mathbf{x_1} \overset{\text{Def.}}{=} A\,\mathbf{x} \in \mathrm{Im}(A)$ and

$$A\,\mathbf{x_0} \overset{\text{Def.}}{=} A\,\mathbf{x} - A^2\,\mathbf{x} \overset{A^2=A}{=} A\,\mathbf{x} - A\,\mathbf{x} \;=\; \mathbf{0}$$

and thus, $\mathbf{x_0} \in \mathrm{Ker}(A)$.

To prove uniqueness of $\mathbf{x_1}$ and $\mathbf{x_0}$, further let $\tilde{\mathbf{x}}_\mathbf{1} \in \mathrm{Im}(A)$ and $\tilde{\mathbf{x}}_\mathbf{0} \in \mathrm{Ker}(A)$ with

(3) $$\tilde{\mathbf{x}}_\mathbf{1} + \tilde{\mathbf{x}}_\mathbf{0} \;=\; \mathbf{x} \;=\; \mathbf{x_1} + \mathbf{x_0}$$

Then, there exists $\tilde{\mathbf{y}} \in \mathbb{R}^n$ with $\tilde{\mathbf{x}}_\mathbf{1} = A\,\tilde{\mathbf{y}}$. Furthermore, $\mathbf{x_0} - \tilde{\mathbf{x}}_\mathbf{0} \in \mathrm{Ker}(A)$ implying

(4) $$\mathbf{0} \;=\; A\left(\mathbf{x_0} - \tilde{\mathbf{x}}_\mathbf{0}\right) \overset{(3)}{=} A\left(\tilde{\mathbf{x}}_\mathbf{1} - \mathbf{x_1}\right) \;=\; A^2\,\tilde{\mathbf{y}} - A^2\,\mathbf{x} \overset{A^2=A}{=} A\,\tilde{\mathbf{y}} - A\,\mathbf{x} \;=\; \tilde{\mathbf{x}}_\mathbf{1} - \mathbf{x_1}$$

(where we used $\tilde{\mathbf{x}}_\mathbf{1} = A\,\tilde{\mathbf{y}}$ and $\mathbf{x_1} = A\,\mathbf{x}$).
We obtain:

$$\mathbf{x_0} - \tilde{\mathbf{x}}_\mathbf{0} \overset{(3)}{=} \tilde{\mathbf{x}}_\mathbf{1} - \mathbf{x_1} \overset{(4)}{=} \mathbf{0}$$

and consequently

$$\tilde{\mathbf{x}}_\mathbf{1} \;=\; \mathbf{x_1} \quad \text{and} \quad \tilde{\mathbf{x}}_\mathbf{0} \;=\; \mathbf{x_0}$$

which together with (2) proves the hint.

Now, let $r := \mathrm{rank}(A) \overset{\text{Def.}}{=} \dim\big(\mathrm{Im}(A)\big)$.

Case 1: $r \in \{1, \ldots, n-1\}$.

Then, there exists an orthonormal basis $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ of $\mathrm{Im}(A)$, and for $i \in \{1, \ldots, r\}$ there exists $\mathbf{u_i} \in \mathbb{R}^n$ with $\mathbf{v_i} = A\,\mathbf{u_i}$. We obtain:

(5) $$A\,\mathbf{v_i} \;=\; A^2\,\mathbf{u_i} \overset{A=A^2}{=} A\,\mathbf{u_i} \;=\; \mathbf{v_i}\;,\quad i \in \{1, \ldots, r\}$$

By (5), $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are eigenvectors of A corresp. to the eigenvalue $\lambda_1 = \ldots = \lambda_r = 1$.

By the dimension formula of Linear algebra, it holds $\dim\big(\mathrm{Ker}(A)\big) = n - r$.

Then, there exists an orthonormal basis $\{\mathbf{v_{r+1}}, \ldots, \mathbf{v_n}\}$ of $\mathrm{Ker}(A)$. We obtain:

(6) $$A\,\mathbf{v_i} \;=\; \mathbf{0} \;=\; 0 \cdot \mathbf{v_i}\;,\quad i \in \{r+1, \ldots, n\}$$

By (6), $\mathbf{v_{r+1}}, \ldots, \mathbf{v_n}$ are eigenvectors of A corresp. to the eigenvalue $\lambda_{r+1} = \ldots = \lambda_n = 0$.

Remark: The existence of the stated orthonormal bases of $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$, respectively, follows by the *Gram/Schmidt*-procedure (see e.g. numerical analysis or next exercise sheet).

Next, the complete system $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a basis of $\mathbb{R}^n$, which will be shown by verifying the two conditions of a vectorspace basis:

(i) First, by (1), each $\mathbf{x} \in \mathbb{R}^n$ can be represented as sum of some (suitable chosen) vectors $\mathbf{x_1} \in \mathrm{Im}(A)$ and $\mathbf{x_0} \in \mathrm{Ker}(A)$. Thus, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with

$$\mathbf{x} \overset{(1)}{=} \mathbf{x_1} + \mathbf{x_0} = \sum_{i=1}^{r} \alpha_i \, \mathbf{v_i} + \sum_{i=r+1}^{n} \alpha_i \, \mathbf{v_i} = \sum_{i=1}^{n} \alpha_i \, \mathbf{v_i}$$

(ii) Second, let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with

$$\sum_{i=1}^{n} \alpha_i \, \mathbf{v_i} = \mathbf{0}$$

Then, we get:

$$\mathbf{0} = A \left(\sum_{i=1}^{n} \alpha_i \, \mathbf{v_i} \right) = \sum_{i=1}^{n} \alpha_i \, A \, \mathbf{v_i} \overset{(5),(6)}{=} \sum_{i=1}^{r} \alpha_i \, \mathbf{v_i}$$

This equation implies $\alpha_1 = \ldots = \alpha_r = 0$ since $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are linear independent (as basis of $\mathrm{Im}(A)$). Togehter with our assumption of (ii), this yields:

$$\sum_{i=r+1}^{n} \alpha_i \, \mathbf{v_i} = \mathbf{0}$$

and then, $\alpha_{r+1} = \ldots = \alpha_n = 0$ since $\mathbf{v_{r+1}}, \ldots, \mathbf{v_n}$ are linear independent (as basis of $\mathrm{Ker}(A)$).

In summation, $\alpha_1 = \ldots = \alpha_n = 0$, and thus $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are linear independent.

In this case, we set $V := [\, \mathbf{v_1}, \ldots, \mathbf{v_n} \,]$.

Case 2: $r \in \{0, n\}$.

Then, either $\mathrm{Ker}(A) = \mathbb{R}^n$ (for $r = 0$) or $\mathrm{Im}(A) = \mathbb{R}^n$ (for $r = n$).
Analogously to the above argumentation for Case 1, we obtain $\lambda_1 = \ldots = \lambda_n = 0$ (for $r = 0$) or $\lambda_1 = \ldots = \lambda_n = 1$ (for $r = n$) as the sole eigenvalue of A.
Notice that this implies $A = 0_{n \times n}$ for $r = 0$ and $A = I_n$ for $r = n$.

In this case, we choose $V := [\, \mathbf{v_1}, \ldots, \mathbf{v_n} \,]$ for some arbitrary basis $\mathbf{v_1}, \ldots, \mathbf{v_n}$ of $\mathbb{R}^n$.

In both cases, $V = [\, \mathbf{v_1}, \ldots, \mathbf{v_n} \,] \in \mathbb{R}^{n \times n}$ is regular, and $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are eigenvectors of A corresponding to the eigenvalues $\lambda_1 = \ldots = \lambda_n = 0$ or $\lambda_1 = \ldots = \lambda_n = 1$.
Let

(7) $$\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \overset{\text{Construction}}{=} \mathrm{diag}(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-r})$$

The eigenvalue-equations $A \, V = V \, \Lambda$ yield the following SVD for A:

(8) $$A = V \, \Lambda \, V^{-1}$$

(Notice that V^{-1} need not be equal to V' in this situation.)

Then, we finally obtain:

$$\mathrm{trace}(A) \overset{(8)}{=} \mathrm{trace}(V \, \Lambda \, V^{-1}) \overset{\text{I.2.4,(10)}}{=} \mathrm{trace}(\underbrace{V^{-1} \, V}_{= I_n} \, \Lambda) = \mathrm{trace}(\Lambda) \overset{(7)}{=} r \overset{\text{Def}}{=} \mathrm{rank}(A)$$

(b) Let $A = A'$ with all eigenvalues in $\{0, 1\}$.
Since A is symmetric, according to Theorem I.2.5, there exists a SVD

$$(9) \qquad A = V \Lambda V'$$

with $\Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of A and $V \in \mathbb{R}^{n \times n}$ is orthogonal. By Exercise 1, (a), it follows:

$$(10) \quad r := \operatorname{rank}(A) \overset{\text{Ex. 1,(a)}}{=} \left|\{i \in \{1, \ldots, n\} \,|\, \lambda_i \neq 0\}\right| \overset{\text{Ass.}}{=} \left|\{i \in \{1, \ldots, n\} \,|\, \lambda_i = 1\}\right|$$

With $\lambda_1 \geq \ldots \geq \lambda_n$, we get by (10) and the assumption:

$$(11) \qquad \Lambda = \operatorname{diag}(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-r})$$

and thus the idempotence of A, as follows:

$$A^2 \overset{(9)}{=} V \Lambda \underbrace{V'V}_{= I_n} \Lambda V' = V \Lambda \Lambda V' \overset{(11)}{=} V \Lambda V' \overset{(9)}{=} A$$

In case that A is not assumed to be symmetric, consider the following counterexample:

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \implies \det\left(A - \lambda I_2\right) = \det\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2$$

That is, $A \neq A'$ and $\lambda = 0$ is the sole eigenvalue of A. Then, it holds

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq A$$

and thus, A is *not* idempotent.

(c) Let $A' = A = A^2$ and $\mathbf{x} \in \mathbb{R}^n$.
Then, we obtain for A:

$$(12) \qquad \mathbf{x}' A \mathbf{x} \overset{A^2 = A}{=} \mathbf{x}' A A \mathbf{x} \overset{A = A'}{=} \mathbf{x}' A A' \mathbf{x} = (A' \mathbf{x})' A' \mathbf{x} = ||A' \mathbf{x}||^2 \geq 0$$

Since $\mathbf{x} \in \mathbb{R}^n$ was chosen arbitrarily, (12) implies the non-negative definiteness of A.
Furthermore, we obtain for $I_n - A$:

$$(13) \qquad (I_n - A)' = I_n' - A' \overset{A = A'}{=} I_n - A$$

$$(14) \qquad (I_n - A)^2 = I_n^2 - I_n A - A I_n + A^2 \overset{A = A^2}{=} I_n - A - A + A = I_n - A$$

By (13) and (14), $I_n - A$ is symmetric and idempotent. Then, analogously to (12), we derive the non-negative definiteness of $I_n - A$.

Exercise 6

(a) First, we get:

$$(1) \qquad E_n' \overset{\text{Def.}}{=} \left(I_n - \frac{1}{n}\mathbb{1}_{n\times n}\right)' = I_n' - \frac{1}{n}\mathbb{1}_{n\times n}' = I_n - \frac{1}{n}\mathbb{1}_{n\times n} \overset{\text{Def.}}{=} E_n$$

Next, it holds:

$$(2) \qquad \mathbb{1}_{n\times n}\,\mathbb{1}_{n\times n} \overset{\text{Def.}}{=} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} n & \dots & n \\ \vdots & & \vdots \\ n & \dots & n \end{pmatrix} = n\,\mathbb{1}_{n\times n}$$

Apllying (2), we obtain:

$$(3) \qquad E_n^2 \overset{\text{Def.}}{=} \left(I_n - \frac{1}{n}\mathbb{1}_{n\times n}\right)\left(I_n - \frac{1}{n}\mathbb{1}_{n\times n}\right) = I_n - \frac{2}{n}\mathbb{1}_{n\times n} + \frac{1}{n^2}\mathbb{1}_{n\times n}\,\mathbb{1}_{n\times n}$$
$$\overset{(2)}{=} I_n - \frac{2}{n}\mathbb{1}_{n\times n} + \frac{1}{n^2}\,n\,\mathbb{1}_{n\times n} = I_n - \frac{1}{n}\mathbb{1}_{n\times n} \overset{\text{Def.}}{=} E_n$$

Thus, (1) yields the symmetry and (3) the idempotence of E_n .

(b) Using (3) and Exercise 5, (a), we obtain:

$$(4) \qquad \operatorname{rank}(E_n) = \operatorname{trace}(E_n) \overset{\text{Def.}}{=} \operatorname{trace}\left(I_n - \frac{1}{n}\mathbb{1}_{n\times n}\right)$$
$$= \sum_{i=1}^{n}\left(1 - \frac{1}{n}\right) = n\left(1 - \frac{1}{n}\right) = n - 1$$

(c) First, we get:

$$(5) \qquad E_n\,\mathbb{1}_{n\times 1} \overset{\text{Def.}}{=} \left(I_n - \frac{1}{n}\mathbb{1}_{n\times n}\right)\mathbb{1}_{n\times 1} = \mathbb{1}_{n\times 1} - \frac{1}{n}\mathbb{1}_{n\times n}\,\mathbb{1}_{n\times 1}$$
$$\overset{\text{cf. (2)}}{=} \mathbb{1}_{n\times 1} - \frac{1}{n}\,n\,\mathbb{1}_{n\times 1} = \mathbf{0}$$

By (5), we obtain:

$$(6) \qquad \left\{\lambda\,\mathbb{1}_{n\times 1} \,\middle|\, \lambda \in \mathbb{R}\right\} \subseteq \operatorname{Ker}(E_n)$$

On the other hand, by the dimension formula of Linear Algebra, it holds:

$$(7) \qquad \dim\bigl(\operatorname{Ker}(E_n)\bigr) = n - \operatorname{rank}(E_n) \overset{\text{(b)}}{=} n - (n-1) = 1$$

Thus, (6) and (7) together yield:

$$(8) \qquad \operatorname{Ker}(E_n) = \left\{\lambda\,\mathbb{1}_{n\times 1} \,\middle|\, \lambda \in \mathbb{R}\right\}$$

Remarks to Exercise 6

(i) Part (c) especially implies that E_n has the eigenvalue $\mu = 0$ with multiciplity 1 and the eigenvalue $\lambda = 1$ with multiciplity $n-1$ (since there are only eigenvalues in $\{0,1\}$ according to part (a) and Exercise 5, (a)).

(ii) Part (a) togehter with Lemma I.2.9 implies that E_n is a othogonal projector on $\mathrm{Im}(\mathrm{E}_n)$. According to part (b) (or according to part (c) and the dimension formula of Linear Algebra), $\mathrm{Im}(\mathrm{E}_n)$ is a hyper-plane in $\mathbb{R}^n$.