Prof. Dr. E. Cramer, Prof. Dr. M. Kateri
T. v. Bentum, M.Sc., L. Kaufmann, M.Sc.

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Applied Data Analysis

Exercise Sheet 1

Exercise 1

Let $A \in \mathbb{R}^{p\times p}$ be a symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_p$. Show the following statements.

(a) The identity $rg(A) = \big|\{i \in \{1,\dots,p\} | \lambda_i \neq 0\}\big|$ holds. rg(A) = rang of A

(b) A is positive semidefinite iff $\lambda_i \geq 0$ for all $i \in \{1,\dots,p\}$.

(c) A is positive definite iff $\lambda_i > 0$ for all $i \in \{1,\dots,p\}$.

Exercise 2

Let $A \in \mathbb{R}^{p\times p}$ be a positive semidefinite matrix with singular value decomposition $A = V\Lambda V'$, where $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_p) \in \mathbb{R}^{p\times p}$, $\lambda_1 \geq \dots \geq \lambda_p \geq 0$ and $V \in \mathbb{R}^{p\times p}$ is orthogonal. Show the following statements.

(a) The matrix $A^{1/2}$ defined by

$$A^{1/2} := V\Lambda^{1/2}V' \quad \text{where} \quad \Lambda^{1/2} := \mathrm{diag}(\lambda_1^{1/2}, \dots \lambda_p^{1/2})$$

is positive semidefinite with $A^{1/2}A^{1/2} = A$.

(b) If additionally A is positive definite, then $A^{1/2}$ is positive definite and the following holds:

(i) $A^{-1/2}A^{-1/2} = A^{-1}$,

(ii) $A^{1/2}A^{-1/2} = I_p = A^{-1/2}A^{1/2}$,

(iii) $A^{1/2}A^{-1}A^{1/2} = I_p = A^{-1/2}AA^{-1/2}$.

Exercise 3

Let $A \in \mathbb{R}^{p\times p}$ be a square matrix. Show the following:

(a) If A is positive semi definite, then there exists exactly one matrix $B \in \mathbb{R}^{p\times p}$ with $A = BB' = B^2$.

(b) If $A = BB'$ for some matrix $B \in \mathbb{R}^{p\times q}$, then A is positive semidefinite.

Hint to (a): To proof uniqueness of B use the following identity for arbitrary matrices $M_1, M_2 \in \mathbb{R}^{p\times p}$:

$$M_1^2 - M_2^2 = \frac{1}{2}\left((M_1 + M_2)(M_1 - M_2) + (M_1 - M_2)(M_1 + M_2)\right).$$

Exercise 4

Let $A \in \mathbb{R}^{p\times p}, B \in \mathbb{R}^{p\times q}, C \in \mathbb{R}^{q\times p}$ and $D \in \mathbb{R}^{q\times q}$.

(a) If A is regular the following identity for the determinant holds:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| \left|D - CA^{-1}B\right| .$$

(b) Define the matrix $\Sigma \in \mathbb{R}^{(p+q)\times(p+q)}$ by

$$\Sigma := \begin{pmatrix} A & B \\ B' & D \end{pmatrix} .$$

Show that if A is symmetric and regular and additionally $E := D - B'A^{-1}B$ is regular, then Σ is regular and the inverse is given by

$$\Sigma^{-1} = \begin{pmatrix} A^{-1} + FE^{-1}F' & -FE^{-1} \\ -E^{-1}F' & E^{-1} \end{pmatrix}$$

where F is defined by $F := A^{-1}B$.

Hint: Use the following identities for partitioned matrices (without proof):

(i) Let $M_1, N_1 \in \mathbb{R}^{p\times p}$, $M_2, N_2 \in \mathbb{R}^{p\times q}$, $M_3, N_3 \in \mathbb{R}^{q\times p}$ and $M_4, N_4 \in \mathbb{R}^{q\times q}$. Then,

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} = \begin{pmatrix} M_1N_1 + M_2N_3 & M_1N_2 + M_2N_4 \\ M_3N_1 + M_4N_3 & M_3N_2 + M_4N_4 \end{pmatrix} .$$

(a) Let $M_1 \in \mathbb{R}^{p\times p}$, $M_2 \in \mathbb{R}^{p\times q}$ and $M_3 \in \mathbb{R}^{q\times q}$. Then,

$$\begin{vmatrix} M_1 & M_2 \\ 0_{q\times p} & M_3 \end{vmatrix} = |M_1| \, |M_3| .$$

Furthermore, in (a) calculate

$$\left| \begin{pmatrix} I_p & 0_{p\times q} \\ -CA^{-1} & I_q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| .$$

Exercise 5

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. Show the following statements.

(a) If A is idempotent, then all its eigenvalues are in $\{0, 1\}$ and $rg(A) = tr(A)$.

(b) If A is symmetric and all its eigenvalues are in $\{0, 1\}$, then A is idempotent.

Proof by counterexample that the condition of symmetry is necessary.

(c) If A is symmetric and idempotent, then both A and $I_n - A$ are positive semidefinite.

Hint to (a): Show that under the assumptions of (a) that for each $x \in \mathbb{R}^n$ there exist unique vectors $x_1 \in Im(A)$ and $x_0 \in Ker(A)$ with $x = x_1 + x_0$, i.e.

$$\mathbb{R}^n = Im(A) \oplus Ker(A).$$

Use this result to derive a singular value decomposition of A analogous to the diagonalization of symmetric matrices.

Exercise 6

Show the following properties of the ortho-projection matrix $E_n := I_n - \frac{1}{n}\mathbb{1}_{n\times n}$.

(a) E_n is symmetric and idempotent.

(b) $rg(E_n) = n - 1$.

(c) The kernel is given by $Ker(E_n) = \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \text{ with } x = \lambda \mathbb{1}_{n\times 1}\}$.