Item 2 **10 points**

For given measurements

$$x_1 = 1\,,\; x_2 = 0\,,\; x_3 = -1\,,\; x_4 = 2\,,$$

consider the following polynomial regression model according to I.5.8:

$$Y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^3 + \varepsilon_i\,,\; i \in \{1,2,3,4\}\,,$$

with (unknown) parameters $\beta_0, \beta_1, \beta_2 \in \mathbb{R}$ and stochastically independent, indentically distributed error terms $\varepsilon_i \sim N(0,\sigma^2)\,,\, i \in \{1,2,3,4\}$, where $\sigma > 0$ is also unknown. Then, it holds:

$$(B'B)^{-1} \approx \begin{pmatrix} 0.33333333 & 0.05555556 & -0.05555556 \\ 0.05555556 & 0.92592593 & -0.25925926 \\ -0.05555556 & -0.25925926 & 0.09259259 \end{pmatrix}$$

with B denoting the corresponding design matrix of the given polynomial regression model.

Further, consider the observation $\boldsymbol{y} = (0,3,0,-3)'$ of $\boldsymbol{Y} = (Y_1,Y_2,Y_3,Y_4)'$ with resulting least-squares estimate

$$\widehat{\boldsymbol{\beta}} = \begin{pmatrix} 1 \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

of the parameter vector $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)'$.

For the following tasks, give your answers by filling the results into the blanks. If necessary, numerical values have to be rounded to three decimal places.

(a) For the given polynomial regression model, consider the following testing problem

$$H_0 : (\beta_1,\beta_2)' = (0,0)' \quad \longleftrightarrow \quad H_1 : (\beta_1,\beta_2)' \neq (0,0)'\,.$$

For this testing problem, the corresponding decision rule based on the F-distribution can be formulated as follows:

$$\text{Rejection of } H_0 : (\beta_1,\beta_2)' = (0,0)' \text{ at significance level } \alpha \in (0,1)\,,\text{ if } \frac{\mathrm{SSR}/\mathrm{df1}}{\mathrm{SSE}/\mathrm{df2}} > F_{1-\alpha}(\mathrm{df1},\mathrm{df2})$$

with the quantities SSR, SSE and the degrees of freedom $\mathrm{df1}$ and $\mathrm{df2}$ being appropriately defined and with $F_{1-\alpha}(\mathrm{df1},\mathrm{df2})$ denoting the $(1-\alpha)-$quantile of the $F(\mathrm{df1},\mathrm{df2})$-distribution.

(i) Determine the degrees of freedom $\mathrm{df1}$ and $\mathrm{df2}$. **1 point**

Give the value of $\mathrm{df1}$. 0.5 points

2

Give the value of $\mathrm{df2}$. 0.5 points

2

(ii) Calculate the values of SSR and SSE for the given observation $\boldsymbol{y}$. **Hint: In order to derive these values, it is *not* neccessary to calculate any orthogonal projector.** **5 points**

Give the value of SSR. 2.5 points

2

Give the value of SSE. 2.5 points

2

(b) Now, for the given polynomial regression model, consider the following testing problem

$$H_0 : \beta_1 = 0 \quad \longleftrightarrow \quad H_1 : \beta_1 \neq 0\,.$$

(i) For the reduced regression model associated with H_0, consider the corresponding design matrix

$$B_0 = \begin{pmatrix} 1 & b_{12} \\ 1 & b_{22} \\ 1 & b_{32} \\ 1 & b_{42} \end{pmatrix} \in \mathbb{R}^{4\times 2}\,.$$

Determine the last entry of the second column of B_0. **1 point**

Give the value of b_{42}. 1 point

2

(ii) For the testing problem considered in part (b), the corresponding decision rule based on the t-distribution can be formulated as follows:

$$\text{Rejection of } H_0 : \beta_1 = 0 \text{ at significance level } \alpha \in (0,1)\,, \text{ if } \frac{|\widehat{\beta}_1|}{\sqrt{6\,c}} > q(\alpha)\,.$$

Here, $\widehat{\beta}_1$ denotes the least-squares estimate of the parameter β_1, $c > 0$ is some appropriate quantity and $q(\alpha)$ denotes the quantile of the corresponding t-distribution.

Determine the quantity c and the quantile $q(\alpha)$ for $\alpha = 0{,}1$. 3 points

1 point

Give the value of $q(\alpha)$, **rounded to threee decimal places**.

2

2 points

Give the value of c, **rounded to threee decimal places**.

2