Applied Data Analysis

Exercise Sheet 1

Exercise 1

Let $A \in \mathbb{R}^{p \times p}$ be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_p$. Show the following statements:

(a) The identity $\operatorname{rank}(A) = \big|\{i \in \{1, \ldots, p\} | \lambda_i \neq 0\}\big|$ holds.

(b) A is non-negative definite iff $\lambda_i \geq 0$ for all $i \in \{1, \ldots, p\}$.

(c) A is positive definite iff $\lambda_i > 0$ for all $i \in \{1, \ldots, p\}$.

If a matrix is symmetric, its rank is the number of eigenvalues
that do not equal to zero
The non-negativity(positiv-definity) depends on whether the eigenvalues
are greather(or equal) to zero

Exercise 2

Let $A \in \mathbb{R}^{p \times p}$ be a non-negative definite matrix with singular value decomposition $A = V\Lambda V'$, where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^{p \times p}$, $\lambda_1 \geq \ldots \geq \lambda_p \geq 0$ and $V \in \mathbb{R}^{p \times p}$ is orthogonal. Show the following statements:

(a) The matrix $A^{1/2}$ defined by

$$A^{1/2} := V\Lambda^{1/2}V' \quad \text{with} \quad \Lambda^{1/2} := \operatorname{diag}(\lambda_1^{1/2}, \ldots, \lambda_p^{1/2})$$

is non-negative definite with $A^{1/2}A^{1/2} = A$.

(b) If additionally A is positive definite, then $A^{1/2}$ is positive definite too and the following equations hold:

(i) $A^{-1/2}A^{-1/2} = A^{-1}$,

(ii) $A^{1/2}A^{-1/2} = I_p = A^{-1/2}A^{1/2}$,

(iii) $A^{1/2}A^{-1}A^{1/2} = I_p = A^{-1/2}AA^{-1/2}$.

Exercise 3

Let $A \in \mathbb{R}^{p \times p}$. Show the following statements:

(a) If A is non-negative definite, then there exists exactly one non-negative definite matrix $B \in \mathbb{R}^{p \times p}$ with $A = BB' = B^2$.

(b) If $A = BB'$ for some (arbitrary) matrix $B \in \mathbb{R}^{p \times q}$, then A is non-negative definite.

Hint to (a): To proof uniqueness of B, use the following identity for arbitrary matrices $M_1, M_2 \in \mathbb{R}^{p \times p}$:

$$M_1^2 - M_2^2 = \frac{1}{2}\big((M_1 + M_2)(M_1 - M_2) + (M_1 - M_2)(M_1 + M_2)\big).$$

Exercise 4

Let $A \in \mathbb{R}^{p\times p}, B \in \mathbb{R}^{p\times q}, C \in \mathbb{R}^{q\times p}$ and $D \in \mathbb{R}^{q\times q}$.

(a) If A is regular the following identity for the considered determinants holds:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\bigl(A\bigr)\,\det\bigl(D - CA^{-1}B\bigr)\ .$$

(b) Define the matrix $\Sigma \in \mathbb{R}^{(p+q)\times(p+q)}$ by

$$\Sigma := \begin{pmatrix} A & B \\ B' & D \end{pmatrix}\ .$$

Show that if A is symmetric and regular and additionally $E := D - B'A^{-1}B$ is regular, then Σ is regular and the inverse is given by

$$\Sigma^{-1} = \begin{pmatrix} A^{-1} + FE^{-1}F' & -FE^{-1} \\ -E^{-1}F' & E^{-1} \end{pmatrix}\ ,$$

where F is defined by $F := A^{-1}B$.

Hints: Use the following identities for partitioned matrices (without proof):

(i) Let $M_1, N_1 \in \mathbb{R}^{p\times p}$, $M_2, N_2 \in \mathbb{R}^{p\times q}$, $M_3, N_3 \in \mathbb{R}^{q\times p}$ and $M_4, N_4 \in \mathbb{R}^{q\times q}$. Then, it holds:

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}\begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} = \begin{pmatrix} M_1N_1 + M_2N_3 & M_1N_2 + M_2N_4 \\ M_3N_1 + M_4N_3 & M_3N_2 + M_4N_4 \end{pmatrix}\ .$$

(ii) Let $M_1 \in \mathbb{R}^{p\times p}$, $M_2 \in \mathbb{R}^{p\times q}$ and $M_3 \in \mathbb{R}^{q\times q}$. Then, it holds:

$$\det\begin{pmatrix} M_1 & M_2 \\ 0_{q\times p} & M_3 \end{pmatrix} = \det\bigl(M_1\bigr)\det\bigl(M_3\bigr)\ .$$

Furthermore, in (a) calculate

$$\det\left(\begin{pmatrix} I_p & 0_{p\times q} \\ -CA^{-1} & I_q \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)\ .$$

Exercise 5

Let $A \in \mathbb{R}^{n \times n}$. Show the following statements:

Eigenvalue 0 -> kernel

(a) If A is idempotent, then all its eigenvalues are in $\{0, 1\}$ and $\operatorname{rank}(A) = \operatorname{trace}(A)$.

(b) If A is symmetric and all its eigenvalues are in $\{0, 1\}$, then A is idempotent. Proof by choice of a counterexample that the condition of symmetry is necessary.

(c) If A is symmetric and idempotent, then both A and $I_n - A$ are non-negative definite.

Hint to (a): Show under the assumptions of (a) that for each $\mathbf{x} \in \mathbb{R}^n$ there exist unique vectors $\mathbf{x_1} \in \operatorname{Im}(A)$ and $\mathbf{x_0} \in \operatorname{Ker}(A)$ with $\mathbf{x} = \mathbf{x_1} + \mathbf{x_0}$, i.e.

$$\mathbb{R}^n = \operatorname{Im}(A) \oplus \operatorname{Ker}(A) .$$

Use this result to derive a singular value decomposition of A analogous to the diagonalization of symmetric matrices.

Exercise 6

For $n \in \mathbb{N}$, show the following properties of the ortho-projection matrix $E_n := I_n - \frac{1}{n}\mathbb{1}_{n \times n}$:

(a) E_n is symmetric and idempotent.

(b) $\operatorname{Rank}(E_n) = n - 1$.

(c) The kernel of E_n is given by $\operatorname{Ker}(E_n) = \{\mathbf{x} \in \mathbb{R}^n \mid \exists\, \lambda \in \mathbb{R} \text{ with } \mathbf{x} = \lambda \mathbb{1}_{n \times 1}\}$.