Matrix formulation of baseline category logit model

Consider the i-th subject in the sample $(i = 1, \ldots, n)$.
The response vector is $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{i,J-1})'$, since
$Y_{iJ} = 1 - \sum_{j=1}^{J-1} Y_{ij}$ (i.e, Y_{iJ} is redundant).

In this case $\boldsymbol{\mu}_i = E(\boldsymbol{Y}_i) = (\pi_{i1}, \ldots, \pi_{i,J-1})'$
and the vector of canonical link functions is

$$\boldsymbol{g} = (g_1, \ldots, g_{J-1})'$$ with

$$g_j(\boldsymbol{\mu}_i) = \log\left(\frac{\mu_{ij}}{\mu_{iJ}}\right) = \log\left(\frac{\mu_{ij}}{1 - \sum_{l=1}^{J-1} \mu_{il}}\right).$$

Thus

$$\boldsymbol{g}(\boldsymbol{\mu}_i) = \log\begin{pmatrix} \mu_{i1}/\mu_{iJ} \\ \mu_{i2}/\mu_{iJ} \\ \vdots \\ \mu_{i,J-1}/\mu_{iJ} \end{pmatrix}_{(J-1)\times 1}$$

Vector of explanatory variables values for subject i:

$$x_i = (x_{i1}, \dots, x_{ip}) \leftarrow \text{row vector}$$

Since for the i-th subject we have $J-1$ equations in the model, the model matrix for the i-th subject is:

$$\mathbb{X}_i \begin{pmatrix} x_i & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & x_i & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & & \vdots \\ & & & & \mathbf{0} \\ \mathbf{0} & \cdots & & \cdots & x_i \end{pmatrix}_{(J-1)\times p\cdot(J-1)}$$

where $\mathbf{0} = (\underbrace{0, \dots, 0}_{p})$ is a p-dimensional row vector of 0's

The parameter vector is

$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{J-1} \end{pmatrix}_{p(J-1)\times 1}$$

the parameter vector of the 1st equation ($j=1$ in slide 127)

For $j = 1, \dots, J-1$:

$$\boxed{\beta_j = (\beta_{j1}, \dots, \beta_{jp})'}$$

Finally:

$$g(\boldsymbol{\mu}_i) = \begin{pmatrix} \log\left(\mu_{i1} / (1 - \sum_{l=1}^{J-1} \mu_{il})\right) \\ \vdots \\ \log\left(\mu_{i,J-1} / (1 - \sum_{l=1}^{J-1} \mu_{il})\right) \end{pmatrix} = \mathbb{X}_i \, \boldsymbol{\beta} = \begin{pmatrix} \mathbf{x}_i \boldsymbol{\beta}_1 \\ \vdots \\ \mathbf{x}_i \boldsymbol{\beta}_{J-1} \end{pmatrix} = \begin{pmatrix} \theta_{i1} \\ \vdots \\ \theta_{i,J-1} \end{pmatrix} = \boldsymbol{\theta}_i$$

(*) In univariate GLMs with canonical link we had:

$g(\mu_i) = \theta_i$ (s. Definition II.2.14)

Remark: θ_i and consequently μ_i are subject specific, since they depend on the explanatory variables values for subject i (vector $\mathbf{x}_i$). It holds

$$\mu_i = E(Y_i \mid \mathbf{x}_i) \quad \text{and} \quad \boldsymbol{\mu}_i = E(\mathbf{Y}_i \mid \mathbf{x}_i)$$

Remark 2: The random sample in univariate GLMs is Y_j, $j = 1, \dots, n$, which is written in vector form

$\boldsymbol{Y} = (Y_1, \ldots, Y_n)'$.

In <u>multivariate GLMs</u> the sample is

$\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{i,J-1})'$, $i = 1, \ldots, n$, and in vector form:

$$\boldsymbol{Y} = (\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_{J-1}')'$$

Analogously, we have

univariate GLM: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)'$

multivariate GLM: $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \ldots, \boldsymbol{\theta}_i', \ldots, \boldsymbol{\theta}_n')'$

with $\boldsymbol{\theta}_i = (\theta_{i1}, \ldots, \theta_{i,J-1})'$

<u>Remark 3</u>: For inferential purposes compare the log-likelihood function for n indep. responses $Y_i \sim EDF(\theta_i, \phi)$ (s. Slide 38) to the log-likelihood for n indep. responses $\boldsymbol{Y}_i \sim MEDF(\boldsymbol{\theta}_i, \phi)$ (s. Definition II.4.8 in Slide 129)

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \ell_i = \sum_{i=1}^{n} \log f(\boldsymbol{y}_i; \boldsymbol{\theta}_i, \phi) = \sum_{i=1}^{n} \frac{\boldsymbol{y}_i' \boldsymbol{\theta}_i - b(\boldsymbol{\theta}_i)}{a(\phi, i)} + \sum_{i=1}^{n} c(\boldsymbol{y}_i, \phi)$$