Mathematical methods of signal and image processing
Winter semester 2021/2022
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Exercise sheet 8 **Due: 17. December 2021**

General information

- Current information will be announced in RWTHmoodle.
- The due date only indicates in which exercise session the solution will be discussed.
- Office hours: By arrangement via Zoom.

Problem 1

Implement the perfect low-pass filter with radius r (cf. Example 3.32) and a ring-shaped high-pass filter with radii r_1 and r_2, i. e. the Fourier transform of the filter is $\frac{1}{(2\pi)^{\frac{d}{2}}}\chi_{B_{r_1,r_2}(0)}$, where $B_{r_1,r_2}(x) := \{y : r_1 < |y - x| < r_2\}$. Test the filters with different radii on the images from the first exercise sheet.

Notes on the implementation: For the Fourier transform, you can use MATLAB's `fft2` and `ifft2`. Implement the filters by directly computing their transfer functions (cf. Remark 3.31). Moreover, `fftshift` has to be applied to the result of `fft2`, while `ifftshift` has to be applied to argument of `ifft2`. The additional shifts are necessary to have the zero frequency $\omega = 0$ in the center of the image domain. Remark 3.57 will explain the interpretation of an index k as high or low frequency as the reason for this shift. Python has equivalent functions of the same name in `numpy.fft`.

Problem 2

Implement the deconvolution with the convolution theorem from Example 3.33. Test the deconvolution by first convolving the images from the first exercise sheet with a Gaussian kernel and then deconvolving the convolved images. Compare the quality of the deconvolution depending on the filter width of the Gaussian kernel, the regularization parameter ϵ and whether the image is quantized to 8-bit after the convolution, but before the deconvolution.

Problem 3

Let $z \in \mathbb{C}$. Show that

$$\int_b^c e^{zt}\,\mathrm{d}t = \frac{1}{z}e^{zt}\Big|_b^c.$$

without using the notion of holomorphic functions / complex differentiability.

Problem 4 (Gram-Schmidt process)

Let X be a pre-Hilbert space, i.e. a $\mathbb{K}$-vector space with a scalar product $(\cdot,\cdot)_X$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $x_1, \ldots, x_n \in X$ linearly independent.

Then, there is a (finite) orthonormal system $\{e_1, \ldots, e_n\} \subset X$, i.e. $(e_i, e_j)_X = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$, and coefficients $c_{ij} \in \mathbb{K}$ for $i = 1, \ldots n$ and $j = 1, \ldots, i$ such that

$$
\begin{aligned}
x_1 &= c_{11} e_1 \\
x_2 &= c_{21} e_1 + c_{22} e_2 \\
&\vdots \\
x_n &= c_{n1} e_1 + c_{n2} e_2 + \ldots + c_{nn} e_n.
\end{aligned}
$$