Mathematical methods of signal and image processing

Winter semester 2021/2022

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Exercise sheet 9 — Due: 14. January 2022

General information

- Current information will be announced in RWTHmoodle.
- The due date only indicates in which exercise session the solution will be discussed.
- Office hours: By arrangement via Zoom.

Problem 1

Let X be a pre-Hilbert space and $(e_n)_n \subset X$ an orthonormal system. Show that the Fourier coefficients converge to zero, i. e. $x_k = (x, e_k)_X \to 0$ for $k \to \infty$ and all $x \in X$. Use this to show that for all $f \in L^2\big((-\pi,\pi),\mathbb{C}\big)$ it holds that

$$\lim_{k\in\mathbb{Z},\ |k|\to\infty} \int_{-\pi}^{\pi} f(x)\mathrm{e}^{ikx}\,\mathrm{d}x = 0.$$

The above means that for any sequence $(k_n)_n \subset \mathbb{Z}$ with $|k_n| \to \infty$ for $n \to \infty$ and $k_n \neq k_m$ for $m \neq n$, it holds that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\mathrm{e}^{ik_n x}\,\mathrm{d}x = 0.$$

Problem 2

We consider the space

$$l^2(\mathbb{C}) = \left\{ (x_n)_{n\in\mathbb{N}} \subset \mathbb{C} \,:\, \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

i. e. the space of square summable sequences of complex numbers equipped with the scalar product

$$(x,y)_{l^2} := \sum_{n=1}^{\infty} x_n \overline{y_n} \text{ for all } x, y \in l^2(\mathbb{C}).$$

First show that this scalar product is well defined by showing that the series $\sum_{n=1}^{\infty} x_n\overline{y_n}$ converges absolutely for all $x, y \in l^2(\mathbb{C})$. Then, show that $(e^k)_{k\in\mathbb{N}} \subset l^2(\mathbb{C})$ given by $e^k_n = \delta_{kn}$, i. e. e^k is a sequence of zeroes, just the k-th element is 1, is a complete orthonormal system in $l^2(\mathbb{C})$ with respect to the scalar product $(\cdot,\cdot)_{l^2}$. To show this also show that the k-th Fourier coefficient of $x \in l^2(\mathbb{C})$ is the same as the k-th element of the series x.

Problem 3

Let X be a Hilbert space and $(e_n)_n \subset X$ an ONS. Show that

$$(e_n)_n \text{ is complete } \Leftrightarrow \left(\underset{k\in\mathbb{N}}{\forall} (x, e_k) = 0 \ \Rightarrow \ x = 0 \right).$$

If X is just a pre-Hilbert space, "$\Rightarrow$" still holds.