Mathematical methods of signal and image processing
Winter semester 2021/2022
Prof. Dr. Benjamin Berkels, Vera Loeser M.Sc.

Exercise sheet 7 — Due: 10. December 2021

General information

- Current information will be announced in RWTHmoodle.
- The due date only indicates in which exercise session the solution will be discussed.
- Office hours: By arrangement via Zoom.

Problem 1

Let $1 \leq p < \infty$ and $q \in \mathbb{N}$ such that $2qp > d$. Then,

$$g : \mathbb{R}^d \to \mathbb{R}, x \mapsto \frac{1}{1 + \|x\|_2^{2q}}$$

is in $L^p(\mathbb{R}^d)$, i. e. $\|g\|_{L^p} < \infty$.

Hint: It is not necessary to actually compute $\|g\|_{L^p}$, we just need to ensure that it is finite. This can be done by computing the L^p-norm of an upper bound of g. Here, Proposition B.14 can be used.

Problem 2

Let $\psi \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ be real-valued such that $\psi \geq 0$ and $\int_{\mathbb{R}^d} \psi(x)\,\mathrm{d}x = 1$. Moreover, let $f \in L^1(\mathbb{R}^d)$ be continuous in $x_0 \in \mathbb{R}^d$ and $\psi_\epsilon(x) := \frac{1}{\epsilon^d}\psi(\frac{x}{\epsilon})$ for $\epsilon > 0$ and $x \in \mathbb{R}^d$. Then,

$$\lim_{\epsilon \to 0} (\psi_\epsilon * f)(x_0) = f(x_0).$$

Note: This completes the proof of Proposition 3.21.

Problem 3

1. Compute the Fourier transform of

   $$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{1}{1 + x^2}.$$

   Hint: Consider the Fourier transform of $g(x) := e^{-a|x|}$ for $a > 0$.

2. Let $(f_a)_{a>0}$ be the family of functions given by

   $$f_a : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{1}{a\pi} \frac{1}{1 + \frac{x^2}{a^2}}.$$

   Show that this family has the folllowning semigroup property with respect to the convolution:

   $$f_a * f_b = f_{a+b} \text{ for all } a, b > 0.$$