Appendix A
Basics of Measure Theory

In this appendix, we give a brief overview of the basics of measure theory. For a detailed discussion of measure theory, a good source is *Real and Complex Analysis* by W. Rudin [33].

A.1 Measurability

Definition A.1 Let X be a set. A collection $\mathcal{A}$ of subsets of X is called a *σ-algebra of sets in X* if the following three statements are true:

(a) The set X is in $\mathcal{A}$.
(b) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$, where A^c is the complement of A in X.
(c) If $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, and if $A = \bigcup_{n=1}^{\infty} A_n$, then $A \in \mathcal{A}$.

A set X together with a σ-algebra $\mathcal{A}$ is called a *measurable space* and is denoted by $(X, \mathcal{A})$, or simply by X when there is no risk of confusion. The elements $A \in \mathcal{A}$ are known as *measurable sets*.

From (a) and (b), we can easily see that the empty set is always measurable; that is, $\emptyset \in \mathcal{A}$. If $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n$ is always measurable, by (b) and (c), because

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c .$$

Among other things, we can now conclude that, for measurable sets A and B, the set $A \backslash B = A \cap B^c$ is measurable.

The σ in the name σ-algebra refers to the countability assumption in (c) of Definition A.1. If we require only finite unions of measurable sets to be in $\mathcal{A}$, then $\mathcal{A}$ is called an *algebra of sets in X*.

Definition A.2 Let $(X, \mathcal{A})$ be a measurable space. A scalar-valued function f with domain X is called *measurable* if $f^{-1}(V) \in \mathcal{A}$ whenever V is an open set in the scalar field.

A. Bowers, N. J. Kalton, *An Introductory Course in Functional Analysis,* Universitext, DOI 10.1007/978-1-4939-1945-1

Example A.3 Let $(X, \mathcal{A})$ be a measurable space. If $E \subseteq X$, then the *characteristic function* (or *indicator function*) of E is the function defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

The characteristic function of E is measurable if and only if $E \in \mathcal{A}$. The characteristic function χ_E is also denoted by 1_E.

Suppose u and v are real-valued functions. Then $f = u + iv$ is a complex measurable function if and only if u and v are real measurable functions. Furthermore, if f and g are measurable functions, then so is $f + g$.

Measurable functions are very well-behaved. For example, if $(f_n)_{n=1}^{\infty}$ is a sequence of real-valued measurable functions, then the functions

$$\sup_{n \in \mathbb{N}} f_n \quad \text{and} \quad \limsup_{n \to \infty} f_n$$

are measurable whenever the function values are finite. In particular, if $(f_n)_{n=1}^{\infty}$ is a convergent sequence of scalar-valued measurable functions, then the limit of the sequence is also a scalar-valued measurable function.

Definition A.4 A *simple function* is any linear combination of characteristic functions; that is

$$\sum_{j=1}^{n} a_j \chi_{E_j},$$

where $n \in \mathbb{N}$, $a_1, \ldots, a_n$ are scalars, and $E_1, \ldots, E_n$ are measurable sets.

Certainly, any simple function is measurable, since it is a finite sum of measurable functions.

The next theorem is key to the further development of the subject.

Theorem A.5 *Let $(X, \mathcal{A})$ be a measurable space. If f is a positive measurable function, then there exists a sequence $(s_n)_{n=1}^{\infty}$ of simple functions such that:*

$$0 \le s_1 \le s_2 \le \cdots \le f,$$

and

$$f(x) = \lim_{n \to \infty} s_n(x), \quad x \in X.$$

A.2 Positive Measures and Integration

In this section, we will consider functions defined on a σ-algebra $\mathcal{A}$. Such functions are called *set functions*. For now, we will restrict our attention to set functions taking values in the interval $[0, \infty]$.

Definition A.6 Let $(X, \mathcal{A})$ be a measurable space. A set function $\mu : \mathcal{A} \to [0, \infty]$ is said to be *countably additive* if

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j), \tag{A.1}$$

whenever $(A_j)_{j=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets. By *pairwise disjoint* we mean $A_j \cap A_k = \emptyset$ whenever $j \neq k$.

A countably additive set function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ is called a *positive measure on* $\mathcal{A}$. In such a case, we call the triple $(X, \mathcal{A}, \mu)$ a *positive measure space*. If the σ-algebra is understood, we often write (X, μ) for the positive measure space and say μ is a positive measure on X.

Notice that we allow μ to attain infinite values. The assumption that $\mu(\emptyset) = 0$ is to avoid the trivial case where the set function is always ∞. Equivalently, we could assume that there exists some $A \in \mathcal{A}$ such that $\mu(A) < \infty$.

Example A.7 The *Borel σ-algebra* on $\mathbb{R}$ is the smallest σ-algebra that contains all of the open intervals in $\mathbb{R}$. A measure defined on the Borel σ-algebra on $\mathbb{R}$ is called a *Borel measure* on $\mathbb{R}$.

Let μ be a positive measure on a σ-algebra. A measurable set that has zero measure with respect to μ is called a *μ-null set* or a *μ-measure-zero set*. A subset of a μ-null set is called *μ-negligible*. A μ-negligible set may or may not be measurable. If every μ-negligible set is measurable (and hence a μ-null set), then the measure μ is called *complete*. (That is, μ is complete if every subset of a μ-null set is μ-measurable and has μ-measure zero.)

Example A.8 (Lebesgue measure) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ and let m be a positive Borel measure on $\mathcal{B}$ defined so that $m((a, b)) = b - a$ whenever a and b are real numbers such that $a < b$. (Such a measure can be shown to exist.) We now define a complete measure on $\mathbb{R}$ that agrees with m for all Borel measurable sets in $\mathcal{B}$. In order to do this, we will enlarge our σ-algebra.

The *Lebesgue σ-algebra* on $\mathbb{R}$ is the collection of all sets of the form $B \cup N$, where $B \in \mathcal{B}$ and N is an m-negligible set. Define a measure λ on this collection of sets by the rule $\lambda(B \cup N) = m(B)$ for any m-measurable set B and any m-negligible set N. It can be shown that λ is a complete countably additive measure on the Lebesgue σ-algebra on $\mathbb{R}$. The measure λ is called *Lebesgue measure* on $\mathbb{R}$.

The Lebesgue σ-algebra on $\mathbb{R}$ is the smallest σ-algebra on $\mathbb{R}$ that contains the Borel measurable sets and all subsets of m-measure-zero sets. Consequently, all Borel measurable sets are Lebesgue measurable, but there are Lebesgue measurable sets that are not Borel measurable.

Example A.9 (Counting measure) Another classic example of a positive measure is the so-called *counting measure* on $\mathbb{N}$. We let $\mathcal{A}$ be the *power set*, which is the family of all subsets of $\mathbb{N}$, often denoted $\mathcal{P}(\mathbb{N})$ or $2^{\mathbb{N}}$. For any subset $A \subseteq \mathbb{N}$, we define $n(A)$ to be the size (or cardinality) of the set A. Then n is a positive measure on $\mathbb{N}$ and $n(\mathbb{N}) = \infty$.

Definition A.10 A positive measure that attains only finite values is called a *finite measure*. If μ is a positive measure on X such that $\mu(X) = 1$, then we call μ a *probability measure*.

Example A.11 The restriction of the measure λ from Example A.8 to the unit interval $[0, 1]$ is a probability measure.

Let us make some obvious comments about positive measures. Suppose $(X, \mathcal{A}, \mu)$ is a positive measure space. If A and B are measurable sets such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$. Also, observe that if μ is a finite positive measure, then $\mu(\,\cdot\,)/\mu(X)$ is a probability measure.

Definition A.12 Let $(X, \mathcal{A}, \mu)$ be a positive measure space and let $s = \sum_{j=1}^{n} a_j\, \chi_{E_j}$ be a simple function. For any $A \in \mathcal{A}$, we define the *integral of s over A with respect to μ* to be

$$\int_A s d\mu = \sum_{j=1}^{n} a_j\, \mu(A \cap E_j).$$

(The value of this integral does not depend on the representation of s that is used.)

If $f : X \to \mathbb{R}$ is a *nonnegative* measurable function, then we define the *integral of f over A with respect to μ* to be

$$\int_A f d\mu = \sup \left\{ \int_A s d\mu \right\},$$

where the supremum is taken over all simple functions s such that $s(x) \leq f(x)$ for all $x \in X$.

In order to meaningfully extend our notion of an integral to a wider class of functions, we first define the class of functions for which our integral will exist.

Definition A.13 Let $(X, \mathcal{A}, \mu)$ be a positive measure space. A scalar-valued measurable function f is called *Lebesgue integrable* (or just *integrable*) with respect to μ if

$$\int_X |f|\, d\mu < \infty.$$

If f is integrable with respect to μ, then we write $f \in L_1(\mu)$.

Now, let $(X, \mathcal{A}, \mu)$ be a positive measure space and let $f : X \to \mathbb{R}$ be a real-valued integrable function. Define the integral of f over A by

$$\int_A f d\mu = \int_A f^+\, d\mu \; - \; \int_A f^-\, d\mu,$$

where

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

is the *positive part of* f, and

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

is the *negative part of* f. If $f : X \to \mathbb{C}$ is a complex-valued integrable function, define the integral of f over A by

$$\int_A f d\mu = \int_A \Re(f)\, d\mu \; + \; i \int_A \Im(f)\, d\mu,$$

where $\Re(f)$ and $\Im(f)$ denote the real and imaginary parts of f, respectively.

To avoid certain arithmetic difficulties which arise from the definitions above, we adopt the convention that $0 \cdot \infty = 0$. Consequently, $\int_A f\, d\mu = 0$ whenever $f(x) = 0$ for all $x \in A$, even if $\mu(A) = \infty$. Furthermore, if $\mu(A) = 0$ for a measurable set $A \in \mathcal{A}$, then $\int_A f\, d\mu = 0$ for all measurable functions f.

In some cases, it is useful to explicitly show the variable dependence in an integration, and in such a situation we write

$$\int_A f d\mu = \int_A f(x)\, \mu(dx).$$

A.3 Convergence Theorems and Fatou's Lemma

Definition A.14 Let $(X, \mathcal{A}, \mu)$ be a positive measure space. Two measurable functions f and g are said to be equal *almost everywhere* if $\mu\{x \in X : f(x) \neq g(x)\} = 0$. If f and g are equal almost everywhere, we write $f = g \text{a.e.}(\mu)$. Sometimes we say $f(x) = g(x)$ for almost every x.

The previous definition is significant because, if $f = g \text{a.e.}(\mu)$, then

$$\int_A f d\mu = \int_A g d\mu, \quad A \in \mathcal{A}.$$

This is true because the integral over a set of measure zero is always zero. Functions that are equal almost everywhere are indistinguishable from the point of view of integration. With this in mind, we extend our notion of a measurable function. Let $(X, \mathcal{A}, \mu)$ be a measure space. A function f is *defined a.e.*(μ) *on* X if the domain D of f is a subset of X and $X \setminus D$ is a μ-negligible set. If f is defined a.e.(μ) on X and there is a μ-null set $E \in \mathcal{A}$ such that $f^{-1}(V) \setminus E$ is measurable for every open set V, then f is called μ*-measurable*. Every measurable function (in the sense of Definition A.2) is μ-measurable and every μ-measurable function is equal almost everywhere (with respect to μ) to a measurable function.

The notion of μ-measurability allows us to extend our constructions to a wider collection of functions. We say that a μ-measurable function f is *integrable* if it is equal a.e.(μ) to a measurable function g that is integrable with respect to μ and we define $\int_A f\,d\mu$ to be $\int_A g\,d\mu$ for every measurable set A.

A sequence of measurable functions $(f_n)_{n=1}^{\infty}$ is said to converge *almost everywhere* to a function f if the set $\{x \in X : f_n(x) \not\to f(x)\}$ has μ-measure zero. In this case, we write $f_n \to f$ a.e.(μ). A function f which is the a.e.(μ)-limit of measurable functions is μ-measurable.

Theorem A.15 (Monotone Convergence Theorem) *Let $(X, \mathcal{A}, \mu)$ be a positive measure space. Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots$$

for almost every x. If $f_n \to f$ a.e.(μ), then f is an integrable function and

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu.$$

The Monotone Convergence Theorem is a very useful tool, and one of the better known consequences of it is Fatou's Lemma, named after the mathematician Pierre Fatou.

Theorem A.16 (Fatou's Lemma) *Let $(X, \mathcal{A}, \mu)$ be a positive measure space. If $(f_n)_{n=1}^{\infty}$ is a sequence of scalar-valued measurable functions, then*

$$\int_X \left(\liminf_{n\to\infty} |f_n|\right) d\mu \leq \liminf_{n\to\infty} \int_X |f_n|\,d\mu.$$

Perhaps the most important use of Fatou's Lemma is in proving the next theorem, which is one of the cornerstones of measure theory.

Theorem A.17 (Lebesgue's Dominated Convergence Theorem) *Let $(X, \mathcal{A}, \mu)$ be a positive measure space and suppose $(f_n)_{n=1}^{\infty}$ is a sequence of scalar-valued measurable functions that converge almost everywhere to f. If there exists a function $g \in L_1(\mu)$ such that $|f_n| \leq g$ a.e.(μ) for all $n \in \mathbb{N}$, then $f \in L_1(\mu)$ and*

$$\lim_{n\to\infty} \int_X |f_n - f|\,d\mu = 0.$$

In particular,

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu.$$

The last equality follows from the fact that $|\int_X f\,d\mu| \leq \int_X |f|\,d\mu$ whenever f is an integrable function.

The next result is a direct consequence of Lebesgue's Dominated Convergence Theorem.

Corollary A.18 (Bounded Convergence Theorem) *Let $(X, \mathcal{A}, \mu)$ be a positive finite measure space and suppose $(f_n)_{n=1}^{\infty}$ is a sequence of uniformly bounded scalar-valued measurable functions. If $f_n \to f$ a.e.(μ), then $f \in L_1(\mu)$ and*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu.$$

A.4 Complex Measures and Absolute Continuity

Definition A.19 Let $(X, \mathcal{A})$ be a measurable space. A countably additive set function $\mu : \mathcal{A} \to \mathbb{C}$ is called a *complex measure*. When we say μ is countably additive, we mean

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j), \tag{A.2}$$

whenever $(A_j)_{j=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets in $\mathcal{A}$, where the series in (A.2) is absolutely convergent. (Compare to Definition A.6). A complex measure which takes values in $\mathbb{R}$ is called a *real measure* or a *signed measure*.

If μ is a complex measure on $\mathcal{A}$, then the triple $(X, \mathcal{A}, \mu)$ is called a (complex) *measure space*. If the σ-algebra is understood, then we may write (X, μ).

More generally, we can define a measure to be a countably additive set function that takes values in any topological vector space, so long as the convergence of the series in (A.2) makes sense. For our purposes, such generality is not necessary, and so we will confine ourselves to complex measures and positive measures.

There is a significant difference between positive measures and complex measures. In Definition A.19, we require that a complex measure μ be finite; that is, $|\mu(A)| < \infty$ for all $A \in \mathcal{A}$. This was not a requirement for a positive measure.

Definition A.20 Let μ be a complex measure on the σ-algebra $\mathcal{A}$. We define the *total variation measure of* μ to be the set function $|\mu| : \mathcal{A} \to \mathbb{R}$ given by

$$|\mu|(E) = \sup\left\{\sum_{j=1}^{n} |\mu(E_j)|\right\},$$

where the supremum is taken over all *finite* sequences of pairwise-disjoint measurable sets $(E_j)_{j=1}^{n}$, for all $n \in \mathbb{N}$, such that $E = \bigcup_{j=1}^{n} E_j$.

As the name suggests, the total variation measure is a measure. Additionally, it is a finite measure—a fact that we state in the next theorem.

Theorem A.21 *If $(X, \mathcal{A}, \mu)$ is a complex measure space, then $|\mu|$ is a positive measure on $\mathcal{A}$ and $|\mu|(X) < \infty$.*

There is, naturally, a relationship between a measure and its total variation measure. For example, if $|\mu|(E) = 0$ for some set $E \in \mathcal{A}$, it must be the case that $\mu(E) = 0$. This is an instance of a property called *absolute continuity*.

Definition A.22 Let μ be a complex measure on $\mathcal{A}$ and suppose λ is a positive measure on $\mathcal{A}$. We say that μ is *absolutely continuous* with respect to λ if, for all

$A \in \mathcal{A}$, we have that $\mu(A) = 0$ whenever $\lambda(A) = 0$. When μ is absolutely continuous with respect to λ, we write $\mu \ll \lambda$.

Shortly, we will state the Radon–Nikodým Theorem, one of the key results in measure theory. Before that, however, we introduce a definition.

Definition A.23 Let $(X, \mathcal{A})$ be a measurable space. A positive measure μ on $\mathcal{A}$ is said to be *σ-finite* if there exists a countable sequence $(E_j)_{j=1}^{\infty}$ of measurable sets such that $X = \bigcup_{j=1}^{\infty} E_j$ and $\mu(E_j) < \infty$ for each $j \in \mathbb{N}$.

Important examples of σ-finite measures include Lebesgue measure on $\mathbb{R}$ and counting measure on $\mathbb{N}$. (See Examples A.8 and A.9, respectively.) One can also have a counting measure on $\mathbb{R}$, but it is not σ-finite.

Theorem A.24 (Radon–Nikodým Theorem) *Suppose $(X, \mathcal{A})$ is a measurable space and let λ be a positive σ-finite measure on $\mathcal{A}$. If μ is a complex measure on $\mathcal{A}$ that is absolutely continuous with respect to λ, then there exists a unique $g \in L_1(\lambda)$ such that*

$$\mu(E) = \int_E g(x)\,\lambda(dx), \quad E \in \mathcal{A}. \tag{A.3}$$

The equation in (A.3) is sometimes written $\mu(dx) = g(x)\,\lambda(dx)$ or $d\mu = g\,d\lambda$. The function $g \in L_1(\lambda)$ is called the *Radon–Nikodým derivative* of μ with respect to λ and is sometimes denoted $d\mu/d\lambda$. When we say the Radon–Nikodým derivative is *unique*, we mean up to a set of measure zero. That is, if g and h are both Radon–Nikodým derivatives of μ with respect to λ, then $g = h$a.e.(λ) (and consequently a.e.(μ)).

The σ-finite assumption on μ in Theorem A.24 cannot be relaxed.

We seek to define an integral with respect to a complex measure. To that end, let $(X, \mathcal{A}, \mu)$ be a complex measure space. Since $\mu \ll |\mu|$, there exists (by the Radon–Nikodým Theorem) a unique (up to sets of measure zero) function $g \in L_1(|\mu|)$ such that $d\mu = g\,d|\mu|$. We can, therefore, define the integral of a measurable function $f : X \to \mathbb{C}$ by

$$\int_E f d\mu = \int_E fgd|\mu|, \quad E \in \mathcal{A}. \tag{A.4}$$

More can be said about the Radon–Nikodým derivative of μ with respect to $|\mu|$. The following proposition is a corollary of Theorem A.24 and is often called the *polar representation* or *polar decomposition* of μ.

Proposition A.25 *If $(X, \mathcal{A}, \mu)$ is a complex measure space, then there exists a measurable function g such that $|g(x)| = 1$ for all $x \in X$ and such that $d\mu = g\,d|\mu|$.*

As a consequence of Proposition A.25 and the definition in (A.4), versions of the Monotone Convergence Theorem, Fatou's Lemma, and Lebesgue's Dominated Convergence Theorem hold for integrals with respect to complex measures. We apply the existing theorems to the positive finite measure $|\mu|$ and observe that

$$\Big|\int_X f\,d\mu\Big| \le \int_X |f|\,d|\mu|,$$

for all measurable functions f on X.

The Radon–Nikodým Theorem is named after Johann Radon, who proved the theorem for $\mathbb{R}^n$ in 1913 ($n \in \mathbb{N}$), and for Otton Nikodým, who proved the theorem for the general case in 1930 [28].

A.5 L_p-spaces

In this section, we consider a measure space $(X, \mathcal{A}, \mu)$, where μ is a positive measure. We will identify certain spaces of measurable functions on X. For $p \in [1, \infty)$, let

$$L_p(\mu) = \Big\{f \text{ a measurable function} : \int_X |f|^p\,d\mu < \infty\Big\}.$$

This is the space of *p-integrable functions*, also known as *L_p-functions*, on X. For each $p \in [1, \infty)$, we let

$$\|f\|_p = \Big(\int_X |f|^p\,d\mu\Big)^{1/p},$$

where f is a μ-measurable function on X. Observe that $\|f\|_p < \infty$ if and only if $f \in L_p(\mu)$. For the case $p = \infty$, let

$$\|f\|_\infty = \inf\big\{K : \mu(|f| > K) = 0\big\},$$

for all measurable functions f. The quantity $\|f\|_\infty$ is called the *essential supremum norm* of f, and is the smallest number having the property that $|f| \le \|f\|_\infty$a.e.(μ).

The set

$$L_\infty(\mu) = \{f \text{ a measurable function} : \|f\|_\infty < \infty\}$$

is the space of *essentially bounded measurable functions* on X.

Theorem A.26 *Let (X, μ) be a positive measure space. If $1 \le p \le \infty$, then $\|\cdot\|_p$ is a complete norm on $L_p(\mu)$. In particular, $L_p(\mu)$ is a Banach space.*

In fact, the L_p-spaces are collections of *equivalence classes* of measurable functions. Two functions f and g in $L_p(\mu)$ are considered equivalent if $f = g$a.e.(μ). In spite of this, we will generally speak of the elements in L_p-spaces as functions, rather than equivalence classes of functions.

The proof of Theorem A.26 relies heavily on the following fundamental inequality, which provides the triangle inequality for $L_p(\mu)$.

Theorem A.27 (Minkowski's Inequality) *Let (X, μ) be a positive measure space. If f and g are in $L_p(\mu)$, where $1 \leq p \leq \infty$, then $f + g \in L_p(\mu)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Before we give another theorem, we must introduce a definition.

Definition A.28 If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then q is called the *conjugate exponent* of p. The conjugate exponent of $p = 1$ is defined to be $q = \infty$ (and *vice versa*).

If $p \in [1, \infty]$ and q is the conjugate exponent of p, then we can also say that q is conjugate to p, or even that p and q are conjugate to each other. If $p \in (1, \infty)$ and q is conjugate to p, it is sometimes convenient to write $q = \frac{p}{p-1}$.

The following theorem, which is ubiquitous in measure theory, can be seen as a generalization of the Cauchy–Schwarz inequality.

Theorem A.29 (Hölder's Inequality) *Let (X, μ) be a positive measure space. Suppose $1 \leq p < \infty$ and let q be conjugate to p. If $f \in L_p(\mu)$ and $g \in L_q(\mu)$, then $fg \in L_1(\mu)$ and*

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q.$$

Hölder's Inequality ensures that, given any $g \in L_q(\mu)$, the map

$$f \mapsto \int_X f(x)\, g(x)\, \mu(dx), \qquad f \in L_p(\mu),$$

defines a bounded linear functional on $L_p(\mu)$ whenever $1 \leq p < \infty$. It turns out that any bounded linear functional on $L_p(\mu)$ can be achieved in this way, which is the content of the next theorem.

Theorem A.30 *Let (X, μ) be a positive σ-finite measure space. If $1 \leq p < \infty$, and if q is conjugate to p, then $L_p(\mu)^* = L_q(\mu)$.*

Frequently, in order to prove something about L_p-spaces, it is sufficient to prove it for simple functions. This is a consequence of the next theorem, which follows from Theorem A.5 and Theorem A.15 (the Monotone Convergence Theorem).

Theorem A.31 (Density of Simple Functions) *Let (X, μ) be a positive measure space. The set of simple functions is dense in $L_p(\mu)$ whenever $1 \leq p \leq \infty$.*

A.6 Borel Measurability and Measures

Definition A.32 Suppose X is a topological space. The smallest σ-algebra on X containing the open sets in X is called the *Borel σ-algebra*, or the *Borel field*, on X. A function which is measurable with respect to the Borel σ-algebra is called a *Borel* *measurable function*, or a *Borel function*. A measure on the Borel σ-algebra is called a *Borel measure*.

We recall that a topological space is said to be *locally compact* if every point has a compact neighborhood. Naturally, all compact spaces are locally compact, but the converse need not be true. For example, the real line $\mathbb{R}$ with its standard topology is locally compact, but not compact.

If X is a locally compact Hausdorff space, then we denote by $C_0(X)$ the collection of all continuous functions that *vanish at infinity*. We say f *vanishes at infinity* if for every $\varepsilon > 0$, there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \notin K$. The set $C_0(X)$ is a Banach space under the supremum norm.

Example A.33 The Banach space $C_0(\mathbb{N})$ is the classical sequence space c_0 of sequences tending to zero.

Definition A.34 Let X be a locally compact Hausdorff space and suppose μ is a positive Borel measure on X. If, for every measurable set E,

$$\mu(E) = \inf\{\mu(V) : E \subseteq V, \quad V \text{ an open set}\},$$

then μ is called *outer regular*. If, for every measurable set E,

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, \quad K \text{ a compact set}\},$$

then μ is called *inner regular*. If μ is both inner regular and outer regular, then we call μ a *regular measure*. A complex measure is called regular if $|\mu|$ is regular.

Definition A.34 allows us to describe the dual space of $C_0(X)$.

Theorem A.35 (Riesz Representation Theorem) *Let X be a locally compact Hausdorff space. If Λ is a bounded linear functional on $C_0(X)$, then there exists a unique regular complex Borel measure μ such that*

$$\Lambda(f) = \int_X f\, d\mu, \quad f \in C_0(X),$$

and $\|\Lambda\| = |\mu|(X)$.

Theorem A.35 is named after F. Riesz, who originally proved the theorem for the special case $X = [0, 1]$ [31]. The next theorem is named after Nikolai Lusin (or Luzin), who also worked in the context of the real line [23].

Theorem A.36 (Lusin's Theorem) *Let X be a locally compact Hausdorff space and suppose μ is a finite positive regular Borel measure on X. The space $C_0(X)$ is dense in $L_p(\mu)$ for all $p \in [1, \infty)$.*

The hypotheses of Lusin's Theorem can be relaxed somewhat. Indeed, the theorem holds for any positive Borel measures that are finite on compact sets. Note that Lusin's Theorem is not true when $p = \infty$. Convergence in the L_∞-norm is the same as uniform convergence, and the uniform limit of a sequence of continuous functions is continuous; however, functions in $L_\infty(\mu)$ need not be continuous.

Proposition A.37 *If X is a locally compact metrizable space, then any finite Borel measure on X is necessarily regular.*

Again, the Borel measure in question need only be finite on compact sets for the conclusion to hold. In light of Proposition A.37, Theorem A.35 is often stated for a metrizable space X, in which case the term "regular" is omitted from the conclusion. (This was done, for example, in Theorem 2.20 of the current text.)

A.7 Product Measures

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. A *measurable rectangle* in $X \times Y$ is any set of the form $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We denote by $\sigma(\mathcal{A} \times \mathcal{B})$ the smallest σ-algebra containing all measurable rectangles in $X \times Y$.

Proposition A.38 *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. If a scalar-valued function f on $X \times Y$ is $\sigma(\mathcal{A} \times \mathcal{B})$-measurable, then the map $x \mapsto f(x, y)$ is $\mathcal{A}$-measurable for all $y \in Y$, and $y \mapsto f(x, y)$ is $\mathcal{B}$-measurable for all $x \in X$.*

Now let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. We define the *product measure* on $\sigma(\mathcal{A} \times \mathcal{B})$ to be the set function $\mu \times \nu$ given by the formula

$$\begin{aligned}(\mu \times \nu)(Q) &= \int_X \left(\int_Y \chi_Q(x, y)\, \nu(dy) \right) \mu(dx) \\ &= \int_Y \left(\int_X \chi_Q(x, y)\, \mu(dx) \right) \nu(dy),\end{aligned}$$

for all $Q \in \sigma(\mathcal{A} \times \mathcal{B})$. As the name implies, $\mu \times \nu$ is a measure on $\sigma(\mathcal{A} \times \mathcal{B})$. Furthermore, the measure $\mu \times \nu$ is such that $(\mu \times \nu)(E \times F) = \mu(E)\nu(F)$ for all $E \in \mathcal{A}$ and $F \in \mathcal{B}$.

The fundamental result in this section is known as Fubini's Theorem. It is named after Guido Fubini, who proved a version of the theorem in 1907 [13].

Theorem A.39 (Fubini's Theorem) *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. If $f \in L_1(\mu \times \nu)$, then*

$$\begin{aligned}\int_{X \times Y} f\, d(\mu \times \nu) &= \int_X \left(\int_Y f(x, y)\, \nu(dy) \right) \mu(dx) \\ &= \int_Y \left(\int_X f(x, y)\, \mu(dx) \right) \nu(dy).\end{aligned}$$

The same conclusion holds if f is a $\sigma(\mathcal{A} \times \mathcal{B})$-measurable function such that

$$\int_X \left(\int_Y |f(x, y)|\, \nu(dy) \right) \mu(dx) < \infty.$$

Fubini's Theorem is sometimes called the Fubini-Tonelli theorem, after Leonida Tonelli, who proved a version of Theorem A.39 in 1909 [35].

Appendix B
Results From Other Areas of Mathematics

Throughout the course of this text, we have invoked important results (usually by name), sometimes without explicitly writing out the statement of the result being used. Many of these come from measure theory, and so appear in Appendix 8.3. We include the rest here, for easy reference. For proofs of these theorems, as well as more discussion on these topics, see (for example) *Topology: a first course* by James Munkres [27] and *Functions of One Complex Variable* by John Conway [7].

B.1 The One-Point and Stone-Čech Compactifications

We will give only a brief discussion of the necessary topological concepts. For more, we refer the interested reader to *Topology: a first course*, by James Munkres [27].

We start by defining the *one-point compactification* of a locally compact Hausdorff space that is not compact. (Recall that a space is *locally compact* if every point has a compact neighborhood containing it.)

Definition B.1 Let X be a locally compact Hausdorff space that is not compact. Adjoin to X an element ∞, called the *point at infinity*, to form a set $Y = X \cup \{\infty\}$. Define a topology on Y by declaring a set U to be open in Y if either U is open in X or $U = Y \backslash K$, where K is compact in X. The space Y is called the *one-point compactification of* X.

The classic example of a one-point compactification is $\mathbb{N} \cup \{\infty\}$, the one-point compactification of the natural numbers, where $\mathbb{N}$ is given the discrete topology. Notice that $\mathbb{N}$ is locally compact but not compact. The space $\mathbb{N} \cup \{\infty\}$ is homeomorphic to the subspace $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$. (See Exercise 2.15.)

A. Bowers, N. J. Kalton, *An Introductory Course in Functional Analysis,*
Universitext, DOI 10.1007/978-1-4939-1945-1

Theorem B.2 *Let X be a locally compact Hausdorff space that is not compact and let Y be its one-point compactification. Then Y is a compact Hausdorff space, $Y\backslash X$ contains exactly one point, X is a subspace of Y, and Y is the closure of X (in Y).*

In general, the one-point compactification of a locally compact metric space is not itself metrizable; however, there are some circumstances under which a metric does exist.

Theorem B.3 *Let X be a locally compact Hausdorff space that is not compact. The one-point compactification of X is metrizable if and only if X is second countable.*

In particular, any locally compact separable metric space that is not compact (such as $\mathbb{N}$) will have a metrizable one-point compactification. We will not provide a proof of Theorem B.3. The inquisitive reader is directed to Theorem 3.44 in [3].

The next type of compactification we wish to define is the Stone-Čech compactification. Before introducing this concept, however, we require some background. We begin by restating Tychonoff's Theorem (Theorem 5.38), which we will use presently.

Theorem B.4 (Tychonoff's Theorem) *Let J be an index set. If $\{K_\alpha\}_{\alpha\in J}$ is a collection of compact topological spaces, then $\prod_{\alpha\in J} K_\alpha$ is compact in the product topology.*

Tychonoff's Theorem is a statement about compactness, and while we do not provide a proof of this theorem, we can profit from some knowledge of the methods used within. The standard proof of Tychonoff's Theorem uses an alternate characterization of compactness, one for which we need a definition.

Definition B.5 (Finite Intersection Property) Let X be a topological space. A collection $\mathcal{C}$ of subsets of X is said to satisfy the *Finite Intersection Property* if for every finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

We now state an alternative formulation of compactness.

Theorem B.6 *A topological space X is compact if and only if every collection $\mathcal{C}$ of closed sets in X satisfying the Finite Intersection Property is such that the intersection $\bigcap_{C\in\mathcal{C}} C$ of all elements in $\mathcal{C}$ is nonempty.*

The above theorem leads directly to the following useful corollary.

Corollary B.7 (Nested Interval Property) *Let X be a topological space. If $(C_n)_{n=1}^{\infty}$ is a sequence of nonempty compact sets such that $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$, then the intersection $\bigcap_{n=1}^{\infty} C_n$ of all sets in the sequence is nonempty.*

A space X is called *completely regular* if all singletons (single-point sets) in X are closed and if for each $x \in X$ and each closed subset A not containing x there exists a bounded continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(A) = \{0\}$.

In particular, if X is completely regular, then the collection of bounded real-valued continuous functions on X *separates the points of* X. That is, whenever $x \neq y$, there is some continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ but $f(y) = 0$.

We remark here that a space X is called *regular* if any closed subset A and point $x \notin A$ can be separated by open neighborhoods. That is to say, there exist disjoint open neighborhoods containing A and x, respectively. A completely regular space is regular, but there exist regular spaces that are not completely regular.

Proposition B.8 *A topological space X is completely regular if and only if it is homeomorphic to a subspace of a compact Hausdorff space.*

A *compactification* of a space X is a compact Hausdorff space Y that contains X as a dense subset. (An example is the one-point compactification given earlier.) In order for a noncompact space X to have a compactification, it is both necessary and sufficient that X be completely regular. There are generally many compactifications for a completely regular space X. The one we consider now is the Stone-Čech compactification.

Let X be a completely regular space. Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded continuous real-valued functions on X, indexed by some (possibly uncountable) set J. For each $\alpha \in J$, let I_α be any closed interval containing the range of f_α, say

$$I_\alpha = \left[\inf_{x \in X} f_\alpha(x),\ \sup_{x \in X} f_\alpha(x) \right],$$

or

$$I_\alpha = \big[-\|f_\alpha\|_\infty,\ \|f_\alpha\|_\infty \big].$$

Each of the intervals I_α is compact in $\mathbb{R}$, and so, by Tychonoff's Theorem, the product $\prod_{\alpha \in J} I_\alpha$ is a compact space.

Define a map $\phi : X \to \prod_{\alpha \in J} I_\alpha$ by

$$\phi(x) = (f_\alpha(x))_{\alpha \in J}, \quad x \in X.$$

It can be shown that ϕ is an embedding. (This follows from the complete regularity of X, because the collection $\{f_\alpha\}_{\alpha \in J}$ separates the points of X. See Theorem 4.4.2 in [27] for the details.) We conclude that $\overline{\phi(X)}$ is a compact Hausdorff space.

Let $A = \overline{\phi(X)} \backslash \phi(X)$ and define a space Y by $Y = X \cup A$. We have a bijective mapping $\Phi : Y \to \overline{\phi(X)}$ given by the rule

$$\Phi(y) = \begin{cases} \phi(y) & \text{if } y \in X, \\ y & \text{if } y \in A. \end{cases}$$

Define a topology on Y by letting U be open in Y whenever $\Phi(U)$ is open in $\overline{\phi(X)}$. It follows that Φ is a homeomorphism, and X is a subspace of the compact Hausdorff space Y.

The topological space Y constructed above is known as the *Stone-Čech compactification* of X, and is generally denoted $\beta(X)$. It may seem that $\beta(X)$ is not well-defined, since one could construct it with a different choice of sets; however, the Stone-Čech compactification is unique up to homeomorphism (and a homeomorphism can be chosen so that it is the identity on X).

While a full discussion of the Stone-Čech compactification is beyond the scope of this appendix, we will state a very important property that it possesses.

Theorem B.9 *Let X be a completely regular space with Stone-Čech compactification $\beta(X)$. Any bounded continuous real-valued function on X can be uniquely extended to a continuous real-valued function on $\beta(X)$.*

The Stone-Čech compactification is named after Marshall Harvey Stone and Eduard Čech. Tychonoff's Theorem is named after Andrey Nikolayevich Tychonoff. It is not clear that these theorems are named quite as they should be, a fact which was remarked upon by Walter Rudin in his *Functional Analysis* [34, Note to Appendix A]:

> . . . Thus it appears that Čech proved the Tychonoff theorem, whereas Tychonoff found the Čech compactification—a good illustration of the historical reliability of mathematical nomenclature.

B.2 Complex Analysis

The subject of complex analysis is overflowing with fantastic and improbable theorems. Unfortunately, we only encounter a small portion of the subject in our current undertaking. Let us begin with some definitions. A complex-valued function is called *differentiable* at z_0 in the complex plane $\mathbb{C}$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists (and thus is a complex number). In this context, the notation $z \to z_0$ means that $|z - z_0| \to 0$. If f is differentiable at every point in a set, then we say f is differentiable in the set (or on the set). A complex-valued function is called *holomorphic* (or *analytic*) if it is differentiable in a neighborhood of every point in its domain. The next theorem is one of the key results of complex analysis.

Theorem B.10 *A holomorphic function is infinitely differentiable.*

The next theorem is a frequently cited result and is used to show, for example, that the disk algebra $A(\mathbb{D})$ is a subalgebra of $C(\mathbb{T})$, the space of continuous functions on the unit circle. (See Example 8.6(b).)

Theorem B.11 (Maximum Modulus Theorem) *Let D be a bounded open set in the complex plane. If f is a holomorphic function in D that is extendable to a continuous function on $\overline{D}$, then*

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)|.$$

Holomorphic functions display a variety of interesting and useful properties. One such property is stated in the next theorem.

Theorem B.12 (Cauchy's Integral Formula) *Let $f : D \to \mathbb{C}$ be a holomorphic function on the simply connected domain D. If γ is a closed curve in D, then $\int_\gamma f(z)\,dz = 0$.*

The integral appearing in Theorem B.12 is a standard line integral over a path γ. When we say D is *simply connected*, we mean that any two points in D can be connected by a path, and that any path connecting those two points can be continuously transformed into any other. A less precise way of saying that is to say D has no "holes" in it.

Cauchy's Integral Formula has a converse, which is named after the mathematician and engineer Giacinto Morera.

Theorem B.13 (Morera's Theorem) *Let D be a connected open set in the complex plane. If $f : D \to \mathbb{C}$ is a continuous function such that $\int_\gamma f(z)\,dz = 0$ for every closed piecewise continuously differentiable curve γ in D, then f is holomorphic on D.*

Morera's Theorem is used to show (among other things) that the uniform limit of holomorphic functions is holomorphic: Suppose $(f_n)_{n=1}^{\infty}$ is a uniformly convergent sequence of holomorphic functions, and suppose $f = \lim_{n\to\infty} f_n$. By Cauchy's Integral Formula, if γ is any continuously differentiable closed curve, then we have $\int_\gamma f_n(z)\,dz = 0$ for all $n \in \mathbb{N}$. Since $\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0$ (by uniform convergence), it follows from Morera's Theorem that f is holomorphic.

A function that is holomorphic on all of $\mathbb{C}$ is called an *entire* function. There are some truly remarkable theorems related to entire functions.

Theorem B.14 (Liouville's Theorem) *A bounded entire function is constant.*

According to Liouville's Theorem, if an entire function is not constant, then it must be unbounded. In fact, something even stronger can be said.

Theorem B.15 (Picard's Lesser Theorem) *If an entire function is not constant, then its image is the entire complex plane, with the possible exception of one point.*

The classic example of such a non-constant function is $f(z) = e^z$. The range of this function is $\mathbb{C}\backslash\{0\}$. Theorem B.15 was proved by Charles Émile Picard in 1879. Another significant theorem due to Picard concerns *essential singularities*.

Theorem B.16 (Picard's Greater Theorem) *Let f be an analytic function with an essential singularity at z_0. On any neighborhood of z_0, the function f will attain every value of $\mathbb{C}$, with the possible exception of one point, infinitely often.*

A function is said to have a *singularity* at the point z_0 if f is analytic in a neighborhood containing z_0, except $f(z_0)$ does not exist. If $\lim_{z\to z_0} f(z)$ exists, then we call it a *removable singularity*. A classic example is the function $f(z) = \frac{\sin z}{z}$, which has a removable singularity at $z = 0$.

A singularity z_0 is a *pole* of f if there is some $n \in \mathbb{N}$ such that

$$\lim_{z\to z_0} (z - z_0)^n f(z) \tag{B.5}$$

exists. We call z_0 a *pole of order n* if n is the smallest integer such that (B.5) exists. A simple example of a function with a polar singularity of order $n \in \mathbb{N}$ at $z = 0$ is $f(z) = \frac{1}{z^n}$.

If the limit in (B.5) does not exist for any $n \in \mathbb{N}$, we call z_0 an *essential singularity* of f. Examples of functions with an essential singularity at $z = 0$ are $f(z) = e^{1/z}$ and $f(z) = \sin(1/z)$.

An alternate characterization of singularities can be obtained from the Laurent series of the function f. A function f has a *Laurent series about* z_0 if there exists a doubly infinite sequence of scalars $(a_n)_{n\in\mathbb{Z}}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

There is a formula to compute a_n for $n \in \mathbb{Z}$:

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$

where γ is a circle centered at z_0 and contained in an annular region in which f is holomorphic. Any function with a singularity at z_0 will have a Laurent series there. Such a function has a pole of order $n \in \mathbb{N}$ at z_0 if $a_{-n} \neq 0$, but $a_{-k} = 0$ for all $k > n$. If $a_{-k} \neq 0$ for infinitely many $k \in \mathbb{N}$, then f has an essential singularity at z_0.

Augustin-Louis Cauchy did fundamental research in complex analysis in the first half of the nineteenth century. He and Joseph Liouville, after whom Theorem B.14 is named, were contemporaries. Charles Émile Picard came later, not being born until 1856, only one year before Cauchy passed away (on 23 May 1857).

References

1. L. Alaoglu, Weak topologies of normed linear spaces. Ann. Math. (2), **41**, 252–267 (1940)
2. F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, vol. 233 of Graduate Texts in Mathematics (Springer, New York, 2006)
3. C.D. Aliprantis, K.C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd edn. (Springer, Berlin, 2006)
4. J.L. Bell, D.H. Fremlin, A geometric form of the axiom of choice. Fundam. Math. **77**, 167–170 (1972)
5. J.W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space. Ann. Math. (2) **42**, 839–873 (1941)
6. P.G. Casazza, A tribute to Nigel J. Kalton (1946–2010). Notices Am. Math. Soc. **59**, 942–951 (2012). (Peter G. Casazza, coordinating editor)
7. J.B. Conway, *Functions of One Complex Variable*, vol. 11 of Graduate Texts in Mathematics, 2nd edn. (Springer-Verlag, New York, 1978)
8. J.B. Conway, *A Course in Operator Theory*, vol. 21 of Graduate Studies in Mathematics (American Mathematical Society, Providence, 2000)
9. P. Enflo, A counterexample to the approximation problem in Banach spaces. Acta Math. **130**, 309–317 (1973)
10. M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. Math. Zeitschrift **17**, 228–249 (1923)
11. E. Fischer, Sur la convergence en moyenne. C. R. Acad. Sci. (Paris) **144**, 1022–1024 (1907)
12. I. Fredholm, Sur une classe d'équations fonctionnelles. Acta. Math. **27**, 365–390 (1903)
13. G. Fubini, Sugli integrali multipli. Rom. Accad. Lincei. Rend. (5) **16**(1), 608–614 (1907)
14. I. Gelfand, Normierte Ringe. Rec. Math. N.S. (Mat. Sbornik) **9**(51), 3–24 (1941)
15. I. Gelfand, A. Kolmogoroff, On rings of continuous functions on topological spaces. Dokl. Akad. Nauk. SSSR **22**, 11–15 (1939)
16. I. Gelfand, M. Neumark, On the imbedding of normed rings into the ring of operators in Hilbert space. Rec. Math. N.S. (Mat. Sbornik) **12**(54), 197–213 (1943)
17. A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. Mem. Am. Math. Soc. **1955**, 140 (1955)
18. W. Hodges, Krull Implies Zorn. J. London Math. Soc. (2) **19**, 285–287 (1979)
19. R.C. James, Characterizations of Reflexivity. Studia Math. **23**, 205–216 (1963/1964)
20. Y. Katznelson, *An Introduction to Harmonic Analysis*, Cambridge Mathematical Library, 3rd edn. (Cambridge University Press, Cambridge, 2004)
21. M. Krein, D. Milman, On extreme points of regular convex sets. Studia Math. **9**, 133–138 (1940)
22. W. Krull, Idealtheorie in Ringen ohne Endlichkeitsbedingung. Math. Ann. **101**, 729–744 (1929)
23. N. Lusin, Sur les propriétés des fonctions mesurables. C. R. Acad. Sci. (Paris) **154**, 1688–1690 (1912)

A. Bowers, N. J. Kalton, *An Introductory Course in Functional Analysis,*
Universitext, DOI 10.1007/978-1-4939-1945-1

24. R.D. Mauldin (ed.), *The Scottish Book*, Birkhäuser Boston, Mass., 1981. Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex. (May 1979)
25. S. Mazur, Sur les anneaux linéaires. C. R. Acad. Sci. (Paris) **207**, 1025–1027 (1938)
26. G.H. Moore, *Zermelo's Axiom of Choice. Its Origins, Development, and Influence*, vol. 8 of Studies in the History of Mathematics and Physical Sciences (Springer-Verlag, New York, 1982)
27. J.R. Munkres, *Topology: A First Course* (Prentice-Hall Inc., Englewood Cliffs, 1975)
28. O. Nikodym, Sur une généralisation des intégrales de M. J. Radon. Fundam. Math. **15**, 131–179 (1930)
29. J.C. Oxtoby, *Measure and Category: A Survey of the Analogies Between Topological and Measure Spaces*, vol. 2 of Graduate Texts in Mathematics, 2nd edn. (Springer-Verlag, New York, 1980)
30. F. Riesz, Sur les systèmes orthogonaux de fonctions. C. R. Acad. Sci. (Paris) **144**, 615–619 (1907)
31. F. Riesz, Sur les opérations fonctionnelles linéaires. C. R. Acad. Sci. (Paris) **149**, 974–977 (1909)
32. J.W. Roberts, A compact convex set with no extreme points. Studia Math. **60**, 255–266 (1977)
33. W. Rudin, *Real and Complex Analysis*, 3rd edn. (McGraw-Hill Book Co., New York, 1987)
34. W. Rudin, *Functional Analysis*. International Series in Pure and Applied Mathematics, 2nd edn. (McGraw-Hill Inc., New York, 1991)
35. L. Tonelli, Sull' integrazione per parti. Rom. Accad. Lincei. Rend. (5). **18**(2), 246–253 (1909)
36. M. Väth, The dual space of L_∞ is L_1. Indag. Math. N.S. **9**, 619–625 (1998)
37. W. Żelazko, *Banach Algebras* (Elsevier Publishing Co., Amsterdam, 1973). (Translated from the Polish by Marcin E. Kuczma.)

Index

A. Bowers, N. J. Kalton, *An Introductory Course in Functional Analysis,*
Universitext, DOI 10.1007/978-1-4939-1945-1