Solution

1. Let $a > 0$ and $g(x) := e^{-a|x|}$. Then, for $\omega \in \mathbb{R}$, we get

$$
\begin{aligned}
(\mathcal{F}g)(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-ix\omega} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{ax} e^{-ix\omega} \, \mathrm{d}x + \int_{0}^{\infty} e^{-ax} e^{-ix\omega} \, \mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\lim_{b \to -\infty} \int_{b}^{0} e^{(a-i\omega)x} \, \mathrm{d}x + \lim_{c \to \infty} \int_{0}^{c} e^{-(a+i\omega)x} \, \mathrm{d}x \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\lim_{b \to -\infty} \frac{1}{a - i\omega} e^{(a-i\omega)x} \Big|_{b}^{0} + \lim_{c \to \infty} \frac{1}{-(a+i\omega)} e^{-(a+i\omega)x} \Big|_{0}^{c} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a - i\omega} - 0 + 0 - \frac{1}{-(a+i\omega)} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a - i\omega} + \frac{1}{a + i\omega} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{a + i\omega + a - i\omega}{a^2 + \omega^2} \right) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + \omega^2} = \frac{2}{\sqrt{2\pi} a} \frac{1}{1 + \frac{\omega^2}{a^2}} \\
&= \sqrt{2\pi} f_a(\omega).
\end{aligned}
$$

Here, we have used that for $z \in \mathbb{C}$, it holds that (exercise)

$$
\int_{b}^{c} e^{zt} \, \mathrm{d}t = \frac{1}{z} e^{zt} \Big|_{b}^{c}.
$$

Thus, we have shown

$$
(\mathcal{F}g)(\omega) = \sqrt{2\pi} f_a(\omega) \text{ for all } \omega \in \mathbb{R}.
$$

Since $f_a, g \in L^1(\Omega)$, the Fourier inversion theorem (Proposition 3.21) implies

$$
\begin{aligned}
e^{-a|-x|} = g(-x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}g)(\omega) e^{-ix\omega} \, \mathrm{d}\omega = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{2\pi} f_a(\omega) e^{-ix\omega} \, \mathrm{d}\omega \\
&= \sqrt{2\pi} (\mathcal{F} f_a)(x)
\end{aligned}
$$

$$
\Rightarrow \quad (\mathcal{F} f_a)(x) = \frac{1}{\sqrt{2\pi}} e^{-a|x|}.
$$

Since $f = \pi f_1$, we get

$$
(\mathcal{F} f)(x) = \pi (\mathcal{F} f_1)(x) = \frac{\pi}{\sqrt{2\pi}} e^{-1|x|} = \sqrt{\frac{\pi}{2}} e^{-|x|}.
$$

2. Using the convolution theorem and the results from the proof above, we get

$$
\begin{aligned}
\mathcal{F}(f_a * f_b)(x) &= \sqrt{2\pi} \mathcal{F}(f_a)(x) \mathcal{F}(f_b)(x) = \sqrt{2\pi} \frac{1}{\sqrt{2\pi}} e^{-a|x|} \frac{1}{\sqrt{2\pi}} e^{-b|x|} \\
&= \frac{1}{\sqrt{2\pi}} e^{-(a+b)|x|} = \mathcal{F}(f_{a+b})(x).
\end{aligned}
$$

Since the Fourier transform is injective (Corollary 3.23), this implies $f_a * f_b = f_{a+b}$.