Mathematical methods of signal and image processing
Winter semester 2021/2022
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Exercise sheet 2 **Due: 5. November 2021**

General information

- Current information will be announced in RWTHmoodle.
- The due date only indicates in which exercise session the solution will be discussed.
- Office hours: By arrangement via Zoom.

Problem 1

Let $1 \leq p < \infty$ and $g \in L^p(\mathbb{R}^d)$. Show

$$\lim_{h \to 0} \|T_h g - g\|_{L^p} = 0 \qquad (T_h g \text{ is the translation operator: } T_h g(x) = g(x+h)).$$

Hint: Use that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ and the Minkowski inequality. Both can be used without proof.

Problem 2 (Heat equation)

Let $g_\sigma(x)$ be the Gaussian filter with filter width σ from **??**, i. e.

$$g_\sigma(x) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^d} \exp\left(\frac{-\|x\|^2}{2\sigma^2}\right).$$

1. Show that

   $$\partial_t(g_{\sqrt{2t}}) = \Delta g_{\sqrt{2t}}$$

   for all $t > 0$.

2. Let $f \in C(\Omega)$ be bounded. Show that

   $$u : (0,\infty) \times \Omega, (t,x) \mapsto u(t,x) := (f * g_{\sqrt{2t}})(x)$$

   solves the PDE

   $$\partial_t u(t,x) - \Delta_x u(t,x) = 0 \text{ in } (0,\infty) \times \Omega$$

   and that $\lim_{t \to 0} u(t,x) = f(x)$ for all $x \in \Omega$.

   Show the claimed limit without using the corresponding property of the convolution!