A. Some elements of measure theory

Definition A.1 (Measurable space)**.** Let Ω be a nonempty set, $\mathcal{P}(\Omega)$ the power set of Ω and $\mathcal{A} \subset \mathcal{P}(\Omega)$. Then, $\mathcal{A}$ is called *σ-algebra*, if the following conditions hold:

(i) $\Omega \in \mathcal{A}$.

(ii) $A \in \mathcal{A} \Rightarrow \Omega \setminus A \in \mathcal{A}$.

(iii) $A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

The pair $(\Omega, \mathcal{A})$ is called *measurable space*. The elements of $\mathcal{A}$ are called *measurable*. If Ω is a topological space, the smallest σ-algebra that contains the open sets of the topological space Ω is called *Borel algebra* and denoted by $\mathcal{B}(\Omega)$.

Definition A.2 (Positive measure)**.** Let $(\Omega, \mathcal{A})$ be a measurable space. A function $\mu : \mathcal{A} \to [0, \infty]$ is called *(positive) measure*, if it fulfills the following conditions:

(i) $\mu(\emptyset) = 0$.

(ii) If $A_1, A_2, \ldots \in \mathcal{A}$ are pairwise disjoint, then

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

The triple $(\Omega, \mathcal{A}, \mu)$ is called *measure space*. If $\mathcal{A} = \mathcal{B}(\Omega)$, then μ is called *Borel measure*. If additionally $\mu(K) < \infty$ holds for all compact sets $K \in \mathcal{B}(\Omega)$, then μ is called *positive Radon measure*. Note that there are also other definitions for term Radon measure in the literature.

If $(\Omega, \mathcal{A}, \mu)$ is a measurable space and it holds that

- there is a sequence $(\Omega_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\mu(\Omega_n) < \infty$ and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$,

μ is called *σ-finite*.

Example A.3.

(i) Let $\Omega = \mathbb{R}$ and $\mathcal{A} = \mathcal{B}(\mathbb{R})$. Then, the *counting measure*

$$\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty], A \mapsto \begin{cases} \#(A) & \text{if } A \text{ contains finitely many elements} \\ \infty & \text{else} \end{cases}$$

is a ein Borel measure, but no positive Radon measure, since $[0, 1]$ is compact but $\mu([0, 1]) = \infty$. It is aldo not σ-finite.

(ii) Let $\Omega \subset \mathbb{R}^d$ and $x \in \Omega$. Then,

$$\delta_x : \mathcal{B}(\Omega) \to [0, \infty], A \mapsto \begin{cases} 1 & x \in A \\ 0 & \text{else} \end{cases}$$

is a positive Radon measure and called *Dirac measure at x*. Moreover, δ_x is σ-finite ($\Omega_n = \Omega$).

(iii) For half-open cuboids $[a_1, b_1) \times \ldots \times [a_d, b_d)$, let

$$\mu([a_1, b_1) \times \ldots \times [a_d, b_d)) = \prod_{i=1}^{d} (b_i - a_i).$$

This can be uniquely extended to a positive Radon measure on $\mathcal{B}(\mathbb{R}^d)$. This measure is called *(d-dimensional) Lebesgue measure* and is σ-finite ($\Omega_n = B_n(0)$). For $d = 1, 2$ and 3, the Lebesgue measure coincides with the standard notion of length, area and volume, respectively.

Definition A.4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

- Let $N \subset \Omega$. If there is a set $A \in \mathcal{A}$ with $N \subset A$ and $\mu(A) = 0$, then N is called *μ-null set.*
- Let $P(x)$ be a property (e. g. convergence of a sequence of functions in a point x). If $P(x)$ holds for all $x \in A \subset \Omega$ and $\Omega \setminus A$ is a μ-null set, then we say $P(x)$ *μ-almost everywhere in* Ω.
- The σ-algebra

$$\mathcal{A}_\mu = \{B \subset \Omega \,:\, B = A \cup N \text{ with } A \in \mathcal{A}, N \text{ } \mu\text{-null set}\}$$

is called *completion of $\mathcal{A}$ with respect to μ*. The elements of $\mathcal{A}_\mu$ are called *μ-measurable.*

- The measure μ can be extended to $\mathcal{A}_\mu$ by $\mu(A \cup N) \coloneqq \mu(A)$ and then is a measure on $\mathcal{A}_\mu$.

Definition A.5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and X a Banach space. A function $f : \Omega \to X$ is called *step function*, if $f(\Omega)$ contains only finitely many elements. A step function f is called *μ-measurable*, if $f^{-1}(x) \in \mathcal{A}_\mu$ for all $x \in X$. If f is a μ-measurable step function, then its *integral* is defined as

$$\int_\Omega f \mathrm{d}\mu \coloneqq \sum_{x \in f(\Omega)} \mu(f^{-1}(x))x.$$

An arbitrary mapping $f : \Omega \to X$ is called *μ-measurable*, if there is a sequence of measurable step functions f_n that converges pointwise μ-almost everywhere to f. The integral of a μ-measurable mapping f is defined as

$$\int_\Omega f \mathrm{d}\mu \coloneqq \lim_{n \to \infty} \int_\Omega f_n \mathrm{d}\mu.$$

If $\int_\Omega \|f\| \,\mathrm{d}\mu < \infty$, then f is called *μ-integrable.*

Remark A.6. One can show that the integral is independent of the choice of the sequence f_n. Thus, the integral is well defined. Moreover, it fulfills the typical properties one would expect from an integral, e. g. linearity, monotonicity and that the norm of an integral is smaller than the integral of the point-wise norm of a function.

Definition A.7. Let $(\Omega, \mathcal{A})$ and $(\Sigma, \mathcal{B})$ be a measurable spaces. A function $f : \Omega \to \Sigma$ is called *$\mathcal{A}$-$\mathcal{B}$ measurable* or just *measurable*, if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.

Proposition A.8 (Push-forward measure)**.** *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space and* $(\Sigma, \mathcal{B})$ *be a measurable space. If* $f : \Omega \to \Sigma$ *is* $\mathcal{A}$-$\mathcal{B}$ *measurable,*

$$\nu : \mathcal{B} \to [0, \infty], B \mapsto \mu(f^{-1}(B))$$

is a measure on $\mathcal{B}$*.* ν *is called* push-forward measure *or* image measure *and denoted by* $f(\mu)$*.*

Proof. Since f is $\mathcal{A}$-$\mathcal{B}$ measurable, ν is well defined. Moreover, we have

$$\nu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0.$$

Let $B_1, B_2, \ldots \in \mathcal{B}$ be pairwise disjoint. Since the preimage fulfills

$$f^{-1}(M \cap N) = f^{-1}(M) \cap f^{-1}(N) \text{ for all } M, N \subset \Sigma,$$

the $f^{-1}(B_1), f^{-1}(B_2), \ldots \in \mathcal{A}$ are also pairwise disjoint. We get

$$\nu\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \mu\Big(f^{-1}\Big(\bigcup_{n\in\mathbb{N}} B_n\Big)\Big) = \mu\Big(\bigcup_{n\in\mathbb{N}} f^{-1}(B_n)\Big) = \sum_{n\in\mathbb{N}} \mu(f^{-1}(B_n)) = \sum_{n\in\mathbb{N}} \nu(B_n). \quad \square$$

Proposition A.9. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space and* $(\Sigma, \mathcal{B})$ *be a measurable space. If* $f : \Omega \to \Sigma$ *is* $\mathcal{A}$-$\mathcal{B}$ *measurable and* $g : \Sigma \to X$ *is* $f(\mu)$*-integrable, then*

$$\int_\Sigma g \mathrm{d}f(\mu) = \int_\Omega (g \circ f) \mathrm{d}\mu.$$

Proof. Let $B \in \mathcal{B}$. Then,

$$\begin{aligned}\int_\Sigma \chi_B \mathrm{d}f(\mu) &= \sum_{x\in\chi_B(\Sigma)} f(\mu)(\chi_B^{-1}(x))x = f(\mu)(\chi_B^{-1}(1)) = f(\mu)(B) = \mu(f^{-1}(B)) \\ &= \mu(\chi_{f^{-1}(B)}^{-1}(1)) = \int_\Omega \chi_{f^{-1}(B)} \mathrm{d}\mu = \int_\Omega (\chi_B \circ f) \mathrm{d}\mu.\end{aligned}$$

Thus, the statement is true for characteristic functions. Due to the linearity of the integral, it also follows for step functions. Now let $g_n : \Sigma \to X$ be a sequence of measurable step functions that converges pointwise $f(\mu)$-almost everywhere to g. Then, we get

$$\int_\Sigma g \mathrm{d}f(\mu) = \lim_{n\to\infty} \int_\Sigma g_n \mathrm{d}f(\mu) = \lim_{n\to\infty} \int_\Omega (g_n \circ f) \mathrm{d}\mu = \int_\Omega (g \circ f) \mathrm{d}\mu.$$

Here, we have used that $g_n \circ f : \Omega \to X$ is a sequence of measurable step functions that converges pointwise μ-almost everywhere to $g \circ f$. $\square$

Definition A.10 (Distributions)**.** Let X be a normed vector space with norm $\|\cdot\|$. The set

$$X' \coloneqq \big\{x' : X \to \mathbb{R} \,:\, x' \text{ linear and continuous wrt. } \|\cdot\|\big\}$$

is called *(topological) dual space* of X.

Let $\Omega \subset \mathbb{R}^d$. The elements of $(C_c^\infty(\Omega))'$, i. e. the continuous, linear mappings from $C_c^\infty(\Omega)$ (cf. Definition B.1) to $\mathbb{R}$, are called *distributions*.

Remark A.11. Every function $g \in L^1(\Omega)$ can be interpreted as distribution using

$$C_c^\infty(\Omega) \to \mathbb{R}, f \to \int_\Omega fg \,\mathrm{d}x\,.$$

This distribution is called *induced* by g. In this sense, distributions are sometimes also called *generalized functions*. Distributions that are induced by functions are called *regular distributions*. Moreover, every positive Radon measure μ induces a distribution by

$$C_c^\infty(\Omega) \to \mathbb{R}, f \to \int_\Omega f \mathrm{d}\mu.$$

In general, such distributions are not regular.

The distribution induced by the Dirac measure δ_x is called *Dirac delta function centered at* x. For $f \in C_c^\infty(\Omega)$, we have (exercise)

$$\delta_x(f) = \int_\Omega f \mathrm{d}\delta_x = f(x).$$

Theorem A.12 (Riesz representation theorem)**.** *Let X be a Hilbert space. For all $x' \in X'$ there is a $y \in X$ such that*

$$x'(x) = (x,y)_X \text{ for all } x \in X$$

(cf. [1, Theorem 2.35]).

B. Important spaces and statements

Definition B.1. Let $\Omega \subset \mathbb{R}^d$. Then,

$$\operatorname{supp}(g) \coloneqq \overline{\{x \in \Omega \,:\, g(x) \neq 0\}}$$ intersection of sets of which elements does not result in g(x) = 0

is called the *(closed) support* of g.

Remark B.2. The spaces of continuous and continuously differentiable functions are denoted by

$$\begin{aligned} C^0(\Omega) &\coloneqq C(\Omega) \coloneqq \{f : \Omega \to \mathbb{R} \,:\, f \text{ continuous}\}\,, \\ C^m(\Omega) &\coloneqq \{f : \Omega \to \mathbb{R} \,:\, f\ m\text{-times continuously differentiable}\}\,, \\ C^\infty(\Omega) &\coloneqq \bigcap_{m=1}^{\infty} C^m(\Omega). \end{aligned}$$

The space of continuous functions with compact support is denoted by

$$C_c^0(\Omega) \coloneqq C_c(\Omega) \coloneqq \left\{f \in C^0(\Omega) \,:\, \operatorname{supp} f \subset \Omega \text{ and } \operatorname{supp} f \text{ compact}\right\}.$$

$C_c^m(\Omega)$ and $C_c^\infty(\Omega)$ are defined analogously.

Definition B.3 (Lebesgue spaces)**.** For $\Omega \subset \mathbb{R}^d$ μ-measurable and $1 \le p \le \infty$, the Lebesgue spaces $L^p(\Omega)$ is defined as

$$L^p(\Omega) \coloneqq \left\{y : \Omega \to \mathbb{R} \,:\, y\ \mu\text{-measurable } \wedge \|y\|_{L^p(\Omega)} < \infty\right\}$$

with the equivalence relation

$$f = g \text{ in } L^p(\Omega) :\Leftrightarrow f = g \ \mu\text{-almost everywhere.}$$

Here,

$$\|y\|_{L^p(\Omega)} := \left(\int_\Omega |y|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \text{ for } 1 \le p < \infty, \quad \|y\|_{L^\infty(\Omega)} := \inf_{\mu(N)=0} \sup_{x \in \Omega \setminus N} |y(x)|$$

and μ is the Lebesgue measure on $\mathbb{R}^d$.

Remark B.4. With the usual addition and scaling of mappings, $L^p(\Omega)$ is a vector space and $\|\cdot\|_{L^p(\Omega)}$ is a norm on $L^p(\Omega)$. The triangle inequality for the L^p-norm is also called *Minkowski inequality.* On can show that L^p is complete with respect to this norm (Riesz–Fischer theorem).

Lemma B.5 (Hölder's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be μ-measurable and $1 \le p, q \le \infty$ with $1 = \frac{1}{p} + \frac{1}{q}$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, we have $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} .$$

In particular, for $1 < p, q < \infty$, we have

$$\int_\Omega |f(x)g(x)| \, \mathrm{d}x \le \left(\int_\Omega |f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \, \mathrm{d}x\right)^{\frac{1}{q}} .$$

Theorem B.6 (Fubini's theorem)**.** *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ both be μ-measurable. If $f \in L^1(X \times Y)$, then*

$$\int_X \left(\int_Y f(x,y) \, \mathrm{d}y\right) \mathrm{d}x = \int_Y \left(\int_X f(x,y) \, \mathrm{d}x\right) \mathrm{d}y = \int_{X \times Y} f(x,y) \mathrm{d}(x,y)$$

(cf. [1, Theorem 2.66 2.]).

Theorem B.7 (Substitution rule for multiple variables)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and $\phi : \Omega \to \mathbb{R}^d$ be a diffeomorphism, i. e. a bijective, continuously differentiable mapping whose inverse is also continuously differentiable. Let $f : \phi(\Omega) \to \mathbb{R}$ be measurable. Then f is integrable on $\phi(\Omega)$ if and only if the mapping*

$$\Omega \to \mathbb{R}, x \mapsto f(\phi(x)) \left|\det(D\phi(x))\right|$$

is integrable on Ω. In this case, we have

$$\int_{\phi(\Omega)} f(y) \, \mathrm{d}y = \int_\Omega f(\phi(x)) \left|\det(D\phi(x))\right| \mathrm{d}x$$

(cf. [1, Theorem 2.69]).

Proposition B.8. *$C_c(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$, i. e. for all $f \in L^p(\Omega)$ and $\epsilon > 0$, there is $g \in C_c(\Omega)$ with $\|f - g\|_{L^p(\Omega)} < \epsilon$ (cf. [1, Theorem 2.55]).*

Definition B.9. For measurable $\psi : \Omega \to \mathbb{R}$ and $f : \mathbb{R}^d \to \mathbb{R}$,

$$(\psi * f)(x) = \int_\Omega \psi(y) f(x-y)\,\mathrm{d}y$$

is called *convolution* of f and ψ at position $x \in \mathbb{R}^d$.

Proposition B.10 (Properties of the convolution)**.**

(i) Let $1 \leq p, q, r \leq \infty$ with $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$, $f \in L^p(\Omega)$ and $g \in L^q(\mathbb{R}^d)$. Then, $(f * g) \in L^r(\mathbb{R}^d)$ and

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\mathbb{R}^d)}.$$

This is called Young's convolution inequality.

If $\Omega = \mathbb{R}^d$, we have $(f * g)(x) = (g * f)(x)$ for all $x \in \mathbb{R}^d$ and

$$\|f * g\|_{L^r(\mathbb{R}^d)} = \|g * f\|_{L^r(\mathbb{R}^d)}.$$

(ii) Let $\psi \in C_c^k(\mathbb{R}^d)$ and $f \in L^p(\Omega)$ with $1 \leq p \leq \infty$. Then, $(f * \psi) \in C^k(\mathbb{R}^d)$. If α is a multi-index with $|\alpha| \leq k$, we have

$$\frac{\partial^\alpha}{\partial x^\alpha}(f * \psi) = f * \frac{\partial^\alpha}{\partial x^\alpha}\psi.$$

If $\Omega = \mathbb{R}^d$, we also have

$$\frac{\partial^\alpha}{\partial x^\alpha}(\psi * f) = \frac{\partial^\alpha}{\partial x^\alpha}\psi * f.$$

(iii) Let $\psi \in L^1(\mathbb{R}^d)$ be such that $\psi \geq 0$ and $\int_{\mathbb{R}^d} \psi(x)\,\mathrm{d}x = 1$. Moreover, for $\epsilon > 0$, let

$$\psi_\epsilon : \mathbb{R}^d \to \mathbb{R}, x \mapsto \tfrac{1}{\epsilon^d}\psi(\tfrac{x}{\epsilon})$$

and let $f \in L^\infty(\mathbb{R}^d)$. If f is continuous in $x \in \mathbb{R}^d$, we have

$$\lim_{\epsilon \to 0}(\psi_\epsilon * f)(x) = f(x).$$

If f is uniformly continuous, $\psi_\epsilon * f$ converges to f on each compact subset of $\mathbb{R}^d$ uniformly.

The properties of the convolution can be shown analogously to the properties cross-correlation in Proposition 2.3. There are just some minor sign differences.

Theorem B.11 (Dominated convergence theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable. Moreover, let $f_n : \Omega \to \mathbb{C}$ be a sequence of μ-measurable functions that converges pointwise almost everywhere in Ω to a measurable function $f : \Omega \to \mathbb{C}$. If there is $g \in L^1(\Omega)$ such that for all $n \in \mathbb{N}$, it holds that*

$$|f_n(x)| \leq g(x) \text{ for almost all } x \in \Omega,$$

then $f_n, f \in L^1(\Omega, \mathbb{C})$,

$$\lim_{n\to\infty}\int_\Omega |f_n - f|\,\mathrm{d}x = 0 \text{ and } \lim_{n\to\infty}\int_\Omega f_n\,\mathrm{d}x = \int_\Omega f\,\mathrm{d}x$$

(cf. [1, Theorem 2.47]). Roughly speaking, if there is an integrable majorant for a pointwise a.e. convergent sequence, integration and limit can be interchanged.

Corollary B.12. *Let $X \subset \mathbb{R}$ be open and let $\Omega \subset \mathbb{R}^d$ be measurable. Moreover, let $f : X \times \Omega \to \mathbb{C}$ be such that*

(i) $\omega \mapsto f(x, \omega)$ *is in* $L^1(\Omega, \mathbb{C})$ *for each* $x \in X$

(ii) $x \mapsto f(x, \omega)$ *is differentiable for almost all* $\omega \in \Omega$

(iii) there is a $g \in L^1(\Omega)$ *with* $|\partial_x f(x, \omega)| \le g(\omega)$ *for all* $x \in X$ *and almost all* $\omega \in \Omega$*,*

then, for all $x \in X$*,*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_\Omega f(x, \omega) \,\mathrm{d}\omega = \int_\Omega \partial_x f(x, \omega) \,\mathrm{d}\omega \,.$$

Roughly speaking, if the derivative is uniformly bounded by an integrable function, differentiation and integration may be interchanged.

Proof. Let $x \in X$ be arbitrary, but fixed. We consider the sequence

$$f_n : \Omega \to \mathbb{C}, \omega \mapsto \begin{cases} n(f(x + \frac{1}{n}, \omega) - f(x, \omega)) & x + \frac{1}{n} \in X \\ 0 & \text{else} \end{cases} .$$

Since X is open, there is $N \in \mathbb{N}$ such that $x + \frac{1}{n} \in X$ for all $n \ge N$. Thus, we have

$$\lim_{n \to \infty} f_n(\omega) = \lim_{n \to \infty} n(f(x + \tfrac{1}{n}, \omega) - f(x, \omega)) = \partial_x f(x, \omega)$$

for almost all $\omega \in \Omega$.

By the mean value theorem, for every $x \in X$, $n \ge N$ and almost all $\omega \in \Omega$, there is a $\zeta(x, n, \omega) \in (x, x + \frac{1}{n}) \subset X$ such that

$$n(f(x + \tfrac{1}{n}, \omega) - f(x, \omega)) = \partial_x f(\zeta(x, n, \omega), \omega).$$

Thus, we get

$$|f_n(\omega)| = |\partial_x f(\zeta(x, n, \omega), \omega)| \le g(\omega).$$

That means we have an integrable majorant for a pointwise a.e. convergent sequence, such we can apply the dominated convergence theorem (note that it is sufficient if the conditions on f_n hold for $n \ge N$), which implies

$$\begin{aligned} \int_\Omega \partial_x f(x, \omega) \,\mathrm{d}\omega &= \lim_{n \to \infty} \int_\Omega f_n(x, \omega) \,\mathrm{d}\omega = \lim_{n \to \infty} n \left(\int_\Omega f(x + \tfrac{1}{n}, \omega) \,\mathrm{d}\omega - \int_\Omega f(x, \omega) \,\mathrm{d}\omega \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \int_\Omega f(x, \omega) \,\mathrm{d}\omega \,. \end{aligned}$$

□

Corollary B.13 (Dominated convergence in L^p)**.** *Let $\Omega \subset \mathbb{R}^d$ be measurable. Moreover, let $f_n : \Omega \to \mathbb{C}$ be a sequence of μ-measurable functions that converges pointwise almost everywhere in Ω to a measurable function $f : \Omega \to \mathbb{C}$. If there is $g \in L^p(\Omega)$ for $p \in 1 \le p < \infty$ such that for all $n \in \mathbb{N}$, it holds that*

$$|f_n(x)| \le g(x) \text{ for almost all } x \in \Omega,$$

then $f_n, f \in L^p(\Omega, \mathbb{C})$*,*

$$\lim_{n \to \infty} \int_\Omega |f_n - f|^p \,\mathrm{d}x = 0.$$

Proof. We consider the sequence $g_n := |f_n - f|^p$. Since f_n converges pointwise a.e. to f, g_n converges pointwise a.e. to 0. Noting that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and the pointwise convergence of f_n to f implies $|f(x)| \leq g(x)$ almost everywhere, we get

$$|f_n(x) - f(x)| \leq |f_n(x)| + |f(x)| \leq 2g(x) \;\Rightarrow\; g_n(x) = |f_n(x) - f(x)|^p \leq (2g(x))^p.$$

Thus, $(2g(x))^p \in L^p(\Omega)$ is an integrable majorant for g_n, and Theorem B.11 gives

$$0 = \lim_{n\to\infty} \int_\Omega |g_n - 0| \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega |f_n - f|^p \,\mathrm{d}x \,. \qquad \square$$

Proposition B.14 (Integrals of rotationally symmetric functions)**.** *Let $f : [0,\infty) \to \mathbb{R}$ be measurable. Then, the rotationally symmetric function*

$$\mathbb{R}^d \to \mathbb{R}, x \mapsto f(\|x\|_2)$$

is in $L^1(\mathbb{R}^d)$, if and only if the function

$$[0,\infty) \to \mathbb{R}, r \mapsto f(r) r^{d-1}$$

is in $L^1([0,\infty))$. If this is the case, then

$$\int_{\mathbb{R}^d} f(\|x\|_2) \,\mathrm{d}x = d\tau_d \int_{[0,\infty)} f(r) r^{d-1} \mathrm{d}r.$$

Here, $\tau_d = \mathrm{Vol}(B_1(0))$ is the volume of the d-dimensional unit ball.