$\beta = 1/2$

$\gamma, \delta \quad \left.\begin{matrix}\gamma = 1 \\ \delta = 3\end{matrix}\right\}$

$$f_{MF} = a + btM^2 + cM^4 + dHM$$

$$f_{MF} = \frac{1}{2} JM^2 - \frac{1}{\beta} \ln\left[\cosh(\beta[H + JM])\right] \quad \ldots (1)$$

$$\frac{\partial f}{\partial M} = 0 \Rightarrow M = \tanh\left[\beta(JM + H)\right]$$

$$\boxed{T_c = J/K_B}, \; H = 0$$

$$t = \frac{T - J/K_B}{J/K_B} = \frac{K_B T - J}{J}$$

start from $M = 0$

$$G(r) = \langle s(r)\, s(0) \rangle \qquad \underline{\underline{H = 0}}$$

$$= \left.\frac{\mathrm{Tr}\left[s(r)\, s(0) \exp[-\beta H]\right]}{\mathrm{Tr}\left[e^{-\beta H}\right]}\right\}$$

$$\mathrm{Tr} = \sum_{s(1)=\pm 1} \sum_{s(2)=\pm 1} \cdots\cdots \sum_{s(N)=\pm 1}$$

$$\left.\begin{aligned} -\beta H = \beta \sum_{r,r'} \frac{J(r-r')}{2} s(r)\, s(r') \\ + \beta H \sum_{r} s(r) \end{aligned}\right\}$$

$$s'(r) = -s(r) \qquad \forall r$$

$$Q = \sum_{s(1)=\pm 1} \cdots \sum_{s(N)=\pm 1} s(r) \exp[-\beta H] / Z$$

$$+ \sum_{s'(1)=\pm 1} \cdots \sum_{s'(N)=\pm 1} (-s'(r))(-1) \frac{\exp[-\beta H]}{Z}$$

$$\langle s(r) s(0) \rangle = \frac{\sum_{(i\neq 0)\{s(i)=\pm 1\}} s(r)\, e^{-\beta H[\{s(r)\}]}}{Z}, \quad s(0)=+1$$

$$= M(r)$$

For the calculation of $G(r)$,
let $M(r)$ depend on r.

$$s(r) = M(r) + \delta s(r)$$

$$\delta s(r) = s(r) - M(r)$$

$$\sum_{r,r'} \frac{J(r-r')}{2} s(r) s(r')$$

$$= \sum_{r,r'} \frac{J(r-r')}{2} [M(r)+\delta s(r)] [M(r')+\delta s(r')]$$

$$\approx \sum_{r,r'} \frac{J(r-r')}{2} \Big[M(r) M(r') + [s(r) - M(r)] M(r') + M(r) [s(r') - M(r')] \Big]$$

$$\approx -\sum_{r,r'} \frac{J(r-r')}{2} M(r) M(r')$$

$$+ \sum_{r, r'} \frac{J(r-r')}{2} \left\{ \delta(r)\, M(r') + \delta(r')\, M(r) \right\}$$

$$J(r-r') \equiv J(r'-r)$$

(2) ... $$\boxed{M(r) = \tanh\left[\beta\left(\sum_{r'} J(r-r')\, M(r')\right)\right]}$$

$$\sum_{ij} J_{ij} M_j = \sum_i \left(\sum_j J_{ij} M_j\right)$$

$$M_j = M \quad \forall j$$

$$= M \sum_i \underbrace{\sum_j J_{ij}}_{J}$$

$$= MJN$$

$$\rightarrow M(r) = \tanh\left[\beta \sum_{r'} J(r-r')\, M(r')\right]$$

if $M(r') = M \quad \forall r'$,

$$M = \tanh\left[\beta M \sum_{r'} J(r-r')\right]$$

$$= \tanh\left[\beta M J\right]$$

$$\left(M(r) \approx \beta \sum_{r'} J(r-r')\, M(r')\right) \boxed{+ \text{ correction terms}}$$

<u>Fourier transform</u>

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(*) ... $M(p) = \frac{1}{N}\sum_{k} e^{-ikp}\,\tilde{M}(k)$

$\overrightarrow{p-p'} = \rho$ $\quad J(p-p') = \frac{1}{N}\sum_{k_1} e^{-ik_1(p-p')}\,\tilde{J}(k_1)$ (**)

$$\sum_{p'} e^{i(k_1-k_2)p'} = \delta_{k_1,k_2}\,N$$

$$\left[\delta_{i,j} = 1 \text{ if } i=j,\quad = 0 \text{ if } i\neq j\right]$$

$$\text{RHS} = \beta\sum_{p'} J(p-p')\,M(p')$$

$$= \beta\sum_{k_1}\sum_{k}\frac{1}{N^2}\sum_{p'} e^{-ik_1(p-p')}\,\tilde{J}(k_1) \times e^{-ikp'}\,\tilde{M}(k)$$

$$= \beta\sum_{k_1}\sum_{k}\frac{1}{N^2}\,\tilde{J}(k_1)\,\tilde{M}(k)\,\boxed{e^{-ik_1 p}}\ \underbrace{\left[\sum_{p'} e^{+i(k_1-k)p'}\right]}_{N\delta_{k_1,k}}$$

$$= \frac{\beta}{N}\sum_{k}\tilde{J}(k)\,\tilde{M}(k)\,\boxed{e^{-ikp}}$$

$$\text{LHS} = M(p) = \frac{1}{N}\sum_{k} e^{-ikp}\,\tilde{M}(k)$$

$$\frac{1}{N}\sum_{k} e^{-ikp}\,\underline{\underline{\tilde{M}(k)}} = \frac{\beta}{N}\sum_{k} e^{-ikp}\,\underline{\underline{\tilde{J}(k)\,\tilde{M}(k)}}$$

$$\boxed{\tilde{M}(k) = \beta \tilde{J}(k)\tilde{M}(k)} \quad \forall k \;\ldots (2)$$
$$+ \text{ const.}$$

$$G(r) \longmapsto M(r) \xrightarrow[\text{F.T.}]{} \tilde{M}(k)_{MFT}$$
$$\downarrow$$
$$G(k)$$

$$\frac{1}{N}\sum_{p} e^{+iqp}$$

[Use $J(r-r') = J(r'-r)$ to connect $\tilde{J}(k)$ and $\tilde{J}(-k)$].

$$\tilde{J}(-k) = [\tilde{J}(+k)]^*$$

If $\tilde{J}$ is real, $\tilde{J}$ = even function of k

Expand $\tilde{J}(k)$ for small k, about $k \approx 0$.

$$\lim_{r \to \infty} G(r) \qquad e^{ikr}$$

$$\tilde{J}(k) \approx J(1 - R^2k^2) + O(k^4)$$

$$\tilde{M}(k) = \beta J(1-R^2k^2)\tilde{M}(k) + \text{const.}$$

$$\tilde{M}(k) \approx \frac{\text{const.}}{[1-\beta J(1-R^2k^2)]}$$

$$e^{-r/\xi_c}$$

$$G(r) \equiv M(r) \sim \frac{\cdots}{r} \xrightarrow{\text{H.W.}} \frac{1}{k^2+\xi_c^{-2}}$$

$$\int dr\, \frac{e^{ikr}\, e^{-r/\xi_c}}{r} \sim \left(ik - \frac{1}{\xi_c}\right) r$$

$$\frac{\text{const.}}{1-\frac{T_c}{T}\left(1-R^2k^2\right)}$$

$$\approx \frac{\text{const.}\; R^{-2}}{k^2+\frac{1}{\xi_c^2}}$$

$$\xi_c \approx R\, t^{-1/2}$$

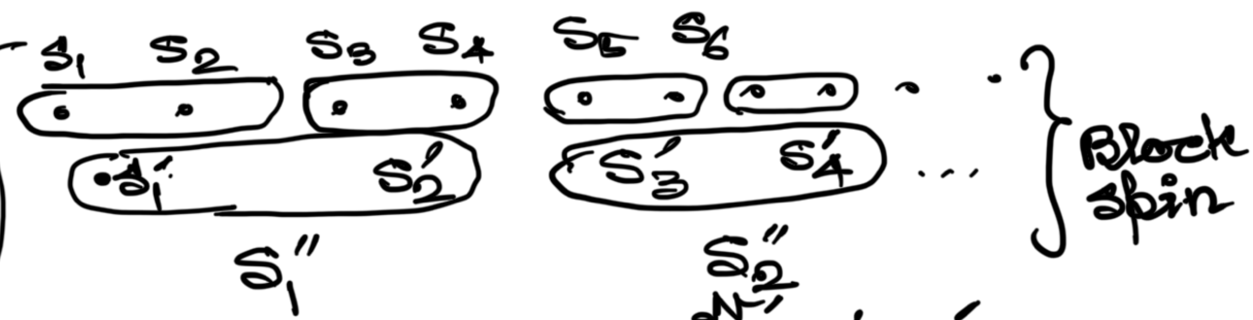
$$\approx t^{-\nu} \qquad \boxed{\nu = \tfrac{1}{2}}$$

Renormalization Group

$$H = -K \sum_{i=1}^{N} S_i S_{i+1}, \quad K = \beta J, \quad \beta = \frac{1}{k_B T}$$

Block Spin

RG



e.g., $$H(\{s'\}) = -K' \sum_{i=1}^{N'} s_i' s_{i+1}' - \lambda \sum_{i=1}^{N'} s_i' s_{i-1}' s_{i+1}'$$

$\vdots$

$W(K)$ $Q(K)$

$K'(K)$, $\lambda(K)$

critical phenomenon $\Longleftrightarrow$ long-distance physics

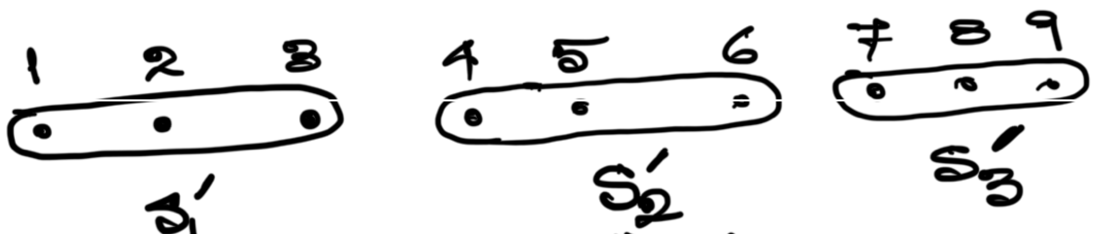
$H(\beta, J_g)$

$s \longmapsto s'$

$$\mathcal{H}(\{S\}) \xrightarrow{R.G.} \mathcal{H}'(\{S'\})$$

$$Z = \operatorname*{Tr}_{s} e^{-\beta \mathcal{H}(\{s\})}$$

Reduced Hamiltonian



'Majority spin-method'

Projection operator:

$$T(s_1'; s_1, s_2, s_3) = 1 \text{ if } s_1' \left[\sum_{i=1}^{3} s_i\right] > 0$$

$$= 0 \quad \text{otherwise}$$

$$e^{-\mathcal{H}'(s')} \equiv \operatorname*{Tr}_{s} \prod_{\text{blocks}} T(s'; \{s_i\})\, e^{-\mathcal{H}(s)}$$

$$\boxed{\sum_{s'} T(s'; s_i) = 1}$$

$$\operatorname*{Tr}_{s} \equiv \sum_{\substack{s_i = \pm 1 \\ i = 1 \ldots N}}$$

$$\sum_{s'} e^{-\mathcal{H}'(\{s'\})}$$

$$-\mathcal{H}(\{s\}) \qquad \operatorname*{Tr}_{s'} \equiv \sum_{s_i' = \pm 1}$$

$$\equiv \mathrm{Tr}_{s}\, e^{\cdots} \qquad {}_{i=1\ldots N}$$

$$= \sum_{s} e^{-\mathcal{H}(\{s\})}$$

$$Z(\{s'\}) = Z(\{s\})$$

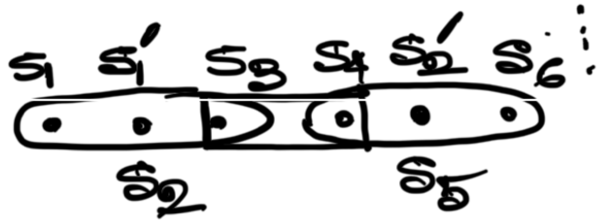
1-d Ising Model

$$\mathcal{H} = -\sum_{i=1}^{N} K s_i s_{i+1}$$

Decimation Rule :-

$$T(s'; s_1, s_2, s_3) = \delta_{s', s_2}$$

$$T(s'; s_4, s_5, s_6) = \delta_{s', s_5}$$



$$\ln Z, \ldots \left[\sum_{\substack{s_3 = \pm 1 \\ s_4 = \pm 1}} e^{K s_1' s_3} e^{K s_3 s_4} e^{K s_4 s_2'}\right] \ldots,$$

$$e^{K s_3 s_4} = \cosh K + \sinh K \; s_3 s_4$$

$$x = \tanh K \qquad = \cosh K \left(1 + (\tanh K) s_3 s_4\right)$$

$$\sum_{s_3 = \pm 1} \sum_{s_4 = \pm 1} (\cosh K)^3 (1 + x s_1' s_3)(1 + x s_3 s_4)(1 + x s_4 s_2')$$

$$(1 + x s_4 s_2)$$

$$= \sum_{\substack{s_3=\pm 1 \\ s_4=\pm 1}} (\cosh K)^3 \left[1 + x s_3 s_4 + x s_1' s_3 + x^2 s_1' s_4\right] (1 + x s_4 s_2')$$

$$= (\cosh K)^3 \sum_{\substack{s_3=\pm 1 \\ s_4=\pm 1}} \Big\{ 1 + \cancel{x s_4 s_2'} + \cancel{x s_3 s_4} + \cancel{x^2 s_3 s_2'} + \cancel{x s_1' s_3} + \cancel{x^2 s_1' s_2' s_3 s_4} + \cancel{x^2 s_1' s_4} + x^3 s_1' s_2' \Big\}$$

$$= \cdots\cdots\; 2^2 (\cosh K)^3 (1 + x^3 s_1' s_2') \;\cdots$$

$$e^{K' s_1' s_2'} = \left[(\cosh K')\left(1 + (\tanh K') s_1' s_2'\right)\right]$$

$$H = \cdots + (-K') s_1' s_2' + \cdots$$

$$(\cosh K)^3 = \cosh K'$$

$$\tanh K' = [\tanh K]^3$$

$$K' = \tanh^{-1}\left[\tanh^3 (K)\right]$$

$$x' = x^3$$

$$x' = \tanh K'$$
$$x = \tanh K$$

$$x_1 = x_0^3$$
$$x_2 = x_1^3 = x_0^9$$
$$x_3 = x_2^3 = (x_0^9)^3$$
$$= x_0^{27}$$

$$\sum^{N/3} K' s_i' s_j' + \text{const.}$$

$$\mathcal{H} = \sum_{i=1} -K\, s_i s_{i+1} + \text{const.}$$

$$Z = \mathrm{Tr}_{s'} \left[e^{-\mathcal{H}(s')} \right]$$

H.W: $\mathcal{H}'(s') = N' g(K') - \sum_{i=1}^{N'} K' s'_i s'_{i+1}$

$$g(K') = \frac{-1}{3} \ln \left[\frac{\cosh^3 K}{\cosh K'} \right] - \frac{2}{3} \ln 2$$

$x = \tanh(K)$

$$x' = x^3$$

High T; $K = J/k_B T$

$$x = \tanh(J/k_B T)$$

$$x \longmapsto 0^+$$

Low T; $x \longrightarrow 1^-$

Zero Temperature : $T = 0$

$x = 1$

fixed point for K : $T = 0$

ordered state

$0 < x < 1$ If $x_{\text{initial}} < 1$

$$x_{\text{final}} = 0$$

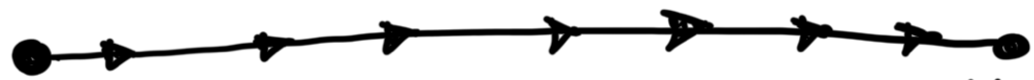
For any $K < \infty$, eventually $K \longmapsto 0$

Only for $T = 0$, Ferromagnet

$T > 0$, paramagnet

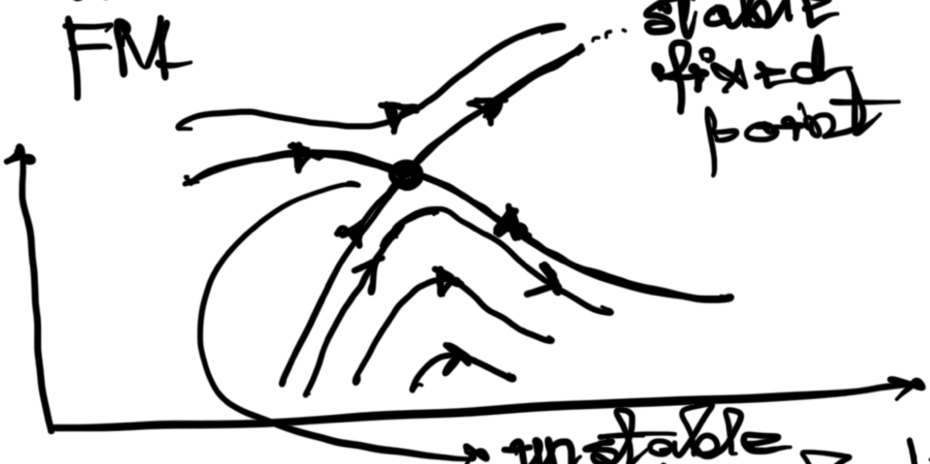
$\boxed{x' = x^3}$ → stable: paramag-net

→ unstable: FM



$K = \infty$
$x = 1$
$T = 0$
unstable
FM

$K = 0$
$x = 0$
$T \to \infty$
stable
PM



Correlation length ξ_c :

$[\xi]$ $\qquad \xi^\# = \xi / a$
(dimensionless)

$a' = ba$

$\boxed{b = 3}$

$$\xi(x') = \xi(x)$$

$$\xi^\#(x')\, a' = \xi^\#(x)\, a$$

or $$\xi^\#(x') = \frac{1}{b}\,\xi^\#(x)$$

$\boxed{x' = x^b}$ $\qquad \boxed{\xi^\#(x') = \frac{1}{b}\,\xi^\#(x)}$

$$\xi^*(x) = \frac{\text{const.}}{\ln x}$$

$$\xi^*(x') = \frac{\text{const.}}{\ln x'}$$

$$= \frac{1}{b} \cdot \frac{\text{const}}{\ln x}$$

$$= \frac{1}{b} \xi^*(x)$$

$$\xi^* = \frac{\text{const.}}{\ln[\tanh(K)]}$$

FM critical point

$\xi^* \longmapsto \infty$

$K \longmapsto \infty$

$x = +1$

<u>Monte Carlo</u>

Each spin : k

$k^N \doteq e^{N \ln k}$

cannot efficiently sample <u>all</u> config.

$P(C) = e^{-\beta E_c} / Z$

- Do not sample high-E_c configurations at low T / high β.

$$M = \langle s_j \rangle = \frac{1}{N}\sum_i \langle s_i \rangle$$

$$= \frac{1}{N}\sum_i \left[\sum_c e^{-\beta E_c} s_i / Z \right]$$

$$= \sum_c \frac{e^{-\beta E_c}}{Z} \left[\sum_i \frac{(s_i)_c}{N} \right]$$

$$= \sum$$

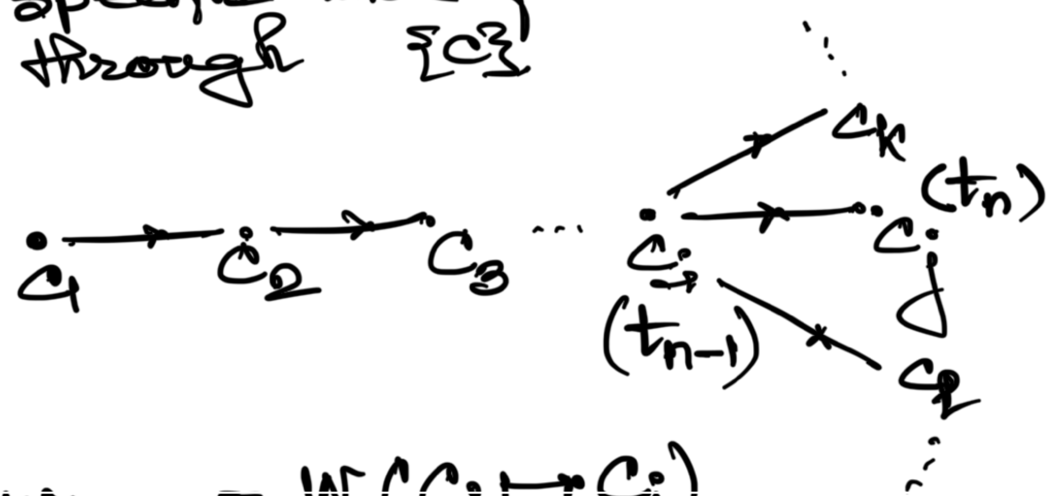
$$\cancel{M =} \quad \langle \theta \rangle = \sum_c p_c \theta_c$$

$$= \sum_c \left(\frac{e^{-\beta E_c}}{Z} \right) \theta_c$$

$$\frac{N_c}{N_{tot}} = e^{-\beta E_c}/Z \quad \text{(Monte Carlo)}$$

'Importance sampling'

- specific kind of movement through $\{c\}$



$$W_{i \to j} = W(C_i \to C_j)$$

$$= P(X_{t_n} = C_j \mid X_{t_{n-1}} = C_i)$$

$$P(X_{t_n} = C_j) \equiv P(C_j, t)$$

$$\frac{dP(C_j,t)}{dt} = -\sum_i W_{j\mapsto i}\, P(C_j,t) + \sum_i W_{i\mapsto j}\, P(C_i,t)$$

: Equilibrium $\frac{dP(C_j,t)}{dt} = 0$

$$P_{eq}(C_j) = e^{-\beta E_{C_j}}/Z$$

$$\frac{dP_{eq}}{dt} = 0 \Rightarrow -\sum_i W_{j\mapsto i}\, P_{eq}(C_j) + \sum_i W_{i\mapsto j}\, P_{eq}(C_i) = 0 \quad (*)$$

<u>Detailed Balance</u>:
(sufficient, not necessary)

$$W_{i\mapsto j}\, P_{eq}(C_i) = W_{j\mapsto i}\, P_{eq}(C_j)$$

$$\frac{W_{i\mapsto j}}{W_{j\mapsto i}} = \frac{P_{eq}(C_j)}{P_{eq}(C_i)}$$

$$= e^{-\beta E_{C_j}}/e^{-\beta E_{C_i}}$$

$\Delta E = E_{C_j} - E_{C_i}$

$$\ldots = e^{-\beta(E_{c_j} - E_{c_i})} \quad \ldots (!)$$

$$W_{j \to i} = T_0^{-1} e^{-\beta(E_{c_j} - E_{c_i})} \text{ o/w},$$

$$= T_0^{-1} \quad \text{if } E_{c_i} < E_{c_j}$$

$$\text{or } \Delta E > 0$$

Metropolis algorithm:

$$W_{i \to j} = 1 \quad \text{if } E_{c_i} > E_{c_j}$$

$$= \exp[-\beta(E_{c_j} - E_{c_i})]$$

$$\text{if } E_{c_j} > E_{c_i}$$

$$\boxed{T_0 = T_0^{-1} = 1}$$

Metropolis algorithm :

- Start with some configuration I_0.
- Choose some site i
- If σ_i is flipped, then calculate $\Delta E = E_F - E_I$

FE

- If $E_F < E_I$, go to C_{new}
- If $E_F > E_I$, go to C_{new} with $p = \exp[-\beta \Delta E]$

[· Generate a random # p, $0 < p < 1$, $p \sim \mathcal{U}(0,1)$]

· If $p < \exp[-\beta \Delta E]$, go to

$<$new.

o/w, stay on C_{I}.]

- Move to next site.

exponential in N $\longrightarrow$ linear in N

exact $\longrightarrow$ convergent, but statistical

$$M_f = \sum_{c=1}^{N_{tot}} s_{f,c} / N_{tot} = \langle s_f \rangle$$

$$M = \sum_{i=1}^{N_s} \langle s_i \rangle / N_s$$

$$= \frac{1}{N_{tot}} \sum_{c=1}^{N_{tot}} \left[\frac{1}{N_s} \sum_{i=1}^{N_s} s_{i,c} \right]$$

<u>H.W</u> : simulate, by Metropolis algorithm, a d=2 Ising Model at temperature T.

$$C_V = \frac{\partial \langle E \rangle}{\partial T} \propto \langle E^2 \rangle - \langle E \rangle^2$$

$$\chi = \frac{\partial M}{\partial H} \propto \langle s^2 \rangle - \underbrace{\langle s \rangle^2}_{M^2}$$

$$H_{2\text{-}d} = -J \sum_{\langle ij \rangle} s_i s_j - B \sum_i s_i$$

i) <u>Input</u> : dimensions of lattice

N_x, N_y

$T(\beta)$, J, B

<u>$k_B = 1$</u>

T (energy)

$N_s = N_x N_y$

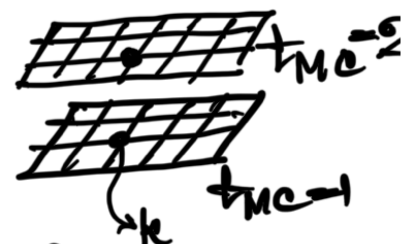
K, H

$\left\{ \begin{array}{l} J/T, \; K \\ B/T \; ; \; H \end{array} \right]$

N_{warms} (warm-up sweep #)

· Do not measure. N_{warms}

$$\frac{1}{N_s}\sum_{k=1}^{N_s}\frac{1}{N_{warms}}\sum_{t_{MC}=1}^{N_{warms}} s^k(t_{MC})\, s^k(t_{MC}+\tau)$$



$\sim F(\tau)$

$$\frac{1}{N_s}\sum_{k=1}^{N_s}\frac{1}{N_{warms}}\Big[s^k(1)s^k(4)+s^k(2)s^k(5)+\dots+s^k(997)s^k(1000)+s^k(998)s^k(1)+\dots+s^k(1000)s^k(3)\Big]$$

$\sim F(3)$

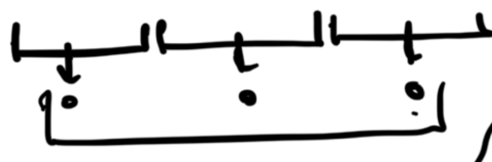
$\boxed{N_{sweeps}} \gg N_{warms}$.

M $(s_i)_c$, $i = 1, \dots, N_s$

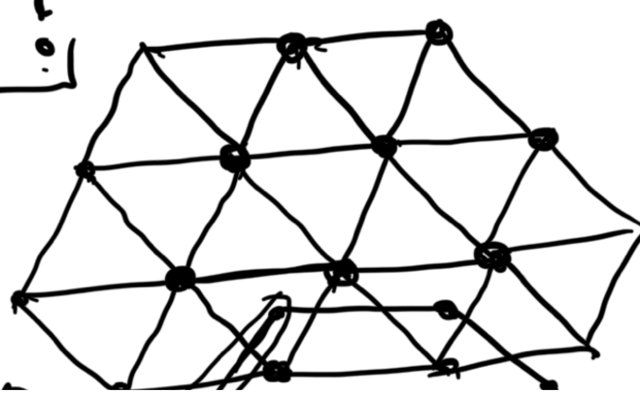
$$s_c = \frac{1}{N_s}\sum_{i=1}^{N_s}(s_i)_c \longrightarrow \text{Store it!}$$

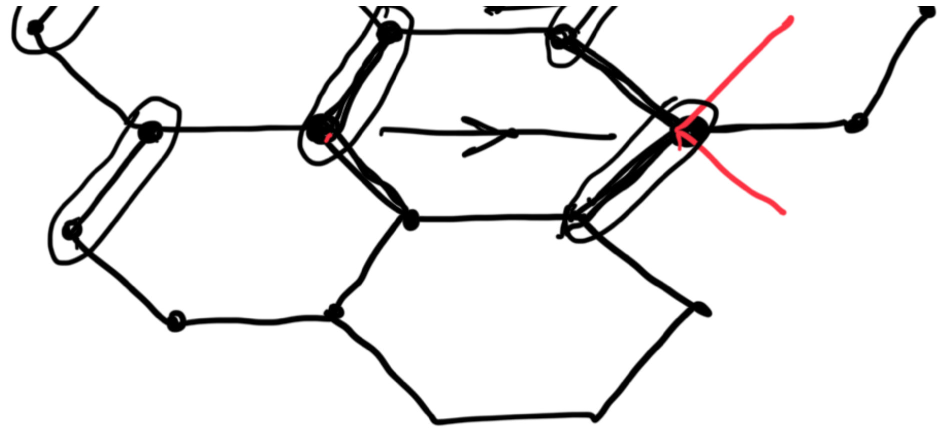
$$\frac{1}{N_{sweeps}}\sum_{c=1}^{N_{sweeps}} s_c = \langle M \rangle$$

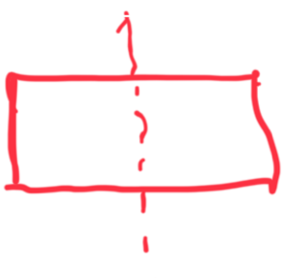
Binning



Bravais Lattices







$$-\left(\frac{F-\langle E\rangle}{T}\right) = S$$

$$\chi \propto \langle S^2\rangle - \langle S\rangle^2$$

C_V;

$\alpha, \beta, \gamma, \delta, \nu, \eta$;

$$T \to T_c \leftarrow T$$

$$h \to 0^+, 0^-$$

$$J_{ij} \sim U\left[-J/2, J/2\right]$$