Statistics II: Introduction to Inference

Problem set 6

1. Show that the following distributions belong to either location, or scale, or location scale families. Hence find the appropriate transformation of a random variable belonging to these distributions, such that the transformed random variable has a distribution free of the parameters θ.
   (a) $\texttt{normal}(\mu, \sigma^2)$,
   (b) $\texttt{exponential}(\lambda)$,
   (c) $\texttt{uniform}(-\theta, \theta)$,
   (d) $\texttt{exponential}(\lambda)$,
   (e) $\texttt{Gamma}(n, \theta)$, with n known,
   (f) $\texttt{normal}(\theta, \theta^2)$,
   (g) $\texttt{uniform}(\theta - 1/2, \theta + 1/2)$.

2. (a) Let $X_1, \ldots, X_n$ be a random sample of size n from a location family of distribution with location parameter θ. Show that the distribution of any function of $Y_{i,j} = \{X_i - X_j\}$, $i, j = 1, \ldots, n$ is free of θ.
   (b) Let $X_1, \ldots, X_n$ be a random sample of size n from a scale family of distribution with scale parameter θ. Show that the distribution of any function of $Y = X_1^2 / \sum_{j=1} X_j^2$ and $Z = X_{(1)}/X_{(n)}$ are free of θ.

3. The independent random variables $X_1, \ldots, X_n$ have common distribution

$$P(X_i \leq x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ (x/\beta)^\gamma & \text{if} \quad 0 < x < \beta \\ 1 & \text{if} \quad x \geq \beta \end{cases}.$$

   Find a $(1 - \alpha)100\%$ confidence interval for β based on the MLE of β, when γ is known.

4. Find a $(1 - \alpha)$ confidence interval for θ, given $X_1, \ldots, X_n$ iid with pdf
   (a) $f_X(x; \theta) = 1$, $\theta - 0.5 \leq x \leq \theta + 0.5$.
   [Instead of the exact confidence interval, you may find a approximate confidence interval based on the asymptotic distribution of an appropriate pivot.]
   (b) $f_X(x; \theta) = 2x\theta^{-2}$, $0 < x < \theta$, $\theta > 0$.

5. Let $X_1, \cdots, X_n$ be iid from $\texttt{Uniform}(\theta, 1)$ distribution, $\theta < 1$.
   (a) Obtain a suitable pivot for finding a confidence interval for θ.
   (b) Based on this test find a $(1 - \alpha)$ confidence set.

6. Let T be a statistic with continuous strictly decreasing CDF $F_T(\cdot; \theta)$, and α_1, α_2 be such that $\alpha_1 + \alpha_2 = \alpha$, for some fixed $\alpha \in (0, 1)$. Suppose that for each t the functions $L(t)$ and $U(t)$ are defined as

$$F_T(T; U(T)) = \alpha_1, \quad \text{and} \quad F_T(T; L(T)) = 1 - \alpha_2.$$

   Then show that the random interval $[L(t), U(t)]$ is a $(1 - \alpha)$ confidence interval for θ.

7. Let X be a single observation from a $\texttt{beta}(\theta, 1)$ distribution. Let $Y = -(\log X)^{-1}$. Evaluate the confidence coefficient, β, of the interval $[y/2, y]$.

8. A confidence interval $[L(\mathbf{X}), U(\mathbf{X})]$ for the parameter θ with confidence coefficient at least $1 - \alpha$ is called *unbiased* if $P_\theta\left(L(\mathbf{X}) < \theta < U(\mathbf{X})\right) \geq 1 - \alpha$, and $P_\theta\left(L(\mathbf{X}) < \theta' < U(\mathbf{X})\right) \leq 1 - \alpha$ for all $\theta' \neq \theta$.

   Based on a random sample of size n from $\texttt{uniform}(0, \theta)$, show that the symmetric confidence interval obtained from the pivot $X_{(n)}/\theta$ is unbiased for sufficiently large n.

9. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu_0, \sigma^2)$ where μ_0 is known.

   (a) Find the UMVUE of σ^2.

   (b) Using the UMVUE, find an appropriate pivot for σ^2, and its distribution.

   (c) Using this pivot find a $(1 - \alpha)$-confidence interval for σ^2.

10. Let $X_1, \ldots, X_n \overset{iid}{\sim} \texttt{exponential}(\theta)$.

    (a) Find an appropriate transformation $W_i = T(X_i, \theta)$ such that the distribution of W_i does not depend on $1/\theta$.

    (b) Consider two pivots, one based on W_1 only, and another involving the UMVUE of $1/\theta$.

    (c) Find the symmetric $(1 - \alpha)$-confidence intervals using the two pivots obtained in part (b).

    (d) Generate $n = 10$ IID samples from $\texttt{exponential}(5)$ distribution. Based on the samples obtain realizations of the two confidence intervals for $\alpha = 0.1, 0.05, 0.025$.