Show all your work. Justify your solutions. Answers without justification will not receive full marks.

Only hand in the problems on page 2.

Practice Problems

Question 1. Prove that if $a \mid b$ and $a \mid 3c$ then $a \mid 6(a+b+c)$.

Question 2. Find integers x and y that solve each of the following equations, or explain why no solution exists.

(a) $30x + 26y = 2$.
(b) $30x + 26y = 16$.
(c) $60x + 42y = 8$.
(d) $927979x + 823543y = 1$

Question 3. Without using the Fundamental Theorem prove the following: If $\gcd(a,b) = \gcd(a,c) = 1$ and $a \mid bcd$, then $a \mid d$.

Question 4. Let $\gcd(w,x,y,z)$ be the largest integer dividing all of w, x, y, z.

(a) Prove that $\gcd(\gcd(w,x), \gcd(y,z)) = \gcd(w,x,y,z)$.
(b) Hence find $\gcd(252, 112, 147, 98)$.

Question 5. Calculate the last decimal digit of $(1997^{1997})^{1997}$.

Question 6. Prove that for all $a \in \mathbb{Z}$, $8 \mid [a^2 + (a-2)^2 - 2]$ or $8 \mid [a^2 + (a-2)^2 - 4]$.

Assignment Problems

Question 1. For each of the equations below, either find a solution in integers x, y, or prove that no solution exists.

(a) $30x + 21x = 3$.
(b) $30x + 21x = 15$.
(c) $30x + 21x = 10$.
(d) $41x + 42x = 2010$.

Question 2. Calculate $d = \gcd(1234567890,\ 987654321)$, and find integers x and y with $1234567890\,x + 987654321\,y = d$.

Question 3.

(a) Define $F_0 = 1$, $F_1 = 1$ and recursively define $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. These are the *Fibonacci* numbers. Calculate F_{10} and F_{11}.
(b) Calculate $d = \gcd(F_{10}, F_{11})$, and write d as a linear combination of F_{10} and F_{11}. What do you notice?
(c) Prove that $\gcd(F_{n+1}, F_n) = 1$ for all $n \geq 0$.

Question 4.

(a) Prove that if $d \mid a$ and $d \mid b$ then $d \mid (ax + by)$ for any integers x and y.
(b) Prove that last decimal digit of any perfect square must be 0, 1, 5, 6 or 9. Hint: compare example 1.4.4.
(c) Prove that for any integer a the number $a(a^2 - 7)$ is a multiple of 6.

Question 5.

(a) Prove that $\sqrt{3}$ is irrational.
(b) Prove that $\sqrt[3]{2}$ is irrational.
(c) Prove that the equation $x^5 - 3y^5 = 2008$ has no solution in integers x, y. Hint: mod 11.

Question 6. I have an unlimited stock of 4 cent and 7 cent stamps. Which postages can I make? Which postages can I not make? Prove your answer. Note: negative numbers of stamps cannot be used. In other words, which $n \in \mathbb{N}$ can be written in the form $n = 4x + 7y$ with x, $y \geq 0$?