Math AM tut 6 or 7

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8th Nov 2024

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How to combine two types of improper Riemann integrals?

When $f : (0,\infty) \to \mathbb{R}$ is such that it blows up near zero, and $f \in \mathbf{R}[\infty)$, then we say that f is Riemann Integrable on $(0,\infty)$ (and write $f \in \mathbf{R}(0,\infty)$), and define

$$\int_{(0,\infty)} f(x)\mathrm{d}x = \int_0^\infty f(x)\mathrm{d}x = \int_{(0,1]} f(x)\mathrm{d}x + \int_{[1,\infty)} f(x)\mathrm{d}x \tag{1}$$

Ex: For each $\alpha \in (0,\infty)$, define a map $f : (0,\infty) \to (0,\infty)$ by $f(x) := x^{\alpha-1}e^{-x}$, $x \in (0,\infty)$. Clearly $f \in \mathbf{R}[0,\infty)$.

Qn: is $\int_0^1 f(x)\mathrm{d}x < \infty$
case 1: $\alpha \geq 1$
then $f(x) = x^{\alpha-1}e^{-x} : x \in [0,1]$ is a continous function on $[0,1]$ and hence $f \in \mathbf{R}[0,1] \implies \int_0^1 f(x)\mathrm{d}x < \infty$, hence if $\alpha = 1$, then $f \in \mathbf{R}(0,\infty)$
case 2: $\alpha < 1$ means $\alpha \in (0,1)$.
Define

$$g(x) = \frac{1}{x^{1-\alpha}} : x \in (0,1] \tag{2}$$

and look at $f|_{[0,1]}$, ie,

$$f(x) = \frac{e^{-x}}{x^{1-\alpha}} : x \in (0,1] \tag{3}$$

Note: $f, g \in \mathbf{R}[\epsilon, 1]\, \forall\, \epsilon \in (0,1)$ and both are positive valued functions and

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-x} = 1 \in (0,\infty) \tag{4}$$

Moreover $g(x) = \frac{1}{x^5}\ x \in (0,1]$ where $s = 1-\alpha \in (0,1)$ is done as an excercise. hence $g \in (0,1]$, hence $\int_0^1 g(x)\mathrm{d}x < \infty$
so by ratio test. $\int_0^1 f(x)\mathrm{d}x < \infty$ Now, combining everything, we get that $\forall \alpha \in (0,\infty)$, $\int_0^\infty x^{\alpha-1}e^{-x}\mathrm{d}x$=$\int_0^1 x^{\alpha-1}e^{-x}\mathrm{d}x$+$\int_1^\infty x^{\alpha-1}e^{-x}\mathrm{d}x$

2 Gamma Function

$\forall\alpha \in (0,\infty)$, $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\mathrm{d}x$ (Gamma function), Note that $\Gamma : (0,\infty) \to (0,\infty)$
Show that :

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \;\; \alpha \in (0,\infty) \tag{5}$$

use by parts
Show that

$$\Gamma(n) = (n-1)! \;\; \forall n \in \mathbb{N} \tag{6}$$

Use the first one and induction

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Show that $\int_0^\infty \frac{1}{x^2+\sqrt{x}}\mathrm{d}x < \infty$
ans:
(Hint: Compare with $\frac{1}{\sqrt{x}}$ on 0 to 1 and with $\frac{1}{x^2} on 1 to \infty$)