Statistics II: Introduction to Inference

Module 10: Interval Estimation

Sometimes providing a point estimate, or testing a hypothesis is not the ideal method of inference. Rather, one may be interested in an interval (or a set), which *efficiently* captures the underlying parameter. For example, in a diagnostic test for a particular disease, one requires an interval (range) of the possible test results, with efficiently detects the occurrence or non-occurrence of a particular disease. In other words we require a random set which captures the underlying parameter with high probability (rather than a random point which is close to the underlying parameter in appropriate sense). This type of estimates are called interval estimates.

Definition 1. *An interval estimate for a real valued parameter θ is a pair of functions of sample observations $L(\mathbf{x}) = L(x_1, \ldots, x_n))$, $U(\mathbf{x}) = U(x_1, \ldots, x_n))$ that satisfy $L(\mathbf{x}) \leq U(\mathbf{x})$ for each $\mathbf{x}$. If realization of $\mathbf{X}$ is $\mathbf{x}$, then we infer that the interval $[L(\mathbf{x}), U(\mathbf{x})]$ contains θ with high probability. The random interval $[L(\mathbf{X}), U(\mathbf{X})]$ is called a interval estimator of θ.*

Note: For some particular examples, $L(\mathbf{x})$ can be $-\infty$, or $U(\mathbf{x})$ can be ∞. In such cases we obtain a one sided interval estimate. Further, instead of closed interval, one may obtain an open, or a semi-closed interval estimate as well.

Definition 2 (Interval Estimate)**.** *The confidence coefficient of an interval estimator $[L(\mathbf{x}), U(\mathbf{x})]$ of a parameter θ, usually denoted by $(1-\alpha)$, is highest probability with which the random interval captures the true parameter θ, for any $\theta \in \Theta$. Notationally, the confidence coefficient is $(1-\alpha)$ satisfying*

$$\inf_{\theta} P_{\theta \in \Theta}\left([L(\mathbf{x}), U(\mathbf{x})] \ni \theta\right) = (1-\alpha), \quad \textit{so that,} \quad P_{\theta \in \Theta}\left([L(\mathbf{x}), U(\mathbf{x})] \ni \theta\right) \geq (1-\alpha), \quad \textit{for all } \ \theta \in \Theta.$$

Interval estimators together with confidence coefficient are called confidence intervals.

Note: We can generalize the idea of confidence intervals to *confidence sets.* A random set $S(\mathbf{X})$ is said to be a confidence set for a parameter vector $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^k$, with confidence coefficient $(1-\alpha)$, if $\inf_{\theta \in \Theta} P_\theta\left(S(\mathbf{X}) \ni \theta\right) = (1-\alpha)$.

A confidence interval can be interpreted as a special type of confidence set, where $S(\mathbf{X})$ is an interval.

Interpretation of Confidence Sets: A confidence set $S(\mathbf{X})$ with confidence coefficient $(1-\alpha)$ can be interpreted as follows: If repeated random samples, that is, repeated realizations of $\mathbf{X}$, are taken for a large (theoretically, infinite) number of times, then in about $(1-\alpha)100\%$ cases, the realization of the confidence set, $S(\mathbf{x})$, will contain the true parameter $\boldsymbol{\theta}$.

Example 1. Let $X_1, \ldots, X_n$ be a random sample from $\texttt{Normal}(\mu, \sigma^2)$ distribution. Consider the interval estimate $[\bar{X}_n - c, \bar{X}_n + c]$ of μ for some constant $c \geq 0$. Find c such that the confidence coefficient is $(1-\alpha)$.

When σ^2 is known, then it can be seen that $c = \sigma\tau_{\alpha/2}/\sqrt{n}$ where $\tau_{\alpha/2}$ is the upper $\alpha/2$-th point of the standard normal distribution.

Remark 1. One could also choose an interval estimate of the type $[\bar{X}_n - c_1, \bar{X}_n + c_2]$ where $c_1, c_2 \geq 0$. Then any c_1, c_2, satisfying $\Phi\left(c_1\sqrt{n}/\sigma\right) - \Phi\left(-c_2\sqrt{n}/\sigma\right) = (1-\alpha)$ would provide a valid confidence interval with confidence coefficient $(1-\alpha)$.

Remark 2. When σ^2 is unknown then one can use the fact that $\sqrt{n}(\bar{X}_n - \mu)/S_n^\star \sim t_{(n-1)}$ to obtain find c_1, c_2, where $S_n^{\star\ 2} = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$.

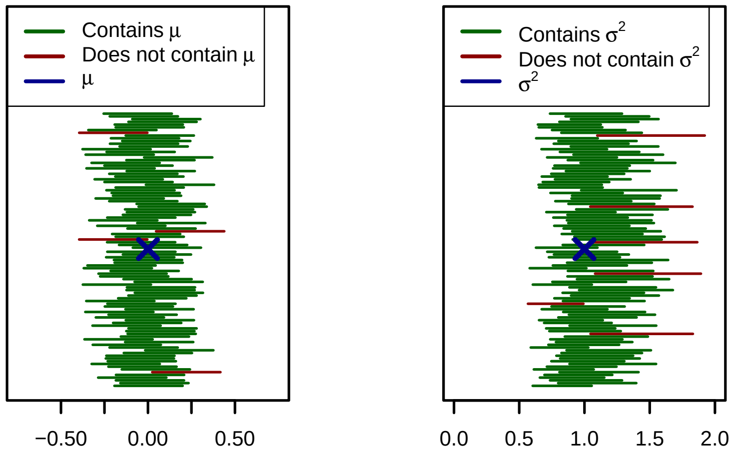

Figure 1: Figure contains 100 realizations of the 95% confidence intervals from normal mean μ (left) and variance σ^2 (right) as obtained in Examples 1 and 2. The realizations of the interval estimates which captures (does not capture) the true parameter are indicated in green (red). Observe that, in about 95% cases, the realization of the interval contains the true parameter.

Example 2 Let $X_1, \ldots, X_n$ be a random sample from $\texttt{Normal}(\mu, \sigma^2)$ distribution. Consider the interval estimate $[c_1 S_n^{\star\,2}, c_2 S_n^{\star\,2}]$ of σ^2 for some constants $0 < c_1 \leq c_2$, where $S_n^{\star\,2} = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Find c_1, c_2 such that the confidence coefficient is $(1-\alpha)$.

Using the result that $(n-1)\sigma^{-2}S_n^{\star\,2} \sim \chi^2_{(n-1)}$, one can show that any c_1, c_2 satisfying

$$P\left(\frac{(n-1)}{c_2} < W \leq \frac{(n-1)}{c_1} \mid W \sim \chi^2_{(n-1)}\right) = (1-\alpha),$$

leads to a valid confidence interval.

In particular, one may choose c_1, c_2 such that

$$P\left(W \geq \frac{(n-1)}{c_1} \mid W \sim \chi^2_{(n-1)}\right) = P\left(W \leq \frac{(n-1)}{c_2} \mid W \sim \chi^2_{(n-1)}\right) = \frac{\alpha}{2},$$

which leads to the interval $\left[(n-1)S_n^{\star\,2}/\chi^2_{(n-1),1-\alpha/2}, (n-1)S_n^{\star\,2}/\chi^2_{(n-1),\alpha/2}\right]$.

1 Method of Finding Confidence Interval

1.1 Method of Pivots

Definition 3 (Pivot)**.** *Let $\mathbf{X} \sim f_{\mathbf{X}}(\cdot;\theta)$. A random variable $T(\mathbf{X};\theta)$ is called a pivot if the distribution of $T(\mathbf{X};\theta)$ does not depend on θ.*

Example 3. Let $X_1, \cdots, X_n$ be a random sample from a location family with location parameter θ, i.e., $X_i = W_i + \theta$ where $W_i,\ i = 1, \ldots, n$ are iid from a distribution free of θ. Then any function of $X_i - \theta;\ i = 1, \ldots, n$ is a pivot.

Example 4. Let $X_1, \cdots, X_n$ be a random sample from a scale family with scale parameter θ, i.e., $X_i = \theta W_i$ where $W_i,\ i = 1, \ldots, n$ are iid from a distribution free of θ. Then any function of $X_i/\theta;\ i = 1, \ldots, n$ is a pivot.

Example 5. If X is distribution as a continuous distribution, then the distribution function $F_X(\cdot;\theta)$ has a $\texttt{Uniform}(0,1)$ distribution. When n i.i.d. samples $X_1,\ldots,X_n$ are available, then one may take $T(\mathbf{X};\theta) = -\sum_{i=1}^n \log F_X(X_i;\theta)$ as a pivot. It can be shown that $T(\mathbf{X};\theta)$ distributed as a $\texttt{Gamma}(n,1)$ distribution.

A pivot may yield a confidence interval when supported with some additional features. The following theorem provides a set of sufficient conditions for a pivot to yield a confidence interval.

Theorem 1. *Let $T(\mathbf{X};\theta)$ be a pivot such that for each θ, $T(\mathbf{X};\theta)$ is a statistic and as a* **function of** θ*,* $T(\mathbf{X};\theta)$ **is strictly monotone at each** $\mathbf{x}\in\mathbb{R}^n$. *Let $\Lambda\in\mathbb{R}$ be the range of $T(\mathbf{X};\theta)$, and for each λ and $\mathbf{x}\in\mathbb{R}$ the equation $\lambda = T(\mathbf{x};\theta)$ is* **solvable** *with respect to θ. Then one can construct a confidence interval for θ at any level.*

Procedure of obtaining confidence interval from a pivot. Suppose we are interested in finding a confidence interval for the parameter of interest θ.

1. First find a pivot $T(\mathbf{X},\theta)$ such that $T(\mathbf{X},\theta)$ is a function of θ and a *good* statistic of θ.

2. Find constants a,b such that $P\left(a\le T(\mathbf{X},\theta)\le b\right) = (1-\alpha)$. Note that, as the distribution of $T(\mathbf{X},\theta)$ is completely known, the constants (a,b) are also known (does not depend on θ).

3. Next, solve the equations $T(\mathbf{X},\theta)=a$ and $T(\mathbf{X},\theta)=b$ with respect to θ. Let $\widehat{\theta}_n(a)$ and $\widehat{\theta}_n(b)$ be the solutions, i.e., $T\big(\mathbf{X},\widehat{\theta}_n(a)\big)=a$ and $T\big(\mathbf{X},\widehat{\theta}_n(b)\big)=b$. Note that, $\big(\widehat{\theta}_n(a),\widehat{\theta}_n(b)\big)$ are functions of $\mathbf{X}$ and (a,b) only. Thus, they are pair of statistics.

4. Let $T(\mathbf{X},\theta)$ be strictly monotonically increasing function of θ. Then

$$\{a\le T(\mathbf{X},\theta)\le b\} \quad\Longleftrightarrow\quad \left\{\widehat{\theta}_n(a)\le\theta\le\widehat{\theta}_n(b)\right\},$$

   so that $P_\theta\left(\widehat{\theta}_n(a)\le\theta\le\widehat{\theta}_n(b)\right) = (1-\alpha)$ for all $\theta\in\Theta$. Thus, $[\widehat{\theta}_n(a),\widehat{\theta}_n(b)]$ is a confidence interval with confidence coefficient $(1-\alpha)$.

   Further, let $T(\mathbf{X},\theta)$ be strictly monotonically decreasing function of θ. Then

$$\{a\le T(\mathbf{X},\theta)\le b\} \quad\Longleftrightarrow\quad \left\{\widehat{\theta}_n(b)\le\theta\le\widehat{\theta}_n(a)\right\},$$

   so that $P_\theta\left(\widehat{\theta}_n(b)\le\theta\le\widehat{\theta}_n(a)\right) = (1-\alpha)$ for all $\theta\in\Theta$. Thus, $[\widehat{\theta}_n(b),\widehat{\theta}_n(a)]$ is a confidence interval with confidence coefficient $(1-\alpha)$.

Remark 3. A sufficient condition for the equation $\lambda = T(\mathbf{x};\theta)$ to be solvable is T is continuous and strictly monotone w.r.t. θ.

Remark 4. The monotonicity assumption in the above theorem ensures that the confidence set obtained from the pivot T is of interval type. In case all the other assumptions in Theorem 1 are satisfied, except the monotonicity assumption, then one would still obtain a confidence set, but it may not be of interval type.

Example 6. Let $X_1,\cdots,X_n$ be a random sample from location exponential distribution, with location parameter θ and scale parameter 1. Then obtain a $(1-\alpha)$ confidence interval based on the complete sufficient statistic $X_{(1)}$ of θ.

Example 7. Revisit Examples 1,2.

1.2 Test Inversion

There is a strong correspondence between testing of a hypothesis and interval estimation. From a test one can obtain a confidence set, and conversely, from a confidence interval one can obtain a test. We will see this with an example first.

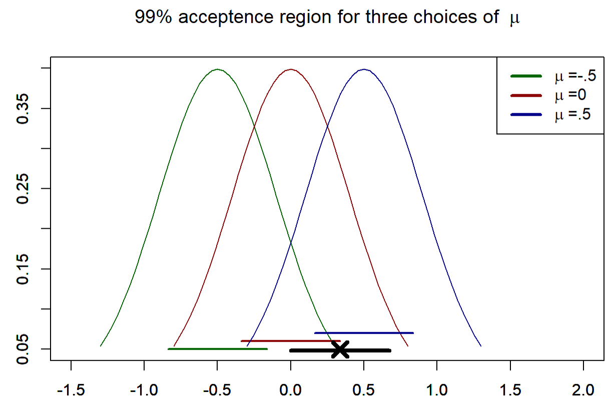

Figure 2: The figure shows PDFs of $\bar{X}_n$ under 3 possible values of μ_0, viz., $\{-0.5, 0, 0.5\}$, and the corresponding acceptance regions $[\mu_0 - \tau_{\alpha/2}/\sqrt{n}, \mu_0 + \tau_{\alpha/2}/\sqrt{n}]$ shown in green, red and blue, respectively. The thick black cross indicates a realization $\bar{x}_n$. Observe that, $\bar{x}_n$ falls inside the acceptance regions of $\mu_0 = 0, 0.5$, but falls outside that of $\mu_0 = -0.5$. In fact, the thick black interval contains all possible values of μ_0 whose corresponding acceptance regions contains $\bar{x}_n$, and hence provides a realization of the test-inverted confidence interval.

Example 8. Consider the problem of testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ for a `Normal`$(\mu, 1)$ population at level α, based on a random sample of size n. Recall the test function of the form

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad |T(\mathbf{x})| > c \\ 0 & \quad \text{otherwise} \end{cases}, \quad \text{where} \quad T(\mathbf{x}) = \sqrt{n}(\bar{X}_n - \mu_0),$$

and

$$P_{H_0}(|T(\mathbf{X})| > c) = \alpha.$$

As $T(\mathbf{X}) \sim N(0, 1)$ under H_0, we have the choice $c = \tau_{\alpha/2}$, where $\tau_{\alpha/2}$ is the upper $\alpha/2$ point of $N(0, 1)$ distribution. This together implies that

$$P_{\mu_0}\left(-\tau_{\alpha/2} \leq \sqrt{n}(\bar{X}_n - \mu_0) \leq \tau_{\alpha/2}\right) = P_{\mu_0}\left(\bar{X}_n - \frac{\tau_{\alpha/2}}{\sqrt{n}} \leq \mu_0 \leq \bar{X}_n + \frac{\tau_{\alpha/2}}{\sqrt{n}}\right) = (1 - \alpha).$$

However, observe that the above probability statement is true for any choice of μ_0. Thus we can rewrite the above statement as

$$P_{\mu}\left(\bar{X}_n - \frac{\tau_{\alpha/2}}{\sqrt{n}} \leq \mu \leq \bar{X}_n + \frac{\tau_{\alpha/2}}{\sqrt{n}}\right) = (1 - \alpha),$$

which yields a confidence interval for μ.

Remark 5. In the above example, we are crucially using the fact that, when θ takes a particular value θ_0 and the samples are generated from f_{θ_0}, then the probability that the test statistic $T(\mathbf{X})$ lies in the acceptance region, say $A(\theta_0)$, is at least $(1 - \alpha)$.

Remark 6. If we fix a realization $\mathbf{x}$, then (given $\theta = \theta_0$), the test statistic $T(\mathbf{x})$ either falls inside or outside of the acceptance region. However, we keep $\mathbf{x}$ fixed and vary θ, then the acceptance region varies.

Remark 7. Now, consider the set of possible values of θ such that a particular realization $\mathbf{x}$ belongs to the $A(\theta)$. In the above example, it is the set of μ values such that $\sqrt{n}(\bar{x}_n - \mu) \in [-\tau_{\alpha/2}, \tau_{\alpha/2}]$, i.e., $\mu \in \left[\bar{x}_n - \tau_{\alpha/2}/\sqrt{n},\ \bar{x}_n + \tau_{\alpha/2}/\sqrt{n}\right]$.

Note that, this set does not depend on a particular θ, but on $\mathbf{x}$ only. Let us call this $C(\mathbf{x})$. One can interpret $C(\mathbf{x})$ as $A^{-1}(\mathbf{x})$. Thus $C(\mathbf{X})$ is a random set, based on $\mathbf{X}$ only.

The following theorem states that this set $C(\mathbf{X})$ forms a valid confidence set with confidence coefficient $(1 - \alpha)$.

Theorem 2. *For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level-α test of $H_0 : \theta = \theta_0$. For each $\mathbf{x}$, define $C(\mathbf{x}) \subseteq \Theta$ such that $C(\mathbf{x}) = \{\theta : \mathbf{x} \in A(\theta)\}$. Then the random set $C(\mathbf{X})$ is a $(1-\alpha)$ confidence set.*

Remark 8. As we have seen in testing of hypothesis, the alternative hypothesis plays a crucial role in determining the acceptance region of an UMP test. Consequently, the form of the confidence set obtained from the acceptance region of a test is also determined by the type of alternative hypothesis.

Remark 9. The above procedure does not guarantee in general that the confidence set obtained by inverting an acceptance region would be interval type.

Example 9. Let $X_1, \ldots, X_n$ be a random sample from $\texttt{Uniform}(0, \theta)$ distribution. A UMP level α test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$ has the form

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad x_{(n)} > c \\ 0 & \quad \text{otherwise.} \end{cases}, \quad \text{where} \quad P_{\theta_0}(X_{(n)} > c) = \alpha.$$

From the size α condition, we get $c = \theta_0(1-\alpha)^{1/n}$. Now, if we fix $\mathbf{x}$, then

$$c(\mathbf{x}) = \{\theta : x_{(n)} \leq \theta(1-\alpha)^{1/n}\} = \left[(1-\alpha)^{-1/n} x_{(n)}, \infty\right].$$

Therefore, by the above theorem, $\left[(1-\alpha)^{-1/n} X_{(n)}, \infty\right]$ is a $(1-\alpha)$ confidence interval.