PROBABILITY models in Optimization

Part 2: Measures of Risk and Models in Stochastic Optimization

Joydeep Dutta

Department of Economic Sciences

IIT Kanpur.

- Measures of Risk (Royset, SIREV, 2025)

A measure of risk $\mathcal{R}$ assigns to a random variable ξ a number $\mathcal{R}(\xi) \in [-\infty, +\infty]$, as a risk quantification, with $\mathcal{R}(\xi) = \xi$ if $\mathbb{P}(\xi = \xi) = 1$

- Examples of Risk Measures

$\mathcal{R}(\xi) = E_{\mathbb{P}}(\xi)$. This is also used as a risk measure. This is risk-neutral

For the portfolio optimization problem, set $r = \xi$ where r denotes the relative return on the total portfolio.

One can pose it as

$$\max E(r) = \sum_{i=1}^{n} \bar{r}_i \omega_i$$

$$\frac{1}{2} \langle \omega, \Sigma \omega \rangle \leq \tau$$

$$\sum_{i=1}^{n} \omega_i = 1.$$

- Mean + Plus Std dev

$$\mathcal{R}(\xi) = E_{\mathbb{P}}(\xi) + \lambda \sqrt{Var(\xi)}$$

- <u>Worst Case Risk</u>

$$\mathcal{R}(\xi) = \sup \xi = \sup_{\xi \in \mathcal{U}} \xi = \sup_{\omega \in \Omega} \xi(\omega)$$

represents conservative outlook.

$\mathcal{U}$ can be viewed as a support.

If ξ has a normal or exponential distribution $\underline{\mathcal{R}(\xi) = \infty}$

<u>More flexible measures are required.</u>

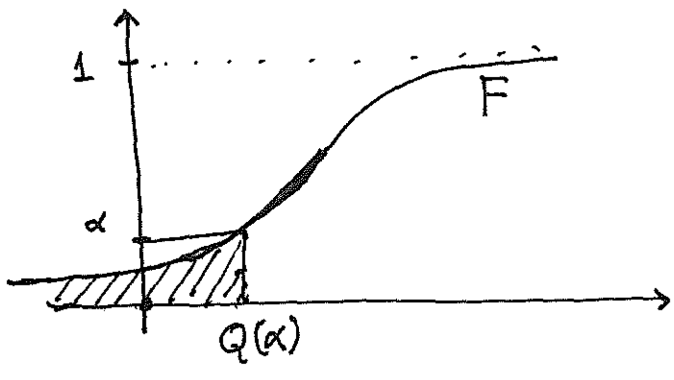
That is done using quantiles.

They allows high and low-values treated differently

- Quantile Risk (α- quantile)

Let F be the distribution function of the random-variable $\xi : \Omega \longrightarrow \mathbb{R}^m$

α - quantile, $\quad Q(\alpha) = \min \{ \xi \in \mathbb{R} \mid F(\xi) \geq \alpha \}, \ \alpha \in (0,1)$



Called VaR

Value-at-Risk.

(Used heavily in

α - Superquantile (C-VaR)

$\downarrow$ conditional.

$$\bar{Q}(\alpha) = Q(\alpha) + \frac{1}{1-\alpha} E_{\mathbb{P}}\left\{\max\{0, \xi - Q(\alpha)\}\right\}$$

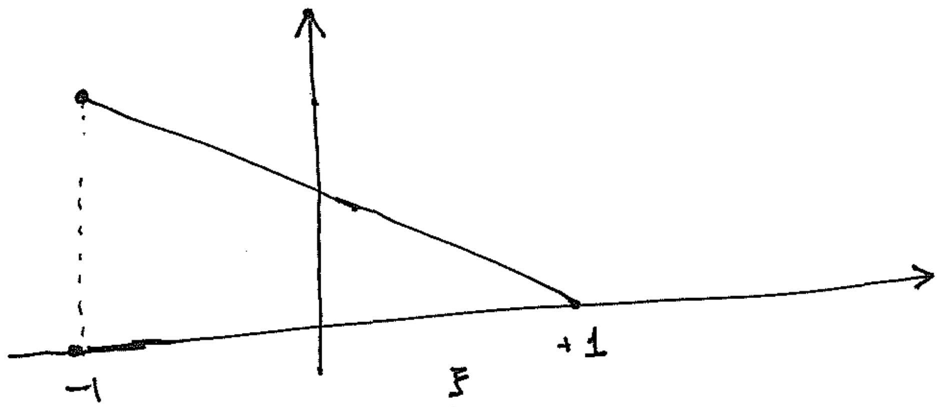
We define $\bar{Q}(0) = E(\xi)$ & $\bar{Q}(1) = \lim_{\alpha \nearrow 1} \bar{Q}(\alpha)$

Example: Computing α-superquantile. (from: Royset and Wets Primer)

Let $\Xi = [-1, +1]$, and the density of ξ is given as

$$\boxed{p(\xi) = -\xi/2 + 1/2}$$

Triangular distribution



$$F(\xi)\ \cancel{Q(\alpha)} = \int_{-1}^{\xi} p(x)\,dx = \alpha$$

$$= \int_{-1}^{\xi} \left(-\frac{x}{2} + \frac{1}{2}\right) dx = \alpha.$$

This shows that $\xi^2 - 2\xi + (4\alpha - 3) = 0$

Hence $\xi = 1 \pm 2\sqrt{1-\alpha}$

Thus for us $\boxed{Q(\alpha) = 1 - 2\sqrt{1-\alpha}}$

$$\bar{Q}(\alpha) = Q(\alpha) + \frac{1}{1-\alpha}\int_{-1}^{1} \max\{0, \xi - Q(\alpha)\}\, p(\xi)\, d\xi$$

$$= Q(\alpha) + \frac{1}{1-\alpha}\int_{Q(\alpha)}^{1} \xi\, p(\xi)\, d\xi - \frac{Q(\alpha)}{1-\alpha}\int_{Q(\alpha)}^{1} p(\xi)\, d\xi$$

$$= Q(\alpha) + \frac{1}{1-\alpha}\int_{Q(\alpha)}^{1} \xi\, p(\xi)\, d\xi - \frac{Q(\alpha)}{1-\alpha}\cdot(1-\alpha)$$

$$= \frac{1}{1-\alpha}\int_{Q(\alpha)}^{1} \xi\, p(\xi)\, d\xi$$

$$= \frac{1}{1-\alpha}\int_{Q(\alpha)}^{1} \xi\left(-\frac{\xi}{2} + \frac{1}{2}\right) d\xi$$

$$\therefore \boxed{\bar{Q}(\alpha) = 1 - \frac{4}{3}\sqrt{1-\alpha}}$$

- If $\xi \sim N(\mu, \sigma^2)$, then

$$Q(\alpha) = \mu + \sigma\,\Phi^{-1}(\alpha)$$

$$\bar{Q}(\alpha) = \mu + \frac{\sigma\, F_\xi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}$$

where $F_\xi(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}$

VaR, C-VaR and Optimization

$$Q(\alpha) \in \operatorname{argmin}\left[\gamma + \frac{1}{1-\alpha} E_{\mathbb{P}}\left[\max\{0, \xi - \gamma\}\right]\right]$$

$$\bar{Q}(\alpha) = \min_{\gamma \in \mathbb{R}} \left[\gamma + \frac{1}{1-\alpha} E_{\mathbb{P}}\left[\max\{0, \xi - \gamma\}\right]\right]$$

In practice we write

$$\text{s-rsk}_\alpha(\xi) = \bar{Q}(\alpha)$$

The new approach to stochastic optimization

$$\underset{x \in S}{\text{minimize}} \ \text{s-rsk}_\alpha(f(x, \xi)).$$

random variable
not
random vector

Thus for a portfolio optimization, case we shall consider

$$\xi = r = \sum r_i w_i$$

In this case the portfolio optimization problem is

$$\underset{w}{\text{minimize}} \quad \text{s-rsk}_\alpha\left((\xi - \bar{\xi})^2\right) = \text{s-rsk}_\alpha\left[(r - \bar{r})^2\right]$$

Sub to

$$E(r) = \rho$$

$$\sum_{i=1}^{n} w_i = 1.$$

Is this meaningful

The reason for using the C-VaR or Super-quantile is the following reason

- If $f(x, \xi)$ be such that ξ is a random ~~vari~~ vector then consider $\{f(x,\xi) : \xi \in U\}$ to the values of the random variable and compute

$$S\text{-}rsk_\alpha(\hat{\xi}) = S\text{-}rsk_\alpha(f(x,\xi))$$

In fact if we take $\xi = (\xi_1 \dots \xi_n) = (r_1, \dots, r_n)$, then take

$$f(x, \xi) = \langle \xi, \underset{\omega}{x} \rangle - \langle \bar{\xi}\, x \rangle$$

$$= \langle \xi - \bar{\xi}, x \rangle$$

A more meaningful formalism is as follows

$$S\text{-}rsk_\alpha(\langle \xi - \bar{\xi}, x \rangle)$$

Subject to $E(\overset{\langle \bar{\xi}, x\rangle}{\bullet}) = \rho$

$$\langle e, x \rangle = 1.$$

The reason that the above formulation is more meaningful since $f(x, \xi)$ above follows the following

- $f(x, \xi)$ is integrable for each x.
- $f(\cdot, \xi)$ is convex for any $\xi \in \mathbb{R}^m$

Then

$$x \longmapsto S\text{-}rsk_\alpha(f(x,\xi))$$

is convex