Normal Tail Bounds.

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1 Gaussian Tail Bounds

Definition 1. *The distribution of a Standard Normal random variable Z with mean $\mu = 0$ and variance $\sigma^2 = 1$ satisfies*

$$\mathbb{P}(Z \leq a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{-x^2}{2}} dx.$$

Notice you also have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx = 1.$$

Hence you can also write after subtracting 1

$$\mathbb{P}(Z \geq a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{\frac{-x^2}{2}} dx.$$

Theorem 1. *For any $a > 0$, the following bounds holds*

$$\frac{a}{a^2+1} e^{\frac{-a^2}{2}} \leq \int_{a}^{\infty} e^{\frac{-x^2}{2}} dx \leq \frac{1}{a} e^{\frac{-a^2}{2}}.$$

Corollary 1. *For any $a > 0$, the Standard Normal random variable Z satisfies*

$$\frac{1}{\sqrt{2\pi}} \frac{a}{a^2+1} e^{\frac{-a^2}{2}} \leq \mathbb{P}(Z \geq a) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{\frac{-a^2}{2}}.$$

Corollary 2. *For any $a > 0$, the Standard Normal random variable Z also satisfies*

$$1 - \frac{1}{\sqrt{2\pi}} \frac{a}{a^2+1} e^{\frac{-a^2}{2}} \leq \mathbb{P}(Z \leq a) \leq 1 - \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{\frac{-a^2}{2}}.$$

2 Proof

The proof is not expected from you, but yet provided here for completeness' sake. We will only outline the proof of (1).

Proof. Note that we have since $x \geq a$

$$\int_a^\infty e^{\frac{-x^2}{2}} dx < \int_a^\infty e^{\frac{-x^2}{2}} \frac{x}{a} dx = \frac{1}{a} e^{\frac{-a^2}{2}} \tag{1}$$

But after substituting $x^2 = y$ and integrating we readily obtain

$$\int_a^\infty e^{\frac{-x^2}{2}} \frac{x}{a} dx = \frac{1}{a} \int_a^\infty e^{\frac{-x^2}{2}} x dx = \frac{1}{2a} \int_{a^2}^\infty e^{\frac{-y}{2}} dy = \frac{1}{a} e^{\frac{-a^2}{2}}.$$

Combining this with the inequality (1) immediately implies

$$\int_a^\infty e^{\frac{-x^2}{2}} dx \leq \frac{1}{a} e^{\frac{-a^2}{2}}.$$

Now also note for $x \geq a$

$$\int_a^\infty e^{\frac{-x^2}{2}} \frac{1}{x^2} dx < \int_a^\infty e^{\frac{-x^2}{2}} \frac{1}{a^2} dx. \tag{2}$$

we use integrate by parts

$$\int_a^\infty e^{\frac{-x^2}{2}} \frac{1}{x^2} dx = e^{\frac{-x^2}{2}} \frac{-1}{x} \bigg|_a^\infty - \int_a^\infty -x e^{\frac{-x^2}{2}} \frac{-1}{x} dx = \frac{1}{a} e^{\frac{-a^2}{2}} - \int_a^\infty e^{\frac{-x^2}{2}} dx. \tag{3}$$

Combining (2) with (3) and rearranging shows that

$$\frac{a}{a^2+1} e^{\frac{-a^2}{2}} \leq \int_a^\infty e^{\frac{-x^2}{2}} dx.$$

□

Question 1. *Can you prove* ***Corollary 1*** *and* ***Corollary 2****?*