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# Assignment - 7

1. (e) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{y}{x}$ is continuous $\forall\ x \neq 0$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)$ is continuous $\forall\ x \neq 0$ & $\forall\ y$ (composition of continuous functions is continuous

limit $(x,y) \to (0,0)$ through line $y = mx$

checking limit:

$$\lim_{x\to 0} \tan^{-1}\left(\frac{mx}{x}\right) = \tan^{-1} m$$

put $m = 1 \Rightarrow$ limit $= \frac{\pi}{4}$

put $m = -1 \Rightarrow$ limit $= -\frac{\pi}{4}$

$\Rightarrow$ limit of the scalar field at zero does not exist.

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)$ is continuous on $\mathbb{R}^2 - \{(0,0)\}$

(h) $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$

$\dfrac{1}{\sqrt{x^2+y^2}}$ is continuous $\forall\ (x,y) \neq (0,0)$

$\Rightarrow \dfrac{x}{\sqrt{x^2+y^2}}$ is continuous $\forall\ (x,y) \neq (0,0)$

as $\frac{f}{g}$ is continous if $f$ & $g$ are continuous $\forall\ g(\bar{x}) \neq 0$

We need to check at $(0,0)$

limit $(x,y) \to (0,0)$ through line $y = mx$

Checking limit :

$$\lim_{x \to 0} \frac{x}{\sqrt{x^2+m^2x^2}} = \frac{1}{\sqrt{1+m^2}}$$

put $m = 1$ , limit $= \frac{1}{\sqrt{2}}$

put $m = 0$ , limit $= 1$

$\Rightarrow$ limit of scalar field at $(0,0)$ does not exist

$=$ $\frac{x}{\sqrt{x^2+y^2}}$ is continuous on $\mathbb{R}^2 - \{(0,0)\}$

take $\delta' = \sqrt{2}\,\delta$

2. Given, $\forall\, \varepsilon > 0$ $\exists\, \delta' > 0$ s.t when $\|(x,y) - (a,b)\| < \sqrt{2}\,\delta$

$\|f(x,y) - L\| < \varepsilon$ $\quad\Rightarrow \sqrt{(x-a)^2+(y-b)^2} < \sqrt{2}\,\delta$

$\Rightarrow (x-a)^2 + (y-b)^2 < 2\delta^2$

for a fixed $x_0 \in (a-\delta, a+\delta)$ $\forall\, |(y-b)| < \delta$

$$\Rightarrow \lim_{y \to b} f(x,y) = L \quad \text{for fixed } x_0 \in (a-\sqrt{\delta}, a+\sqrt{\delta})$$

$$\therefore \lim_{x \to a} \lim_{y \to b} f(x,y) = \lim_{x \to a} L = L$$

(Consider only $x \in (a-\delta, a+\delta)$ for any $\delta > 0$)

Similarly for fixed $y_0 \in (b-\delta, b+\delta)$ $\forall\, (x-a)^2 < \delta^2$

$\Rightarrow |x-a| < \delta$

$$= \lim_{x \to a} f(x,y) = L \quad \text{for fixed } y_0 \in (b-\delta, b+\delta)$$

$$\lim_{y \to b} \lim_{x \to a} f(x,y) = \lim_{y \to b} L = L$$

(Consider only $y \in (b-\delta, b+\delta)$ for any $\delta > 0$)

3. $f(x,y) = \dfrac{x-y}{x+y}$

$$\lim_{x \to 0} \lim_{y \to 0} \frac{x-y}{x+y} = \lim_{x \to 0} \frac{x-0}{x+0} = \lim_{x \to 0} 1 = 1$$

$$\lim_{y \to 0} \lim_{x \to 0} \frac{x-y}{x+y} = \lim_{y \to 0} \frac{0-y}{0+y} = \lim_{y \to 0} -1 = -1$$

Exc. 2 says.

if $\lim_{(x,y) \to (a,b)} f(x,y) = L$ (i.e limit exists)

then $\lim_{x \to a} \lim_{y \to b} f(x,y) = \lim_{y \to b} \lim_{x \to b} f(x,y)$

This is logically equivalent to

if $\lim_{x \to a} \lim_{y \to b} f(x,y) \neq \lim_{y \to b} \lim_{x \to b} f(x,y)$

then $\lim_{(x,y) \to (a,b)} f(x,y)$ does not exist

$\therefore$ $\lim_{(x,y) \to (0,0)} \dfrac{x-y}{x+y}$ does not exist.

4. $f(x,y) = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$ $\quad x^2y^2 + (x-y)^2 \neq 0$

$$\lim_{y \to 0} \lim_{x \to 0} \frac{x^2y^2}{x^2y^2 + (x-y)^2} = \lim_{y \to 0} 0 = 0$$

$$\lim_{x \to 0} \lim_{y \to 0} \frac{x^2y^2}{x^2y^2 + (x-y)^2} = \lim_{x \to 0} 0 = 0$$

$\lim_{(x,y) \to (0,0)} \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$ consider when $(x,y) \to (0,0)$ through $y = x$

$$= \lim_{x \to 0} \frac{x^4}{x^4} = 1$$

through $y = 2x$

$$= \lim_{x \to 0} \frac{4x^4}{4x^4 + x^2} = 0$$

$\Rightarrow$ limit at $(0,0)$ Does not exist

5. $f(x,y) = \begin{cases} x \sin\frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}$ We know $\lim_{(x,y)\to(0,0)} x = 0$

$\Rightarrow \forall\ \varepsilon > 0\ \exists\ \delta$ s.t

$|x \sin\frac{1}{y}| < |x| < \delta$

$\Rightarrow \forall\ \varepsilon > 0\ \exists\ \delta$ s.t $|x \sin\frac{1}{y}| < \delta$

$$\Rightarrow \lim_{(x,y)\to(0,0)} x \sin\frac{1}{y} = 0$$

The reason this does not violate Q 2. is that the condition $\lim_{y \to b} f(x,y)$ exist is not met.

$\lim_{y \to 0} x \sin\frac{1}{y}$ does not exist $\forall x$

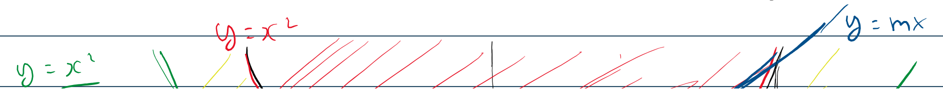
6. limit of $f(x,y)$ at $(0,0)$ along $y = mx$

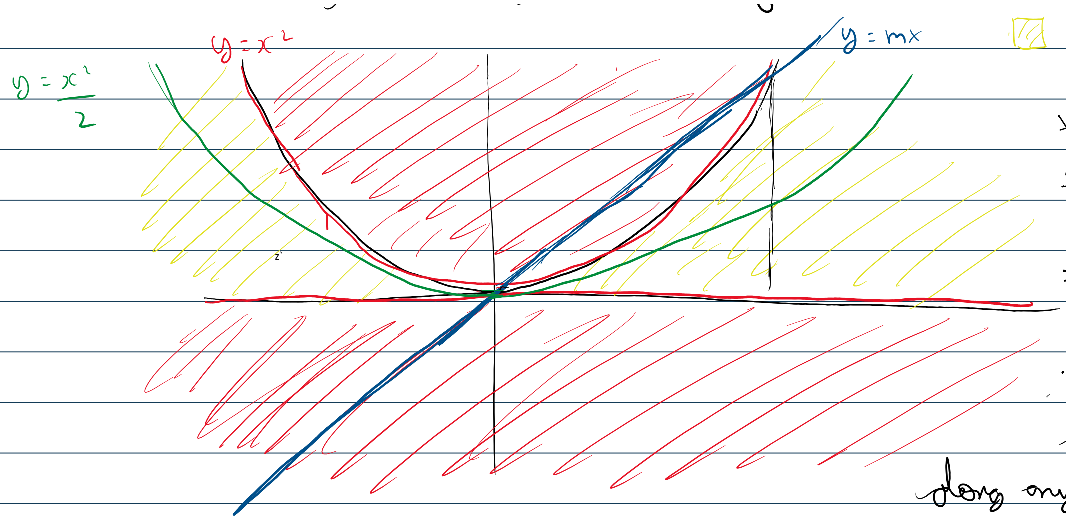
$$\Rightarrow \lim_{x \to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m^2}{1 + m^2}$$

clearly limit is non-existent and cannot be made continuous by defining $f(0,0)$ as the limit varies with m which does not depend on $f(0,0)$.

7. $f : \mathbb{R}^2 \to \mathbb{R}$, Sketching Domain of $f$

- $f$ takes Value 0
- $f$ takes Value 1






– $f$ takes value 1

$\forall x < m \Rightarrow x^2 < mx$ $\Downarrow$ $y$

$\underline{\underline{f(x,y) = 0}}$

$\Rightarrow$ on $y = mx$

$f(x,y) = 0$

$\therefore$ $f(x,y) \to 0$

as $(x,y) \to 0$

along any straight line through $(0,0)$

Consider curve $y = x^2/2$ since $\frac{1}{2} < 1$

$\frac{x^2}{2} < x^2$

$\therefore$ on $y = x^2/2$ $\quad 0 \leq y < x^2$

$\Rightarrow f(x,y) = 1$.

Except at origin as there $y = 0 \Rightarrow f(x,y) = 0$

It is clear from above that $f$ is not continuous at $(0,0)$.

8. The Question implies that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists

$\therefore$ finding limit through $y = x$ is enough

$$\Rightarrow \lim_{x \to 0} \frac{\sin 2x^2}{2x^2} = 1$$

$\therefore$ define $f(0,0) = 1$ to make it continuous at $(0,0)$

9. $f$ is continuous at $a$

$\Rightarrow \forall \varepsilon > 0 \ \exists \ \delta > 0$

S.t $\forall$ if $\|x-a\| < \delta$

$$|f(x) - f(a)| < \varepsilon$$

take $\varepsilon = \frac{|f(a)|}{3}$

$\Rightarrow$

$$-\frac{|f(a)|}{3} < f(x) - f(a) < \frac{|f(a)|}{3}$$

$$\Rightarrow \quad f(a) - \frac{|f(a)|}{3} < f(x) < f(a) + \frac{|f(a)|}{3}$$

if $f(a) > 0$

$$f(x) > \frac{2}{3} f(a) > 0$$

if $f(a) < 0$

$$f(x) < \frac{2}{3} f(a) < 0$$

$\Rightarrow$ $\forall$ $x \in B(a;\delta)$

$f(x)$ & $f(a)$ have same sign.