Statistics II: Introduction to Inference

Problem set 5

1. Consider the testing problem $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$, where

x :	-4	-3	0	1	2	5
f_0 :	0.05	0.20	0.30	0.15	0.25	0.05
f_1 :	0.15	0.30	0.05	0.05	0.25	0.20

Using Neyman Pearson (NP) lemma find the most powerful (MP) test at level 0.05 and level 0.075 for testing H_0 and H_1.

2. Let $\phi^\star$ be an MP size α test for $H_0 : \mathbf{X} \sim f_0(\mathbf{x})$ against $H_1 : \mathbf{X} \sim f_1(\mathbf{x})$, and let $\beta^\star$ be the power of $\phi^\star$, $0 < \beta^\star < 1$.
   (a) Show that $\phi^{\star\star} = (1 - \phi^\star)$ is an MP test for testing $H_0 : \mathbf{X} \sim f_1(\mathbf{x})$ against $H_1 : \mathbf{X} \sim f_0(\mathbf{x})$ at level $(1 - \beta^\star)$.
   (b) Further, show that if $\phi^\star$ is an unbiased test, then $\phi^{\star\star}$ is also unbiased.

3. Let ϕ_1 and ϕ_2 be 2 size-α tests for testing $H_0 : \theta \in \Theta_0$ against $H_A : \theta \in \Theta_A$ and $\phi^\star$ is a convex combination of ϕ_1 and ϕ_2. Show that $\phi^\star$ is a level-α test. What can say about the power function of $\phi^\star$?

4. The lifetime of equipment is normally distributed with mean θ and standard distribution 5. For testing the null hypothesis $H_0 : \theta \leq 30$ against the alternative hypothesis $H_A : \theta > 30$, a random sample of size n is chosen. Determine n and the cutoff c such that the test
$$\phi(\mathbf{x}) = 1, \ \text{ if } \ \bar{x} \geq c, \qquad \phi(\mathbf{x}) = 0, \ \text{ if } \ \bar{x} < c$$
has power function values 0.1 and 0.9 at the points $\theta = 30$ and $\theta = 35$ respectively. Draw the power function of the resultant test.

5. Let X be distributed as $U(0, \theta)$ and $X_{(n)}$ denote the largest order statistic based on a random sample of size n from this distribution. We reject $H_0 : \theta = 1$ and accept $H_1 : \theta \neq 1$ if either $x_{(n)} \leq 1/2$ or $x_{(n)} \geq 1$. Find the power function of the test.

6. Based on a random sample $X_1, \ldots, X_n$, derive the MP size-α test for testing $H_0 : \theta = \theta_0$ against $H_A : \theta = \theta_1 \, (> \theta_0)$ for the population with the following pdf
$$f_X(x; \theta) = (\sqrt{2\pi}\theta)^{-1} e^{-x^2/2\theta^2}; -\infty < x < \infty; \ \theta > 0, \quad \text{and} \quad f_X(x; \theta) = 0, \quad \text{otherwise.}$$
Will the MP test obtained above be UMP for testing $H_0 : \theta \leq \theta_0$, against $H_1 : \theta > \theta_0$.

[Hint: To generalize to $H_0 : \theta \leq \theta_0$ you need to show that the power function of the MP test obtained above is of the form
$$\beta_\phi(\theta) = P\left(T_n \geq K_0^2/\theta^2 \mid T_n \sim \chi^2_n\right),$$
for some fixed value of K_0. Then from the monotonicity of the CDF you can show that $\beta_\phi(\theta)$ is an increasing function of θ. This in turn will lead to the fact that the size of the generalized test is $\beta_\phi(\theta_0)$.]

7. Let X be distributed as $f_X(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1, \theta > 0$, and $f_X(x;\theta) = 0$, otherwise, $\theta > 0$. To test $H_0 : \theta = \theta_1$ against $H_A : \theta > \theta_1$ based on a sample of size n , the following critical region is proposed $\mathbb{C} = \{\mathbf{x} : \prod_{i=1}^n x_i \geq 0.5\}$. Find the power function of the above test.

   [Hint: It might be helpful to consider the transformation $Y = -\log X$.]

8. Let X be an observation in $(0, 1)$. Find an

   MP level-α test of $H_0 : X \sim f_0$ against $H_A : X \sim \texttt{Uniform}(0, 1)$, where, $f_0(x) = 4x$ if $0 < x < \dfrac{1}{2}$, or $f_0(x) = 4(1 - x)$ if $\dfrac{1}{2} \leq x < 1$.

9. Suppose that $X_1, \ldots, X_n$ are iid with a common pdf $f(x)$, which takes one of the following forms:

$$f_0(x) = \begin{cases} 3x^2/64 & 0 < x < 4 \\ 0 & \text{otherwise,} \end{cases} \qquad f_1(x) = \begin{cases} 3\sqrt{x}/16 & 0 < x < 4 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the MP level-α test for testing $H_0 : X \sim f_0(x)$ against $H_A : X \sim f_1(x)$.

10. Based on a random sample $X_1, \ldots, X_n$, derive the UMP size- α test for testing $H_0 : \theta \leq \theta_0$ against $H_A : \theta > \theta_0$ for the location exponential population with pdf $f_X(x;\theta) = \theta^{-1} \exp\{-(x - \mu)/\theta\}$; with $x > \mu$, $\mu \in \mathbb{R}$ and $\theta > 0$, when μ is known.

    [Hint: First consider the MP test for testing $H_0 : \theta = \theta_0$ against $H_0 : \theta = \theta_1 (> \theta_0)$, and then generalize. Towards that, show that the power function of the MP test is

$$\beta_\phi(\theta) = P\left(T_n > k_0/\theta \mid T_n \sim \texttt{Gamma}(n, 1)\right).$$

    Hence show that $\beta_\phi(\theta)$ is an increasing function of θ.]

11. Let X be an observation from $\texttt{Poisson}(\theta)$. Find an UMP level-α test for testing $H_0 : \theta \leq \theta_0$ against $H_A : \theta > \theta_0$.

12. Suppose $X_1, \cdots, X_m$ be a random sample of size m from $B(n, p)$, find the UMP level- α test for testing $H_0 : p \leq p_0$ against $H_A : p > p_0$.