# Statistics II: Introduction to Inference

## Problem set 7

1. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, 1)$.

   (a) Consider testing the hypotheses
   $$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu > \mu_0.$$
   Suppose we want the power of the MP test of size $\alpha$ to be at least $(1-\beta)$, for some fixed $0 < \beta < 1$ when $\mu = \mu_1$ ($\mu_1 > \mu_0$). Determine the minimum sample size $n$ required to achieve this power level.

   (b) Find the confidence interval for $\mu$ based on the sample. Then, determine the minimum sample size $n$ such that the length of the confidence interval for $\mu$ is at most $l$.

2. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$ and $Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$, where $\sigma_1^2, \sigma_2^2$ known.

   (a) Consider testing the hypotheses
   $$H_0 : \mu_1 = \mu_2 \quad \text{vs} \quad H_1 : \mu_1 > \mu_2.$$
   Suppose we want the power of the MP test of size $\alpha$ to be at least $(1-\beta)$, for some fixed $0 < \beta < 1$ when $\mu_1 - \mu_2 = \delta$ ($\delta > 0$). Determine the minimum sample size $n$ required to achieve this power level.

   (b) Construct a confidence interval for the difference $\mu_1 - \mu_2$. Determine the minimum sample size $n$ such that length of the confidence interval does not exceed a specified value $l$.

3. Let $X_1, \ldots, X_n \overset{iid}{\sim} \texttt{Uniform}(0, \theta)$.

   (a) Consider testing the hypotheses
   $$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0.$$
   Find the minimum sample size $n$ such that the most powerful (MP) test of size $\alpha$ has power at least $1 - \beta$ for some fixed $0 < \beta < 1$ when $\theta = \theta_1$ ($\theta_1 > \theta_0$).

   (b) Construct a confidence interval $(L, U)$ for $\theta$. Determine the minimum sample size $n$ such that the ratio of the upper bound (U) to the lower bound (L) of the confidence interval does not exceed a specified value $l$.

4. Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. Construct a confidence interval for $\sigma^2$. Determine the minimum sample size $n$ (numerically) such that the ratio of the upper bound and lower bound of the confidence interval does not exceed a specified value $l = 1.5$. The following table provides critical values of the chi-squared distribution for various sample sizes:

| Sample Size $n$ | 187 | 188 | 189 | 190 | 191 | 192 | 193 |
|---|---|---|---|---|---|---|---|
| $\chi^2_{n-1;\,1-\alpha/2}$ | 150.126 | 151.024 | 151.923 | 152.822 | 153.721 | 154.621 | 155.521 |
| $\chi^2_{n-1;\,\alpha/2}$ | 225.660 | 226.761 | 227.863 | 228.964 | 230.064 | 231.165 | 232.265 |

Here, $\chi^2_{n-1;\,\alpha/2}$ and $\chi^2_{n-1;\,1-\alpha/2}$ denote the upper and lower critical values, respectively, of the chi-squared distribution with $n-1$ degrees of freedom, respectively.