Statistics II: Introduction to Inference

Problem set 4

1. For a random variable X, the following are known

$$P(X \geq 0) = 1 \qquad \text{and} \qquad P(X \geq 10) = 1/5.$$

   Prove that $E(X) \geq 2$.

2. Suppose that X is a random variable for which $E(X) = 10$, $P(X \leq 7) = 0.2$ and $P(X \geq 13) = 0.3$. Then show that $\text{var}(X) \geq 9/2$.

3. Consider a probability distribution with CDF F, expectation μ and variance σ^2. Find the minimum size, say n, of random samples which ensures at least 0.99 probability of the event that the sample mean $\bar{X}_n$ will lie within 2σ limit of the expectation μ, i.e., $\mu - 2\sigma \leq \bar{X}_n \leq \mu + 2\sigma$.

4. Let $Z_1, Z_2, \ldots$ be a sequence of random variables, and suppose that $n = 1, 2, \ldots$, the distribution Z_n is as follows

$$P(Z_n = 0) = 1 - \frac{1}{n}, \qquad \text{and} \qquad P(Z_n = n^2) = \frac{1}{n}, \qquad \text{for} \quad n = 1, 2, \ldots.$$

   Show that

$$\lim_{n\to\infty} E(Z_n) = \infty, \quad \text{and} \quad Z_n \xrightarrow{p} 0, \quad \text{i.e., for any } \epsilon > 0, \quad P(|Z_n| > \epsilon) \to 0, \quad \text{as} \quad n \to \infty.$$

   [Note: This example shows that the sufficient condition for consistency of a sequence of estimators is not *necessary* (i.e., the converse is not true).]

5. Suppose that 75% of the people in a certain metropolitan area live in the city and 25% of the people live in the suburbs. If 1200 people attending a certain concert represent a random sample from the metropolitan area, what is the probability that the number of people from the suburbs attending the concert will be fewer than 270?

6. Suppose that a random sample of size n is to be taken from a distribution with mean is μ and the standard deviation is $\sigma = 3$. Use the central limit theorem to determine approximately the smallest value of n for which the following relation will be satisfied:

$$P\left(|\bar{X}_n - \mu| < 0.3\right) \geq 0.95.$$

7. Suppose that the proportion of defective items in a large manufactured lot is 0.1. What is the smallest random sample of items that must be taken from the lot in order for the probability to be at least 0.99 that the proportion of defective items in the sample will be less than 0.13?

8. Let $X_1, \ldots, X_n$ be a random sample from $\texttt{uniform}(0, \theta)$ distribution and $T_n = X_{(n)}$ be the maximum order statistic. Show that $Z_n = n(T_n - \theta) \xrightarrow{d} Z$ where Z has the CDF F_Z

$$F_Z = \begin{cases} \exp\{z/\theta\} & \text{if} \quad z < 0, \\ 1 & \text{if} \quad z \geq 0. \end{cases}$$

9. Consider the random variable with X with the following specifications $E(X) = \mu$ and the r-th order central moments μ_r are $\mu_2 = 5/4,\ \ \mu_4 = 125/2$. What is the best possible upperbound of the probability of the event that $\bar{X}_n \in [\mu - 1, \mu + 1]$ which can be obtained using a random sample of size $n = 20$.

10. Let $X_1, \ldots, X_n$ be a random sample from some distribution with expectation μ, variance σ^2 and finite forth order raw moment μ'_4. Determine the sequence of real numbers $\{a_n\}$ and the random variable Z such that the sequence of random variables $Z_n = \sum_{i=1}^{n} X_i^2/\sqrt{n} + \bar{X}_n^2$ satisfies $Z_n - a_n \xrightarrow{d} Z$. What is the distribution of Z?