DEPARTMENT OF MATHEMATICS

MATH2301 Assignment 5 Solutions Semester 1, 2010

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(1) We need to check all 10 axioms of a vector space.

1. $\forall x, y \in V$, by definition
$$x \hat{+} y = x + y + a \in V.$$

2. $\forall x \in V$ and $k \in \mathbb{R}$,
$$k \hat{\cdot} x = k \cdot x + (k-1) \cdot a \in V.$$

3.
$$\begin{aligned} x \hat{+} y &= x + y + a, \\ y \hat{+} x &= y + x + a = x + y + a \end{aligned}$$
since the operation $+$ is commutative on V. Hence
$$x \hat{+} y = y \hat{+} x.$$

4.
$$\begin{aligned} x \hat{+} (y \hat{+} z) &= x \hat{+} (y + z + a) = x + y + z + a + a = x + y + z + 2a, \\ (x \hat{+} y) \hat{+} z &= (x + y + a) \hat{+} z = x + y + a + z + a = x + y + z + 2a. \end{aligned}$$
In the first line we have used the fact that $+$ is associative on V, and in the second line we have used the fact that $+$ is associative and commutative on V. Hence
$$x \hat{+} (y \hat{+} z) = (x \hat{+} y) \hat{+} z.$$

5. For there to be a zero vector, say h, we must have
$$x \hat{+} h = x = h \hat{+} x.$$
By definition, this would mean that $x + h + a = x = h + x + a$. Since we can use the cancellation law with $+$, along with commutativity and associativity, this can be true if $h + a = 0$ or $h = -a$. We take this to be the zero vector under the addition $\hat{+}$.

6. For any $w \in V$, we need to find a vector $g \in V$ such that
$$w \hat{+} g = -a = g \hat{+} w.$$
By definition, this means
$$w + g + a = -a = g + w + a.$$
Once again using, commutativity, associativity and the cancellation law, the above tells us that we require $g = -w - 2a$. We take the vector $g = -w - 2a$ to be the additive inverse of w under $\hat{+}$.

7.

$$\begin{aligned}
k\hat{\cdot}(l\hat{\cdot}x) &= k\hat{\cdot}(l\cdot x + (l-1)\cdot a) \\
&= k\cdot(l\cdot x + (l-1)\cdot a) + (k-1)\cdot a \\
&= (kl)\cdot x + k\cdot((l-1)\cdot a) + (k-1)\cdot a \\
&= (kl)\cdot x + (kl-k)\cdot a + k\cdot a + (-a) \\
&= (kl)\cdot x + (kl)\cdot a + (-k)\cdot a + k\cdot a + (-a) \\
&= (kl)\cdot x + (kl-1)\cdot a \\
&= (kl)\hat{\cdot}x
\end{aligned}$$

8.

$$\begin{aligned}
k\hat{\cdot}(x\hat{+}y) &= k\hat{\cdot}(x+y+a) \\
&= k\cdot(x+y+a) + (k-1)\cdot a \\
&= k\cdot x + k\cdot y + k\cdot a + k\cdot a + (-a).
\end{aligned}$$

Also,

$$\begin{aligned}
k\hat{\cdot}x &= k\cdot x + (k-1)\cdot a, \\
k\hat{\cdot}y &= k\cdot y + (k-1)\cdot a, \\
\Rightarrow \quad k\hat{\cdot}x\hat{+}k\hat{\cdot}y &= k\cdot x + (k-1)\cdot a + k\cdot y + (k-1)\cdot a + a \\
&= k\cdot x + k\cdot y + k\cdot a + (-a) + k\cdot a + (-a) + a \\
&= k\cdot x + k\cdot y + k\cdot a + k\cdot a + (-a) \\
&= k\hat{\cdot}(x\hat{+}y).
\end{aligned}$$

9.

$$\begin{aligned}
(k+l)\hat{\cdot}x &= (k+l)\cdot x + (k+l-1)\cdot a, \\
k\hat{\cdot}x\hat{+}l\hat{\cdot}x &= k\cdot x + (k-1)\cdot a + l\cdot x + (l-1)\cdot a + a \\
&= (k+l)\cdot x + (k-1+l-1+1)\cdot a \\
&= (k+l)\cdot x + (k+l-1)\cdot a \\
&= (k+l)\hat{\cdot}x.
\end{aligned}$$

10.

$$\begin{aligned}
1\hat{\cdot}x &= 1\cdot x + (1-1)\cdot a \\
&= x + 0\cdot a = x.
\end{aligned}$$

Therefore V is a vector space under $\hat{+}$ and $\hat{\cdot}$.

(2) To show that $M_n(\mathbb{F}) = V_n^s \oplus V_n^{\text{S-S}}$, we need to show the following:

1. V_n^s and $V_n^{\text{S-S}}$ are subspaces of $M_n(\mathbb{F})$.
2. Every $a \in M_n(\mathbb{F})$ can be written in the form $a = b + c$ where $b \in V_n^s$ and $c \in V_n^{\text{S-S}}$.
3. $V_n^s \cap V_n^{\text{S-S}} = \{0\}$

To this end, consider the following.

1. To prove that $V_n^{\text{S-S}}$ is a subspace of $M_n(\mathbb{F})$, we need $V_n^{\text{S-S}}$ to be closed under addition and scalar multiplication.
   Firstly, if $A, B \in V_n^{\text{S-S}}$,
   $$(A+B)^T = A^T + B^T = -A - B = -(A+B),$$
   so $V_n^{\text{S-S}}$ closes under addition. For $A \in V_n^{\text{S-S}}$, $k \in \mathbb{F}$,
   $$(kA)^T = kA^T = -kA,$$
   so $V_n^{\text{S-S}}$ closes under scalar multiplication. Hence $V_n^{\text{S-S}}$ is a subspace.
   Now let $A, B \in V_n^s$. Again, using the property of the transpose $(kA + lB)^T = kA^T + lB^T$ we have
   $$\begin{aligned} (A+B)^T = A^T + B^T &= A + B \\ \Rightarrow\ A + B &\in V_n^s, \end{aligned}$$
   and
   $$\begin{aligned} (kA)^T = kA^T &= kA \\ \Rightarrow V_n^s &\text{ is a subspace.} \end{aligned}$$
   (This example was outlined in lectures.)
2. Let $A \in V_n^{\text{S}} \cap V_n^{\text{S-S}}$. We know the intersection of two subspaces is a subspace. So, for $k \in \mathbb{F}$, we then have $(kA)^T = kA = -kA \Rightarrow 2kA = 0$. Since $\mathbb{F}$ is not of characteristic $2 \Rightarrow A = 0$ (this highlights the assumption that $\mathbb{F}$ is not of characteristic 2). Hence $V_n^{\text{S}} \cap V_n^{\text{S-S}} = \{0\}$.
3. Every matrix in $M_n(\mathbb{F})$ can be written as the sum of a matrix in V_n^{S} and a matrix in $V_n^{\text{S-S}}$. Note that every matrix in V_n^{S} is of the form
   $$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$
   and every matrix in $V_n^{\text{S-S}}$ is of the form
   $$\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{pmatrix}.$$
   Every matrix $B \in M_n(\mathbb{F})$ can be written in the form
   $$\begin{aligned} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} &= \begin{pmatrix} b_{11} & \frac{1}{2}(b_{12}+b_{21}) & \cdots & \frac{1}{2}(b_{1n}+b_{n1}) \\ \frac{1}{2}(b_{12}+b_{21}) & b_{22} & \cdots & \frac{1}{2}(b_{2n}+b_{n2}) \\ \vdots & & \ddots & \vdots \\ \frac{1}{2}(b_{1n}+b_{n1}) & \frac{1}{2}(b_{2n}+b_{n2}) & \cdots & b_{nn} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \frac{1}{2}(b_{12}-b_{21}) & \cdots & \frac{1}{2}(b_{1n}-b_{n1}) \\ -\frac{1}{2}(b_{12}-b_{21}) & 0 & \cdots & \frac{1}{2}(b_{2n}-b_{n2}) \\ \vdots & & \ddots & \vdots \\ -\frac{1}{2}(b_{1n}-b_{n1}) & -\frac{1}{2}(b_{2n}-b_{n2}) & \cdots & 0 \end{pmatrix} \end{aligned}$$

$\Rightarrow$ any matrix $B \in M_n(\mathbb{F})$ can be written in the form $B = B^s + B^{s-s}$ where $B^s \in V_n^{\text{S}}$ and $B^{s-s} \in V_n^{\text{S-S}}$.

Another way would be to expand in terms of basis vectors. We would need a basis for V_n^{S}. Such a set in terms of the standard basis of $M_n(\mathbb{F})$ is

$$\{\ e_{ii},\ e_{ij} + e_{ji}\ ;\ i \neq j, i, j = 1, \ldots, n\ \}.$$

We could then observe that $\forall B \in M_n(\mathbb{F})$,

$$\begin{aligned} B &= \sum_{1 \leq i,j \leq n} b_{ij} e_{ij} \\ &= \sum_{1 \leq i \leq n} b_{ii} e_{ii} + \sum_{1 \leq i < j \leq n} \left(\frac{1}{2} b_{ij} + \frac{1}{2} b_{ij} + \frac{1}{2} b_{ji} - \frac{1}{2} b_{ji} \right) e_{ij} \\ &\quad + \sum_{1 \leq i < j \leq n} \left(\frac{1}{2} b_{ji} + \frac{1}{2} b_{ji} + \frac{1}{2} b_{ij} - \frac{1}{2} b_{ij} \right) e_{ji} \\ &= \sum_{1 \leq i \leq n} b_{ii} e_{ii} + \sum_{1 \leq i < j \leq n} \left(\frac{1}{2} b_{ij} + \frac{1}{2} b_{ji} \right) (e_{ij} + e_{ji}) \\ &\quad + \sum_{1 \leq i < j \leq n} \left(\frac{1}{2} b_{ij} - \frac{1}{2} b_{ji} \right) (e_{ij} - e_{ji}). \end{aligned}$$

Hence we have written B as a linear combination of vectors in the bases of V_n^{S} and $V_n^{\text{S-S}}$ which is sufficient to prove part 3.

(3) (a) Let $A = (a_{ij})$, then $\text{tr}(A) = 0$ gives

$$a_{11} + a_{22} + \ldots + a_{nn} = 0.$$

Therefore $a_{nn} = -a_{11} - a_{22} - \ldots - a_{n-1\ n-1}$, which we interpret in terms of the standard basis of $M_n(\mathbb{F})$. Let e_{ij} denote the matrix with a 1 in the ith row and jth column and 0 everywhere else. A basis is

$$\{e_{ij}\ (1 \leq i \neq j \leq n), e_{ii} - e_{nn}\ (1 \leq i \leq n-1\}$$

and therefore the dimension $= (n^2 - n) + (n - 1) = n^2 - 1$.

(b) Again, we write $A = (a_{ij})$. If we impose the condition $A^T = -A$, we must have that

$$\begin{aligned} a_{ij} &= -a_{ji}, \quad i \neq j, \\ a_{ii} &= -a_{ii}. \end{aligned}$$

Since $a_{ij} \in \mathbb{F}$, which is a field not of charateristic 2, the second equation tells us that $a_{ii} = 0$. Note if $\mathbb{F}$ *was* a field of characteristic 2, the a_{ii} would remain unrestricted since in that case $2a_{ii} = 0$. We then have

$$V_n^{\text{S-S}} = \{\ (a_{ij}) = A \in M_n(\mathbb{F}) | a_{ij} = -a_{ji},\ a_{ii} = 0\ \}.$$

In terms of the standard basis of $M_n(\mathbb{F})$ (let e_{ij} denote the matrix with a 1 in the i-j entry and 0 elsewhere), a basis of $V_n^{\text{S-S}}$ must be

$$\{\ e_{ij} - e_{ji};\ 1 \leq i < j \leq n\ \}.$$

Note for any matrix $C \in M_n(\mathbb{F})$, we have $(C - C^T)$ is skew-symmetric. The dimension is $\frac{1}{2}n(n-1)$.

(c) For a vector to be palindromic, the ith entry of the vector must coincide with the $(n-i+1)$th entry. In particular when n is odd we must take care not to count these entries twice, since the $\frac{n+1}{2}$th entry appears in the "middle entry". In terms of the standard basis of $\mathbb{R}^n$, we let e_i deonte the vector with a 1 in the ith entry and 0 everywhere else. A basis for the space of palindromic vectors is therefore

$$\{e_i + e_{n-i+1}, 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil\}$$

and the dimension is therefore $\lceil \frac{n}{2} \rceil$. (Here we have used the "ceiling function" $\lceil x \rceil =$ smallest integer $\geq x$).

(4) (a) Applying the Gram-Schmidt process to $\{1, x, x^2, x^3\}$ using $< f, g >= \displaystyle\int_{-1}^{1} f(t)g(t)\, dt$ leads to the following calculation.

Set $v_1 = 1$.

$$v_2 = x - \frac{< x, 1 >}{< 1, 1 >}1 = x - \frac{\int_{-1}^{1} t\, dt}{\int_{-1}^{1} dt} = x - \frac{0}{2} = x.$$

$$\begin{aligned} v_3 &= x^2 - \frac{< x^2, 1 >}{< 1, 1 >}1 - \frac{< x^2, x >}{< x, x >}x = x^2 - \frac{\int_{-1}^{1} t^2\, dt}{\int_{-1}^{1} dt} - x\frac{\int_{-1}^{1} t^3\, dt}{\int_{-1}^{1} t^2\, dt} \\ &= x^2 - \frac{2/3}{2} - x\frac{0}{2/3} = x^2 - \frac{1}{3}. \end{aligned}$$

$$\begin{aligned} v_4 &= x^3 - \frac{< x^3, 1 >}{< 1, 1 >}1 - \frac{< x^3, x >}{< x, x >}x - \frac{< x^3, x^2 - \frac{1}{3} >}{< x^2 - \frac{1}{3}, x^2 - \frac{1}{3} >}(x^2 - \frac{1}{3}) \\ &= x^3 - \frac{\int_{-1}^{1} t^3\, dt}{2} - x\frac{\int_{-1}^{1} t^4\, dt}{\int_{-1}^{1} t^2\, dt} - (x^2 - \frac{1}{3})\frac{\int_{-1}^{1}(t^5 - \frac{1}{3}t^3)dt}{\int_{-1}^{1}(t^4 - \frac{2}{3}t^2 + \frac{1}{9})dt} \\ &= x^3 - x\frac{[\frac{1}{5}t^5]_{-1}^{1}}{[\frac{1}{3}t^3]_{-1}^{1}} = x^3 - x\frac{2/5}{2/3} = x^3 - \frac{3}{5}x. \end{aligned}$$

$\Rightarrow \{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x\}$ is an orthogonal basis.

(b)

Apply the Gram-Schmidt process to $\left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & 17 \\ 2 & -6 \end{pmatrix} \right\}$.

Set $v_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$.

$$v_2 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} -1 & 9 \\ 5 & 1 \end{pmatrix}, v_1 \right\rangle}{\langle v_1, v_1 \rangle}.$$

Note that

$$\left\langle \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \right\rangle = \mathrm{tr}\left(\begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} \right) = \mathrm{tr}\begin{pmatrix} -8 & * \\ * & 44 \end{pmatrix} = 36.$$

and

$$\left\langle \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \right\rangle = \mathrm{tr}\left(\begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \right) = \mathrm{tr}\begin{pmatrix} 10 & * \\ * & 26 \end{pmatrix} = 36.$$

Here the $*$ denotes the unimportant off-diagonal terms. Therefore

$$v_2 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} - \frac{36}{36} \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}.$$

We also have

$$v_3 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix}, v_1 \right\rangle}{\langle v_1, v_1 \rangle} - \frac{\left\langle \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix}, v_2 \right\rangle}{\langle v_2, v_2 \rangle}.$$

Note that

$$\left\langle \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} \right\rangle = \mathrm{tr} \left(\begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right) = \mathrm{tr} \begin{pmatrix} 19 & * \\ * & -91 \end{pmatrix} = -72,$$

$$\left\langle \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix}, \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \right\rangle = \mathrm{tr} \left(\begin{pmatrix} -4 & 6 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right) = \mathrm{tr} \begin{pmatrix} -16 & * \\ * & -56 \end{pmatrix} = -72$$

and

$$\left\langle \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}, \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \right\rangle = \mathrm{tr} \left(\begin{pmatrix} -4 & 6 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \right) = \mathrm{tr} \begin{pmatrix} 52 & * \\ * & 20 \end{pmatrix} = 72.$$

We then have

$$\begin{aligned} v_3 &= \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} - \frac{-72}{36} \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} - \frac{-72}{72} \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} + \begin{pmatrix} 6 & 10 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}. \end{aligned}$$

This gives the orthogonal basis

$$\left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}, \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix} \right\}$$

(5) (a)

$$\begin{aligned} T\left(\lambda \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) &= T\left(\begin{pmatrix} \lambda a_1 + a_2 & \lambda b_1 + b_2 \\ \lambda c_1 + c_2 & \lambda d_1 + d_2 \end{pmatrix} \right) \\ &= \begin{pmatrix} \lambda a_1 + a_2 & \lambda d_1 + d_2 \\ \lambda a1 + a_2 + \lambda d_1 + d_2 & \lambda b_1 + b_2 - \lambda c_1 - c_2 \end{pmatrix} \\ &= \lambda \begin{pmatrix} a_1 & d_1 \\ a_1 + d_1 & b_1 - c_1 \end{pmatrix} + \begin{pmatrix} a_2 & d_2 \\ a_2 + d_2 & b_2 - c_2 \end{pmatrix} \\ &= \lambda T \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + T \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \end{aligned}$$

$\Rightarrow T$ is linear.

(b) In $M_{m\times n}(\mathbb{F})$, we make the association between the standard basis β and $\mathbb{F}^{mn}$ as $e_{ij} \longrightarrow e_{n(i-1)+j}$. The action of T on the standard basis is

$$\begin{aligned} T(e_{11}) &= e_{11} + e_{21} \\ T(e_{12}) &= e_{22} \\ T(e_{21}) &= -e_{22} \\ T(e_{22}) &= e_{12} + e_{21} \end{aligned}$$

The matrix representation of T with respect to β is

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

It is easy to verify that the characteristic polynomial is $-t^3(1-t)$. Therefore the eigenvalues are 0 and 1. Now we determine the corresponding eigenvectors.
For eigenvalue 0, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow\ a, d = 0,\ b = c.$$

$\Rightarrow c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to eigenvalue 0 for $c \neq 0$.

For eigenvalue 1, we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow\ b = d,\ c = 0,\ a = -d$$

$\Rightarrow d \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to eigenvalue 1 for $d \neq 0$.

(c) $\ker(T)$ is the set of all $A \in M_2(\mathbb{R})$ such that $T(A) = 0$. This space corresponds to eigenspace E_0, so a basis is $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

$$\text{Im}(T) = \text{span}\left\{ T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$\Rightarrow$ a basis for $\text{Im}(T)$ is $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

The T-cyclic subspace generated by $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$:

$$T(A) = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$T^2(A) = T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

$$T^3(A) = T\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$T^4(A) = T\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = T\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

A basis is given by $\{A, T(A), T^2(A), T^3(A)\}$.

The T-cyclic subspace generated by $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$:

$$T(B) = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T^2(B) = T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T^3(B) = T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

A basis is given by $\{B, T(B), T^2(B)\}$.

The T-cyclic subspace generated by $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$:

$$T(C) = T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

$$T^2(C) = T\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$T^3(C) = T\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

A basis is given by $\{C, T(C), T^2(C)\}$.

The T-cyclic subspace generated by $D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$:

$$T(D) = T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T^2(D) = T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

A basis is given by $\{D, T(D)\}$.

The eigenspace E_0 is the same as $\ker(T)$ given above.

From the answer to part (b), the eigenspace E_1 has basis $\left\{\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$