BDS Lecture

I

B

Ising Model in $d=2$ and phase transitions in $d=1$, $d=2$, critical exponents, RG, MC etc.

$$\uparrow_{s_1} \quad \uparrow_{s_2} \qquad s_1 = \pm 1, \quad s_2 = \pm$$

$$\mathcal{H} = E(s_1, s_2) = -J s_1 s_2 - B(s_1 + s_2)$$

$J > 0$

$$(4) \quad \begin{cases} s_1 = 1, \; s_2 = 1 \\ s_1 = 1, \; s_2 = -1 \\ s_1 = -1, \; s_2 = +1 \\ s_1 = -1, \; s_2 = -1 \end{cases}$$

$$P(E_i) = \frac{e^{-E_i/k_BT}}{Z} \qquad Z(J, B, T)$$

$$(k_BT)^{-1} = \beta$$

$$= \frac{1}{Z} e^{-\beta E_i} \qquad k_B = 1.3 \times 10^{-23} \; J/K$$

$$F = \langle E \rangle - TS$$

E : average energy

$$\langle E \rangle = \frac{\sum_c E_c \, e^{-\beta E_c}}{Z}$$

$$Z = \sum_c e^{-\beta E_c}$$

Consider $B = 0$.

'energy – entropy argument'

low T: tendency to 'order'

high T: tendency to 'disorder'

low T — high T

magnetism ⟷ paramagnet

solid ← melting → liquid

liquid ← boiling → gas

T_c : critical temperature

$$H = -J s_1 s_2 - B(s_1 + s_2)$$

$$J > 0, \quad B > 0$$

$$(T, \beta)$$

$$Z(K = \beta J, h = \beta B) = Z(K, h)$$

$$Z = \sum_{\{C\}} e^{+(\beta J) s_1 s_2 + (\beta B)(s_1 + s_2)}$$

$$Z(K, h) = 2\cosh(2h) \cdot e^{K} + 2e^{-K}$$

$$Z = \sum_{\{C\}} \exp\left[K s_1 s_2 + h(s_1 + s_2)\right]$$

$s_1 = s_2 = 1$, e.g.,

$$\exp[K + 2h]$$

$s_1 = s_2 = -1$, $\quad \exp[K - 2h]$

$$\begin{cases} s_1 = +1,\ s_2 = -1, & \exp[-K] \\ s_1 = -1,\ s_2 = +1, & \exp[-K] \end{cases}$$

$$Z(K, h) = 2\exp[K]\cosh(2h) + 2e^{-K}$$

$$H_{\text{Potts}} = -J s_1 s_2 - B(s_1 + s_2)$$

$J > 0$ $\quad s_i = -2, -1, +1, +2$

$(B = 0)$

$Z(K, h)$ Two

HW : Find $Z(K, h)$ for the Potts at T.

HW : Plot $Z(K, h)$ for the Potts as a function of (K, h)

$$M = \frac{s_1 + s_2}{2}$$

1-d Ising Model

$$M = \frac{1}{N}\sum_{i=1}^{N} s_i \quad \dots (1)$$

$B \neq 0$

$$\langle M \rangle = \left\langle \frac{s_1 + s_2}{2} \right\rangle$$

$$= \frac{1}{2}\left[\langle s_1 \rangle + \langle s_2 \rangle\right]$$

$$\langle M \rangle = \frac{1}{N}\sum_{i=1}^{N} \langle s_i \rangle$$

$$\langle M \rangle = \frac{\sum_{\{c\}} \frac{1}{2}(s_1 + s_2)_c e^{-\beta E_c}}{Z} \quad \dots (2)$$

$$= \sum_{\{c\}} \frac{1}{2}(s_1 + s_2)_c P_c$$

C.W.

$$\langle M \rangle (K, h) = \frac{1}{2}\frac{\partial}{\partial h}(\ln Z)$$

(Please show)

Hint :
$$\frac{\partial}{\partial h}(\ln Z) = \frac{1}{Z}\frac{\partial Z}{\partial h}$$

$$Z = \sum_{\{c\}} e^{K s_1 s_2 + h(s_1 + s_2)}$$

$$\frac{1}{2Z}\frac{\partial Z}{\partial h}$$

$$= \sum_{\{c\}} \frac{1}{Z}\frac{(s_1 + s_2)}{2} e^{K s_1 s_2 + h(s_1 + s_2)}$$

$\ln Z$

$F = -k_B$

$= \langle M \rangle$

$\Rightarrow \ln Z = -\beta F$

$$\langle M \rangle = \frac{1}{2}\frac{\partial(-\beta F)}{\partial h} \quad \ldots\ldots (3)$$

show: $$\langle M \rangle = \frac{\sinh(2h)}{\cosh(2h) + e^{-2K}} \quad \ldots\ldots (4)$$

H.W: Define M for $\mathcal{H}_{Potts}$ exactly as for $\mathcal{H}_{Ising}$. Calculate $\langle M \rangle$ using (3) as well as your code! ($\langle M \rangle$ as a function of (K, h))

$$\langle s_1 \rangle = \sum_{\{c\}} (s_1)_c \, p_c$$

$$p_c = \frac{e^{-\beta E_c}}{Z}$$

$$H = -J s_1 s_2 + h_1 s_1 + h_2 s_2$$

$\langle s_i \rangle$, $i = 1$ or 2

$$\frac{\partial(\ln Z)}{\partial h_1} = \frac{1}{Z}\sum_{\{c\}} s_1 \, e^{-\beta H_c}$$

$$= \langle s_1 \rangle$$

$$\frac{\partial(-\beta F)}{\partial h_i} = \langle s_i \rangle$$

C.W.

$$\frac{\partial^2(-\beta F)}{\partial h_1 \partial h_2} = \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle$$

$$\frac{\partial}{\partial h_1}\left(\frac{\partial}{\partial h_2}(\ln Z)\right)$$

$$= \frac{\partial}{\partial h_1}\left[\frac{1}{Z}\frac{\partial Z}{\partial h_2}\right]$$

$$= \frac{-1}{Z^2}\frac{\partial Z}{\partial h_1}\frac{\partial Z}{\partial h_2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial h_1 \partial h_2}$$

$$-\left(\frac{\partial Z}{\partial h_2}\frac{1}{Z}\right)\left(\frac{\partial Z}{\partial h_1}\frac{1}{Z}\right)$$

$$= -\langle s_1\rangle\langle s_2\rangle$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial h_1 \partial h_2}$$

$$= \frac{1}{Z}\frac{\partial^2}{\partial h_1 \partial h_2}\left(\sum_c e^{-\beta E_c}\right)$$

$$= \frac{1}{Z}\frac{\partial^2}{\partial h_1 \partial h_2}\sum_c\left[e^{K s_1 s_2 + h_1 s_1 + h_2 s_2}\right]$$

$$= \frac{1}{Z}\sum_c \frac{\partial}{\partial h_1}\left[s_2\, e^{K s_1 s_2 + h_1 s_1 + h_2 s_2}\right]$$

$$= \frac{1}{Z}\sum_c s_1 s_2\, e^{K s_1 s_2 + h_1 s_1 + h_2 s_2}$$

$$= \langle s_1 s_2\rangle$$

$$\frac{\partial^2(-\beta F)}{\partial h_1 \partial h_2} = \langle s_1 s_2\rangle - \langle s_1\rangle\langle s_2\rangle$$

$$\langle s_1\rangle,\ \langle s_2\rangle$$

$$\tilde{s}_1 = s_1 - \langle s_1\rangle$$

$$\tilde{s}_2 = s_2 - \langle s_2\rangle$$

$$\langle \tilde{s}_1 \tilde{s}_2\rangle = \frac{\sum_c \tilde{s}_1 \tilde{s}_2\, e^{-\beta E_c}}{\quad}$$

$$\overline{\qquad\qquad Z \qquad}$$

$$= \langle (S_1 - \langle S_1\rangle)(S_2 - \langle S_2\rangle)\rangle$$

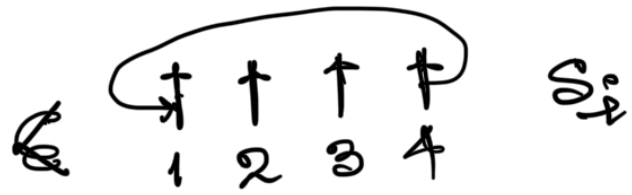
$$= \langle S_1 S_2\rangle - \underbrace{\langle \langle S_1\rangle S_2\rangle}_{-\langle S_1\rangle\langle S_2\rangle}$$

$$- \langle S_1 \langle S_2\rangle\rangle = -\langle S_1\rangle\langle S_2\rangle$$

$$+ \langle S_1\rangle\langle S_2\rangle$$

$$= \langle S_1 S_2\rangle - \langle S_1\rangle\langle S_2\rangle$$

$$= \langle S_1 S_2\rangle_c$$



$$\frac{\partial^4(-\beta F)}{\partial h_1\, \partial h_2\, \partial h_3\, \partial h_4}$$

$$H = -JS_1S_2 - JS_2S_3 - JS_3S_4 - JS_4S_1$$
$$-B_1S_1 - B_2S_2 - B_3S_3 - B_4S_4$$

$K = \beta J,\quad h_i = \beta B_i$

$i = 1, 2, 3, 4$

$$\frac{\partial^4(-\beta F)}{\partial h_1\, \partial h_2\, \partial h_3\, \partial h_4}$$

$$= \langle S_1 S_2 S_3 S_4\rangle$$
$$- \langle S_1 S_2\rangle\langle S_3 S_4\rangle$$
$$- \langle S_1 S_3\rangle\langle S_2 S_4\rangle$$
$$- \langle S_1 S_4\rangle\langle S_2 S_3\rangle \;\ldots\ldots\; (*)$$

<u>H.W.</u> <u>Proof of (*)</u>.

<u>Wick's theorem</u>

2-spin problem

$$\langle s_1 s_2 \rangle = \frac{e^{K}\cosh(2h) - e^{-K}}{e^{K}\cosh(2h) + e^{-K}} \quad \ldots (**)$$

C.W.

$$= \frac{\partial(-\beta F)}{\partial K} \quad \text{(Works for 2 spins)}$$

$$= \frac{\partial \ln Z}{\partial K} = \frac{1}{Z}\frac{\partial \sum_c e^{K s_1 s_2 + h(s_1+s_2)}}{\partial K}$$

$$= \frac{\sum_c s_1 s_2 e^{K s_1 s_2 + h(s_1+s_2)}}{\sum_c e^{K s_1 s_2 + h(s_1+s_2)}} \Big\downarrow \sum_c$$

$$= \frac{e^{K+2h} - 2e^{-K} + e^{K-2h}}{e^{K+2h} + 2e^{-K} + e^{K-2h}}$$

$$= \frac{e^{K}\cosh(2h) - e^{-K}}{e^{K}\cosh(2h) + e^{-K}}$$

$$= \langle s_1 s_2 \rangle$$

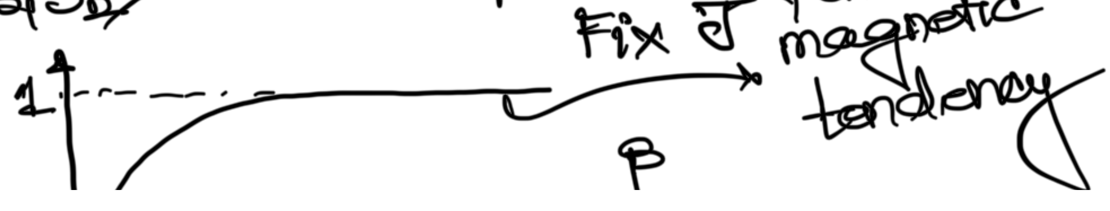
$h=0 \Rightarrow B=0$

$\cosh(2h) = 1$

$$\langle s_1 s_2 \rangle = \frac{e^{K} - e^{-K}}{e^{K} + e^{-K}}$$

$$= \tanh(K)$$

$$\langle s_1 s_2 \rangle = \tanh(\beta J)$$



$0 \longrightarrow \infty$

paramagnetic tendency

$$-\beta \frac{\partial^2 F}{\partial h_1 \partial h_2}\bigg|_{h_1=h_2=0} = \langle s_1 s_2 \rangle_c$$

x

$$\langle M\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle s_i\rangle$$

$\underline{N=2}$

$$\langle M\rangle = \frac{1}{2}\left(\langle s_1\rangle + \langle s_2\rangle\right)$$

$$\langle M\rangle(\beta, J, B) \equiv \langle M\rangle(K, h)$$

$$\chi = \frac{1}{2}\frac{\partial\langle M\rangle}{\partial h}$$

$$= \frac{1}{4}\frac{\partial^2(-\beta F)}{\partial h^2}$$

$$= \frac{1}{4}\left[\frac{\partial}{\partial h^2}(\ln Z)\right]$$

$$= \langle M^2\rangle - \langle M\rangle^2$$

$\boxed{\text{show this as H.W.}}$

$\underline{N=2}$ $\quad \langle s_1 s_2\rangle - \langle s_1\rangle\langle s_2\rangle$

$\langle M\rangle$: order parameter

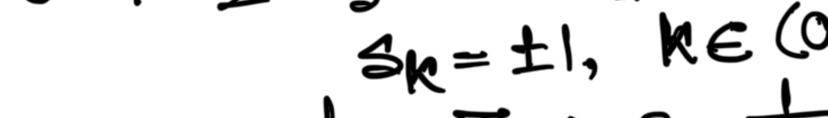
χ : susceptibility

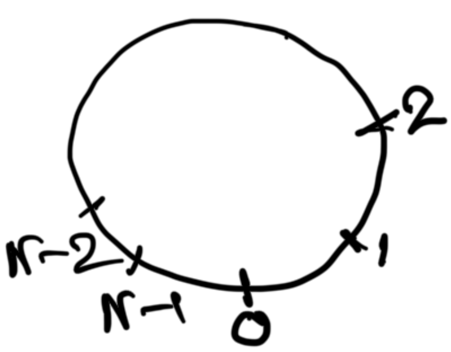
Shankar : (Quantum Field theory in Condensed Matter Physics)

: M. Kardar, I & II

: J. P. Sethna

Ising Model $(d=1)$





$s_k = \pm 1, \quad k \in (0, N)$

$B = 0 = h, \quad T \Rightarrow \beta = \frac{1}{k_B T}$

$J, \quad K = \beta J$

$s_{N-2} s_{N-1}$

$$H = E = -J\sum_{i=0}^{N-2} s_i s_{i+1} + s_{N-1}s_0$$

$\underline{\text{PBC}}$: $s_N = s_0$

$\underline{\text{OBC}}$: Boundaries are distinct

$\underline{\text{PBC}}$: No boundary (all spins are equivalent)

: Translational invariance

$\underline{\text{PBC}}$: $\langle s_{17} s_{11} \rangle = \langle s_{32} s_{26} \rangle$

$\underline{\text{OBC}}$: $\langle s_{17} s_{11} \rangle \neq \langle s_{32} s_{26} \rangle$

$$Z(K) = \sum_{s_i=\pm 1} \exp\left[\sum_{i=0}^{N-1} K(s_i s_{i+1} - 1)\right]$$

. shifts βF by NK

. Fix s_0 ; $t_i = s_i s_{i+1}$

$t_{16} = s_{16} s_{17}$

$t_{15} = s_{15} s_{16}$

$$Z = \sum_{t_i=\pm 1} \exp\left[\sum_{i=1}^{N-1} (t_i - 1)K\right] \quad \left. \begin{matrix} t_i = +1 \\ t_i = -1 \end{matrix} \right\}$$

$$Z = \sum_{t_i=\pm 1} \exp\left[\sum_{i=0}^{N-1} K(t_i - 1)\right]$$

$$= \sum_{t_i=\pm 1} \prod_{i=0}^{N-1} \exp\left[K(t_i - 1)\right]$$

$$= \prod_{i=0}^{N-1} \underbrace{\left\{\sum_{t_i=\pm 1} \exp\left[K(t_i - 1)\right]\right\}}$$

$$= \prod^{N-1} \left(1 + e^{-2K}\right)$$

$i=0$

$$= (1+e^{-2K})^{N}$$

$$Z = 2(1+e^{-2K})^{N}$$

correct

$$Z = 2(1+e^{-2K})^{N}$$

$$\Rightarrow f(K) = \lim_{N\to\infty}\left(\frac{-1}{N}\right)\ln Z$$

$$\beta f = \lim_{N\to\infty} -\frac{1}{N}\ln Z$$

$$f(K) = \lim_{N\to\infty}\left[-\frac{\ln 2}{N} - \ln(1+e^{-2K})\right]$$

If $N\to\infty$, $\frac{\ln 2}{N}\to 0$

(physics of a chain as $N\to\infty$ (thermodynamic limit))

$$f(K) = -\ln(1+e^{-2K})$$

$$\langle s_j s_i\rangle = \sum_{s_k=\pm1}\frac{s_j s_i \exp[K(s_k s_{k+1}-1)]}{Z}$$

WLOG, $j>i$

$$0\le m\le N,\quad s_m^2\Big| = 1$$

$$s_i s_j = \underbrace{s_i s_{i+1}}_{t_i}\,\underbrace{s_{i+1} s_{i+2}}_{t_{i+1}}\, s_{i+2}\cdots s_{j-1}\Big]\,\underbrace{s_{j-1} s_j}_{t_{j-1}}$$

$$\langle s_i s_j\rangle = \langle t_i t_{i+1}\cdots t_{j-1}\rangle$$

$$= \langle t_i \rangle \langle t_{i+1} \rangle \dots \langle t_{j-1} \rangle$$

$$\langle t_{i*} \rangle = \frac{\overbrace{1\cdot e^{0\cdot K}}^{t_{i*}=+1} + \overbrace{(-1)\, e^{-2K}}^{t_{i*}=-1}}{e^{0\cdot K} + e^{-2K}}$$

$$= \frac{1 - e^{-2K}}{1 + e^{-2K}}$$

$$= \tanh(K)$$

$$\therefore \langle s_i s_j \rangle = \left[\tanh(K)\right]^{(j-i)}$$

$$= \exp\left[(j-i) \ln \tanh(K)\right]$$

$\langle s_i s_j \rangle$ (any general i, j)

$$= \exp\left[\,|j-i| \ln \tanh K\,\right]$$

$$= \exp\left[\frac{-|j-i|}{(1/\ln \tanh K)}\right]$$

$\tanh(K) =$

$T = 0,\ K \mapsto \infty$

$\tanh(K) \mapsto 1$

$|\ln(\tanh K)| \mapsto 0$

$\Rightarrow \langle s_i s_j \rangle$ is independent of $|j-i|$

Magnet

for any $T > 0$, K finite

$0 < \tanh(K) < 1$

$-\infty < \ln(\tanh K) < 0$

$$\boxed{\langle s_i s_j \rangle = \exp\left[\frac{-|i-j|}{\xi_c}\right]}$$

ξ_c : correlation length

$$\xi_c = \frac{1}{|\ln(\tanh K)|}$$

As $K \mapsto \infty$ $(T \mapsto 0)$

$\tanh K \mapsto 1$

$|\ln(\tanh K)| \mapsto 0$

$$\boxed{\xi_c \mapsto \infty}$$

Phase transition from the disordered/paramagnetic side :-

$$\xi_c \mapsto \infty$$

(Phase trans^n 1d Ising Model

$T_c = 0$ (critical temperature)

Atypical with OBC :-

$$\langle s_i s_j \rangle = \exp\left[\frac{-|i-j|}{\xi_c(K)}\right]$$

depends ONLY on $|i-j|$, NOT on i, j separately. (Translational invariance).

H.W. carry out this same exercise with $\langle s_i s_j \rangle$ for 1-d Potts Model $s_i = -2, -1, +1, +2$. Calculate $\langle s_i s_j \rangle$.

Landau & Binder Young : Monte Carlo simulations in statistical Physics

$$\langle s_i s_j \rangle = \exp\left[\frac{-|i-j|}{\xi_c}\right]$$

1-d $\Gamma(r) = \langle S(r)\, S(0) \rangle = (A) \exp\left[-r/\xi_c \right]$

$$\xi_c = \frac{1}{|\ln(\tanh K)|}$$

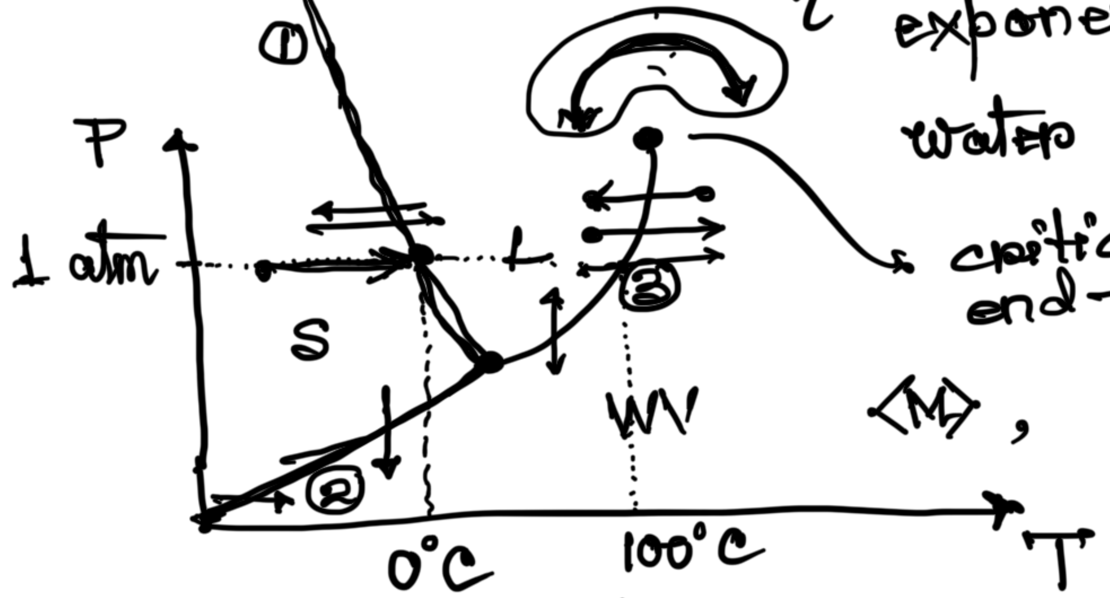
$K = \beta J$

In 1-d Ising Model, at $T = 0 = T_c$,
$\Gamma(r)$ = independent of r.

<u>General case</u>: $\lim_{r \to \infty} \Gamma(r) = \Gamma_0\, r^{-\kappa}$

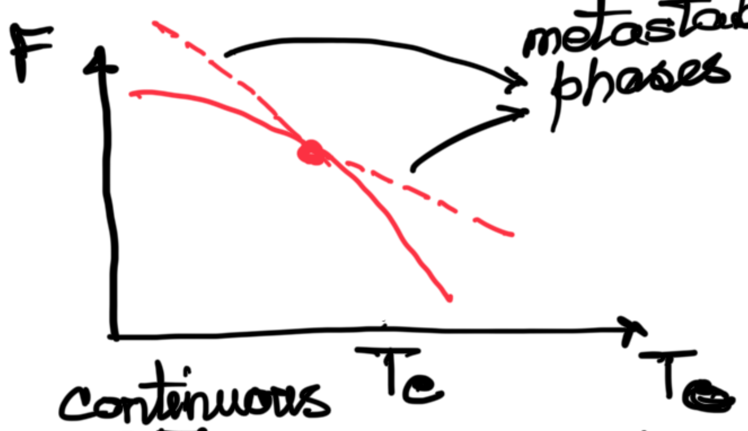
$T_c \neq 0$ $\quad \kappa = +(d-2+\eta)$

$\boxed{T = T_c}$ $\quad \eta$: critical exponent

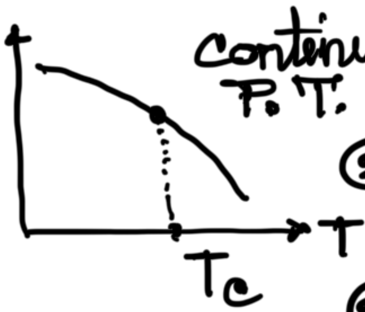


$\langle M \rangle$, $\rho_L - \rho_G$

(1) melting line
(1): 1st order transition



Free-energy
$\Rightarrow$ discontinuous derivative
$\Rightarrow \left.\frac{\partial F}{\partial T}\right|_{T_c}$ is discontinuous

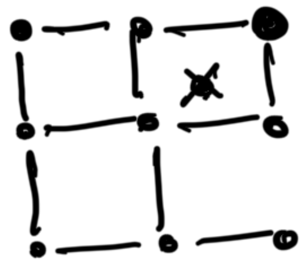


(2): sublimation phase transition
(3): boiling

- WV, L share the same symmetry
- S / L does not.

full translational symmetry

WV, L : full ...

S :  Translational symmetry is reduced.

E_3, $SO(3)$

(G : symmetry groups of the crystal)

↑ ↓ ↑ ↑ ↓ ↑ ↓
full $\mathbb{Z}_2$-symmetry
(Paramagnet)

↑ ↑ ↑ ↑ ↑ ↑
$\mathbb{Z}_2$-symmetry is broken

$\mathbb{Z}_2$: Name of a very simple group

<u>What is a group:</u>

$G = \{g_1, g_2, \ldots, \}$ • $g_1 \cdot g_2 = g_3$ $\forall g_1, g_2 \in G$

'$\cdot$' : $G \times G \to G$

• $\exists$ 'e', $g_1 \cdot e = e \cdot g_1 = g_1$ $\forall g_1 \in G$

'e' : identity element

$\forall g_1 \in G$, $\exists g_2$ s.t.

$g_2 \cdot g_1 = g_1 \cdot g_2 = e$

$g_2 = g_1^{-1}$

• Integers under addition
• Reals $- \{0\}$ under multiplication

Ordered state (FM, solid) : smaller group $\cap$
(Dis)ordered state (PM, WV/L) : Bigger symm. group

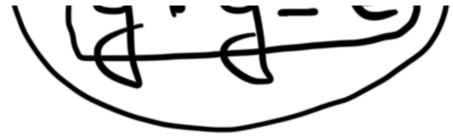
$\mathbb{Z}_2$: $\{e, g\}$

$\begin{cases} \mathbb{R}_0 : \vec{p} \mapsto -\vec{p} \\ \{\mathbb{I}, \mathbb{R}_0\} \end{cases}$

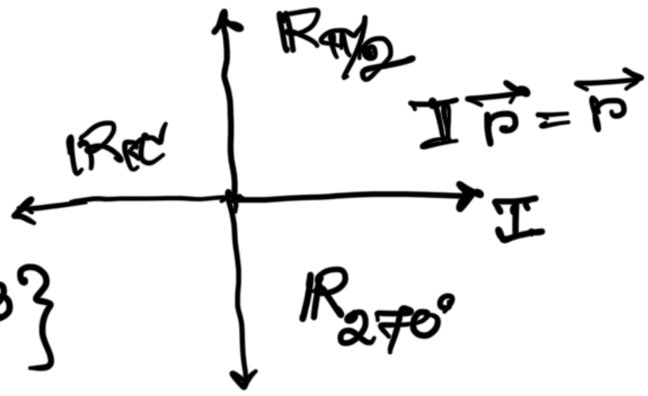
$e \cdot e = e$
$g \cdot e = g$
$e \cdot g = g$
$g \cdot g = e$

	e	g
e	e	g
g	g	e

$\mathbb{Z}_2$



$\mathbb{Z}_N : g^N = e$

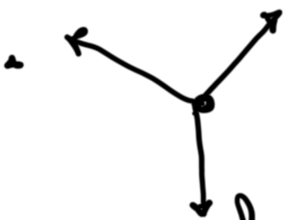


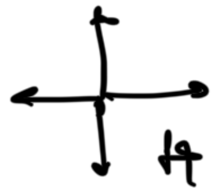
$\mathbb{Z}_4 : \{e, g, g^2, g^3\}$
$\{g^4 = e\}$

$g^2 = g \cdot g$

$g^3 = g^2 \cdot g = g \cdot g \cdot g$

$\mathbb{Z}_N : \{e, g, g^2, \ldots, g^{N-1}\}$
$g^N = e$

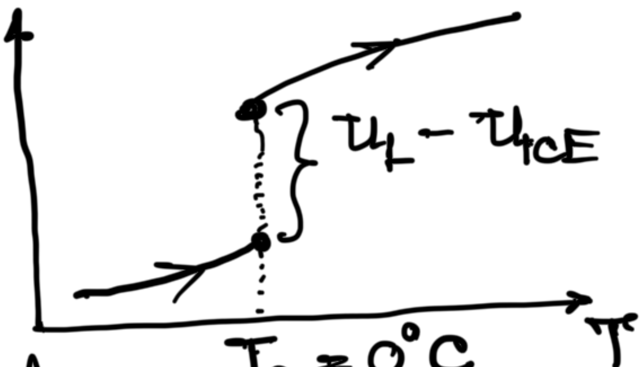




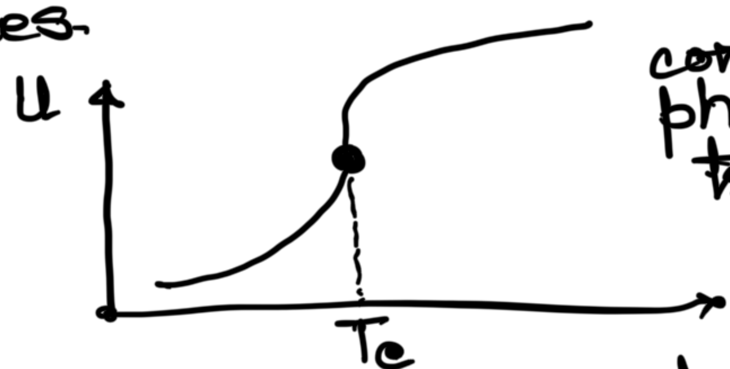
Back to 1st order Transitions

$p = 1$ atm

$U = \langle H \rangle = \langle E \rangle$



can be calculated using stat. mech. techniques.



continuous phase transition

1st order	continuous
F: continuous @ T_c differentiable	F: Differentiable at T_c

non-differentiable at T_c

$\frac{\partial F}{\partial T}$: discontinuous @ T_c

$\frac{\partial F}{\partial T}$ (or, $\frac{\partial F}{\partial h}$) is continuous but non-differentiable at T_c (or (h_c))

$\frac{\partial^2 F}{\partial T^2}$ (or $\frac{\partial^2 F}{\partial h^2} = \chi$) is divergent at T_c (or h_c)

$$\varepsilon = \left|1 - \frac{T}{T_c}\right| = \left|\frac{T_c - T}{T_c}\right|$$

lim $\varepsilon \mapsto 0$

$$\langle M \rangle = M_0 \varepsilon^{\beta}$$

$$\chi = \left.\frac{\partial \langle M \rangle}{\partial h}\right|_{h=0} = \chi_0 \varepsilon^{-\gamma}$$

$$C_V = \frac{\partial \langle E \rangle}{\partial T} = C_0 \varepsilon^{-\alpha}$$

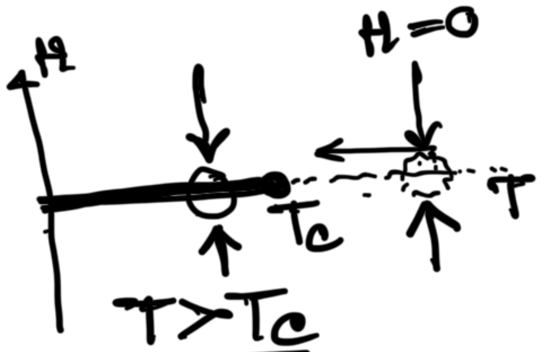
$$\xi_c \longmapsto \xi_0 \varepsilon^{-\nu}$$

$\{\alpha, \gamma, \nu, \beta\}$ Ising Model LG, solid Magnets

'Renormalization Group'

Correlation length

$H = 0$, $T \mapsto T_c$



$T < T_c$

ξ_c : infinite

$\lim_{r \to \infty} G(r) \sim$ const.

$\langle s(r) s(0) \rangle$

$T = T_c$

$G(r) \sim r^{-\eta}$

$\eta > 0$

$T > T_c$

ξ_c : finite

$G(r) \sim \exp\left(\frac{-r}{\xi_c}\right)$

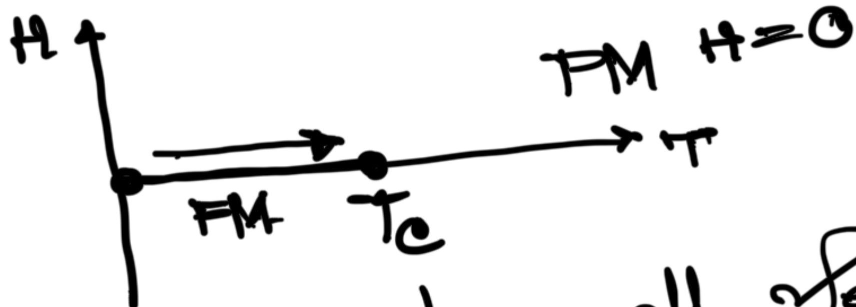
$\lim\limits_{p \to \infty} \langle s(i)\, s(j) \rangle \qquad [\xi_c \to \infty]$
$|i-j| = p$

<u>Scaling theory</u> → Economics (Financial) → } Econo physics
→ Network theory
→ Biological networks

$$F = \frac{G M_1 M_2}{p^2}, \quad V \sim \frac{1}{p}$$

$$T \sim p^{3/2} \text{ (Kepler)}$$

$a \ll R$, $\frac{a}{R} \to 0$
(separation of scales)



<u>Critical point</u> : all length-scales are important.

$\exp(-p/\xi)$ $\quad \xi \to ?$
At $T = T_c$, $p^{-\nu} \sim G(p)$
$\nu > 0$.

<u>1-d (2-d) Ising Model</u>

(T, H) : $L(N)$, a, ξ_c → correlation length
$L(N)$ → System size; a → Microscopic

<u>Phenomenology</u>

$$\lim_{p \to 0} \langle s(p)\, s(0) \rangle \simeq p^{-\alpha}\, f\left(\frac{p}{\xi_c}, \frac{a}{\xi_c}\right)$$

(2-d, 3-d, IM, $T = T_c$, $H = 0$)

$p^{-\alpha}$ ↑ power-law; f → dimensionless variables

At $T = T_c$, $\xi_c \longmapsto \infty$, $a/\xi_c \ll 1$ vanishes

$$f(r/\xi_c, a/\xi_c) \approx \left(\frac{a}{\xi_c}\right)^{\eta} g(r/\xi_c)$$

$\alpha, \beta, \gamma, \delta, \nu, \eta$: universal

$\langle M \rangle$ $\quad S_i \quad e^{(\;\;) + hS}$

$\downarrow$

$\langle S_i S_j \rangle \sim \dfrac{\partial \langle M \rangle}{\partial h}$

$$\chi \approx \int G(r)\, d^d r$$

$\underline{d=3}$

$$\chi \approx \int G(r)\, d^3 r$$

$$\approx \int \left(\frac{a}{\xi_c}\right)^{\eta} g\left(\frac{r}{\xi_c}\right) r^2 r^{-\alpha} \, dr \, d\Omega$$

$d^3 r = r^2 \, dr \, d\Omega$

$$\approx \int g\left(\frac{r}{\xi_c}\right) r^{2-\alpha} \, dr \, d\Omega$$

$\tilde{r} = r/\xi_c$

$$\approx \int g(\tilde{r}) \; \tilde{r}^{2-\alpha} \xi_c^{2-\alpha} \; d\tilde{r}\, \xi_c \; d\Omega$$

$$\approx \xi_c^{2-\alpha} \underbrace{\int g(\tilde{r})\, \tilde{r}^{2-\alpha} \, d\tilde{r} \, d\Omega}_{K}$$

$$\chi \sim \xi_c^{2-\alpha}, \quad \boxed{\alpha < 2}$$

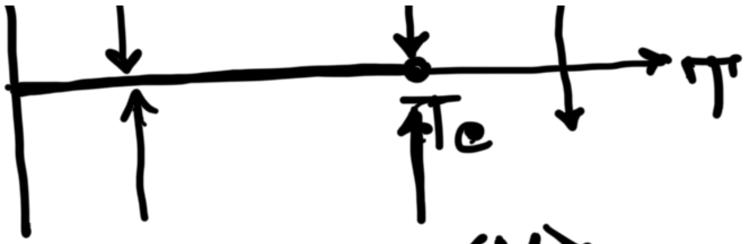
As $T \longmapsto T_c$, $(H=0)$,

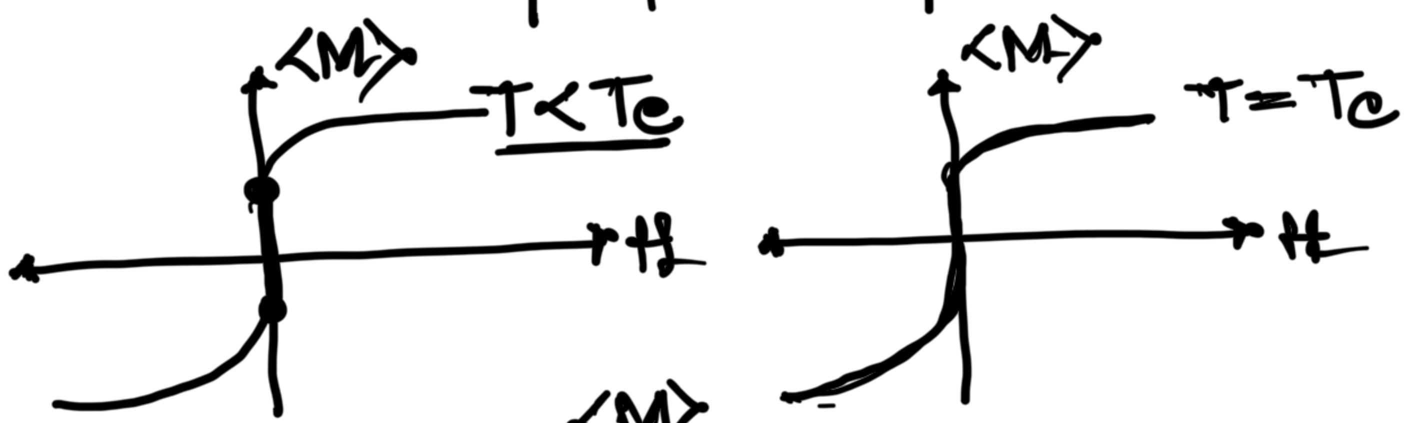
$$\left.\begin{array}{l} \xi_c \longmapsto \infty \\ \chi \longmapsto \infty \end{array}\right\}.$$

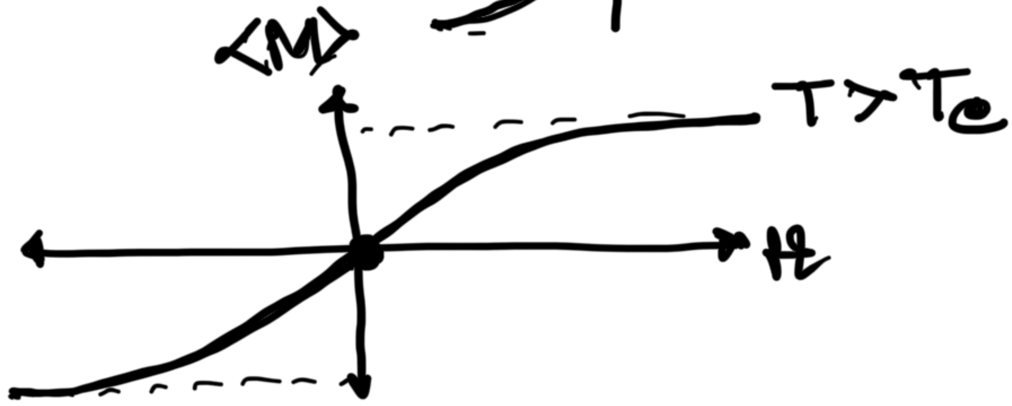
$(\alpha, \beta, \gamma, \delta, \nu, \eta)$

H

t $\quad$ | $\quad$ | $\quad$ | PM







$$\lim_{H\to 0^+}\langle M\rangle \neq \lim_{H\to 0^-}\langle M\rangle$$

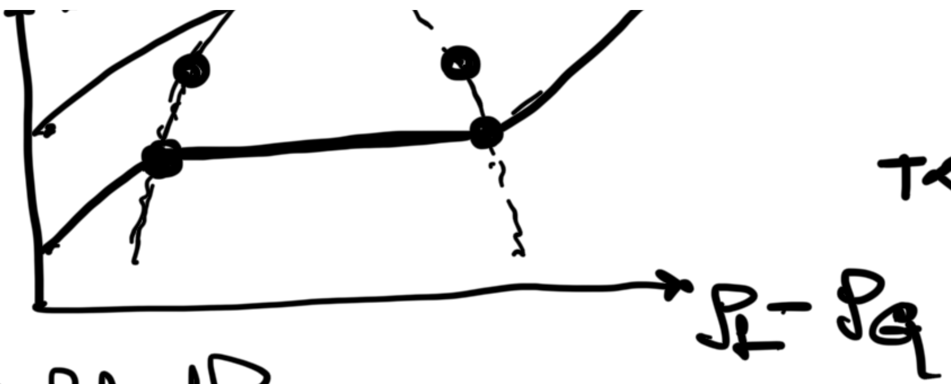
for $T<T_c$.

(depends on previous history)

[spontaneous symmetry breaking] ($H=0$ limit)

$\left\{ \begin{array}{l} H: \ S_i \to -S_i \ \forall i \\ \text{Eqbm. state is } \underline{\text{not}} \text{ invariant} \end{array} \right.$

<u>Simple Fluid</u>



Mean-Field theory :
(MFT)

Ising Model

d=1 Exactly solvable
$T_c = 0$

d=2 : Exactly solvable
(~ very hard)
$T_c \neq 0$

d=3 : Unsolved in 2025

d>3 (4, 5, 6, 7, ..., ∞)
: described (near CP) by MFT

$$\hat{H} = \sum_{i,j} \frac{J_{ij}\, s_i s_j}{2} + \sum_i h\, s_i$$

$$Z =$$