

In[]:=

```
(* Useful abbreviations *)
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```
X = x1 - y1; Y = x2 - y2;
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In[]:=

```
(* Linearity of the integral *)
```

```
integrate[a_ - b_, x_] := integrate[a, x] - integrate[b, x];
```

```
integrate[a_ + b_, x_] := integrate[a, x] + integrate[b, x];
```

```
integrate[c_ a_, x_] := c integrate[a, x] /; FreeQ[c, x];
```

```
integrate[a_, x_] := integrate[Expand[a, x], x];
```

In[]:=

```
(* Rules for the x1 integration *)

integrate[x1^n_Integer, x1] :=  $\frac{1}{n+1} x1^{n+1}$ ;

integrate[x1^n_Integer * Log[X^2 + Y^2], x1] := integrate[x1^n * Log[X^2 + Y^2], x1] = `

$$\frac{x1^n * X}{n+1} * \text{Log}[X^2 + Y^2] - \frac{2 * x1^{n+1}}{(n+1)^2} - \frac{2 * x1^n * Y}{n+1} * \text{ArcTan}\left[\frac{Y}{X}\right] +$$


$$\frac{n * y1}{n+1} * \text{integrate}[x1^{n-1} * \text{Log}[X^2 + Y^2], x1] + \frac{2 * Y * n}{n+1} * \text{integrate}\left[x1^{n-1} * \text{ArcTan}\left[\frac{Y}{X}\right], x1\right];$$


integrate[x1^n_Integer * ArcTan[X/Y], x1] := integrate[x1^n * ArcTan[X/Y], x1] = `

$$\frac{x1^n * X}{n+1} * \text{ArcTan}\left[\frac{X}{Y}\right] - \frac{x1^n * Y}{2 * (n+1)} * \text{Log}[X^2 + Y^2] +$$


$$\frac{n * y1}{n+1} * \text{integrate}\left[x1^{n-1} * \text{ArcTan}\left[\frac{X}{Y}\right], x1\right] + \frac{n * Y}{2 * (n+1)} * \text{integrate}[x1^{n-1} * \text{Log}[X^2 + Y^2], x1];$$


integrate[x1^n_Integer * ArcTan[Y/X], x1] := integrate[x1^n * ArcTan[Y/X], x1] = `

$$\frac{x1^n * X}{n+1} * \text{ArcTan}\left[\frac{Y}{X}\right] + \frac{x1^n * Y}{2 * (n+1)} * \text{Log}[X^2 + Y^2] +$$


$$\frac{n * y1}{n+1} * \text{integrate}\left[x1^{n-1} * \text{ArcTan}\left[\frac{Y}{X}\right], x1\right] - \frac{n * Y}{2 * (n+1)} * \text{integrate}[x1^{n-1} * \text{Log}[X^2 + Y^2], x1];$$


(* The recursion ends. *)

integrate[Log[X^2 + Y^2], x1] = X * Log[X^2 + Y^2] - 2 * X - 2 * Y * ArcTan[Y/X];

integrate[ArcTan[X/Y], x1] = X * ArcTan[X/Y] - Y/2 * Log[X^2 + Y^2];

integrate[ArcTan[Y/X], x1] = X * ArcTan[Y/X] + Y/2 * Log[X^2 + Y^2];

(* One level above the ends of the recursion *)

integrate[x1 * Log[X^2 + Y^2], x1] =

$$\frac{X^2 + Y^2}{2} * \text{Log}[X^2 + Y^2] - \frac{X^2 + Y^2}{2} + y1 * \text{integrate}[\text{Log}[X^2 + Y^2], x1];$$


integrate[x1 * ArcTan[X/Y], x1] =  $\frac{X^2 + Y^2}{2} * \text{ArcTan}\left[\frac{X}{Y}\right] - \frac{X * Y}{2} + y1 * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{Y}\right], x1\right];$ 

integrate[x1 * ArcTan[Y/X], x1] =  $\frac{X^2 + Y^2}{2} * \text{ArcTan}\left[\frac{Y}{X}\right] + \frac{X * Y}{2} + y1 * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{X}\right], x1\right];$ 
```

In[]:=

(* Rules for the x2 integration *)

$$\text{integrate}[x2^{n_Integer}, x2] := \frac{1}{n+1} x2^{n+1};$$

$$\text{integrate}[x2^{n_Integer} * \text{Log}[X^2 + Y^2], x2] := \text{integrate}[x2^n * \text{Log}[X^2 + Y^2], x2] = \cdot:$$

$$\frac{x2^n * Y}{n+1} * \text{Log}[X^2 + Y^2] - \frac{2 * x2^{n+1}}{(n+1)^2} + \frac{2 * x2^n * X}{n+1} * \text{ArcTan}\left[\frac{Y}{X}\right] +$$

$$\frac{n * y2}{n+1} * \text{integrate}[x2^{n-1} * \text{Log}[X^2 + Y^2], x2] - \frac{2 * X * n}{n+1} * \text{integrate}\left[x2^{n-1} * \text{ArcTan}\left[\frac{Y}{X}\right], x2\right];$$

$$\text{integrate}\left[x2^{n_Integer} * \text{ArcTan}\left[\frac{X}{Y}\right], x2\right] := \text{integrate}\left[x2^n * \text{ArcTan}\left[\frac{X}{Y}\right], x2\right] = \cdot:$$

$$\frac{x2^n * Y}{n+1} * \text{ArcTan}\left[\frac{X}{Y}\right] + \frac{x2^n * X}{2 * (n+1)} * \text{Log}[X^2 + Y^2] +$$

$$\frac{n * y2}{n+1} * \text{integrate}\left[x2^{n-1} * \text{ArcTan}\left[\frac{X}{Y}\right], x2\right] - \frac{n * X}{2 * (n+1)} * \text{integrate}[x2^{n-1} * \text{Log}[X^2 + Y^2], x2];$$

$$\text{integrate}\left[x2^{n_Integer} * \text{ArcTan}\left[\frac{Y}{X}\right], x2\right] := \text{integrate}\left[x2^n * \text{ArcTan}\left[\frac{Y}{X}\right], x2\right] = \cdot:$$

$$\frac{x2^n * Y}{n+1} * \text{ArcTan}\left[\frac{Y}{X}\right] - \frac{x2^n * X}{2 * (n+1)} * \text{Log}[X^2 + Y^2] +$$

$$\frac{n * y2}{n+1} * \text{integrate}\left[x2^{n-1} * \text{ArcTan}\left[\frac{Y}{X}\right], x2\right] + \frac{n * X}{2 * (n+1)} * \text{integrate}[x2^{n-1} * \text{Log}[X^2 + Y^2], x2];$$

(* The recursion ends. *)

$$\text{integrate}[\text{Log}[X^2 + Y^2], x2] = Y * \text{Log}[X^2 + Y^2] - 2 * Y + 2 * X * \text{ArcTan}\left[\frac{Y}{X}\right];$$

$$\text{integrate}\left[\text{ArcTan}\left[\frac{X}{Y}\right], x2\right] = Y * \text{ArcTan}\left[\frac{X}{Y}\right] + \frac{X}{2} * \text{Log}[X^2 + Y^2];$$

$$\text{integrate}\left[\text{ArcTan}\left[\frac{Y}{X}\right], x2\right] = Y * \text{ArcTan}\left[\frac{Y}{X}\right] - \frac{X}{2} * \text{Log}[X^2 + Y^2];$$

(* One level above the ends of the recursion *)

$$\text{integrate}[x2 * \text{Log}[X^2 + Y^2], x2] =$$

$$\frac{X^2 + Y^2}{2} * \text{Log}[X^2 + Y^2] - \frac{X^2 + Y^2}{2} + y2 * \text{integrate}[\text{Log}[X^2 + Y^2], x2];$$

$$\text{integrate}\left[x2 * \text{ArcTan}\left[\frac{X}{Y}\right], x2\right] = \frac{X^2 + Y^2}{2} * \text{ArcTan}\left[\frac{X}{Y}\right] + \frac{X * Y}{2} + y2 * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{Y}\right], x2\right];$$

$$\text{integrate}\left[x2 * \text{ArcTan}\left[\frac{Y}{X}\right], x2\right] = \frac{X^2 + Y^2}{2} * \text{ArcTan}\left[\frac{Y}{X}\right] - \frac{X * Y}{2} + y2 * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{X}\right], x2\right];$$

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```
(* Rules for the y1 integration *)
integrate[2, y1] := 2 y1;
integrate[y1n_Integer, y1] :=  $\frac{1}{n+1} y1^{n+1}$ ;
integrate[y1n_Integer * Log[X2 + Y2], y1] := integrate[y1n * Log[X2 + Y2], y1] = `
-  $\frac{y1^n * X}{n+1} * \text{Log}[X^2 + Y^2] - \frac{2 * y1^{n+1}}{(n+1)^2} + \frac{2 * y1^n * Y}{n+1} * \text{ArcTan}\left[\frac{Y}{X}\right] +$ 
 $\frac{n * x1}{n+1} * \text{integrate}[y1^{n-1} * \text{Log}[X^2 + Y^2], y1] - \frac{2 * Y * n}{n+1} * \text{integrate}\left[y1^{n-1} * \text{ArcTan}\left[\frac{Y}{X}\right], y1\right];$ 
integrate[y1n_Integer * ArcTan[X/Y], y1] := integrate[y1n * ArcTan[X/Y], y1] = `
-  $\frac{y1^n * X}{n+1} * \text{ArcTan}\left[\frac{X}{Y}\right] + \frac{y1^n * Y}{2 * (n+1)} * \text{Log}[X^2 + Y^2] +$ 
 $\frac{n * x1}{n+1} * \text{integrate}\left[y1^{n-1} * \text{ArcTan}\left[\frac{X}{Y}\right], y1\right] - \frac{n * Y}{2 * (n+1)} * \text{integrate}[y1^{n-1} * \text{Log}[X^2 + Y^2], y1];$ 
integrate[y1n_Integer * ArcTan[Y/X], y1] := integrate[y1n * ArcTan[Y/X], y1] = `
-  $\frac{y1^n * X}{n+1} * \text{ArcTan}\left[\frac{Y}{X}\right] - \frac{y1^n * Y}{2 * (n+1)} * \text{Log}[X^2 + Y^2] +$ 
 $\frac{n * x1}{n+1} * \text{integrate}\left[y1^{n-1} * \text{ArcTan}\left[\frac{Y}{X}\right], y1\right] + \frac{n * Y}{2 * (n+1)} * \text{integrate}[y1^{n-1} * \text{Log}[X^2 + Y^2], y1];$ 
(* The recursion ends. *)
integrate[Log[X2 + Y2], y1] = -X * Log[X2 + Y2] + 2 * X + 2 * Y * ArcTan[Y/X];
integrate[ArcTan[X/Y], y1] = -X * ArcTan[X/Y] + Y/2 * Log[X2 + Y2];
integrate[ArcTan[Y/X], y1] = -X * ArcTan[Y/X] - Y/2 * Log[X2 + Y2];
(* One level above the ends of the recursion *)
integrate[y1 * Log[X2 + Y2], y1] =
 $\frac{X^2 + Y^2}{2} * \text{Log}[X^2 + Y^2] - \frac{X^2 + Y^2}{2} + x1 * \text{integrate}[\text{Log}[X^2 + Y^2], y1];$ 
integrate[y1 * ArcTan[X/Y], y1] =  $\frac{X^2 + Y^2}{2} * \text{ArcTan}\left[\frac{X}{Y}\right] - \frac{X * Y}{2} + x1 * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{Y}\right], y1\right];$ 
integrate[y1 * ArcTan[Y/X], y1] =  $\frac{X^2 + Y^2}{2} * \text{ArcTan}\left[\frac{Y}{X}\right] + \frac{X * Y}{2} + x1 * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{X}\right], y1\right];$ 
```

In[]:=

```
(* Rules for the y2 integration *)
integrate[2, y2] := 2 y2;
integrate[y2n_Integer, y2] :=  $\frac{1}{n+1} y2^{n+1}$ ;
integrate[y2n_Integer * Log[X2 + Y2], y2] := integrate[y2n * Log[X2 + Y2], y2] = ∴

$$-\frac{y2^n * Y}{n+1} * \text{Log}[X^2 + Y^2] - \frac{2 * y2^{n+1}}{(n+1)^2} - \frac{2 * y2^n * X}{n+1} * \text{ArcTan}\left[\frac{Y}{X}\right] +$$


$$\frac{n * x2}{n+1} * \text{integrate}[y2^{n-1} * \text{Log}[X^2 + Y^2], y2] + \frac{2 * X * n}{n+1} * \text{integrate}\left[y2^{n-1} * \text{ArcTan}\left[\frac{Y}{X}\right], y2\right];$$

integrate[y2n_Integer * ArcTan[X/Y], y2] := integrate[y2n * ArcTan[X/Y], y2] = ∴

$$-\frac{y2^n * Y}{n+1} * \text{ArcTan}\left[\frac{X}{Y}\right] - \frac{y2^n * X}{2 * (n+1)} * \text{Log}[X^2 + Y^2] +$$


$$\frac{n * x2}{n+1} * \text{integrate}\left[y2^{n-1} * \text{ArcTan}\left[\frac{X}{Y}\right], y2\right] + \frac{n * X}{2 * (n+1)} * \text{integrate}[y2^{n-1} * \text{Log}[X^2 + Y^2], y2];$$

integrate[y2n_Integer * ArcTan[Y/X], y2] := integrate[y2n * ArcTan[Y/X], y2] = ∴

$$-\frac{y2^n * Y}{n+1} * \text{ArcTan}\left[\frac{Y}{X}\right] + \frac{y2^n * X}{2 * (n+1)} * \text{Log}[X^2 + Y^2] +$$


$$\frac{n * x2}{n+1} * \text{integrate}\left[y2^{n-1} * \text{ArcTan}\left[\frac{Y}{X}\right], y2\right] - \frac{n * X}{2 * (n+1)} * \text{integrate}[y2^{n-1} * \text{Log}[X^2 + Y^2], y2];$$

(* The recursion ends. *)
integrate[Log[X2 + Y2], y2] = -Y * Log[X2 + Y2] + 2 * Y - 2 * X * ArcTan[Y/X];
integrate[ArcTan[X/Y], y2] = -Y * ArcTan[X/Y] -  $\frac{X}{2} * \text{Log}[X^2 + Y^2]$ ;
integrate[ArcTan[Y/X], y2] = -Y * ArcTan[Y/X] +  $\frac{X}{2} * \text{Log}[X^2 + Y^2]$ ;
(* One level above the ends of the recursion *)
integrate[y2 * Log[X2 + Y2], y2] =

$$\frac{X^2 + Y^2}{2} * \text{Log}[X^2 + Y^2] - \frac{X^2 + Y^2}{2} + x2 * \text{integrate}[\text{Log}[X^2 + Y^2], y2];$$

integrate[y2 * ArcTan[X/Y], y2] =  $\frac{X^2 + Y^2}{2} * \text{ArcTan}\left[\frac{X}{Y}\right] + \frac{X * Y}{2} + x2 * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{Y}\right], y2\right];$ 
integrate[y2 * ArcTan[Y/X], y2] =  $\frac{X^2 + Y^2}{2} * \text{ArcTan}\left[\frac{Y}{X}\right] - \frac{X * Y}{2} + x2 * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{X}\right], y2\right];$ 
```