

In[]:=

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(* Useful abbreviations *)
X = x1 - y1; Y = x2 - y2; Z = x3 - y3;
Ξ = Y2 + Z2; Υ = X2 + Z2; Θ = X2 + Y2;
R =  $\sqrt{X^2 + Y^2 + Z^2}$ ;
```

In[]:=

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(* Linearity of the integral *)
integrate[a_ - b_, x_] := integrate[a, x] - integrate[b, x];
integrate[a_ + b_, x_] := integrate[a, x] + integrate[b, x];
integrate[c_ a_, x_] := c integrate[a, x] /; FreeQ[c, x];
integrate[a_, x_] := integrate[Expand[a, x], x];
```

In[]:=

```
(* Rules for the x1 integration *)
(* First, the actual recursions *)

integrate[ $\frac{x1^{n\_Integer}}{R}$ , x1] := integrate[ $\frac{x1^n}{R}$ , x1] = `.`

$$\frac{x1^{n-1} R}{n} + \frac{(2n-1)y1}{n} * integrate[\frac{x1^{n-1}}{R}, x1] - \frac{(n-1)*(y1^2 + \Xi)}{n} * integrate[\frac{x1^{n-2}}{R}, x1];$$

integrate[x1n\_Integer * R, x1] := integrate[x1n * R, x1] = `.`

$$\frac{x1^{n-1} * (X^2 + \Xi) * R}{n+2} + \frac{(2 * n + 1) y1}{n+2} * integrate[x1^{n-1} * R, x1] -$$


$$\frac{(n-1) * (y1^2 + \Xi)}{n+2} * integrate[x1^{n-2} R, x1];$$

integrate[x1n\_Integer * ArcTanh[ $\frac{X}{R}$ ], x1] := integrate[x1n * ArcTanh[ $\frac{X}{R}$ ], x1] = `.`

$$\frac{x1^{n+1}}{n+1} * ArcTanh[\frac{X}{R}] - \frac{1}{n+1} * integrate[\frac{x1^{n+1}}{R}, x1];$$

integrate[x1n\_Integer * ArcTanh[ $\frac{Y}{R}$ ], x1] := integrate[x1n * ArcTanh[ $\frac{Y}{R}$ ], x1] = `.`

$$\frac{x1^n}{n+1} * integrate[ArcTanh[\frac{Y}{R}], x1] + \frac{n}{n+1} * y1 * integrate[x1^{n-1} * ArcTanh[\frac{Y}{R}], x1] -$$


$$\frac{n}{n+1} * Y * integrate[x1^{n-1} * ArcTanh[\frac{X}{R}], x1] +$$


$$\frac{n}{n+1} * Z * integrate[x1^{n-1} * ArcTan[\frac{Y}{R} * \frac{X}{R} * \frac{R}{Z}], x1];$$

integrate[x1n\_Integer * ArcTanh[ $\frac{Z}{R}$ ], x1] := integrate[x1n * ArcTanh[ $\frac{Z}{R}$ ], x1] = `.`

$$\frac{x1^n}{n+1} * integrate[ArcTanh[\frac{Z}{R}], x1] + \frac{n}{n+1} * y1 * integrate[x1^{n-1} * ArcTanh[\frac{Z}{R}], x1] -$$


$$\frac{n}{n+1} * Z * integrate[x1^{n-1} * ArcTanh[\frac{X}{R}], x1] +$$

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$$\begin{aligned}
& \frac{n}{n+1} * Y * \text{integrate}\left[x1^{n-1} * \text{ArcTan}\left[\frac{Z}{R} * \frac{X}{R} * \frac{R}{Y}\right], x1\right]; \\
& \text{integrate}\left[x1^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x1\right] := \text{integrate}\left[x1^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x1\right] = \cdot; \\
& \frac{x1^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x1\right] + \frac{n}{n+1} * y1 * \\
& \quad \text{integrate}\left[x1^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x1\right] - \frac{n}{n+1} * Z * \text{integrate}\left[x1^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], x1\right]; \\
& \text{integrate}\left[x1^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x1\right] := \text{integrate}\left[x1^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x1\right] = \cdot; \\
& \frac{x1^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x1\right] + \frac{n}{n+1} * y1 * \\
& \quad \text{integrate}\left[x1^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x1\right] - \frac{n}{n+1} * Y * \text{integrate}\left[x1^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], x1\right]; \\
& \text{integrate}\left[x1^{n_Integer} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x1\right] := \text{integrate}\left[x1^n * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x1\right] = \cdot; \\
& \frac{x1^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x1\right] + \\
& \quad \frac{n}{n+1} * y1 * \text{integrate}\left[x1^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x1\right] + \frac{n}{n+1} * Y * \\
& \quad \text{integrate}\left[x1^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], x1\right] + \frac{n}{n+1} * Z * \text{integrate}\left[x1^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], x1\right];
\end{aligned}$$

(* The recursion ends. *)

$$\begin{aligned}
& \text{integrate}\left[\frac{1}{R}, x1\right] = \text{ArcTanh}\left[\frac{X}{R}\right]; \\
& \text{integrate}[R, x1] = \frac{X * R}{2} + \frac{\pi}{2} * \text{ArcTanh}\left[\frac{X}{R}\right]; \\
& \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x1\right] = X * \text{ArcTanh}\left[\frac{X}{R}\right] - R; \\
& \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x1\right] = X * \text{ArcTanh}\left[\frac{Y}{R}\right] + Y * \text{ArcTanh}\left[\frac{X}{R}\right] - Z * \text{ArcTan}\left[\frac{Y}{R} * \frac{X}{R} * \frac{R}{Z}\right]; \\
& \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x1\right] = X * \text{ArcTanh}\left[\frac{Z}{R}\right] + Z * \text{ArcTanh}\left[\frac{X}{R}\right] - Y * \text{ArcTan}\left[\frac{Z}{R} * \frac{X}{R} * \frac{R}{Y}\right]; \\
& \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x1\right] = X * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right] + Z * \text{ArcTanh}\left[\frac{Y}{R}\right]; \\
& \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x1\right] = X * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right] + Y * \text{ArcTanh}\left[\frac{Z}{R}\right]; \\
& \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x1\right] = \\
& \quad X * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right] - Y * \text{ArcTanh}\left[\frac{Z}{R}\right] - Z * \text{ArcTanh}\left[\frac{Y}{R}\right];
\end{aligned}$$

(* One level above the ends of the recursion *)

$$\text{integrate}\left[\frac{x1}{R}, x1\right] = R + y1 \text{ArcTanh}\left[\frac{X}{R}\right];$$

$$\text{integrate}[x1 * R, x1] = \left(\frac{X^2 + \Xi}{3} + \frac{y1 * X}{2}\right) R + \frac{y1 * \Xi}{2} \text{integrate}\left[\frac{1}{R}, x1\right];$$

$$\text{integrate}\left[x1 * \text{ArcTanh}\left[\frac{X}{R}\right], x1\right] = \frac{x1^2}{2} * \text{ArcTanh}\left[\frac{X}{R}\right] - \frac{1}{2} * \text{integrate}\left[\frac{x1^2}{R}, x1\right];$$

$$\begin{aligned} \text{integrate}\left[x1 * \text{ArcTanh}\left[\frac{Y}{R}\right], x1\right] &= \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x1\right] - \\ &\frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x1\right] + \frac{Z}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{X}{R} * \frac{R}{Z}\right], x1\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[x1 * \text{ArcTanh}\left[\frac{Z}{R}\right], x1\right] &= \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x1\right] - \\ &\frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x1\right] + \frac{Y}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Z}{R} * \frac{X}{R} * \frac{R}{Y}\right], x1\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[x1 * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x1\right] &= \\ \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x1\right] &- \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x1\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[x1 * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x1\right] &= \\ \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x1\right] &- \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x1\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[x1 * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x1\right] &= \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x1\right] + \\ \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x1\right] &+ \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x1\right]; \end{aligned}$$

In[]:=

(* Rules for the y1 integration *)

(* First the actual recursions *)

$$\text{integrate}\left[\frac{y1^{n_Integer}}{R}, y1\right] := \text{integrate}\left[\frac{y1^n}{R}, y1\right] = \cdot$$

$$\frac{y1^{n-1} R}{n} + \frac{(2n-1)x1}{n} * \text{integrate}\left[\frac{y1^{n-1}}{R}, y1\right] - \frac{(n-1)*(x1^2 + \Xi)}{n} * \text{integrate}\left[\frac{y1^{n-2}}{R}, y1\right];$$

$$\text{integrate}[y1^{n_Integer} * R, y1] := \text{integrate}[y1^n * R, y1] = \cdot$$

$$\frac{y1^{n-1} * (X^2 + \Xi) * R}{n+2} + \frac{(2*n+1)x1}{n+2} * \text{integrate}[y1^{n-1} * R, y1] -$$

$$\frac{(n-1)*(x1^2 + \Xi)}{n+2} * \text{integrate}[y1^{n-2} * R, y1];$$

$$\text{integrate}[y1^{n_Integer} * \text{ArcTanh}\left[\frac{X}{R}\right], y1] := \text{integrate}[y1^n * \text{ArcTanh}\left[\frac{X}{R}\right], y1] = \cdot$$

$$\frac{y1^{n+1}}{n+1} * \text{ArcTanh}\left[\frac{X}{R}\right] + \frac{1}{n+1} * \text{integrate}\left[\frac{y1^{n+1}}{R}, y1\right];$$

$$\text{integrate}[y1^{n_Integer} * \text{ArcTanh}\left[\frac{Y}{R}\right], y1] := \text{integrate}[y1^n * \text{ArcTanh}\left[\frac{Y}{R}\right], y1] = \cdot$$

$$\begin{aligned} & \frac{y1^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y1\right] + \frac{n}{n+1} * x1 * \text{integrate}\left[y1^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], y1\right] + \\ & \frac{n}{n+1} * Y * \text{integrate}\left[y1^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], y1\right] - \\ & \frac{n}{n+1} * Z * \text{integrate}\left[y1^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{X}{R} * \frac{R}{Z}\right], y1\right]; \end{aligned}$$

$$\text{integrate}[y1^{n_Integer} * \text{ArcTanh}\left[\frac{Z}{R}\right], y1] := \text{integrate}[y1^n * \text{ArcTanh}\left[\frac{Z}{R}\right], y1] = \cdot$$

$$\begin{aligned} & \frac{y1^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y1\right] + \frac{n}{n+1} * x1 * \text{integrate}\left[y1^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], y1\right] + \\ & \frac{n}{n+1} * Z * \text{integrate}\left[y1^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], y1\right] - \\ & \frac{n}{n+1} * Y * \text{integrate}\left[y1^{n-1} * \text{ArcTan}\left[\frac{Z}{R} * \frac{X}{R} * \frac{R}{Y}\right], y1\right]; \end{aligned}$$

$$\text{integrate}[y1^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y1] := \text{integrate}[y1^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y1] = \cdot$$

$$\begin{aligned} & \frac{y1^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y1\right] + \frac{n}{n+1} * x1 * \\ & \text{integrate}\left[y1^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y1\right] + \frac{n}{n+1} * Z * \text{integrate}\left[y1^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], y1\right]; \end{aligned}$$

$$\text{integrate}[y1^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y1] := \text{integrate}[y1^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y1] = \cdot$$

$$\begin{aligned} & \frac{y1^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y1\right] + \frac{n}{n+1} * x1 * \\ & \text{integrate}\left[y1^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y1\right] + \frac{n}{n+1} * Y * \text{integrate}\left[y1^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], y1\right]; \end{aligned}$$

$$\text{integrate}[y1^{n_Integer} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y1] := \text{integrate}[y1^n * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y1] = \cdot$$

$$\begin{aligned} & \frac{y1^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y1\right] + \\ & \frac{n}{n+1} * x1 * \text{integrate}\left[y1^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y1\right] - \frac{n}{n+1} * Y * \\ & \text{integrate}\left[y1^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], y1\right] - \frac{n}{n+1} * Z * \text{integrate}\left[y1^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], y1\right]; \end{aligned}$$

(* The recursion ends. *)

$$\begin{aligned} \text{integrate}\left[\frac{1}{R}, y1\right] &= -\text{ArcTanh}\left[\frac{X}{R}\right]; \\ \text{integrate}[R, y1] &= -\frac{X * R}{2} - \frac{\Xi}{2} * \text{ArcTanh}\left[\frac{X}{R}\right]; \\ \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y1\right] &= R - X * \text{ArcTanh}\left[\frac{X}{R}\right]; \\ \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y1\right] &= Z * \text{ArcTan}\left[\frac{Y}{R} * \frac{X}{R} * \frac{R}{Z}\right] - X * \text{ArcTanh}\left[\frac{Y}{R}\right] - Y * \text{ArcTanh}\left[\frac{X}{R}\right]; \\ \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y1\right] &= Y * \text{ArcTan}\left[\frac{Z}{R} * \frac{X}{R} * \frac{R}{Y}\right] - X * \text{ArcTanh}\left[\frac{Z}{R}\right] - Z * \text{ArcTanh}\left[\frac{X}{R}\right]; \\ \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y1\right] &= -X * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right] - Z * \text{ArcTanh}\left[\frac{Y}{R}\right]; \\ \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y1\right] &= -X * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right] - Y * \text{ArcTanh}\left[\frac{Z}{R}\right]; \\ \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y1\right] &= \\ & -X * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right] + Y * \text{ArcTanh}\left[\frac{Z}{R}\right] + Z * \text{ArcTanh}\left[\frac{Y}{R}\right]; \end{aligned}$$

(* One level above the ends of the recursion *)

$$\begin{aligned} \text{integrate}\left[\frac{y1}{R}, y1\right] &= R - x1 \text{ArcTanh}\left[\frac{X}{R}\right]; \\ \text{integrate}[y1 * R, y1] &= \left(\frac{X^2 + \Xi}{3} - \frac{x1 * X}{2}\right) R + \frac{x1 * \Xi}{2} \text{integrate}\left[\frac{1}{R}, y1\right]; \\ \text{integrate}\left[y1 * \text{ArcTanh}\left[\frac{X}{R}\right], y1\right] &= \frac{y1^2}{2} * \text{ArcTanh}\left[\frac{X}{R}\right] + \frac{1}{2} * \text{integrate}\left[\frac{y1^2}{R}, y1\right]; \\ \text{integrate}\left[y1 * \text{ArcTanh}\left[\frac{Y}{R}\right], y1\right] &= \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y1\right] + \\ & \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y1\right] - \frac{Z}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{X}{R} * \frac{R}{Z}\right], y1\right]; \\ \text{integrate}\left[y1 * \text{ArcTanh}\left[\frac{Z}{R}\right], y1\right] &= \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y1\right] + \\ & \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y1\right] - \frac{Y}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Z}{R} * \frac{X}{R} * \frac{R}{Y}\right], y1\right]; \\ \text{integrate}\left[y1 * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y1\right] &= \\ & \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y1\right] + \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y1\right]; \\ \text{integrate}\left[y1 * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y1\right] &= \\ & \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y1\right] + \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y1\right]; \end{aligned}$$

$$\text{integrate}\left[y1 * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y1\right] = \frac{x1 + y1}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y1\right] -$$

$$\frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y1\right] - \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y1\right];$$

In[]:=

(* Rules for the x2 integration *)

(* First the actual recursions *)

$$\text{integrate}\left[\frac{x2^{n_Integer}}{R}, x2\right] := \text{integrate}\left[\frac{x2^n}{R}, x2\right] = \cdot$$

$$\frac{x2^{n-1} R}{n} + \frac{(2n-1)y2}{n} * \text{integrate}\left[\frac{x2^{n-1}}{R}, x2\right] - \frac{(n-1)*(y2^2 + Y)}{n} * \text{integrate}\left[\frac{x2^{n-2}}{R}, x2\right];$$

$$\text{integrate}[x2^{n_Integer} * R, x2] := \text{integrate}[x2^n * R, x2] = \cdot$$

$$\frac{x2^{n-1} * (Y^2 + Y) * R}{n+2} + \frac{(2*n+1)y2}{n+2} * \text{integrate}[x2^{n-1} * R, x2] -$$

$$\frac{(n-1)*(y2^2 + Y)}{n+2} * \text{integrate}[x2^{n-2} R, x2];$$

$$\text{integrate}\left[x2^{n_Integer} * \text{ArcTanh}\left[\frac{X}{R}\right], x2\right] := \text{integrate}\left[x2^n * \text{ArcTanh}\left[\frac{X}{R}\right], x2\right] = \cdot$$

$$\frac{x2^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x2\right] + \frac{n}{n+1} * y2 * \text{integrate}\left[x2^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], x2\right] -$$

$$\frac{n}{n+1} * X * \text{integrate}\left[x2^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], x2\right] +$$

$$\frac{n}{n+1} * Z * \text{integrate}\left[x2^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right];$$

$$\text{integrate}\left[x2^{n_Integer} * \text{ArcTanh}\left[\frac{Y}{R}\right], x2\right] := \text{integrate}\left[x2^n * \text{ArcTanh}\left[\frac{Y}{R}\right], x2\right] = \cdot$$

$$\frac{x2^{n+1}}{n+1} * \text{ArcTanh}\left[\frac{Y}{R}\right] - \frac{1}{n+1} * \text{integrate}\left[\frac{x2^{n+1}}{R}, x2\right];$$

$$\text{integrate}\left[x2^{n_Integer} * \text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] := \text{integrate}\left[x2^n * \text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] = \cdot$$

$$\frac{x2^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] + \frac{n}{n+1} * y2 * \text{integrate}\left[x2^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] -$$

$$\frac{n}{n+1} * Z * \text{integrate}\left[x2^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], x2\right] +$$

$$\frac{n}{n+1} * X * \text{integrate}\left[x2^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right];$$

$$\text{integrate}\left[x2^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right] := \text{integrate}\left[x2^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right] = \cdot$$

$$\frac{x2^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right] + \frac{n}{n+1} * y2 *$$

$$\begin{aligned}
& \text{integrate}\left[x2^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right] - \frac{n}{n+1} * Z * \text{integrate}\left[x2^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], x2\right]; \\
\text{integrate}\left[x2^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x2\right] &:= \text{integrate}\left[x2^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x2\right] = \\
& \frac{x2^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x2\right] + \\
& \frac{n}{n+1} * y2 * \text{integrate}\left[x2^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x2\right] + \frac{n}{n+1} * X * \\
& \text{integrate}\left[x2^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] + \frac{n}{n+1} * Z * \text{integrate}\left[x2^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], x2\right]; \\
\text{integrate}\left[x2^{n_Integer} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right] &:= \text{integrate}\left[x2^n * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right] = \\
& \frac{x2^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right] + \frac{n}{n+1} * y2 * \\
& \text{integrate}\left[x2^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right] - \frac{n}{n+1} * X * \text{integrate}\left[x2^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], x2\right];
\end{aligned}$$

(* The recursion ends. *)

$$\begin{aligned}
\text{integrate}\left[\frac{1}{R}, x2\right] &= \text{ArcTanh}\left[\frac{Y}{R}\right]; \\
\text{integrate}[R, x2] &= \frac{Y * R}{2} + \frac{Y}{2} * \text{ArcTanh}\left[\frac{Y}{R}\right]; \\
\text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x2\right] &= Y * \text{ArcTanh}\left[\frac{X}{R}\right] + X * \text{ArcTanh}\left[\frac{Y}{R}\right] - Z * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right]; \\
\text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x2\right] &= Y * \text{ArcTanh}\left[\frac{Y}{R}\right] - R; \\
\text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] &= Y * \text{ArcTanh}\left[\frac{Z}{R}\right] + Z * \text{ArcTanh}\left[\frac{Y}{R}\right] - X * \text{ArcTan}\left[\frac{Z}{R} * \frac{Y}{R} * \frac{R}{X}\right]; \\
\text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right] &= Y * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right] + Z * \text{ArcTanh}\left[\frac{X}{R}\right]; \\
\text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x2\right] &= Y * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right] - X * \text{ArcTanh}\left[\frac{Z}{R}\right] - Z * \text{ArcTanh}\left[\frac{X}{R}\right]; \\
\text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right] &= Y * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right] + X * \text{ArcTanh}\left[\frac{Z}{R}\right];
\end{aligned}$$

(* One level above the ends of the recursion *)

$$\begin{aligned}
\text{integrate}\left[\frac{x2^2}{R}, x2\right] &= R + y2 * \text{ArcTanh}\left[\frac{Y}{R}\right]; \\
\text{integrate}[x2 * R, x2] &= \left(\frac{Y^2 + Y}{3} + \frac{y2 * Y}{2}\right) R + \frac{y2 * Y}{2} * \text{integrate}\left[\frac{1}{R}, x2\right]; \\
\text{integrate}\left[x2 * \text{ArcTanh}\left[\frac{X}{R}\right], x2\right] &= \frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x2\right] -
\end{aligned}$$

$$\begin{aligned}
& \frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x2\right] + \frac{Z}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right]; \\
& \text{integrate}\left[x2 * \text{ArcTanh}\left[\frac{Y}{R}\right], x2\right] = \frac{x2^2}{2} * \text{ArcTanh}\left[\frac{Y}{R}\right] - \frac{1}{2} * \text{integrate}\left[\frac{x2^2}{R}, x2\right]; \\
& \text{integrate}\left[x2 * \text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] = \frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] - \\
& \quad \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x2\right] + \frac{X}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right]; \\
& \text{integrate}\left[x2 * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right] = \\
& \quad \frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x2\right] - \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x2\right]; \\
& \text{integrate}\left[x2 * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x2\right] = \frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x2\right] + \\
& \quad \frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x2\right] + \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x2\right]; \\
& \text{integrate}\left[x2 * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right] = \\
& \quad \frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x2\right] - \frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x2\right];
\end{aligned}$$

In[]:=

```
(* Rules for the y2 integration *)
(* First the actual recursions *)

integrate[ $\frac{y2^{n\_Integer}}{R}$ , y2] := integrate[ $\frac{y2^n}{R}$ , y2] = `:

 $\frac{y2^{n-1} R}{n} + \frac{(2n-1)x2}{n} * \text{integrate}\left[\frac{y2^{n-1}}{R}, y2\right] - \frac{(n-1)*(x2^2 + Y)}{n} * \text{integrate}\left[\frac{y2^{n-2}}{R}, y2\right];$ 

integrate[y2n_Integer * R, y2] := integrate[y2n * R, y2] = `:

 $\frac{y2^{n-1} * (Y^2 + Y) * R}{n+2} + \frac{(2*n+1)x2}{n+2} * \text{integrate}[y2^{n-1} * R, y2] -$ 
 $\frac{(n-1)*(x2^2 + Y)}{n+2} * \text{integrate}[y2^{n-2} R, y2];$ 

integrate[y2n_Integer * ArcTanh[ $\frac{X}{R}$ ], y2] := integrate[y2n * ArcTanh[ $\frac{X}{R}$ ], y2] = `:

 $\frac{y2^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y2\right] + \frac{n}{n+1} * x2 * \text{integrate}\left[y2^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], y2\right] +$ 
 $\frac{n}{n+1} * X * \text{integrate}\left[y2^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], y2\right] -$ 
 $\frac{n}{n+1} * Z * \text{integrate}\left[y2^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right];$ 

integrate[y2n_Integer * ArcTanh[ $\frac{Y}{R}$ ], y2] := integrate[y2n * ArcTanh[ $\frac{Y}{R}$ ], y2] = `:
```


$$\frac{y2^{n+1}}{n+1} * \text{ArcTanh}\left[\frac{Y}{R}\right] + \frac{1}{n+1} * \text{integrate}\left[\frac{y2^{n+1}}{R}, y2\right];$$

$$\text{integrate}\left[y2^{n_Integer} * \text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] := \text{integrate}\left[y2^n * \text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] = \cdot.$$

$$\begin{aligned} & \frac{y2^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] + \frac{n}{n+1} * x2 * \text{integrate}\left[y2^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] + \\ & \frac{n}{n+1} * Z * \text{integrate}\left[y2^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], y2\right] - \\ & \frac{n}{n+1} * X * \text{integrate}\left[y2^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right]; \end{aligned}$$

$$\text{integrate}\left[y2^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right] := \text{integrate}\left[y2^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right] = \cdot.$$

$$\begin{aligned} & \frac{y2^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right] + \frac{n}{n+1} * x2 * \\ & \text{integrate}\left[y2^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right] + \frac{n}{n+1} * Z * \text{integrate}\left[y2^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], y2\right]; \end{aligned}$$

$$\text{integrate}\left[y2^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y2\right] := \text{integrate}\left[y2^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y2\right] = \cdot.$$

$$\begin{aligned} & \frac{y2^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y2\right] + \\ & \frac{n}{n+1} * x2 * \text{integrate}\left[y2^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y2\right] - \frac{n}{n+1} * X * \\ & \text{integrate}\left[y2^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] - \frac{n}{n+1} * Z * \text{integrate}\left[y2^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], y2\right]; \end{aligned}$$

$$\text{integrate}\left[y2^{n_Integer} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right] := \text{integrate}\left[y2^n * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right] = \cdot.$$

$$\begin{aligned} & \frac{y2^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right] + \frac{n}{n+1} * x2 * \\ & \text{integrate}\left[y2^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right] + \frac{n}{n+1} * X * \text{integrate}\left[y2^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], y2\right]; \end{aligned}$$

(* The recursion ends. *)

$$\text{integrate}\left[\frac{1}{R}, y2\right] = -\text{ArcTanh}\left[\frac{Y}{R}\right];$$

$$\text{integrate}[R, y2] = -\frac{Y * R}{2} - \frac{Y}{2} * \text{ArcTanh}\left[\frac{Y}{R}\right];$$

$$\text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y2\right] = Z * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right] - Y * \text{ArcTanh}\left[\frac{X}{R}\right] - X * \text{ArcTanh}\left[\frac{Y}{R}\right];$$

$$\text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y2\right] = R - Y * \text{ArcTanh}\left[\frac{Y}{R}\right];$$

$$\text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] = X * \text{ArcTan}\left[\frac{Z}{R} * \frac{Y}{R} * \frac{R}{X}\right] - Y * \text{ArcTanh}\left[\frac{Z}{R}\right] - Z * \text{ArcTanh}\left[\frac{Y}{R}\right];$$

$$\begin{aligned} \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right] &= -Y * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right] - Z * \text{ArcTanh}\left[\frac{X}{R}\right]; \\ \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y2\right] &= -Y * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right] + X * \text{ArcTanh}\left[\frac{Z}{R}\right] + Z * \text{ArcTanh}\left[\frac{X}{R}\right]; \\ \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right] &= -Y * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right] - X * \text{ArcTanh}\left[\frac{Z}{R}\right]; \end{aligned}$$

(* One level above the ends of the recursion *)

$$\text{integrate}\left[\frac{y2}{R}, y2\right] = R - x2 \text{ArcTanh}\left[\frac{Y}{R}\right];$$

$$\text{integrate}[y2 * R, y2] = \left(\frac{Y^2 + Y}{3} - \frac{x2 * Y}{2}\right) R + \frac{x2 * Y}{2} \text{integrate}\left[\frac{1}{R}, y2\right];$$

$$\begin{aligned} \text{integrate}\left[y2 * \text{ArcTanh}\left[\frac{X}{R}\right], y2\right] &= \frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y2\right] + \\ &\frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y2\right] - \frac{Z}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right]; \end{aligned}$$

$$\text{integrate}\left[y2 * \text{ArcTanh}\left[\frac{Y}{R}\right], y2\right] = \frac{y2^2}{2} * \text{ArcTanh}\left[\frac{Y}{R}\right] + \frac{1}{2} * \text{integrate}\left[\frac{y2^2}{R}, y2\right];$$

$$\begin{aligned} \text{integrate}\left[y2 * \text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] &= \frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] + \\ &\frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y2\right] - \frac{X}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[y2 * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right] &= \\ &\frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y2\right] + \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y2\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[y2 * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y2\right] &= \frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y2\right] - \\ &\frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y2\right] - \frac{Z}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y2\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[y2 * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right] &= \\ &\frac{x2 + y2}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y2\right] + \frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y2\right]; \end{aligned}$$

In[]:=

(* Rules for the x3 integration *)

(* First the actual recursions *)

$$\text{integrate}\left[\frac{x3^{n_Integer}}{R}, x3\right] := \text{integrate}\left[\frac{x3^n}{R}, x3\right] = \cdot$$

$$\frac{x3^{n-1} R}{n} + \frac{(2n-1)y3}{n} * \text{integrate}\left[\frac{x3^{n-1}}{R}, x3\right] - \frac{(n-1)*(y3^2 + \Theta)}{n} * \text{integrate}\left[\frac{x3^{n-2}}{R}, x3\right];$$

$$\text{integrate}[x3^{n_Integer} * R, x3] := \text{integrate}[x3^n * R, x3] = \cdot$$

$$\frac{x3^{n-1} * (Z^2 + \Theta) * R}{n+2} + \frac{(2 * n + 1) y3}{n+2} * \text{integrate}[x3^{n-1} * R, x3] -$$

$$\frac{(n-1) * (y3^2 + \Theta)}{n+2} * \text{integrate}[x3^{n-2} R, x3];$$

$$\text{integrate}[x3^{n_Integer} * \text{ArcTanh}\left[\frac{X}{R}\right], x3] := \text{integrate}[x3^n * \text{ArcTanh}\left[\frac{X}{R}\right], x3] = \cdot$$

$$\frac{x3^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x3\right] + \frac{n}{n+1} * y3 * \text{integrate}\left[x3^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], x3\right] -$$

$$\frac{n}{n+1} * X * \text{integrate}\left[x3^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], x3\right] +$$

$$\frac{n}{n+1} * Y * \text{integrate}\left[x3^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x3\right];$$

$$\text{integrate}[x3^{n_Integer} * \text{ArcTanh}\left[\frac{Y}{R}\right], x3] := \text{integrate}[x3^n * \text{ArcTanh}\left[\frac{Y}{R}\right], x3] = \cdot$$

$$\frac{x3^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x3\right] + \frac{n}{n+1} * y3 * \text{integrate}\left[x3^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], x3\right] -$$

$$\frac{n}{n+1} * Y * \text{integrate}\left[x3^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], x3\right] +$$

$$\frac{n}{n+1} * X * \text{integrate}\left[x3^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x3\right];$$

$$\text{integrate}[x3^{n_Integer} * \text{ArcTanh}\left[\frac{Z}{R}\right], x3] := \text{integrate}[x3^n * \text{ArcTanh}\left[\frac{Z}{R}\right], x3] = \cdot$$

$$\frac{x3^{n+1}}{n+1} * \text{ArcTanh}\left[\frac{Z}{R}\right] - \frac{1}{n+1} * \text{integrate}\left[\frac{x3^{n+1}}{R}, x3\right];$$

$$\text{integrate}[x3^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x3] :=$$

$$\text{integrate}[x3^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x3] = \cdot$$

$$\frac{x3^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x3\right] +$$

$$\frac{n}{n+1} * y3 * \text{integrate}\left[x3^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x3\right] + \frac{n}{n+1} * X *$$

$$\text{integrate}\left[x3^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], x3\right] + \frac{n}{n+1} * Y * \text{integrate}\left[x3^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], x3\right];$$

$$\text{integrate}[x3^{n_Integer} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x3] := \text{integrate}[x3^n * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x3] = \cdot$$

$$\frac{x3^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x3\right] + \frac{n}{n+1} * y3 *$$

$$\text{integrate}\left[x3^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x3\right] - \frac{n}{n+1} * Y * \text{integrate}\left[x3^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], x3\right];$$

$$\text{integrate}[x3^{n_Integer} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x3] := \text{integrate}[x3^n * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x3] = \cdot$$

$$\frac{x^3^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x^3\right] + \frac{n}{n+1} * y^3 * \text{integrate}\left[x^3^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x^3\right] - \frac{n}{n+1} * X * \text{integrate}\left[x^3^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], x^3\right];$$

(* The recursion ends. *)

$$\text{integrate}\left[\frac{1}{R}, x^3\right] = \text{ArcTanh}\left[\frac{Z}{R}\right];$$

$$\text{integrate}[R, x^3] = \frac{Z * R}{2} + \frac{\Theta}{2} * \text{ArcTanh}\left[\frac{Z}{R}\right];$$

$$\text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x^3\right] = Z * \text{ArcTanh}\left[\frac{X}{R}\right] + X * \text{ArcTanh}\left[\frac{Z}{R}\right] - Y * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right];$$

$$\text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x^3\right] = Z * \text{ArcTanh}\left[\frac{Y}{R}\right] + Y * \text{ArcTanh}\left[\frac{Z}{R}\right] - X * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right];$$

$$\text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x^3\right] = Z * \text{ArcTanh}\left[\frac{Z}{R}\right] - R;$$

$$\text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x^3\right] =$$

$$Z * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right] - X * \text{ArcTanh}\left[\frac{Y}{R}\right] - Y * \text{ArcTanh}\left[\frac{X}{R}\right];$$

$$\text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x^3\right] = Z * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right] + Y * \text{ArcTanh}\left[\frac{X}{R}\right];$$

$$\text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x^3\right] = Z * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right] + X * \text{ArcTanh}\left[\frac{Y}{R}\right];$$

(* One level above the ends of the recursion *)

$$\text{integrate}\left[\frac{x^3}{R}, x^3\right] = R + y^3 \text{ArcTanh}\left[\frac{Z}{R}\right];$$

$$\text{integrate}[x^3 * R, x^3] = \left(\frac{Z^2 + \Theta}{3} + \frac{y^3 * Z}{2}\right) R + \frac{y^3 * \Theta}{2} \text{integrate}\left[\frac{1}{R}, x^3\right];$$

$$\text{integrate}\left[x^3 * \text{ArcTanh}\left[\frac{X}{R}\right], x^3\right] = \frac{x^3 + y^3}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x^3\right] - \frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x^3\right] + \frac{Y}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x^3\right];$$

$$\text{integrate}\left[x^3 * \text{ArcTanh}\left[\frac{Y}{R}\right], x^3\right] = \frac{x^3 + y^3}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x^3\right] - \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], x^3\right] + \frac{X}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x^3\right];$$

$$\text{integrate}\left[x^3 * \text{ArcTanh}\left[\frac{Z}{R}\right], x^3\right] = \frac{x^3^2}{2} * \text{ArcTanh}\left[\frac{Z}{R}\right] - \frac{1}{2} * \text{integrate}\left[\frac{x^3^2}{R}, x^3\right];$$

$$\text{integrate}\left[x^3 * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x^3\right] = \frac{x^3 + y^3}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], x^3\right] +$$

$$\begin{aligned} & \frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x3\right] + \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x3\right]; \\ \text{integrate}\left[x3 * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x3\right] = \\ & \frac{x3 + y3}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], x3\right] - \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], x3\right]; \\ \text{integrate}\left[x3 * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x3\right] = \\ & \frac{x3 + y3}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], x3\right] - \frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], x3\right]; \end{aligned}$$

In[]:=

(* Rules for the y3 integration *)

(* First the actual recursions *)

$$\begin{aligned} \text{integrate}\left[\frac{y3^{n_Integer}}{R}, y3\right] &:= \text{integrate}\left[\frac{y3^n}{R}, y3\right] = \cdot \\ \frac{y3^{n-1} R}{n} + \frac{(2n-1)x3}{n} * \text{integrate}\left[\frac{y3^{n-1}}{R}, y3\right] &- \frac{(n-1)*(x3^2 + \Theta)}{n} * \text{integrate}\left[\frac{y3^{n-2}}{R}, y3\right]; \\ \text{integrate}[y3^{n_Integer} * R, y3] &:= \text{integrate}[y3^n * R, y3] = \cdot \\ \frac{y3^{n-1} * (Z^2 + \Theta) * R}{n+2} + \frac{(2*n+1)x3}{n+2} * \text{integrate}[y3^{n-1} * R, y3] &- \\ \frac{(n-1)*(x3^2 + \Theta)}{n+2} * \text{integrate}[y3^{n-2} R, y3]; \\ \text{integrate}[y3^{n_Integer} * \text{ArcTanh}\left[\frac{X}{R}\right], y3] &:= \text{integrate}[y3^n * \text{ArcTanh}\left[\frac{X}{R}\right], y3] = \cdot \\ \frac{y3^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y3\right] + \frac{n}{n+1} * x3 * \text{integrate}[y3^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], y3] &+ \\ \frac{n}{n+1} * X * \text{integrate}[y3^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], y3] - \\ \frac{n}{n+1} * Y * \text{integrate}\left[y3^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y3\right]; \\ \text{integrate}[y3^{n_Integer} * \text{ArcTanh}\left[\frac{Y}{R}\right], y3] &:= \text{integrate}[y3^n * \text{ArcTanh}\left[\frac{Y}{R}\right], y3] = \cdot \\ \frac{y3^n}{n+1} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y3\right] + \frac{n}{n+1} * x3 * \text{integrate}[y3^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], y3] &+ \\ \frac{n}{n+1} * Y * \text{integrate}\left[y3^{n-1} * \text{ArcTanh}\left[\frac{Z}{R}\right], y3\right] - \\ \frac{n}{n+1} * X * \text{integrate}\left[y3^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y3\right]; \\ \text{integrate}[y3^{n_Integer} * \text{ArcTanh}\left[\frac{Z}{R}\right], y3] &:= \text{integrate}[y3^n * \text{ArcTanh}\left[\frac{Z}{R}\right], y3] = \cdot \\ \frac{y3^{n+1}}{n+1} * \text{ArcTanh}\left[\frac{Z}{R}\right] + \frac{1}{n+1} * \text{integrate}\left[\frac{y3^{n+1}}{R}, y3\right]; \end{aligned}$$

```

integrate[y3^n_Integer * ArcTan[ $\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}$ ], y3] :=
integrate[y3^n * ArcTan[ $\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}$ ], y3] = `:
 $\frac{y3^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y3\right] +$ 
 $\frac{n}{n+1} * x3 * \text{integrate}\left[y3^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y3\right] - \frac{n}{n+1} * X *$ 
 $\text{integrate}\left[y3^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], y3\right] - \frac{n}{n+1} * Y * \text{integrate}\left[y3^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], y3\right];$ 
integrate[y3^n_Integer * ArcTan[ $\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}$ ], y3] := integrate[y3^n * ArcTan[ $\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}$ ], y3] = `:
 $\frac{y3^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y3\right] + \frac{n}{n+1} * x3 *$ 
 $\text{integrate}\left[y3^{n-1} * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y3\right] + \frac{n}{n+1} * Y * \text{integrate}\left[y3^{n-1} * \text{ArcTanh}\left[\frac{X}{R}\right], y3\right];$ 
integrate[y3^n_Integer * ArcTan[ $\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}$ ], y3] := integrate[y3^n * ArcTan[ $\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}$ ], y3] = `:
 $\frac{y3^n}{n+1} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y3\right] + \frac{n}{n+1} * x3 *$ 
 $\text{integrate}\left[y3^{n-1} * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y3\right] + \frac{n}{n+1} * X * \text{integrate}\left[y3^{n-1} * \text{ArcTanh}\left[\frac{Y}{R}\right], y3\right];$ 

(* First the recursion ends. *)
integrate[ $\frac{1}{R}$ , y3] = -ArcTanh[ $\frac{Z}{R}$ ];
integrate[R, y3] = - $\frac{Z * R}{2} - \frac{\Theta}{2} * \text{ArcTanh}\left[\frac{Z}{R}\right];$ 
integrate[ArcTanh[ $\frac{X}{R}$ ], y3] = Y * ArcTan[ $\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}$ ] - Z * ArcTanh[ $\frac{X}{R}$ ] - X * ArcTanh[ $\frac{Z}{R}$ ];
integrate[ArcTanh[ $\frac{Y}{R}$ ], y3] = X * ArcTan[ $\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}$ ] - Z * ArcTanh[ $\frac{Y}{R}$ ] - Y * ArcTanh[ $\frac{Z}{R}$ ];
integrate[ArcTanh[ $\frac{Z}{R}$ ], y3] = R - Z * ArcTanh[ $\frac{Z}{R}$ ];
integrate[ArcTan[ $\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}$ ], y3] =
X * ArcTanh[ $\frac{Y}{R}$ ] + Y * ArcTanh[ $\frac{X}{R}$ ] - Z * ArcTan[ $\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}$ ];
integrate[ArcTan[ $\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}$ ], y3] = -Z * ArcTan[ $\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}$ ] - Y * ArcTanh[ $\frac{X}{R}$ ];
integrate[ArcTan[ $\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}$ ], y3] = -Z * ArcTan[ $\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}$ ] - X * ArcTanh[ $\frac{Y}{R}$ ];

```

(* One level above the ends of the recursion *)

$$\text{integrate}\left[\frac{y^3}{R}, y3\right] = R - x^3 \text{ArcTanh}\left[\frac{Z}{R}\right];$$

$$\text{integrate}[y^3 * R, y3] = \left(\frac{Z^2 + \Theta}{3} - \frac{x^3 * Z}{2}\right) R + \frac{x^3 * \Theta}{2} \text{integrate}\left[\frac{1}{R}, y3\right];$$

$$\begin{aligned} \text{integrate}\left[y^3 * \text{ArcTanh}\left[\frac{X}{R}\right], y3\right] &= \frac{x^3 + y^3}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y3\right] + \\ &\frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y3\right] - \frac{Y}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y3\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[y^3 * \text{ArcTanh}\left[\frac{Y}{R}\right], y3\right] &= \frac{x^3 + y^3}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y3\right] + \\ &\frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Z}{R}\right], y3\right] - \frac{X}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y3\right]; \end{aligned}$$

$$\text{integrate}\left[y^3 * \text{ArcTanh}\left[\frac{Z}{R}\right], y3\right] = \frac{y^3^2}{2} * \text{ArcTanh}\left[\frac{Z}{R}\right] + \frac{1}{2} * \text{integrate}\left[\frac{y^3^2}{R}, y3\right];$$

$$\begin{aligned} \text{integrate}\left[y^3 * \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y3\right] &= \frac{x^3 + y^3}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], y3\right] - \\ &\frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y3\right] - \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y3\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[y^3 * \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y3\right] &= \\ &\frac{x^3 + y^3}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], y3\right] + \frac{Y}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{X}{R}\right], y3\right]; \end{aligned}$$

$$\begin{aligned} \text{integrate}\left[y^3 * \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y3\right] &= \\ &\frac{x^3 + y^3}{2} * \text{integrate}\left[\text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right], y3\right] + \frac{X}{2} * \text{integrate}\left[\text{ArcTanh}\left[\frac{Y}{R}\right], y3\right]; \end{aligned}$$

In[]:= **expr** =

$$\begin{aligned} &\text{FullSimplify}\left[\text{Collect}\left[\text{integrate}\left[\text{integrate}\left[\text{integrate}\left[\frac{1}{R}, y1\right], y2\right], y3\right], \left\{R, \text{ArcTanh}\left[\frac{X}{R}\right], \right. \right. \right. \\ &\quad \left. \left. \text{ArcTanh}\left[\frac{Y}{R}\right], \text{ArcTanh}\left[\frac{Z}{R}\right], \text{ArcTan}\left[\frac{X}{R} * \frac{Y}{R} * \frac{R}{Z}\right], \text{ArcTan}\left[\frac{X}{R} * \frac{Z}{R} * \frac{R}{Y}\right], \text{ArcTan}\left[\frac{Y}{R} * \frac{Z}{R} * \frac{R}{X}\right]\right\}\right] \end{aligned}$$

$$\begin{aligned}
\text{Out}[*] = & \frac{1}{2} \left((x_3 - y_3)^2 \operatorname{ArcTan} \left[\frac{(x_1 - y_1)(x_2 - y_2)}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} (x_3 - y_3)} \right] + \right. \\
& (x_2 - y_2)^2 \operatorname{ArcTan} \left[\frac{(x_1 - y_1)(x_3 - y_3)}{(x_2 - y_2) \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} \right] + \\
& (x_1 - y_1)^2 \operatorname{ArcTan} \left[\frac{(x_2 - y_2)(x_3 - y_3)}{(x_1 - y_1) \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} \right] + \\
& 2(x_2 - y_2)(x_3 - y_3) \operatorname{ArcTanh} \left[\frac{-x_1 + y_1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} \right] + \\
& 2(x_1 - y_1) \left((x_3 - y_3) \operatorname{ArcTanh} \left[\frac{-x_2 + y_2}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} \right] + \right. \\
& \left. \left. (x_2 - y_2) \operatorname{ArcTanh} \left[\frac{-x_3 + y_3}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} \right] \right) \right)
\end{aligned}$$

In[] :=* (* The triple anti-derivative w.r.t. y1,y2,
y3 thus is given as follows. A check below. *)

```

expr = 1/2 (X^2 * ArcTan[Y/R, Z/R, X] + Y^2 * ArcTan[X/R, Z/R, Y] + Z^2 * ArcTan[X/R, Y/R, Z]) -
X * Y * ArcTanh[Z/R] - X * Z * ArcTanh[Y/R] - Y * Z * ArcTanh[X/R];
expr = Simplify[D[expr, y1]];
expr = Simplify[D[expr, y2]];
expr = Simplify[D[expr, y3]];
expr

```

$$\text{Out}[*] = \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}}$$