Color Blind Cellular Automata*

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Abstract. We introduce the classes of color blind and typhlotic cellular automata, that is, cellular automata that commute with all symbol permutations and all symbol mappings, respectively. We show that color blind cellular automata form a relatively large subclass of all cellular automata which contains an intrinsically universal automaton. On the other hand, we give simple characterizations for the color blind CA which are also group homomorphisms, and for general typhlotic CA, showing that both must be trivial in most cases.

Keywords: cellular automata, commutation, symbol permutations, homomorphisms

1 Introduction

Suppose we wish to study a cellular automaton f, that is, a continuous shift invariant function from $S^{\mathbb{Z}}$ to itself, where S is a finite alphabet. A natural direction of study would be to consider its relation to some other cellular automata, for example to find the commutator of f, the set of all cellular automata it commutes with. This is known as the *commuting block maps problem*, and it has a long history, dating back to the 70s [2]. Algebraically, the commutator of f can also be viewed as the set of homomorphisms of the unary algebra $(S^{\mathbb{Z}}, f)$ that are also cellular automata. To generalize this notion, one defines the commutator of a whole family of cellular automata as the set of those CA that commute with all of them.

In this article, we study so-called color blind (typhlotic) cellular automata, that is, automata which commute with all symbol permutations (all symbol mappings, respectively), on full shifts and their subshifts. In other words, color blind cellular automata form the commutator of the family of all cellwise permutations. We give a natural logical characterization of color blind cellular automata and show that there exists an intrinsically universal color blind CA. Perhaps

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somewhat surprisingly, we show that intrinsic universality is also possible in typhotic CA if the full shift is binary, but that every typhlotic CA must be a shift map on other full shifts. We also show that in a quantitative sense, the class of color blind cellular automata is relatively as large as possible in the class of all cellular automata.

We then consider the case of full shifts over a finite group alphabet. The natural self-maps of such objects are the cellular automata that are also group homomorphisms for the product group structure, and we call them homomorphic cellular automata. We investigate cellular automata that are both color blind and homomorphic. This turns out to be very restrictive, and the situation is similar to that of typhlotic CA without the group structure: if the alphabet group is sufficiently simple ($\mathbb{Z}_2$, $\mathbb{Z}_3$, or $\mathbb{Z}_2^2$), then there exist nontrivial color blind homomorphic CA, but on more complicated full group shifts, all color blind homomorphic CA are shift maps.

2 Definitions

Let S be a finite set, called the *alphabet*. The *full shift* is the space $S^{\mathbb{Z}}$ of infinite configurations over S endowed with the product topology. For $x \in S^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, we denote by x_n the symbol of x at coordinate n. For a word $w \in S^*$ and $s \in S$, we denote by $|w|$ the length of w, and by $|w|_s$ the number of occurrences of s in w. For a configuration $x \in S^{\mathbb{Z}}$, we say *w occurs in x* if $w = x_{[n,n+|w|-1]}$ for some $n \in \mathbb{Z}$.

A subset $X \subset S^{\mathbb{Z}}$ is called a *subshift* if it is closed in the topology and invariant under the *shift map* $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$, defined by $\sigma(x)_n = x_{n+1}$. Alternatively, a subshift X is defined by a set $\mathcal{F} \subset S^*$ of *forbidden words* as the set of configurations in which no $w \in \mathcal{F}$ occurs. If $\mathcal{F}$ can be taken finite, X is a *subshift of finite type* (SFT for short). A configuration $x \in S^{\mathbb{Z}}$ is *spatially periodic* if $\sigma^p(x) = x$ for some $p > 0$.

A continuous mapping $f : X \to X$ in a subshift that commutes with σ is called a *cellular automaton*. All cellular automata f are defined by *local functions* $F : S^N \to S$, where $N \subset \mathbb{Z}$ is the finite *neighborhood* of f, by the formula $f(x)_n = F(x_{n+N})$ for all $n \in \mathbb{Z}$ [4]. We usually define $N = [-r, r]$ for some $r \in \mathbb{N}$, called the *radius* of f. To each CA f we associate a local function f_{loc}, which in general is not uniquely defined, but this should not cause any confusion. A configuration $x \in X$ is called *temporally periodic (with respect to f)* if $f^p(x) = x$ for some $p > 0$. A symbol mapping $\pi : S \to S$ can also be seen as a cellular automaton on $S^{\mathbb{Z}}$ by $\pi(x)_n = \pi(x_n)$.

Let S be a finite algebra. Then, $S^{\mathbb{Z}}$ becomes an algebra when the operations are applied cellwise. We say a CA $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is *homomorphic* (with respect to the algebraic structure of S) if it is a homomorphism of $S^{\mathbb{Z}}$. From the results of [10] we know that this is the case exactly if the local rule $F : S^N \to S$ is a homomorphism. If S is a group and f is of the form $\sum_{i=0}^{k-1} \sigma^{n_i}$ for some $n_i \in \mathbb{Z}$, we say f is a *sum of shifts*, and if the n_i are pairwise distinct, f is a *sum of distinct shifts*.

Let X be a nonempty set, and let $Q \subset 2^X$. If $\emptyset \notin Q$ and for all $A, B \in Q$ we have $A \subset C \implies C \in Q$ and $A \cap B \in Q$, then Q is a *filter* on X. If additionally $Q \subset Q' \implies Q = Q'$ for all filters Q' of X, then Q is an *ultrafilter*.

Remark 1. In the literature, the terminology related to cellular automata that are also group homomorphisms varies wildly. For example, in [7], the authors use the terms additive CA and group CA for cellular automata that are homomorphic with respect to an abelian group alphabet, and the term k-rule for CA that are sums of k distinct shifts. In [8] the term linear CA is used for homomorphic cellular automata. On the other hand, in [6] and many subsequent articles, the term linear CA refers to cellular automata on $\mathbb{Z}_p^{\mathbb{Z}}$ that we would call sums of shifts. Of course, over the alphabet $\mathbb{Z}_p$ the notions of homomorphic CA and sum of shifts coincide, but not over general abelian group alphabets. Even worse, the term linear is sometimes used to refer to *one-dimensional* CA. We have chosen our terminology in the hope of being as unambiguous as possible.

3 Color Blind Cellular Automata

We begin with the definition of our objects of interest, the color blind cellular automata.

Definition 1. *Let $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be a CA such that for all symbol permutations $\pi : S \to S$ we have $f \circ \pi = \pi \circ f$. Then we say f is* color blind*. If f commutes with all symbol mappings, we say f is* typhlotic*.*

In other words, the set of color blind (typhlotic) cellular automata on $S^{\mathbb{Z}}$ is exactly the commutator of the set of all permutations on S (functions from S to itself, respectively). We use the somewhat obscure term typhlotic, meaning blind, to avoid cluttering the global namespace of cellular automata: we will soon see that these automata are rather trivial (Proposition 4), and presumably do not have much theory beyond what we prove in this article.

A CA f on $S^{\mathbb{Z}}$ is called *captive* if the local rule f_{loc} satisfies $f_{\mathrm{loc}}(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ for all $a_1, \ldots, a_n \in S$. Color blind CA are 'almost captive' in the following sense.

Lemma 1. *Let $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be a color blind CA. Then $f_{\mathrm{loc}}(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$ whenever $|\{a_1, \ldots, a_n\}| < |S| - 1$.*

Proof. Suppose that we have $|\{a_1, \ldots, a_n\}| < |S| - 1$, but $a = f_{\mathrm{loc}}(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$. Then, there exists

$$b \in S \setminus \{a, a_1, \ldots, a_n\}.$$

Now, f does not commute with the transposition $(a\ b)$. $\square$

Typhlotic CA are in fact captive, which we will obtain as a corollary of Proposition 4. We continue with a simple logical characterization of color blind cellular automata which motivates their definition.

Definition 2. *Fix a set of variables $V = \{v_1, \dots, v_n\}$. A* color blind equation *over V is a boolean combination of basic equations of the form $v_i = v_j$. For a symbol equation E over V, an alphabet S and a word $w \in S^n$, we denote by $E(w)$ the equation obtained by replacing each v_i by w_i in E. The equation E defines a set of words $E(S) \subset S^n$ by $E(S) = \{w \in S^n \mid E(w) \text{ holds}\}$. We say E is* captive on S *if the last letter of w occurs at least twice in w for all $w \in E(S)$, and* captive*, if it is captive on S for all finite S. If $n = 2r+2$ and $E(S)$ defines a function from S^{2r+1} to S, we let $f_E^S : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be the cellular automaton whose local function it is. We say f_E^S is* defined by a color blind equation*.*

Lemma 2. *A set of words $W \subset S^n$ is defined by a color blind equation if and only if it is closed under symbol permutations.*

Proof. First, let $W = E(S)$ for an equation E, and consider an arbitrary symbol permutation $\pi : S \to S$. It is clear that if $E(w)$ holds for a word $w \in S^n$, then so does $E(\pi(w))$, and thus W is closed under symbol permutations.

Suppose then that W is closed under symbol permutations. For all $w \in W$, define the equation $E_w = \bigwedge_{i,j \in [0,n-1]} t(i,j)$, where $t(i,j)$ is $v_i = v_j$ if $w_i = w_j$, and $\neg(v_i = v_j)$ otherwise. We let $E = \bigvee_{w \in W} E_w$. Now, it is clear that $W \subset E(S)$. On the other hand, let $v \in E(S)$. This means that $v \in E_w(S)$ for some $w \in W$. It is easy to see that there then exists a symbol permutation $\pi : S \to S$ with $\pi(w) = v$, and since W is closed under symbol permutations, we have $v \in W$. □

As a cellular automaton commutes with symbol permutations if and only if its local rule does, we obtain the following corollary.

Corollary 1. *A CA $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is (captive and) color blind if and only if it is defined by a (captive and) color blind equation.*

The characterization essentially says that a cellular automaton is color blind if and only if it can be defined without referring to any particular colors, but only their arrangements on the neighborhood. Note that the reason such a cellular automaton may not be captive, but only almost captive in the sense of Lemma 1, is that one may output the 'last remaining color' unambiguously when all but one color appear in the neighborhood (as the alphabet size is known). If we restrict to captive color blind cellular automata, the equation defining the color blind CA can be chosen so that it defines a CA on any subshift containing the original. This is essentially the content of Proposition 1. To prove it, we need a few definitions and lemmas.

Definition 3. *Let $X \subset Y \subset S^{\mathbb{Z}}$ be subshifts. If $X = Y \cap Z$ for some SFT $Z \subset S^{\mathbb{Z}}$, we say X is a* subSFT *of Y.*

Lemma 3. *If X is a subSFT of Y and $X = \bigcap_{i \in \mathbb{N}} X_i$ for some subshifts X_i of Y such that $X_{i+1} \subset X_i$ for all $i \in \mathbb{N}$, then $X = X_i$ for some i.*

Proof. Let $X = Y \cap Z$ where Z is an SFT. Then, $\bigcap_{i \in \mathbb{N}} X_i = X \subset Z$, so $X_i \subset Z$ for all large enough i, since Z is an SFT (all the finitely many forbidden patterns of Z must be absent in some X_i). But $X_i = X$ for all such i.

Definition 4. *Let $X \subset S^{\mathbb{Z}}$ be a subshift, and let $f : X \to X$ be a CA. Suppose that whenever $\pi : S \to S$ is a symbol permutation and $x \in X$ satisfies $\pi(x) \in X$, then $\pi(f(x)) = f(\pi(x))$. Then we say f is* color blind *on X. If there exists $r \in \mathbb{N}$ such that for all $x \in X$ there exists $k \in [-r, r]$ such that $f(x) = x_k$, we say f is* captive *on X.*

Proposition 1. *Let $X \subset S^{\mathbb{Z}}$ be a subshift. Then a CA $f : X \to X$ is captive and color blind on X iff $f = f_E^S|_X$ for a captive color blind equation E.*

Proof. If $f = f_E^S|_X$ for some captive color blind equation E and $\pi : S \to S$ is any symbol permutation, then $\pi(f_E^S(x)) = f_E^S(\pi(x))$ for all $x \in S^{\mathbb{Z}}$, so in particular this is the case when $x, \pi(x) \in X$. In this case f is also clearly captive on X.

For the other direction, the idea is to take the subshift

$$Y = \{(x, y) \in X^2 \mid f(x) = y\},$$

and define a decreasing sequence of subshifts $Y_i \subset (S^2)^{\mathbb{Z}}$ such that $Y = \bigcap_{i \in \mathbb{N}} Y_i$, and the Y_i are all defined by color blind equations of a certain form. Since the subshift Y is a subSFT of $Z = X^2$ and $Y_i \subset Z$, Lemma 3 implies that $Y = Y_i$ for some $i \in \mathbb{N}$.

So suppose that $f : X \to X$ is captive and color blind on X, and let $[-r, r]$ be the neighborhood of f. For all $i \in \mathbb{N}$, define $W_i = \{x_{[-r-i, r+i]} \cdot f(x)_0 \mid x \in X\}$. Let E_i be a set of equations defined by the W_i, as in Lemma 2, so that $E_i(v \cdot a)$ holds if and only if there is a permutation $\pi : S \to S$ and some $w \cdot b \in W_i$ with $\pi(w \cdot b) = v \cdot a$. For large enough i the E_i are captive, since f is captive on X. Define then

$$Y_i = \{(x, y) \in X^2 \mid \forall j \in \mathbb{Z} : E_i(x_{[j-r-i, j+r+i]} \cdot y_j)\} \subset (S^2)^{\mathbb{Z}}.$$

Now, we claim that $Y = \{(x, y) \in X^2 \mid f(x) = y\} = \bigcap_{i \in \mathbb{N}} Y_i$. Clearly, $Y \subset Y_i$ for all $i \in \mathbb{N}$, so suppose $(x, y) \in \bigcap_{i \in \mathbb{N}} Y_i$ but $(x, y) \notin Y$. We may assume $f(x)_0 \neq y_0$. Now, as $E_i(x_{[-r-i, r+i]} \cdot y_0)$ holds for all i, there exist symbol permutations $\pi_i : S \to S$ and words $w_i \cdot b_i \in W_i$ such that $\pi_i(w_i \cdot b_i) = x_{[-r-i, r+i]} \cdot y_0$.

Take an infinite subsequence where the π_i and b_i are fixed, and denote these by π and b. Then, extract a subsequence where the w_i converge to some configuration $z \in S^{\mathbb{Z}}$. Now we have $z \in X$, $\pi(z) = x \in X$, $f(z)_0 = b$ and thus $\pi(f(z))_0 = \pi(b) = y_0$, but $f(\pi(z))_0 = f(x)_0$. This is a contradiction, and thus $Y = \bigcap_{i \in \mathbb{N}} Y_i$.

Since Y is a subSFT of X^2 and the sequence $(Y_i)_{i \in \mathbb{N}}$ is decreasing, we have $Y = Y_i$ for some $i \in \mathbb{N}$ by Lemma 3, and the captive color blind equation E_i thus defines a local function $f_{\mathrm{loc}} : \mathcal{B}_{2(r+i)+1}(X) \to S$ for f. Let F be the equation defined by $\mathcal{B}_{2(i+r)+1}(X)$ and let

$$E = E_i \vee (\neg F \wedge v_{\mathrm{out}} = v_1).$$

Then, f_E^R is a captive color blind cellular automaton for any alphabet R: If the input of the local function is a word of X up to renaming the symbols, then f_E^R chooses the output from the inputs as f would. Otherwise, the word does not satisfy F, and f_E^R chooses the leftmost input as the output. $\square$

Example 1. The restriction of captivity in the above result is necessary: the symbol permutation $(0\ 1)$ on $\{0,1\}^{\mathbb{Z}}$ cannot be extended, as a cellular automaton, to a color blind cellular automaton on $\{0,1,2\}^{\mathbb{Z}}$.

We also show by another example that the color blind equation defined by f_{loc} may not be sufficient even if it is captive in the sense that it always outputs a symbol seen in the neighborhood. Let X consist of the configurations $x = {}^{\infty}(0122)^{\infty}$ and $y = {}^{\infty}(0022)^{\infty}$ and their shifts, and define $f_{\mathrm{loc}} : \{0,1,2\}^3 \to \{0,1,2\}$ by $f_{\mathrm{loc}}(a,b,c) = b$, except for $f_{\mathrm{loc}}(0,1,2) = 0$. Now, f is captive and color blind on X (the only symbol permutation we need to check is $(0\ 2)$). However, the color blind equation defined by f_{loc} does not extend to a cellular automaton on $\{0,1,2\}^{\mathbb{Z}}$, since $f_{\mathrm{loc}}(012) = 0 = f_{\mathrm{loc}}(201)$. One can check that the local rule with neighborhood $N = [-1,2]$ suffices though.

4 Constructing Color Blind Cellular Automata

In this section, we give concrete examples of color blind cellular automata, and prove some results that require explicit construction of such objects.

Definition 5. *Let $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be a cellular automaton with neighborhood size n. We say f is a* majority CA *if, whenever $f_{\mathrm{loc}}(s_1,\dots,s_n) = s$, we have $|\{i \in [1,n] \mid s_i = s\}| \geq |\{i \in [1,n] \mid s_i = s'\}|$ for all $s' \in S$*

This means that the local rule of a majority CA always outputs a symbol that occurs a maximal number of times in the input. All majority CA are of course captive. In the binary case, there is a unique majority CA for each odd neighborhood size, and this CA is color blind. In other cases, the CA must have a tie-breaking rule. To make such a CA color blind we can, for example, always choose the leftmost input symbol s_m that maximizes $|\{i \in [1,n] \mid s_i = s_m\}|$.

Of the 256 elementary cellular automata (see [11] for the definitions and the numbering scheme), 16 rules are color blind. The even-numbered rules are summarized in Table 1, while the odd-numbered rules are obtained by subtracting their numbers from 255, effectively composing them with the symbol permutation $(0\ 1)$. We show the even-numbered color blind rules, as they are exactly the captive ones. Of these 8 elementary automata, the most interesting ones are 150 and 142. Rule 150 is a sum of three distinct shifts, and some properties of rule 142 are studied in at least [3].

In the next result, the term intrinsically universal is understood with respect to simulation by injective bulking. See [9] for the precise definitions; the main message of the proposition is that captive color blind cellular automata can be very complex both from the computational and the dynamical points of view.

Table 1. The even-numbered color blind elementary CA. The variables v_1, v_2 and v_3 denote inputs to the local rule, and v_4 is its output.

CA	Color blind equation	Description
142	$(v_4 \neq v_2) \iff (v_1 = v_2 \neq v_3)$	Left shift with 'barriers'
150	$(v_1 = v_2) \iff (v_4 = v_3)$	Sum of neighborhood mod 2
170	$v_4 = v_3$	Left shift
178	$(v_4 = v_2) \iff (v_1 = v_2 = v_3)$	Flip unless all inputs equal
204	$v_4 = v_2$	Identity
212	$(v_4 \neq v_2) \iff (v_1 \neq v_2 = v_3)$	Mirrored 142
232	$(v_4 \neq v_2) \iff (v_1 \neq v_2 \neq v_3)$	Majority
240	$v_4 = v_1$	Right shift

Proposition 2. *For any alphabet S with $|S| \geq 2$, there exists an intrinsically universal captive color blind cellular automaton on $S^{\mathbb{Z}}$.*

Proof. We need to show that there exists an intrinsically universal captive color blind CA on $S^{\mathbb{Z}}$. For this, it is enough to show that any single CA can be simulated, as there exists an intrinsically universal CA and injective simulations are composable. Let thus $g : \{1, n-1\}^{\mathbb{Z}} \to \{1, n-1\}^{\mathbb{Z}}$ be any cellular automaton, choose distinct symbols $a, b \in S$ and for all $i \in \{1, n-1\}$, let $w_i = a^i b^{2n-i}$. Define the injection $h : \{1, n-1\}^{\mathbb{Z}} \to S^{\mathbb{Z}}$ by

$$h(x) = w_{x_{-2}} w_{x_{-1}} . w_{x_0} w_{x_1} \cdots .$$

Let $Z = h(\{1, n-1\}^{\mathbb{Z}})$ and $Y = \bigcup_{i=0}^{2n-1} \sigma^i(Z)$. It is easy to see that $Y \subset S^{\mathbb{Z}}$ is an SFT. Define $f : Z \to Z$ by $f \circ h = h \circ g$. Now, f has a unique shift-commuting extension to a function $\hat{f} : Y \to Y$, which is then a cellular automaton simulating g. We may assume $\hat{f}$ has neighborhood $[-r, r]$ for some $r > 2n$. Then $\hat{f}$ is trivially captive and commutes with all symbol permutations of Y, since both symbols are always visible in the neighborhood, and no nontrivial symbol permutation keeps any configuration of Y inside it. Thus, $\hat{f}$ has a color blind extension to $S^{\mathbb{Z}}$ by Proposition 1. ◻

In [1], the set $\mathrm{Aut}(X)$ of bijective cellular automata on a mixing SFT $X \subset S^{\mathbb{Z}}$ is considered (see the article for the precise definition). The *symmetry of* X is defined as the relative asymptotic density of $\mathrm{Aut}(X)$ in the set of all cellular automata on X: $s(X) = \limsup_{n \to \infty} \frac{1}{n} \log \log |\mathrm{Aut}(X)_n|$, where $\mathrm{Aut}(X)_n$ denotes the set of bijective cellular automata on X that can be defined on the neighborhood $[-\lfloor n/2 \rfloor, \lceil n/2 \rceil]$. Inspired by this, we define the following.

Definition 6. *Let $\mathcal{C}$ be a family of cellular automata on $S^{\mathbb{Z}}$. The* density of $\mathcal{C}$ *is defined as*

$$d(\mathcal{C}) = \limsup_{n \to \infty} \frac{1}{n} \log_{|S|} \log_{|S|} |\mathcal{C}_n|, \qquad (1)$$

where $\mathcal{C}_n$ denotes the set of cellular automata in $\mathcal{C}$ that can be defined on the neighborhood $[-\lfloor n/2 \rfloor, \lceil n/2 \rceil]$.

We now show that color blind cellular automata are abundant in the sense of the previous definition. Note that the set $\mathcal{CA}$ of all cellular automata on $S^{\mathbb{Z}}$ has density 1, as $|\mathcal{CA}_n| = |S|^{|S|^n}$ for all $n \in \mathbb{N}$.

Proposition 3. *Denote by $\mathcal{CB}$ the set of captive color blind cellular automata on $S^{\mathbb{Z}}$. Then $d(\mathcal{CB}) = 1$.*

Proof. Let $S = \{s_1, \ldots, s_{|S|}\}$, and let $n \in \mathbb{N}$ be arbitrary. We define an injective map $\phi : \mathcal{CA}_n \to \mathcal{CB}_{n+|S|}$, which shows that $|\mathcal{CA}_n| \leq |\mathcal{CB}_{n+|S|}|$. For that, let $f \in \mathcal{CA}_n$ have neighborhood size n. The local function $\phi(f)_{\mathrm{loc}} : S^{n+|S|} \to S$ works as follows on the inputs $a_1, \ldots, a_{n+|S|} \in S$. If the symbols $a_{n+1}, \ldots, a_{n+|S|}$ are pairwise distinct, we let $\pi : S \to S$ be the symbol permutation that maps each a_{n+i} to s_i. The local function then returns $\pi^{-1}(f_{\mathrm{loc}}(\pi(a_1), \ldots, \pi(a_n)))$. If the symbols $a_{n+1}, \ldots, a_{n+|S|}$ are not pairwise distinct, $\phi(f)_{\mathrm{loc}}$ returns a_1. Then $\phi(f)$ is captive and color blind, and ϕ is injective.

Now, we calculate

$$\begin{aligned}
&\frac{1}{n+|S|} \log_{|S|} \log_{|S|} |\mathcal{CB}_{n+|S|}| \geq \frac{1}{n+|S|} \log_{|S|} \log_{|S|} |\mathcal{CA}_n| \\
&= \frac{1}{n+|S|} \log_{|S|} \log_{|S|} |S|^{|S|^n} = \frac{n}{n+|S|} \overset{n\to\infty}{\longrightarrow} 1,
\end{aligned}$$

which proves the claim. ◻

5 Typhlotic Cellular Automata

We now turn our attention to typhlotic cellular automata, and start with the observation that they are not necessarily trivial. For example, the intrinsically universal CA given in Proposition 2 is in fact typhlotic in the case $|S| = 2$. Furthermore, every binary majority CA is typhlotic. These CA are already color blind, so we only need to check that they commute with the symbol maps that are not permutations, namely the constant maps $s \mapsto 0$ and $s \mapsto 1$. But this easily follows from the fact that both the intrinsically universal CA and majority CA are captive.

Somewhat curiously, if the alphabet S has more than two elements, the situation changes drastically. For example, as a corollary of Proposition 4, a ternary color blind majority CA can not be typhlotic unless it has a neighborhood of size 1. The proof of Proposition 4 follows from some rather general set theory. We start with two characterizations of ultrafilters. The first one is just the observation that the well-known partition property of ultrafilters characterizes them, as also the filter axioms follow from it. This result has already appeared in at least [5]. The second one is rather specific to typhloticity, and is in fact just the first part of Proposition 4 in thin disguise.

Lemma 4 (Corollary 1.6 of [5]). *Let X be a nonempty set, let $k \in \mathbb{N}$ with $k \geq 3$, and let $Q \subset 2^X$ have the property that for all partitions $(A_1, \ldots, A_k)$ of X, exactly one A_i is in Q. Then Q is an ultrafilter. Furthermore, every ultrafilter satisfies the property for every $k \geq 1$.*

Proof. First, from the partition $(X, \emptyset, \ldots, \emptyset)$ we deduce that $\emptyset \notin Q$ and $X \in Q$. Now, Q cannot contain two disjoint subsets $A, B \subset X$, as otherwise the partition $(A, B, X \setminus (A \cup B), \emptyset, \ldots, \emptyset)$ would contradict the assumptions.

Suppose then that $A \in Q$ and $A \subset B$. The partition $(X \setminus B, A, B \setminus A, \emptyset, \ldots, \emptyset)$ proves that $X \setminus B \notin Q$, so by the above $B \in Q$. Next, if $A, B \in Q$, then neither of $A \setminus B$ or $B \setminus A$ can be in Q, and then the partition $(A \setminus B, B \setminus A, A \cap B, \emptyset, \ldots, \emptyset)$ shows that $A \cap B \in Q$. Finally, if $A \subset X$, then exactly one of X and $X \setminus A$ is in Q, by the partition $(A, X \setminus A, \emptyset, \ldots, \emptyset)$.

The converse claim is a well known property of ultrafilters. □

For the next lemma, we need a more general definition of typhloticity.

Definition 7. *Let S and T be sets with S finite, and let $f : S^T \to S$ be a function. Then we say f is* typhlotic *if for every function $g : S \to S$, we have $f \circ g = g \circ f$, where g is applied coordinatewise on the left side of the equation.*

Lemma 5. *Let T be a set, and S a finite set with $|S| \geq 3$. Suppose further that $f : S^T \to S$ is typhlotic. Then, there exists an ultrafilter $Q \subset 2^T$ such that $\{i \in T \mid x_i = f(x)\} \in Q$ for all $x \in S^T$, and this ultrafilter is unique. Conversely, every ultrafilter arises this way.*

Proof. Without loss of generality, let $S = [1, k]$. For all $x \in S^T$ and $s \in S$ we define $x|_s = \{i \in T \mid x_i = s\}$.

First, we show that if $x|_s = y|_s$, then $f(x) = s$ if and only if $f(y) = s$. Assume the contrary, that $f(x) = s$ and $f(y) = r \neq s$. Consider the function $g : S \to S$ defined by $g(s) = s$, and $g(t) = r$ for all $t \neq s$. Then, $g(x) = g(y)$, so

$$s = g(f(x)) = f(g(x)) = f(g(y)) = g(f(y)) = r,$$

a contradiction.

For all $s \in S$, define $Q_s = \{x|_s \mid f(x) = s\}$. By definition, we have $x|_{f(x)} \in Q_{f(x)}$. Also, since f in particular commutes with permutations of S, we have $Q_s = Q_r$ for all $s, r \in S$, and we denote this set by Q. Now, we observe that $Q \subset 2^T$ has the property that if we partition T into k parts, exactly one of these sets is in Q: Let $(A_1, \ldots, A_k)$ be a partition of T, and define $x \in S^T$ by $x|_s = A_s$. By the previous paragraph, $A_{f(x)} \in Q$, but the other $k - 1$ sets are not in Q.

By Lemma 4, Q is an ultrafilter, and it is clear that it has the required property. The uniqueness of Q is clear, as

$$Q = \{\{i \in T \mid x_i = f(x)\} \mid x \in S^T\}.$$

Conversely, let Q be an ultrafilter on T. Define $f : S^T \to S$ by $f(x) = a$ iff $\{i \in T \mid x_i = a\} \in Q$. Again by Lemma 4 (the converse direction), f is then well-defined. Because ultrafilters are closed under supersets, f is easily seen to be typhlotic. As the ultrafilter corresponding to f is Q, this concludes the claim. □

The following is also a well known property of ultrafilters (for instance, it appears as Example 1.3 in [5]).

Lemma 6. *Let T be finite and let Q be an ultrafilter on T. Then Q is principal, that is, $Q = \{A \subset T \mid j \in A\}$ for some $j \in T$.*

Proposition 4. *If $|S| \geq 3$, the typhlotic CA $f : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ are exactly the shift maps. If $|S| = 2$, they are exactly the captive color blind CA.*

Proof. First, suppose $|S| \geq 3$, and let $N \subset \mathbb{Z}$ be the neighborhood of f. The local rule $f_{\mathrm{loc}} : S^N \to S$ is typhlotic since f is. Let Q be the ultrafilter on N that defines it, given by Lemma 5. Since N is finite, $Q = \{A \subset N \mid j \in A\}$ for some $j \in N$ by Lemma 6, which means

$$f(x)_0 = a \iff \{i \in N \mid x_i = a\} \in Q \iff x_j = a.$$

Thus f is a shift map.

In the case $|S| = 2$, a CA is captive if and only if it commutes with constant maps, and all symbol maps are either permutations or constant maps. This concludes the proof. □

6 Homomorphic Color Blind Automata

In Section 4, we saw that color blind cellular automata can do almost anything a general cellular automaton can do, with any alphabet size. On the other hand, typhlotic cellular automata turned out to be almost the same objects as color blind CA in the binary case, but shift maps for larger alphabets. In this section, we show that cellular automata that are color blind and *homomorphic* satisfy a similar property: if the group is very simple, the color blind homomorphic CA form a large subclass of all homomorphic CA, but when the group is larger, they are all shift maps.

Color blindness of homomorphic CA was also studied in [7], and there, the term k-rule was used for a sum of k distinct shifts. In the article, two particular cases of our main result Theorem 1 were proven. We prove Theorem 1 in a long series of simple lemmas, starting with the fact that every CA that is a group homomorphism is a sum of symbol endomorphisms.

Lemma 7. *Let G be a finite group and let $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be an homomorphic CA with neighborhood $N = [-r, r]$. For all $i \in N$, there exists a group endomorphism $f_i : G \to G$ such that*

- *$f_i(g)f_j(h) = f_j(h)f_i(g)$ whenever $h, g \in G$ and $i \neq j \in N$, and*
- *$f_{\mathrm{loc}}(g_1, \ldots, g_n) = f_1(g_1) \cdot f_2(g_2) \cdots f_n(g_n)$ for all $g_1, \ldots, g_n \in G$.*

Note that the order of multiplication in the above formula for f_{loc} is irrelevant by the first item.

Proof. For all $i \in N$, define the function $f_i : G \to G$ by

$$f_i(g) = f_{\mathrm{loc}}(\underbrace{1, \ldots, 1}_{i-1}, g, \underbrace{1, \ldots, 1}_{m-i}),$$

and note that this is an endomorphism of G. Let $i < j \in N$ and $g, h \in G$. Since f_{loc} is a homomorphism, we have

$$\begin{aligned} f_i(g)f_j(h) &= f_{\mathrm{loc}}(1, \dots, g, \dots, 1, \dots, 1)f_{\mathrm{loc}}(1, \dots, 1, \dots, h, \dots, 1) \\ &= f_{\mathrm{loc}}(1, \dots, g, \dots, h, \dots, 1) \\ &= f_{\mathrm{loc}}(1, \dots, 1, \dots, h, \dots, 1)f_{\mathrm{loc}}(1, \dots, g, \dots, 1, \dots, 1) \\ &= f_j(h)f_i(g), \end{aligned}$$

and for all $g_1, \dots, g_n \in G$,

$$\begin{aligned} f_{\mathrm{loc}}(g_1, \dots, g_n) &= \prod_{i=-r}^{r} f_{\mathrm{loc}}(\underbrace{1, \dots, 1}_{i-1}, g_i, \underbrace{1, \dots, 1}_{m-i}) \\ &= f_1(g_1) \cdot f_2(g_2) \cdots f_n(g_n). \end{aligned}$$

□

We call the endomorphisms f_i the *symbol endomorphisms of* f. If $n \geq 1$, all endomorphisms of $\mathbb{Z}_n$ are multiples of the identity map, so we have the following.

Lemma 8. *Let G be a finite abelian group with decomposition $G = \prod_{i=1}^{m} \mathbb{Z}_{p_i^{m_i}}$, where the p_i are prime numbers and $m_i \geq 1$. Then every homomorphic cellular automaton on $G^{\mathbb{Z}}$ is a sum of shifts if and only if the primes p_i are distinct.*

Thus, in general every group homomorphic CA is a *sum of shifted endomorphisms*, and for certain abelian groups the endomorphisms can be taken to be identity maps. Note that the fact that the images of distinct symbol endomorphisms commute means that the local rule of a homomorphic cellular automaton first projects its inputs to subgroups of G which commute with each other, and then multiplies them together. In particular, we have the following.

Lemma 9. *Let G be a group and let the CA $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be homomorphic. If at least two of the symbol endomorphisms of f are surjective, then G is abelian.*

We now see that in the case of color blind homomorphic CA, there is no loss of generality in retricting to the abelian case.

Lemma 10. *Let G be a finite group and let the CA $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be color blind and homomorphic with minimal neighborhood size at least 2. If $|G| \geq 4$, then f is a sum of distinct shifts, and in any case, G is abelian.*

Proof. The groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are abelian, so we may assume $|G| \geq 4$. Let $1 \neq g \in G$, and consider the configuration $z(g) = {}^{\infty}1g1^{\infty}$. Since the local rule sees at most 2 symbols in its neighborhood, the image $f(z(g))$ must also be a configuration over $\{1, g\}$ by Lemma 1. Since f commutes with the transposition $(g\ h)$, we have $I = \{i \in \mathbb{Z} \mid f(z(g)) = g\} = \{i \in \mathbb{Z} \mid f(z(h)) = h\}$ for all $1 \neq h \in G$. From this we deduce that the symbol endomorphisms of f are either trivial or identity maps, and since at least two of them must be nontrivial, G is abelian by Lemma 9. □

From now on, all alphabets will be abelian groups, so we switch to additive notation. Lemma 8 and Lemma 10 now give us the following.

Corollary 2. *Let G be a finite abelian group and $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ a color blind homomorphic CA. Then f is a sum of shifts, which are distinct if $|G| \geq 4$.*

The radius-1 CA f with local rule $(a, b, c) \mapsto a + 2b + c$ is an example of a color blind homomorphic CA on $\mathbb{Z}_3^{\mathbb{Z}}$ which is not a sum of distinct shifts.

Lemma 11. *Let G be a finite abelian group and $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ an homomorphic CA. The f commutes with the symbol permutation $\phi_g(h) = h + g$ if and only if $f({}^\infty g^\infty) = {}^\infty g^\infty$.*

Proof. Having $f({}^\infty g^\infty) = {}^\infty g^\infty$ is equivalent to $f(x) + {}^\infty g^\infty = f(x) + f({}^\infty g^\infty)$ for all $x \in G^{\mathbb{Z}}$, which is simply commutation with ϕ_g, since $f(x) + f({}^\infty g^\infty) = f(x + {}^\infty g^\infty)$. $\square$

We now proceed with a case analysis on the small groups $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_2^2$.

Lemma 12. *Let the CA $f : \mathbb{Z}_2^{\mathbb{Z}} \to \mathbb{Z}_2^{\mathbb{Z}}$ be homomorphic with minimal neighborhood size $m \in \mathbb{N}$. Then f is color blind if and only if f fixes ${}^\infty 1^\infty$, if and only if m is odd.*

Proof. The only nontrivial permutation of $\mathbb{Z}_2$ is ϕ_1, so it follows from Lemma 11 that f is color blind if and only if it fixes ${}^\infty 1^\infty$. Since f is a sum of shifts by Corollary 2, this is the case if and only if m is odd. $\square$

Lemma 13. *Denote $G = \mathbb{Z}_2^2$, and let the CA $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be homomorphic. Then f is color blind iff it is a sum of an odd number of distinct shifts.*

Proof. The proof relies on the facts that $2ng = 0$ for all $g \in G$ and $n \in \mathbb{N}$, and if $G = \{a, b, c, d\}$ then $a + b + c = d$.

Suppose first that f is color blind. Corollary 2 applies, so that f is an sum of m distinct shifts for some $m \in \mathbb{N}$. This means that $X = \{(0, 0), (0, 1)\}^{\mathbb{Z}} \cong \mathbb{Z}_2^{\mathbb{Z}}$ is closed under f, and the restriction of f to X is also a sum of shifts. If $f|_X$ were not color blind then f would not be either, so m must be odd by Lemma 12.

On the other hand, let f be a sum of m distinct shifts for odd m, and consider an arbitrary transposition $\phi = (g\ h)$. Denote $G = \{a, b, g, h\}$. Let $g_1, \ldots, g_m \in G$, and for $c \in G$, let n_c be the number of $i \in \{1, \ldots, m\}$ such that $g_i = c$.

If both n_g and n_h are even, then exactly one of n_a and n_b is odd, let us say n_a. Then $f_{\mathrm{loc}}(\phi(g_1), \ldots, \phi(g_m)) = a = \phi(f_{\mathrm{loc}}(g_1, \ldots, g_m))$. If both n_g and n_h are odd, we may again assume n_a is odd and n_b is even, so that $f_{\mathrm{loc}}(\phi(g_1), \ldots, \phi(g_m)) = a + g + h = \phi(f_{\mathrm{loc}}(g_1, \ldots, g_m))$, since $a + g + h = b \notin \{g, h\}$ is a fixed point of ϕ.

If $n_g + n_h$ is odd, we may assume n_g is odd and n_h is even. Then, $n_a + n_b$ is even, and the cases left to consider are that both n_a and n_b are odd or both are even. If n_a and n_b are both odd, then $f_{\mathrm{loc}}(g_1, \ldots, g_m) = a + b + g = h$, which implies $f_{\mathrm{loc}}(\phi(g_1), \ldots, \phi(g_m)) = a+b+h = g$ and $\phi(f_{\mathrm{loc}}(g_1, \ldots, g_m)) = \phi(h) = g$. If both are even, then $f_{\mathrm{loc}}(\phi(g_1), \ldots, \phi(g_m)) = h = \phi(g) = \phi(f_{\mathrm{loc}}(g_1, \ldots, g_m))$. This finishes the proof since transpositions generate the group of permutations. $\square$

Lemma 14. *Let the CA $f : \mathbb{Z}_3^{\mathbb{Z}} \to \mathbb{Z}_3^{\mathbb{Z}}$ be homomorphic. Then f is color blind if and only if it fixes ${}^{\infty}1^{\infty}$.*

Proof. By Lemma 11, f fixes ${}^{\infty}1^{\infty}$ if and only if it commutes with the symbol permutation ϕ_1. We prove that all such homomorphic CA are color blind, for which it is enough to show that they also commute with the transposition $(1\ 2)$. By Corollary 2, f is a sum of shifts $\sum_{i=1}^{m} \sigma^{k_i}$ for some $m \in \mathbb{N}$ and $k_i \in \mathbb{Z}$. For all $x \in \mathbb{Z}_3^{\mathbb{Z}}$, we then have

$$\begin{aligned}((1\ 2) \circ f)(x) = (1\ 2)\left(\sum_{i=1}^{m} \sigma^{k_i}(x)\right) &= \sum_{i=0}^{m} (1\ 2)(\sigma^{k_i}(x))) \\ &= \sum_{i=0}^{m} \sigma^{k_i}((1\ 2)(x)) = (f \circ (1\ 2))(x),\end{aligned}$$

where the second equality follows from the fact that $(1\ 2)$ is an automorphism of $\mathbb{Z}_3$ and the third one directly from the fact that $(1\ 2)$ is a cellular automaton. $\square$

Finally, we handle the remaining cases in a single lemma.

Lemma 15. *Let G be a finite abelian group such that $|G| > 3$ and $G \ncong \mathbb{Z}_2^2$, and let the CA $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be homomorphic. Then f is color blind if and only if it is a shift map.*

Proof. First, a shift map is trivially a color blind homomorphic CA for any group alphabet.

As for the nontrivial direction, Corollary 2 again applies, so that f_{loc} returns the sum of the values in the neighborhood N of f. If $|N| = 0$, then f does not commute with symbol permutations, as it sends everything to ${}^{\infty}0^{\infty}$. Assume then that $|N| \geq 2$.

We first suppose $|G| > 4$. In this case, we take $0 \neq g \in G$ and $h \in G$ such that $h \notin \{0, g, -g\}$. Now, $g + h \notin \{0, g, h\}$, so that $f_{\mathrm{loc}}(g, h, 0, \ldots, 0) = g + h \notin \{0, g, h\}$, which is a contradiction by Lemma 1. Now, let $|G| = 4$, so by the assumption that $G \ncong \mathbb{Z}_2^2$, we have that $G \cong \mathbb{Z}_4$. But now $f_{\mathrm{loc}}(1, 1, 0, \ldots, 0) = 2$, again contradicting Lemma 1.

Of course, in the remaining case that $|N| = 1$, f is a shift map.

We collect the results of Lemma 10, Lemma 12, Lemma 13, Lemma 14 and Lemma 15 into a single statement.

Theorem 1. *Let G be a finite group, and let $f : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be a homomorphic cellular automaton. Then, f is color blind iff one of the following (partially overlapping) conditions holds.*

- *$G = \mathbb{Z}_2$, $G = \mathbb{Z}_2^2$ or $G = \mathbb{Z}_3$, and f fixes unary points,*
- *$G = \mathbb{Z}_2$ or $G = \mathbb{Z}_2^2$, and f is a sum of an odd number of distinct shifts,*
- *$G = \mathbb{Z}_3$, and f is a sum of $3k + 1$ shifts for some k,*
- *$|G| > 4$ or $G = \mathbb{Z}_4$, and f is a shift map.*

This gives a complete characterization of homomorphic color blind cellular automata on full shifts whose alphabet is a finite group. We also note that in our arguments we mainly manipulated the local functions of cellular automata, so the result should hold as such for multidimensional automata with exactly the same proofs. Thus Theorem 1 is a generalization of the results of [7], which state that for all dimensions $d \geq 1$, any sum of 4 distinct shifts on $\mathbb{Z}_3^{\mathbb{Z}^d}$ is color blind, and no sum of m distinct shifts on $\mathbb{Z}_n^{\mathbb{Z}^d}$ is color blind if $n \geq m > 1$.

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