What can oracles teach us about the ultimate fate of life?

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Abstract

We settle two long-standing open problems about Conway's Life, a two-dimensional cellular automaton. We solve the Generalized grandfather problem: for all $n \geq 0$, there exists a configuration that has an nth predecessor but not an $(n+1)$st one. We also solve (one interpretation of) the Unique father problem: there exists a finite stable configuration that contains a finite subpattern that has no predecessor patterns except itself. In particular this gives the first example of an unsynthetizable still life. The new key concept is that of a spatiotemporally periodic configuration (agar) which has a unique chain of preimages; we show that this property is semidecidable, and find examples of such agars using a SAT solver.

Our results about the topological dynamics of Game of Life are as follows: it never reaches its limit set; its dynamics on its limit set is chain-wandering, in particular it is not topologically transitive and does not have dense periodic points; and the spatial dynamics of its limit set is non-sofic, and does not admit a sublinear gluing radius in the cardinal directions (in particular it is not block-gluing). Our computability results are that Game of Life's reachability problem, as well as the language of its limit set, are PSPACE-hard.

1 Introduction

Conway's Game of Life is a famous two-dimensional cellular automaton defined by John Conway in 1970 and popularized by Martin Gardner [7]. In this paper, we study Game of Life through its *agars*, which are the Game of Life community's term for spatiotemporally periodic points. More specifically, we observe that a simple algorithm (essentially Wang's partial algorithm from [20]),

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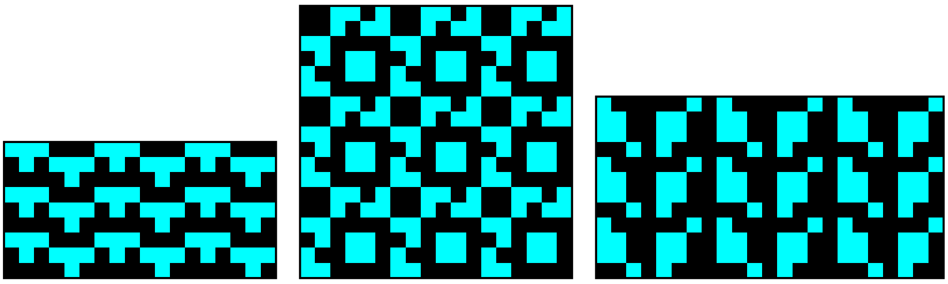
combined with a SAT solver, can be used to find all agars with small enough periodicity parameters which have no other preimage than themselves. We then study finite patches of these agars, and find some with interesting backwards forcing properties.

In Section 1.1, we show three agars that suffice to obtain all the results. The dynamical and Game of Life results obtained are listed in Section 1.2.

1.1 The protagonists



(a) Köynnös. (b) Kynnös. (c) Marching band.

Figure 1: Patches of the agars. A 3-by-3 grid of the repeating patterns is shown for each.

We begin with a brief discussion of the agars that we use to prove our results. These will be explained in more detail in separate sections.

Figure 1a shows the pattern we call köynnös[1]. The infinite agar obtained by repeating this pattern infinitely in each direction has no preimage other than itself; we say it is *self-enforcing*. Up to symmetries there are exactly 11 self-enforcing agars of size 6×3. Köynnös has the special property that it is impossible to stabilize a finite difference to this configuration, meaning if one modifies the agar in finitely many coordinates, the difference spreads at the speed of light. We say it *cannot be stabilized from the inside*.

Figure 1b shows the pattern we call kynnös[2]. The corresponding infinite agar is again self-enforcing. Up to symmetries there are at least 52 self-enforcing agars of size 6×6 (we were unable to finish the search, so it is possible that more exist). Kynnös has two special properties. First, there is a finite patch of this agar such that if a configuration has this patch in its image, then the configuration already had that patch in place, i.e. one cannot synthetize this patch from any other patch. Second, unlike köynnös, it can be stabilized from the inside.

Figure 1c shows the pattern we call marching band. This agar has temporal period two. The most important property of this pattern is that an infinite south half-plane of this pattern must shrink if there is a difference on its border, meaning that in the nondeterministic inverse dynamics of Game of Life, an infinite south half-plane of this pattern "marches" to the north.

[1] Finnish for vine.

[2] Finnish for that which is tilled.

To find the marching band, we searched through all $w \times h$-patterns such that the corresponding agar with periods $(w, 0)$ and $(0, h)$ is temporally (exactly) t-periodic, for the parameter range $2 \leq w \leq 9$, $2 \leq h \leq 5$, $2 \leq t \leq 3$. There were no self-enforcing agars with temporal period 3 in this range, and there were exactly 14 self-enforcing agars with period 2. The marching band is the only one that has the marching property in any direction.

1.2 Results

The following solves the Generalized grandfather problem: for all $n \geq 0$, there exists a configuration that has an nth predecessor but not an $(n+1)$st one.

Theorem 1. *For each $n \geq 0$, there exists $x \in \{0,1\}^{\mathbb{Z}^2}$ with $g^{-n}(x) \neq \emptyset$ and $g^{-(n+1)}(x) = \emptyset$.*

The case of $n = 0$ was resolved by R. Banks in 1971, only a year after the introduction of Game of Life. Conway stated the Grandfather problem, namely the case $n = 1$ of the above, in 1972, and promised \$50 in the *Lifeline* newsletter [19] for its solution. This stayed open until 2016, when the cases $n \in \{1, 2, 3\}$ by the user mtve of the ConwayLife forum.

Cellular automata satisfying the conclusion of the above theorem are sometimes called "unstable" [12], though we avoid this terminology here, as "stable" has another meaning in Game of Life jargon.

More specifically, we prove the following two results, which strengthen Theorem 1 in different directions. The first result is proved using köynnös and is based on the fact it cannot be stabilized from the inside.

Theorem 1.1. *There exists a polynomial time algorithm that, given $n \geq 0$ in unary, produces a finite pattern p which appears in the nth image of Game of Life, but not in the $(n+1)$th image.*

The algorithm is very simple: change the value of one cell in the agar, apply the Game of Life rule n times, and pick the central $[-30-6n, 30+6n] \times [-27-8n, 27+8n]$-patch of the resulting configuration as p. An example with $n = 28$ is shown in Figure 2 (with insufficient padding).

The second result is proved using kynnös, and is based on the facts that kynnös admits a self-enforcing patch and that it can be stabilized from the inside.

Theorem 2. *For any large enough n, there exists an $n \times n$ pattern which appears in the kth image of Game of Life, but does not appear in its $(k+1)$st image, where $48 \cdot 4^{(n-O(1))^2/736} \leq k \leq 2^{n^2}$.*

This corollary was already explained in [8]. This is (up to a suitable equivalence relation) the optimally slow growth rate for higher level orphans. The same idea can be used to obtain the following results about the limit set of Game of Life.

Theorem 3. *The limit set of Game of Life has PSPACE-hard language.*

The language might well be much harder. In fact, even for one-dimensional cellular automata it can be Π^0_1-complete [9]. We also obtain information about the symbolic dynamical nature of the limit set, namely that it is not covered by a tiling system.

Figure 2: A "level 29 orphan" obtained by perturbing köynnös: these angry deities could be found 28 seconds after the Big Bang, then went extinct.

Theorem 4. *The limit set of Game of Life is not sofic.*

Besides illuminating the iterated images of Game of Life and its limit set, the self-enforcing kynnös patch itself solves a second open problem, namely the Unique father problem stated by John H. Conway in [19, 3]. Thus this property is worth a theorem block.

Theorem 5. *There exists a finite still-life configuration x which contains a finite subpattern p such that every preimage of x also has subpattern p.*

One can also imagine a stronger variant of the Unique father problem, which requires that the still life enforces all its live cells backwards. This stays open. Theorem 5 also tells us something about the dynamics of Game of Life *restricted to* its limit set, i.e. its limit dynamics.

Theorem 6. *Game of Life is chain-wandering on its limit set.*

Much is known about the kinds of things that can happen in Game of Life orbits, in particular it is well-known that Game of Life is computationally universal and can simulate any cellular automaton. Nevertheless, to our knowledge the following was not known before our results.

Theorem 7. *The reachability problem of Game of Life is PSPACE-hard, i.e. given two cylinders $[p], [q]$ where p, q have polynomial extent, it is PSPACE-hard to tell whether $g^n([p]) \cap [q] \neq \emptyset$ for some n.*

Finally, the properties of the marching band's backwards dynamics imply the following result.

Theorem 8. *For all large enough n there exist patterns $p, q \subset \{0,1\}^{[0,n-1]^2}$ such that p and p appear in the limit set, but $p \sqcup \sigma_{(0,\lfloor \frac{1}{15} n \rfloor)}(q)$ does not.*

Corollary 1. *The limit set of Game of Life is not block-gluing (thus has none of the gluing properties listed in [2]).*

1.3 Programs

Some of the programs we used can be found on GitHub at [18]. We have included Python scripts enumerating self-enforcing agars, and scripts checking the claimed properties of our three agars. In particular one can find implementations of algorithms 1 and 2. The scripts use the PySAT [16] library to call the Minisat [14] SAT solver.

2 Definitions

Our intervals are discrete. We use several notations for areas of the discrete plane $\mathbb{Z}^2$: $B_{m,n} = [0, m-1] \times [0, n-1]$ is a rectangle rooted on the positive quadrant, $\begin{bmatrix} a & b & c & d \end{bmatrix} = [-a, b] \times [-c, d]$ is a general rectangle (note the signs!), and $\left[\begin{smallmatrix} a & b & c & d \\ e & f & g & h \end{smallmatrix}\right] = \begin{bmatrix} a & b & c & d \end{bmatrix} \setminus \begin{bmatrix} e & f & g & h \end{bmatrix}$, denotes a (possibly thick) annulus when the second rectangle fits full inside the first.

We assume some familiarity with topological and symbolic dynamics and give only brief definitions, see e.g. [11] for a basic reference. We denote by S a finite *alphabet*. A *configuration* or *point* is an element of $S^{\mathbb{Z}^d}$. More generally, a *pattern* (or sometimes *patch* in more informal contexts) is a function $p : \mathrm{dom}(p) \to S$, where $\mathrm{dom}(p) \subset \mathbb{Z}^d$ is the *domain* of p. If $S \subset \mathbb{N}$, then by $\sum p$ we denote the sum $\sum_{\vec{v} \in \mathrm{dom}(p)} p(\vec{v})$. For $\vec{v} \in \mathbb{Z}^d$, a pattern p and $D \subset \mathrm{dom}(p)$, we write $q = p|D$ for the restriction $\mathrm{dom}(q) = D, q(\vec{v}) = p(\vec{v})$. A pattern is *finite* if its domain is, and a configuration is *finite* if its sum as a pattern is finite. If p, q are patterns with disjoint domains, define $p \sqcup q = r$ by $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q), r|\mathrm{dom}(p) = p, r|\mathrm{dom}(q) = q$. The *extent* of a pattern is the minimal hypercube containin the origin and its domain. For two patterns, write $\mathrm{eq}(q, q')$ for the set of vectors $\vec{v} \in \mathrm{dom}(q) \cap \mathrm{dom}(q')$ such that $q(\vec{v}) = q'(\vec{v})$, and $\mathrm{diff}(q, q')$ for those that satisfy $q(\vec{v}) \neq q'(\vec{v})$. For computer science purposes, we note that patterns with polynomial extent have an efficient encoding.

The *full shift* is the set of all configurations $S^{\mathbb{Z}^d}$ with the product topology, under the action of $\mathbb{Z}^d$ by homeomorphisms $\sigma_{\vec{v}}(x)_{\vec{u}} = x_{\vec{v}+\vec{u}}$ called *shifts*. We use the same formula to define $\sigma_{\vec{v}}(p)$ for patterns p (of course shifting the domain correspondingly). A pattern p defines a *cylinder* $[p] = \{x \in S^{\mathbb{Z}^d} \mid x|\mathrm{dom}(p) = p\}$. Cylinders defined by finite patterns are a base of the topology, and their finite unions are exactly the clopen sets. The *symbol partition* is the clopen partition $\{[s] \mid s \in S\}$ where s is identified with the pattern $p : \{\vec{0}\} \to S$ with $p(\vec{0}) = s$. The space $S^{\mathbb{Z}^d}$ is homeomorphic to Cantor space, and is metrizable. We pick some metric $\mathrm{dist} : (S^{\mathbb{Z}^2})^2 \to \mathbb{R}$ for it.

A *cellular automaton* (or *CA*) is a continuous self-map $f : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ that commutes with the shifts. The *minimal neighborhood* is the minimal $N \subset \mathbb{Z}^d$ such that $f(x)_{\vec{0}}$ is determined by $x|N$. A *subshift* is a closed subset X of $S^{\mathbb{Z}^d}$ invariant under shifts. Its *language* is the set of finite patterns p such that $[p] \cap X \neq \emptyset$, and we say these patterns *appear* or *occur* in the subshift. Patterns that do not appear in $f(S^{\mathbb{Z}^d})$ are usually called *orphans*. We say p is a *level n orphan* if it appears in $f^{n-1}(S^{\mathbb{Z}^d})$ but not in $f^n(S^{\mathbb{Z}^d})$ (so the usual orphans are of level 1). The *limit set* of a cellular automaton f is $\Omega(f) = \bigcap_n f^n(S^{\mathbb{Z}^d})$. It is a subshift invariant under f. An *SFT* is a subshift of the form $\bigcap \sigma_{\vec{v}}(C)$ where C is clopen. A *sofic shift* is a subshift which is the image of an SFT under a shift-commuting continuous function.

We are mainly interested in $d = 2$, $S = \{0, 1\}$, and the *Game of Life* cellular automaton $g : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$ defined by

$$g(x)_{\vec{v}} = 1 \iff (x_{\vec{v}} = 0 \wedge \sum (x|\vec{v} + [-1,1]^2) = 3) \\ \vee (x_{\vec{v}} = 1 \wedge \sum (x|\vec{v} + [-1,1]^2) \in \{3,4\}).$$

A *fixed point* (of a CA f) is $x \in S^{\mathbb{Z}^d}$ such that $f(x) = x$. In the context of Game of Life these are also called *stable configurations* or *still lives*. *Spatial* and *temporal* generally refer respectively to the $\mathbb{Z}^d$-action of shifts and the action of a CA. In particular a *spatially periodic point* is a configuration $x \in S^{\mathbb{Z}^d}$ which has a finite orbit under the shift dynamics, and *temporal periodicity* means $f^n(x) = x$ for some $n \geq 1$. Spatiotemporal periodicity means that both hold; in the Game of Life context spatiotemporal points are also called *agars*.

If $f : X \to X$ is a continuous function, an *ϵ-chain from x to y* is $x = x_0, x_1, \ldots, x_k = y$ with $k \geq 1$ such that $\mathrm{dist}(f(x_i)), x_{i+1} < \epsilon$ for $0 \leq i < k$. We say f is *chain-nonwandering* if for all $\epsilon > 0$ and $x \in X$ there is an ϵ-chain from x to itself; otherwise f is *chain-wandering*. We say f is *topologically transitive* if for all nonempty open sets U, V we have $f^n(U) \cap V \neq \emptyset$ for some n. It is *sensitive* (to initial conditions) if there exists $\epsilon > 0$ such that for all $x \in X$ and $\delta > 0$ there exists $y \in X$ with $\mathrm{dist}(x, y) < \delta$ and $n \in \mathbb{N}$ such that $\mathrm{dist}(f^n(x), f^n(y)) \geq \epsilon$. We say f has *dense periodic points* if its set of temporally periodic points is dense.

3 Proofs

The general topological idea is the following: if we have a zero-dimensional space X and a family of closed sets $\mathcal{I}$ which is closed under arbitrary intersections and contains the empty set, then to any continuous $f : X \to X$ we can associate a map from $\mathcal{I}$ to $\mathcal{I}$ by $\hat{f}(A) = \bigcap \{B \in \mathcal{I} \mid f^{-1}(A) \subset B\}$.

In our situation, $f : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is our cellular automaton. Let $\mathcal{F}$ be all cylinders defined by finite patterns, $\mathcal{I}$ their arbitrary intersections, and $\mathcal{C}$ the (full) configurations. Observe that $\mathcal{F} \cap \mathcal{C} = \emptyset$ and $\mathcal{F}, \mathcal{C} \subset \mathcal{I}$. We denote $\emptyset \in \mathcal{I}$ as $\top$. Then elements of $\mathcal{I}$ other than $\top$ can be identified with patterns, and from now on we think of $\mathcal{I}$ as simply the set of patterns plus $\top$. Through this identification, $\mathcal{F}$ is naturally stratified into finite subsets $\mathcal{F}_M = S^M$ where $M \subset \mathbb{Z}^d$ ranges over finite sets. The image $\hat{f}(p)$ consists of those cells whose values are the same in all f-preimages of p, or $\top$ if p is an orphan.

Say $p \leq q$ for two elements $p, q \in \mathcal{I}$ if neither is equal to $\top$ and $\mathrm{dom}(p) \subset \mathrm{dom}(q) \wedge q|\mathrm{dom}(p) = p$, or $q = \top$. Note that the empty pattern defines the full cylinder $S^{\mathbb{Z}^d}$ since any configuration agrees with it, and we denote it as $\perp$. One may feel that the opposite convention for $\top$ and $\perp$ is more appropriate, but our ordering on patterns is that $p \leq q$ if q specifies the values of more cells than p (which we find more intuitive when actually working with patterns – patterns growing in this order are literally growing). The empty set in a sense specifies the maximal amount of information: it specifies a contradiction.

We give $\mathcal{I}$ the topology with basis $\{q \in \mathcal{I} \mid p \leq q\}$ for $p \in \mathcal{F}$, and $\top$ is isolated. Note that this space is not Hausdorff (T_2) but is Fréchet (T_1). The induced topology on $\mathcal{C}$ is the standard compact Cantor topology, and $\mathcal{F}$ is a dense subset of $\mathcal{I}\backslash\{\top\}$. Every nonempty open set contains $\top$: "the contradiction is dense".

Lemma 1 (*semicontinuity*)**.** *The dual map $\hat{f} : \mathcal{I} \to \mathcal{I}$ is continuous.*

Note that because of our choice of topology, this should be seen as a kind of semicontinuity rather than continuity. We are simply saying that if a (possibly infinite) pattern forces some particular value in some cell in the preimage, then actually some finite patch already forces it. Because $\top$ (the empty set) is isolated, we are also saying that if a pattern in $\mathcal{I}$ has no preimage, then a finite subpattern already does not (which of course is a well-known fact).

We list some other easy properties of $\hat{f}$. Write $p \parallel q$ for $p \vee q < \top$, where $\vee$ denotes the least common upper bound; in other words p and q are compatible. The empty set $\top$ is fixed under translations. For a pattern p, write $f(p)$ for the pattern obtained by applying the local rule of f (with the minimal neighborhood) in every position.[3]

Lemma 2.

- *$\mathcal{F} \cup \{\top\}$ is preserved under $\hat{f}$. Indeed, we have $\hat{f}(\mathcal{F}_M) \subset \mathcal{F}_{M+N} \cup \{\top\}$ where N is the minimal neighborhood of f.*
- *$\hat{f}$ is monotone, i.e. $a \leq b$ implies $\hat{f}(a) \leq \hat{f}(b)$.*
- *$\hat{f} \circ \hat{g} \leq \widehat{f \circ g}$ pointwise.*
- *for all $p \in \mathcal{I}$, either $\hat{f}(p) = \top$ or $f(\hat{f}(p)) \leq p$; in particular $f(\hat{f}(p)) \parallel p$ in the latter case.*
- *for all $p \in \mathcal{I}$, we have $\hat{f}(\sigma_{\vec{v}}(p)) = \sigma_{\vec{v}}(\hat{f}(p))$.*

Images of $\hat{f}$ are computable:

Lemma 3. *Given $p \in \mathcal{F}$, the images of p under $\hat{f}$ can be computed in* $\mathrm{FP}^{\mathrm{NP}}$.

Proof. Since $\hat{f}(\mathcal{F}_M) \subset \mathcal{F}_{M+N}$, we only need to figure out which of the coordinates $M+N$ are forced in preimages, and this obviously requires at most $|M+N|$ calls to the oracle (simply ask the oracle for a pair of preimages which differ at a particular position). $\square$

The proof above is the easiest way to get the theoretical result, but for practical purposes we give another algorithm that tends to find the $\hat{f}$-image much quicker (and is just as quick to implement). The following algorithm is written for an "incremental oracle", meaning we can only *add* constraints to it when we make a new query. This is because modern SAT solvers tend to support such incremental access – of course, on the side of theory it is easy to see that the class $\mathrm{FP}^{\mathrm{NP}}$ is the same whether or not queries are restricted to be incremental.

Lemma 4. *The set of all self-enforcing agars is recursively enumerable.*

[3]One could also define $f(p)$ to be the maximal pattern that is forced by f, analogously to how $\hat{f}$ is defined, but we prefer to use the standard definition here; this is only used in two formulas.

Algorithm 1 Finding the $\hat{f}$-image of a finite pattern $p \in S^M$.

function $\textsc{HatCA}(f, p)$
  Describe to the oracle the constraints for being an f-preimage of p.
  if there is a preimage **then**
    Receive $q \in S^{M+N}$ from the oracle.
  else
    return $\top$
  Let $D \leftarrow M + N$.
  Add constraint: the preimage should differ from q somewhere in D.
  loop
    if there is a preimage **then**
      Receive $q' \in S^{M+N}$ of p from the oracle.
    else
      return $q|_D$
    Let $D \leftarrow \mathrm{eq}(q, q')$.
    Add constraint: the preimage should differ from q somewhere in D.

Proof. It suffices to fix a temporal period t, consider an arbitrary self-enforcing agar $x \in S^{\mathbb{Z}^d}$, and show that there exists a finite algorithmically checkable certificate that it is self-enforcing. Our algorithm is also uniform in the CA, so by considering the CA f^t, we may assume $t = 1$, and now we simply need to find a certificate for $\hat{f}(x) = x$, where x is some $\langle (m, 0), (0, n) \rangle$-periodic configuration.

For this, observe that by semicontinuity there is a finite pattern $p < x$ such that $p \leq y$ implies $\hat{f}(y) \geq x|B_{m,n}$ for any configuration y. In particular $\hat{f}(p) \geq x|B_{m,n}$. By Lemma 3, checking $\hat{f}(p) \geq x|B_{m,n}$ is in $\mathrm{FP}^{\mathrm{NP}}$.

We claim that p is a certificate that x is a self-enforcing agar. This is a direct calculation: for any $i, j \in \mathbb{Z}$ we have

$$\begin{aligned}\hat{f}(x)|((im, jn) + B_{m,n}) &= \sigma_{(im,jn)}(\hat{f}(x))|B_{m,n} = \hat{f}(\sigma_{(im,jn)}(x))|B_{m,n} \\ &= \hat{f}(x)|B_{m,n} \geq \hat{f}(p)|B_{m,n} \geq x|B_{m,n} = x|((im, jn) + B_{m,n}),\end{aligned}$$

so x is a fixed point of $\hat{f}$, as desired. $\square$

Remark 1. *This algorithm is not very practical: given an agar, we have no information about how large the certificate could be, so for each agar we either need to simply guess some certificate size, or we have to keep trying arbitrary sizes. Thus, in parallel we run a search for other periodic preimages for the agar – if such a preimage exists, then clearly the agar does not enforce itself, and we can stop looking for a certificate. Most agars in the range we searched were either self-enforcing or had another periodic preimage. We omit the pseudocode.*

Say a pattern $p \subset S^M$ is *locally fixed* for the CA f if there exists a pattern $q \in S^{M+N}$ (where N is the minimal neighborhood of f) such that $q|M = p$ and $f(q) = p$.

Lemma 5. *For a CA f and locally fixed pattern $p \in S^M$, there exists a unique maximal subpattern q with $\hat{f}(q) \geq q$. For every fixed CA g, given p and integers $a, b \in \mathbb{Z}$ and $n \geq 1$ in unary, it is in* $\mathrm{FP}^{\mathrm{NP}}$ *to compute q for the CA $f = \sigma_{(a,b)} \circ g^n$.*

Proof. We first show existence. If $D, D' \subset M$ satisfy $\hat{f}(p|_D) \geq p|_D$ and $\hat{f}(p|_{D'}) \geq p|_{D'}$, then $\hat{f}(p|_{D \cup D'}) \geq p|_{D \cup D'}$ by monotonicity of $\hat{f}$. Thus $q = p|_E$ for $E = \bigcup \{D \subset M \mid \hat{f}(p|_D) \geq p|_D\}$ is the desired subpattern.

Then fix g, and let a, b, p be given. We apply Algorithm 2 to the CA $f = \sigma_{(a,b)} \circ g^n$. On each iteration of the loop, the algorithm replaces p with the maximal subpattern forced by p (here we use the fact that p is locally fixed). Since q is a subpattern forced by itself, by monotonicity it is also forced by each of these subpatterns, and thus remains a subpattern on each iteration. Since q is maximal, the algorithm eventually converges on q.

Finally, Algorithm 2 is in $\mathrm{FP}^{\mathrm{NP}}$, since the number of iterations of the loop is at most $|M|$, and $\textsc{HatCA}(f, p)$ is in $\mathrm{FP}^{\mathrm{NP}}$ with respect to these parameters. $\square$

Algorithm 2 Finding the maximal subpattern q satisfying $\hat{f}(q) \geq q$ of a locally fixed pattern $p \in S^M$.

```
function SelfEnforcingSubpattern(p)
    loop
        Let q ← HatCA(f, p)|_M.
        if q = p then
            return q
        else
            Let p ← q.
```

3.1 Köynnös

We begin by studying köynnös, which we recall is obtained from the 6×3 pattern

$$P = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}$$

by repeating P horizontally and vertically to define an infinite 6×3-periodic configuration $x^P \in \{0,1\}^{\mathbb{Z}^2}$. We have $g(x^P) = x^P$, so that x^P is indeed an agar. Spatial periodicity is true by definition, for temporal periodicity observe that every 0 in x^P is surrounded by exactly four 1s, and every 1 by exactly three 1s.

Moreover, x^P has no other predecessors than itself: $g^{-1}(x^P) = \{x^P\}$. This follows from Lemma 4, and the following lemma.

Lemma 6. *Let x be in the spatial orbit of köynnös. Then $\hat{g}(x|([-12,17] \times [-12,14])) \geq x|[-8,13] \times [-9,10]$.*

Proof. Applying Algorithm 1 to $\sigma_{\vec{v}}(x)|([-11,17] \times [-8,11])$ for all $\vec{v} \in B_{6,3}$ gives the result. The intersection of the domains of the patterns $\hat{g}(x|([-12,17] \times [-12,14]))$ is shown in Figure 3, and clearly contains the rectangle $[-8,13] \times [-9,10]$. $\square$

Put concretely, the lemma states that if R is a periodic continuation of P of size 30×27 and Q is its predecessor, then P must occur at the center of Q (and indeed many more cells are forced, even beyond what we state in the

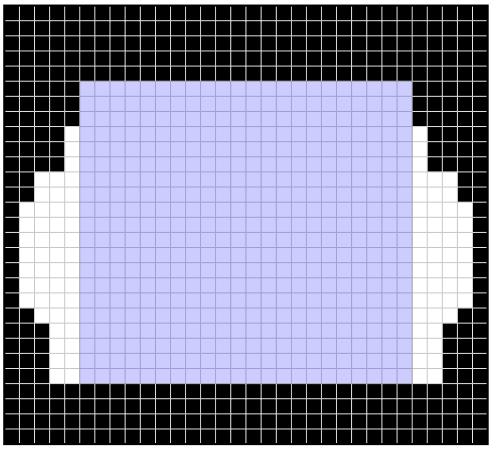


Figure 3: The intersection of the domains of $\hat{g}(x|([-12,17]\times[-12,14]))$ for x in the spatial orbit of köynnös, drawn in white inside $[-13,18]\times[-13,15]$. The area $[-8,13]\times[-9,10]$ is highlighted in blue.

lemma). Note that this is indeed a certificate for being a self-enforcing agar because $B_{6,3}\subset[-8,13]\times[-9,10]$.

As a corollary of Lemma 6, finite perturbations of x^P can never be erased by g. We prove a stronger claim: all finite perturbations spread to the left and right at a speed of one column per time step. In particular, köynnös cannot be stabilized from the inside.

Lemma 7. *Let* $R=\begin{bmatrix} n_W & n_E & n_S & n_N\end{bmatrix}$ *be a rectangle and* $A=\left[\begin{smallmatrix} n_W+1 & n_E+1 & n_S+1 & n_N+1 \\ n_W & n_E & n_S & n_N\end{smallmatrix}\right]$ *the surrounding annulus of thickness 1. Let* p *be a pattern such that the domain of* $g(p)$ *contains* $A\cup R$*, and suppose* $p|A=g(p)|A=x^P|A$*. If* $\mathrm{diff}(x^P,g(p))\cap R\subset[a,b]\times\mathbb{Z}$*, then* $\mathrm{diff}(x^P,p)\cap R\subset[a+1,b-1]\times\mathbb{Z}$*.*

Note that we may have $b-a\le 1$, in which case the conclusion becomes $\mathrm{diff}(x^P,p)\cap R=\emptyset$, or equivalently, $x^P|R=p|R$.

Proof. Since köynnös and g are left-right symmetric, it is enough to prove that $\mathrm{diff}(x^P,p)\cap R\subset(-\infty,b-1]$. We prove the contrapositive: suppose there exists $(i,j)\in\mathrm{diff}(x^P,p)\cap R$ for some $i\ge b$, and let i be maximal. We split into cases based on the congruence class of i modulo 6, that is, the column of P that i lies in. Note that the bottom left cell of P is at the origin in x^P, and the domain of P is the rectangle $[0,5]\times[0,2]$. If $i\in\{0,2,3\}+6\mathbb{Z}$, we choose j as maximal, and otherwise we choose it as minimal.

We handle the case $i\in 2+6\mathbb{Z}$, the others being similar or easier. If $j\in 3\mathbb{Z}$, then $p_{(i+1,j+1)}=x^P_{(i+1,j+1)}=1$ has four other 1s in its neighborhood, and becomes 0 in $g(p)$. If $j\in 1+3\mathbb{Z}$, then $p_{(i+1,j)}=x^P_{(i+1,j+1)}=1$ has four or five other 1s in its neighborhood, and becomes 0 in $g(p)$. If $j\in 2+3\mathbb{Z}$, then $p_{(i+1,j+1)}=x^P_{(i+1,j+1)}=0$ has three 1s in its neighborhood, and becomes 1 in $g(p)$. In each case $\mathrm{diff}(x^P,g(p))$ intersects $\{i+1\}\times\mathbb{Z}$. $\square$

For any $D\subset\mathbb{Z}^2$, the subpattern of köynnös of shape $D+B_{30,27}$ forces the subpattern of shape $D+B_{22,20}+(4,3)$ to occur in its g-preimage, by Lemma 6.

By Lemma 7, we force more: a non-köynnös area inside a hollow patch of köynnös expands horizontally under g, so under $\hat{g}$ the horizontal extent of the hole must shrink. We do not give a precise statement for this general fact, and only apply the lemma in the case of annuli.

Lemma 8. *Let x be in the orbit of köynnös. Suppose the following inequalities hold:*

$$m_W - n_W \geq 30, m_E - n_E \geq 30, m_S - n_S \geq 27, m_N - n_N \geq 27.$$

Denote $Q = x| \left[\begin{smallmatrix} m_W & m_E & m_S & m_N \\ n_W & n_E & n_S & n_N \end{smallmatrix}\right]$. If $n_E + n_W \geq 2$, then

$$\hat{g}(Q) \geq x| \begin{bmatrix} m_W - 4 & m_E - 4 & m_S - 3 & m_N - 4 \\ n_W - 1 & n_E - 1 & n_S + 4 & n_N + 3 \end{bmatrix}$$

while if $n_E + n_W \in \{0, 1\}$ we have

$$\hat{g}(Q) \geq x| \begin{bmatrix} m_W - 4 & m_E - 4 & m_S - 3 & m_N - 4 \end{bmatrix}$$

One may consider the latter case a special case of the former: there too, the hole shrinks horizontally by two steps, and since its width is at most two it disappears.

Proof. The inequalities simply state that the annulus Q is thick enough that each of its cells is part of a 30×27 rectangle contained in Q. From Lemma 6, we get $\hat{g}(Q) \geq x|A$, where $A = \left[\begin{smallmatrix} m_W-4 & m_E-4 & m_S-3 & m_N-4 \\ n_W+4 & n_E+4 & n_S+4 & n_N+3 \end{smallmatrix}\right]$ is a slightly thinner annulus. If there is no g-preimage for Q, then $\hat{g}(Q) = \top$ and we are done. Suppose then that it has a preimage R. Since $R \geq \hat{g}(Q) \geq x|A$, both Q and R agree with x on the thickness-1 annulus $\left[\begin{smallmatrix} n_W+5 & n_E+5 & n_S+5 & n_N+4 \\ n_W+4 & n_E+4 & n_S+4 & n_N+3 \end{smallmatrix}\right] \subset A$. Lemma 7 implies that R agrees with x on $\left[\begin{smallmatrix} m_W-4 & m_E-4 & m_S-3 & m_N-4 \\ n_W-1 & n_E-1 & n_S+4 & n_N+3 \end{smallmatrix}\right]$, as claimed. $\square$

We now prove theorem 1.1, and thus give the first proof of Theorem 1. In fact, we give a simple formula that produces configurations that have an nth preimage, but no $(n+1)$st one.

Theorem 9. *Let x be in the orbit of köynnös, and suppose $\emptyset \neq \mathrm{diff}(y, x) \subset B = [0, a] \times [0, n]$ where $a \in \{0, 1\}$. Then*

$$p = g^k(y)|[-30 - 6k, 30 + a + 6k] \times [-27 - 8k, 27 + n + 8k]$$

appears in the kth image of g, but not in the $(k+1)$st.

Proof. By definition, p in the kth image of g. It suffices to show its $\hat{g}^{k+1}$-image is $\top$. Namely, we then have $\widehat{g^{k+1}}(p) \geq \hat{g}^{k+1}(p) = \top$ by Lemma 2, which means precisely that p has no g^{k+1}-preimage.

Let q be the restriction of p to

$$\begin{bmatrix} 30 + 6k & 30 + a + 6k & 27 + 8k & 27 + n + 8k \\ k & a + k & k & n + k \end{bmatrix}.$$

Observe that q agrees with x because g has radius 1, so by Lemma 8 and induction, we can deduce that

$$\hat{g}^j(q) \geq x| \begin{bmatrix} 30 + 6k - 4j & 30 + a + 6k - 4j & 27 + 8k - 3j & 27 + n + 8k - 4j \\ k - j & a + k - j & k + 4j & n + k + 3j \end{bmatrix}$$

for all $j \leq k$. This is because for $j \leq k-1$ we have

$$30 + 6k - 4j - (k-j) \geq 30, \qquad 30 + a + 6k - 4j - (a + k - j) \geq 30,$$
$$27 + 8k - 3j - (k + 4j) \geq 27, \qquad 27 + n + 8k - 4j - (n + k + 3j) \geq 27,$$

and thus we can inductively apply the lemma. But in

$$\hat{g}^k(q) \geq x| \begin{bmatrix} 30+2k & 30+a+2k & 27+5k & 27+n+4k \\ 0 & a & 5k & n+4k \end{bmatrix}$$

the annulus still has sufficient thickness (i.e. the inequalities still hold for $j = k$), so we can apply the second case of the lemma to get

$$\hat{g}^{k+1}(q) \geq x| \begin{bmatrix} 26+2k & 26+2k & 24+5k & 23+n+4k \end{bmatrix} = r.$$

By Lemma 2 we have $\hat{g}^{k+1}(p) \geq \hat{g}^{k+1}(q) \geq r$, and by the same lemma we either have $\hat{g}^{k+1}(p) = \top$ (as desired), or

$$g^{k+1}(\hat{g}^{k+1}(p)) \leq g^{k+1}(\widehat{g^{k+1}}(p)) \parallel p.$$

But since the speed of light is 1 and x is a fixed point, we have

$$g^{k+1}(r) \geq x| \begin{bmatrix} 25+k & 25+k & 23+4k & 22+n+3k \end{bmatrix} \geq x|[-k, a+k] \times [-k, n+k]$$

and $x|[-k, a+k] \times [-k, n+k] \parallel p$. But Lemma 7 applied k times to y implies that $\mathrm{diff}(x, g^k(y))$, and thus $\mathrm{diff}(x, p)$, intersects $[-k, a+k] \times [-k, n+k]$, a contradiction. Thus we indeed must have $\hat{g}^{k+1}(p) = \top$. $\square$

3.2 Kynnös

Denote by

$$Q = \begin{matrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{matrix}$$

the fundamental domain of kynnös, and by $x^Q \in \{0,1\}^{\mathbb{Z}^2}$ the associated 6×6-periodic configuration with $g(x^Q) = x^Q$. The following lemma states that it contains a self-enforcing patch (it is essentially a more precise stement of Theorem 5).

Lemma 9. *There is a finite set $D \subset \mathbb{Z}^2$ such that $p = x^Q|_D$ satisfies $\hat{g}(p) = p$. Furthermore, there is a finite-support configuration $x \in [p]$ with $g(x) = x$.*

The patch p is shaped like a 22×28 rectangle with 8 cells missing from each corner. It is depicted in Figure 4, together with the still life x containing it. The patch was found by simply applying the function of Algorithm 2 to the 70×70-patches of the agars we found during our searches. Kynnös was the first configuration that yielded a nonempty self-enforcing patch, which we then optimized to its current size. This lemma directly implies Theorem 5, and almost directly Theorem 6.

Proof of Theorem 6. Let y be the finite-support configuration obtained by taking x from the previous lemma and adding a glider that is just about to hit the kynnös patch. It can be checked by simulation that the patch can be annihilated this way. Observe that y is in the limit set $\Omega(g)$: simply shoot the glider from infinity. If $\epsilon > 0$ is very small, in any ϵ-chain starting from y we see the patch destroyed. It is impossible to reinstate it, as the existence of a first step in the chain where it appears again contradicts Lemma 9. □

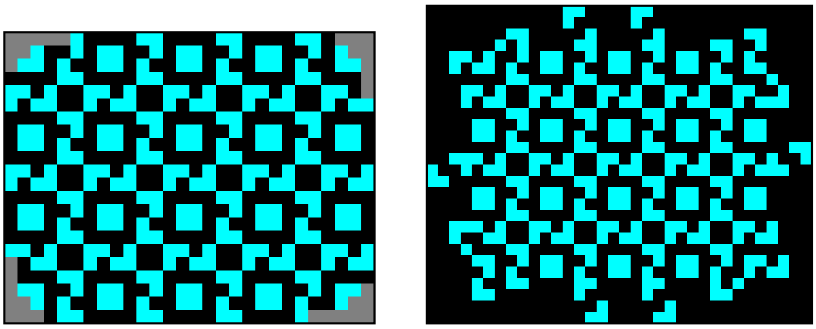


Figure 4: A self-enforcing patch of kynnös and a still life containing it. The still life has the minimal number of live cells, 306, of any still life containing the patch. [4]

As stated, kynnös can be stabilized from both inside and outside. Figure 5 shows a still life configuration containing a "ring" of kynnös with a hole of 0-cells inside it. From the figure it is easy to deduce the existence of such rings of arbitrary size and thickness.

Note that if the ring is at least 22 cells thick, then its interior is completely surrounded by a ring-shaped self-enforcing pattern consisting of translated, rotated and partially overlapping copies of the 28×22 self-enforcing patch, which no information can pass without destroying it forever. If we then replace the empty cells inside the ring with an arbitrary pattern, the resulting finite pattern P occurs in the limit set $\Omega(g)$ if and only if the interior pattern evolves periodically under g. Namely, if the pattern occurs in $\Omega(g)$, then it has an infinite sequence of preimages, each of which must contain the self-forcing kynnös ring. The interior of the ring has a finite number of possible contents (2^m for an interior of m cells), so it must evolve into a periodic cycle, of which P is part. From this idea, and some engineering with gadgets found by other researchers and Life enthusiasts, we will obtain Theorems 2, 3 and 4. The first one was essentially proved by Adam Goucher [8].

Proof of Theorem 2. Given integers $k, m \geq 1$ with k odd, we construct a configuration $x \in \{0,1\}^{\mathbb{Z}^2}$ such that the support of $g^n(x)$ is contained in $B_{32k+74,46m+1}$ for all $n \geq 0$, and $g^{48 \cdot 4^{(2k+1)m}}(x)$ is not g-periodic. When the support of $g^{48 \cdot 4^{(2k+1)m}}(x)$ is surrounded by a kynnös ring of width 22, the resulting pattern has a $48 \cdot 4^{(2k+1)m}$th preimage, but not arbitrarily distant preimages. If we choose $k = 23s$ and $m = 16s$ for some $s \geq 0$, the resulting pattern has size $(736s + O(1)) \times (736s + O(1))$, and the chain of preimages has length

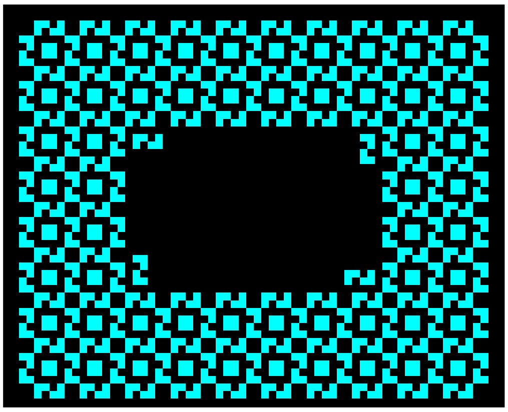


Figure 5: A stable ring of kynnös.

$48 \cdot 4^{736s^2+16s}$. Choosing $s = n/736 - O(1)$ yields lower bound, and the upper bound is the trivial one (even ignoring the fact we do not modify the boundaries).

The main components of x are the *period-48 glider gun* [15], which produces one glider every 48 time steps, and the *quadri-snark* [17], which emits one glider at a 90 degree angle for every 4 gliders it receives. The support of x consists of a single period-48 gun aimed at a sequence of $(2k+1)m$ quadri-snarks, each of which receives the gliders the previous one emits. They effectively implement a quaternary counter with values in $[0, 4^{(2k+1)m} - 1]$. The glider emitted by the final quasi-snark will collide with the kynnös ring, ensuring that the pattern right before the impact does not occur in the limit set $\Omega(g)$, but has a chain of preimages of length at least $48 \cdot 4^{(2k+1)m}$. An example pattern and a schematic for $m = n = 2$ are given in Figure 6. It is easy to extrapolate to arbitrary $m, n \geq 1$ from the figure. □

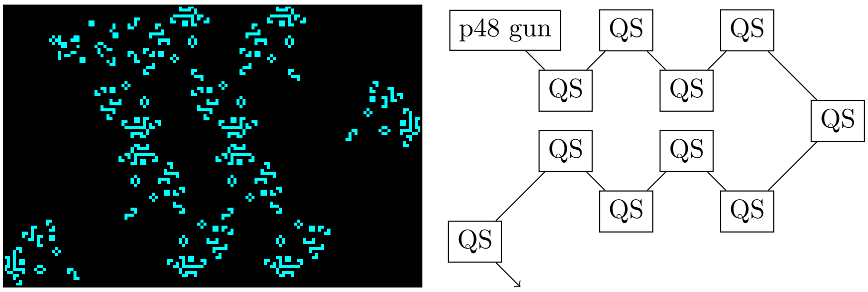


Figure 6: A configuration corresponding to $k = m = 2$ in the proof of Theorem 2.

To implement more complex Life patterns with desired properties, we use the fact that Life is *intrinsically universal*, that is, capable of simulating all $\mathbb{Z}^2$ cellular automata. Formally, for any other CA $f : \Sigma^{\mathbb{Z}^2} \to \Sigma^{\mathbb{Z}^2}$, there are numbers $K, T \geq 1$ and an injective function $\tau : \Sigma \to \{0,1\}^{K \times K}$ such that for all configurations $x \in \Sigma^{\mathbb{Z}^2}$ we have $g^T(\tau(x)) = \tau(f(x))$, where τ is applied cellwise in the natural way. We use the simulation technique of [6], which allow us to easily simulate patterns with *fixed boundary conditions*. This means that any rectangular pattern $R \in \Sigma^{a \times b}$ can be simulated by a finite-support configuration of g in such a way that simulated cells whose f-neighborhood is not completely contained in the rectangle $B_{a,b}$ are forced to retain their value.

Proof of Theorem 3. Let M be a Turing machine that has a PSPACE-hard language and is limited to the tape cells that contain its input string, for example one that decides TQBF in place. We simulate M by a cellular automaton f in a standard way: each cell is either empty, or contains a tape symbol and possibly the state of the computation head. We additionally require that if M accepts, it will write the input back onto the tape and reset its computation. If M rejects, it will stay in a rejecting state forever.

Next, we simulate the CA f by g. Given a word $w \in \{0,1\}^*$, let $P(w)$ be the pattern corresponding to a simulated initial configuration of M on input w with fixed boundary conditions, surrounded by a kynnös ring. If M accepts w, then $P(w)$ occurs in the limit set $\Omega(g)$, since it can be completed into a g-periodic configuration in which the simulated M repeatedly accepts w. If M rejects w, then $P(w)$ does not occur in $\Omega(g)$, since the interior of the ring eventually evolves into a simulated configuration with M in a rejecting state, never returning to $P(w)$. $\square$

Proof of Theorem 7. We use the same construction as in the previous proof, but this time use any Turing machine with a PSPACE-hard reachability problem on bounded-length tapes, for example decide TQBF and write down a special tape content if the instance is positive. As long as the ring is included in both p and q, the reachability problem of g exactly mimics that of the Turing machine. $\square$

Proof of Theorem 4. Let M be a two-dimensional Turing machine whose tape alphabet has two distinguished values, denoted a and b. When M is initialized on a rectangular tape containing only as and bs, it repeatedly checks whether its left and right halves are equal, destroying the tape is they are not. We again simulate M by a CA f, and then f by g. Then a simulated rectangular tape with the head of M in its initial state, surrounded by a kynnös ring, is in $\Omega(g)$ if and only if the two halves of the tape are equal.

It was proved in [10] that for all sofic shifts $X \subset \Sigma^{\mathbb{Z}^2}$ there exists an integer $C > 1$ with the following property. For all $n \geq 1$ and configurations $x^1, \ldots, x^{C^n} \in X$, there exist $i \neq j$ such that the configuration $y = x^i|_{B_{n,n}} \sqcup x^j|_{\mathbb{Z}^2 \setminus B_{n,n}}$ is in X. Assuming for a contradiction that $\Omega(g)$ is sofic, consider the configurations $x(P) \in \Omega(g)$ for $P \in \{a,b\}^{n \times n}$ that contain a kynnös ring and a simulated tape of M with two identical P-halves. Based on the above, when n is large enough that $2^{n^2} > C^{Kn}$, we can swap the right half of one $x(P)$ with that of another to obtain a configuration $y \in \Omega(g)$ containing a simulated tape of M with unequal halves inside a kynnös ring, a contradiction. $\square$

3.3 The marching band

Let $h = g^2$. Denote by

$$R = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{matrix}$$

the fundamental domain of the marching band, and by $x^R \in \{0,1\}^{\mathbb{Z}^2}$ the associated 8×4-periodic configuration with $h(x^R) = x^R$. The following is proved just like Lemma 6. Note that the forced region extends outside the original pattern.

Lemma 10. *Let x be in the spatial orbit of the marching band. Then $\hat{h}(x|[0,47] \times [0,43]) \geq x|[10,29] \times [-1,48]$.*

Proof of Theorem 8. Let x be in the orbit of x^R, and let $p = x| \begin{bmatrix} a & b & c & d \end{bmatrix}$. By the previous lemma, as long as $a + b \geq 48$ and $c + d \geq 44$ we have $\hat{h}(p) = x| \begin{bmatrix} a-10 & b-18 & c+1 & d+1 \end{bmatrix}$. Iterating this we get

$$\hat{h}^n(x| \begin{bmatrix} 10n & 18n+47 & 0 & 43 \end{bmatrix}) \geq x| \begin{bmatrix} 0 & 47 & n & n+43 \end{bmatrix}.$$

Denote $S = \begin{bmatrix} 10n & 18n+47 & 0 & 43 \end{bmatrix}$. Let $P = x|S$ and $Q = \sigma_{\vec{u}}(x)|\vec{v} + S$ for some $\vec{u} \in \mathbb{Z}^2$ and $\vec{v} = (0, 2n)$. Both patterns appear in the limit set of g, since they are extracted from a fixed point of $h = g^2$.

Observe that since the domains of $\hat{h}^n(P)$ and $\hat{h}^n(Q)$ intersect, we can pick the shift $\vec{u}$ so that one of the forced bits is different in some position in $\hat{h}^n(P)$ and $\hat{h}^n(Q)$, which clearly means $\hat{h}^n(P \sqcup Q) = \top$.

Now, P and Q each fit inside a $29n \times 29n$ rectangle (if $n \geq 47$), and the patterns cannot be glued in the limit set with gluing distance at most $2n$, since the glued pattern should have an nth h-preimage. This gives the statement. □

4 Chaotic conclusions

There are several definitions of topological chaos. We refer the reader to [1] for a survey. Briefly, a system is called Auslander-Yorke chaotic if it is topologically transitive and is sensitive to initial conditions, and Devaney chaotic if it is Auslander-Yorke chaotic and additionally has dense periodic points. As far as we know, before our results it was open whether Game of Life exhibits these types of chaos on its limit set; the following corollary shows that it does not.

Theorem 10. *The Game of Life restricted to its limit set is not topologically transitive, and does not have dense periodic points.*

Proof. Either of these properties clearly implies chain-nonwanderingness, contradicting Theorem 6. □

Two other standard notions of chaos are Li-Yorke chaos and positive entropy (we omit the definitions). Game of Life exhibits these trivially, since it admits a glider. More generally, intrinsical universality implies that it exhibits any property of spatiotemporal dynamics of cellular automata that is inherited from subsystems of finite-index subactions of the spacetime subshift. Sensitivity in itself is also sometimes considered a notion of chaos. This remains wide open.

Question 1. *Is Game of Life sensitive to initial conditions?*

One can also ask about chaos on "typical configurations". For example, take the uniform Bernoulli measure (or some other distribution) as the starting point, and consider the trajectories of random configurations. We can say essentially nothing about this setting.

In our topological dynamical context, a natural way to formalize this problem is through the *generic limit set* as defined in [13]. It is a subset of the phase space of a dynamical system that captures the asymptotic behavior of topologically large subsets of the space. We omit the exact definition, but for a cellular automaton f, this is a subshift invariant under f [5]. It follows that the generic limit set is contained in the limit set, and that the language of the generic limit set of g contains the letter 0 (because the singleton subshift $\{1^{\mathbb{Z}^2}\}$ is not g-invariant).

We say a cellular automaton is *generically nilpotent* if its generic limit set contains only one point, called the *zero point*. By the previous observation, if Game of Life were generically nilpotent, its zero point would necessarily be $0^{\mathbb{Z}^2}$. We strongly suspect that it is not generically nilpotent, i.e. the symbol 1 occurs in the generic limit set. However, we have been unable to show this.

Question 2. *Is Game of Life generically nilpotent?*

Chaos is usually discussed for one-dimensional dynamical system, but we find its standard ingredients, such as topological transitivity and periodic points, quite interesting. We have been unable to resolve most of these.

Question 3. *Is the limit set of Game of Life topologically transitive as a subshift?*

Question 4. *Does the limit set of Game of Life have dense totally periodic points as a subshift?*

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