

Distortion element in the automorphism group of a full shift

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Abstract

We show that there is a distortion element in a finitely-generated subgroup G of the automorphism group of the full shift, namely an element of infinite order whose word norm grows polylogarithmically. As a corollary, we obtain a lower bound on the entropy dimension of any subshift containing a copy of G , and that a sofic shift's automorphism group contains a distortion element if and only if the sofic shift is uncountable. We obtain also that groups of Turing machines and the higher-dimensional Brin-Thompson groups mV admit distortion elements. The distortion element is essentially the SMART machine.

1 Introduction

We begin with an introduction to automorphism groups and the topic of distortion in Section 1.1, as this is the motivation and context for our main results listed in Section 1.2.

The proofs of the main results are based on rather different ideas, namely conveyor belts, ducking, dynamics of Turing machines, permutation groups, and also some ideas from computer science, namely reversible computation and logical gates. Some background for these ideas is given in Section 1.3, and Section 1.4 gives a high-level sketch of the proof.

1.1 Automorphism groups and distortion

A recent trend in symbolic dynamics is the study of automorphism groups of subshifts. Typical activities include the study of restrictions that dynamical properties of the subshift put on these groups, and in turn constructing complicated automorphism groups or subgroups thereof. The former activity has

been most successful in the low-complexity setting, see [39] for a recent account of the state of the art. The latter activity has been most successful on sofic shifts, in particular a lot is known about the finitely-generated subgroups of automorphism groups of full shifts, see [45] for a listing.

In this paper, we study the group-theoretic notion of distortion, introduced by Gromov [22], in the context of automorphism groups of subshifts. If G is a finitely-generated group, we say $g \in G$ is a *distortion element*, or *distorted*, if the word norm $|g^n|$ grows sublinearly (with respect to some, or equivalently any, finite generating set). For groups that are not finitely-generated, we say that an element is distorted if it is distorted in some finitely-generated subgroup.

Two basic examples of groups with distortion elements are the Heisenberg group with presentation $\langle a, b \mid [[a, b], a], [[a, b], b] \rangle$, where the element $[a, b]$ has quadratic distortion, meaning we can represent an element of the form $[a, b]^{\Omega(n^2)}$ by composing n generators; and the Baumslag-Solitar group $\text{BS}(1, 2)$ with presentation $\langle a, b \mid a^b = a^2 \rangle$ where a is easily seen to be exponentially distorted, meaning the word norm of a^n grows logarithmically.

The previous examples show that distortion elements can appear in nilpotent and metabelian linear groups. It is known that they cannot appear in biautomatic groups, mapping class groups, and the outer automorphism group of the free group [11]. See [50, 32, 11, 23, 12, 21, 40, 36] for other distortion-related works.

Getting back to automorphism groups, it is an open problem whether the automorphism group of any subshift can contain a distortion element [16]. It is not known whether the Heisenberg group [29] or the Baumslag-Solitar group $\text{BS}(1, 2)$ embed in $\text{Aut}(A^\mathbb{Z})$, or indeed in the automorphism group of any subshift. (It is also open whether the additive group of dyadic rationals $\mathbb{Z}[\frac{1}{2}] \leq \text{BS}(1, 2)$ embeds in $\text{Aut}(A^\mathbb{Z})$ [9].)

Besides being an interesting group-theoretic notion, the quest for distortion elements in automorphism groups of subshifts is motivated by several purely symbolic dynamical considerations. First, [17, Theorem 1.2] shows that finitely-generated torsion-free subgroups of the automorphism group of a subshift of polynomial complexity are virtually nilpotent. See [19, Theorem 5.5] for a similar conclusion for inverse limits of bounded step nilsystems. If we could rule out distortion in such examples, we could conclude virtual abelianness.

Second, it is known that the Baumslag-Solitar group, more generally any group with an exponentially distorted element, does not embed in the automorphism group of a zero-entropy subshift [16]. More precisely, it was observed there that the Morse-Hedlund theorem allows one to translate a distortion element into a lower bound on the complexity of a subshift. This is notable, as this is the only known restriction for automorphism groups of general zero-entropy subshifts. Thus, distortion looks like a natural candidate for restrictions on automorphism groups of general subshifts (as far as the authors know, no restrictions are known on countable subgroups).

Third, distortion is tied to an intrinsic notion in automorphism group theory, namely the growth of the *radius* (a.k.a. range) of the automorphism, when seen as a cellular automaton. Namely, distortion in the group sense implies sublinear growth of the radius [15]. It is not immediately obvious that even sublinear radius growth is possible (indeed this was left open in [15]), but several examples of sublinear radius growth have been constructed. The most relevant for us is

the observation from [24] that one can even obtain sublinear radius growth in the automorphism group of a full shift: the so-called *SMART machine*, when simulated by an automorphism, gives rise to such growth.

While distortion elements have not previously been exhibited in automorphism groups of subshifts, some facts are known about their dynamics (mostly related the notion of radius). Links to expansive directions and Lyapunov exponents are shown in [15]. A related result is shown in [5], namely distortion elements of automorphism groups of general expansive systems can not themselves be expansive. Links to the dimension group action and inertness are discussed in [15, 48].

1.2 Results

The main result of the present paper is that the automorphism group of some full shift (thus any full shift by standard embedding theorems [29]), contains a distortion element with “quasi-exponential” distortion, in the sense that the distortion function grows like $\exp(\sqrt[4]{\Omega(n)})$. It is more convenient to work directly with word norms than with the distortion function, so we take this approach in the paper. Note that for well-behaved functions, the word norm growth is just the inverse of the distortion function.

Theorem A. *For any non-trivial alphabet A , the group $\text{Aut}(A^{\mathbb{Z}})$ has an element g of infinite order such that $|g^n|_F = O(\log^4 n)$ for some finite set F .*

(We use the standard shorthand $\log^4 n = (\log n)^4$.)

A simple counting argument that the word norms of n th powers of a group element cannot be $o(\log n)$ with respect to a fixed finite generating set. For our specific automorphism, one can strengthen this: the radius of g^n as a cellular automaton is $\Theta(\log n)$, so the true growth of word norms of powers of our automorphism is between $\Omega(\log n)$ and $O(\log^4 n)$.

Most of the present paper deals with the proof of this theorem. This solves the second subquestion of [16, Question 5.1] in the affirmative. The element g in this theorem is essentially the SMART machine [13], so morally this also confirms a conjecture of [24], although the embedding we use is slightly more involved than the specific one considered in [24]. The group we use in the proof is given in Lemma 5.8.

The generators of our group are relatively simple, but we have little idea what kind of group they generate. The finitely-generated group $\langle F \rangle$ can of course be taken to be larger (the distortion function can only become faster-growing this way), so one can take a more canonical choice, say, all reversible cellular automata with biradius 1 (on the huge alphabet we use).

One can perform some further massage to get a simpler-sounding example: it is known that the automorphism group of a full shift contains so-called finitely-generated f.g.-universal subgroups, namely ones containing copies of all f.g. groups of reversible cellular automata [42]. Any such group can be used in the result (although the element g will be more complicated). In particular, one can pick as F the symbol permutations and the partial shift $\sigma \times \text{id}$ on the product full shift $\{0, 1\}^{\mathbb{Z}} \times \{0, 1, 2\}^{\mathbb{Z}}$.

From the main theorem, we obtain several corollaries of interest, which are proved in Section 6. First, we obtain the characterization of the class of sofic shifts whose automorphism groups have distortion elements.

Theorem B. *Let X be a sofic shift. Then $\text{Aut}(X)$ contains a distortion element if and only if X is uncountable.*

It is well-known that for sofic shifts, uncountability is equivalent to having positive entropy.

As another immediate consequence, using the argument of [16] we obtain that the automorphism group of a full shift cannot be embedded in the automorphism group of a low-complexity subshift. Recall that the *lower entropy dimension* [33] of a subshift is defined by the formula

$$\underline{D}(X) = \liminf_{k \rightarrow \infty} \frac{\log(\log N_k(X))}{\log k},$$

where $N_k(X)$ is the number of words of length k that appear in X . The entropy dimension of a (one-dimensional) subshift with positive entropy is of course 1.

Lemma 1.1. *Let X be a subshift with lower entropy dimension less than $1/d$. If $f \in \text{Aut}(X)$ satisfies $|f^n| = O(\log^d n)$, then f is periodic.*

Theorem C. *The group $\text{Aut}(A^\mathbb{Z})$ has a finitely-generated subgroup G such that every subshift X with $G \leq \text{Aut}(X)$ has lower entropy dimension at least $1/4$.*

Theorem C is of course an immediate corollary of Lemma 1.1. It states a “low-complexity restriction” on the automorphism group, i.e. it states that automorphism groups of subshifts with low enough complexity (growth of the number of admissible words) cannot have some property. The above theorem seems to be the first “low-complexity restriction” on automorphism groups where

1. the complexity bound is superpolynomial,
2. there are no additional dynamical restrictions, and
3. the prevented behavior can be exhibited in the automorphism group of another subshift.

There are previously known restrictions satisfying any two of these items. For 1.&2., zero entropy prevents exponential distortion [16]; for 1.&3., [17] shows that if X is minimal and has upper entropy dimension less than $1/2$, then it is amenable (while $\text{Aut}(A^\mathbb{Z})$ is not); for 2.&3. (very low complexity restrictions) there are many results, see [39].

The subgroup where our distortion element lies can itself be seen as a group of Turing machines, indeed restricting its action to a certain sofic subshift directly gives rise to a subgroup of the group $\text{RTM}(n, k)$ studied in [2], leading to the following theorem.

Theorem D. *Let $n \geq 2, k \geq 1$. Then the group of Turing machines $\text{RTM}(n, k)$ contains a distortion element; indeed there is a finitely-generated subgroup $G = \langle F \rangle$ and an element f such that $|f^n|_F = O(\log^4 n)$.*

All groups of Turing machines in turn embed in the higher-dimensional Brin-Thompson mV for $m \geq 2$ introduced by Brin [10], and we obtain the following.

Theorem E. *The Brin-Thompson group mV contains a distortion element; indeed there is an element f such that $|f^n| = O(\log^4 n)$.*

This theorem provides a new restriction for geometries of $2V$. Namely, it is known that Thompson’s group V admits a proper action by isometries on a CAT(0) cube complex [20]. By [25, Theorem 1.5], a group with distortion elements does not admit such an action, thus:

Corollary 1.2. *The Brin-Thompson group mV does not act properly on a CAT(0) cube complex for $m \geq 2$.*

Of course, a similar fact is true for the other groups where we exhibit distortion elements.

We conclude with previously known (but possibly not well-known) related distortion facts that are easy to prove. First, the automorphism group of a full shift does contain finitely-generated subgroups that are distorted, namely $F_2 \times F_2$ embeds in $\text{Aut}(A^{\mathbb{Z}})$ [29] and has subgroups with arbitrarily bad (recursive) distortion essentially by [34]. To give a more down-to-earth example, $\mathbb{Z}_2 \wr \mathbb{Z}^2$, which embeds in $\text{Aut}(A^{\mathbb{Z}})$ ([45]), contains a polynomially distorted copy of itself, by a nice geometric argument [18]. One can also construct distorted subgroups directly by more intrinsic automorphism group techniques.

Second, in the setting of general expansive homeomorphisms, finding distortion elements is very easy. Namely, if \mathbb{S} is the invertible natural extension of the $\times 2$ -map on the circle, \mathbb{S}^n contains a natural copy of $\text{GL}(n, \mathbb{Z})$ by simply summing tracks to each other [30]. For $n = 3$, the group $\text{GL}(n, \mathbb{Z})$ contains the Heisenberg group, thus has distortion elements.

1.3 Turing machines and gates

While our results in the previous section are stated fully in terms of homeomorphism groups, our *proof methods* rather belong to the theories of dynamical Turing machines and of reversible gates. In this section, we outline some history of these ideas.

1.3.1 Turing machines

As mentioned in Section 1.1, our automorphism group element simulates a “Turing machine”, i.e. a dynamical system where a single head moves over an infinite tape of arbitrary data (over a fixed finite alphabet), and all the action happens near the head (which may move around the tape, such movement depending on the content of the tape; or modify said content). The dynamics of Turing machines, also known as one-head machines, is an important branch of symbolic dynamics. This can be seen as initiated in the 1997 paper of Kůrka [31], which explicitly defined the moving-head and moving-tape dynamics of Turing machines (although many relevant dynamical ideas appeared in the literature before this [26, 41, 35]).

One of the most-studied behaviors of Turing machines is aperiodicity, meaning that the action of the Turing machine has no periodic points. This property is particularly interesting in the moving tape model, where the head is seen as fixed and only the tape moves. Kůrka originally conjectured that Turing machines cannot be aperiodic, but an explicit aperiodic Turing machine was exhibited in 2002 by Blondel, Cassaigne and Nichitiu [6] (inspired by a technique of Hooper from 1966 [26]). Later, reversible aperiodic Turing machines

(ones whose action is a homeomorphism) were found, the first by Kari and Ollinger [28]. This culminated in the discovery of the SMART machine \mathcal{S} [13], a machine with only four states and three tape-letters, which is reversible and aperiodic, and whose moving-tape dynamics is a minimal homeomorphism on the Cantor space, see also [37].

Turing machines, in the moving-head dynamics where the tape is not shifted and the head moves over it, can be directly seen as automorphisms of a sofic shift [2]. In fact, it is well-known that Turing machines can be “embedded” into automorphism groups of full shifts $\text{Aut}(A^{\mathbb{Z}})$. There are multiple ways of doing so; in this paper, we use the conveyor belt technique similar to the one used in [24].

For the purpose of establishing distortion, the first important consideration, already discussed in Section 1.1, is the “speed” of a Turing machine: a Turing machine with positive speed, meaning the existence of tape contents such that the head moves to infinity at a positive rate, could not possibly give rise to a distortion element. This is because a linear movement of the head means that the radius of the corresponding automorphism must grow at a linear rate as well, which prevents distortion.

It was shown in [27] that all aperiodic Turing machines have zero average speed, and in [24] this was strengthened by proving that the maximal offset by which such a machine can move in t time steps is $O(t/\log t)$. For the SMART machine \mathcal{S} , more is known: in t steps, it can only move by an offset of at most $O(\log t)$. This makes \mathcal{S} a perfect candidate for a distortion element of a subshift automorphism group, and indeed it was conjectured in [24] that it is one.

1.3.2 Gates

The next ideas come from the study of reversible gates. By this, we refer to the study of permutation groups acting on (a sublanguage of) A^n , where A is a finite alphabet, that are generated by “reversible gates”: i.e. permutations that only touch one subset of coordinates at a time. More precisely, if $k \leq n$ and $\pi \in \text{Sym}(A^k)$, then we can apply π to the subword starting at i by the formula $\hat{\pi}(u \cdot v \cdot w) = u \cdot \pi(v) \cdot w$ where $u \in A^i, v \in A^k, w \in A^{n-k-i}$. From now on, we use the term “gate” for reversible gates, and “classical gate” to refer to the usual not necessarily reversible gates (in the few places where they are needed).

A fundamental lemma in this topic is that $\text{Alt}(A^n)$ admits a generating set with bounded k , namely it is generated by the even permutations of A^2 if the cardinality $\#A$ is at least 3 (when we consider them as gates, and allow their applications at any position $i = 0, \dots, n-2$), and is generated by $\text{Alt}(A^3)$ if $|A| = 2$. For odd alphabets, the same is true for symmetric groups, i.e. $\text{Sym}(A^n)$ is generated by $\text{Sym}(A^2)$ -gates. This statement appears in [46], while earlier proofs are given in [49, 8, 7].

The connection between gates and Turing machines is as follows. Let us consider generalized Turing machines in the sense of [2], meaning the machine can look at and modify multiple cells at once, although only at a bounded distance from the head. Now, walking on a cyclic tape containing an element of A^n , we can apply permutations of A^k at different relative positions i : simply move by i steps, apply the permutation locally, and then move back by $-i$ steps. The above paragraph translates to the fact that there is a finite set of

generalized Turing machines that can perform any even permutation of the tape content (relative to the head position). Actually, it turns out that since Turing machines carry a state, $k = 1$ suffices, i.e. the generating Turing machines need not be of a generalized type.

1.4 The ideas and steps of the proof

In this section, we describe the new ideas in our proof, and give a high-level outline of it. The reader interested in precise statements may simply skip ahead to Section 2 and Section 3, which do not refer to this discussion.

At a very high level (high enough to be almost nonsensical), the idea of the proof is to show that the SMART machine \mathcal{S} is distorted by the following sequence of steps:

1. embed \mathcal{S} in $\text{Aut}(A^{\mathbb{Z}})$ in a suitable way,
2. take a high power of it,
3. “view” the resulting map as a permutation of a suitable set A^n ,
4. construct the resulting permutation from permutations of A^2 using the fundamental lemma.

An analogous sequence of steps was used in [44], namely it was shown that such steps can be used to construct any (suitably encoded) periodic Turing machine from scratch, without having “direct access” to its local rule.

Unfortunately, for the SMART machine \mathcal{S} , the movement rate $O(\log t)$ is too high to already prove that such implementation of (the encoding of) \mathcal{S}^t is sublinear. Indeed, in t steps of computations, the head moves over $O(\log t)$ cells, so we are dealing with the permutation group of words of length $O(\log t)$. There are $\exp(O(\log t)) = t^{O(1)}$ such words, so this permutation group acts on a set of polynomial size in t . As the diameter of the full symmetric group is of course superlinear in n over any fixed-cardinality generating set, we cannot conclude a small word norm of powers of \mathcal{S}^t on purely abstract grounds.

Worse yet, the fact that the action of \mathcal{S} is aperiodic in a sense translates to the fact that the number of distinct words we see when iterating it always grows at a rate linear in t . Actually, since the action of \mathcal{S} is minimal, we always eventually see *all* possible words near the head, and as we will show, it is also very “minimal” when seen as a permutation on finite configurations (having only four distinct orbits). Long story short, as proofs of the fundamental lemma of gates are based on constructing small cycles or transpositions of a bounded number of words from the generators, they can not be directly used for efficiently implementing permutations with such massive cycles.

1.4.1 Programming periodic SMART efficiently

The gist of our paper is find a way to program SMART efficiently, using the fact that we have a precise description of its action [13]. (In the body of this paper, we slightly modify the SMART machine as to make some proofs easier, but in this section we work with the original.)

We first study the SMART machine on periodic tapes, seeing it as a permutation of the Cartesian product $\{0, 1, 2\}^n \times [\![n]\!] \times Q$. Here, $\{0, 1, 2\}^n$ is the tape

content, $C = \llbracket n \rrbracket = \{0, 1, \dots, n-1\}$ is the position of the head, and $|Q| = 4$ gives the current state. When studying the t -th power of SMART, an appropriate choice of n is $n = \Theta(\log t)$: indeed, a much larger choice will not lead to a small implementation (i.e. a small word norm), while we cannot learn anything about the action of SMART on infinite tapes with a much smaller choice.

We show that the permutation \mathcal{S} gives on C is very simple. We have $\#C = 4 \cdot 3^n \cdot n$, and under the action of \mathcal{S} , C splits into four cycles of length exactly $3^n \cdot n$. Furthermore, there are very simple representatives for these cycles, namely the tuples $(0^n, 0, q)$ are in distinct \mathcal{S} -cycles for distinct $q \in Q$.

Our first task is to show that we can jump immediately to the t -th power of this periodic SMART action, without computing the intermediate steps. To do this, we conjugate configurations in C to their position in their respective cycles. More precisely, if a configuration (u, j, q') is equal to $\mathcal{S}^t(0^n, 0, q)$, then we construct a permutation that rewrites (u, j, q') into (w, h, q) where (h, w) is the natural presentation of the number t in “base $n \times 3^n$ ”. This permutation should be generated by polynomially many gates in n (so that it is polylogarithmic in t).

Then the application of \mathcal{S}^t is conjugated into simply adding the number t to a counter, and we are left with performing such an addition in polynomially many steps in n . This latter task is much easier, so we now focus our explanations on the conjugation.

The proof that we can indeed conjugate configurations to their position in their cycle is quite technical, although the ideas follow naturally from the description of the machine’s behavior in [13]. More precisely, in [13], it is shown that the SMART machine is “typically” performing “sweeps” of four types, which represent the head moving over a sequence of zeroes. Sweeps split recursively into subsweeps (separated by individual moves), whose lengths are deterministic and known. Furthermore, if we know the type of “sweep of level k ” that is currently performed, and the current temporal position inside this sweep, then we can check a few local symbols outside the swept area to deduce the same information about the level- $(k+1)$ sweep.

Based on these ideas, it is not difficult to find a polynomial time classical algorithm that computes the action of the t -th power of SMART on periodic tapes, in time polynomial in n (recall $n = \Theta(\log t)$).

1.4.2 Programming periodic SMART efficiently with gates

As discussed in the previous section, our method to efficiently program SMART essentially consists in conjugating configurations to their temporal positions in their cycles. As mentioned, it is not difficult to do this in the sense of classical computation. The main problem is that a polynomial time classical algorithm does not immediately imply an efficient implementation with reversible gates. There are in fact two somewhat orthogonal problems that come into play.

First, we need to perform the computation with no extra space, and fit all the information about the k -th level sweep into the exact area of the sweep. This is quite technical, since the temporal position of the sweep depends in a complicated manner not only on the initial position of the head, but also on the initial content of the tape.

Second, we need to perform the computation reversibly. This is the harder issue of the two, and we explain how we proceeded in the following paragraphs.

The first idea to perform reversible computation consists in “controlled permutations”, used to prove the fundamental lemma in Section 1.3.2, and Barrington’s theorem [3]. The gist is the “commutator trick”: if $\pi|_E$ denotes a permutation π “conditioned on an event” E (so π permutes some set if an event E holds), then $[\pi|_E, \pi'|_F] = [\pi, \pi']|_{E \cap F}$. Performing basic Boolean algebra, and using the fact that $\text{Alt}(A)$ is equal to its commutator subgroup for $|A| \geq 5$, it becomes possible to condition permutations on intersections, unions and negations of events.

In fact, we can write short formulas for even permutations on length- k segments of A^n conditioned on arbitrary properties on the remaining A -symbols. Importantly, this is efficient: if the condition deals with n bits and belongs in the complexity class NC^1 , i.e. it admits an efficient implementation (logarithmic depth in n) by classical gates, then the word norm of the corresponding conditional permutation will be polynomial in n .

The second idea to perform reversible computation is a classical idea that we call *ducking*. This is a simple trick that allows to efficiently compute a piecewise defined function, if we know how to efficiently describe the pieces and perform the correct maps on these pieces. This trick is useful in this paper because the recursive description of how a level- k sweep of SMART splits into level- $(k - 1)$ sweeps (essentially following the formulas of [13]) gives rise to a piecewise-defined function.

We describe this idea of “ducking” in some detail here. Suppose we want to perform a permutation ϕ on a set C , for simplicity let us say $C = \{0, 1\}^n$. Suppose that we have finite partitions $\bigsqcup_{i=1}^p C_i = C$ and $\bigsqcup_{i=1}^p D_i = C$, bijections $\phi_i : C_i \rightarrow D_i$, and that ϕ can be written as the union $\phi = \bigsqcup_i \phi_i$ (of graphs of functions). Suppose now that for each i ,

1. we can “efficiently implement” some element $\psi_i : C \rightarrow C$ (i.e. such ψ_i has a small norm with respect to some set of generators) such that $\psi_i|_{C_i} = \phi_i$, and
2. the sets D_i have a simple description,

then can we implement the bijection ϕ efficiently?

To do this, we add some auxiliary bits to C , and consider instead permutations on $C \times \{0, 1\}^3$. We may see ϕ_i and ψ_i as permutations on $C \times \{0, 1\}$ by identifying $C \times \{0\} \cong C$, and fixing $C \times \{1\}$ pointwise; and then as permutations of $C \times \{0, 1\}^3$ by completely ignoring the last two bits (so we act on $C \times \{0\} \times \{0, 1\}^2$ and fix all of $C \times \{1\} \times \{0, 1\}^2$).

Observe now that $|\{0, 1\}^3| \geq 5$ and is even, and consider the permutation π_i that flips the first of the three auxiliary bits, sometimes called the *duck*, if and only if the word w in the C -component belongs to D_i . By a previous discussion, if we take “simple description” to mean the existence of an NC^1 circuit computing membership in D_i , then the element π_i has small word norm.

Now conjugate π_i by ψ_i to get $\chi_i = \psi_i^{-1} \circ \pi_i \circ \psi_i$. It turns out that that χ_i exchanges $C_i \times \{0\} \times \{0, 1\}^2$ and $D_i \times \{1\} \times \{0, 1\}^2$, and the correspondence between C_i and D_i is precisely ϕ_i , while on other $C_j \times \{0\}$ and $D_j \times \{1\}$ it has no effect.

One may check this by a case analysis, but the simple way to understand this is that conjugation by ψ_i translates the “preimage side” (duck equal to 0) forward, and does not act on the “image side” (duck equal to 1). So, when we conjugate the exchange $D_i \times \{0\} \times \{0, 1\}^2 \leftrightarrow D_i \times \{1\} \times \{0, 1\}^2$ by ψ_i , in $D_i \times \{0\} \times \{0, 1\}^2$ we now see ψ_i -elements of $C_i \times \{0\} \times \{0, 1\}^2$, so the exchange becomes the desired $C_i \times \{0\} \times \{0, 1\}^2 \leftrightarrow D_i \times \{1\} \times \{0, 1\}^2$.

Now, the χ_i have disjoint supports, so they commute. Composing them in any order, by the assumption on the ϕ_i , we obtain the permutation that exchanges $C \times \{0\} \times \{0, 1\}^2$ and $C \times \{1\} \times \{0, 1\}^2$, and for all i , points that belong in $C_i \times \{0\} \times \{0, 1\}^2$ are moved to $D_i \times \{1\} \times \{0, 1\}^2$ by applying ϕ_i in the C -component (and flipping the bit). Finally, when composing this permutation with a flip of the duck, we obtain a map that performs the original map ψ on $C \times \{0\} \times \{0, 1\}^2$, and performs ψ^{-1} on $C \times \{1\} \times \{0, 1\}^2$.

Using this “ducking” idea, more precisely the analog of it for Turing machines on cyclic tapes, we can perform, for each k , the map χ^k that computes, from the current sweep type and the temporal position on the sweep encoded on the tape, the encoding of the type level- $(k+1)$ sweep and the temporal position on it. Of course, we will only apply χ^k when the duck equals 0; while if the duck equals 1, we will apply $(\chi^k)^{-1}$.

Composing all these χ^k (and some additional pre- and post-processing), we obtain a map that correctly implements the encoding procedure when the initial duck equals 0, as the position on the highest-level sweep is essentially equivalent to the temporal position of the initial configuration in its cycle. When the initial duck equals 1, we have actually applied $(\chi^k)^{-1} \circ \dots \circ (\chi^2)^{-1} \circ (\chi^1)^{-1}$, which is meaningless (these operations do not commute, so this is not the inverse of $\chi^k \circ \dots \circ \chi^2 \circ \chi^1$).

This is not a problem, as this encoding is only used to conjugate the SMART machine to the addition of t : if we choose to perform this addition only when the duck equals 0, the meaningless part of the encoding (when the initial duck equals 1) simply cancels itself in the conjugation. The resulting map performs SMART when duck equals 0, and otherwise perform no operation.

1.4.3 Programming SMART efficiently with gates

In the previous sections, we outlined how to implement SMART efficiently on periodic tapes. To implement SMART efficiently on a full shift (using the conveyor belt technique), we do the following. First, we go through small tapes in order, and implement them separately (though for optimization purposes, we actually do perform some steps in parallel).

We then deal with all large enough conveyor belts, and also infinite ones, by actually introducing inside them new temporary conveyor belts of a well-chosen smaller size. Indeed, as long as the head does not meet the border of these temporary conveyor belts when moving, the movement actually coincides with the action of SMART in the original very large conveyor belt.

To implement this idea, we start with ducking: we introduce a new duck, and using Barrington’s theorem on NC¹ conditions, we create a permutation of small word norm that flips this duck if and only if the head sits on a large enough conveyor belt. Now, if we make the head move forward $t/2$ steps if the duck equals 0, and backward $-t/2$ steps if the duck equals 1, conjugating the

duck flip with this map will lead to the machine being applied, if the conveyor belt is long enough, $t/2$ steps forward when the duck equals 0 and $-t/2$ steps backward when the duck equals 1. On short conveyor belts, nothing happens.

Of course, we can only apply the SMART machine efficiently when it sits on a small enough periodic tape. Thus, as mentioned above, we have to conjugate the above duck flip not just by an application of SMART, but also by a map that cuts the conveyor belt into smaller pieces, where the machine can then be applied. Unfortunately, it is impossible to reversibly cut the conveyor belt, then move the head, and then *glue the belt back together*. Indeed, we do not know the exact distance to the boundaries of the conveyor belt after the application of the machine has moved the head, and gluing a short belt back into a longer tape loses the information about the exact position of the head inside the conveyor belt, so it is inherently irreversible.

To get around this, we use a technical trick that we call the “two-scale trick”. It consists in conjugating the duck flip by the following sequence of operations:

1. cut the conveyor belt at distance n (from the head) on both sides,
2. apply the machine for t steps on the small tape this cut introduces,
3. cut the conveyor belt at distance $2n$ on both sides,
4. apply the machine for $-t$ steps on the small tape,
5. glue the conveyor belt back together at distance n on both sides,
6. apply the machine for t steps on the bigger tape introduced in step 3,
7. glue the conveyor belt back together at distance $2n$ on both sides.

The point of the small tape is that the boundaries of the bigger conveyor belt (introduced in the third step) are at distance $2n$ with respect to the “future location” of the head after applying t steps of SMART, so they can indeed be glued back together in the last step of this procedure. If n is big enough (bigger than $O(\log t)$, the movement of the head in t steps), then this action actually coincides with applying t steps of SMART, as the head never encounters the border of the temporary conveyor belts.

This somewhat technical feat is performed in Section 5.4, see Figure 6.

Acknowledgements

We would like to thank Anthony Genevois for pointing out Corollary 1.2. We thank Pierre Guillou for helpful discussions.

2 Definitions

2.1 General notions

For S a finite set, we denote by $\#S$ the cardinal of S . For $i, j \in \mathbb{N}$, denote $\llbracket i, j \rrbracket = \{n \in \mathbb{N} : i \leq n \leq j\}$ and $\llbracket n \rrbracket = \llbracket 0, n - 1 \rrbracket$. If $w \in \{0, 1, \dots, k - 1\}^*$, write $v_k(w)$ for the value w represents in base k (the leftmost digit having the highest significance by default), and $n_{(k)}$ for the number n written in base k (with length determined from context or specified in text).

For Σ a finite set, called an *alphabet*, denote $\Sigma^* = \bigcup_{n=0}^{+\infty} \Sigma^n$ the set of finite words over Σ . For $w \in \Sigma^*$, denote $\text{len}(w)$ the length of w , i.e. the integer n such that $w \in \Sigma^n$. For a word $w \in \Sigma^*$, denote \overline{w} the reverse (or “mirror”) of w , i.e. if $w = w_0 \cdot w_1 \cdots w_{n-1}$, then $\overline{w} = w_{n-1} \cdot w_{n-2} \cdots w_0$. For $w \in \Sigma^n$ and $J \subseteq \llbracket 0, n-1 \rrbracket$, define $w|_J = w_{j_0} \cdot w_{j_1} \cdots w_{j_k}$ the restriction of w to J , for $J = \{j_0, \dots, j_k\}$ and $j_0 \leq \cdots \leq j_k$.

In Lemma 4.5, we denote NC^1 for “Nick’s Class” of complexity of level 1, i.e. the class of languages $L \subseteq \Sigma^*$ such that L is decidable by Boolean circuits with a polynomial number of gates, with at most two inputs and depth $O(\log n)$ (see for example [1]). The reader need not be familiar with this class to follow our argument. The main technical result we need is Barrington’s theorem from [3] (but this is also proved from scratch in our context).

For a, b elements of a group, the commutator of a and b is $[a, b] = a^{-1}b^{-1}ab$. The conjugation convention is $a^b = b^{-1} \circ a \circ b$. If $\pi \in \text{Sym}(A)$ is a permutation, we may regard it as a permutation of $A \times B$ by $\pi((a, b)) = (\pi(a), b)$. Slightly abusing the English language, we say a permutation of $A \times B$ is *B-ignorant*, or *ignores B*, if it comes from a permutation of A through this formula. Our groups always act from the left. If g_1, \dots, g_n are commuting elements of a group, we write $\prod_{i=1}^n g_i$ for their ordered product $g_n \cdots g_1$. In groups of bijections on a set (which almost all our groups are), we denote composition by \circ .

2.2 Subshifts and cellular automata

Let Σ be a finite alphabet. An element $x \in \Sigma^{\mathbb{Z}}$ is called a *configuration*. An element $w \in \Sigma^*$ is called a *word* or a *pattern*, and a pattern $w \in \Sigma^*$ is said to *appear* in a configuration $x \in \Sigma^{\mathbb{Z}}$, denoted $w \sqsubseteq x$, if there exists some $i \in \mathbb{Z}$ such that $x_{i+j} = w_j$ for every $j \in \llbracket 0, \text{len}(w) - 1 \rrbracket$.

We endow $\Sigma^{\mathbb{Z}}$ with the product topology. This topology is generated by the cylinders $[a]_j = \{x \in \Sigma^{\mathbb{Z}} : x_j = a\}$ for $a \in \Sigma$ and $j \in \mathbb{Z}$. The *left shift* $\sigma : \Sigma^{\mathbb{Z}} \mapsto \Sigma^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ is a \mathbb{Z} action on $\Sigma^{\mathbb{Z}}$. Closed and shift-invariant subsets X of $\Sigma^{\mathbb{Z}}$ are called *subshifts*. For X a subshift and $n \in \mathbb{N}$, we denote $\mathcal{L}_n(X)$ the set of finite words of length n that appear in X , and $\mathcal{L}(X) = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n(X)$ its *language*.

We say that a subshift X is *sofic* if $\mathcal{L}(X)$ is a regular language.

If X and Y are subshifts, a continuous and shift-invariant map $f : X \mapsto Y$ is called a *morphism*. It is an *endomorphism* if $X = Y$ and an *automorphism* if, in addition, it is bijective (in which case f^{-1} is also an endomorphism). Endomorphisms are sometimes called *cellular automata*, and automorphisms *reversible cellular automata*. For $f : X \rightarrow Y$ a morphism between two subshifts, its *radius* (as a cellular automaton) is the minimal r such that $f(x)_i$ is a function of $x_{\llbracket i-r, i+r \rrbracket}$. The *biradius* of an automorphism is the maximum of the radii of f and f^{-1} .

2.3 One-head machines

Let Q be a finite set called the *state set*, and Γ be a finite set called the *tape alphabet*. In the model¹ of [28], a *Turing machine* is a triple $\mathcal{M} = (\Gamma, Q, \Delta)$,

¹This model is equivalent to the usual definition of Turing machines, but handles reversibility better.

where $\Delta \subseteq (Q \times \{+1, -1\} \times Q) \cup (Q \times \Gamma \times Q \times \Gamma)$ is the *transition table*. A transition $(q, \delta, q') \in Q \times \{+1, -1\} \times Q$ is called a *move transition*, and a transition $(q, a, q', b) \in Q \times \Gamma \times Q \times \Gamma$ is called a *matching transition*.

In this article, we focus on the action of Turing machines on two objects: bi-infinite tapes, and finite cyclic tapes.

Bi-infinite tapes

On the alphabet $\Gamma \cup (Q \times \Gamma)$, elements of $Q \times \Gamma$ are called *heads*. Denote

$$X_{Q,\Gamma} = \{x \in (\Gamma \cup (Q \times \Gamma))^{\mathbb{Z}} \mid \forall i, j \in \mathbb{Z} : i \neq j \implies x_i \in \Gamma \vee x_j \in \Gamma\}$$

the set of bi-infinite tapes with at most one head somewhere. We can associate to \mathcal{M} its so-called *moving-head model* [31], i.e. the binary relation $\rightarrow_{\mathcal{M}}$ on $X_{Q,\Gamma}$ defined by $x \rightarrow_{\mathcal{M}} x'$ if $x \in \Gamma^{\mathbb{Z}}$, and if $x_i = (q, a_i)$ for $i \in \mathbb{Z}$, then $x \rightarrow_{\mathcal{M}} x'$ if there exists $t \in \Delta$ such that:

$$\begin{aligned} \text{If } t = (q, a_i, q', b) \in \Delta : \quad x'_j &= \begin{cases} (q', b) & \text{if } j = i \\ x_j & \text{otherwise} \end{cases} \\ \text{If } t = (q, \delta, q') \in \Delta : \quad x'_j &= \begin{cases} a_i & \text{if } j = i \\ (q', x_{i+\delta}) & \text{if } j = i + \delta \\ x_j & \text{otherwise} \end{cases} \end{aligned}$$

The machine \mathcal{M} is *deterministic* if $\rightarrow_{\mathcal{M}}$ defines a partial function, *complete deterministic* if it defines a total function (which is then continuous and, obviously, shift-commuting), and *complete reversible* (or *reversible* for short) if it defines a bijection (which is then a homeomorphism). When \mathcal{M} is complete deterministic (which all our machines are), when using the relation $\rightarrow_{\mathcal{M}}$ as a function we write it as $T_{\mathcal{M}} : X_{Q,\Gamma} \mapsto X_{Q,\Gamma}$, which is an endomorphism of the subshift $X_{Q,\Gamma}$. Similarly, when \mathcal{M} is reversible, it is an automorphism of $X_{Q,\Gamma}$.

Finite cyclic tapes

The set of *cyclic configurations of length ℓ* is the set $C_{\ell,Q,\Gamma} = \mathcal{L}_{\ell}(X_{Q,\Gamma})$ of finite configurations containing at most one head. Then \mathcal{M} defines a binary relation $\rightarrow_{\mathcal{M}}$ on $C_{\ell,Q,\Gamma}$ by considering these finite tapes cyclic, i.e. we define $x \rightarrow_{\mathcal{M}} x'$ if $x \in \Gamma^{\ell}$, and if $x_i = (q, a_i)$ for $i \in \llbracket 0, \ell - 1 \rrbracket$, then $x \rightarrow_{\mathcal{M}} x'$ if there exists $t \in \Delta$ such that:

$$\begin{aligned} \text{If } t = (q, a_i, q', b) \in \Delta : \quad x'_j &= \begin{cases} (q', b) & \text{if } j = i \\ x_j & \text{otherwise} \end{cases} \\ \text{If } t = (q, \delta, q') \in \Delta : \quad x'_j &= \begin{cases} a_i & \text{if } j = i \\ (q', \pi_{\Gamma}(x_{i+\delta \bmod \ell})) & \text{if } j = i + \delta \bmod \ell \\ x_j & \text{otherwise} \end{cases} \end{aligned}$$

where $\pi_{\Gamma} : \Gamma \sqcup (Q \times \Gamma) \mapsto \Gamma$ is the natural projection. If \mathcal{M} is complete deterministic, the function $\rightarrow_{\mathcal{M}}$ will be denoted by $T_{\ell,\mathcal{M}} : C_{\ell,Q,\Gamma} \mapsto C_{\ell,Q,\Gamma}$. Note that it is an endomorphism of the shift action of \mathbb{Z} (or \mathbb{Z}_{ℓ} which translates the cyclic tape around).

These conditions (determinism, completeness and reversibility) are characterized by obvious combinatorial properties. In particular, $\mathcal{M} = (\Gamma, Q, \Delta)$ is complete deterministic if and only if exactly one transition applies at any time:

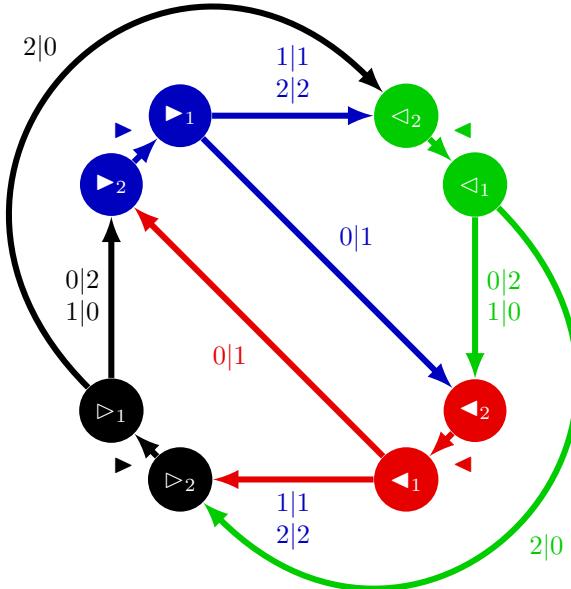
$$\forall (q, a) \in (Q \times \Gamma) : \#\{t \in \Delta \mid t = (q, a, \cdot, \cdot) \text{ or } t = (q, \cdot, \cdot, \cdot)\} = 1.$$

Defining the reverse of a transition by $(q, \delta, q')^{-1} = (q', -\delta, q)$ and $(q, a, q', b)^{-1} = (q', b, q, a)$, this reverse relation extends to transition tables with $\Delta^{-1} = \{t^{-1} \mid t \in \Delta\}$: the *reverse* of \mathcal{M} is then defined by $\mathcal{M}^{-1} = (\Gamma, Q, \Delta^{-1})$, and \mathcal{M} is reversible if both \mathcal{M} and \mathcal{M}^{-1} are complete deterministic.

Finally, for any machine $\mathcal{M} = (Q, \Gamma, \Delta)$, denote $m : \mathbb{N} \mapsto \mathbb{N}$ its *movement function*, i.e. $m(n)$ is the maximal number of cells the machine can visit in n steps.

3 The SMART machine on cyclic tapes

Let SMART be the Turing machine (Q, Γ, Δ) , where $Q = \{\blacktriangleright_1, \blacktriangleleft_1, \triangleright_1, \triangleleft_1\} \cup \{\blacktriangleright_2, \blacktriangleleft_2, \triangleright_2, \triangleleft_2\}$, $\Gamma = \{0, 1, 2\}$ and Δ is the following transition table:



We refer to $\blacktriangleright_1 \blacktriangleright_2, \blacktriangleleft_1, \blacktriangleleft_2$ (resp. $\triangleright_1, \triangleright_2, \triangleleft_1, \triangleleft_2$) as *filled* and *hollow triangles*.

Remark 3.1. The SMART machine was introduced with a slightly different formalism in [13], and slightly revised in [37] (states were renamed and permuted). The machine above adapts the latter in the model of [28] for Turing machines: in other words, we duplicate the states. We kindly advise readers already familiar with the SMART machine to read these definitions and propositions carefully.

Namely, while our SMART machine is in a sense completely equivalent, in the formulas in Proposition 3.2 describing traversals of SMART over zeroes, the patterns corresponding to filled and hollow initial states are of the same length (unlike the corresponding ones in [13]). This will be helpful later, when we encode the position in the sweep into the corresponding area on the tape without any extra space.

Proposition 3.2. (Adapted from [13, Lemma 1]) Let $f(k) = 3^{k+1} - 2$. For all k , $s_* \in \{0, 1, 2\}$ and $s_+ \in \{1, 2\}$, the following moves hold:

$$\begin{array}{ll} M_{\blacktriangleright}(k) : \left(\begin{smallmatrix} s_+ & 0^k & s_* \\ \blacktriangleright_2 & & \end{smallmatrix} \right) \vdash^{f(k)} \left(\begin{smallmatrix} s_+ & 0^k & s_* \\ \blacktriangleright_1 & & \end{smallmatrix} \right) & M_{\blacktriangleleft}(k) : \left(\begin{smallmatrix} s_* & 0^k & s_+ \\ \blacktriangleleft_2 & & \end{smallmatrix} \right) \vdash^{f(k)} \left(\begin{smallmatrix} s_* & 0^k & s_+ \\ \blacktriangleleft_1 & & \end{smallmatrix} \right) \\ M_{\triangleright}(k) : \left(\begin{smallmatrix} s_* & 0^k & s_+ \\ \triangleright_2 & & \end{smallmatrix} \right) \vdash^{f(k)} \left(\begin{smallmatrix} s_* & 0^k & s_+ \\ \triangleright_1 & & \end{smallmatrix} \right) & M_{\triangleleft}(k) : \left(\begin{smallmatrix} s_+ & 0^k & s_* \\ \triangleleft_2 & & \end{smallmatrix} \right) \vdash^{f(k)} \left(\begin{smallmatrix} s_+ & 0^k & s_* \\ \triangleleft_1 & & \end{smallmatrix} \right) \end{array}$$

Additionally, the cell containing s_* is only visited at the last (resp. first) step of the sequences of transitions M_{\blacktriangleright} and M_{\blacktriangleleft} (resp. M_{\triangleright} and M_{\triangleleft}). And the cell containing s_+ is never modified.

Proof. This proof adapts the proof of [13, Lemma 1], and highlights the recursive/nested aspects of these moves. In the case $k = 0$ one can check that indeed the formula describes a single transition. We reason by induction, and assume $M_{\blacktriangleright}(k)$, $M_{\blacktriangleleft}(k)$, $M_{\triangleright}(k)$ and $M_{\triangleleft}(k)$ hold. We only prove $M_{\blacktriangleright}(k + 1)$ and $M_{\triangleright}(k + 1)$, by symmetry between \blacktriangleright and \blacktriangleleft (resp. \triangleright and \triangleleft). Since $f(k + 1) = 3f(k) + 4$ we should find 3 recursions, and 4 extra steps. This is what happens:

$$\begin{array}{ll} M_{\blacktriangleright}(k + 1) & M_{\triangleright}(k + 1) \\ \left(\begin{smallmatrix} s_+ & 0^k & 0 & s_* \\ \blacktriangleright_2 & & & \end{smallmatrix} \right) & \left(\begin{smallmatrix} s_* & 0 & 0^k & s_+ \\ \triangleright_2 & & & \end{smallmatrix} \right) \\ \text{Apply } M_{\blacktriangleright}(k) & \text{Apply one step} \\ \vdash^{f(k)} \left(\begin{smallmatrix} s_+ & 0^k & 0 & s_* \\ \blacktriangleright_1 & & & \end{smallmatrix} \right) & \vdash \left(\begin{smallmatrix} s_* & 0 & 0^k & s_+ \\ \triangleright_1 & & & \end{smallmatrix} \right) \\ \text{Apply one step} & \text{Apply one step} \\ \vdash \left(\begin{smallmatrix} s_+ & 0^k & 1 & s_* \\ \blacktriangleleft_2 & & & \end{smallmatrix} \right) & \vdash \left(\begin{smallmatrix} s_* & 2 & 0^k & s_+ \\ \blacktriangleright_2 & & & \end{smallmatrix} \right) \\ \text{Apply } M_{\blacktriangleleft}(k) & \text{Apply } M_{\blacktriangleright}(k) \\ \vdash^{f(k)} \left(\begin{smallmatrix} s_+ & 0^k & 1 & s_* \\ \blacktriangleleft_1 & & & \end{smallmatrix} \right) & \vdash^{f(k)} \left(\begin{smallmatrix} s_* & 2 & 0^k & s_+ \\ \blacktriangleright_1 & & & \end{smallmatrix} \right) \\ \text{Apply one step} & \text{Apply one step} \\ \vdash \left(\begin{smallmatrix} s_+ & 0^k & 1 & s_* \\ \triangleright_2 & & & \end{smallmatrix} \right) & \vdash \left(\begin{smallmatrix} s_* & 2 & 0^k & s_+ \\ \triangleright_1 & & & \end{smallmatrix} \right) \\ \text{Apply } M_{\triangleright}(k) & \text{Apply } M_{\triangleleft}(k) \\ \vdash^{f(k)} \left(\begin{smallmatrix} s_+ & 0^k & 1 & s_* \\ \triangleright_1 & & & \end{smallmatrix} \right) & \vdash^{f(k)} \left(\begin{smallmatrix} s_* & 2 & 0^k & s_+ \\ \triangleleft_1 & & & \end{smallmatrix} \right) \\ \text{Apply one step} & \text{Apply one step} \\ \vdash \left(\begin{smallmatrix} s_+ & 0^k & 0 & s_* \\ \blacktriangleright_2 & & & \end{smallmatrix} \right) & \vdash \left(\begin{smallmatrix} s_* & 0 & 0^k & s_+ \\ \triangleright_2 & & & \end{smallmatrix} \right) \\ \text{Apply one step} & \text{Apply } M_{\triangleright}(k) \\ \vdash \left(\begin{smallmatrix} s_+ & 0^k & 0 & s_* \\ \blacktriangleright_1 & & & \end{smallmatrix} \right) & \vdash^{f(k)} \left(\begin{smallmatrix} s_* & 0 & 0^k & s_+ \\ \triangleright_1 & & & \end{smallmatrix} \right) \end{array}$$

□

3.1 Action of SMART on cyclic tapes

This section studies the action of SMART on cyclic tapes of length $\ell \geq 2$. We call *initial configurations* the four following cyclic configurations:

$$\begin{array}{ll} C_{\blacktriangleright} = \left(\begin{smallmatrix} 0 & 0^{\ell-1} \\ \blacktriangleright_1 & \end{smallmatrix} \right) & C_{\blacktriangleleft} = \left(\begin{smallmatrix} 0 & 0^{\ell-1} \\ \blacktriangleleft_1 & \end{smallmatrix} \right) \\ C_{\triangleright} = \left(\begin{smallmatrix} 0 & 0^{\ell-1} \\ \triangleright_1 & \end{smallmatrix} \right) & C_{\triangleleft} = \left(\begin{smallmatrix} 0 & 0^{\ell-1} \\ \triangleleft_1 & \end{smallmatrix} \right) \end{array}$$

Proposition 3.3. Let $\ell \geq 1$. The action of the $(2 \cdot 3^\ell)$ -th power of SMART on C_\blacktriangleright and C_\blacktriangleleft (resp. C_\blacktriangleleft and C_\blacktriangleleft) is a right-shift (resp. left-shift). Furthermore, the intermediate configurations are all distinct.

Proof. By symmetries between \blacktriangleright and \blacktriangleleft (resp. \blacktriangleright and \blacktriangleleft), we prove the result for C_\blacktriangleright and C_\blacktriangleleft .

$$\begin{array}{ccc}
\begin{array}{c} (0 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_1 \end{array} & \xrightarrow{\text{Apply one step}} & \begin{array}{c} (0 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_1 \end{array} \\
\vdash & & \vdash \\
\begin{array}{c} (1 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleleft_2 \end{array} & & \begin{array}{c} (2 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_2 \end{array} \\
\vdash & & \vdash \\
\begin{array}{c} (1 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleleft_2 \end{array} & \xrightarrow{\text{Apply } M_\blacktriangleleft(\ell-1)} & \begin{array}{c} (2 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_1 \end{array} \\
\vdash^{f(\ell-1)} & & \vdash^{f(\ell-1)} \\
\begin{array}{c} (1 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleleft_1 \end{array} & & \begin{array}{c} (2 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_1 \end{array} \\
\vdash & & \vdash \\
\begin{array}{c} (1 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_2 \end{array} & \xrightarrow{\text{Apply one step}} & \begin{array}{c} (2 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleleft_2 \end{array} \\
\vdash & & \vdash \\
\begin{array}{c} (1 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_1 \end{array} & \xrightarrow{\text{Apply } M_\blacktriangleright(\ell-1)} & \begin{array}{c} (2 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleleft_1 \end{array} \\
\vdash^{f(\ell-1)} & & \vdash^{f(\ell-1)} \\
\begin{array}{c} (1 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_1 \end{array} & & \begin{array}{c} (2 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleleft_1 \end{array} \\
\vdash & & \vdash \\
\begin{array}{c} (0 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_2 \end{array} & \xrightarrow{\text{Apply one step}} & \begin{array}{c} (0 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_2 \end{array} \\
\vdash & & \vdash \\
\begin{array}{c} (0 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_1 \end{array} & \xrightarrow{\text{Apply one step}} & \begin{array}{c} (0 \quad 0 \quad 0^{\ell-2}) \\ \blacktriangleright_1 \end{array}
\end{array}$$

We used moves $M_\blacktriangleright(\ell-1)$, $M_\blacktriangleleft(\ell-1)$, $M_\blacktriangleright(\ell-1)$ and $M_\blacktriangleleft(\ell-1)$ in patterns that overlap themselves on their first and last letters in the cyclic tape. This is valid, because the cell containing s_* is only visited at the last (resp. first) step of M_\blacktriangleright and M_\blacktriangleleft (resp. M_\blacktriangleright and M_\blacktriangleleft).

For the last claim, it is enough to show that the last (shifted) configuration does not appear before the last step. This is clear from looking at the first columns, which have positive values on all but the first step and the two last steps. \square

Lemma 3.4. For $\ell \geq 1$, the action of SMART on cyclic tapes of length ℓ is composed of four disjoint cycles of length $2\ell \cdot 3^\ell$, which are the orbits of the four initial configurations. Additionally, the action of the $(2 \cdot 3^\ell)$ -th power of SMART on a cyclic tapes consists in a right-shift (resp. left-shift) on the orbits of C_\blacktriangleright and C_\blacktriangleright (resp. C_\blacktriangleleft and C_\blacktriangleleft).

Proof. By Proposition 3.3, the orbits of C_\blacktriangleright , C_\blacktriangleright , C_\blacktriangleleft and C_\blacktriangleleft are each of length $2\ell \cdot 3^\ell$ (number of shifts \times number of steps for each shift). As there are $8\ell \cdot 3^\ell$ different cyclic configurations containing a head (eight different states with ℓ possible positions, and a ternary tape of length ℓ), we now only need to prove that the orbits of C_\blacktriangleright , C_\blacktriangleright , C_\blacktriangleleft and C_\blacktriangleleft are disjoint.

For this, observe again that in the proof of Proposition 3.3, the positive ternary letter 1 or 2 are never erased in the searches $M_\blacktriangleright(\ell-1)$, $M_\blacktriangleleft(\ell-1)$, $M_\blacktriangleright(\ell-1)$ and $M_\blacktriangleleft(\ell-1)$. Considering the orbit of C_q for some fixed $q \in \{\blacktriangleright, \blacktriangleleft, \blacktriangleright, \blacktriangleleft\}$, the word 0^ℓ appears only at the first or last step of this proof (or

their shifts): in particular, they are all in states either q_1 or q_2 (with this fixed q). This implies that these orbits cannot intersect. \square

3.2 Analysis of SMART configurations

We now explain how, given a cyclic SMART configuration of length ℓ , we determine which orbit it belongs in and its position in this orbit, i.e. the number of steps required to obtain it from its corresponding initial configuration $C_\blacktriangleright, C_\blacktriangleleft, C_\triangleright$ or C_\triangleleft .

We say that a cyclic configuration is performing the j -th step of computation of $M_\blacktriangleright(k)$ (resp. $M_\blacktriangleleft(k), M_\triangleright(k), M_\triangleleft(k)$), for $0 \leq j \leq f(k)$, if it contains the j -th pattern of the sequence of transitions $M_\blacktriangleright(k)$ (resp. ...) of Proposition 3.2. At this point, it may not be clear that this is unique, but this will follow from our argument.

If a configuration is performing some step of computation from one of the moves $M_\blacktriangleright(k), M_\blacktriangleleft(k), M_\triangleright(k)$ or $M_\triangleleft(k)$, we refer to this move as its *computation of level k*.

Initialization We call the following patterns *special patterns of level k* ≥ 1 :

$$\begin{array}{ll} s(\blacktriangleright_2, k) = \begin{pmatrix} s_+ & 0^{k-1} & 0 \\ & \blacktriangleright_2 \end{pmatrix} & s(\blacktriangleright_1, k) = \begin{pmatrix} s_+ & 0^k & s_* \\ & \blacktriangleright_1 \end{pmatrix} \\ s(\blacktriangleleft_2, k) = \begin{pmatrix} 0 & 0^{k-1} & s_+ \\ & \blacktriangleleft_2 \end{pmatrix} & s(\blacktriangleleft_1, k) = \begin{pmatrix} s_* & 0^k & s_+ \\ & \blacktriangleleft_1 \end{pmatrix} \\ s(\triangleright_2, k) = \begin{pmatrix} s_* & 0^k & s_+ \\ & \triangleright_2 \end{pmatrix} & s(\triangleright_1, k) = \begin{pmatrix} 0 & 0^{k-1} & s_+ \\ & \triangleright_1 \end{pmatrix} \\ s(\triangleleft_2, k) = \begin{pmatrix} s_+ & 0^k & s_* \\ & \triangleleft_2 \end{pmatrix} & s(\triangleleft_1, k) = \begin{pmatrix} s_+ & 0^{k-1} & 0 \\ & \triangleleft_1 \end{pmatrix} \end{array}$$

By the proof of Proposition 3.2, we see that if a cyclic configuration contains a special pattern $s(\blacktriangleright_2, k), s(\blacktriangleright_1, k), s(\blacktriangleleft_2, k)$ or $s(\blacktriangleleft_1, k)$ (resp. $s(\triangleright_2, k), s(\triangleright_1, k), s(\triangleleft_2, k)$ or $s(\triangleleft_1, k)$), then it performs the last two steps of $M_\blacktriangleright(k)$ or $M_\blacktriangleleft(k)$ respectively (resp. the first two steps of $M_\triangleright(k)$ or $M_\triangleleft(k)$).

Claim 3.5. *Given a cyclic configuration c of length ℓ containing a head, exactly one of the following holds:*

- *c is on the full-zero word 0^ℓ .*
- *c performs some step of computation of level 0 from either $M_\blacktriangleright(0), M_\blacktriangleleft(0), M_\triangleright(0)$ or $M_\triangleleft(0)$.*
- *there exists some unique $1 \leq k \leq \ell - 1$ such that c either performs the last two steps of $M_\blacktriangleright(k)$ or $M_\blacktriangleleft(k)$, or the first two steps of $M_\triangleright(k)$ or $M_\triangleleft(k)$.*

Proof. The patterns of $M_\blacktriangleright(0), M_\blacktriangleleft(0), M_\triangleright(0)$ and $M_\triangleleft(0)$ (eight in total), along with the special patterns of every level, disjointly cover all the non-zero configurations with a head. \square

Inductive analysis: $k \mapsto k + 1 \leq \ell - 1$ Assume a cyclic configuration of length ℓ is performing some computation of level k , for $k < \ell - 1$. Then it performs some computation of level $k + 1$, and Figure 1 details a case-analysis to figure out which computation it performs out of $M_\blacktriangleright(k + 1), M_\blacktriangleleft(k + 1), M_\triangleright(k + 1)$ or $M_\triangleleft(k + 1)$. (This analysis also implies that the computation step performed at a particular level is unique, if one exists.)

$M_{\blacktriangleright}(k)$ is performed in:	
$M_{\blacktriangleright}(k+1)$ (at step 0):	$\left(\begin{smallmatrix} s_+ & 0^k & \textcircled{0} & s_* \\ \blacktriangleright_2 & & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} s_+ & 0^k & \textcircled{0} & s_* \\ & \blacktriangleright_1 & & \end{smallmatrix} \right)$
$M_{\blacktriangleleft}(k+1)$ (at step $f(k)+1$):	$\left(\begin{smallmatrix} s_* & \textcircled{1} & 0^k & \textcircled{s_+} \\ & \blacktriangleright_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} s_* & \textcircled{1} & 0^k & \textcircled{s_+} \\ & \blacktriangleright_1 & & \end{smallmatrix} \right)$
$M_{\triangleright}(k+1)$ (at step 2):	$\left(\begin{smallmatrix} s_* & \textcircled{2} & 0^k & \textcircled{s_+} \\ & \blacktriangleright_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} s_* & \textcircled{2} & 0^k & \textcircled{s_+} \\ & \blacktriangleright_1 & & \end{smallmatrix} \right)$
$M_{\blacktriangleleft}(k)$ is performed in:	
$M_{\blacktriangleleft}(k+1)$ (at step 0):	$\left(\begin{smallmatrix} s_* & \textcircled{0} & 0^k & s_+ \\ & \blacktriangleleft_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} s_* & \textcircled{0} & 0^k & s_+ \\ & \blacktriangleleft_1 & & \end{smallmatrix} \right)$
$M_{\blacktriangleright}(k+1)$ (at step $f(k)+1$):	$\left(\begin{smallmatrix} \textcircled{s_+} & 0^k & \textcircled{1} & s_* \\ & \blacktriangleleft_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} \textcircled{s_+} & 0^k & \textcircled{1} & s_* \\ & \blacktriangleleft_1 & & \end{smallmatrix} \right)$
$M_{\triangleleft}(k+1)$ (at step 2):	$\left(\begin{smallmatrix} \textcircled{s_+} & 0^k & \textcircled{2} & s_* \\ & \blacktriangleleft_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} \textcircled{s_+} & 0^k & \textcircled{2} & s_* \\ & \blacktriangleleft_1 & & \end{smallmatrix} \right)$
$M_{\triangleright}(k)$ is performed in:	
$M_{\triangleright}(k+1)$ (at step $2f(k)+4$):	$\left(\begin{smallmatrix} s_* & \textcircled{0} & 0^k & s_+ \\ & \triangleright_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} s_* & \textcircled{0} & 0^k & s_+ \\ & \triangleright_1 & & \end{smallmatrix} \right)$
$M_{\blacktriangleright}(k+1)$ (at step $2f(k)+2$):	$\left(\begin{smallmatrix} \textcircled{s_+} & 0^k & \textcircled{1} & s_* \\ & \triangleright_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} \textcircled{s_+} & 0^k & \textcircled{1} & s_* \\ & \triangleright_1 & & \end{smallmatrix} \right)$
$M_{\triangleleft}(k+1)$ (at step $f(k)+3$):	$\left(\begin{smallmatrix} \textcircled{s_+} & 0^k & \textcircled{2} & s_* \\ & \triangleright_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} \textcircled{s_+} & 0^k & \textcircled{2} & s_* \\ & \triangleright_1 & & \end{smallmatrix} \right)$
$M_{\triangleleft}(k)$ is performed in:	
$M_{\triangleleft}(k+1)$ (at step $2f(k)+4$):	$\left(\begin{smallmatrix} s_+ & 0^k & \textcircled{0} & s_* \\ & \triangleleft_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} s_+ & 0^k & \textcircled{0} & s_* \\ & \triangleleft_1 & & \end{smallmatrix} \right)$
$M_{\blacktriangleleft}(k+1)$ (at step $2f(k)+2$):	$\left(\begin{smallmatrix} s_* & \textcircled{1} & 0^k & \textcircled{s_+} \\ & \triangleleft_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} s_* & \textcircled{1} & 0^k & \textcircled{s_+} \\ & \triangleleft_1 & & \end{smallmatrix} \right)$
$M_{\triangleright}(k+1)$ (at step $f(k)+3$):	$\left(\begin{smallmatrix} s_* & \textcircled{2} & 0^k & \textcircled{s_+} \\ & \triangleleft_2 & & \end{smallmatrix} \right) \vdash^* \left(\begin{smallmatrix} s_* & \textcircled{2} & 0^k & \textcircled{s_+} \\ & \triangleleft_1 & & \end{smallmatrix} \right)$

Figure 1: Bottom-up analysis of SMART configurations: $k \mapsto k+1$

We write configurations performing sub-computations (of level k) of computations of level $k+1$. We darken the part of the configuration performing the sub-computation of level k . Circled ternary bits are not modified by the sub-computations of level k , and can be used to perform a case-analysis.

Conclusion: This proves that every cyclic configuration of length ℓ is either on the full zero word, or performs the j -th step of computation of a move of level $\ell - 1$ for some $0 \leq j \leq f(\ell - 1)$. With the proof of Proposition 3.3, we then conclude whether the input configuration belongs in the orbit of C_\triangleright , C_\triangleleft , C_\rhd or C_\lhd , and its position in this orbit.

3.3 Encoding cyclic configurations into their orbit positions in $C_{\ell,Q,\Gamma}$

Remark: in this section, we work with the SMART machine introduced above; this will define encodings for the decorated machine (introduced below) by simply carrying the decorations in the state, unmodified.

Denote \mathcal{S} the SMART machine introduced above. We define $\Phi : C_{\ell,Q,\Gamma} \mapsto C_{\ell,Q,\Gamma}$ the following bijection:

$$\Phi(\mathcal{S}^n(C_q)) = \begin{cases} (q_1, w_1) \cdot w_2 \dots w_\ell, & \text{if } n < 3^\ell, \text{ for } w = n_{(3)} \\ & \text{and } q \in \{\triangleright, \triangleleft, \rhd, \lhd\} \\ (q_2, w_1) \cdot w_2 \dots w_l, & \text{if } 3^\ell \leq n < 2 \cdot 3^\ell, \text{ for } w = [n - 3^\ell]_{(3)} \\ & \text{and } q \in \{\triangleright, \triangleleft, \rhd, \lhd\} \\ \sigma^{-j} (\Phi(\mathcal{S}^{n'}(C_q))) & \text{where } q \in \{\triangleright, \rhd\}, \quad 2j \cdot 3^\ell + n' = n \\ \sigma^j (\Phi(\mathcal{S}^{n'}(C_q))) & \text{where } q \in \{\triangleleft, \lhd\}, \quad 2j \cdot 3^\ell + n' = n \end{cases}$$

In other words, given as input a configuration $\mathcal{S}^n(C_q)$ of length ℓ , for some $q \in \{\triangleright, \triangleleft, \rhd, \lhd\}$, Φ encodes the tuple (q, n) in base $4 \cdot (2\ell \cdot 3^\ell)$: the head has state ranging in q_1 or q_2 for $q \in \{\triangleright, \triangleleft, \rhd, \lhd\}$, the position of the head ranges from 0 to $\ell - 1$, and the ternary counter of length ℓ ranges from 0 to $3^\ell - 1$. Note that this bijection is shift-commuting, because on each tape C_q , applying SMART $2 \cdot 3^\ell$ times performs a shift map (in the same direction as we do in the above formula).

Inductive encoding In this section, we use the analysis performed in Section 3.2 to provide a linear-time algorithm that breaks this complicated bijection into smaller and easier steps, which we then “implement” in Section 4.3 in some finitely generated group of permutations.

Let c be a cyclic configuration of length ℓ of SMART, and $k \leq \ell - 1$. If c is not special of level $> k$ or on the full zero word, there exists some $q \in \{\triangleright, \triangleleft, \rhd, \lhd\}$ and some unique word $w(k)$ of length $k + 2$ such that $w(k) \sqsubseteq c$ and $w(k)$ computes the j -th step of $M_q(k)$, for some $0 \leq j \leq f(k)$. For such k , we define the encoding of level k of c by replacing the occurrence of $w(k)$ in c by the word $w'(k)$ defined by:

If $q = \triangleright$: $w'(k) = c_0 \dots c_k \cdot (q', w(k)_{k+1})$, where $c \in \{0, 1, 2\}^{k+1}$ is a ternary counter with $v_3(c) = j + 1$, and $q' = \triangleright_1$ if $w(k)_0 = 1$ (resp. $q' = \triangleright_2$ if $w(k)_0 = 2$).

If $q = \triangleleft$: $w'(k) = (q', w(k)_0) \cdot c_0 \dots c_k$, where $c \in \{0, 1, 2\}^{k+1}$ is a ternary counter with $v_3(c) = j + 1$, and $q' = \triangleleft_1$ if $w(k)_{k+1} = 1$ (resp. $q' = \triangleleft_2$ if $w(k)_{k+1} = 2$).

Special \blacktriangleright_1 higher level:	$\begin{pmatrix} 0 & s_* \\ \blacktriangleright_1 & \end{pmatrix} \mapsto \begin{pmatrix} 0 & s_* \\ \blacktriangleright_1 & \end{pmatrix}$
Special \blacktriangleright_2 higher level:	$\begin{pmatrix} 0 & s_* \\ \blacktriangleright_2 & \end{pmatrix} \mapsto \begin{pmatrix} 0 & s_* \\ \blacktriangleright_2 & \end{pmatrix}$
$M_{\blacktriangleright}(0) :$	$\begin{pmatrix} 1 & s_* \\ \blacktriangleright_2 & \end{pmatrix} \mapsto \begin{pmatrix} 1 & s_* \\ \blacktriangleright_1 & \end{pmatrix}$
	$\begin{pmatrix} 2 & s_* \\ \blacktriangleright_2 & \end{pmatrix} \mapsto \begin{pmatrix} 1 & s_* \\ \blacktriangleright_2 & \end{pmatrix}$
	$\begin{pmatrix} 1 & s_* \\ \blacktriangleright_1 & \end{pmatrix} \mapsto \begin{pmatrix} 2 & s_* \\ \blacktriangleright_1 & \end{pmatrix}$
	$\begin{pmatrix} 2 & s_* \\ \blacktriangleright_1 & \end{pmatrix} \mapsto \begin{pmatrix} 2 & s_* \\ \blacktriangleright_2 & \end{pmatrix}$
Special \blacktriangleleft_1 higher level:	$\begin{pmatrix} s_* & 0 \\ \blacktriangleleft_1 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 0 \\ \blacktriangleleft_1 & \end{pmatrix}$
Special \blacktriangleleft_2 higher level:	$\begin{pmatrix} s_* & 0 \\ \blacktriangleleft_2 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 0 \\ \blacktriangleleft_2 & \end{pmatrix}$
$M_{\blacktriangleleft}(0) :$	$\begin{pmatrix} s_* & 1 \\ \blacktriangleleft_2 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 1 \\ \blacktriangleleft_1 & \end{pmatrix}$
	$\begin{pmatrix} s_* & 2 \\ \blacktriangleleft_2 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 1 \\ \blacktriangleleft_2 & \end{pmatrix}$
	$\begin{pmatrix} s_* & 1 \\ \blacktriangleleft_1 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 2 \\ \blacktriangleleft_1 & \end{pmatrix}$
	$\begin{pmatrix} s_* & 2 \\ \blacktriangleleft_1 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 2 \\ \blacktriangleleft_2 & \end{pmatrix}$
Special \triangleright_1 higher level:	$\begin{pmatrix} s_* & 0 \\ \triangleright_1 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 0 \\ \triangleright_1 & \end{pmatrix}$
Special \triangleright_2 higher level:	$\begin{pmatrix} s_* & 0 \\ \triangleright_2 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 0 \\ \triangleright_2 & \end{pmatrix}$
$M_{\triangleright}(0) :$	$\begin{pmatrix} s_* & 1 \\ \triangleright_2 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 1 \\ \triangleright_1 & \end{pmatrix}$
	$\begin{pmatrix} s_* & 2 \\ \triangleright_2 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 1 \\ \triangleright_2 & \end{pmatrix}$
	$\begin{pmatrix} s_* & 1 \\ \triangleright_1 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 2 \\ \triangleright_1 & \end{pmatrix}$
	$\begin{pmatrix} s_* & 2 \\ \triangleright_1 & \end{pmatrix} \mapsto \begin{pmatrix} s_* & 2 \\ \triangleright_2 & \end{pmatrix}$
Special \triangleleft_1 higher level:	$\begin{pmatrix} 0 & s_* \\ \triangleleft_1 & \end{pmatrix} \mapsto \begin{pmatrix} 0 & s_* \\ \triangleleft_1 & \end{pmatrix}$
Special \triangleleft_2 higher level:	$\begin{pmatrix} 0 & s_* \\ \triangleleft_2 & \end{pmatrix} \mapsto \begin{pmatrix} 0 & s_* \\ \triangleleft_2 & \end{pmatrix}$
$M_{\triangleleft}(0) :$	$\begin{pmatrix} 1 & s_* \\ \triangleleft_2 & \end{pmatrix} \mapsto \begin{pmatrix} 1 & s_* \\ \triangleleft_1 & \end{pmatrix}$
	$\begin{pmatrix} 2 & s_* \\ \triangleleft_2 & \end{pmatrix} \mapsto \begin{pmatrix} 1 & s_* \\ \triangleleft_2 & \end{pmatrix}$
	$\begin{pmatrix} 1 & s_* \\ \triangleleft_1 & \end{pmatrix} \mapsto \begin{pmatrix} 2 & s_* \\ \triangleleft_1 & \end{pmatrix}$
	$\begin{pmatrix} 2 & s_* \\ \triangleleft_1 & \end{pmatrix} \mapsto \begin{pmatrix} 2 & s_* \\ \triangleleft_2 & \end{pmatrix}$

Figure 2: Encoding SMART configurations: $\Phi_{\text{init}} : c \mapsto c_0$

From a cyclic SMART configuration, compute its encoding of level 0.
Rewriting words of length 2.

From $M_{\blacktriangleright}(k)$ to $(k + 1)$ -level encoding:

$$\begin{array}{ll}
M_{\blacktriangleright}(k+1)(\text{if } v_3(c) \neq 0) : & \left(\begin{array}{cccc} * & c & 0 & * \\ & \blacktriangleright_1 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{ccc} * & 0 & c \\ & \blacktriangleright_1 & * \end{array} \right) \\
& \left(\begin{array}{cccc} * & c & 0 & * \\ & \blacktriangleright_2 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{ccc} * & 0 & c \\ & \blacktriangleright_2 & * \end{array} \right) \\
M_{\blacktriangleleft}(k+1)(\text{if } v_3(c) \neq 0) : & \left(\begin{array}{cccc} * & c & 1 & * \\ & \blacktriangleright_1 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{c} * [v_3(c) + f(k) + 1]_{(3)} \\ \blacktriangleleft_1 \\ * \end{array} \right) \\
& \left(\begin{array}{cccc} * & c & 2 & * \\ & \blacktriangleright_1 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{c} * [v_3(c) + f(k) + 1]_{(3)} \\ \blacktriangleleft_2 \\ * \end{array} \right) \\
M_{\triangleright}(k+1)(\text{if } v_3(c) \neq 0) : & \left(\begin{array}{cccc} * & c & 1 & * \\ & \blacktriangleright_2 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{c} * [v_3(c) + 2]_{(3)} \\ \triangleright_1 \\ * \end{array} \right) \\
& \left(\begin{array}{cccc} * & c & 2 & * \\ & \blacktriangleright_2 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{c} * [v_3(c) + 2]_{(3)} \\ \triangleright_2 \\ * \end{array} \right)
\end{array}$$

From $M_{\blacktriangleleft}(k)$ to $(k + 1)$ -level encoding:

$$\begin{aligned}
M_{\blacktriangleleft}(k+1)(\text{if } v_3(c) \neq 0) : & \quad \left(\begin{array}{cccc} * & 0 & c & * \\ & \blacktriangle_1 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{ccc} * & 0 & c \\ \blacktriangle_1 & & * \end{array} \right) \\
& \quad \left(\begin{array}{cccc} * & 0 & c & * \\ & \blacktriangle_2 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{ccc} * & 0 & c \\ \blacktriangle_2 & & * \end{array} \right) \\
M_{\blacktriangleright}(k+1)(\text{if } v_3(c) \neq 0) : & \quad \left(\begin{array}{cccc} * & 1 & c & * \\ & \blacktriangle_1 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{c} * [v_3(c) + f(k) + 1]_{(3)} \\ \blacktriangleright_1 \end{array} \right) \\
& \quad \left(\begin{array}{cccc} * & 2 & c & * \\ & \blacktriangle_1 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{c} * [v_3(c) + f(k) + 1]_{(3)} \\ \blacktriangleright_2 \end{array} \right) \\
M_{\triangleleft}(k+1)(\text{if } v_3(c) \neq 0) : & \quad \left(\begin{array}{cccc} * & 1 & c & * \\ & \blacktriangle_2 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{c} * [v_3(c) + 2]_{(3)} \\ \triangleleft_1 \end{array} \right) \\
& \quad \left(\begin{array}{cccc} * & 2 & c & * \\ & \blacktriangle_2 & & \\ \end{array} \right) \rightarrow \left(\begin{array}{c} * [v_3(c) + 2]_{(3)} \\ \triangleleft_2 \end{array} \right)
\end{aligned}$$

From $M_{\triangleright}(k)$ to $(k + 1)$ -level encoding:

$$\begin{array}{ll}
M_{\triangleright}(k+1)(\text{if } v_3(c) \neq 0) : & \left(\begin{array}{cccc} * & 0 & c & * \\ & \triangleright_1 & & \end{array} \right) \rightarrow \left(\begin{array}{c} * \\ \triangleright_1 \\ [v_3(c) + 2f(k) + 4]_{(3)} * \end{array} \right) \\
& \left(\begin{array}{cccc} * & 0 & c & * \\ & \triangleright_2 & & \end{array} \right) \rightarrow \left(\begin{array}{c} * \\ \triangleright_2 \\ [v_3(c) + 2f(k) + 4]_{(3)} * \end{array} \right) \\
M_{\blacktriangleright}(k+1)(\text{if } 0 \ c \neq 0) : & \left(\begin{array}{cccc} * & 1 & c & * \\ & \triangleright_1 & & \end{array} \right) \rightarrow \left(\begin{array}{c} * \\ \triangleright_1 \\ [v_3(c) + 2f(k) + 2]_{(3)} * \end{array} \right)_{\blacktriangleright_1} \\
& \left(\begin{array}{cccc} * & 2 & c & * \\ & \triangleright_1 & & \end{array} \right) \rightarrow \left(\begin{array}{c} * \\ \triangleright_1 \\ [v_3(c) + 2f(k) + 2]_{(3)} * \end{array} \right)_{\blacktriangleright_2} \\
M_{\triangleleft}(k+1)(\text{if } v_3(c) \neq 0) : & \left(\begin{array}{cccc} * & 1 & c & * \\ & \triangleright_2 & & \end{array} \right) \rightarrow \left(\begin{array}{c} * \\ \triangleright_2 \\ [v_3(c) + f(k) + 3]_{(3)} * \end{array} \right)_{\triangleleft_1} \\
& \left(\begin{array}{cccc} * & 2 & c & * \\ & \triangleright_2 & & \end{array} \right) \rightarrow \left(\begin{array}{c} * \\ \triangleright_2 \\ [v_3(c) + f(k) + 3]_{(3)} * \end{array} \right)_{\triangleleft_2}
\end{array}$$

From $M_{\leq}(k)$ to $(k + 1)$ -level encoding:

$$\begin{aligned}
M_{\triangleleft}(k+1)(\text{if } v_3(c) \neq 0) : & \quad \left(\begin{array}{ccccc} * & c & 0 & * \\ & & \triangleleft_1 & \\ \left(\begin{array}{ccccc} * & c & 0 & * \\ & & \triangleleft_2 & \end{array} \right) & \rightarrow & \left(\begin{array}{ccccc} * & [v_3(c) + 2f(k) + 4]_{(3)} & * \\ & & \triangleleft_1 & \end{array} \right) \\
& \rightarrow & \left(\begin{array}{ccccc} * & [v_3(c) + 2f(k) + 4]_{(3)} & * \\ & & \triangleleft_2 & \end{array} \right) \\
M_{\blacktriangleleft}(k+1)(\text{if } v_3(c) \neq 0) : & \quad \left(\begin{array}{ccccc} * & c & 1 & * \\ & & \triangleleft_1 & \end{array} \right) \rightarrow \left(\begin{array}{ccccc} * & [v_3(c) + 2f(k) + 2]_{(3)} & * \\ & \blacktriangleleft_1 & \end{array} \right) \\
& \left(\begin{array}{ccccc} * & c & 2 & * \\ & & \triangleleft_1 & \end{array} \right) \rightarrow \left(\begin{array}{ccccc} * & [v_3(c) + 2f(k) + 2]_{(3)} & * \\ & \blacktriangleleft_2 & \end{array} \right) \\
M_{\triangleright}(k+1)(\text{if } v_3(c) \neq 0) : & \quad \left(\begin{array}{ccccc} * & c & 1 & * \\ & & \triangleleft_2 & \end{array} \right) \rightarrow \left(\begin{array}{ccccc} * & [v_3(c) + f(k) + 3]_{(3)} & * \\ & \triangleright_1 & \end{array} \right) \\
& \left(\begin{array}{ccccc} * & c & 2 & * \\ & & \triangleleft_2 & \end{array} \right) \rightarrow \left(\begin{array}{ccccc} * & [v_3(c) + f(k) + 3]_{(3)} & * \\ & \triangleright_2 & \end{array} \right)
\end{aligned}$$

Figure 3: Encoding SMART configurations: $\Phi_{k \mapsto k+1}$ (Part 1: $k \mapsto k+1$)

From an encoding of level k of a configuration, compute its $(k + 1)$ -encoding, following Figure 1. Cells with $*$ are unmodified, their point is to describe the head movement. We are rewriting words of length $k + 4$ (including the two gray symbols). Note that at $k + 1 = \ell - 2$ (resp. $k + 1 = \ell - 1$), the $*$ -cells overlap or each other (resp. the counter), because we reach the length of the cyclic tape.

Encoding special $M_{\blacktriangleright}(k+1)$:					
Special \blacktriangleright_1 :	$(^* \ 1 \ 0^{k+1} * \ \blacktriangleright_1)$	\rightarrow	$(^* \ [f(k+1)+1]_{(3)} \ ^* \ \blacktriangleright_1)$		
	$(^* \ 2 \ 0^{k+1} * \ \blacktriangleright_1)$	\rightarrow	$(^* \ [f(k+1)+1]_{(3)} \ ^* \ \blacktriangleright_2)$		
Special \blacktriangleright_2 :	$(^* \ 1 \ 0^{k+1} * \ \blacktriangleright_2)$	\rightarrow	$(^* \ [f(k+1)]_{(3)} \ ^* \ \blacktriangleright_1)$		
	$(^* \ 2 \ 0^{k+1} * \ \blacktriangleright_2)$	\rightarrow	$(^* \ [f(k+1)]_{(3)} \ ^* \ \blacktriangleright_2)$		
Special \blacktriangleright_1 higher level :	$(^* \ 0 \ 0^{k+1} * \ \blacktriangleright_1)$	\mapsto	$(^* \ 0^{k+2} \ ^* \ \blacktriangleright_1)$		
Special \blacktriangleright_2 higher level :	$(^* \ 0 \ 0^{k+1} * \ \blacktriangleright_2)$	\mapsto	$(^* \ 0^{k+2} \ ^* \ \blacktriangleright_2)$		
Encoding special $M_{\blacktriangleleft}(k+1)$:					
Special \blacktriangleleft_1 :	$(^* \ 0^{k+1} 1 \ ^* \ \blacktriangleleft_1)$	\rightarrow	$(^* \ [f(k+1)+1]_{(3)} \ ^* \ \blacktriangleleft_1)$		
	$(^* \ 0^{k+1} 2 \ ^* \ \blacktriangleleft_1)$	\rightarrow	$(^* \ [f(k+1)+1]_{(3)} \ ^* \ \blacktriangleleft_2)$		
Special \blacktriangleleft_2 :	$(^* \ 0^{k+1} 1 \ ^* \ \blacktriangleleft_2)$	\rightarrow	$(^* \ [f(k+1)]_{(3)} \ ^* \ \blacktriangleleft_1)$		
	$(^* \ 0^{k+1} 2 \ ^* \ \blacktriangleleft_2)$	\rightarrow	$(^* \ [f(k+1)]_{(3)} \ ^* \ \blacktriangleleft_2)$		
Special \blacktriangleleft_1 higher level :	$(^* \ 0^{k+1} 0 \ ^* \ \blacktriangleleft_1)$	\mapsto	$(^* \ 0^{k+2} \ ^* \ \blacktriangleleft_1)$		
Special \blacktriangleleft_2 higher level :	$(^* \ 0^{k+1} 0 \ ^* \ \blacktriangleleft_2)$	\mapsto	$(^* \ 0^{k+2} \ ^* \ \blacktriangleleft_2)$		
Encoding special $M_{\triangleright}(k+1)$:					
Special \triangleright_1 :	$(^* \ 0^{k+1} 1 \ ^* \ \triangleright_1)$	\rightarrow	$(^* \ [2]_{(3)} \ ^* \ \triangleright_1)$		
	$(^* \ 0^{k+1} 2 \ ^* \ \triangleright_1)$	\rightarrow	$(^* \ [2]_{(3)} \ ^* \ \triangleright_2)$		
Special \triangleright_2 :	$(^* \ 0^{k+1} 1 \ ^* \ \triangleright_2)$	\rightarrow	$(^* \ [1]_{(3)} \ ^* \ \triangleright_1)$		
	$(^* \ 0^{k+1} 2 \ ^* \ \triangleright_2)$	\rightarrow	$(^* \ [1]_{(3)} \ ^* \ \triangleright_2)$		
Special \triangleright_1 higher level :	$(^* \ 0^{k+1} 0 \ ^* \ \triangleright_1)$	\mapsto	$(^* \ 0^{k+2} \ ^* \ \triangleright_1)$		
Special \triangleright_2 higher level :	$(^* \ 0^{k+1} 0 \ ^* \ \triangleright_2)$	\mapsto	$(^* \ 0^{k+2} \ ^* \ \triangleright_2)$		
Encoding special $M_{\triangleleft}(k+1)$:					
Special \triangleleft_1 :	$(^* \ 1 \ 0^{k+1} * \ \triangleleft_1)$	\rightarrow	$(^* \ [2]_{(3)} \ ^* \ \triangleleft_1)$		
	$(^* \ 2 \ 0^{k+1} * \ \triangleleft_1)$	\rightarrow	$(^* \ [2]_{(3)} \ ^* \ \triangleleft_2)$		
Special \triangleleft_2 :	$(^* \ 1 \ 0^{k+1} * \ \triangleleft_2)$	\rightarrow	$(^* \ [1]_{(3)} \ ^* \ \triangleleft_1)$		
	$(^* \ 2 \ 0^{k+1} * \ \triangleleft_2)$	\rightarrow	$(^* \ [1]_{(3)} \ ^* \ \triangleleft_2)$		
Special \triangleleft_1 higher level :	$(^* \ 0 \ 0^{k+1} * \ \triangleleft_1)$	\mapsto	$(^* \ 0^{k+2} \ ^* \ \triangleleft_1)$		
Special \triangleleft_2 higher level :	$(^* \ 0 \ 0^{k+1} * \ \triangleleft_2)$	\mapsto	$(^* \ 0^{k+2} \ ^* \ \triangleleft_2)$		

Figure 4: Encoding SMART configurations: $\Phi_{k \mapsto k+1}$ (Part 2: special $\mapsto k+1$)

Encode the special configurations of level $k+1$ into their $(k+1)$ -encodings (see the proof of Proposition 3.2 for their correct orbit positions), and preserve the special configurations of level $> k+1$. Rewriting words of length $k+4$.

$\begin{pmatrix} 0 & 0^{\ell-2} & 0 \\ \blacktriangleright_1 & & \\ 0 & 0^{\ell-2} & 0 \\ \blacktriangleright_2 & & \end{pmatrix}$	\rightarrow	$\begin{pmatrix} 0 & 0^{\ell-2} & 0 \\ \blacktriangleright_1 & & \\ 2 & 2^{\ell-2} & 2 \\ \blacktriangleright_2 & & \end{pmatrix}$
$\begin{pmatrix} c_0 & \cdots & c_{\ell-1} \\ \blacktriangleright_1 & & \\ c_0 & \cdots & c_{\ell-1} \\ \blacktriangleright_2 & & \end{pmatrix}$	\rightarrow	$\begin{pmatrix} c & & \\ \blacktriangleleft_1 & & \\ c & & \\ \triangleright_1 & & \end{pmatrix}$
$\begin{pmatrix} 0 & 0^{\ell-2} & 0 \\ \blacktriangleleft_1 & & \\ 0 & 0^{\ell-2} & 0 \\ \blacktriangleleft_2 & & \end{pmatrix}$	\rightarrow	$\begin{pmatrix} 0 & 0^{\ell-2} & 0 \\ \blacktriangleleft_1 & & \\ 2 & 2^{\ell-2} & 2 \\ \blacktriangleleft_2 & & \end{pmatrix}$
$\begin{pmatrix} c_{\ell-1} & c_0 & \cdots & c_{\ell-2} \\ \blacktriangleleft_1 & & & \\ c_{\ell-1} & c_0 & \cdots & c_{\ell-2} \\ \blacktriangleleft_2 & & & \end{pmatrix}$	\rightarrow	$\begin{pmatrix} c & & \\ \blacktriangleright_1 & & \\ c & & \\ \triangleleft_1 & & \end{pmatrix}$
$\begin{pmatrix} 0 & 0^{\ell-2} & 0 \\ \triangleright_1 & & \\ 0 & 0^{\ell-2} & 0 \\ \triangleright_2 & & \end{pmatrix}$	\rightarrow	$\begin{pmatrix} 0 & 0^{\ell-2} & 0 \\ \triangleright_1 & & \\ 2 & 2^{\ell-2} & 2 \\ \triangleright_2 & & \end{pmatrix}$
$\begin{pmatrix} c_{\ell-1} & c_0 & \cdots & c_{\ell-2} \\ \triangleright_1 & & & \\ c_{\ell-1} & c_0 & \cdots & c_{\ell-2} \\ \triangleright_2 & & & \end{pmatrix}$	\rightarrow	$\begin{pmatrix} [v_3(c) - 1]_{(3)} & & \\ \blacktriangleright_2 & & \\ [v_3(c) - 1]_{(3)} & & \\ \triangleleft_2 & & \end{pmatrix}$
$\begin{pmatrix} 0 & 0^{\ell-2} & 0 \\ \triangleleft_1 & & \\ 0 & 0^{\ell-2} & 0 \\ \triangleleft_2 & & \end{pmatrix}$	\rightarrow	$\begin{pmatrix} 0 & 0^{\ell-2} & 0 \\ \triangleleft_1 & & \\ 2 & 2^{\ell-2} & 2 \\ \triangleleft_2 & & \end{pmatrix}$
$\begin{pmatrix} c_0 & \cdots & c_{\ell-1} \\ \triangleleft_1 & & \\ c_0 & \cdots & c_{\ell-1} \\ \triangleleft_2 & & \end{pmatrix}$	\rightarrow	$\begin{pmatrix} [v_3(c) - 1]_{(3)} & & \\ \blacktriangleleft_2 & & \\ [v_3(c) - 1]_{(3)} & & \\ \triangleright_2 & & \end{pmatrix}$

Figure 5: Final encoding step of SMART configurations: $\Phi_{\ell,\text{final}} : c_{\ell-1} \mapsto c_*$

This transformation refers directly to the proof of Proposition 3.3, and maps encodings of level $\ell - 1$ to their positions in the orbits of C_q for $q \in \{\blacktriangleright, \blacktriangleleft, \triangleright, \triangleleft\}$. It also ‘corrects’ the position the head and shifts the counter in the encodings of $M_{\blacktriangleleft}(\ell - 1)$ and $M_{\triangleright}(\ell - 1)$, and shifts heads in full-zero configurations whose heads were moved during the encoding of level 0. Rewriting words of length ℓ .

If $q = \triangleright$: $w'(k) = (q', w(k)_0) \cdot c_0 \dots c_k$, where $c \in \{0, 1, 2\}^{k+1}$ is a ternary counter with $v_3(c) = j + 1$, and $q' = \triangleright_1$ if $w(k)_{k+1} = 1$ (resp. $q' = \triangleright_2$ if $w(k)_{k+1} = 2$).

If $q = \triangleleft$: $w'(k) = c_0 \dots c_k \cdot (q', w(n)_{k+1})$, where $c \in \{0, 1, 2\}^{k+1}$ is a ternary counter with $v_3(c) = j + 1$, and $q' = \triangleleft_1$ if $w(k)_0 = 1$ (resp. $q' = \triangleleft_2$ if $w(k)_0 = 2$).

Words with an all-zero counter value, i.e. $(0^{k+1} \cdot (\blacktriangleright_1, w_{k+1}))$, $(0^{k+1} \cdot (\blacktriangleright_2, w_{k+1}))$, $((\blacktriangleleft_1, w_0) \cdot 0^{k+1})$, $((\blacktriangleleft_2, w_0) \cdot 0^{k+1})$ are not used in this encoding. Conveniently, they are exactly the special patterns of levels strictly greater than k .

Once we have specified the encodings at each level, we can define a sequence of transformations implicitly, by simply saying that we decode the level- k encoding and recode it on level $k + 1$; composing these for all values of k , we get the encoding of level ℓ , which one may easily check is simply Φ . In Figures 2, 3, 4 and 5, we perform the somewhat tedious exercise of describing these transformations in detail, as we need to know the form of these transformations in order to implement them with permutations later. Not surprisingly, it turns out that they require only ternary addition and basic rewriting of symbols.

Define Φ_{init} the bijective transformation given in Figure 2, $\Phi_{k \mapsto k+1}$ the bijective transformation of Figures 3 and 4, and $\Phi_{\ell, \text{final}}$ the bijection of Figure 5. Given a cyclic configuration c of length ℓ , denote $c_0 = \Phi_{\text{init}}(c)$ its encoding of level 0, $c_{k+1} = \Phi_{k \mapsto k+1}(c_k)$ for $k \leq \ell - 2$, and $c_* = \Phi_{\ell, \text{final}}(c_{\ell-1})$.

Lemma 3.6. *Let c be a cyclic configuration of length ℓ . Then $c_* = \Phi(c)$.*

Proof. By induction, one sees from Figure 1 that c_k is the encoding of level k of c , for $k \leq \ell - 1$. We then conclude that $c_* = \Phi(c)$ by the proof of Proposition 3.3 by comparing the formulas, and recalling the structure of the orbits of $C_\blacktriangleright, C_\blacktriangleleft, C_\triangleright, C_\triangleleft$ (to check that we shift in the correct direction when the counter overflows). \square

4 Finitary distortion for SMART

This chapter introduces a slightly altered version of $T_{\ell, S}$ (called the *decorated SMART*) acting on the cyclic tapes of some $C_{\ell, Q_{\text{dec}}, \Gamma}$, and establishes Lemma 4.2: this automorphism is “distorted” in the corresponding $G_{\ell, Q_{\text{dec}}, \Gamma}$, in the sense that all its powers (including ones exponential in ℓ) have word norm polynomial in ℓ under the fixed generators (which is exponentially lower than the order of the group would suggest).

4.1 Turing machines on finite cyclic tapes

Recall that $C_{\ell, Q, \Gamma}$ is the set of finite cyclic tapes of length ℓ (see Section 2), with states Q and tape-alphabet Γ , containing at most one head (i.e. a letter in $Q \times \Gamma$).

Let $G_{\ell, Q, \Gamma}$ be the finitely generated subgroup of $\text{Sym}(C_{\ell, Q, \Gamma})$ generated by state-dependent moves, and the unary gates permuting the contents of cells containing heads. Formally, for $g \in \text{Sym}(Q \times \Gamma)$, define the *unary gate* $\pi_g \in$

$\text{Sym}(C_{\ell,Q,\Gamma})$ as

$$\pi_g(w)_j = \begin{cases} w_j & \text{if } w_j \in \Gamma \\ g(w_j) & \text{if } w_j \in (Q \times \Gamma) \end{cases}$$

and for $q \in Q$ the *state-dependent (right) move* $\rho_q \in \text{Sym}(C_{\ell,Q,\Gamma})$ as:

$$\rho_q(w)_j = \begin{cases} \pi_\Gamma(w_j) & \text{if } w_{j-1} \bmod \ell \notin (\{q\} \times \Gamma) \wedge w_j \notin ((Q \setminus \{q\}) \times \Gamma) \\ w_j & \text{if } w_j \in ((Q \setminus \{q\}) \times \Gamma) \\ (q, \pi_\Gamma(w_j)) & \text{if } w_{j-1} \bmod \ell \in (\{q\} \times \Gamma) \end{cases}$$

Then we define:

$$G_{\ell,Q,\Gamma} = \langle \{\pi_g : g \in \text{Sym}(Q \times \Gamma)\} \cup \{\rho_q : q \in Q\} \rangle$$

We can see the group $G_{\ell,Q,\Gamma}$ as the group generated by the instructions of Turing-machines: moving heads based on their states, or permuting their values. To ease notations, we denote $\prod_{q \in Q} \rho_q$ by ρ .

It is easy to see that for any reversible Turing machine \mathcal{M} , $T_{\ell,\mathcal{M}}$ is an element of $G_{\ell,Q,\Gamma}$. Indeed, a step of computation is the composition of a head permutation α of $Q \times \Gamma$, followed with state-dependent moves β_{+1} and β_{-1} :

$$\begin{aligned} \alpha(q, a) &= \begin{cases} (q', b) & \text{if } (q, a, q', b) \in \Delta \\ (q', a) & \text{if } (q, \pm 1, q') \in \Delta \end{cases} \\ \beta_{+1} &= \prod_{q' \mid \exists q, (q, +1, q') \in \Delta} \rho_{q'} \\ \beta_{-1} &= \prod_{q' \mid \exists q, (q, -1, q') \in \Delta} \rho_{q'}^{-1} \end{aligned}$$

We denote by $\delta(\ell, n)$ the word norm of $(T_{\ell,\mathcal{M}})^n$ in $G_{\ell,Q,\Gamma}$.

Remark 4.1. *It can be shown that $G_{\ell,Q,\Gamma}$ is, for large enough ℓ , $|Q|$ and $|\Gamma|$ (in particular for all versions of the SMART machine we consider and for $\ell \geq 2$), of finite index in the automorphism group of $C_{\ell,Q,\Gamma}$ under the shift action of \mathbb{Z}_n . This is not particularly useful, however, as what we need it for is to provide a group where the SMART machine corresponds to an element of small radius (far smaller than the radius of the group).*

4.2 Decorated SMART on $C_{\ell,Q_{\text{dec}},\Gamma}$

Let $\mathcal{S} = (Q, \Gamma, \Delta)$ be the SMART machine (see Section 3). Define the *decorated version* of the SMART machine as $\mathcal{S}_{\text{dec}} = (Q_{\text{dec}}, \Gamma, \Delta_{\text{dec}})$, with

$$\begin{aligned} Q_{\text{dec}} &= Q \times D \times G \\ \Delta_{\text{dec}} &= \bigcup_{(d,x) \in D \times G} \left\{ \left((q, d, x), a, (q', d, x), b \right) : (q, a, q', b) \in \Delta \right\} \\ &\quad \cup \bigcup_{(d,x) \in D \times G} \left\{ \left((q, d, x), \delta, (q', d, x) \right) : (q, \delta, q') \in \Delta \right\} \end{aligned}$$

for $D = \{d_1, d_2\}$ and $G = \llbracket 0, 5 \rrbracket \times \Gamma$, i.e. the states of Q_{dec} now carry a state $q \in Q$ of the original machine \mathcal{S} , a special symbol $d \in D$ called the *duck*, and a *ghost symbol* $x \in G$. We have $|Q_{\text{dec}}| = 288$.

The decorated SMART machine \mathcal{S}_{dec} behaves exactly like \mathcal{S} , in the sense that it ignores the new decorations. While the duck component D and ghost component G are completely ignored, they will be used by the generators we use to build powers of the machine efficiently. The point of the *hex component* $\llbracket 0, 5 \rrbracket$ in the ghost is to allow us to condition the application of gates, and to build the permutations we perform in Section 4.3. The *transport component* Γ in the ghost allows us to carry letters around. The duck will be important during intermediate steps of computation in Section 4.3, in order to realize piecewise defined functions.

Recall that $m : \mathbb{N} \mapsto \mathbb{N}$ is the *movement function*, i.e. $m(n)$ is the maximal number of cells the machine S_{dec} can visit in n steps; and that $\delta(\ell, n)$ is the word norm of $(T_{\ell, \mathcal{S}_{\text{dec}}})^n$ in $G_{\ell, Q_{\text{dec}}, \Gamma}$.

Lemma 4.2. *Let \mathcal{S}_{dec} be the decorated version of SMART introduced above:*

1. $T_{\mathcal{S}_{\text{dec}}}$ has infinite order.
2. There exist some $C, C' > 0$ such that $m(n) \leq C \log n + C'$.
3. There exists some $d > 0$ such that $\delta(\ell, n) = O(\ell^d)$.

In fact, for \mathcal{S}_{dec} , we can take $C = \ln(2)/\ln(3)$ and $d = 4$.

Any finite order T satisfies the latter two items, and any non-trivial state-dependent shift satisfies the first and the third items. Achieving the first two items is already difficult, and to our knowledge these properties have only been explicitly shown for the SMART machine and the binary SMART machine [14]. We expect that the Kari-Ollinger construction in [28] can be used to produce more examples of machines satisfying these two properties (at least $m(n) = O(n/\log n)$ follows from general principles for all these machines [24]).

The second item is proved with the following computation: after less than 18 steps, the head of SMART is in state \triangleright_1 or \triangleleft_1 reading a 0 (ignoring the ghost and the duck). Then SMART is either at the left (for \triangleright_1) or right (for \triangleleft_1) extremity of some word 0^m for some $m \geq \log_3(k) + 2$, or by [13, Lemma 4] it builds around this position some pattern in the set C_m for $m \geq \log_3(k) + 2$ (with the notations of [13, Lemma 4]). Either way, after this point, k steps of computations cannot read more than $\log_3(k) + 2$ different cells. We omit a detailed proof, as the logarithmic speed of SMART is well known.

The proof of the third item is a matter of programming powers of the machine efficiently with primitive reversible operations. The cyclic configurations acted on by SMART form four disjoint cycles, and the overall idea is to turn a configuration into a direct encoding of its position in the cycle (in ternary notation), using the formulas from Section 3.3. This reduces computing powers of SMART to performing ternary additions. The two following sections are dedicated to the proof of this third item.

4.3 Permutation engineering in $C_{\ell, Q_{\text{dec}}, \Gamma}$

Let $g \in \text{Sym}(Q_{\text{dec}} \times \Gamma)$ be a permutation, and $C \subseteq (Q_{\text{dec}} \times \Gamma) \times \Gamma^{\ell-1}$ a subset called a *condition*. Define $\pi_{g,C} : C_{\ell, Q_{\text{dec}}, \Gamma} \mapsto C_{\ell, Q_{\text{dec}}, \Gamma}$ as $\pi_{g,C}(w) = w$ if there is no head in w , and if $w_i \in Q_{\text{dec}} \times \Gamma$, then:

$$\pi_{g,C}(w)_j = \begin{cases} w_j & \text{if } j \neq i \\ g(w_j) & \text{if } j = i \text{ and } w_j \cdot w_{j+1} \dots w_{n-1} \cdot w_0 \dots w_{j-1} \in C \\ w_j & \text{if } j = i \text{ and } w_j \cdot w_{j+1} \dots w_{n-1} \cdot w_0 \dots w_{j-1} \notin C \end{cases}$$

We call $\pi_{g,C}$ the *conditional application of g under condition C* .

We say C is *g -invariant* if $w_j \dots w_{n-1} \cdot w_0 \dots w_{j-1} \in C$ if and only if $g(w_j) \cdot w_{j+1} \dots w_{n-1} \cdot w_0 \dots w_{j-1} \in C$. If C is g -invariant, then $\pi_{g,C}$ is a bijection. We say that C is *ghost-invariant* if there exists $C' \subseteq ((Q \times D) \times \Gamma) \times \Gamma^{\ell-1}$ such that $C = \pi^{-1}(C')$ where $\pi : ((Q \times D) \times \Gamma) \times \Gamma^{\ell-1} \rightarrow ((Q \times D) \times \Gamma) \times \Gamma^{\ell-1}$ is the natural projection. Equivalently, it is g -invariant for all permutations $g \in \text{Sym}(G)$, seen as permutations of the G -component of the head.

We say that a head-permutation $g \in \text{Sym}(Q_{\text{dec}} \times \Gamma)$ is *ghost-ignorant* if it factors through the projection that forgets the ghost symbol. In other words, for any choice of $h \in Q_{\text{dec}} \times \Gamma$, w and $g(w)$ carry the same ghost symbol if w has the state-symbol pair h at the head position, and furthermore the ghost symbol does not affect how the other symbols change.

4.3.1 Permutation conditioning

We now prove that if $g \in \text{Alt}(Q_{\text{dec}} \times \Gamma)$ is ghost-ignorant and C is a condition that is both ghost-ignorant and g -invariant, then $\pi_{g,C} \in G_{\ell, Q_{\text{dec}}, \Gamma}$. We also provide upper bounds on its word norm depending on C . We begin with what is essentially Barrington's theorem [3].

As a first step, we consider permutations that are the opposite of ghost-ignorant, i.e. only touch the ghost-component G of the head. Let $g \in \text{Alt}(G)$, considered as a subgroup of $\text{Alt}(Q_{\text{dec}} \times \Gamma)$ only permuting the ghost-information in Q_{dec} , and C a ghost-ignorant subset of $(Q_{\text{dec}} \times \Gamma) \times \Gamma^{\ell-1}$. Then note that $\pi_{g,C}$ belongs in $G_{\ell, Q_{\text{dec}}, \Gamma}$.

Lemma 4.3. Denote $T : \mathcal{GI} \rightarrow \mathbb{N}$ be the optimal function such that $\|\pi_{g,C}\| \leq T(C)$ for all $g \in \text{Alt}(G)$ (considered as a subgroup of $\text{Sym}(Q_{\text{dec}} \times \Gamma)$), where $\mathcal{GI} \subset \mathcal{P}((Q_{\text{dec}} \times \Gamma) \times \Gamma^{\ell-1})$ is the set of all ghost-invariant sets. Then T satisfies the following inequalities:

$$\begin{aligned} T([a]_j) &\leq |\min(j, \ell - j)| \\ T(C \cap C') &\leq 2(T(C) + T(C')) \\ T(C \cup C') &\leq \begin{cases} T(C) + T(C') & \text{if } C \cap C' = \emptyset \\ 2(T(C) + T(C')) + 5 & \text{otherwise} \end{cases} \\ T(C^c) &\leq T(C) + 1 \end{aligned}$$

Proof. We prove by induction over C that every $\pi_{g,C}$, for $g \in \text{Alt}(G)$ (considered as a subgroup of $\text{Sym}(Q_{\text{dec}} \times \Gamma)$), has word norm that checks the aforementioned inequalities.

Case 1. If $C = C' \times \Gamma^{\ell-1}$ for some $C' \subseteq (Q \times D \times G)$ that is ghost-invariant, then any such $\pi_{g,C}$ already appears in the set of generators of $G_{\ell,Q_{\text{dec}},\Gamma}$.

Case 2. If $C = [x]_j$ for some $j \in \llbracket -\ell, \ell \rrbracket$, $j \neq 0$ and $x \in \Gamma$, define $C'' = (Q \times D \times G) \times \{x\}$ and $C' = C'' \times \Gamma^{\ell-1}$. Then one can conjugate $\pi_{g,C'}$ (which belongs to $G_{\ell,Q_{\text{dec}},\Gamma}$ by the first item) with ρ^j : the resulting permutation applies g on the ghost symbol if and only if j cells away from the head, the content of the tape is x .

Case 3. If $C = C_1^c$, then $\pi_{g,C_1^c} = \pi_{g^{-1},C_1} \circ \pi_g$.

Case 4. If $C = C_1 \cap C_2$. We use the “commutator trick”. As G has cardinality greater than 5, g is a commutator by Ore’s theorem [38, Theorem 7]: there exist g_1, g_2 such that $g = [g_1, g_2]$. By the induction hypothesis and a straightforward calculation, we conclude that:

$$\pi_{g,C_1 \cap C_2} = [\pi_{g_1,C_1}, \pi_{g_2,C_2}]$$

Case 5. If $C = C_1 \cup C_2$, with $C_1 \cap C_2 = \emptyset$. Then $\pi_{g,C_1 \cup C_2} = \pi_{g,C_1} \circ \pi_{g,C_2}$.

Case 6. If $C = C_1 \cup C_2$, then $\pi_{g,C_1 \cup C_2} = \pi_{g,(C_1^c \cap C_2^c)^c}$.

We conclude that $\pi_{g,C} \in G_{\ell,Q_{\text{dec}},\Gamma}$, and that the provided upper-bounds are correct. \square

Note that any ghost-ignorant permutation $g \in \text{Sym}(Q_{\text{dec}} \times \Gamma)$ is even since $|G|$ is. Combining this with the previous lemma, we obtain:

Lemma 4.4. *Let $T : (Q_{\text{dec}} \times \Gamma) \times \Gamma^{\ell-1} \mapsto \mathbb{N}$ be given by the previous lemma. For any ghost-ignorant permutation $g \in \text{Sym}(Q_{\text{dec}} \times \Gamma)$, and any ghost- and g -invariant condition $C \subseteq (Q_{\text{dec}} \times \Gamma) \times \Gamma^{\ell-1}$, the permutation $\pi_{g,C}$ belongs to $G_{\ell,Q_{\text{dec}},\Gamma}$ with word norm $O(T(C))$.*

Proof. Consider $G = \llbracket 0, 5 \rrbracket \times \Gamma$ as $G = \{0, 1\} \times G'$, for $G' = \{0, 1, 2\} \times \Gamma$. As a remark, the structure of G' has no importance in this proof and the reader can consider G' as a black box – in particular Γ never refers to the transport Γ -component of G' in what follows, but instead to a tape letter. For the rest of the proof, fix C some ghost-invariant condition.

Define $H'' = (Q \times D) \times \Gamma$ and $H' = (Q \times D \times \{0, 1\}) \times \Gamma$, considered as projections of $H = (Q \times D \times \{0, 1\} \times G') \times \Gamma = Q_{\text{dec}} \times \Gamma$. Then any permutation $g'' \in \text{Sym}(H'')$ lifts into a permutation $g' = \chi(g'') \in \text{Sym}(H')$ (by ignoring the bit), and any $g' \in \text{Sym}(H')$ lifts into a permutation $g = \xi(g') \in \text{Sym}(H)$ (by ignoring G'). We first prove that the ghost-ignorant permutations $g \in \text{Sym}(H)$ such that C is g -invariant are generated by the $\xi(\sigma') \in \text{Sym}(H)$, for σ' the 3-cycles of $\text{Sym}(H')$ such that C is $\xi(\sigma')$ -invariant.

For any ghost-ignorant $g \in \text{Sym}(H)$, there exists some $g'' \in \text{Sym}(H'')$ such that $g = \xi(\chi(g''))$. Additionally, $g'' \in \text{Sym}(H'')$ decomposes into a product of cycles of disjoint support. Without any loss of generality, we can assume that g'' is a cycle of support $S'' \subseteq H''$. So g'' can be considered as a permutation of $\text{Sym}(S'')$. Then g'' lifts into a permutation $g' = \chi(g'') \in \text{Alt}(H')$ that can be considered as a permutation of $\text{Alt}(S')$, for $S' = S'' \times \{0, 1\}$. The group $\text{Alt}(S')$ is generated by its 3-cycles, and any permutation h' of support S' lifts into a

permutation $h = \xi(h') \in \text{Sym}(H)$ such that C is h -invariant, because S'' is the support of the cycle g'' and C is g -invariant.

Now, for any 3-cycle $\sigma' \in \text{Sym}(H')$ with C being $\xi(\sigma')$ -invariant, we build $\pi_{\xi(\sigma'), C}$. According to the previous paragraph, this will conclude the proof. First, let $r = (0 \cdot z_1, 0 \cdot z_2, 0 \cdot z_3) \in \text{Alt}(G)$ (for some three arbitrary distinct $z_i \in G'$) and pick an arbitrary $x = ((q, d), a) \in S'' \subseteq H''$. Now $\pi_{r, C \cap [x]_0}$ belongs to $G_{\ell, Q_{\text{dec}}, \Gamma}$ with word norm $O(T(C))$, where T is defined in the previous lemma. Here we see $\pi_{r, [x]_0}$ as the 3-cycle

$$(((q, d, 0, z_1), a), ((q, d, 0, z_2), a), ((q, d, 0, z_3), a))$$

in $\text{Alt}(H)$.

Denote $(h'_1, h'_2, h'_3) = \sigma'$ with $h'_i = ((q_i, d_i, g_i), a_i) \in S' \subseteq H'$. By conjugating $\pi_{r, C \cap [x]_0}$ with the three following permutations of $\text{Sym}(H)$ for $i = 1, 2, 3$:

$$h'_i^{(z)} \leftrightarrow (((q, d, 0, z_i), a))$$

we build in $G_{\ell, Q_{\text{dec}}, \Gamma}$ with word norm $O(T(C))$ the element $\pi_{(h'_1^{(z)}, h'_2^{(z)}, h'_3^{(z)}), C}$, where $h'_i^{(z)} \in H$ is the lift of h'_i whose second ghost-component is z , i.e. $h'_i^{(z)} = ((q_i, d_i, (g_i, z)), a_i)$. We obtain:

$$\pi_{\sigma, C} = \prod_{z \in G'} \pi_{(h'_1^{(z)}, h'_2^{(z)}, h'_3^{(z)}), C}$$

□

Readers with a background in complexity theory will find the following version of the statement useful; it is immediate from the definition of NC^1 and the previous lemmas.

Lemma 4.5. *Let L be a language in NC^1 , and L_n its words of size n . For any ghost-ignorant $g \in \text{Alt}(Q_{\text{dec}} \times \Gamma)$, if L_n is ghost- and g -invariant, then $f_{g, L_n} \in G_{\ell, Q_{\text{dec}}, \Gamma}$ has polynomial word norm in n .*

We now provide upper bounds on Lemma 4.4 for a few specific conditions, following the proof of Lemma 4.3.

Lemma 4.6. *Let $C \subseteq (Q_{\text{dec}} \times \Gamma)$ be a condition of the form $\sim c$, for $c \in (\{0, 1, 2\} \cup \{\bullet\})^j$ with $j \leq \ell$ and $\sim \in \{=, <, \leq, >, \geq\}$ some lexicographic (in)-equality, i.e. $w \in C$ if and only if, if $J \subseteq \llbracket 0, j-1 \rrbracket$ is the ordered set of non-“ \bullet ” indices of c , then $w|_J \sim c|_J$.*

Then $T(C) = O(|c|^2)$.

Proof. We use the fact that the permutations of Lemma 4.3 are only permuting the ghost to reduce the head movement. This is a matter of divide and conquer: for $w \in \{0, 1, 2\}^j$ and $(c \in \{0, 1, 2\} \cup \{\bullet\})^j$, if J is the set of non-“ \bullet ” indices of c , then:

$$\begin{aligned} w = c &\iff \left(w_{\llbracket 0, j'-1 \rrbracket \cap J} = c_{\llbracket 0, j'-1 \rrbracket \cap J} \right) \wedge \left(w_{\llbracket j', j-1 \rrbracket \cap J} = c_{\llbracket j', j-1 \rrbracket \cap J} \right) \\ w < c &\iff \left(w_{\llbracket 0, j'-1 \rrbracket \cap J} < c_{\llbracket 0, j'-1 \rrbracket \cap J} \right) \\ &\vee \left(\left(w_{\llbracket 0, j'-1 \rrbracket \cap J} = c_{\llbracket 0, j'-1 \rrbracket \cap J} \right) \wedge \left(w_{\llbracket j', j-1 \rrbracket \cap J} < c_{\llbracket j', j-1 \rrbracket \cap J} \right) \right) \end{aligned}$$

So if C is the condition $= c$, and $g_1, g_2 \in \text{Alt}(G)$ (considered as a subgroup of $\text{Alt}(Q_{\text{dec}} \times \Gamma)$), then:

$$\pi_{[g_1, g_2], =c} = \left[\pi_{g_1, =c_{[0, j'-1]}}, \rho^{-j'} \circ \pi_{g_2, =c_{[j', j-1]}} \circ \rho^{j'} \right]$$

Taking $j' = \lfloor k/2 \rfloor$, we obtain $T(C) = O(|c|^2)$. Similarly, if C is the condition $< c$, we obtain $T(C) = O(|c|^2)$. The other cases follow using elementary Boolean algebra. \square

4.3.2 Using ducks to implement the encoding with involutions

In this section, we use the previous lemmas to implement the inductive encoding of Section 3.3 in $G_{\ell, Q_{\text{dec}}, \Gamma}$. More precisely, we use the duck of $D = \{d_1, d_2\}$ to “decompose” each step of the encoding into products of transpositions that swap ducks d_1 and d_2 . These involutions are built using the *ducking* trick mentioned in Section 1.4.2.

Lemma 4.7. *For $k \in \{0, 1, 2\}^*$, $|k| \leq \ell$ and $q_d \in Q \times D$, let $s(k, [q_d])$ be the permutation on the cyclic tapes of $C_{\ell, Q_{\text{dec}}, \Gamma}$ that does nothing on configurations w containing no head in $[q_d] = \{q\} \times \{d\} \times G$, and adds the ternary number k to the $|k|$ tape-letters to the right of heads in state $[q_d]$, i.e. if $w_{[i, i+|k|-1]} = ([q_d], c_0) \cdot c_1 \dots c_{|k|-1}$:*

$$s(k, [q_d])(w)_{[i, i+|k|-1]} = ([q_d], c'_0) \cdot c'_1 \dots c'_{|k|-1}$$

where $v_3(c'_0 \dots c'_{|k|-1}) \equiv v_3(c_0 \dots c_{|k|-1}) + k \pmod{3^{|k|}}$

and $s(k, [q_d])(w)_j = w_j$ if $j \notin [i, i+|k|-1]$.

Then $s(k, [q_d])$ has word norm $O(|k|^3)$ in $G_{\ell, Q_{\text{dec}}, \Gamma}$.

Proof. Performing addition modulo $3^{|k|}$ reduces to rotating digits and performing carries. Define $r_{[q_d]} \in \text{Alt}(Q_{\text{dec}})$ (considered as a permutation of $\text{Alt}(Q_{\text{dec}} \times \Gamma)$) the tape-invariant permutation that rotates the letter $x \in \Gamma$ at the head (on the tape, not in the transport component) with the 3-cycle $(0, 1, 2) \in \text{Alt}(\Gamma)$ if the head is in state $[q_d]$. Below, we perform the standard “school algorithm” for addition.

First, move the head $|k|$ letters to the right by applying $\rho^{|k|-1}$. Apply $\pi_{r_{[q_d]}}$ if $k_{|k|-1} = 1$, $\pi_{r_{[q_d]}^2}$ if $k_{|k|-1} = 2$, or the identity if $k_{|k|-1} = 0$. Then, for $j \in [1, |k|-2]$, do (with the notations of Lemma 4.6):

1. Move the head to the left with ρ^{-1} .
2. Denoting $k' = k_{[|k|-j, |k|-1]}$, apply $\pi_{r_{[q_d]}} \circ \pi_{r_{[q_d]}, <\bullet k'}$ if $k_{|k|-j-1} = 1$, $\pi_{r_{[q_d]}^2} \circ \pi_{r_{[q_d]}, <\bullet k'}$ if $k_{|k|-j-1} = 2$, and $\pi_{r_{[q_d]}, <\bullet k'}$ if $k_{|k|-j-1} = 0$ (in other words, unconditionally add the corresponding digit of k , and perform a carry if the word to the right proves it necessary).

So $s(k, [q_d])$ is the composition of $O(|k|)$ permutations of $G_{\ell, Q_{\text{dec}}, \Gamma}$ whose word norms are $O(|k|^2)$, which concludes the proof. \square

For Φ_{\dots} one of the encoding steps in Figure 2, 3 and 4, or 5, let $W \mapsto W'$ be one of its cases (i.e. one line of rewriting in one of the aforementioned figures), with $W, W' \subseteq C_{\ell, Q, \Gamma}$.

Denote by $W_{\text{dec}}, W'_{\text{dec}}$ their decorated versions and $W_{\text{dec}}^{d_1}$ (resp. $W_{\text{dec}}^{d_2}$) the subset of W_{dec} whose heads bear duck d_1 (resp. d_2). Finally, let

$$(w \in W_{\text{dec}}^{d_1}) \leftrightarrow (\Phi_{\dots}(w) \in W'_{\text{dec}}^{d_2})$$

be the involution of $\text{Sym}(C_{\ell, Q_{\text{dec}}, \Gamma})$ that swaps words $w \in W_{\text{dec}}^{d_1}$ with words $\Phi_{\dots}(w) \in W'_{\text{dec}}^{d_2}$. More precisely, we rewrite the symbols as indicated in the respective figure, swap the duck and we copy all other symbols unchanged.

Lemma 4.8. *For any such case $W \mapsto W'$, the permutation $(w \in W_{\text{dec}}^{d_1}) \leftrightarrow (\Phi_{\dots}(w) \in W'_{\text{dec}}^{d_2})$ belongs in $G_{\ell, Q_{\text{dec}}, \Gamma}$ with norm $O(\ell^3)$.*

Proof. For any configuration $w \in C_{\ell, Q_{\text{dec}}, \Gamma}$, the condition $w \in W_{\text{dec}}^{d_1}$ checks the state of the head in w , the values of some tape-letters, and a counter being non-zero (in the case of Figure 3) or full-zero (in the case of Figure 4). By Lemma 4.6, denoting $\pi_{d, W_{\text{dec}}}$ the *ducking* permutation that swaps ducks d_1 and d_2 on the heads of words w if and only if $w \in W_{\text{dec}}$, then $\pi_{d, W_{\text{dec}}}$ belongs in $G_{\ell, Q_{\text{dec}}, \Gamma}$ with norm $O(\ell^2)$.

By conjugating $\pi_{d, W_{\text{dec}}}$ by a sequence of permutations conditioned on the duck being d_2 , we can then build $(w \in W_{\text{dec}}^{d_1}) \leftrightarrow (\Phi_{\dots}(w) \in W'_{\text{dec}}^{d_2})$ with norm $O(\ell^3)$ (this $O(\ell^3)$ comes from the addition of some number to a counter).

On an example, consider $W \mapsto W'$ being the following transformation between levels k and $k+1$ (this is the third rewrite in Figure 3):

$$\left(\begin{array}{ccccc} * & c & 1 & * & \\ & \blacktriangleright_1 & & * & \end{array} \right) \rightarrow \left(\begin{array}{ccccc} * & & [v_3(c) + f(n) + 1]_{(3)} & * & \\ & \blacktriangleleft_1 & & & \end{array} \right)$$

Then the condition W_{dec} is the conjunction of the head being in state \blacktriangleright_1 , on top of the tape-letter 1, and the $k+1$ tape-letters at its left being non-zero. By Lemma 4.6, $\pi_{d, W_{\text{dec}}}$ belongs in $G_{\ell, Q_{\text{dec}}, \Gamma}$ with norm $O(k^2)$. Then, one can conjugate $\pi_{d, W_{\text{dec}}}$ with the following sequence of permutations, which we all apply *conditioned on the head having duck d_2* :

1. Apply the transposition $(1, 0) \in \text{Sym}(\Gamma)$ on the tape-letter under the head to change the letter 1 into 0 if the head has duck d_2 (norm $O(1)$). At the moment, we have the permutation:

$$\left(\begin{array}{ccccc} * & c & 1 & * & \\ & \blacktriangleright_1^{d_1} & & * & \end{array} \right) \leftrightarrow \left(\begin{array}{ccccc} * & c & 0 & * & \\ & \blacktriangleright_1^{d_2} & & * & \end{array} \right)$$

2. Move the counter c one step to its right, and move the tape-letter 0 under the head $k+1$ steps to its left. To do so (word norm $O(k)$), just shift each letter of the counter one step to its right while using the transport component of the ghost as a temporary buffer, and move the 0 to the leftmost position (applying all these permutations only if the duck is d_2). At the moment, we have the permutation:

$$\left(\begin{array}{ccccc} * & c & 1 & * & \\ & \blacktriangleright_1^{d_1} & & * & \end{array} \right) \leftrightarrow \left(\begin{array}{ccccc} * & 0 & c & * & \\ & \blacktriangleright_1^{d_2} & & * & \end{array} \right)$$

3. Apply $s([f(k) + 1]_{(3)}, [\blacktriangleright_1^{d_2}])$, namely the permutation that adds $f(k) + 1$ to the ternary number to the right of the head (norm $O(k^3)$). Note that we take $[f(k) + 1]_{(3)}$ of length $k+2$. Then move the head one step left (norm $O(1)$) if the duck is d_2 . At the moment, we have the permutation:

$$\left(\begin{smallmatrix} * & c & 1 \\ & \blacktriangleright_1^{d_1} & * \end{smallmatrix} \right) \leftrightarrow \left(\begin{smallmatrix} * & [v_3(c) + f(n) + 1]_{(3)} & * \\ \blacktriangleleft_1^{d_2} & & \end{smallmatrix} \right)$$

4. Apply the transposition that swaps $\blacktriangleright_1^{d_2}$ and $\blacktriangleleft_1^{d_2}$. We finally obtain:

$$\left(\begin{smallmatrix} * & c & 1 \\ & \blacktriangleright_1^{d_1} & * \end{smallmatrix} \right) \leftrightarrow \left(\begin{smallmatrix} * & [v_3(c) + f(n) + 1]_{(3)} & * \\ \blacktriangleleft_1^{d_2} & & \end{smallmatrix} \right) \quad \square$$

Define $C_{\ell, Q_{\text{dec}}, \Gamma}^{d_1}$ and $C_{\ell, Q_{\text{dec}}, \Gamma}^{d_2}$ the cyclic tapes whose heads respectively have ducks d_1 and d_2 . We complete the “ducking” process and obtain as an immediate corollary:

Lemma 4.9. Define Π_{init} , $\Pi_{n \mapsto n+1}$ and $\Pi_{\ell, \text{final}}$ as:

$$\Pi_{...} : (w \in C_{\ell, Q_{\text{dec}}, \Gamma}^{d_1}) \leftrightarrow (\Phi_{...}(w) \in C_{\ell, Q_{\text{dec}}, \Gamma}^{d_2})$$

where $\Phi_{...}$ is a transformation defined in Section 3.3. Then Π_{init} , $\Pi_{k \mapsto k+1}$ and $\Pi_{\ell, \text{final}}$ all belong in $G_{\ell, Q_{\text{dec}}, \Gamma}$ with word norm $O(\ell^3)$.

Proof. Each of these transformations $\Phi_{...}$ is composed of finitely many cases $W \mapsto W'$ of disjoint support and images, i.e. each $\Pi_{...}$ can be written as the product of finitely many $(w \in W_{\text{dec}}^{d_1}) \leftrightarrow (\Phi_{...}(w) \in W'_{\text{dec}}^{d_2})$. \square

4.4 Proof of Lemma 4.2

Recall that S_{dec} denotes the decorated version of SMART: we prove that $(T_{\ell, S_{\text{dec}}})^n$, the action of the n -th power of S_{dec} on the cyclic tapes of $C_{\ell, Q_{\text{dec}}, \Gamma}$, has word norm $O(\ell^4)$.

Lemma 4.10. Denoting π_d the involution that swaps ducks d_1 and d_2 , let

$$\Pi_\ell = \pi_d \circ \Pi_{\ell, \text{final}} \circ \left(\prod_{k=0}^{\ell-2} \pi_d \circ \Pi_{k \mapsto k+1} \right) \circ \pi_d \circ \Pi_{\text{init}}$$

Then $\Pi_\ell \in G_{\ell, Q_{\text{dec}}, \Gamma}$ with norm $O(\ell^4)$. And, assuming $w \in C_{\ell, Q_{\text{dec}}, \Gamma}^{d_1}$, we have $\Pi_\ell(w) = \Phi_{\text{dec}}(w)$, where Φ_{dec} denotes the decorated version of the encoding described in Section 3.3.

Proof. This is an immediate consequence of Lemma 4.9, the stability of $C_{\ell, Q_{\text{dec}}, \Gamma}^{d_1}$ and $C_{\ell, Q_{\text{dec}}, \Gamma}^{d_2}$ by every $\pi_d \circ \Pi_{...}$, and of the correctness of the encoding proved in Lemma 3.6. \square

One should note that, on configurations $w \in C_{\ell, Q_{\text{dec}}, \Gamma}^{d_2}$, Π_ℓ “produces garbage”: we claim no meaningful interpretation for the image of such w .

Lemma 4.11. Define $+\ell, n, d_1$ as the bijection of $C_{\ell, Q_{\text{dec}}, \Gamma}$ that performs the addition of n in base $\ell \cdot 2 \cdot 3^\ell$ in the sense of our encoding, i.e. $+\ell, n, d_1$ is the shift-commuting bijection defined by $+\ell, n, d_1(w) = w$ if w contains a head with duck d_2 or no head at all, and if $w = ([t], c_0) \cdot c_1 \dots c_{\ell-1}$ with $t \in Q_{\text{dec}}$ has duck d_1 and Q -projection $q_i \in \{\blacktriangleright, \blacktriangleleft, \triangleright, \triangleleft\} \times \{1, 2\}$, then $+\ell, n, d_1(w) = w'$, where:

$$w' = \begin{cases} \sigma^{-j}(([t'], c'_0) \cdot c'_1 \dots c'_{\ell-1}) & \text{if } q \in \{\blacktriangleright, \triangleright\} \\ \sigma^j(([t'], c'_0) \cdot c'_1 \dots c'_{\ell-1}) & \text{if } q \in \{\blacktriangleleft, \triangleleft\} \end{cases}$$

where t' is equal to t with the Q -component q_i replaced by $q_{i'}$, j is the quotient of $(i-1) \cdot 3^\ell + v_3(c) + n$ by $2 \cdot 3^\ell$, and $(i'-1) \cdot 3^\ell + v_3(c')$ is the remainder.

Then $+_{\ell,n,d_1}$ belongs in $G_{\ell,Q_{\text{dec}},\Gamma}$ with word norm $O(\ell^3)$.

Proof. As rotating the whole tape at most ℓ times can be done with word norm $O(\ell^2)$, we can assume that $0 \leq n < 2 \cdot 3^\ell$.

First, the addition of n modulo $2 \cdot 3^\ell$ (and without rotating the tape if the addition overflows), conditioned on the head having duck d_1 , can be implemented with norm $O(\ell^3)$. The proof is very similar to Lemma 4.7, as we consider the configuration w as counters of $\{1, 2\} \cdot \{0, 1, 2\}^\ell$, the only difference being the bit $\{1, 2\}$ being carried by the state q_1 or q_2 .

We now need to perform a rotation of the tape if the addition modulo $2 \cdot 3^\ell$ of the previous paragraph overflowed, which is equivalent to $(i'-1) \cdot 3^\ell + v_3(c') \equiv (i-1) \cdot 3^\ell + v_3(c) + n \bmod 2 \cdot 3^\ell$ being strictly smaller than n . Define $\pi_{\blacktriangleright, \blacktriangleleft}$ (resp. $\pi_{\triangleright, \triangleleft}$) the involution that exchanges states \blacktriangleright and \blacktriangleleft (resp. \triangleright , \triangleleft) if $(i'-1) \cdot 3^\ell + v_3(c') < n$ (and, of course, if they both have duck d_1). By Lemma 4.6, this involution has word norm $O(\ell^2)$.

Then, if r_\blacktriangleright (resp. r_\triangleright) rotates the whole tape right if the head is in state $(\blacktriangleright_i, d_1, \cdot)$ (resp. $(\triangleright_i, d_1, \cdot)$) (this permutation belongs in $G_{\ell,Q_{\text{dec}},\Gamma}$: use the transport component of the ghost as a temporary buffer), the commutator $[\pi_{\blacktriangleright, \blacktriangleleft}, r_\blacktriangleright]$ rotates the whole tape right if the state is \blacktriangleright and $(i'-1) \cdot 3^\ell + v_3(c') < n$, and rotates the whole tape left if the state is \blacktriangleleft and $(i'-1) \cdot 3^\ell + v_3(c') < n$. Similarly, $[\pi_{\triangleright, \triangleleft}, r_\triangleright]$ rotates \triangleright -tapes right and \triangleleft -tapes left if $(i'-1) \cdot 3^\ell + v_3(c') < n$.

We conclude that $+_{\ell,n,d_1}$ (which is the composition of these steps) indeed belongs in $G_{\ell,Q_{\text{dec}},\Gamma}$ with norm $O(\ell^3)$. \square

Lemma 4.12. *Define:*

$$T_{\ell,S_{\text{dec}}}^{d_1}(c) = \begin{cases} T_{\ell,S_{\text{dec}}}(c) & \text{if } c \in C_{\ell,Q_{\text{dec}},\Gamma}^{d_1} \\ c & \text{otherwise} \end{cases}$$

Then $T_{\ell,S_{\text{dec}}}^{d_1} \in G_{\ell,Q_{\text{dec}},\Gamma}$ with word norm $O(\ell^4)$.

Proof. Let $+_{\ell,n,d_1}$ be given by Lemma 4.11. With Lemma 4.10, one can conjugate $+_{\ell,n,d_1}$ with Π_ℓ and obtain a bijection on $C_{\ell,Q_{\text{dec}},\Gamma}$ that maps configurations of $C_{\ell,Q_{\text{dec}},\Gamma}^{d_1}$ to their n -th iterate by S_{dec} , and is the identity on $C_{\ell,Q_{\text{dec}},\Gamma}^{d_2}$. In other words:

$$\left(T_{\ell,S_{\text{dec}}}^{d_1} \right)^n = (+_{\ell,n,d_1})^{\Pi_\ell}$$

Indeed, the addition of $+_{n,k,d_1}$ is only performed on heads having duck d_1 . And the shift of the tape, when the addition modulo $2 \cdot 3^\ell$ overflows, is performed to the right (resp. to the left), exactly like $\Phi^{2 \cdot 3^\ell}$ performs it on configurations C_\blacktriangleright and C_\triangleright (resp. C_\blacktriangleleft and C_\triangleleft). \square

We can now prove the third point of our main Lemma 4.2 about SMART being distorted on finite cyclic tapes: for S_{dec} , we have $\delta(\ell, n) = O(\ell^4)$.

Proof. Denoting by π_d the “ducking” involution that swaps ducks d_1 and d_2 , we have

$$(T_{\ell,S_{\text{dec}}})^n = (\pi_d \circ \left(T_{\ell,S_{\text{dec}}}^{d_1} \right)^n \circ \pi_d) \circ \left(T_{\ell,S_{\text{dec}}}^{d_1} \right)^n$$

because $(T_{\ell,S_{\text{dec}}}^{d_1})^{\pi_d} = T_{\ell,S_{\text{dec}}}^{d_2}$. \square

5 Distortion of automorphisms of the full shift

This chapter contains a proof of Theorem A. By [29], the automorphism groups of full shifts with different (nontrivial) alphabets embed in each other, so we only need to prove that *there exists Σ and some $g \in \text{Aut}(\Sigma^{\mathbb{Z}})$ such that $|g^n|_F = O(\log^4 n)$ for some finite set F .*

Sections 5.3 and 5.4 focus on the proof of the general distortion result Lemma 5.1, which states that any machine that satisfies the properties from Lemma 4.2 gives rise to a distorted automorphism on a full shift. Section 5.5 concludes the proof of Theorem A with Lemma 5.8, which shows how to optimize the degree to four in the case of the SMART machine.

5.1 Conveyor belts

Recall that a Turing machine $\mathcal{M} = (Q, \Gamma, \Delta)$ acts on the subshift $X_{Q, \Gamma}$ (see Section 2), i.e. on the set of bi-infinite tapes containing at most one head (i.e. one letter of $Q \times \Gamma$).

To make a Turing machine $\mathcal{M} = (Q, \Gamma, \Delta)$ act on a full shift instead, we use the conveyor belt trick (see for example [24, Lemma 3]). Let

$$\Sigma_{Q, \Gamma} = (\Gamma^2 \times \{+1, -1\}) \sqcup ((Q \times \Gamma) \times \Gamma) \sqcup (\Gamma \times (Q \times \Gamma))$$

where we call *conveyor bits* the bits of $\{+1, -1\}$ in $(\Gamma^2 \times \{+1, -1\})$, and define the action of \mathcal{M} on $(\Sigma_{Q, \Gamma})^{\mathbb{Z}}$ as the following automorphism $f_{\mathcal{M}}$.

First, any $x \in (\Sigma_{Q, \Gamma})^{\mathbb{Z}}$ uniquely splits into $x = \dots w_{-2}w_{-1}w_0w_1w_2\dots$ such that for every $i \in \mathbb{Z}$, we have either

$$w_i \in (\Gamma^2 \times \{+1\})^* \left(((Q \times \Gamma) \times \Gamma) \cup (\Gamma \times (Q \times \Gamma)) \right) (\Gamma^2 \times \{-1\})^*$$

or $w_i \in (\Gamma^2 \times \{+1\})^+ (\Gamma^2 \times \{-1\})^+$

with the exception that on some configurations, there might exist a leftmost or rightmost word with an infinite number of $+1$ or -1 .² We describe the action of $f_{\mathcal{M}}$ on such finite words w : as configurations made of infinitely many finite w_i are dense, and $f_{\mathcal{M}}$ will be uniformly continuous on these, $f_{\mathcal{M}}$ will uniquely extend to an automorphism on the full shift.

- On words $w \in (\Gamma^2 \times \{+1\})^+ (\Gamma^2 \times \{-1\})^+$, we do nothing.
- On words $w \in (\Gamma^2 \times \{+1\})^* \left(((Q \times \Gamma) \times \Gamma) \cup (\Gamma \times (Q \times \Gamma)) \right) (\Gamma^2 \times \{-1\})^*$, let

$$w' \in (\Gamma^2)^* \left(((Q \times \Gamma) \times \Gamma) \cup (\Gamma \times (Q \times \Gamma)) \right) (\Gamma^2)^*$$

be the word obtained by erasing the conveyor bits $+1$ and -1 from w . We see w' as a conveyor belt of length $2|w|$, that is the superimposition of a top word u and a bottom word v , glued together at their borders as if the words were laid down on a conveyor belt.

²The corresponding decomposition claim in [24] has a mistake, as it uses Kleene stars also in the second form.

More precisely, let $u = \pi_1(w')$ and $v = \overline{\pi_2(w')}$ the reversal of $\pi_2(w')$. Then one of these words is in Γ^+ , and the other is in $\Gamma^*(Q \times \Gamma)\Gamma^*$. Then we make \mathcal{M} act on $(uv)^\mathbb{Z}$ (despite it having infinitely many heads, it should be clear what this means, as all the heads move with the same transition), i.e. define

$$u'v' = T_{\mathcal{M}}((uv)^\mathbb{Z})_{[0,2|w|-1]}$$

Note that $u'v'$ also contains exactly one head. We then rewrap $u'v'$ into a conveyor belt of $(\Gamma^2)^*((Q \times \Gamma) \times \Gamma) \cup (\Gamma \times (Q \times \Gamma))(\Gamma^2)^*$, and add conveyor bits $+1$ (resp. -1) to the cell symbols in Γ^2 to the left of the head (resp. right).

This defines how $f_{\mathcal{M}}$ acts on such words w . This can be summarized as: $f_{\mathcal{M}}$ considers such words as a cyclic tape folded in the shape of a conveyor belt, and acts on the cyclic tape.

Note that x and $f_{\mathcal{M}}(x)$ have the same decomposition into a product of conveyor belts, and that if \mathcal{M} is reversible, then $f_{\mathcal{M}}$ is an automorphism of $\Sigma_{Q,\Gamma}^\mathbb{Z}$.

Another way to see Turing machines in conveyor belts is the following. For $g \in \text{Alt}(Q \times \Gamma)$, define f_g^{up} , f_g^{down} and f_g as the following automorphisms of $\text{Aut}(\Sigma_{Q,\Gamma}^\mathbb{Z})$:

$$\begin{aligned} f_g^{\text{up}}(x)_i &= \begin{cases} x_i & \text{if } x_i \in \Gamma^2 \times \{+1, -1\} \text{ or } x_i \in (\Gamma \times (Q \times \Gamma)) \\ (g(q, a), b) & \text{if } x_i = ((q, a), b) \in ((Q \times \Gamma) \times \Gamma) \end{cases} \\ f_g^{\text{down}}(x)_i &= \begin{cases} x_i & \text{if } x_i \in \Gamma^2 \times \{+1, -1\} \text{ or } x_i \in ((Q \times \Gamma) \times \Gamma) \\ (b, g(q, a)) & \text{if } x_i = (b, (q, a)) \in (\Gamma \times (Q \times \Gamma)) \end{cases} \\ f_g(x)_i &= \begin{cases} x_i & \text{if } x_i \in \Gamma^2 \times \{+1, -1\} \\ (g(q, a), b) & \text{if } x_i = ((q, a), b) \in ((Q \times \Gamma) \times \Gamma) \\ (b, g(q, a)) & \text{if } x_i = (b, (q, a)) \in (\Gamma \times (Q \times \Gamma)) \end{cases} \end{aligned}$$

For every $q \in Q$, define ρ_q as the right movement of heads in state q *inside their respective conveyor belts*, and $\rho = \prod_{q \in Q} \rho_q$. And define $G_{Q,\Gamma}$ the group they generate:

$$G_{Q,\Gamma} = \langle \{f_{g,\text{up}}, f_{g,\text{down}}, f_g \mid g \in \text{Sym}(Q \times \Gamma)\} \cup \{\rho_q \mid q \in Q\} \rangle$$

The generators of $G_{\ell,Q,\Gamma}$ introduced in Section 4 are in direct correspondence with the generators $\{f_g \mid g \in \text{Sym}(Q \times \Gamma)\}$ and $\{\rho_q \mid q \in Q\}$ of $G_{Q,\Gamma}$, and the latter can also be seen as the basic instructions of Turing machines: moving heads based on their states, or permuting their values.

In a similar way as for the cyclic tapes in Section 4, it is easy to see that for

any machine $\mathcal{M} = (Q, \Gamma, \Delta)$, $f_{\mathcal{M}}$ is defined as the composition $\beta_{-1} \circ \beta_{+1} \circ f_\alpha$:

$$\begin{aligned}\alpha(q, a) &= \begin{cases} (q', b) & \text{if } (q, a, q', b) \in \Delta \\ (q', a) & \text{if } (q, \pm 1, q') \in \Delta \end{cases} \\ \beta_{+1} &= \prod_{q' | \exists q, (q, +1, q') \in \Delta} \rho_{q'} \\ \beta_{-1} &= \prod_{q' | \exists q, (q, -1, q') \in \Delta} \rho_{q'}^{-1}\end{aligned}$$

5.2 Main result on $(\Sigma_*)^{\mathbb{Z}}$

For an arbitrary Turing machine $\mathcal{M} = (Q, \Gamma, \Delta)$, define the symmetrized Turing machine $\mathcal{M}_s = (Q_s, \Gamma_s, \Delta_s)$, with $Q_s = Q \times D$ and $D = \{d_{\rightarrow}, d_{\leftarrow}\}$, the same tape alphabet Γ , and the transitions $\Delta_s = \Delta_{d_{\rightarrow}} \times (\Delta^{-1})_{d_{\leftarrow}}$, i.e. the machine whose heads carry ducks $D = \{d_{\rightarrow}, d_{\leftarrow}\}$, and acts forward in time on heads having duck d_{\rightarrow} and backward in time on heads with duck d_{\leftarrow} .

Define

$$\begin{aligned}\Gamma_* &= \Gamma \times \{0, 1\} \\ Q_* &= Q_s \times \{+1, -1\} \times \{0, 1\}^2 \\ \Sigma_* &= \Sigma_{Q_*, \Gamma_*} = \left(\Gamma_*^2 \times \{+1, -1\} \right) \sqcup \left((Q_* \times \Gamma_*) \times \Gamma_* \right) \sqcup \left(\Gamma_* \times (Q_* \times \Gamma_*) \right)\end{aligned}$$

where the bit of $\{0, 1\}$ in Γ_* is the *tape-ghost*, and the value $\{+1, -1\} \times \{0, 1\}^2$ in Q_* is the *state-ghost*. Both are technicalities: the tape-ghosts are used as temporary markings to check the lengths of conveyor belts, and the state-ghosts make permutations even and increase the cardinality of our sets of even permutations to make them perfect groups. Note that the $\{+1, -1\}$ in Σ_* are not ghosts, but *conveyor bits*: they point in the direction of the unique head in the conveyor belt containing the cell, if there is one.

Finally, define θ the right movement of the heads *disregarding the conveyor belt structure*, i.e. for $x \in (\Sigma_*)^{\mathbb{Z}}$, if $\pi_{\text{up}} : \Sigma_* \mapsto \Gamma_*$ returns the top tape-letter, $\pi_{\text{down}} : \Sigma_* \mapsto \Gamma_*$ returns the bottom tape-letter, and $\pi_{\text{sign}} : \Sigma_* \mapsto \{+1, -1\}$ returns the conveyor bit of the head or the tape cell:

$$\theta(x)_i = \begin{cases} \left(\left((q, \pi_{\text{sign}}(x_i), \text{ghost}), \pi_{\text{up}}(x_i) \right), \pi_{\text{down}}(x_i) \right) & \text{if } x_{i-1} \in (\{(q, \pm 1, \text{ghost})\} \times \Gamma) \times \Gamma \\ \left(\pi_{\text{up}}(x_i), \left((q, \pi_{\text{sign}}(x_i), \text{ghost}), \pi_{\text{down}}(x_i) \right) \right) & \text{if } x_{i-1} \in \Gamma \times (\{(q, \pm 1, \text{ghost})\} \times \Gamma) \\ ((\pi_{\text{up}}(x_i), \pi_{\text{down}}(x_i)), \pi_{\text{sign}}(x_i)) & \text{if } x_{i-1} \in \Gamma^2 \times \{+1, -1\} \end{cases}$$

And define

$$G_* = \langle G_{Q_*, \Gamma_*} \cup \{\theta\} \rangle$$

the finitely-generated group generated by θ and the Turing-machine instructions of G_{Q_*, Γ_*} . The automorphism θ is only used in the proof of Lemma 5.7, and is the only generator that can modify the conveyor-belt structure.

We can now establish the second part of the proof of Theorem A. Considering $f_{\mathcal{M}_s} \in \text{Aut}((\Sigma_{Q_s, \Gamma_s})^{\mathbb{Z}})$ as an element of $\text{Aut}((\Sigma_*)^{\mathbb{Z}})$:

Lemma 5.1. *If some Turing machine \mathcal{M} satisfies the three properties of Lemma 4.2, then $(f_{\mathcal{M}_s})^n$ has word norm $O(\log^{d+1} n + \log^2 n)$ in $G_* \leq \text{Aut}(\Sigma_*^\mathbb{Z})$.*

Remark 5.2. *Note that the condition in this lemma is about \mathcal{M} (and not \mathcal{M}_s) verifying the properties of Lemma 4.2, but the conclusion of this lemma is about $(f_{\mathcal{M}_s})^n$ (and not $(f_{\mathcal{M}})^n$).*

After the proof of this lemma, we provide some additional tricks to lower the upper bound on the word norm of $(f_{S_{dec,s}})^n$, to get the claimed degree 4 in Theorem A.

5.3 Permutation engineering in $(\Sigma_*)^\mathbb{Z}$

5.3.1 From C_{ℓ,Q_*,Γ_*} to conveyor belts in $(\Sigma_*)^\mathbb{Z}$

Lemma 5.3. *Let $g \in \text{Sym}(Q_* \times \Gamma_*)$ be a ghost-ignorant permutation.*

Then for any $\ell \in \mathbb{N}$ and $\sim \in \{\langle, \leq, =, \geq, \rangle\}$, $f_{g,\text{len}\sim\ell}^{\text{up}}$ (resp. $f_{g,\text{len}\sim\ell}^{\text{down}}$, $f_{g,\text{len}\sim\ell}$), the automorphism that applies f_g^{up} (resp. f_g^{down} , f_g) in conveyor belts of length $\sim \ell$, has word norm $O(\ell^2)$.

Proof. Let $g \in \text{Alt}(Q_* \times \Gamma_*)$ be an even permutation that is ignorant of the tape-ghost bit of Γ_* and of one state-ghost bit $\{0, 1\} \subseteq Q_*$, in the usual sense that it factors through the projection that forgets these bits. Any ghost-ignorant permutation of $\text{Sym}(Q_* \times \Gamma_*)$ fits this condition. We assume that the second state-ghost is the one ignored. We first prove that, for any $\ell \in \mathbb{N}$, the permutation $f_{g,\text{ghost}_0=\text{ghost}_\ell}^{\text{up}}$ (resp. $f_{g,\text{ghost}_0=\text{ghost}_\ell}^{\text{down}}$), that applies f_g^{up} (resp. f_g^{down}) if the tape-ghost under the head and the tape-ghost ℓ steps to its right have equal bit values (and otherwise does nothing), has word norm $O(\ell)$.

As the alphabet $(Q_s \times \{+1, -1\} \times \{0, 1\}) \times \Gamma$ has cardinality greater than five, by Ore's theorem [38, Theorem 7], there exist $g_1, g_2 \in G$ such that $g = [g_1, g_2]$. Define $f_{g,\text{ghost}_0=\text{ghost}_\ell=0}^{\text{up}}$ the permutation that applies f_g^{up} if and only if both the tape-ghost of Γ_* under the head and the tape-ghost of Γ_* that is ℓ steps to its right are equal to 0.

Then, denoting $\pi_{\text{ghost}} \in \text{Alt}(Q_* \times \Gamma_*)$ the involution that exchanges the second (ignored) state-ghost $\{0, 1\} \subseteq Q_*$ and the tape-ghost of Γ_* under the head, we have:

$$f_{g,\text{ghost}_0=\text{ghost}_\ell=0}^{\text{up}} = \left[\left(f_{g_1,\text{ghost}_h=0}^{\text{up}} \right)^{f_{\pi_{\text{ghost}}}^{\text{ghost}}}, \left(f_{g_2,\text{ghost}_h=0}^{\text{up}} \right)^{\rho^{-\ell} \circ f_{\pi_{\text{ghost}}} \circ \rho^\ell} \right]$$

where $f_{g,\text{ghost}_h=0}^{\text{up}}$ applies g on the head if the second ghost bit of Q_* is equal to 0.

To see this, observe that $\left(f_{g,\text{ghost}_h=0}^{\text{up}} \right)^{\rho^{-i} \circ f_{\pi_{\text{ghost}}} \circ \rho^i}$ applies g if the tape-ghost at (relative) cell i is 0.

Using a similar trick with the value 1 instead of 0, and composing both, we obtain $f_{g,\text{ghost}_0=\text{ghost}_\ell}^{\text{up}}$ that applies f_g^{up} if and only if both the tape-ghost under the head and the tape-ghost ℓ steps to its right are equal. The same reasoning proves the same statement for $f_{g,\text{ghost}_0=\text{ghost}_\ell}^{\text{down}}$.

For $g \in \text{Alt}(Q_* \times \Gamma_*)$ an even permutation that leaves the ghost bit of Γ_* and one ghost bit of Q_* unchanged, we now prove that $f_{g,\text{len}\mid\ell}^{\text{up}}$ (resp. $f_{g,\text{len}\mid\ell}^{\text{down}}$)

that applies f_g^{up} (resp. f_g^{down}) inside conveyor belts whose length divides ℓ , and is the identity otherwise, has word norm $O(\ell)$.

Indeed, by Ore's theorem [38, Theorem 7] once again, there exists g_1, g_2 such that $g = [g_1, g_2]$. Now, denoting $r_{\text{ghost}} \in \text{Alt}(Q_* \times \Gamma_*)$ the involution that increases (modulo 2) the tape-ghost of Γ_* , applying $f_{r_{\text{ghost}}}$ once modifies both the ghost under the head and the ghost ℓ steps to its right if and only if the length of the conveyor belt divides ℓ . Using this, we obtain:

$$f_{g,\text{len}|\ell}^{\text{up}} = \left[f_{g_1,\text{ghost}_0=\text{ghost}_\ell}^{\text{up}}, \left(f_{r_{\text{ghost}}} \circ f_{g_2,\text{ghost}_0=\text{ghost}_\ell}^{\text{up}} \circ f_{r_{\text{ghost}}} \right) \right]$$

As any power of g belongs in $\text{Alt}(Q_* \times \Gamma_*)$, by going through the divisors of ℓ in decreasing order, we can build any $f_{g,\text{len}=\ell}^{\text{up}}$ with word norm $O(\ell^2)$.³ For example, if $\ell = 6$,

$$f_{g,\text{len}=6} = f_{g,\text{len}|1} \circ f_{g^{-1},\text{len}|2} \circ f_{g^{-1},\text{len}|3} \circ f_{g,\text{len}|6}.$$

(Our conveyor belts cannot actually have length 1, so $f_{g,\text{len}|1}$ may be dropped.)

We also build any $f_{g,\text{len}\leq\ell}^{\text{up}}$ with word norm $O(n^2)$ by going through the interval $\llbracket 1, \ell \rrbracket$ in decreasing order and picking suitable powers of g . In particular, we get $f_{g,\text{len}\sim\ell}^{\text{up}}$ for the relations $\sim \ell$ with $\sim \in \{<, \leq, =\}$. From this, the automorphisms with $\sim \in \{\geq, >\}$ are easy to obtain. \square

Remark 5.4. One may view the above proof as an instance of Möbius inversion. If g has order m , take $K = \oplus_{\mathbb{Z}_+} \mathbb{Z}_m$ the commutative ring of infinitely many copies of \mathbb{Z}_m . We see K as keeping track of how many times g is applied at each conveyor belt length. Define functions $\iota, \gamma : \mathbb{Z}_+ \rightarrow K$ where $\iota(n)$ as the indicator function of n (as an element of K), and $\gamma(n)$ the indicator function of the divisor poset of n . Then $\gamma(n) = \sum_{d|n} \iota(d)$ so by Möbius inversion $\iota(n) = \sum_{d|n} \gamma(d)\mu(d,n)$ where μ is the Möbius function of the divisibility poset; thus $\mu(d,n)$ tells us which power of g we should use for each divisor to get $\iota(n)$. The values of ι are a basis of K , so we can get other conditional applications of g with linear combinations.

Consider now any $T \in G_{\ell,Q_*,\Gamma_*}$. There exists $T_1, \dots, T_N \in \{\pi_g \mid g \in \text{Alt}(Q_* \times \Gamma_*)\} \cup \{\rho_q \mid q \in Q_*\}$ such that $T = T_N \circ \dots \circ T_1$. Then, as each generator π_g with $g \in \text{Alt}(Q_* \times \Gamma_*)$ corresponds to a generator f_g of G_* , T defines an automorphism f_T of $\text{Aut}((\Sigma_*)^\mathbb{Z})$. The choice of f_T is not canonical, but we can always take it to be defined by the shortest possible formula, or use the formula we used to define T . By construction, if $T \in G_{\ell,Q_*,\Gamma_*}$, then f_T acts like T on conveyor belts of length ℓ (and produces garbage on conveyor belts of length $\neq \ell$).

We now use the previous lemma to condition f_T so that it acts only in conveyor belts of length ℓ , by “symmetrizing” T .

Lemma 5.5. Assume $T \in G_{\ell,Q,\Gamma}$. Then $f_{T_s,\text{len}=\ell} \in \text{Aut}((\Sigma_*)^\mathbb{Z})$, which acts like (the lift of) T on conveyor belts of length ℓ having duck d_\rightarrow , like (the lift of) T^{-1} on conveyor belts of length ℓ with duck d_\leftarrow , and is the identity otherwise, has word norm $O(\|T\| + \ell^2)$ in G_* .

³The number of divisors function d satisfies $d(\ell) = o(\ell^\epsilon)$ for any $\epsilon > 0$, so we even get word norm $O(\ell^{1+\epsilon})$ for $f_{g,\text{len}=\ell}^{\text{up}}$.

Remark 5.6. Once again, this lemma requires conditions on T as an element of $G_{\ell, Q, \Gamma}$ (and not G_{ℓ, Q_*, Γ_*}), but its conclusion is about $f_{T_s, \text{len}=\ell}$ in $\text{Aut}((\Sigma_*)^{\mathbb{Z}})$ (and not about $f_{T, \text{len}=\ell}$ in $\text{Aut}(\Sigma_{Q, \Gamma})$).

Proof. Assume $T \in G_{\ell, Q, \Gamma}$. Then T lifts into an automorphism of G_{ℓ, Q_*, Γ_*} , and $T^{d_{\rightarrow}}$ which acts like T on cyclic tapes with duck d_{\rightarrow} , and is the identity otherwise, is also an automorphism of G_{ℓ, Q_*, Γ_*} with word norm $O(\|T\|)$: indeed, any presentation of T in $G_{\ell, Q, \Gamma}$ lifts into a presentation in G_{ℓ, Q_*, Γ_*} , in which we then restrict every generator to apply only on heads with duck d_{\rightarrow} .

Let $d \in \text{Sym}(Q_* \times \Gamma_*)$ be the involution that swaps ducks d_{\rightarrow} and d_{\leftarrow} on the head. By Lemma 5.3, $f_{d, \text{len}=\ell}$ has word norm $O(\ell^2)$ and:

$$f_{T_s, \text{len}=\ell} = f_{d, \text{len}=\ell} \circ ((f_{T^{d_{\rightarrow}}})^{-1} \circ f_{d, \text{len}=\ell} \circ (f_{T^{d_{\leftarrow}}})) \quad \square$$

5.3.2 A few specific automorphisms

Define $f_{d, \rightarrow t}$ (resp. $f_{d, \leftarrow t}$ and $f_{d, t \leftrightarrow t}$) the automorphism of $\text{Aut}((\Sigma_*)^{\mathbb{Z}})$ that swaps ducks d_{\rightarrow} and d_{\leftarrow} on the head if the head is at distance less than t from the left border of its conveyor belt, and at least t from the right border of its conveyor belt (resp. distance at least t from the left border and less than t from the right border, resp. distance at least t from both borders).

Define $f_{cb, \rightarrow t}$ (resp. $f_{cb, \leftarrow t}$ and $f_{cb, t \leftrightarrow t}$) the automorphism of $\text{Aut}((\Sigma_*)^{\mathbb{Z}})$ that applies the involution $(+1) \leftrightarrow (-1)$ on the conveyor bits carried by $\Gamma_*^2 \times \{+1, -1\}$ or the sign $\{+1, -1\}$ in Q_* , at distance t to the right of the heads (resp. left of the heads, resp. both left and right of the heads), independently of the conveyor belt structures. Note that, these three automorphisms modify the conveyor-belt structures of the configurations they are applied on.

Lemma 5.7. We have:

1. $f_{d, \rightarrow t}$, $f_{d, \leftarrow t}$ and $f_{d, t \leftrightarrow t}$ have word norm $O(t^2)$ in G_* .
2. $f_{cb, \rightarrow t}$, $f_{cb, \leftarrow t}$ and $f_{cb, t \leftrightarrow t}$ have word norm $O(t^2)$ in G_* .

Proof. For the first item, for $g \in \text{Alt}(Q_*)$ (considered as a subgroup of $\text{Alt}(Q_* \times \Gamma_*)$), there exists $g_1, g_2 \in \text{Alt}(Q_*)$ such that $g = [g_1, g_2] \in \text{Alt}(Q_*)$. With the commutator trick,

$$[f_{g_j}^{\text{up}}, \rho^{-1} \circ f_{g'_j}^{\text{down}} \circ \rho]$$

applies f_g on heads that are exactly in the top-right corner of a conveyor belt. Similar formulas exist for bottom-right, top-left and bottom-left corners of conveyor belts, so that one can condition any such f_g to be applied on heads that are not in the left, right or both corners of their conveyor belts, with word norm $O(1)$.

Then, using a divide and conquer approach with the commutator trick, one can implement $f_{d, \rightarrow t}$, $f_{d, \leftarrow t}$ and $f_{d, t \leftrightarrow t}$ in G_* with word norm $O(t^2)$, as in the proof of Lemma 4.6.

Implementing the second item is a bit more ad-hoc, but very simple: we use the right shift θ that moves heads to their right while disregarding the structure of the conveyor belts. Then, if $g \in \text{Sym}(Q_* \times \Gamma_*)$ is the involution that swaps the sign $\{+1, -1\}$ carried by the head regardless of its state,

$$f_{cb, \rightarrow t} = \theta^{-t} \circ f_g \circ \theta^t$$

Similar formulas exist for $f_{cb, \leftarrow t}$ and $f_{cb, t \leftrightarrow t}$. \square

5.4 Proof of Lemma 5.1

We prove that under the assumptions of Lemma 5.1, the automorphisms $(f_{\mathcal{M}_s})^n$ have word norm $O(\log^{d+1} n + \log^2 n)$ in G_* .

Proof. Fix an integer n . Without any loss of generality, we assume that n is even. Indeed, if n is odd, then $n-1$ is even, and $\|(f_{\mathcal{M}_s})^n\| = \|(f_{\mathcal{M}_s})^{n-1}\| + O(1)$. We denote $n = 2n'$.

With the notations of Lemma 4.2, define $L = C \cdot \log n' + C'$. By hypothesis, every $(T_{\ell, \mathcal{M}})^{n'}$ has word norm $O(\ell^d)$ in G_{ℓ, Q_*, Γ_*} .

First, using Lemma 5.5, we can manage all conveyor belts of length $< 12L$ with word norm $O(L \cdot L^d)$, as:

$$(f_{\mathcal{M}_s, \text{len} < 12L})^{2n'} = \prod_{\ell=1}^{6L-1} f_{(T_{2\ell, \mathcal{M}_s})^{2n'}, \text{len}=2\ell}$$

Then, to manage bigger conveyor belts, we use what we call the “two-scale trick”. The idea is to generate temporary conveyor belts of size $> L$ around heads in large conveyor belts, so that, when applying $(f_{\mathcal{M}_s})^{2n'}$ inside these temporary conveyor belts, the heads will never see any border, and the resulting cyclic configuration (in the temporary conveyor belt) matches the cyclic configuration that would have been obtained by applying $(f_{\mathcal{M}_s})^{2n'}$ in the original larger conveyor belt. We give a visual explanation of this two-scale trick in Figure 6. Intuitively, the difficulty lies in properly removing the temporary conveyor belt once the machine has been applied: to solve this, we will actually use temporary conveyor belts twice, with different sizes (hence the name “two-scale trick”).

Define $L_1 = 4L - 2$, $L_2 = 8L - 2$, $L_3 = 12L - 2$. Note that L_1 (resp. L_2) is the length of a conveyor belt constructed by $f_{\text{cb}, L \leftrightarrow L}$ (resp. $f_{\text{cb}, 2L \leftrightarrow 2L}$). With Lemma 5.7 and Lemma 5.5, define:

$$\lambda_{n'} = \left(f_{\text{cb}, 2L \leftrightarrow 2L} \circ \left(f_{(T_{L_2, \mathcal{M}_s})^{n'}, \text{len}=L_2} \right) \circ f_{\text{cb}, L \leftrightarrow L} \circ \left(f_{(T_{L_1, \mathcal{M}_s})^{n'}, \text{len}=L_1} \right)^{-1} \circ f_{\text{cb}, 2L \leftrightarrow 2L} \circ \left(f_{(T_{L_1, \mathcal{M}_s})^{n'}, \text{len}=L_1} \right) \circ f_{\text{cb}, L \leftrightarrow L} \right).$$

Let f_i denote the composition of the first i automorphisms on this list, i.e. $f_1 = f_{\text{cb}, 2L \leftrightarrow 2L}, \dots, f_7 = \lambda_{n'}$. The actions of the inverses of the automorphisms f_i are illustrated in the left column in Figure 6 (on a certain subset of configurations).

Now denote $d \in \text{Alt}(Q_* \times \Gamma_*)$ the ducking involution, i.e. the permutation that flips ducks d_\rightarrow and d_\leftarrow , and consider:

$$f_{\mathcal{M}_s, 2n', 3L \leftrightarrow 3L} = (\lambda_{n'})^{-1} \circ (f_{d, 3L \leftrightarrow 3L}) \circ (\lambda_{n'}).$$

We see from Figure 6 that, letting $f = f_{d, 3L \leftrightarrow 3L}$ and reading the successive partial conjugations f^{f_i} top-down, $f_{d, 3L \leftrightarrow 3L}$ gets conjugated to a map that applies our machine $(\mathcal{M}_s)^{n'}$ twice if it is on a conveyor belt that extends sufficiently both left and right, and flips the duck as a side product.

Similar formulas $f_{\mathcal{M}_s, 2n', 3L \leftarrow}$ (resp. $f_{\mathcal{M}_s, 2n', \rightarrow 3L}$) exist, managing heads in large conveyor belts at distance less than $3L$ from their right (resp. left) border. The latter two have word norm $O(L^{d+1} + L^2)$, as we have to replace occurrences

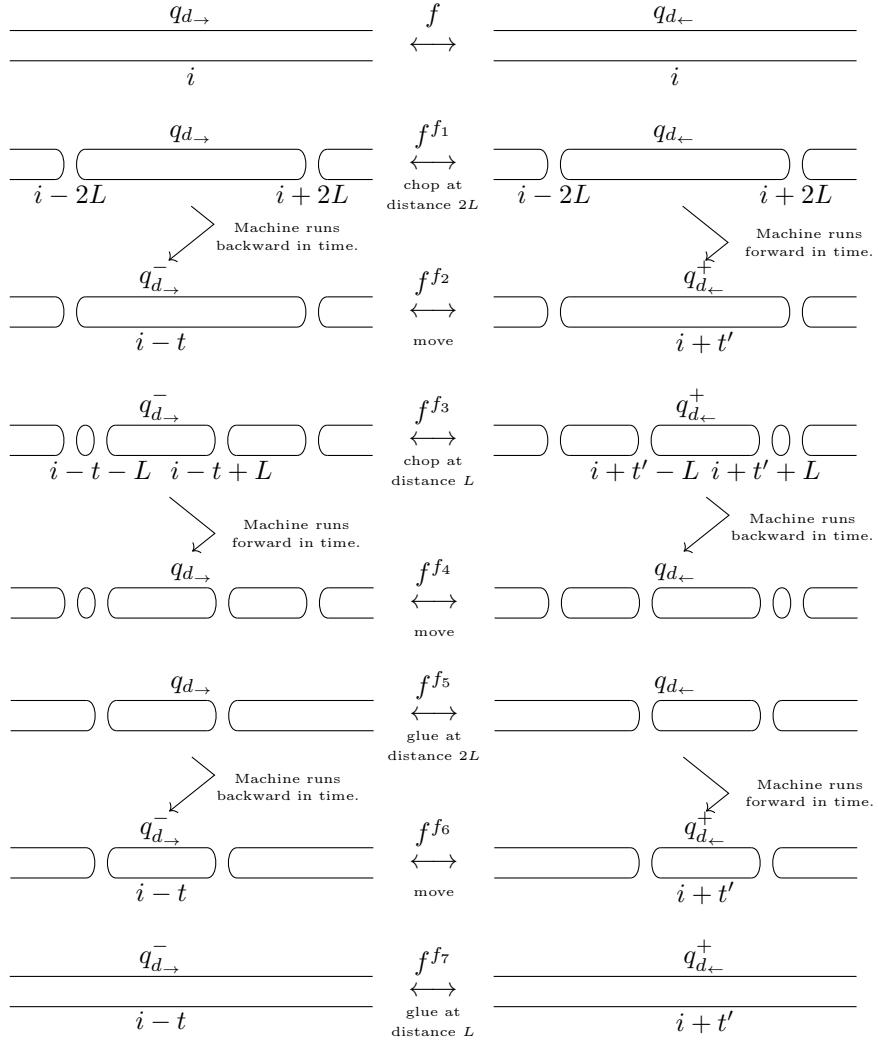


Figure 6: The “two-scale trick”

Illustration of the two-scale trick. We show the conveyor belts as lines (without tape letters). The head may be on either track. The head is in position i in state q initially, was in position $i - t$ in state q^- at time $-n'$, and will be in position $i + t'$ in state q^+ at time $+n'$. The automorphism f acts trivially unless we are in the situation of the first line, so the conjugated automorphisms also act nontrivially only in the shown situation. In particular f^{f_7} behaves as expected.

of $(f_{(T_{\ell, \mathcal{M}_s})^{n'}, \text{len}=\ell})$ with $\prod_{j=1}^{\ell} (f_{(T_{j, \mathcal{M}_s})^{n'}, \text{len}=j})$. In any case, these permutations have disjoint support (because the distance to a conveyor belt border is checked “after n' steps of computation”) and word norm $O(L^{d+1} + L^2)$.

Now if $d \in \text{Alt}(Q_* \times \Gamma_*)$ denotes the involution that flips ducks d_{\rightarrow} and d_{\leftarrow} , we have:

$$(f_{\mathcal{M}_s})^n = \left(f_{d, \text{len} \geq L_3} \circ f_{\mathcal{M}_s, n, \rightarrow 3L} \circ f_{\mathcal{M}_s, n, 3L \leftarrow} \circ f_{\mathcal{M}_s, n, 3L \leftrightarrow 3L} \right) \circ \left(\prod_{n=1}^{6L-1} f_{T_{\ell, \mathcal{M}}^n, \text{len}=2l} \right)$$

Which concludes the proof. \square

5.5 Improving the upper bound on SMART

As the computation of $(T_{\ell, \mathcal{S}_{\text{dec}}})^n$ is “uniform” in ℓ (the k -th step of the encoding, and the k -th step of the addition, are the same on all conveyor belts of length $\geq k$), we can compute the action of SMART on all conveyor belts in parallel. This drops the word norm of $(T_{\mathcal{S}_{\text{dec}, s}})^n$ from $O(\log^5 n)$ with Lemma 5.1, to $O(\log^4 n)$ with this new method. We also make minor optimizations to the alphabet, by joining some of the auxiliary symbols used on the Turing machine and automorphism side.

All in all, Lemma 5.8 concludes the proof of Theorem A.

Lemma 5.8. *Let \mathcal{S} be the SMART machine introduced in Section 3, and define $\mathcal{S}_* = (\Gamma_*, Q_*, \Delta_*)$ as follows:*

$$\begin{aligned} \Gamma_* &= \Gamma \times \{0, 1\} \\ Q_* &= Q \times \{d_{\rightarrow}, d_{\leftarrow}\} \times \left(\{+1, -1\} \times \{0, 1\}^2 \times \Gamma \right) \\ \Delta_* &= \left\{ (q_*, a_*, q'_*, b_*) : (q_*, \delta, q'_*) \mid \right. \\ &\quad \exists d \in \{d_{\rightarrow}, d_{\leftarrow}\}, g \in (\{+1, -1\} \times \{0, 1\}^2 \times \Gamma), g' \in \{0, 1\} \\ &\quad q_* = (q, d, g), q'_* = (q', d, g), a_* = (a, g'), b_* = (b, g'), \\ &\quad d = d_{\rightarrow} \implies (q, a, q', b) \in \Delta \text{ or } (q, \delta, q') \in \Delta \\ &\quad d = d_{\leftarrow} \implies (q, a, q', b) \in \Delta^{-1} \text{ or } (q, \delta, q') \in \Delta^{-1} \left. \right\} \end{aligned}$$

and

$$\Sigma_* = \Sigma_{Q_*, \Gamma_*} = \left(\Gamma_*^2 \times \{+1, -1\} \right) \sqcup \left((Q_* \times \Gamma_*) \times \Gamma_* \right) \sqcup \left(\Gamma_* \times (Q_* \times \Gamma_*) \right)$$

Then $(f_{\mathcal{S}_*})^n$ has word norm $O(\log^4 n)$ in $\text{Aut}((\Sigma_*)^{\mathbb{Z}})$.

Proof. Once again, we prove that $(f_{\mathcal{S}_*})^n$ has word norm $O(\log^4 n)$ in the following subgroup of $\text{Aut}((\Sigma_*)^{\mathbb{Z}})$

$$\begin{aligned} G_{Q_*, \Gamma_*} &= \langle \{f_{g, \text{up}}, f_{g, \text{down}}, f_g : g \in \text{Sym}(Q_* \times \Gamma_*) \} \\ &\quad \cup \{\rho_q \mid q \in Q_*\} \cup \{\theta\} \rangle \end{aligned}$$

We start by explaining the new alphabet. We have dropped the duck $\{d_1, d_2\}$ to fuse it with $\{d_{\rightarrow}, d_{\leftarrow}\}$, and we have dropped the hex component $\llbracket 0, 5 \rrbracket$ of the

ghost. We can use the set $\{+1, -1\} \times \{0, 1\}^2$ in its place, as the only thing we used was that the cardinality is large enough, and that 6 is even. The hex component was also only used temporarily, always returning to its original value after modifications to other components of the state, so it is safe to reuse this set for it.

The reason we have applied the construction without the duck $\{d_1, d_2\}$ is of course that this is the exact same duck; our assumption in the previous section is that we can efficiently apply the Turing machine when the duck is d_{\rightarrow} , while doing nothing on ducks d_{\leftarrow} , and this is exactly how the duck was used in Section 4 (see Lemma 4.12).

Now we explain the optimization; we only give a high-level explanation, as this is completely analogous to what was done in the previous section. We only amend the proof above by proving that both $(f_{M_s, \text{len} < L_3})^{2n'}$ and $(f_{M_s, \text{len} \leq L_3})^{2n'}$ have word norm $O(L^4)$, as they are sufficient to manage both small conveyor belts of length $< L_3$ and large conveyor belts in the two-scale trick.

1. The encoding (word norm $O(L^4)$). Let Π_{init} , $\Pi_{k \mapsto k+1}$ and $\Pi_{\ell, \text{final}}$ be the steps of encoding of \mathcal{S} defined in Lemma 4.9, with Π_{\dots} also swapping ducks d_{\rightarrow} and d_{\leftarrow} :

$$\Pi_{\dots} : (w \in C_{\ell, Q_{\star}, \Gamma_{\star}}^{d_{\rightarrow}}) \leftrightarrow (\Phi_{\dots}(w) \in C_{\ell, Q_{\star}, \Gamma_{\star}}^{d_{\leftarrow}})$$

As explained above, the generators of $G_{l, Q_{\star}, \Gamma_{\star}}$ correspond to generators of $G_{Q_{\star}, \Gamma_{\star}}$, so that each Π_{\dots} of $G_{l, Q_{\star}, \Gamma_{\star}}$ can also be considered as an element $f_{\Pi_{\dots}}$ of $G_{Q_{\star}, \Gamma_{\star}}$.

The key point of this proof consists in understanding that $\Pi_{k \mapsto k+1}$ behaves as expected on every conveyor belt of length $\geq k+2$, and produces garbage on conveyor belts of length $\leq k+1$.

In other words, let d be the “ducking” involution that swaps ducks d_{\rightarrow} and d_{\leftarrow} on heads, and define:

$$\begin{aligned} f_{\text{init}} &= f_{d, \text{len} < L_3} \circ (f_{\Pi_{\text{init}}}^{-1} \circ f_{d, \text{len} < L_3} \circ f_{\Pi_{\text{init}}}) \\ f_{k \mapsto k+1} &= f_{d, k+2 \leq \text{len} < L_3} \circ (f_{\Pi_{k \mapsto k+1}}^{-1} \circ f_{d, k+2 \leq \text{len} < L_3} \circ f_{\Pi_{k \mapsto k+1}}) \\ f_{\ell, \text{final}} &= f_{d, \text{len} = l} \circ (f_{\Pi_{\ell, \text{final}}}^{-1} \circ f_{d, \text{len} = l} \circ f_{\Pi_{\ell, \text{final}}}) \end{aligned}$$

Each of these elements has word norm $O(L^2 + L^3)$. Then, define:

$$f_{\text{encode}} = \left(\prod_{\ell=1}^{L_3-1} f_{2\ell, \text{final}} \right) \circ \left(\prod_{k=1}^{L_3-1} f_{k \mapsto k+1} \right) \circ f_{\text{init}}$$

The automorphism f_{encode} , which acts on conveyor belts of length $< L_3$ and encodes SMART configurations with ducks d_{\rightarrow} into their correct encoding, and produces garbage on ducks d_{\leftarrow} , has word norm $O(L^4)$. (A similar automorphism exists for conveyor belts of length $\leq L_3$).

2. The addition of k on ducks d_{\rightarrow} (word norm $O(L^3)$). We follow the proof of Lemma 4.11. First, we manage addition modulo 3^{ℓ} in every conveyor

belt of length $\ell < L_3 = 12L - 2$. To do so, we use the commutator trick on the proofs of Lemma 4.7 and Lemma 5.3, so that we can build the ternary addition of each of the first ℓ digits of n in base 3 on conveyor belts of length $\leq \ell$ with norm $O(\ell^2)$, which means that adding $n \bmod 3^\ell$ to every conveyor belt of length ℓ for $\ell < L_3$ can be done in word norm $O(L^3)$.

Let us now focus on addition modulo $2 \cdot 3^\ell$. Let b_ℓ be the integer division of n by 3^ℓ modulo 2, and q_ℓ be the quotient of n by $2 \cdot 3^\ell$. In conveyor belts of length exactly ℓ , we can manage the addition of b_ℓ to the head, while considering the carry of the previous ternary addition, with word norm $O(\ell^2)$. Considering the sizes of conveyor belts one by one, this step can be done on all conveyor belts of length $\ell < L_3$ in word norm $O(L^3)$.

Finally, one has to shift the tape by $q_\ell \bmod \ell$ steps (or $q_\ell + 1 \bmod \ell$ in case of an overflow) left or right (depending on the state) in conveyor belts of size ℓ . Using the same commutator trick on states $\blacktriangleright, \blacktriangleleft$ and \lhd, \rhd as in the proof of Lemma 4.11, this can be done with word norm $O(\ell^2)$. Considering the sizes of conveyor belts one by one, this step can be done on all conveyor belts of length $\ell < L_3$ in word norm $O(L^3)$.

All in all, the automorphism f_{+n} that performs the addition of n in conveyor belts of all sizes $< L_3$ with ducks d_\rightarrow , has word norm $O(L^3)$.

Then, conjugating f_{+n} by f_{encode} performs $(f_{S_*})^n$ on heads with duck d_\rightarrow on conveyor belts of length $< L_3$, and is the identity otherwise. Adding the same automorphism conjugated with f_d and composing, we obtain $(f_{M_s, \text{len} < L_3})^{2n'}$ with word norm $O(L^4)$. Similar formulas exist for $(f_{M_s, \text{len} \leq L_3})^{2n'}$. \square

As a final remark, the word norm of this implementation of $(f_{S_*})^n$ is $O(\log n \cdot (\omega_{\leq}(\log n) + \omega_+(\log n)))$, where $\omega_{\leq}(N)$ is the complexity of the lexicographic inequalities on words of length N , and $\omega_+(N)$ is the complexity of the ternary addition on words of length N . While we could not find a way to perform $\omega_+(N)$ with complexity less than $O(N^3)$, it would be interesting to optimize this specific operation by itself.

6 Corollaries

We prove the other theorems listed in the introduction, all of which are straightforward corollaries of Theorem A.

6.1 Distortion in other subshifts

Theorem B. *Let X be a sofic shift. Then $\text{Aut}(X)$ contains a distortion element if and only if X is uncountable.*

Proof. If X is uncountable, then $\text{Aut}(A^\mathbb{Z}) \leq \text{Aut}(X)$ [43, 29]. If X is countable, then the proof of Proposition 2 in [47] shows that every automorphism $f \in \text{Aut}(X)$ is either periodic or admits a *spaceship*, namely a configuration of the form $x = \dots uuuuvwww\dots$ which is not spatially periodic, and $f^n(x) = \sigma^m(x)$ for some $m \neq 0$. Clearly this prevents distortion. \square

Recall that the lower entropy dimension [33] is

$$\underline{D}(X) = \liminf_{k \rightarrow \infty} \frac{\log(\log N_k(X))}{\log k}$$

We recall and prove Lemma 1.1 (which was used to prove Theorem C).

Lemma 1.1. *Let X be a subshift with lower entropy dimension less than $1/d$. If $f \in \text{Aut}(X)$ satisfies $|f^n| = O(\log^d n)$, then f is periodic.*

Proof. Suppose we have $|f^n| = O(\log^d n)$ for large n . Then the radius of f^n is also $O(\log^d n)$. It follows that the trace subshift of f has complexity function at most $n \mapsto N_{\lfloor C \log^d n \rfloor}(X)$ for some constant C . If f is not of finite order, by the Morse-Hedlund theorem we must have $N_{\lfloor C \log^d n \rfloor}(X) > n$ for all n . Substituting $\lfloor e^{\sqrt[4]{n/C}} \rfloor$ for n we get $N_n(X) \geq e^{\sqrt[4]{an}}$ for some constant $a > 1$. Substituting this lower bound into the definition of lower entropy dimension, we get $\underline{D}(X) \geq \frac{1}{d}$. \square

6.2 Distortion in the group of Turing machines

We recall the definition of the group of Turing machines from [2].

Definition 6.1. *Let $n \geq 2$ and $k \geq 1$. Let Y_n be the full shift on n letters, and $X_k = \{x \in \{0, 1, \dots, k\}^{\mathbb{Z}} \mid 0 \notin \{x_i, x_j\} \implies i = j\}$. Then*

$$\text{RTM}(n, k) = \{f \in \text{Aut}(Y_n \times X_k) \mid f|_{Y_n \times \{0^{\mathbb{Z}}\}} = \text{id}|_{Y_n \times \{0^{\mathbb{Z}}\}}\}.$$

Theorem D. *Let $n \geq 2, k \geq 1$. Then the group of Turing machines $\text{RTM}(n, k)$ contains a distortion element; indeed there is a finitely-generated subgroup $G = \langle F \rangle$ and an element f such that $|f^n|_F = O(\log^4 n)$.*

Proof. We show that it immediately follows from the main theorem that $\text{RTM}(72, 384)$ has a distortion element. We then explain how to conclude this for all $\text{RTM}(n, k)$.

Recall that our automorphisms use the alphabet

$$\Sigma_{\star} = \left(\Gamma_{\star}^2 \times \{+1, -1\} \right) \sqcup \left((Q_{\star} \times \Gamma_{\star}) \times \Gamma_{\star} \right) \sqcup \left(\Gamma_{\star} \times (Q_{\star} \times \Gamma_{\star}) \right)$$

where $\Gamma_{\star} = \Gamma \times \{0, 1\}$ and $Q_{\star} = Q \times \{d_{\rightarrow}, d_{\leftarrow}\} \times \left(\{+1, -1\} \times \{0, 1\}^2 \times \Gamma \right)$.

We may instead view this as

$$(\Gamma_{\star}^2 \times \{+1, -1\}) \sqcup (Q_{\star} \times \{\uparrow, \downarrow\} \times \Gamma_{\star}^2),$$

by grouping $(Q_{\star} \times \Gamma_{\star}^2)$ and $(\Gamma_{\star} \times (Q_{\star} \times \Gamma_{\star}))$ together and replacing the choice with an arrow from $\{\uparrow, \downarrow\}$. Next, moving $\{\uparrow, \downarrow\}$ to the state and dropping $\{+1, -1\}$ out of it, we may view this as

$$(\Gamma_{\star}^2 \times \{+1, -1\}) \sqcup (Q' \times \{+1, -1\} \times \Gamma_{\star}^2),$$

for a certain set of states Q' with $|Q'| = |Q_{\star}| = 384$.

Consider the sofic subshift Z where a symbol of $(Q' \times \{+1, -1\} \times \Gamma^2)$ can appear at most once. We clearly have a conjugacy $Z \cong X_{384} \times Y_{72}$, since $|\Gamma_{\star}^2 \times \{+1, -1\}| = 72$.

It is easy to see that all of the generators F defined in Lemma 5.8 fix Z . Furthermore, our generators only act near the head, so by definition this restricted action makes them elements of $\text{RTM}(72, 384)$. The element $f_{\mathcal{M}_s}$ coming from the SMART machine clearly has infinite order, since it acts as the SMART machine on infinite configurations. The word norm of $f_{\mathcal{M}_s}$ w.r.t. F of course cannot grow faster after restricting these elements to an invariant subspace, so we obtain that the subgroup of $\text{RTM}(72, 384)$ generated by F still has a distortion element, and the distortion is at least as bad as on the full shift.

Now, we describe some minor modifications to the main construction that allow to conclude the result for $\text{RTM}(n, k)$. In the construction of the main theorem, in place of the alphabet recalled above, take any finite set S and use instead

$$((\Gamma^2 \times \{+1, -1\}) \sqcup S) \sqcup (Q' \times ((\Gamma^2 \times \{+1, 1\}) \sqcup S)).$$

Imagining elements of S as new empty conveyor belts of size 1, it is clear how most generators of F should act, as their action is defined by how they act on finite conveyor belts. The element θ does not respect the conveyor belts, but it is also clear how it should act (now that we have moved $\{+1, -1\}$ out of the state onto the tape) – it simply moves all heads.

Now recall that the only use of θ was to make sure that the automorphisms $f_{cb, \rightarrow t}$ (resp. $f_{cb, t \leftarrow}$ and $f_{cb, t \leftrightarrow t}$) are in our group. These automorphisms apply the involution $(+1) \leftrightarrow (-1)$ on the sign carried either by $\Gamma_*^2 \times \{+1, -1\}$ or by a head, at distance t to the right of the heads (resp. left of the heads, resp. both left and right of the heads). The correct extension of these is simply that the flip $(+1) \leftrightarrow (-1)$ does nothing on symbols in S . Then θ allows the implementation of natural analogs of the automorphisms $f_{cb, \rightarrow t}$, $f_{cb, t \leftarrow}$ and $f_{cb, t \leftrightarrow t}$ (with the exact same description).

Next, we recall that the automorphisms $f_{cb, \rightarrow t}$ are only used “through conjugation”, i.e. they are used during the two-scale trick in very specific situations where we already know the head is on a large conveyor belt, in particular there are no S -symbols in the affected part. Thus the proof goes through without any modifications.

The introduction of S with $|S| = t$ changes the group of Turing machines from $\text{RTM}(72, 284)$ to $\text{RTM}(72 + t, 384)$, and $\text{RTM}(72 + t, 384)$ clearly embeds in $\text{RTM}(72 + t, 384 + \ell)$ for any $\ell \geq 0$ (by behaving as the identity when the head is in one of the ℓ many new states). In particular for large enough m , we can pick t, ℓ such that $72 + t = n^m$ and $384 + \ell = kn^m$, to get a distortion element in a subgroup of $\text{RTM}(n^m, kn^m)$. Finally, there is an embedding of $\text{RTM}(n^m, kn^m)$ into $\text{RTM}(n, k)$, by considering m -blocks of cells as individual cells, and considering the word on the tape at the origin as part of the state (indeed this is an isomorphism). \square

6.3 Distortion in the Brin-Thompson group mV

It was shown by Belk and Bleak that classical reversible Turing machines embed in the Brin-Thompson group $2V$. More generally, the group of Turing machines embeds in $2V$, and indeed this embedding is entirely transparent. For this, we recall the *moving-tape model* of Turing machines.

Definition 6.2. Write $\text{RTM}_{\text{fix}}(n, k)$ for the family of homeomorphisms $f : [k] \times [n]^{\mathbb{Z}} \rightarrow [k] \times [n]^{\mathbb{Z}}$ such that for some radius $r \geq 1$ and local rule $f_{\text{loc}} :$

$\{0, 1\}^r \times \{0, 1\}^r \times [k] \rightarrow \{0, 1\}^* \times \{0, 1\}^* \times [k]$ we have

$$f(xu.vy, a) = (xu'.v'y, b) \text{ whenever } f_{\text{loc}}(u, v, a) = (u', v', b)$$

and for all u, v , $f_{\text{loc}}(u, v) = (u', v', n)$ satisfies $|u'| + |v'| = 2r$.

A proof of the following easy fact was outlined in [2]; one simply translates tape shifts into head movement into the opposite direction.

Lemma 6.3. *The family of homeomorphisms $\text{RTM}_{\text{fix}}(n, k)$ forms a group under composition, and there is a canonical group isomorphism $\text{RTM}_{\text{fix}}(n, k) \cong \text{RTM}(n, k)$.*

Lemma 6.4. *The group $\text{RTM}(n, k)$ embeds in the Brin-Thompson group mV for all $m \geq 2, n \geq 2, k \geq 1$.*

Proof. The group $2V$ embeds in mV , so it is enough to show this for $m = 2$. This is very similar to the proof in Belk-Bleak [4], and was also essentially stated in [2], so we only outline the proof. First, it is enough to embed $\text{RTM}(n, 1)$, since $\text{RTM}(n, k)$ embeds in $\text{RTM}(n, k + \ell)$ for all $\ell \geq 0$, thus in particular in $\text{RTM}(n, n^m)$ for sufficiently large m , and this group is isomorphic to $\text{RTM}(n, 1)$ (see the end of the proof of Theorem D).

Now pick a complete suffix code $C \subset \{0, 1\}^*$ of cardinality n , and a complete prefix code $D \in \{0, 1\}^*$ of cardinality n . One can uniquely parse any $x.y \in \{0, 1\}^{\mathbb{Z}}$ as $\cdots u_{-2}u_{-1}.v_0v_1v_2 \cdots$ with $u_{-i} \in C, v_i \in D$ for all applicable i , which gives a homeomorphism $\phi : \{0, 1\}^{\mathbb{Z}} \rightarrow [n]^{\mathbb{Z}}$. For $g \in \text{RTM}(n, 1)$, the map g^ϕ is easily seen to be in $2V$, so this gives a group-theoretic embedding of $\text{RTM}(n, 1)$ into $2V$. \square

Dynamically, the proof gives a topological conjugacy between the natural action of $\text{RTM}(n, 1)$ and the natural action of the subgroup of $2V$ that respects the encoding.

Theorem E. *The Brin-Thompson group mV contains a distortion element; indeed there is an element f such that $|f^n| = O(\log^4 n)$.*

Proof. The embedding of the group $\text{RTM}(n, k)$ in particular embeds the group where we constructed a distortion element. Adding the finite generating set of mV clearly cannot make the element less distorted. \square

7 Open questions

Question 7.1. *Are there ever distortion elements in $\text{Aut}(X)$ for X a minimal subshift? What about X of zero entropy?*

Minimal subshifts are interesting, because at present we do not know any restrictions on their automorphism groups, yet all known examples are locally virtually abelian. Zero entropy is interesting because on the one hand there are many known examples of interesting behaviors in their automorphism groups, but [16] shows that exponential distortion is impossible.

Next, it seems worth recalling the remaining parts of [16, Questions 5.1–3].

Question 7.2. Are there more natural subgroups having distortion elements in $\text{Aut}(A^{\mathbb{Z}})$, or even in $\text{Aut}(X)$, where $X \subset A^G$ is an arbitrary subshift on an abelian group G ? For example, can we embed the Heisenberg group (or more generally $\text{SL}(3, \mathbb{Z})$), or the Baumslag-Solitar group $\text{BS}(1, n)$?

The Heisenberg group was originally asked about in [29] (though not explicitly due to distortion concerns). One important note about this group is that every (infinite f.g. torsion-free nonabelian) nilpotent group contains a copy of it. Nilpotent groups are considered some of the simplest (in the non-technical sense) kinds of infinite groups after abelian groups; in the case of automorphism groups of subshifts we can implement a wide variety of behaviors, yet embeddability of nilpotent groups remains a mystery.

Embedding the Baumslag-Solitar group is the same as finding an element of infinite order that is conjugate to a higher power of itself. We believe the SMART machine does not have this property (before or after an embedding into $\text{Aut}(A^{\mathbb{Z}})$), but we have not proved this.

A slightly more abstract question of interest is whether there exists an amenable subgroup of the automorphism group of a subshift which has distortion elements. One thing amenability rules out is groups that are too large, e.g. f.g.-universal subgroups. The Heisenberg group and $\text{BS}(1, n)$ are of course amenable (even solvable).

Question 7.3. Can a one-sided subshift have distortion elements in its automorphism group? Does $\text{Aut}(A^{\mathbb{N}})$ have distortion elements?

Note that $\text{Aut}(A^{\mathbb{N}})$ is simply the subgroup of $\text{Aut}(A^{\mathbb{Z}})$ consisting of automorphisms f such that both f and f^{-1} have “one-sided radius”, i.e. $f(x)_i, f^{-1}(x)_i$ depend only on $x_{[i, i+r]}$ for some r . We do not even know whether one-sided automorphisms of subshifts can have sublinear radius growth.

As mentioned in the introduction, the true word norm growth of our automorphism is between $\Omega(\log n)$ and $O(\log^4 n)$. It would be of great interest to pinpoint the growth up to a multiplicative constant for our automorphism, or for any other automorphism with sublinear growth.

Question 7.4. What are the distortion functions (word norm growth rates) of elements of $\text{Aut}(A^{\mathbb{Z}})$ (or $\text{Aut}(X)$ for more general subshifts)?

Of course, in the case of a non-finitely generated group like $\text{Aut}(A^{\mathbb{Z}})$, the distortion function depends on the finite generating set chosen. While it is of interest to implement distortion functions with respect to subgroups, a more natural object to consider is the directed set of distortion functions with increasing generating sets.

Similar questions can be asked about groups of Turing machines and the Brin-Thompson $2V$, where we also exhibit elements whose word norm grows polylogarithmically, but have no further control on the distortion.

A natural idea for getting control over the distortion function would be to use, in place of SMART, a general-purpose Turing machine, which is made to have sublinear movement using the reversible Hooper trick from [28] (and finally embedded in some natural way into the automorphism group of a subshift). It is known that this construction always produces Turing machines with zero Lyapunov exponents, i.e. with sublinear movement [24, 27].

Question 7.5. Does the Kari-Ollinger construction in [28] always produce distortion elements?

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