

# Conjugacy of transitive SFTs minus periodics

Ville Salo  
vosalo@utu.fi

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## Abstract

It is a question of Hochman whether any two one-dimensional mixing SFTs with the same entropy are topologically conjugate when their periodic points are removed. We give a negative answer, in fact we prove the stronger result that there is a canonical correspondence between topological conjugacies of transitive SFT and topological conjugacies between the systems obtained by removing the periodic points. This stronger result does not extend to topological conjugacies between mixing sofic shifts, or mixing SFTs with the eventually periodic points removed (for any reasonable definition of the term). In the one-sided case this extends to general shift-commuting continuous maps between mixing SFTs, while in the two-sided case it does not.

## 1 Introduction

In [3], Hochman proved that for many systems (including mixing SFTs), when their periodic points are removed entropy becomes a complete invariant for Borel isomorphism off null sets. In the same paper, Hochman asks about the possibility of the following very strong topological variant of this result [3, Problem 1.9]. The problem is quoted also in [1, 4].

**Problem.** *Let  $X, Y$  be topologically mixing SFTs on finite alphabets, and  $h(X) = h(Y)$ . Let  $X', Y'$  denote the sets obtained by removing all periodic points from  $X, Y$ . Is there a topological conjugacy between the (non-compact) systems  $X'$  and  $Y'$ ?*

We prove that this does not always happen.<sup>1</sup>

**Theorem 1.** *Let  $X, Y$  be infinite transitive SFTs. Every topological conjugacy  $\phi' : X' \rightarrow Y'$  is a domain-codomain restriction of a topological conjugacy  $\phi : X \rightarrow Y$ , and the choice of this  $\phi$  is unique.*

We initially proved the following theorem, to answer a technical question about  $\text{Aut}(X')$  by Nishant Chandgotia (who was interested in this group due to Hochman's problem). It is an easy corollary of Theorem 1.

**Theorem 2.** *Let  $X$  be an infinite transitive SFT. Then  $\text{Aut}(X)$  is isomorphic to  $\text{Aut}(X')$ .*

As in Hochman's paper [3], our main interpretation of these statements is in the two-sided case, and with this interpretation we prove the theorems under slightly weaker assumptions in Section 3. In Section 4 we prove a one-sided variant; in this case the shift action is only partial, so the setting is slightly less natural, but we obtain a stronger statement applying to all shift-commuting continuous maps (and in the one-sided variant of Theorem 2 one therefore obtains  $\text{End}(X) \cong \text{End}(X')$ ).

In Section 5 we show that the statement of Theorem 1 does not extend to topological conjugacies between two-sided mixing sofic shifts, or subshifts where the eventually periodic points are removed (for various meanings of the term), or to general shift-commuting continuous maps between two-sided mixing SFTs. We have no counterexample to Hochman's original problem or to the statement of Theorem 2 in any of these variants.

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<sup>1</sup>It is well known that two mixing SFTs can have the same entropy without being topologically conjugate, see for example Exercise 4.1.9 or Example 7.3.4 in [5].

In higher-dimensional settings, we do not even know to what extent variants of Theorem 1 hold, although some parts of our argument have direct analogs. In the two-dimensional case, it is known that the set of totally aperiodic points in the binary full shift does not admit a continuous shift-commuting map to the space of proper 3-colorings of the standard grid [2].

## 2 Definitions

We have  $0 \in \mathbb{N}$ ,  $\subset$  means the same as  $\subseteq$  and  $[a, b] = \{c \mid c \in \mathbb{Z}, a \leq c \leq b\}$ . Let  $A$  denote a finite set, called an *alphabet*. The elements of  $A$  are called *symbols*. Write  $A^* = \bigcup_{n \in \mathbb{N}} A^n$  for the set of all finite words over the alphabet  $A$ . These can be seen as the elements of the free monoid on the generating set  $A$ , and we write  $uv$  for the product of words  $u, v \in A^*$ , i.e. their concatenation. One-sided (left- or right-) infinite words  $u \in A^\mathbb{N} \cup A^{-\mathbb{N}}$  can also be concatenated with finite words, and the meaning should be clear. We index words (and one- or two-sided infinite words) with subscripts.

See [5] for a basic reference on symbolic dynamics. The set  $A^\mathbb{Z}$  of two-way infinite words over a finite alphabet  $A$  is called the *full shift* (on alphabet  $A$ ). Words that belong to a subshift are also called *points*. The *shift*  $\sigma : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$  defined by  $\sigma(x)_i = x_{i+1}$  makes it a compact dynamical system. If  $x \in A^\mathbb{Z}$ , we write  $u \sqsubset x$  if  $x_{[i, i+|u|-1]} = u$  for some  $i \in \mathbb{Z}$ . If  $X \subset A^\mathbb{Z}$  write  $u \sqsubset X$  if  $\exists x \in X : u \sqsubset x$ . A *subshift* is a subset  $X$  of  $A^\mathbb{Z}$  defined by forbidding a set of subwords  $F \subset A^*$ , in the sense that  $X = \{x \in A^\mathbb{Z} \mid \forall u \in F : u \not\sqsubset x\}$ . Subshifts are exactly the closed  $\sigma$ -invariant sets in  $A^\mathbb{Z}$ . An *SFT* is a subshift where the defining set of forbidden words can be taken to be finite, and a *sofic shift* is an image of an SFT under a shift-commuting continuous function between two subshifts.

A subshift  $X \subset A^\mathbb{Z}$  is *transitive* if for all  $u, v \sqsubset X$ , there exists  $w$  such that  $uvw \sqsubset X$ , and *mixing* if for any  $u, v$  such a word  $w$  can be found of any large enough length. A *topological conjugacy* between two subshifts is a shift-commuting homeomorphism between them. If  $X \subset A^\mathbb{Z}$  is a subshift, then we can see the dynamical system  $(X, \sigma^m)$  as a subshift by blocking consecutive  $A$ -words of length  $m$  into a new product alphabet  $A^m$  to get a subshift in  $(A^m)^\mathbb{Z}$ .

More generally a (non-compact) *dynamical system* is a topological space with a  $\mathbb{Z}$ -action, and we use the same definition of topological conjugacy for such systems. Hochman's original problem deals with the systems  $X' = \{x \in X \mid x \text{ is not periodic}\}$  where  $X$  is a subshift, (the restriction of)  $\sigma$  is of course well-defined and continuous on this set when we take the subspace topology for  $X'$ .

If  $X \subset A^\mathbb{Z}$  is a subshift, a point  $s \in X$  is *synchronizing* if there exists  $\ell \in \mathbb{N}$  such that whenever  $x, y \in X$  and  $x_{[-\ell, \ell]} = s_{[-\ell, \ell]} = y_{[-\ell, \ell]}$ , the point  $z$  defined by

$$z_i = \begin{cases} x_i & \text{if } i < \ell, \\ s_i & \text{if } -\ell \leq i \leq \ell, \\ y_i & \text{if } i > \ell \end{cases}$$

is also in  $X$ . If  $X \subset A^\mathbb{Z}$  is a subshift,  $X_\mathbb{N} = \{y \in A^\mathbb{N} \mid \exists x \in X : \forall i \in \mathbb{N} : y_i = x_i\}$  is a closed set which is invariant under the map  $\sigma : A^\mathbb{N} \rightarrow A^\mathbb{N}$  defined by the same formula as in the two-sided case, i.e. it is an  $\mathbb{N}$ -subshift. We similarly define  $X_{-\mathbb{N}}$ , and define the functions  $x \mapsto x_\mathbb{N} : X \rightarrow X_\mathbb{N}$  and  $x \mapsto x_{-\mathbb{N}} : X \rightarrow X_{-\mathbb{N}}$  in the obvious way.

A *periodic point* in  $X \subset A^\mathbb{Z}$  is  $x \in X$  satisfying  $\sigma^n(x) = x$  for some  $n \geq 0$ . We similarly define periodic points of  $X \subset A^\mathbb{N}$  (using the same formula for  $\sigma$ ) and  $X \subset A^{-\mathbb{N}}$  (shifting in the other direction). An *aperiodic point* is a point that is not periodic. For  $a \in A$  write  $a^\mathbb{Z}$  for the unique  $\sigma$ -fixed point  $x \in A^\mathbb{Z}$  satisfying  $x_0 = a$ . Define  $a^\mathbb{N}$  and  $a^{-\mathbb{N}}$  similarly. An *isolated point* in a subshift  $X_\mathbb{N}$  is one that is topologically isolated.

The *language* of a subshift  $X \subset A^\mathbb{Z}$  is the set of words  $\{u \in A^* \mid u \sqsubset X\}$ . See any standard reference on formal languages for the definition of a regular expression (this is only used in the last section).

## 3 Two-sided subshifts

**Theorem 3.** *Let  $X$  and  $Y$  be two-sided subshifts. Suppose every periodic point  $s \in X$  is synchronizing and neither  $X_\mathbb{N}$  nor  $X_{-\mathbb{N}}$  has an isolated periodic point, and the same assumptions hold for*

$Y$ . Then every topological conjugacy  $\phi' : X' \rightarrow Y'$  is a domain-codomain restriction of a topological conjugacy  $\phi : X \rightarrow Y$ , and the choice of this  $\phi$  is unique.

*Proof.* Suppose  $X, Y \subset A^{\mathbb{Z}}$  for some finite alphabet  $A$ . The non-trivial direction is to show that a topological conjugacy  $\phi' : X' \rightarrow Y'$  extends uniquely to a conjugacy  $\phi : X \rightarrow Y$ . Consider

$$Z = \{(x, \phi'(x)) \mid x \in X\} \subset X' \times Y'$$

Let  $R$  be the closure  $\bar{Z} \subset X \times Y$ , and let us think of it as a relation. Note that the system  $\bar{Z}$ , under the diagonal action  $\sigma(x, y) = (\sigma(x), \sigma(y))$ , can be seen naturally as a subshift over the alphabet  $A^2$  (the closure of a shift-invariant set is shift-invariant).

We use relation notation, and write  $xR = \{y \in Y \mid (x, y) \in \bar{Z}\}$  and  $Ry = \{x \in X \mid (x, y) \in \bar{Z}\}$ , and similarly for subsets in place of  $x$  and  $y$ . Note that we have  $XR = Y$  and  $RY = X$  because there are no isolated periodic points in  $X$  or  $Y$  (which follows from the corresponding assumptions on  $X_{\mathbb{N}}$  and  $Y_{\mathbb{N}}$ ). Our first claim is an immediate consequence of the fact  $\phi'$  is a homeomorphism.

**Claim 1.** *The sets  $xR$  and  $Ry$  are singletons for any  $x \in X'$  and  $y \in Y'$ .*

We now show that proving the same for periodic points suffices to prove the theorem.

**Claim 2.** *If  $xR$  and  $Ry$  are singletons for every periodic point  $x \in X$  and every periodic point  $y \in Y$ , then the result holds.*

*Proof.* Extend  $\phi'$  to  $X$  by mapping each periodic point  $x$  to the unique element in  $xR$ . This gives a bijection:  $Ry$  is a singleton for any  $y \in Y$  by symmetry, and we can extend  $(\phi')^{-1}$  by mapping  $y$  to this point, and the two extensions are clearly inverses. It remains to show that the extension  $\phi$  is shift-commuting and continuous. Shift-commutation is assumed on aperiodic points, and by continuity it then holds on periodic points. Continuity follows from the closed graph theorem, since by definition the graph  $R$  of  $\phi$  is closed, and  $Y$  is a compact Hausdorff space. Finally, it is clear that the choice of the extension  $\phi$  is unique, simply because aperiodic points are dense.  $\square$

We now prove that  $xR$  and  $Ry$  are indeed singletons for any periodic points  $x, y$ , under the assumptions of the theorem. Since we prove this for all systems at once and the roles of  $X$  and  $Y$  are symmetric, it suffices to prove this for sets of the form  $xR$ . Furthermore, since  $(X, \sigma^m)$  satisfies the dynamical assumptions whenever  $(X, \sigma)$  does, it is enough to prove it for fixed points for the dynamics, by replacing  $\sigma$  by  $\sigma^m$  (for both  $X$  and  $Y$ ).

**Claim 3.** *If  $s \in X$  is a fixed point for  $\sigma$ , then  $sR$  is a finite subshift.*

*Proof of claim.* It is clear that  $W = (\{s\} \times Y) \cap \bar{Z}$  is a subshift (under the diagonal action) as the intersection of two subshifts, and  $sR$  is just the projection of  $W$  to  $Y$ , thus it is a subshift. Suppose for a contradiction that this subshift is infinite. Any infinite subshift contains an aperiodic point, so there is an aperiodic point  $y \in sR$ . It follows that  $Ry$  contains both  $s$  and the aperiodic point  $(\phi')^{-1}(y)$ , contradicting Claim 1.  $\blacksquare$

Now observe that if  $sR$  is a finite subshift, there exists  $m > 0$  such that  $\sigma^m(y) = y$  for all  $y \in sR$ . If we further replace  $\sigma$  by  $\sigma^m$  (again both for  $X$  and  $Y$ ),  $s$  will stay a fixed point, and  $sR$  becomes pointwise stabilized by the dynamics. It is thus enough to prove that  $sR$  is a singleton under this assumption.

The next crucial observation is that if  $s = a^{\mathbb{Z}}$  and  $sR$  is fixed pointwise by  $\sigma$ , then every long  $a^*$  segment of an aperiodic point must be mapped under  $\phi'$  to a long  $b^*$  segment, for some symbol  $b \in A$ , and the length difference is bounded. This is a compactness argument.<sup>2</sup>

**Claim 4.** *Suppose  $a \in A$ ,  $s = a^{\mathbb{Z}}$  and  $\sigma(y) = y$  for all  $y \in sR$ . Then there exists  $n$  such that for any  $x \in X'$ , if  $x_{[i-n, i+1+n]} = a^{2n+2}$  for  $i \in \mathbb{Z}$ , then  $\phi'(x)_i = \phi'(x)_{i+1}$ .*

*Proof of claim.* Suppose not. Then shifting such points  $x$  by  $\sigma^i$ , for each  $n$  we can find an aperiodic point  $x^n$  such that  $x_{[-n, 1+n]}^n = a^{2n+2}$ , and  $\phi'(x)_0 \neq \phi'(x)_1$ . But  $x^n \rightarrow s$  as  $n \rightarrow \infty$ , and clearly no limit point of the sequence  $\phi'(x^n)$  is fixed by  $\sigma$ , contradicting the assumption that  $sR$  contains only fixed points.  $\blacksquare$

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<sup>2</sup>Dynamically, we are using the fact that every finite dynamical system has the pseudo-orbit tracing property.

It is useful to once again replace  $\sigma$  by a power: replacing it with  $\sigma^n$  changes  $n$  to 1 in the previous claim: the symbol  $a$  is replaced by the symbol  $a^n \in A^n$ , and if we see  $a^n a^n a^n a^n$  in a configuration, then by applying the previous claim in every position, we see that the corresponding word in the image must be some  $ub^{2n}v$  where  $|u| = |v| = n$ , i.e. the symbols are  $u, b^n, b^n, v$ , of which the central two are equal.

At this point, we may therefore assume that whenever  $x_{[i,i+3]} = aaaa$ , we have  $\phi'(x)_{i+1} = \phi'(x)_{i+2}$ . So far, we have not used any property of the subshift  $X$ . The proof of the next claim crucially depends on the two dynamical properties assumed.

**Claim 5.** *Suppose  $s \in X$  is periodic. Then  $sR$  is a singleton.*

*Proof of claim.* As we have deduced, we may suppose  $s = a^{\mathbb{Z}}$  for  $a \in A$ ,  $\sigma(y) = y$  for all  $y \in sR$ , and that whenever  $x_{[i,i+3]} = aaaa$ , we have  $\phi'(x)_{i+1} = \phi'(x)_{i+2}$ .

Consider the set  $W$  of all points  $x \in X'$  such that  $x_i = a$  for all  $i \in \mathbb{N}$  and  $x_{-1} \neq a$ . Note that  $W$  is nonempty, as otherwise the point  $a^{-\mathbb{N}}$  is clearly isolated in  $X_{-\mathbb{N}}$ , contradicting the assumption. Observe that every point  $z \in \phi'(W)$  satisfies  $z_i = z_{i+1}$  for all  $i \geq 1$ . Let  $B \subset A$  be the set of all possible symbols  $z_1$  that appear. The set  $W$  is compact so  $\phi' : X' \rightarrow X'$  is uniformly continuous in  $W$ .<sup>3</sup> This means that there exists  $m$  such that whenever  $x \in X'$  and  $x_{[-m,m]} = w_{[-m,m]}$  for some  $w \in W$ , we have  $\phi'(x)_1 = \phi'(w)_1$ . Note that the assumption “ $\exists w \in W : x_{[-m,m]} = w_{[-m,m]}$ ” just means  $x_{[0,m]} = a^{m+1}$ ,  $x_{-1} \neq a$ .

Replacing  $\sigma$  with  $\sigma^m$ , we may assume that for all  $x \in X'$ , if  $x_{[-1,2]} = a'aaa$  and  $a' \neq a$  then  $\phi'(x)_1 \in B$  is determined by  $a'$ . More precisely, there exists a function  $F : A \setminus \{a\} \rightarrow A$  such that

$$\forall x \in X' : x_{[-1,2]} = a'aaa \wedge a' \neq a \implies \phi'(x)_1 = F(a').$$

Now perform a left-right symmetric argument (and possibly replace  $\sigma$  by a power yet again) to obtain that whenever  $x_{[-2,1]} = aaaa'$  where  $a' \neq a$ , the symbol  $\phi'(x)_{-1}$  is determined by the choice of  $a'$ . Let  $C \subset A$  be, symmetrically with how  $B$  was defined, the symbols that appear as such  $\phi'(x)_{-1}$ .

Applying the conclusions of the two paragraphs above on both sides of a finite interval (the latter through a shift), and also applying the property from the first paragraph (which does not disappear when passing to a power of the shift, as explained above the present claim), we deduce that whenever  $x_{[j,j+k]} = a^{k+1}$  with  $k \geq 2$  and  $a \notin \{x_{j-1}, x_{j+k+1}\}$ , we have  $\phi'(x)_i = \phi'(x)_{i+1}$  for all  $i \in [j+1, j+k-2]$ , and the symbol  $\phi'(x)_{j+1} = \phi'(x)_{j+k-1}$  is determined uniquely by both  $x_{[j-1,j+2]}$  and by  $x_{[j+k-2,j+k+1]}$ .

Now we use the fact that  $a^{\mathbb{Z}}$  is synchronizing. We obtain  $\ell$  such that if  $x, y \in X$  satisfy  $x_{[0,\ell-1]} = y_{[0,\ell-1]} = a^{\ell}$ , then the point  $z$  defined by

$$z_i = \begin{cases} x_i & \text{if } i < 0, \\ a & \text{if } 0 \leq i < \ell, \\ y_i & \text{if } i \geq \ell \end{cases}$$

is also in  $X$ . (The definition of synchronization considers a two-sided interval  $[-\ell, \ell]$ , but for a point of the form  $a^{\mathbb{Z}}$  this makes no difference up to changing  $\ell$  and conjugating by a power of the shift.) By replacing  $\sigma$  with  $\sigma^\ell$ , we may assume  $\ell = 1$ , and clearly we can then apply the construction also for higher  $\ell$ . We will apply this with  $\ell \in \{2, 3\}$ .

Suppose that either  $B$  or  $C$  is not a singleton. Then in particular we can find  $b \in B$  and  $c \in C$  with  $b \neq c$  such that for some  $x, x' \in X$  we have  $x_i = x'_{-i} = a$  for all  $i \geq 0$  and  $x_{-1} \neq a$ ,  $x'_1 \neq a$ , and  $\phi'(x)_1 = b \neq c = \phi'(x')_{-1}$ . Now apply the synchronization assumption (with  $\ell = 2$ ) to  $x$  and  $\sigma^{-2}(x')$ , to obtain a point  $x''$  with  $x''_{[-1,2]} = x_{[-1,2]}$  and  $x''_{[0,3]} = \sigma^{-2}(x')_{[0,3]} = x'_{[-2,1]}$ . Suppose first that  $x''$  is aperiodic. Then we have

$$\phi'(x'')_1 = \phi'(x)_1 = b \neq c = \phi'(x')_{-1} = \phi'(\sigma^2(x''))_{-1} = \sigma^2(\phi'(x''))_{-1} = \phi'(x'')_1,$$

a contradiction. If  $x''$  is periodic, then apply synchronization to  $x$  and  $\sigma^{-3}(x')$  with  $\ell = 3$ . If  $x''$  is periodic, this point clearly cannot be, and an analogous calculation gives a contradiction.

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<sup>3</sup>To spell out a topological “subtlety” here, we mean more than just  $\phi'|_W : W \rightarrow X'$  being uniformly continuous, namely  $\forall \epsilon > 0 : \exists \delta > 0 : \forall w \in W, x \in X' : d(w, x) < \delta \implies d(\phi'(w), \phi'(x)) < \epsilon$ , where  $d$  is the induced metric from any metric on  $X$ .

We now have  $B = C = \{b\}$  for some  $b \in A$ . This clearly implies  $sR = \{b^{\mathbb{Z}}\}$ , since in any aperiodic point close to  $s$  the sequence of  $as$  containing the origin eventually breaks either on the left or on the right, and a run of  $bs$  reaching the origin is forced by the assumptions. ■

By Claim 2, Claim 5 concludes the proof of the theorem. □

Theorem 1 follows directly, because any long enough word is synchronizing in an SFT, and one-sided infinite transitive SFTs do not have isolated points. For Theorem 2, map  $f \in \text{Aut}(X)$  to  $f|_{X', X'} : X' \rightarrow X'$ . This is clearly well-defined and a group homomorphism. It is surjective by the first claim in Theorem 1, and injective by the second, therefore it is a group isomorphism.

## 4 One-sided subshifts

Subshifts are two-sided in the paper of Hochman, so our Theorem 1 indeed solves his problem. However, as written, one can also interpret the problem in the one-sided category. Directly removing the periodic points typically leads to a set which is not invariant for the shift action, so  $\sigma$  is only a partial action. We again set  $X' = \{x \in X \mid x \text{ is aperiodic}\}$ . In this case, we take topological conjugacy to mean a homeomorphism that commutes with  $\sigma$  whenever  $\sigma$  is defined on  $X'$ , i.e. if both  $x$  and  $\sigma(x)$  are aperiodic, then  $\sigma(\phi'(x)) = \phi'(\sigma(x))$  (in particular  $\sigma(\phi'(x))$  must be aperiodic when the codomain of  $\phi'$  is another subshift with periodic points removed).

Whether or not this interpretation is natural can be debated, but we nevertheless solve the problem of Hochman in a strong form with this interpretation.

**Theorem 4.** Suppose  $Z \subset A^{\mathbb{Z}}$  is a subshift where every periodic point  $s \in Z$  is synchronizing and neither  $Z_{-\mathbb{N}}$  nor  $Z_{\mathbb{N}}$  has an isolated periodic point, and let  $X = Z_{\mathbb{N}}$ . Then every shift-commuting continuous function  $\phi' : X' \rightarrow Y'$  is a domain-codomain restriction of a shift-commuting continuous function  $\phi : X \rightarrow Y$ , and the choice of this  $\phi$  is unique.

*Proof.* Define the relation  $R \subset X \times Y$  analogously to the two-sided case. Again it is enough to show that for every periodic point  $x$ ,  $xR$  is a singleton (whose unique element is then automatically a periodic point). Suppose that this is not the case. Again passing to a power of the shift we may assume that for some  $a \in A$  and some  $i \in \mathbb{N}$ , for arbitrarily large  $n$  we can find  $x, x' \in X'$  such that  $x_{[0, n-1]} = x'_{[0, n-1]} = a^n$ ,  $x_n \neq x'_n$ , and  $\phi'(x)_i \neq \phi'(x')_i$ .<sup>4</sup>

Consider such a fixed  $n$ , and observe that by uniform continuity of  $\phi'$  on  $\{x, x'\}$  we can find  $m \in \mathbb{N}$  and words  $u, v \in A^m$  such that  $u_0 \neq v_0$ , for any two points  $y, y' \in X'$  such that  $y_{[0, n+m-1]} = a^n u, y'_{[0, n+m-1]} = a^n v$  we have  $\phi'(y)_i \neq \phi'(y')_i$ , and at least one such pair  $y, y'$  exists (i.e.  $a^n u$  and  $a^n v$  appear in the language of  $X$ ).

By the assumption that  $Z_{-\mathbb{N}}$  has no isolated periodic points, there exists a point of the form  $ba^{\mathbb{N}}$  in  $X$ , for some  $b \neq a$ . By synchronization of  $a^{\mathbb{Z}}$ , we have  $ba^{\ell}uz, ba^{\ell}vz' \in X'$ , for some  $z, z' \in X$  and some  $\ell \geq n$ . By repeatedly applying the assumption that  $Z_{\mathbb{N}}$  has no isolated periodic points, and synchronization of periodic points, we may assume  $z, z' \in X'$ .

The same  $b \neq a$  appears for infinitely many  $n$ , and from

$$\phi'(ba^{\ell}uz)_{i+1} = \phi'(a^{\ell}uz)_i \neq \phi'(a^{\ell}vz')_i = \phi'(ba^{\ell}vz')_{i+1}$$

it follows that  $\phi'$  is not continuous at  $ba^{\mathbb{N}} \in X'$ . □

## 5 Counterexamples

Theorem 3 does not cover mixing sofic shifts, and indeed the conclusion fails there:

**Example 1:** For a two-sided mixing sofic shift  $X$ , an automorphism  $f' \in \text{Aut}(X')$  may not be the restriction of an automorphism  $f \in \text{Aut}(X)$ : Consider the smallest subshift whose language contains  $((0^* + 1^*)2(0^* + 1^*)3)^*$ . Define  $f'$  by swapping each occurrence of  $2a^n$  with the word  $2(1-a)^n$  and each (possibly overlapping) occurrence of  $a^n3$  with the word  $(1-a)^n3$ , for  $a \in \{0, 1\}$ . ○

<sup>4</sup>The fact  $i$  can be taken to be 0 does follow from shift-commutation, but this needs a small additional argument since  $x, x'$  might be eventually periodic, and it does not simplify the rest of the proof.

Note that the proof of Theorem 1 shows in general that the “image subshift”  $sR$  must be finite for a periodic point  $s$ . The above example shows it can indeed have positive cardinality, but in the example the period is the same. We give another simple example where the period is different.

**Example 2:** Consider the two-sided even shift  $X \subset \{0,1\}^{\mathbb{Z}}$ , namely the smallest subshift whose language contains  $(1(00)^*)^*$ , and  $Y \subset \{0,1,2\}^{\mathbb{Z}}$  the smallest subshift whose language contains  $(2(01)^*)^*$ . These are mixing sofic shifts. Define  $f' : X \rightarrow Y$  by rewriting each subword  $0^{2n}1$  to  $(01)^n2$ , and  $10^{2n}$  to  $2(01)^n$ . Clearly this gives a topological conjugacy between  $X'$  and  $Y'$ , but  $f'$  is not the restriction of a topological conjugacy between  $X$  and  $Y$ .  $\circ$

The one-sided result in Theorem 4 does not extend to the two-sided case. In fact in the two-sided case, the subshift  $sR$  need not even be finite when invertibility of  $\phi'$  is not assumed.

**Example 3:** There are two-sided mixing SFTs  $X, Y$  and a shift-commuting continuous map  $f' : X' \rightarrow Y'$  which is not the restriction of any continuous shift-commuting map  $f : X \rightarrow Y$ . From the full shift on symbols  $\{0,2\}$ , map to the full shift on symbols  $\{0,1,2\}$  by mapping 2 to 2, and between two 2s, in a segment  $20^n2$  write 1 at the midpoint between the 2s (with an arbitrary tie-breaking rule). Tails of the form  $20^{\mathbb{N}}$  and  $0^{-\mathbb{N}}2$  are fixed. This clearly defines a shift-commuting continuous map from  $\{0,2\}^{\mathbb{Z}}$  to  $\{0,1,2\}^{\mathbb{Z}}$ , but no extension of  $f'$  to  $X$  is continuous at  $0^{\mathbb{Z}}$ .  $\circ$

One may also ask what happens if we remove a larger set than just the periodic points. In the one-sided case  $X \subset A^{\mathbb{N}}$ , it is tempting to remove all *eventually periodic points*, i.e. points  $x \in X$  whose orbit is finite, as the resulting set  $X''$  is the largest shift-invariant set not containing any periodic points. We give a stronger example.

**Example 4:** Let  $X = \{0,1,2\}^{\mathbb{N}}$  and let

$$Y = X \setminus \{x \in X \mid \sum_i x_i < \infty\}.$$

Let  $g \in \text{Aut}(Y)$  be map that rewrites  $a \in \{1,2\}$  to  $3 - a$  if the distance to the nearest symbol from  $\{1,2\}$  on the right is odd, and otherwise does not. On  $Y$ , this map is continuous, because for all  $x \in Y$  we can find an open neighborhood that specifies the positions of at least  $n$  symbols from the alphabet  $\{1,2\}$ , and this determines the new values at at least the first  $n - 1$  positions. Clearly this map has no continuous extension to any point  $x \in X \setminus Y$ .

It is easy to see that the restriction of  $g$  to  $X''$  gives an element of  $\text{Aut}(X'')$ , where  $X''$  is  $X$  without its eventually periodic points (note that  $g$  is an involution and its definition clearly implies that the image of an eventually periodic point is eventually periodic), and there is clearly no continuous extension to for example  $10^{\mathbb{N}}$  (for this we only need that points in  $X''$  can begin with words  $10^na$  with  $a \neq 0$ , where  $n$  is arbitrarily large and of arbitrary parity).  $\circ$

In the two-sided case, one can remove points that have an eventually periodic right tail or left tail (or one can remove both types of points, or points that are of both types simultaneously), or one can remove points that agree with some periodic point in all but finitely many positions. The following example covers all these cases, by a similar argument as in the previous example.

**Example 5:** Let  $X = \{0,1,2\}^{\mathbb{Z}}$  and let

$$Y = X \setminus \{x \in X \mid \exists n \in \mathbb{Z} : x_n \in \{1,2\} \wedge \forall i \neq n : x_i = 0\}$$

(i.e. we remove only two orbits from  $Y$ ). Let  $g \in \text{Aut}(Y)$  be map that rewrites  $a \in \{1,2\}$  to  $3 - a$  if the distance to the nearest symbol from  $\{1,2\}$  is odd, and otherwise does not. On  $Y$ , this map is continuous, because for all  $x \in Y$  we can find an open neighborhood that specifies at least two non-zero coordinates, and this gives a bound on how far we have to look to deduce the  $i$ th coordinate of the image of  $x$ . Clearly this map has no continuous extension to any point  $x \in X \setminus Y$ .  $\circ$

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