Entropy pair realization

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Abstract

We show that the CPE class α of Barbieri and García-Ramos contains a one-dimensional subshift for all countable ordinals α, i.e. the process of alternating topological and transitive closure on the entropy pairs relation of a subshift can end on an arbitrary ordinal. This is the composition of three constructions: We first realize every ordinal as the length of an abstract "close-up" process on a countable compact space. Next, we realize any abstract process on a compact metric space as the process started from a shift-invariant relation on a subshift, the crucial construction being the implementation of every zero-dimensional compact metrizable space as an open invariant quotient of a subshift. Finally we realize any shift-invariant relation E on a subshift X as the entropy pair relation of a supershift $Y \supset X$, and under strong technical assumptions we can make the CPE process on Y end on the same ordinal as the close-up process of E.

1 Introduction

The *entropy* of a subshift X is the limit of $\frac{\log |\mathcal{L}_n(X)|}{n}$ where $\mathcal{L}_n(X)$ is the set of words in X of length n, as $n \to \infty$. It measures the amount of information in orbits of this dynamical system. There are several ways to quantify how this information is produced, one is the entropy pairs relation, which measures the amount of information in orbits when we obtain information only when the orbit approximates x or y very well.[1] We provide [2] as an entry point to the literature.

The entropy pairs relation (plus the diagonal) is symmetric, reflexive and topologically closed but not necessarily transitive, and a subshift turns out to have only positive entropy (non-trivial) factors if and only if the smallest closed equivalence relation containing the entropy pairs relation is the full relation [3]. The hierarchy generated by alternately closing the relation topologically and transitively is studied in [2], and it is shown that the full relation can arise at any countable ordinal. For ordinals larger than 3, the examples are not expansive. Nishant Chandgotia asked in personal communication whether the same can be done with subshifts.

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[1] This is a convenient intuitive understanding for applying this notion, but I make no claim that this is an epistemologically correct description.

2 The results

By alternating topological and transitive closure on a symmetric reflexive relation E, one obtains a sequence of equivalence relations, and the union of these equivalence relations is of course just the smallest closed equivalence relation containing E.

Definition 1. *Let X be a topological space and $E \subset X^2$ a symmetric reflexive relation. If E is closed, let $\zeta = 1$, otherwise $\zeta = 0$. Define $E^{(\zeta)} = E$ and write any ordinal greater than ζ in the unique way as $\lambda + 2n + b$, λ a limit ordinal (including 0), $n \in \mathbb{N}, b \in \{0,1\}$ and define*

$$E^{(\lambda+2n+1)} = \overline{E^{(\lambda+2n)}} \qquad \text{(topological closure)}$$
$$E^{(\lambda+2(n+1))} = (E^{(\lambda+2n+1)})^* \qquad \text{(transitive closure)}$$
$$E^{(\lim_i \lambda_i)} = \bigcup_i E^{(\lambda_i)}$$

We say the close-up rank *of E is the least λ such that $E^{(\lambda)} = E^{(\lambda+1)}$. We say E is* equalizing *if $E^{(\lambda)} = X^2$ for some λ.*

Ordinals $\lambda + 2n + b$ where $b = 0$ are called *even*, and others *odd*.

The check for topological closedness is made in the beginning so that $E^{(\lambda)}$ is always topologically closed at odd ordinals.

Lemma 2. *Let $\lambda \geq 1$ be a countable ordinal. Then there exists a closed countable set $X \subset [0,1]$ and an equalizing closed symmetric reflexive relation $E \subset X^2$ with rank λ.*

For this, we arrange an ordinal of the form ω^α or $\omega^\alpha + 1$ on the circle and join each point with its successor.

In [2], Barbieri and García-Ramos study the close-up process for the closed equivalence relation $\mathrm{EP}(X) = E \cup \Delta$ where Δ is the diagonal and E is the family of *entropy pairs* of the compact topological dynamical system (G, X) (G an amenable discrete group acting on X), where (x, y) is an *entropy pair* if the open cover (U^c, V^c) has positive entropy [4, Definition 14.5] for all disjoint closed neighborhoods $U \ni x, V \ni y$.

A slight difference between our definition and that of [2] is that the sequence of steps taken is independent of original relation E (apart from the first step), in that we do not check the topological closedness of the relation at any later point. Nevertheless, topological closedness must strictly alternate on successor steps unless the process stops, and at limit ordinals only topological closedness can fail (every limit ordinal is a limit of even ordinals, and transitivity is a universal first-order property), so our process agrees with theirs.

In [2], a dynamical system is defined to be of *CPE class α* if the equivalence relation $E \cup \Delta$ is equalizing with rank α. One of the results of [2] is that among abstract dynamical systems, the CPE class α is nonempty for every group G.

Nishant Chandgotia asked me if there is a subshift realization of their process, i.e. whether there is a subshift in the CPE class α. We prove a positive answer for $G = \mathbb{Z}$. See [11] for more information on subshifts.

Theorem 3. *Let $\alpha \geq 1$ be a countable ordinal. Then there exists a subshift $X \subset A^{\mathbb{Z}}$ in CPE class α.*

We prove this by first implementing closed relations on topological spaces as shift-invariant relations on subshifts, and then implementing such relations as CPE pairs.

A point $x \in A^{\mathbb{Z}}$ is *Toeplitz* is $\forall i \in \mathbb{Z} : \exists n > 0 : \forall m \in \mathbb{Z} : x_{i+mn} = x_i$. A *Toeplitz subshift* is a subshift generated by a Toeplitz point. A *pointwise zero-entropy Toeplitz* subshift is one where every point generates a Toeplitz subshift with zero entropy. Observe that by the variational principle [4] a pointwise zero-entropy Toeplitz subshift has zero-entropy, since any generic point for an ergodic measure of positive entropy would obviously generate a Toeplitz subshift with positive entropy.

If X is a topological space, the *hit-or-miss topology* on its closed subsets has subbasis
$$\mathcal{U}_{U,K} = \{Y \subset X \text{ closed} \mid Y \cap K = \emptyset, Y \cap U \neq \emptyset\}$$
where U ranges over open sets of X and K over compact sets of X. For the space of closed subsets of a subshift, this is equal to the topology given by Hausdorff distance [9], and compares the set of words up to length n. This is the topology we use for the space of subshifts of a subshift X. The minimal subshifts form a subspace,[2] and we give it the induced topology.

Lemma 4. *For any compact metrizable zero-dimensional space Z, there exists a pointwise zero-entropy Toeplitz subshift X and an open quotient map $\phi : X \to Z$ such that $\phi^{-1}(z)$ is a minimal subshift for every $z \in Z$, and ϕ induces a homeomorphism between Z and the space of minimal subshifts of X.*

The important word in the above statement is "open" – intuitively because taking preimages of relations in an open quotient map induces a "close-up process homomorphism", in particular the close-up rank is preserved.

We show that any close-up process on a subshift, satisfying some strong technical assumptions, can be realized as the entropy pairs process. We say a relation $E \subset X^2$ is *doubly shift-invariant* if it is invariant under the action of $\mathbb{Z}^2$ that shifts the two components separately. If λ is a successor ordinal, write $\lambda - 1$ for its predecessor.

If $E \subset X^2$ is a relation, write $E^{[k]} = \{(x_1, x_2, \cdots, x_k) \mid \forall i, j : (x_i, x_j) \in E\}$ (so $E = E^{[2]}$ if and only if E is symmetric and reflexive).

Lemma 5. *Let $X \subset A^{\mathbb{Z}}$ be a subshift with zero entropy and let $E \subset X^2$ be any closed doubly shift-invariant symmetric reflexive relation. Then there exists a subshift Y with $X \subset Y \subset \hat{A}^{\mathbb{Z}} (\supset A^{\mathbb{Z}})$ such that $\mathrm{EP}(Y)^{(\lambda)} \cap X^2 = E^{(\lambda)}$ for all λ. Furthermore, if $E^{(\lambda)} \cap X^2 = X^2$, then $\mathrm{EP}(Y)^{(\lambda+1)} = Y^2$, and we have $\mathrm{EP}(Y)^{(\lambda)} = Y^2$ if*

- *λ is even, or*

- *λ is odd and $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$.*

The intuitive description is that we add an entropy-generating supersystem on top of X, by allowing alternation between words from the two points from X (separated by a special symbol) if and only if they are in the relation E. We want that X originally has no entropy pairs, so we require it has zero entropy.

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[2]This subspace need not be closed, consider for example the space of minimal subshifts of the Grand Sturmian subshift [10], where obviously no sequence of Sturmian subshifts with density of 1s decreasing to zero has no converging subsequence.

For $A \in X^k$ and $B \in X^k$, we can think of the pair (A, B) as an element $(A, B) \in X^{2k}$ in a natural way. For the intuition behind the *odd condition* in Lemma 5 for odd $\lambda > 2$, first note that $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$ means precisely that if $A, B \in E^{[k]}$ then either $(a, b) \in E^{(\lambda-1)}$ for some $a \in A$ and $b \in B$ (equivalently $(A, B) \in (E^{(\lambda-1)})^{[2k]}$ by the transitivity of $E^{(\lambda-1)}$), or we can find $A_i, B_i \in E^{[k]}$ such that $(A_i, B_i) \in (E^{(\lambda-1)})^{[2k]}$ and $(A_i, B_i) \to (A, B)$.

This odd condition holds in the construction proving Lemma 2 (see Lemma 7 for the precise statement), and is preserved under open quotients, so it passes to the subshift implementations of abstract processes given by Lemma 4.

We do not know to what extent doubly shift-invariance, the zero-entropy assumption and the odd condition can be weakened in the previous lemma. For implementing any relation as entropy pairs of a supersystem (with no control on the "new pairs"), we provide a simpler construction.

Theorem 6. *Suppose X has zero entropy and $E \subset X^2$ is a closed symmetric reflexive shift-invariant relation. Then there exists a subshift Y with $X \subset Y \subset \hat{A}^{\mathbb{Z}} \, (\supset A^{\mathbb{Z}})$ such that $\mathrm{EP}(Y) \cap X^2 = E$.*

This is optimal in the sense that for positive entropy X, it fails for E the diagonal relation [3, 8, 2], and for X zero-entropy and for any $Y \supset X$, $\mathrm{EP}(Y) \cap X^2$ is closed symmetric reflexive shift-invariant. One could, however, ask for the aesthetically more pleasing $\mathrm{EP}(Y) = E \cup \Delta_Y$ instead of just $\mathrm{EP}(Y) \cap X^2 = E$, and we conjecture that this can be done.

The constructions are made for $\mathbb{Z}$, and of course one can ask whether they can be generalized to general amenable groups. We conjecture that the question of whether all CPE classes are inhabited by subshifts on all groups has the obvious answer.

In the case of $\mathbb{Z}^d$ in particular, $d \geq 2$, one can also ask if this can be done by SFTs. We conjecture that $\mathbb{Z}^d$-SFTs inhabit exactly the CPE classes α where α is recursive.

3 The proofs

We prove the claims from the previous section. Ordinals are Von Neumann ordinals, see any standard reference (e.g. [7]) for definitions and basic properties of ordinal arithmetic.

Lemma 7. *Let $\lambda \geq 1$ be a countable ordinal. Then there exists a closed countable set $X \subset [0, 1]$ and an equalizing closed symmetric reflexive relation $E \subset X^2$ with rank λ. Furthermore, if $\lambda > 2$ is an odd ordinal, then $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$.*

Proof. Take $\alpha = \omega^\zeta$ or $\alpha = \omega^\zeta + 1$ for $\zeta \geq 1$ with the order topology, take an injective order homomorphism $\phi(\alpha + 1) \subset [0, 1]$, so that $\phi(0) = 0$, $\phi(\alpha) = 1$. Identify $0 \equiv 1$ to obtain a continuous function $\phi : \alpha + 1 \to \phi(\alpha + 1) \subset \mathbb{S}^1 = [0, 1]/\equiv$, and let $\phi(\alpha + 1) = X$. The reason for arranging the ordinal on a sphere is simply to compactify the non-compact ordinals, and the reason for wanting non-compact ordinals is to allow the last step of the close-up process to be a transitive closure step.

While we do the construction on $\mathbb{S}^1 = [0, 1]/\equiv$ rather than $[0, 1]$, we note that any non-full closed subset of $\mathbb{S}^1$ is homeomorphic to a closed subset of $[0, 1]$.

Let $E \subset (\mathbb{S}^1)^2$ be the union of the diagonal relation Δ and the successor and predecessor relations, i.e.

$$E = (\{(\phi(\beta), \phi(\beta)), (\phi(\beta), \phi(\beta+1)), (\phi(\beta+1), \phi(\beta)) \mid \beta + 1 < \alpha\}$$

Observe that if α is a successor, we do not join $\phi(\alpha) = \phi(0)$ with its "predecessor" $\phi(\alpha - 1)$. It is easy to see that X and E are closed. Consider the close-up process for this relation, starting from $E^{(1)} = E$.

Note that at all times, $E^{(\lambda)}$ for $\lambda \geq 1$ is symmetric and reflexive, so we simply have to figure out which pairs $(\phi(\beta), \phi(\beta + \gamma))$ are contained in $E^{(\lambda)}$ for each ordinal $\lambda \geq 1$. What happens is that only the length of the jumps γ matters. Alternately, we are allowed to first take less than ω steps with the current relation (a transitive closure step), and then to close up these jumps into a single superjump consisting of ω many previous jumps (a topological closure step).

Concretely, we prove an exact characterization of $E^{(\lambda)}$ by transfinite induction: Every ordinal can be written in a unique way as $\lambda + 2n$ (even ordinals) or $\lambda + 2n + 1$ (odd ordinals) where $n \in \mathbb{N}$ and λ is a limit ordinal (interpreting 0 as a limit ordinal). For all λ, n, we prove by induction for all ordinals $\lambda + 2n + b$ greater than 0 ($n \in \mathbb{N}, b \in \{0, 1\}$) that

$$E^{(\lambda + 2n)} = \{(\phi(\beta), \phi(\beta + \gamma)), (\phi(\beta + \gamma), \phi(\beta)) \mid \gamma < \omega^{\lambda + n}, \beta + \gamma < \alpha\},$$

$$E^{(\lambda + 2n + 1)} = \{(\phi(\beta), \phi(\beta + \gamma)), (\phi(\beta + \gamma), \phi(\beta)) \mid \gamma \leq \omega^{\lambda + n}, \beta + \gamma < \alpha\}.$$

The claim for $E^{(1)}$ is (setting $\lambda = 0, n = 0, b = 1$) equivalent to having (up to symmetry) $E^{(1)} \ni (\beta, \beta + \gamma)$ if and only if $\gamma < 2$, $\beta + \gamma < \alpha$. This is precisely the definition of $E^{(1)} = E$.

For the induction steps, suppose $E^{(\lambda + 2n)}$ is of the claimed form, and consider $E^{(\lambda + 2n + 1)}$. This is a topological closure step, i.e. $E^{(\lambda + 2n + 1)} = \overline{E^{(\lambda + 2n)}}$. Taking the closure of the pairs given by the inductive assumption, we clearly have

$$\{(\phi(\beta), \phi(\beta + \gamma)) \mid \gamma \leq \omega^{\lambda + n}, \beta + \gamma < \alpha\} \subset E^{(\lambda + 2n + 1)}.$$

If $\beta > 0$, then $(\phi(\beta), \phi(\beta + \gamma)) \notin E^{(\lambda + 2n + 1)}$ for $\gamma \geq \omega^{\lambda + n} + 1$, $\beta + \gamma < \alpha$ since otherwise $(\phi(\theta), \phi(\beta + \gamma)) \in E^{(\lambda + 2n)}$ for $\theta \leq \beta$ arbitrarily close to β and $\gamma \geq \omega^{\lambda + n} + 1$ (note that $\beta + \omega^{\lambda + n} + 1$ is isolated), contradicting the inductive assumption.

If $\beta = 0$, then there may be additional limits on the right end of $[0, 1]/\equiv$. In this case suppose we have $(\phi(\beta), \phi(\beta + \gamma)) \notin E^{(\lambda + 2n + 1)}$ for $\gamma \geq \omega^{\lambda + n} + 1$, $\beta + \gamma < \alpha$. Note that we must have $\omega^{\lambda + n} + 1 < \alpha$. To obtain such pairs, we must have $(\phi(\delta), \phi(\theta)) \in E^{(\lambda + 2n)}$ for $\theta < \alpha$ arbitrarily close to α and $\delta \leq \gamma$ arbitrarily close to γ, for some $\gamma \geq \omega^{\lambda + n} + 1$. If $\alpha = \omega^\zeta + 1$, then α cannot be approximated from below, so this is impossible.

If $\alpha = \omega^\zeta$, then observe that since $\gamma < \alpha$, we have $\omega^{\lambda + n} + 1 \leq \gamma < \alpha$, so since $\alpha = \omega^\zeta$, we have $\omega^{\lambda + n} + 1 \leq \gamma \leq \omega^{\zeta'} \cdot m$ for some $\zeta' < \zeta$, $m \in \mathbb{N}$ (here we simplify the successor and limit case of ζ into one), meaning

$$\gamma + \omega^{\lambda + n} + 1 \leq \omega^{\zeta'} \cdot 2m \leq \omega^{\zeta'} \cdot \omega \leq \omega^{\zeta' + 1} \leq \omega^\zeta$$

by left distributivity and monotonicity of ordinal multiplication. This concludes the analysis of the step from $\lambda + 2n$ to $\lambda + 2n + 1$.

Suppose then that $E^{(\lambda+2n+1)}$ is of the claimed form and consider $E^{(\lambda+2n+2)}$. Since $(\phi(\beta), \phi(\beta + \omega^{\lambda+n})) \in E^{(\lambda+2n+1)}$ for all β, $\beta + \gamma < \alpha$ with $\gamma \leq \omega^{\lambda+n}$, taking the transitive closure we have $(\phi(\beta), \phi(\beta + \gamma)) \in E^{(\lambda+2n+2)}$ for any $m \in \mathbb{N}$ and $\gamma < \omega^{\lambda+n} \cdot m$ with $\beta + \gamma < \alpha$. This is indeed equivalent to having $(\phi(\beta), \phi(\beta + \gamma)) \in E^{(\lambda+2n+2)}$ for any $\gamma < \omega^{\lambda+n} \cdot \omega = \omega^{\lambda+n+1}$ as required.

To see that $E^{(\lambda+2n+2)}$ does not contain any pairs $(\phi(\beta), \phi(\beta + \gamma))$ with $\gamma > \omega^{\lambda+n} \cdot \omega$, $\beta + \gamma < \alpha$, consider any tuple $(\phi(\beta_1), \phi(\beta_2), \cdots, \phi(\beta_k))$ with $(\phi(\beta_i), \phi(\beta_{i+1})) \in E^{(\lambda+2n+1)}$, $\beta_i \neq \beta_{i+1}$ for all i, and $\beta_1 < \beta_k$. If β_j is minimal among of the β_i, then it easily follows from the inductive assumption that

$$\beta_k \leq \beta_j + \omega^{\lambda+n} \cdot k \leq \beta_1 + \omega^{\lambda+n} \cdot k < \beta_1 + \omega^{\lambda+n} \cdot \omega.$$

This concludes the analysis of the step from $\lambda + 2n + 1$ to $\lambda + 2n + 2$.

Finally, we look at the case that λ is an infinite limit ordinal. By definition, $E^{(\lambda)} = \bigcup_{\delta < \lambda} E^{(\delta)}$. We need to show that $E^{(\lambda)}$ contains $(\phi(\beta), \phi(\beta + \gamma))$ for $\beta + \gamma < \alpha$ if and only if $\gamma < \omega^{\lambda}$. Since $\gamma < \omega^{\lambda}$ is equivalent to $\exists \delta < \lambda : \gamma < \omega^{\delta}$, this follows from the inductive assumption on the relations $E^{(\delta)}$. This concludes the analysis of the limit steps.

Now observe that by the above characterization, $E^{(\lambda+2n+b)} = X^2$ if and only if $(0, \gamma) \in E^{(\lambda+2n+b)}$ for all $\gamma < \alpha$. This happens if and only if $b = 0$ and $\gamma < \alpha \implies \gamma < \omega^{\lambda+n}$, i.e. $\alpha \leq \omega^{\lambda+n}$; or $b = 1$ and $\gamma < \alpha \implies \gamma \leq \omega^{\lambda+n}$, i.e. $\alpha \leq \omega^{\lambda+n} + 1$.

Now, to prove the original claim, for $\lambda = 1$ we can realize close-up rank 1 by setting $E = X^2$ for any choice of X. For larger countable ordinals, in the above construction we obtained for the choice $\alpha = \omega^{\lambda+n}$ that the close-up rank of E is $\lambda + 2n$, and that for the choice $\alpha = \omega^{\lambda+n} + 1$ it is $\lambda + 2n + 1$.

For the last sentence, observe that in the case of odd $\lambda + 2n + 1$, the pairs added at the last step are just those joining some $\phi(\beta)$ for $\beta < \alpha - 1$ with $\phi(\alpha - 1)$. We can realize any such limit through pairs $(\phi(\beta), \phi(\beta_i)) \in E^{(\lambda+2n)}$ where $\beta_i \nearrow \alpha - 1$.

Now consider any $(a_1, ..., a_k, b_1, ..., b_k) \in (E^{[k]})^2$, where we can take the a_i to be sorted in increasing order of $\phi^{-1}(a_i)$, and similarly for the b_i. If $(a_1, b_1) \in E^{(\lambda-1)}$ then

$$(a_1, ..., a_k, b_1, ..., b_k) \in (E^{(\lambda+2n)})^{[2k]} \cap (E^{[k]})^2 \subset \overline{(E^{(\lambda+2n)})^{[2k]} \cap (E^{[k]})^2}$$

follows from transitivity of $E^{(\lambda+2n)}$. Otherwise by the above paragraph, and up to symmetry, we can assume (a_1, b_1) is the limit of $(\phi(\beta), \phi(\beta_i))$. We have $b_i = b_1$ for all i since $\phi(\alpha-1)$ is not in E-relation with any other point. We have $a_i \in \{a_1, \phi(\phi^{-1}(a_1) + 1)\}$ for all i (recall that the original relation E is just the successor relation, symmetrized and reflexivized), and by the characterization of $E^{(\lambda+2n)}$ we then have $(a_i, \phi(\beta_i)) \in E^{(\lambda+2n)}$ for any a_i, as required. $\qquad \square$

The *positioned words* over A are $A^{**} = \{w : [a, b] \to A \mid a, b \in \mathbb{Z}\}$. For $w \in A^{**}$, $w : [a, b] \to A$, and $X \subset A^{\mathbb{Z}}$ a subshift (deduced from context), write $[w] = \{x \in X \mid x|_{[a,b]} = w\}$.

We use the shift convention $\sigma(x)_i = x_{i+1}$.

Lemma 4. *For any compact metrizable zero-dimensional space Z, there exists a pointwise zero-entropy Toeplitz subshift X and an open quotient map $\phi : X \to Z$ such that $\phi^{-1}(z)$ is a minimal subshift for every $z \in Z$, and ϕ induces a homeomorphism between Z and the space of minimal subshifts of X.*

Proof. Define the subshift $X' \subset \{0, 1, \#\}^{\mathbb{Z}}$ by requiring that in every configuration, there is an arithmetic progression $\{3k + i \mid k \in \mathbb{Z}\}$ containining only $\#$, such that $\{3k + i + 1 \mid k \in \mathbb{Z}\}$ is a constant sequence containing only 0 or only 1, and recursively $\{3k + i + 2 \mid k \in \mathbb{Z}\}$ is a point of X'.

This indeed describes a unique subshift whose typical points look like

$$\ldots \#z_0 \#\#z_0 z_1 \#z_0 \#\#z_0 \#\#z_0 z_1 \#z_0 z_2 \#z_0 \#\#z_0 z_1 \#z_0 \#\# \ldots$$

where $z \in \{0, 1\}^{\mathbb{N}}$, and we associate a sequence of bits to every point in X' by locating the unique infinite 3-progression of $\#$s (which can be deduced from any word of length 3), outputting the bit to the right of one (thus any) of them, and recursively extracting the rest of the bits from the third 3-progression. Let $\phi : X' \to \{0, 1\}^{\mathbb{N}}$ extract this sequence of bits, so ϕ is continuous, and $\phi(x) = \phi(\sigma(x))$ for all $x \in X'$.

Since Z compact, metrizable and zero-dimensional, we can take (up to homeomorphism) $Z \subset \{0, 1\}^{\mathbb{N}}$ a closed set such that no point in Z is eventually constant (equivalently, 01 appears infinitely many times in every $z \in Z$). Define $X = \phi^{-1}(Z)$, which is clearly a subshift. A simple computation shows that the number of words in X of length n in X is $O(n^{\log_3 6})$, so X has zero entropy.

The map ϕ is surjective and continuous by definition, and it is closed since X is compact and Z Hausdorff, so it is a quotient map from X to Z.

Let us now analyze the dynamical structure of Z. Call the unique infinite 3-progression of $\#$s in a point of X the 0-*skeleton*. Recursively we refer to the 0-skeleton of the configuration of X' obtained after i iterations of the recursive extraction process as the i-skeleton. The *skeleton* refers collectively to the family of all the i-skeletons, more precisely the skeleton of $x \in X$ is $\pi(x)$, where $\pi(\#) = \#$, $\pi(0) = \pi(1) = 0$.

Dynamically, extraction of the skeletons gives an almost-1-to-1 subshift cover of the 3-odometer $(\mathbb{Z}_3, (a \mapsto a + 1))$ (where $\mathbb{Z}_3$ is the space of 3-adic integers) more precisely $\pi(X)$ is a 2-to-1, and almost 1-to-1, cover of the 3-odometer, by the obvious map $\psi : \pi(X) \to \mathbb{Z}_3$ that records the offsets of the $\#$-progressions in an intertwining way. What happens is that a typical $\psi(\pi(x))$ determines the skeleton $\pi(x)$ entirely. If it does not, then there is a single "hole" left after filling in the $\#$s. If there is a hole left, then $|\pi^{-1}(\pi(x))| = 2$.

It is easy to see that if $\psi^{-1}(\psi(\pi(x)))$ is a singleton, then for any $z \in Z$, there is exactly one point $x' \in X$ with $\phi(x') = z$ and $\pi(x') = \pi(x)$, since after filling in the skeleton and the bits to the right of it, there is no hole left to insert a $\#$, thus no hole to insert a bit either. If there is a hole left, then we can insert any of $0, 1, \#$, but all other cells are determined by $\psi(\pi(x))$ and $\phi(x)$.

It follows that the subshift X itself is a 3-to-1 and almost-1-to-1 subshift cover of the system $\mathbb{Z}_3 \times Z$ under $(\alpha, z) \mapsto (\alpha + 1, z)$ where $\mathbb{Z}_3$ denotes the 3-odometer.

We now show openness of ϕ. Suppose $x \in [u]$ and $\phi(x) = z$. Pick $x' \in [u]$ such that

$$\forall y, y' : \pi(y') = \pi(y) = \pi(x') \wedge \phi(y) = \phi(y') \implies y = y'.$$

This is possible because $[u]$ determines only part of the skeleton of x, and we may modify the rest to find x' such that $\pi(x')$ is a singleton ψ-fiber. The bits left in u after determining part of the skeleton do not constrain this since there are no eventually constant points in Z; i.e. if there is a hole left in x after

determining the skeleton and filling in the bits of $\phi(z)$, and this hole is filled with $a \in \{0, 1, \#\}$, then we can pick x' so that it that this hole is filled with a.

Now pick $w \in \{0, 1\}^*$ with $z \in [w] \subset [v]$, long enough so the bits in u are determined by those in w, so that for all $z' \in [w]$ there exists a (unique) point $y \in [u]$ with $\pi(y) = \pi(x') \wedge \phi(y) = z'$. This gives a continuous section for $\phi|_C : C \to [w]$ where $C = \phi^{-1}([w]) \subset [u]$, so we have $\phi([u]) \supset [w] \ni z$. Thus ϕ is open.

To see that every point generates a Toeplitz subshift, let $x \in X$ be arbitrary and suppose $x \in [u]$ for some $u \in A^{**}$, $u : [a, b] \to A$. If $\psi^{-1}(\psi(\pi(x)))$ is a singleton, i.e. $\pi(x)$ and $\phi(x)$ determine x uniquely, then clearly there exists a period $3^n > 0$ such that $\sigma^{i3^n}(x) \in [u]$ for all $i \in \mathbb{Z}$, i.e. x is already Toeplitz. In general, we show that for any $x \in X$, x at least satisfies

$$\forall i \in \mathbb{Z}, k > 0 : \exists j \in \mathbb{Z} : \exists n > 0 : \forall \ell \in [-k, k], m \in \mathbb{Z} : x_{j+\ell+mn} = x_{i+\ell},$$

i.e. every subword appearing in x at i appears in some arithmetic progression in x (but not necessarily one going through position i), which implies that the orbit-closure of x contains a Toeplitz point with the same language as x, so $\overline{\mathcal{O}(x)}$ is Toeplitz.

For this, note that since $\pi(X)$ is an almost-1-to-1 cover of Z_3, we can find m such that the set $\psi^{-1}(\psi(\pi(\sigma^j(x))))$ has very small diameter, concretely meaning that the skeleton and bits to the right of #s in the skeleton determine all of the coordinates in $\sigma^j(x)_{[a,b]}$. Since $\phi(x)$ is not eventually constant, we can pick j so that together with the bits of z, the forced word is $\sigma^j(x)_{[a,b]} = u$. The subword $[j + a, j + b]$ of x then appears in an arithmetic progression because of the equicontinuity properties of the 3-odometer, as required.

Observe now that (since every Toeplitz subshift is minimal) every point generates a minimal subshift, and this subshift maps to a single point in ϕ. Conversely, the language of a minimal subshift is determined by the bits in its image in a continuous way (again use that no point in Z is constant). Thus the fibers of ϕ are precisely the minimal subshifts of X and ϕ induces a homeomorphism between them and Z. □

Next, we realize relations as entropy pairs. We first deal with the easy case where E is not doubly shift-invariant, but we only realize the relation, not the whole process.

For $y \in A^{\mathbb{Z}}$, write $w \sqsubset y \iff \exists i : y_{[i,i+|w|-1]} = w$, and for $Y \subset A^{\mathbb{Z}}$ write $w \sqsubset Y \iff \exists y \in Y : w \sqsubset y$.

Theorem 6. *Suppose X has zero entropy and $E \subset X^2$ is a closed symmetric reflexive shift-invariant relation. Then there exists a subshift Y with $X \subset Y \subset \hat{A}^{\mathbb{Z}} (\supset A^{\mathbb{Z}})$ such that $\mathrm{EP}(Y) \cap X^2 = E$.*

Proof. Let $\hat{A} = A \cup \{\#\}$, fix a sequence $(x^n, y^n) \in E$ such that each pair $(x, y) \in E$ is a limit point of the sequence, and define

$$Y_n = \{\ldots \# u_i \# u_{i+1} \# u_{i+2} \# \ldots \mid \forall i : u_i \in \{x^n_{[-n,n]}, y^n_{[-n,n]}\}\}$$

(i.e. Y_n is the shift-invariant set of all points of the stated form). Let $Y = \overline{\bigcup_n Y_n}$.

If $(x, y) \in E \setminus \Delta$, then obviously[3] by picking larger and larger n such that (x^n, y^n) is very close to (x, y), $Y_n \subset Y$ implies that $(x, y) \in \mathrm{EP}(Y)$. Suppose then that $(x, y) \in \mathrm{EP}(Y) \setminus E$. Since E is closed, for some n we have $([x_{[-k,k]}] \times [y_{[-k,k]}]) \cap E = \emptyset$. We claim that for $U = [x_{[-k,k]}], V = [y_{[-k,k]}]$, the cover (U^c, V^c) has zero topological entropy.

It is enough to give, as a function of m, a set of binary words $U \subset \{0, 1\}^m$, whose size grows subexponentially in m, such that for any $w \in \mathcal{L}_m(Y)$, $(U^c, V^c)^u$ intersects[4] $[w]$, where $(U^c, V^c)^u$ is defined inductively on word length by

$$(U^c, V^c)^0 = U^c, \ \ (U^c, V^c)^1 = V^c, \ \ (U^c, V^c)^{a \cdot u} = (U^c, V^c)^a \cap \sigma^{-1}((U^c, V^c)^u)$$

for $a \in \{0, 1\}, u \in \{0, 1\}^m$. Say $u \in \{0, 1\}^m$ *covers* $w \sqsubset Y$ if $(U^c, V^c)^u \cap [w] \neq \emptyset$.

We first note that the words w containing at most one $\#$ are easy to cover. For such $w \sqsubset Y$, we always have $w \sqsubset X$ or $w = u\#v$ where u and v are words in X. All such words appear in a morphic image of the zero entropy subshift $X^2 \times \overline{\mathcal{O}(...000111...)}$, so they cannot generate entropy w.r.t. any partition.

To cover words with at least two $\#$-symbols, observe that in Y all such words have an arithmetic progression of $\#$s. Consider a progression $\alpha = \{i(2n + 2) + c \mid i \in \mathbb{Z}\} \cap \{0, 1, 2, ...m - 1\}$ where $n \geq 0, c \in [0, 2n + 1]$ and $|\alpha| \geq 2$. Then the set of words where $\#$s appear exactly in this arithmetic progression consists precisely of words obtained by inserting any combination of $x^n_{[-n,n]}$ or $y^n_{[-n,n]}$ between $i(2n + 2) + c$ and $(i + 1)(2n + 2) + c$ for each i (and we have up to 4 choices in total for the prefix and suffix).

Since $(x^n, y^n) \in E$, by shift-invariance also $(\sigma^\ell(x^n), \sigma^\ell(y^n)) \in E$ for all $\ell \in \mathbb{Z}$ and thus from $([x_{[-k,k]}] \times [y_{[-k,k]}]) \cap E = \emptyset$ we obtain that for all ℓ,

$$\{x^n_{[\ell-k,\ell+k]}, y^n_{[\ell-k,\ell+k]}\} \neq \{x_{[-k,k]}, y_{[-k,k]}\},$$

and thus see that there is a single word $u_{n,c}$ of length m that covers every word w with progression α. This gives a polynomial bound on the number of words u needed to cover those w with at least two $\#$s, all in all we have obtained a subexponential bound. $\qquad \square$

Lemma 5. *Let $X \subset A^\mathbb{Z}$ be a subshift with zero entropy and let $E \subset X^2$ be any closed doubly shift-invariant symmetric reflexive relation. Then there exists a subshift Y with $X \subset Y \subset \hat{A}^\mathbb{Z} \ (\supset A^\mathbb{Z})$ such that $\mathrm{EP}(Y)^{(\lambda)} \cap X^2 = E^{(\lambda)}$ for all λ. Furthermore, if $E^{(\lambda)} \cap X^2 = X^2$, then $\mathrm{EP}(Y)^{(\lambda+1)} = Y^2$, and we have $\mathrm{EP}(Y)^{(\lambda)} = Y^2$ if*

- *λ is even, or*
- *λ is odd and $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$.*

Proof. Define $w_0 = 1$ and inductively $w_{i+1} = w_i 0^{i+1} w_i$. Define $w_\omega = \lim_i w_i \in \{0, 1\}^\mathbb{N}$. Another description is that w_ω is obtained from the *ruler sequence* A007814 in OEIS [1] or *universal counterexample* [5]

$$0102010301020104010201030102010 5...$$

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[3]This is obvious from the intuition in the introduction: even if we only obtain information when the orbit is very close to x and y, we obtain an exponential amount of information from points of Y_n if $d((x_n, y_n), (x, y))$ is very small. The concrete computation is also straightforward.

[4]Covering partial orbits is more difficult than covering words in this sense by at most a constant factor, so this is indeed sufficient.

by applying the morphism $n \mapsto 10^{n+1}$ (or $0 \mapsto 1$, $n \mapsto 0^n$). We refer to the words w_i (and later their obvious generalizations) as *full words* and refer to the process of extending a positioned $w_i \in A^{**}$ (abusing notation quite a bit, we mean a positioned word equal to w_i as an unpositioned word) to $w_{i+1} \in A^{**}$ (which can be done in exactly two ways) as the *left* or *right full extension*. We assume familiarity with the process of constructing all words in the orbit closure of the ruler sequence, and how it manifests for subshifts based on the sequence, see e.g. [12].

By induction $n_i = |w_i| = 3 \cdot 2^i - i - 2$ and $\sum_j |w_i|_j = 2^i$. A short computation shows that for every $k \geq 1$, there exists a subword of w_ω of length k (namely its prefix) where the symbol 1 appears at least $k/3$ times (any k/c, for example $k/6$, works just as well for what follows).

Let $\tau(0) = \{0\}, \tau(1) = \{1, 11\}$ be a nondeterministic substitution and define $Y''' = \tau(w_\omega) \subset \{0,1\}^{\mathbb{N}}$, i.e. Y''' is the set of all possible points obtained by replacing 0s in w_ω by 0s, and 1s by either 1 or 11. Let τ' be the deterministic substitution mapping $\# \mapsto 1$, $a \mapsto 0$ for $a \in A$ and $\# \notin A$, and let $Y'' = (\tau')^{-1}(Y''')$. Let Y' be the $\mathbb{Z}$-subshift whose language consists of the left-extendable words in Y''. Let $\hat{A} = A \cup \{\#\}$ and let Y be the subshift of Y' with additional forbidden words

$$\{u_0 \# u_1 \# \cdots \# u_k \mid \nexists x_0, x_1, \cdots x_k \in X : \forall i, j : x_i \in [u_i] \wedge (x_i, x_j) \in E\}$$

(where it is understood that $\forall i : u_i \in A^*$).

Say that a set of words $U \subset A^*$ is a *friendship* if for any finite subset $V \subset U$ there exist points $(b_v)_v \in X^V$ such that $\forall u, v \in V : (b_u, b_v) \in E$, so since E is doubly shift-invariant, the forbidden words of Y precisely require that the set of words $U \subset A^*$ that appear in y is a friendship. (We remark that even for a closed equivalence relation, for a finite set U, in general being a friendship is stronger than all pairs $\{u, v\}$ being friendships for $u, v \in U$.)

For a concrete picture of points in Y', consider the sequence $w_\omega \in \{0,1\}^{\mathbb{N}}$. Its two-sided orbit closure contains points of the form

$$...1010010100010100101000010100101000010100101...$$

Let us allow any of the 1s to be duplicated, e.g.

$$...1011001010001101100101100001010011010000101001011...,$$

and then replace the 1s by $\#$, i.e.

$$...\#0\#\#00\#0\#000\#\#0\#\#00\#0\#\#0000\#0\#00\#\#0\#000\#0\#00\#0\#\#...$$

finally replace maximal 0-sequences by words over A of the same length, e.g.

$$...\#0\#\#01\#0\#011\#\#1\#\#10\#1\#\#0100\#1\#01\#\#1\#101\#1\#11\#1\#\#...$$

if $A = \{0,1\}$. Such are the points of Y'. In points of Y, the finite words should additionally be from X, and should form a friendship.

If $U \subset A^*$ is a friendship, then by compactness to every word $u \in U$, we can associate a point $b_u \in [u]$, so that $(b_u, b_v) \in E$ for all $u, v \in U$. Fix such points for each set U and call b_u the *(friendship) bracelet of u (w.r.t. U)*. Of particular importance in what follows are (sets of) bracelets for sets of words

U that are sets words, not containing #, that appear in some point $y \in Y$, or the union of such sets for two points y, z. We note that in such a situation, the bracelets are not enforced to be in the orbit closure of y or z.

Call the binary word $\tau'(w)$ recording positions of # by 1 in a point w the *skeleton*, similarly for points $x \in Y$, and the word where 11 is further replaced by 1 the *preskeleton*. We call a (positioned or not) word $w \in \hat{A}^* \cup \hat{A}^{**}$ *full* if it is equal to (a shift of) $(\tau')^{-1}(w_n)$ for some n, or if (more generally) its preskeleton is full.

We now explain the technical trick for characterizing the entropy pairs $\mathrm{EP}(Y)$, and show that $(y, z) \notin \Delta_Y$ is an entropy pair if and only if the set

$$ U = \{u \in A^* \mid u \sqsubset y \vee u \sqsubset z\}, $$

containing all non-# subwords from y and z, is a friendship.

Suppose $u', v' \in \hat{A}^{**}$ are any two positioned words that appear in some points of Y, with # as their first and last symbols (note that words beginning and ending with # are dense in Y in the usual topology of $\hat{A}^{**} \cup \hat{A}^{\mathbb{Z}}$), and let U be the set of words containing $u \in A^*$ iff $\#u\# \sqsubset u'$ or $\#u\# \sqsubset v'$. Suppose U is a friendship. Extend u' and v' to full positioned words whose preskeletons are the same length, in an arbitrary way, though so that their finite subwords over A^* remain a friendship (safe continuations for words can always be found in the bracelets proving the friendship). The left and right ends of the resulting words u'' and v'' have some left and right offsets n_1, n_2 respectively, in the sense that $u'' \in A^{[a,b]}$ and $v'' \in A^{[a+n_1, b+n_2]}$.

Now, continue extending the words by alternately turning them into full positioned words by adding a left and a right full extension, so that no new occurrence of ## appears, and all finite subwords over A^* remain a friendship. Note that the left and right ends continue to differ by exactly n_1 and n_2, since the length of the left or right continuation is always the same if no occurrence of ## appears. Extend the words so many times that you have at least $\max(|n_1|, |n_2|)$ undoubled #s on both sides. Now finally double some of the # to get two positioned full words u, v which have u' and v' at the center, have the same preskeleton, and have the same left and right length, that is, $u, v \in A^{[c,d]}$ for some $c, d \in \mathbb{Z}$.

Now suppose u and v have preskeleton w_i. By the analysis of the density of 1s in the ruler sequence, a word of length k can contain at least $k/(6 \cdot 2^i) - 2$ occurrences of u and v, which are *interchangeable*, meaning we can change any subword u into v in any word $w \sqsubset Y$ containing it, to obtain another valid $w' \sqsubset Y$ (or vice versa). To see the density claim, recall there can be $k/3$ symbols 1 in a word of length k from the ruler sequence, and thus at least $k/(3 \cdot 2^i) - 2$ occurrences of w_i. We additionally precompose with $k \mapsto k/2$ since some of the 1s are doubled in the skeleton, compared to the preskeleton.

The interchangeability, together with the lower bound on maximal density, shows that the cover $([u]^c, [v]^c)$ generates entropy in Y, and thus we have shown that any points whose non-# subwords form a friendship are entropy pairs. On the other hand if the union $U \subset A^*$ of the sets of subwords of two words u, v is not a friendship, then these words do not even appear together in any point of Y, by definition, so neither can u and v, and it follows that the cover $([u]^c, [v]^c)$ has zero entropy. This concludes the characterization of $\mathrm{EP}(Y)$. An important corollary of this characterization is that $\mathrm{EP}(Y)$ is doubly shift-invariant. An-

other important fact to keep in mind is that if $(x, y) \in F$ and F is a doubly shift-invariant closed relation, then $(x', y) \in F$ for any x' in $\overline{\mathcal{O}(x)}$.

It is clear from the characterization that $\mathrm{EP}(Y) \cap X^2 = E$ as required. We need show $\mathrm{EP}(Y)^{(\lambda)} \cap X^2 = E^{(\lambda)}$ for all λ, and we do this by transfinite induction. On limit steps, this obviously continues to hold. On even successor steps, i.e. transitive closure steps, use the fact that points of Y can be replaced by any point in their orbit closure, and the orbit-closure of every point contains a point of X. This allows turning any transitivity sequence $(x = y_0, y_1, ..., y_k = x')$ where $x, x' \in X$, $y_i \in Y$, $(y_i, y_{i+1}) \in E^{(\lambda-1)}$, into one where $y_i \in X$ for all i. This shows $(\mathrm{EP}(Y)^{(\lambda-1)})^* \cap X^2 = (\mathrm{EP}(Y)^{(\lambda-1)} \cap X^2)^*$ as required.

Now, note that if $\mathrm{EP}(Y)^{(\lambda)}$ is transitively closed, $y \in Y \setminus X$ and b is any bracelet of y, or more generally any point in X whose set of A^*-subwords together with A^*-subwords of Y forms a friendship, then we have $(y, z) \in \mathrm{EP}(Y)^{(\lambda)}$ if and only if $(b, z) \in \mathrm{EP}(Y)^{(\lambda)}$. This is because $(b, y) \in \mathrm{EP}(Y)$ by the characterization of entropy pairs of Y. In other words, on every transitive step, every $y \in Y$ satisfies the same relations as any of its bracelets or any point that could be chosen as a bracelet for it.

On a topological closure step, i.e. for an odd ordinal λ, suppose $\lim_i (x_i, y_i) = (x, y) \in X^2$ where $(x_i, y_i) \in \mathrm{EP}(Y)^{(\lambda-1)}$ for all i. It is enough to show that the x_i can be replaced by points from X, as then (also applying this idea to the right component) we have $\overline{\mathrm{EP}(Y)^{(\lambda-1)} \cap X^2} = \overline{\mathrm{EP}(Y)^{\lambda-1} \cap X^2}$. If there is a subsequence of the x_i in X, then we can directly restrict to that. Otherwise, restrict to a subsequence such that $x_i \notin X$ for all i.

In x_i, take a maximal central positioned word $u_i : A^{[-a,a]}$ not containing $\#$ (where necessarily $a \to \infty$ as $i \to \infty$) and replace x_i by the bracelet b_{u_i} with respect to the language of x_i, centered to have u_i in the same place as in x_i. We have $(b_{u_i}, y_i) \in \mathrm{EP}(Y)^{(\lambda-1)}$ since $\mathrm{EP}(Y)^{(\lambda-1)}$ is transitively closed and $(b_{u_i}, x_i) \in \mathrm{EP}(Y)$, and clearly $(b_{u_i}, y_i) \to (x, y)$, as required.

Now suppose $E^{(\lambda)} = X^2$ for some λ. If λ is even, then $\mathrm{EP}(Y)^{(\lambda)}$ is transitively closed so since any $y \in Y \setminus X$ appears in the same relations as any bracelet of its, $\mathrm{EP}(Y)^{(\lambda)} = Y^2$. Suppose then that λ is odd. Let $(y, z) \in Y^2$, and suppose $y, z \in Y \setminus X$, the other three cases being similar.[5] Let $(b_u)_u$ and $(c_v)_v$ be the respective bracelets of their subwords w.r.t. their languages. If any two of these bracelets, one from y and one from z, are in relation on step $\lambda - 1$, then already $(y, z) \in \mathrm{EP}(Y)^{(\lambda-1)}$ since $\mathrm{EP}(Y)^{(\lambda-1)}$ is transitive.

Otherwise, let k be arbitrary and let (up to swapping y and z) $m \geq k$ and $n \geq m$ be such that $y_{[-m,m]} = u_1 \# u_2 \# \cdots \# u_k$, $U = (u_1, ..., u_k) \in (A^*)^k$ and $z_{[-m,n]} = v_1 \# v_2 \# \cdots \# v_k$, $V = (v_1, ..., v_k) \in (A^*)^k$. Let $B = (b_u)_{u \in U}$ and $C = (c_v)_{v \in V}$ be bracelets for these words with respect to the subwords of y and z, respectively.

Now, recall that the bracelets of a point are always (by definition) in the E-relation together, and apply the odd condition to $(B, C) \in (E^{(k)})^2$. Since $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$, we can find in $(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2$ arbitrarily good approximations $B' = (b'_u)_{u \in U}$ and $C' = (c'_v)_{v \in V}$ to B and C respectively. In particular, we can find good enough approximations to ensure that b'_u indeed has prefix u and c'_v prefix v for each $u \in U, v \in V$. Then since $\mathrm{EP}(Y)^{(\lambda-1)}$

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[5]If $y \in X$, you can just pick all bracelets from the orbit of y, similarly for z, and you can approximate y and z in the exact same fashion. However, then $u_1 \# u_2 \# \cdots \# u_k$ does not look correct.

is transitive and $(b'_u, c'_v) \in E^{(\lambda-1)}$), any pair of points (y_k, z_k) in Y for which we can take b'_u and c'_v respectively as one of their bracelets, satisfies $(y_k, z_k) \in$ $\mathrm{EP}(Y)^{(\lambda-1)}$.

We construct the pair y_k, z_k so that $(y_k, z_k)_{[-k,k]} = (y, z)_{[-k,k]}$. For this, directly let $(y_k)_{[-m,m]} = u_1 \# u_2 \# \cdots \# u_k$ and $(z_k)_{[-m,n]} = v_1 \# v_2 \# \cdots \# v_k$. Extend y_k to a point of Y by copying the skeleton from y and filling all the remaining gaps arbitrarily so that all words between two #s (and possible infinite tails without an occurrence of #) are subwords of the words in B'. Extend z_k similarly. These are indeed valid points: the skeletons are correct since we copied them from valid points, and the set of A^*-subwords of each point is a friendship, since $B', C' \in E^{[k]}$. Clearly we can take any b'_u and c'_v respectively as one of their bracelets, so $(y_k, z_k) \in \mathrm{EP}(Y)^{(\lambda-1)}$ gives an approximation to (y, z) that is correct in the interval $[-k, k]$. $\qquad \square$

For the main proof, we make some simple topological observations.

Lemma 8. *Suppose X, Y are topological spaces, X is first-countable, suppose $f : X \to Y$ is a quotient map. Then the following are equivalent:*

- *f is open,*

- *whenever $\lim_i y_i = y \in Y$ and $f(x) = y$, then there exist $x_i \in X$ such that $f(x_i) = y_i$ and $\lim_i x_i = x$.*

Proof. Suppose f is open and $f(x) = y$. Then every neighborhood $U \ni x$ maps to a neighborhood $f(U)$ of y, in particular if $\lim_i y_i = y$, then for all large enough i, y_i has a preimage x_i in U. Picking open sets U from a decreasing countable neighborhood basis of x, and always picking preimages for y_i from U until they start having preimages in the next neighborhood of x, the claim is proved.

For the other direction, suppose f is not open, and let U be open such that for some $y \in f(U)$, every neighborhood of y contains a point without a preimage in U. Pick $x \in U$ such that $f(x) = y$, and use first-countability to get a sequence $y_i \to y$ such that the points y_i have no preimage in U. The second condition fails, since we cannot pick $x_i \in U$. $\qquad \square$

The following lemmas explain why we want ϕ to be open.

Lemma 9. *Let X, Y be first-countable. Let $f : X \to Y$ be an open quotient map, $E \subset Y^2$ a symmetric reflexive closed relation, and $F = (f^{-1} \times f^{-1})(E)$. Then $F^{(\lambda)} = (f^{-1} \times f^{-1})(E^{(\lambda)})$ for all ordinals λ.*

Proof. Since X is first-countable, so is X^2, and thus the closure of a set $C \subset X^2$ is just the set of limits of sequences in C.[6]

Write $g = f \times f : X^2 \to Y^2$. We have $F^{(\zeta)} = g^{-1}(E^{(\zeta)})$ where $\zeta = 0$ if E is not closed, and $\zeta = 1$ otherwise, since E is closed if and only if F is closed by continuity, openness and surjectivity of g. We proceed by induction. Suppose λ is an odd ordinal, and $g(x, x') = (y, y')$. If $(x, x') \in F^{(\lambda)}$ then there exist $(x_i, x'_i) \in F^{(\lambda-1)}$ with $(x_i, x'_i) \to (x, x')$ and then $g(x_i, x'_i) \to (y, y')$ by continuity showing $E^{(\lambda)} \supset g(F^{(\lambda)})$, which implies $g^{-1}(E^{(\lambda)}) \supset F^{(\lambda)}$.

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[6]In other words X^2 has the *Fréchet-Urysohn* property. For general topological spaces, even if X is Frechét-Urysohn, X^2 need not be [6].

For the other inclusion, suppose $(y, y') \in E^{(\lambda)}$, so $(y, y') = \lim_i (y_i, y_i')$ for some $(y_i, y_i') \in E^{(\lambda-1)}$. By the previous lemma, there exist $(x_i, x_i') \in X^2$ such that $g(x_i, x_i') = (y, y')$ and $(x_i, x_i') \to (x, x')$. Then by induction we have $(x_i, x_i') \in F^{(\lambda-1)}$ for all i, so $(x, x') \in F^{(\lambda)}$, showing $g^{-1}(E^{(\lambda)}) \subset F^{(\lambda)}$.

If λ is an even successor ordinal, suppose $g(x, x') = (y, y')$. If $(x, x') \in F^{(\lambda)}$, there exist $x = x_0, x_1, ..., x_k = x'$ such that $(x_i, x_{i+1}) \in F^{(\lambda-1)}$ for all i, and then by induction $g(x_i, x_{i+1}) \in E^{(\lambda-1)}$ and since $y = f(x_0), y' = f(x_k)$ we have $(y, y') \in E^{(\lambda)}$.

If on the other hand $(y, y') \in E^{(\lambda)}$, then there exist $y = y_0, y_1, ..., y_k = y'$ with $(y_i, y_{i+1}) \in E^{(\lambda-1)}$. Pick any preimages $f(x_i) = y_i$ with $x_0 = x, x_k = x'$. by induction $(x_i, x_{i+1}) \in F^{(\lambda-1)}$ for all i, and thus $(x, x') \in F^{(\lambda)}$.

For the limit ordinal case, observe that union commutes with preimage. $\square$

Lemma 10. *Let X, Y be first-countable. Let $f : X \to Y$ be an open quotient map, $E \subset Y^2$ a symmetric reflexive relation, and $F = (f^{-1} \times f^{-1})(E)$. Then*

$$(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$$

if and only if

$$(F^{[k]})^2 \subset \overline{(F^{(\lambda-1)})^{[2k]} \cap (F^{[k]})^2}$$

Proof. Suppose $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$ and consider $(A', B') \in (F^{[k]})^2$. Consider $f(A')$ and $f(B')$ (where f is applied diagonally to all elements of the tuple), which are by definition k-tuples $A, B \in E^{[k]}$. In $(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2$ we find arbitrarily good approximations (A_i, B_i) to (A, B). Take any f-preimages, $f(A_i') = A_i$, $f(B_i') = B_i$ which are close to A', B' elementwise, using Lemma 8, obtaining by definition $A_i', B_i' \in F^{[k]}$. By the previous lemma, we have $(F^{(\lambda-1)})^{[2k]} = (f \times f)^{-1}(E^{(\lambda-1)})^{[2k]}$ so we have $(A_i', B_i') \in (F^{(\lambda-1)})^{[2k]}$. This implies $(F^{[k]})^2 \subset \overline{(F^{(\lambda-1)})^{[2k]} \cap (F^{[k]})^2}$.

The other direction is similar, but using continuity instead of openness. $\square$

Theorem 3. *Let $\alpha \geq 1$ be a countable ordinal. Then there exists a subshift $X \subset A^{\mathbb{Z}}$ in CPE class α.*

Proof. If $\alpha = 1$, $X = A^{\mathbb{Z}}$ is an example.

Let $\lambda \geq 2$ be a countable ordinal. By Lemma 7, there exists a closed countable set $Z \subset [0, 1]$ and a closed symmetric reflexive relation $E \subset Z^2$ such that E is equalizing with close-up rank λ. Furthermore, if λ is an odd ordinal, then $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$.

By Lemma 4, since every countable compact subspace of $[0, 1]$ is zero-dimensional, there exists a pointwise zero-entropy Toeplitz subshift X and a doubly shift-invariant closed equivalence relation $K \subset X^2$ such that Z is the open quotient of X, by $\phi : X \to Z$, and $\phi(x) = \phi(y) \iff (x, y) \in K$.

Since ϕ is an open quotient map, Lemma 9 shows that $E^{(\lambda)} = Z^2$ if and only if $(\phi^{-1} \times \phi^{-1})(E) = F$ satisfies $F^{(\lambda)} = X^2$. If λ is an odd ordinal, then $(E^{[k]})^2 \subset \overline{(E^{(\lambda-1)})^{[2k]} \cap (E^{[k]})^2}$, and by the previous lemma we have $(F^{[k]})^2 \subset \overline{(F^{(\lambda-1)})^{[2k]} \cap (F^{[k]})^2}$.

Now, apply Lemma 5 to obtain $Y \supset X$ such that λ is the minimal ordinal satisfying $\mathrm{EP}(Y)^{(\lambda)} = Y^2$. This Y is a subshift in CPE class λ. $\square$

References

[1] The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2014, Sequence A007814.

[2] Sebastián Barbieri and Felipe García-Ramos. A hierarchy of topological systems with completely positive entropy. *arXiv e-prints*, page arXiv:1803.01948, Mar 2018.

[3] François Blanchard. A disjointness theorem involving topological entropy. *Bulletin de la Société Mathématique de France*, 121(4):465–478, 1993.

[4] Manfred Denker, Christian Grillenberger, and Karl Sigmund. *Ergodic theory on compact spaces*. Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin, 1976.

[5] Sébastien Ferenczi. Substitution dynamical systems on infinite alphabets. *Annales de l'Institut Fourier*, 56(7):2315–2343, 2006.

[6] Gary Gruenhage. Products of Fréchet spaces. In *Topology Proceedings*, volume 30, pages 475–499, 2006.

[7] K. Hrbacek and T. Jech. *Introduction to Set Theory, Third Edition, Revised and Expanded*. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1999.

[8] David Kerr and Hanfeng Li. Independence in topological and C^*-dynamics. *Mathematische Annalen*, 338(4):869–926, 2007.

[9] K. Kuratowski. *Topology. Vol. I*. New edition, revised and augmented. Translated from the French by J. Jaworowski. Academic Press, New York, 1966.

[10] Petr Kůrka. *Topological and symbolic dynamics*, volume 11 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2003.

[11] Douglas Lind and Brian Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.

[12] Ville Salo. Decidability and universality of quasiminimal subshifts. *Journal of Computer and System Sciences*, 89:288 – 314, 2017.