A mathematical model for the estimation of stomatal conductance under environmental variables

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Abstract

Global Warming has become a pressing concern for agriculture, with scientists extensively studying its impacts in recent years. This phenomenon is characterized by the gradual increase in temperature on Earth, causing significant climate changes. The stomatal conductance is a mechanism that explains the gas exchange between the plant and the atmosphere, directly related to photosynthesis and the plant's ability to adapt to environmental changes. Studying the reaction of stomatal conductance to different environmental factors and its modeling is essential to characterize the water status of plants and develop adaptation strategies to Global Warming. In light of these challenges, a dynamic, two-dimensional ordinary differential equations model that estimates stomatal conductance is proposed, considering factors such as relative humidity and ambient temperature, to improve the estimation of the stomatal conductance in the context of Global Warming and changing environmental conditions.

Keywords:
Transpiration, vapor pressure deficit, gas exchange, stomatal behavior, dynamic modelling.

1. Introduction

The Mediterranean-type climate areas are characterized by dry, hot summers and warm and wet winters (Rundel & Cowling, 2013; Joffre & Randal, 2001). They are observed in regions worldwide, such as California, Chile, South Africa, and Australia (Rundel & Cowling, 2013). This type of climate facilitates the growth and sustenance of a wide variety of plant species, many of which play a crucial role in human nutrition and the maintenance of ecological equilibrium (del Pozo et al., 2019). The global warming (GW) phenomenon

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has substantially modified this intricately regulated environment, presenting notable obstacles to the proliferation and availability of consumable vegetation. Particularly in the Mediterranean-type climate zones, GW can alter precipitation patterns and cause a general increase in temperatures, resulting in warmer and longer summers, with the intensification of droughts and heatwaves (Meseguer-Ruiz & Olcina, 2022). These factors have considerably impacted plant life in areas, particularly agricultural plants, which are currently facing the challenge of adapting to the various stressors imposed by GW.

Stomatal conductance (g_s) refers to the extent to which stomata are opened or closed to regulate the plant gas exchange (water vapor and CO_2) of plants with their environment based on the soil-plant-atmosphere continuum (SPAC) (see Figure 1). g_s is a crucial mechanism of adaptation of plants to GW, and it can be affected by different environmental factors. In summer, GW has increased air temperature and soil dryness during the plant growing period, generating water stress negatively affecting photosynthesis efficiency and plant water uptakes. g_s is a physiological indicator to assess plants' water status to water stress. In this manner, the adequate characterizing of the plant adjustments through g_s, because of the effect of GW, appears as a reliable theoretical tool to study and simulate how plants adapt to GW, considering the development of adaptive techniques for agriculture. In this context, mathematical modeling for g_s within the SPAC can provide an appealing tool for comprehending and predicting plant responses to GW.

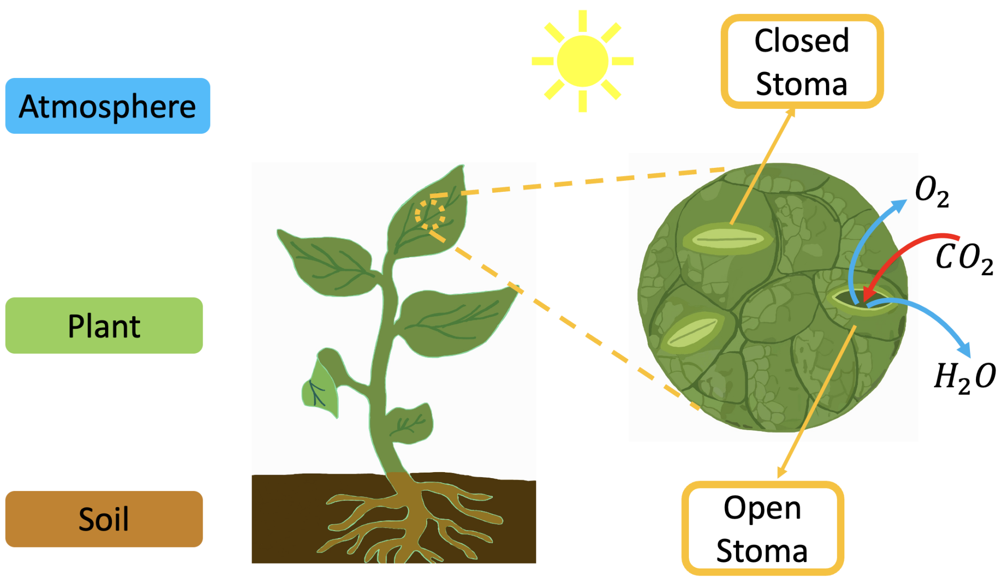

Figure 1: Representation of Stomata in the SPAC.

Several attempts have been made to estimate g_s. In an extensive review, Buckley & Mott, 2013 distinguish four modeling approaches: (1) empirical, (2) optimization, (3) mechanistic, and (4) spatial and kinetic (Buckley & Mott, 2013). Within (1) empirical approaches or based on experimentally measured data, one of the most used models is the one proposed by Jarvis, 1976 which estimates g_s using photon flux density, leaf temperature, vapor pressure deficit, water potential, and ambient concentration of CO_2 as variables. Similarly, Ball et al. (1987) proposed an empirical model that described g_s as a function of net photosynthesis, CO_2 concentration at the surface of the leaf, the relative humidity at the surface of the leaf and the residual g_s when net photosynthesis is zero. This kind of model has been extensively used in simulating g_s because they had a good level of agreement against measured data ($>$ 95%) (Jarvis, 1976; Damour et al., 2010). The optimization models consider the hypothesis that stomata should act to maximize CO_2 assimilation while minimizing water loss (Huang et al., 2021). Within this approach is the model proposed by Cowan and Farquhar, 1977, who establish a relationship between g_s, the carbon assimilation rate, and the difference in water vapor concentration in the leaves of plants (Cowan & Farquhar, 1977). Another approach is mechanistic models, which are based on processes and whose main objective is to provide predictions that can be verified from theories about the mechanics of stomatal control (Suarez & Fernandez, 1984). A mechanistic model approach was proposed by Delwiche and Cooke,1997, who built an analytical model of the hydraulic aspects of stomatal dynamics. They hypothesize that stomatal movements are governed by the guard and subsidiary cells' turgor pressure (Delwiche & Cooke, 1977). Models of this type are often not applicable for predicting g_s if the modeling is considered a function of environmental factors because the parameters used represent biophysical properties that are difficult to measure experimentally (Buckley & Mott, 2013).

The last approach is spatial and kinetic models. These models describe temporal variations in g_s, where several aim to predict photosynthesis under variable conditions (Buckley & Mott, 2013). An example is the one proposed by Vialet-Chabrand et al., 2013, who proposed a dynamic model to predict the temporal response of g_s at the leaf level in the face of a change in irradiance. This model uses a function to describe the dynamic change of g_s and a different one to describe the objective in a steady state (Vialet-Chabrand et al., 2013). The model has an error of less than 50% when estimating g_s, but the validation study was carried out under controlled conditions of environmental variables (Vialet-Chabrand et al., 2013).

Despite advancements in existing models, there are still obstacles to estimating g_s adequately. Some models may not adequately capture the temporal and spatial variability of g_s, making it difficult to apply them to various scales and environmental conditions. In addition, most models mentioned above have a static approach, meaning that they do not consider the dynamic responses of plants to changes in environmental conditions over time. A dynamic model would account for the temporal variability of environmental conditions, such as diurnal and seasonal fluctuations, and the varying physiological responses of plants in response to global warming and drought, enabling an improved estimation of stomatal conductance.

For this reason, this study aimed to formulate a dynamical model that estimates g_s. We construct a bidimensional ordinary differential equations model that describes how mesophyll water content and g_s vary through time. The model includes the interaction between the variables ambient relative humidity and temperature and considers a steady-state target of g_s (Medlyn et al., 2011; Vialet-Chabrand et al.,2013). We use the model proposed by Moualeu-Ngangue et al. (2016) as a basis.

2. The Model

2.1. Model formulation

Moualeu-Ngangue et al. (2016) examined the impact of dynamic properties of stomata on water use efficiency (WUE) in plants. The study uses a dynamic model based on ordinary differential equations that include sub-models of stomatal conductance dynamics, solute accumulation in the mesophyll, mesophyll water content, and water flow to the mesophyll. The model was parameterized for a cucumber leaf, and model outputs were evaluated using climatic data (Moualeu-Ngangue *et al.*, 2016) .

Based on what was proposed by Moualeu-Ngangue et al. (2016), to study how relative humidity and environmental temperature influence g_s, we proposed a model that considers the variables: water content of the mesophyll and g_s and is described as follows:

1. The rate of change of the water content of the mesophyll ($W(t) > 0$) varies through time according to inflows and outflows. The inflow is described in terms of the mesophyll water potential ($\Psi_m < 0$) and the xylem water potential ($\Psi_x < 0$). When Ψ_m is less than Ψ_x, the flow of water is transported from the xylem to the mesophyll according to the available free capacity, being described by the rate

$$\alpha \cdot \Psi \left(\frac{1 - W}{W_{max}} \right),$$

with $\Psi = \Psi_x - \Psi_m > 0$, α positive constant and $W_{max} > 0$ the maximum water content of the mesophyll. On the other hand, the outflow of water from the mesophyll is estimated by the transpiration rate ($E > 0$), which depends on the Vapor pressure deficit (VPD)

$$\delta_e = \left(1 - \frac{h_r}{100} \right) \cdot e_s(T) > 0, \tag{1}$$

the resistance to water vapor transport

$$c(g_s) = g_s^{-1} + c_0 > 0,$$

and the rate of the free space of the mesophyll, given by

$$E(W, g_s, h_r, T) = \beta \cdot \frac{W}{W_{max}} \cdot \frac{1}{c(g_s)} \cdot \left(1 - \frac{h_r}{100} \right) \cdot e_s(T),$$

where $h_r > 0$ is the relative humidity of the environment, β is a positive constant and $e_s(T) > 0$ is the saturation vapor pressure dependent on the ambient temperature $T > 0$, described by

$$e_s(T) = 0.61078 \cdot \exp \left(\frac{17.2694 \cdot T}{237.3 + T} \right).$$

2. g_s is modeled as a sigmoidal response with an initial time lag, then followed by an exponential phase and a stabilization at a steady state (Vialet-Chabrand et al., 2013; Moualeu-Ngangue et al., 2016).
3. Let $G(x)$ be a real function, with $x \in \mathbb{R}^+$, that satisfies
   (i) $G(x)$ is a positive function, $G(x) > 0$ for all $x \in \mathbb{R}^+$,
   (ii) $G(x)$ is a C^2 function, all its partial derivatives of order two exist and are continuous,
   (iii) $G(x)$ is a decreasing function, $G'(x) < 0$ for all $x \in \mathbb{R}^+$.

From the above, the proposed model is:

$$\begin{cases} \dfrac{dW}{dt} &= \alpha \cdot \Psi \cdot \left(1 - \dfrac{W}{W_{max}}\right) - \beta \cdot \dfrac{W}{W_{max}} \cdot \dfrac{g_s}{1 + c_0 \cdot g_s} \cdot \left(1 - \dfrac{h_r}{100}\right) \cdot e_s(T) \\ \dfrac{dg_s}{dt} &= k \cdot (g_s - r_0) \cdot \ln\left(\dfrac{1.6 \cdot G - r_0}{g_s - r_0}\right) \end{cases}, \tag{2}$$

where $k > 0$ is a time constant, $r_0 > 0$ is a parameter that modify the initial time lag, $G = G(W) > 0$ is the objective steady state of g_s, $\Psi_m < \Psi_x$, $g_s \neq r_0$, $1.6 \cdot G \neq r_0$ and W_{max}, g_s, $C_a \neq 0$ (Table 2).

Variable or Parameter	Description	Unit
$W > 0$	Mesophyll water content	$mol\, H_2O\, m^{-2}$
$W_{max} > 0$	Maximum mesophyll water content	$mol\, H_2O\, m^{-2}$
$\alpha > 0$	Constant	$\dfrac{mol\, H_2O\, m^{-2}\, s^{-1}}{kPa}$
$\beta > 0$	Constant	$\dfrac{1}{kPa}$
c_0	Constant	$(mol\, H_2O\, m^{-2}\, s^{-1})^{-1}$
$\Psi_m < 0$	Mesophyll water potential	kPa
Ψ_x	Xylem water potential	kPa
$E > 0$	Transpiration rate	$mol\, H_2O\, m^{-2}$
$g_s > 0$	Stomatal conductance	$mol\, H_2O\, m^{-2}\, s^{-1}$
$\delta_e > 0$	Vapour pressure deficit	kPa
h_r	Ambient relative humidity	—-
$T > 0$	Ambient temperature	$°C$
$k > 0$	Time constant	s^{-1}
r_0	Parameter that modify the initial time lag	$mol\, H_2O\, m^{-2}$
$e_s > 0$	Saturation vapour pressure	kPa

Table 1: Parameters and variables of the model (2)

The model corresponds to a system of nonlinear ordinary differential equations. Let us note that the proposed model considers the interaction between the ambient temperature and the relative humidity of the environment. These are related through the VPD, which is obtained through the equation (1) (Monteith & Unsworth, 2013; Kroshavi et al., 2021).

A change of variables is made to simplify the qualitative study of model (2), obtaining the following model

$$\begin{cases} \dfrac{dU}{dt} &= M_1 \cdot (1 - U) - \beta_1 \cdot U \cdot \dfrac{g_s}{1 + c_0 \cdot g_s} \\ \dfrac{dg_s}{dt} &= k \cdot (g_s - r_0) \cdot \ln\left(\dfrac{1.6 \cdot G - r_0}{g_s - r_0}\right) \end{cases}, \tag{3}$$

where $U = \dfrac{W}{W_{max}}$, $G = G(U)$, $M_1 = \dfrac{\alpha \cdot (\Psi_x - \Psi_m)}{W_{max}}$, $L = \left(1 - \dfrac{h_r}{100}\right) \cdot e_s(T)$ and $\beta_1 = \dfrac{\beta \cdot L}{W_{max}}$.

3. Results

Using methodologies from the theory of ordinary differential equations, we carried out the qualitative study of the model (3), obtaining the following results [cite gucke, perko, kuznet].

3.1. Equilibrium Points

The equilibrium points of model (3) are obtained by solving the following system:

$$\left\{ \begin{array}{rcl} M_1 \cdot (1-U) - \beta_1 \cdot U \cdot \dfrac{g_s}{1+c_0 \cdot g_s} & = & 0 \\ \left(\dfrac{1.6 \cdot G(U) - r_0}{g_s - r_0}\right) & = & 1 \end{array} \right. . \tag{4}$$

From the second equation, we obtain the equality

$$1.6 \cdot G(U) = g_s, \tag{5}$$

then replacing in the first equation, the following equality is obtained

$$\frac{M_1 \cdot (1-U)}{\beta_1 \cdot U} = \frac{1.6 \cdot G(U)}{1 + 1.6 \cdot c_0 \cdot G(U)}. \tag{6}$$

Remark 1. 1. *Studying the right-hand side of the equality (6), it is obtained that*

$$F_1(U) = \frac{M_1 \cdot (1-U)}{\beta_1 \cdot U} \quad \textit{is}$$

a) A positive function ($F_1(U) > 0$) if $0 < U < 1$,
b) A decreasing function ($F_1'(U) < 0$).

2. *The left-hand side of the equality corresponds to a composition of functions $F_2(U) = (H \circ G)(U)$ where*

$$H(x) = \frac{1.6 \cdot x}{1 + 1.6 \cdot c_0 \cdot x}.$$

Given that $H(x)$ is an increasing function , and $G(U)$, by assumption, is a decreasing function, we obtain that $F_2(U)$

a) Is a decreasing function ($F_2'(U) < 0$),
b) Is a positive function ($F_2(U) > 0$), since

$$Dom\, F_2 = Dom\, G \,\cap\, Dom\, H = \mathbb{R}^+,$$

where
$$Dom\, G = \mathbb{R}^{+},$$
and
$$Dom\, H = \left]-\infty, -\frac{1}{1.6 \cdot c_0}\right[\cup \left]-\frac{1}{1.6 \cdot c_0}, \infty\right[.$$

From remark 1, we observe that there are three possible cases of solutions for equality (6), characterized in Figure 2 .

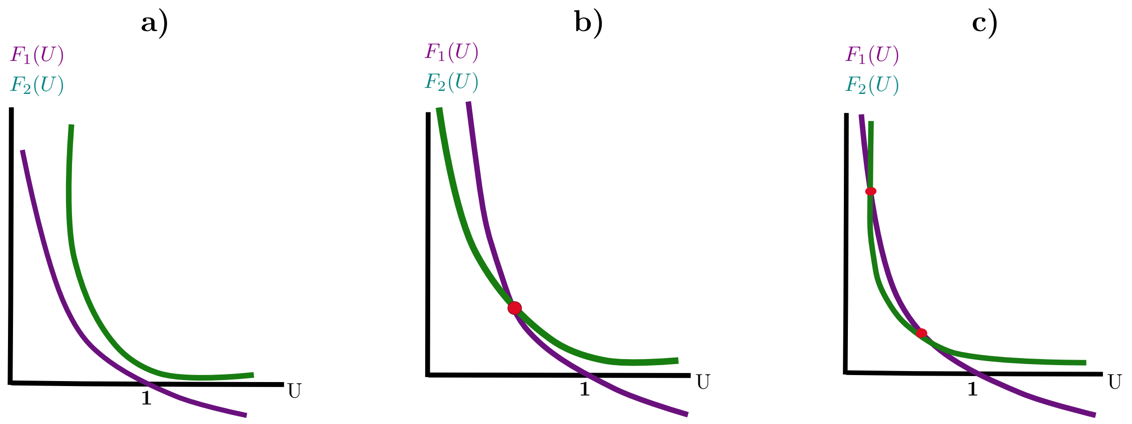

Figure 2: Possible cases of the solutions of equality (6).

Remark 2. *For the purposes of this study, we will assume that*

(i) If $U \in \left]0, \frac{1.6 \cdot M_1 \cdot c_0}{1.6 \cdot M_1 \cdot c_0 + 1.6 \cdot \beta_1}\right[$ *then*
$$G(U) > \frac{M_1 \cdot (U-1)}{1.6 \cdot M_1 \cdot c_0 - (1.6 \cdot M_1 \cdot c_0 + 1.6 \cdot \beta_1) \cdot U},$$

(ii) If $U \in \left]\frac{1.6 \cdot M_1 \cdot c_0}{1.6 \cdot M_1 \cdot c_0 + 1.6 \cdot \beta_1}, 1\right[$ *then*
$$G(U) < \frac{M_1 \cdot (U-1)}{1.6 \cdot M_1 \cdot c_0 - (1.6 \cdot M_1 \cdot c_0 + 1.6 \cdot \beta_1) \cdot U}.$$

From remark 2 we obtain the following lemma.

Lemma 1. *System (3) has at least one equilibrium point.*

Proof. See Appendix

Now, solving equality (6), we obtain

$$\begin{aligned}
\frac{M_1 \cdot (1-U)}{\beta_1 \cdot U} &= \frac{1.6 \cdot G(U)}{1 + 1.6 \cdot c_0 \cdot G(U)} \\
(M_1 - M_1 \cdot U) \cdot (1 + 1.6 \cdot c_0 \cdot G(U)) &= 1.6 \cdot \beta_1 \cdot G(U) \cdot U \\
M_1 + 1.6 \cdot M_1 \cdot c_0 \cdot G(U) &= (1.6 \cdot \beta_1 \cdot G(U) + M_1 + 1.6 \cdot c_0 \cdot M_1 \cdot G(U)) \cdot U.
\end{aligned}$$

Therefore, we have that

$$U = \frac{M_1 + 1.6 \cdot M_1 \cdot c_0 \cdot G(U)}{1.6 \cdot \beta_1 \cdot G(U) + M_1 + 1.6 \cdot c_0 \cdot M_1 \cdot G(U)}. \tag{7}$$

Then from (7) and from (5), we have the equilibrium point

$$E_1 = \left(\frac{M_1 + 1.6 \cdot M_1 \cdot c_0 \cdot G(U)}{1.6 \cdot \beta_1 \cdot G(U) + M_1 + 1.6 \cdot c_0 \cdot M_1 \cdot G(U)}, 1.6 \cdot G(U) \right). \tag{8}$$

3.2. Local stability of equilibrium points.

Let's study the stability of the equilibrium point E_1. First, let us define

$$R_1(U) = \frac{-0.625 \cdot M_1 + 0.25 \cdot \sqrt{-10 \cdot \beta_1 \cdot M_1 \cdot G'(U)}}{c_0 \cdot M_1 + \beta_1}, \tag{9}$$

and

$$R_2(U) = \frac{-0.625 \cdot M_1 - 0.25 \cdot \sqrt{-10 \cdot \beta_1 \cdot M_1 \cdot G'(U)}}{c_0 \cdot M_1 + \beta_1}. \tag{10}$$

Remark 3. *Let us note that*

$$R_2 < 0.$$

Lemma 2. *The equilibrium point E_1 is*

(i) a saddle point if

$$G(U) > R_1(U),$$

(ii) locally asymptotically stable if

$$G(U) < R_1(U).$$

Proof. See Appendix

3.3. Numerical simulations

To evaluate the behaviour of the model, we realize numerical simulations to illustrate some results of the study of the system (2). The following simulations were performed using an application and a script written in Matlab R2022a. Parameter values present in the literature were considered.

For the simulations, we will consider the following function $G(U)$

$$G(W) = N_1 + \left(\frac{N_2}{\sqrt{\beta \cdot \frac{W}{W_{max}}}} \right), \tag{11}$$

which satisfies the assumptions for $G(x)$, $N1 = \dfrac{g_0 \cdot A \cdot f}{C_a}$ and $N1 = \dfrac{g_1 \cdot A \cdot f}{C_a}$.

Variable or Parameter	Description	Unit
$A > 0$	Steady-state net photosynthesis rate	$\mu mol\, C0_2\, m^{-2} s^{-1}$
$C_a > 0$	Ambient CO_2 concentration at the leaf surface	$\mu mol\, C0_2\, m^{-2} s^{-1}$
$f > 0$	Parameter that quantifies the dependency of G to mesophyll water potential	—

Table 2: Parameters of N.

Figure 3 illustrates the phase portrait of model (2), and Figure 4 illustrates the evolution through time of W and g_s. Figure 3 shows how the curves have a stable equilibrium point, which is represented in Figure 4. It is obtained that the stomatal conductance reaches its saturation point and stabilizes over time. This stable equilibrium point could represent a state in which the plant has found a balance between the amount of water it retains in its tissues and the opening of its stomata to allow the entry of CO_2 for photosynthesis.

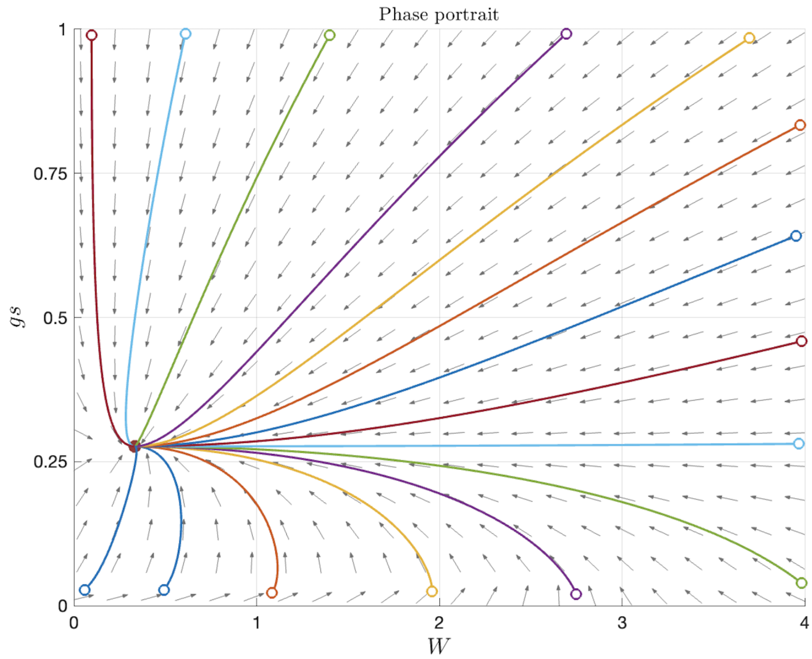

Figure 3: Phase portrait of model (3) for the following parameter values: $g_0 = 0.009$, $g_1 = 0.31$; $M_1 = 0.00025$, $\beta_1 = 0.091203927$; $N = 0.03125$, $c = 0.004$, $r_0 = 0.01$, $k = 0.004051610$.

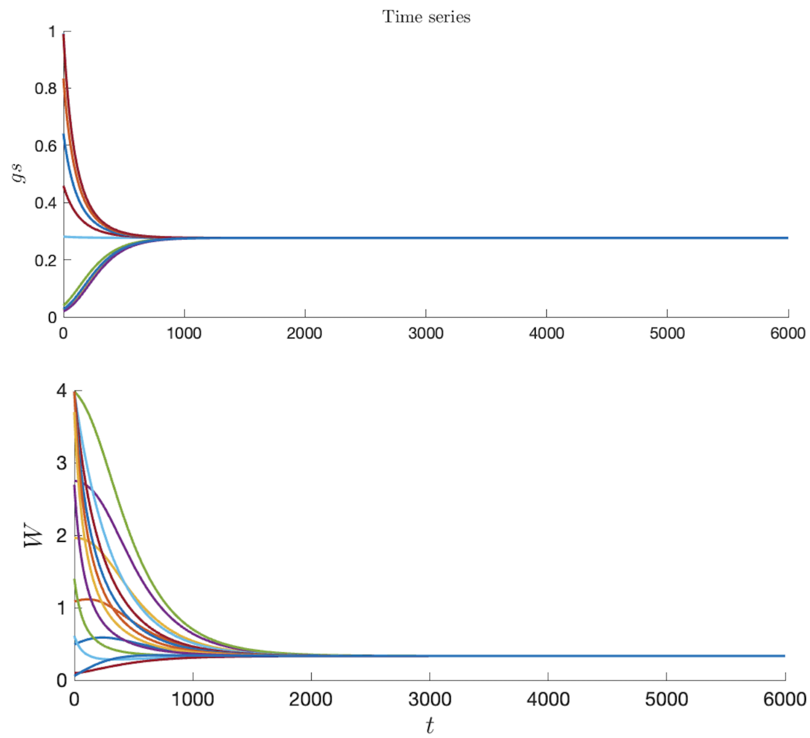

Figure 4: Time series of model (3) for the following parameter values: $g_0 = 0.009$, $g_1 = 0.31$; $M_1 = 0.00025$, $\beta_1 = 0.091203927$; $N = 0.03125$, $c = 0.004$, $r_0 = 0.01$, $k = 0.004051610$.

Figures 5 and 6 illustrate how the water content of the mesophyll and the stomatal conductance vary when considering different values of environmental temperature and relative humidity of the environment. We observe that the value where the stomatal conductance stabilizes increases as the environmental temperature increases and the relative humidity decreases. The last occurs because, in general, stomata open at higher temperatures since plants need more CO2 for photosynthesis in warm conditions. On the other hand, the water content of the mesophyll decreases because as the stomatal conductance increases, the opening of the stomata allows the water vapor in the mesophyll cells to evaporate into the atmosphere, leading to a loss of water in the mesophyll tissue.

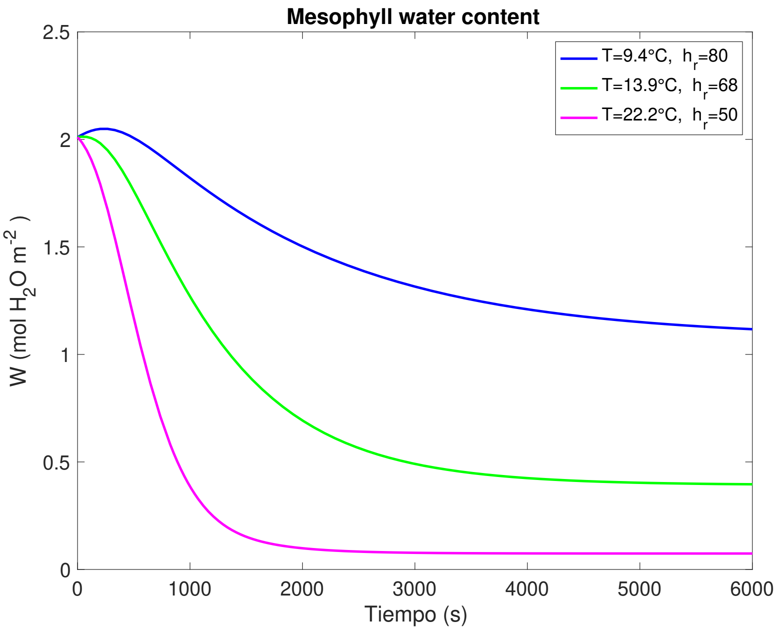

Figure 5: Simulation of model (2) for the variable U. The parameter values are: $g_0 = 0.009$, $g_1 = 0.31$; $M_1 = 0.00025$, $\beta_1 = 0.091203927$; $N = 0.03125$, $c = 0.004$, $r_0 = 0.01$, $k = 0.004051610$.

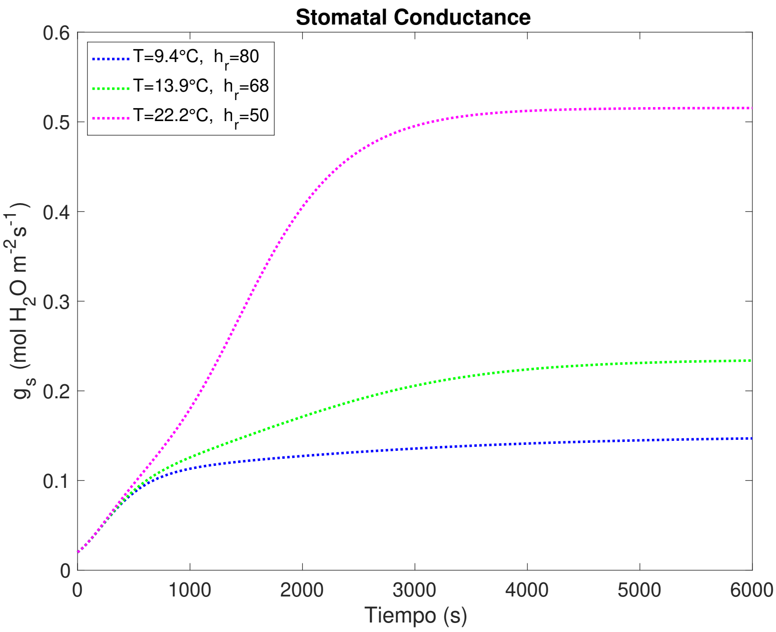

Figure 6: Simulation of model (2) for the variable g_s. The parameter values are: $g_0 = 0.009$, $g_1 = 0.31$; $M_1 = 0.00025$, $\beta_1 = 0.091203927$; $N = 0.03125$, $c = 0.004$, $r_0 = 0.01$, $k = 0.004051610$.

4. Conclusions and discussion

5. Appendix

5.1. Proof of Lemma 1 and Lemma 2

Proof. Lemma 1

Since $F_1(1) = 0$ and $F_2(1) > 0$, we have that $F_1(U) < F_2(U)$ at $U = 1$. So now we have to prove that $F_2(U) < F_1(U)$ for some $U = U^* \in]0, 1[$, that is, only cases b) and c) of Figure 2 can occur.

From remark 2, if U satisfies (i), we have that

$$G(U) > \frac{M_1 \cdot (U - 1)}{1.6 \cdot M_1 \cdot c_0 - (1.6 \cdot M_1 \cdot c_0 + 1.6 \cdot \beta_1) \cdot U}.$$

That is,

$$
\begin{aligned}
M_1 \cdot U - M_1 &< (1.6 \cdot M_1 \cdot c_0 - (1.6 \cdot M_1 \cdot c_0 + 1.6 \cdot \beta_1) \cdot U) \cdot G(U) \\
1.6 \cdot \beta_1 \cdot U \cdot G(U) &< M_1 + 1.6 \cdot M_1 \cdot c_0 \cdot G(U) - M_1 \cdot U - 1.6 \cdot M_1 \cdot c_0 \cdot U \cdot G(U) \\
1.6 \cdot \beta_1 \cdot U \cdot G(U) &< M_1 \cdot (1 - U) \cdot (1 + 1.6 \cdot c_0 \cdot G(U)) \\
\frac{1.6 \cdot G(U)}{1 + 1.6 \cdot c_0 \cdot G(U)} &< \frac{M_1 \cdot (1 - U)}{\beta_1 \cdot U},
\end{aligned}
\tag{12}
$$

Therefore, $F_2(U) < F_1(U)$.

And if U satisfies (ii), then analogously to (12), we obtain that $F_2(U) < F_1(U)$.

Therefore, the above tells us that exists at least one equilibrium point of system (3)$\blacksquare$.

Proof Lemma 2

The Jacobian matrix associated with Model (3) is

$$
J(U, g_s) = \begin{pmatrix} -\dfrac{\beta_1 \cdot g_s}{1 + c_0 \cdot g_s} - M_1 & -\dfrac{\beta_1 \cdot U}{(1 + c_0 \cdot g_s)^2} \\ \dfrac{1.6 \cdot k \cdot (g_s - r_0)}{1.6 \cdot G(U) - r_0} \cdot G'(U) & k \cdot \ln\left(\dfrac{1.6 \cdot G(U) - r_0}{g_s - r_0}\right) - k \end{pmatrix}. \tag{13}
$$

Replacing the equilibrium E_1 in (20), we obtain

$$
J(E_1) = \begin{pmatrix} -\left(\dfrac{1.6 \cdot \beta_1 \cdot G(U)}{1 + 1.6 \cdot c_0 \cdot G(U)} + M_1\right) & j_{12} \\ 1.6 \cdot k \cdot G'(U) & -k \end{pmatrix}, \tag{14}
$$

where j_{12} is

$$
j_{12} = -\left(\frac{\beta_1 \cdot M_1 + 1.6 \cdot M_1 \cdot \beta_1 \cdot c_0 \cdot G(U)}{(1.6 \cdot \beta_1 \cdot G(U) + M_1 + 1.6 \cdot c_0 \cdot M_1 \cdot G(U)) \cdot (1 + 1.6 \cdot c_0 \cdot G(U))^2}\right).
$$

The determinant of matrix (14) is

$$
\begin{aligned}
Det\, J(E_1) &= \frac{1.6\cdot\beta_1\cdot k\cdot G(U)}{1+1.6\cdot c_0\cdot G(U)} + M_1\cdot k + \\
&+ 1.6\cdot k\cdot G'(U)\cdot\left(\frac{\beta_1\cdot M_1 + 1.6\cdot M_1\cdot\beta_1\cdot c_0\cdot G(U)}{(1.6\cdot\beta_1\cdot G(U) + M_1 + 1.6\cdot c_0\cdot M_1\cdot G(U))\cdot(1+1.6\cdot c_0\cdot G(U))^2}\right) \\
&= k\cdot M_1 + \frac{1.6\cdot\beta_1\cdot k\cdot G(U)}{1+1.6\cdot c_0\cdot G(U)} - \\
&- \frac{1.6\cdot\beta_1\cdot k\cdot M_1\cdot(-G'(U))}{(1.6\cdot\beta_1\cdot G(U) + M_1 + 1.6\cdot c_0\cdot M_1\cdot G(U))\cdot(1+1.6\cdot c_0\cdot G(U))}
\end{aligned}
$$

$$
Det\, J(E_1) = \frac{R(U)}{[(c_0\cdot M_1+\beta_1)\cdot G(U) + 0.625\cdot M_1]\cdot(0.625 + c_0\cdot G(U))}, \tag{15}
$$

where $R(U)$ is

$$
\begin{aligned}
R(U) &= k\cdot[(2\cdot M_1\cdot\beta_1\cdot c_0 + c_0^2\cdot M_1^2 + \beta_1^2)\cdot G^2(U) + (1.25\cdot M_1\cdot\beta_1 + 1.25\cdot M_1^2\cdot c_0)\cdot G(U) + \\
&+ 0.390625\cdot M_1^2 + 0.625\cdot\beta_1\cdot M_1\cdot G'(U)]\,.
\end{aligned}
\tag{16}
$$

The denominator of the fraction satisfies that

$$
[(c_0\cdot M_1+\beta_1)\cdot G(U) + 0.625\cdot M_1]\cdot(0.625 + c_0\cdot G(U)) > 0.
$$

Therefore, we only need to study the sign of the numerator. Factoring $R(U)$, we obtain that

$$
R(U) = k\cdot(2\cdot M_1\cdot\beta_1\cdot c_0 + c_0^2\cdot M_1^2 + \beta_1^2)\cdot(G - R_1(U))\cdot(G - R_2(U)). \tag{17}
$$

Considering that

$$
k\cdot(2\cdot M_1\cdot\beta_1\cdot c_0 + c_0^2\cdot M_1^2 + \beta_1^2) > 0
$$

and Lemma (3), upon examining the remaining factors of $R(U)$, it is determined that

(i) when $G(U)$ satisfies the condition $(G(U) > R_1(U)$, it can be deduced that

$$
Det\, J(E_1) < 0.
$$

Therefore, in this case the equilibrium E_1 is a saddle point.

(ii) When $G(U)$ satisfies the condition $(G(U) < R_1(U)$, it can be deduced that

$$
Det\, J(E_1) > 0.
$$

Studying the trace of (14), we have that

$$Tra\, J(E_1) = -\left(\left(\frac{1.6 \cdot \beta_1 \cdot G(U)}{1 + 1.6 \cdot c_0 \cdot G(U)} + M_1\right) + k\right) < 0. \tag{18}$$

Therefore, in this case we have that E_1 is locally asymptotically stable $\blacksquare$.

5.2. Sensitivity analysis

Jacobian matrix of model (2)

$$J(W,g_s)=\begin{pmatrix} j_{11} & \dfrac{0.0061078\cdot\beta\cdot\exp(\frac{17.2694\cdot T}{237.3+T})\cdot(h_r-100)\cdot W}{(1+c_0\cdot g_s)^2\cdot W_{max}} \\ \dfrac{1.6\cdot k\cdot(g_s-r_0)}{1.6\cdot G(W)-r_0}\cdot G'(W) & k\cdot\ln\left(\dfrac{1.6\cdot G(W)-r_0}{g_s-r_0}\right)-k \end{pmatrix}, \tag{19}$$

where

$$j_{11}=\frac{\beta\cdot g_s\cdot(-0.61078+0.0061078\cdot h_r)\cdot\exp(\frac{17.2694\cdot T}{237.3+T})+\alpha\cdot(-1-c_0\cdot g_s)\cdot\Psi}{(1+c_0\cdot g_s)\cdot W_{max}}.$$

Jacobian matrix of model (2) with $G(W)=N_1+\left(\dfrac{N_2}{\sqrt{\beta\cdot\frac{W}{W_{max}}}}\right)$

$$J2(W,g_s)=\begin{pmatrix} j_{11} & j_{12} \\ -\dfrac{0.5\cdot k\cdot N_2\cdot(g_s-r_s)}{W\left(N_2+(N_1-0.625\cdot r_0)\cdot\sqrt{\frac{\beta\cdot W}{W_{max}}}\right)} & j_{22} \end{pmatrix}, \tag{20}$$

where

$$j_{11}=\frac{\beta\cdot g_s\cdot(-0.61078+0.0061078\cdot h_r)\cdot\exp(\frac{17.2694\cdot T}{237.3+T})+\alpha\cdot(-0.999-0.999\cdot c_0\cdot g_s)\cdot\Psi}{(1+c_0\cdot g_s)\cdot W_{max}},$$

$$j_{12}=\frac{0.0061078\cdot\beta\cdot\exp(\frac{17.2694\cdot T}{237.3+T})\cdot(h_r-100)\cdot W}{(1+c_0\cdot g_s)^2\cdot W_{max}},$$

and

$$j_{22}=k\cdot\ln\left(\frac{1.6\cdot\left(N_1+\left(\dfrac{N_2}{\sqrt{\beta\cdot\frac{W}{W_{max}}}}\right)\right)-r_0}{g_s-r_0}\right)-k.$$

Derivatives from the equations of the model (2) with respect to the parameters $\alpha, \Psi, \beta, W_{max}, c_0, h_r, T, k$ and r_0

$$
\begin{aligned}
D\left[\frac{dW}{dt}, \alpha\right] &= P \cdot \left(1 - \frac{W}{W_{max}}\right) \\
D\left[\frac{dW}{dt}, \Psi\right] &= \alpha \cdot \left(1 - \frac{W}{W_{max}}\right) \\
D\left[\frac{dW}{dt}, \beta\right] &= \frac{0.0061078 \cdot \exp(\frac{17.2694 \cdot T}{237.3+T}) \cdot g_s \cdot (h_r - 100) \cdot W}{(1 + c_0 \cdot g_s) \cdot W_{max}} \\
D\left[\frac{dW}{dt}, W_{max}\right] &= \frac{W}{W_{max}} \cdot \left(-\frac{0.0061078 \cdot \beta \cdot \exp(\frac{17.2694 \cdot T}{237.3+T}) \cdot g_s \cdot (-100 + h_r)}{(1 + c_0 \cdot g_s)} + \alpha \cdot \Psi\right) \\
D\left[\frac{dW}{dt}, c_0\right] &= -\frac{0.0061078 \cdot \beta \cdot \exp(\frac{17.2694 \cdot T}{237.3+T}) \cdot g_s^2 \cdot (-100 + h_r) \cdot W}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \\
D\left[\frac{dW}{dt}, h_r\right] &= \frac{0.0061078 \cdot \beta \cdot \exp(\frac{17.2694 \cdot T}{237.3+T}) \cdot g_s \cdot W}{(1 + c_0 \cdot g_s) \cdot W_{max}} \\
D\left[\frac{dW}{dt}, T\right] &= \frac{25.0299 \cdot \beta \cdot \exp(\frac{17.2694 \cdot T}{237.3+T}) \cdot g_s \cdot (-100 + h_r) \cdot W}{(1 + c_0 \cdot g_s) \cdot (237.3 + T)^2 \cdot W_{max}} \\
D\left[\frac{dW}{dt}, k\right] &= 0 \\
D\left[\frac{dW}{dt}, r_0\right] &= 0
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
D\left[\tfrac{dg_s}{dt}, \alpha\right] &= 0 \\
D\left[\tfrac{dg_s}{dt}, \Psi\right] &= 0 \\
D\left[\tfrac{dg_s}{dt}, \beta\right] &= -\frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0)}{\beta \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} \\
D\left[\tfrac{dg_s}{dt}, W_{max}\right] &= \frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0)}{\left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right) \cdot W_{max}} \\
D\left[\tfrac{dg_s}{dt}, c_0\right] &= 0 \\
D\left[\tfrac{dg_s}{dt}, h_r\right] &= 0 \\
D\left[\tfrac{dg_s}{dt}, T\right] &= 0 \\
D\left[\tfrac{dg_s}{dt}, k\right] &= (g_s - r_0) \cdot \ln\left(\frac{1.6 \cdot \left(N_1 + \left(\frac{N_2}{\sqrt{\beta \cdot \frac{W}{W_{max}}}}\right)\right) - r_0}{g_s - r_0}\right) \\
D\left[\tfrac{dg_s}{dt}, r_0\right] &= \frac{k}{-1.6 \cdot N_2 + (-1.6 \cdot N_1 + r_0) \cdot \sqrt{\beta \cdot \frac{W}{W_{max}}}} \cdot \left(-1.6 \cdot N_2 + (g_s - 1.6 \cdot N_1) \cdot \sqrt{\beta \cdot \frac{W}{W_{max}}}\right. \\
&\quad + \left(1.6 \cdot N_2 + 1.6 \cdot (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\beta \cdot \frac{W}{W_{max}}}\right) \\
&\quad \left.\cdot \ln\left(\frac{1.6 \cdot \left(N_1 + \left(\frac{N_2}{\sqrt{\beta \cdot \frac{W}{W_{max}}}}\right)\right) - r_0}{g_s - r_0}\right)\right)
\end{aligned}
\tag{22}
$$

Determination of system $\dot{Z} = f_c + J * Z$

First, lets define the following:

$$
\begin{aligned}
y_1 &= W \\
y_2 &= g_s \\
f(W, g_s, \alpha, \Psi, \beta, h_r, T, k, r_0) &= \begin{pmatrix} \alpha \cdot \Psi \cdot \left(1 - \frac{W}{W_{max}}\right) - \beta \cdot \frac{W}{W_{max}} \cdot \frac{g_s}{1+c_0 \cdot g_s} \cdot \left(2 - \frac{h_r}{100}\right) \cdot e_s(T) \\ k \cdot (g_s - r_0) \cdot \ln\left(\dfrac{1.6 \cdot G(W) - r_0}{g_s - r_0}\right) \end{pmatrix}
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
Z_1 &= \frac{\partial y_1}{\partial \alpha}, \quad Z_2 = \frac{\partial y_2}{\partial \alpha}, \quad Z_3 = \frac{\partial y_1}{\partial \Psi}, \quad Z_4 = \frac{\partial y_2}{\partial \Psi}, \quad Z_5 = \frac{\partial y_1}{\partial \beta} \quad Z_6 = \frac{\partial y_2}{\partial \beta}, \\
Z_7 &= \frac{\partial y_1}{\partial h_r}, \quad Z_8 = \frac{\partial y_2}{\partial h_r}, \quad Z_9 = \frac{\partial y_1}{\partial T}, \quad Z_{10} = \frac{\partial y_2}{\partial T} \quad Z_{11} = \frac{\partial y_1}{\partial k}, \quad Z_{12} = \frac{\partial y_2}{\partial k}, \\
Z_{13} &= \frac{\partial y_1}{\partial r_0}, \quad Z_{14} = \frac{\partial y_2}{\partial r_0} \quad Z_{15} = \frac{\partial y_1}{\partial c_0}, \quad Z_{16} = \frac{\partial y_2}{\partial c_0}, \quad Z_{17} = \frac{\partial y_1}{\partial W_{nax}}, \quad Z_{18} = \frac{\partial y_2}{\partial W_{max}}
\end{aligned} \tag{24}
$$

Now, let us determine the equations

$$
\begin{aligned}
\dot{Z}_1 &= \frac{\partial f_1}{\partial \alpha} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial \alpha} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial \alpha} \\
\dot{Z}_1 &= \frac{1}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \cdot (-\Psi \cdot (1 + c_{0 \cdot g_s}) \cdot ((1 + c_0 \cdot g_s) \cdot W + (-1 - c_0 \cdot g_s) \cdot W_{max} \\
&+\alpha \cdot (1 + c_0 \cdot g_s) \cdot Z_1) \\
&+\beta \cdot \exp\left(\tfrac{17.2694 \cdot T}{237.3 + T}\right) \cdot (-0.61078 + 0.0061078 \cdot h_r) \cdot (g_s \cdot (1 + c_0 \cdot g_s) \cdot Z_1 + W \cdot Z_2)\big)
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\dot{Z}_2 &= \frac{\partial f_2}{\partial \alpha} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial \alpha} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial \alpha} \\
\dot{Z}_2 &= -\frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0) \cdot Z_1}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} + k \cdot Z_2 \cdot \left(-1 + \ln\left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}}\right) - r_0}{g_s - r_0}\right)\right)
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\dot{Z}_3 &= \frac{\partial f_1}{\partial \Psi} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial \Psi} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial \Psi} \\
\dot{Z}_3 &= \alpha - \frac{\alpha \cdot W}{W_{max}} + \frac{0.0061078 * \beta \cdot \exp\left(\frac{17.2694 \cdot T}{237.3+T}\right) \cdot (h_r - 100) \cdot W \cdot Z_2}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \\
&\quad + \frac{\left(\beta \cdot \exp\left(\frac{17.2694 \cdot T}{237.3+T}\right) \cdot g_s \cdot (-0.61078 + 0.0061078 \cdot h_r) + \alpha \cdot (-1 - c_0 \cdot g_s) \cdot \Psi\right) \cdot Z_3}{(1 + c_0 \cdot g_s) \cdot W_{max}}
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\dot{Z}_4 &= \frac{\partial f_2}{\partial \Psi} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial \Psi} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial \Psi} \\
\dot{Z}_4 &= -\frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0) \cdot Z_3}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} + k \cdot Z_4 \cdot \left(-1 + \ln\left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}}\right) - r_0}{g_s - r_0}\right)\right)
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\dot{Z}_5 &= \frac{\partial f_1}{\partial \beta} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial \beta} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial \beta} \\
\dot{Z}_5 &= \frac{1}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \cdot (\alpha \cdot (-1 + c_0 \cdot g_s \cdot (-2 - c_0 \cdot g_s)) \cdot \Psi \cdot Z_5 \\
&\quad + \exp\left(\frac{17.2694 \cdot T}{237.3+T}\right) \cdot (g_s \cdot (-100 + h_r + c_0 \cdot g_s \cdot (-100 + h_r)) \\
&\quad \cdot (0.0061078 \cdot W + 0.0061078 \cdot \beta \cdot Z_5) + \beta \cdot (-0.61078 + 0.0061078 \cdot h_r) \cdot W \cdot Z_6)
\end{aligned} \tag{29}
$$

$$\dot{Z}_6 = \frac{\partial f_2}{\partial \beta} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial \beta} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial \beta}$$

$$\dot{Z}_6 = -\frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0)}{\beta \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} - \frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0) \cdot Z_5}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)}$$
$$+k \cdot Z_6 \cdot \left(-1 + \ln\left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}}\right) - r_0}{g_s - r_0}\right)\right) \tag{30}$$

$$\dot{Z}_7 = \frac{\partial f_1}{\partial h_r} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial h_r} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial h_r}$$

$$\dot{Z}_7 = \frac{1}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \cdot (\alpha \cdot (-1 + c_0 \cdot g_s \cdot (-2 - c_0 \cdot g_s)) \cdot \Psi \cdot Z_7$$
$$+\beta \cdot \exp\left(\frac{17.2694 \cdot T}{237.3 + T}\right) \cdot (g_s \cdot (0.0061078 + 0.0061078 \cdot c_0 \cdot g_s) \cdot W$$
$$+g_s \cdot (-0.61078 + c_0 \cdot g_s \cdot (-0.61078 + 0.0061078 \cdot h_r) + 0.0061078 \cdot h_r) \cdot Z_7$$
$$+(-0.61078 + 0.0061078 \cdot h_r) \cdot W \cdot Z_8)) \tag{31}$$

$$\dot{Z}_8 = \frac{\partial f_2}{\partial h_r} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial h_r} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial h_r}$$

$$\dot{Z}_8 = -\frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0) \cdot Z_7}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} + k \cdot Z_8 \cdot \left(-1 + \ln\left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}}\right) - r_0}{g_s - r_0}\right)\right) \tag{32}$$

$$\dot{Z}_9 = \frac{\partial f_1}{\partial T} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial T} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial T}$$

$$\begin{aligned}\dot{Z}_9 = \; & \frac{1}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \cdot \Bigg(\frac{25.0299 \cdot \beta \exp\left(\frac{17.2694 \cdot T}{237.3+T}\right) \cdot g_s \cdot (1 + c_0 \cdot g_s) \cdot (-100 + h_r) \cdot W}{(237.3 + T)^2} \\ & +0.0061078 \cdot \beta \cdot \exp\left(\frac{17.2694 \cdot T}{237.3+T}\right) \cdot (-100 + h_r) \cdot W \cdot Z_{10} \\ & +(1 + c_0 \cdot g_s) \cdot \left(\beta \cdot \exp\left(\frac{17.2694 \cdot T}{237.3+T}\right) \cdot g_s \cdot (-0.61078 + 0.0061078 \cdot h_r) + \alpha \cdot (-1 - c_0 \cdot g_s) \cdot \Psi\right) \cdot Z_9\Bigg)\end{aligned} \tag{33}$$

$$\dot{Z}_{10} = \frac{\partial f_2}{\partial T} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial T} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial T}$$

$$\dot{Z}_{10} = -\frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0) \cdot Z_9}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} + k \cdot Z_{10} \cdot \left(-1 + \ln\left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}}\right) - r_0}{g_s - r_0}\right)\right) \tag{34}$$

$$\dot{Z}_{11} = \frac{\partial f_1}{\partial k} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial k} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial k}$$

$$\begin{aligned}\dot{Z}_{11} = \; & \frac{1}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \cdot (\alpha \cdot (-1 + c_0 \cdot g_s \cdot (-2 - c_0 \cdot g_s)) \cdot \Psi \cdot Z_{11} \\ & + \beta \cdot \exp\left(\frac{17.2694 \cdot T}{237.3+T}\right) \cdot (-0.61078 + 0.0061078 \cdot h_r) \cdot (g_s \cdot (1 + c_0 \cdot g_s) \cdot Z_{11} + W \cdot Z_{12}))\end{aligned} \tag{35}$$

$$\dot{Z}_{12} = \frac{\partial f_2}{\partial k} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial k} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial k}$$

$$\dot{Z}_{12} = \frac{k \cdot \left(-0.5 \cdot g_s \cdot N_2 \cdot Z_{11} - (N_1 - 0.625 \cdot r_0) \cdot W \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}} \cdot Z_{12} + N_2 \cdot (0.5 \cdot r_0 \cdot Z_{11} - W \cdot Z_{12})\right)}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)}$$
$$+(g_s - r_0 + k \cdot Z_{12}) \cdot \ln \left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}}\right) - r_0}{g_s - r_0} \right) \tag{36}$$

$$\dot{Z}_{13} = \frac{\partial f_1}{\partial r_0} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial r_0} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial r_0}$$

$$\dot{Z}_{13} = \frac{1}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \cdot (\alpha \cdot (-1 + c_0 \cdot g_s \cdot (-2 - c_0 \cdot g_s)) \cdot \Psi \cdot Z_{13}$$
$$+\beta \cdot \exp\left(\frac{17.2694 \cdot T}{237.3 + T}\right) \cdot (-0.61078 + 0.0061078 \cdot h_r) \cdot (g_s \cdot (1 + c_0 \cdot g_s) \cdot Z_{13} + W \cdot Z_{14})\Big) \tag{37}$$

$$\dot{Z}_{14} = \frac{\partial f_2}{\partial r_0} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial r_0} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial r_0}$$

$$\dot{Z}_{14} = \frac{k \cdot W \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}} \cdot (-0.625 \cdot g_s + N_1 - N_1 \cdot Z_{14} + 0.625 \cdot r_0 \cdot Z_{14})}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)}$$
$$+\frac{k \cdot N_2 \cdot (W - 0.5 \cdot g_s \cdot Z_{13} + 0.5 \cdot r_0 \cdot Z_{13} - W \cdot Z_{14})}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} \tag{38}$$
$$+k \cdot (-1 + Z_{14}) \cdot \ln \left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}}\right) - r_0}{g_s - r_0} \right)$$

$$
\begin{aligned}
\dot{Z}_{15} &= \frac{\partial f_1}{\partial c_0} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial c_0} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial c_0} \\
\dot{Z}_{15} &= \frac{1}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}} \cdot \left(\alpha \cdot (-1 + c_0 \cdot g_s \cdot (-2 - c_0 \cdot g_s)) \cdot \Psi \cdot Z_{15} + \beta \cdot \exp\left(\tfrac{17.2694 \cdot T}{237.3 + T}\right)\right. \\
&\quad \cdot (g_s \cdot (g_s \cdot (0.61078 - 0.0061078 \cdot h_r) \cdot W + (-0.61078 + 0.0061078 \cdot h_r) \cdot Z_{15} \\
&\quad \left. + c_0 \cdot g_s \cdot (-0.61078 + 0.0061078 \cdot h_r) \cdot Z_{15}) + (-0.61078 + 0.0061078 \cdot h_r) \cdot W \cdot Z_{16})\right)
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
\dot{Z}_{16} &= \frac{\partial f_2}{\partial c_0} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial c_0} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial c_0} \\
\dot{Z}_{16} &= -\frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0) \cdot Z_{15}}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} + k \cdot Z_{16} \cdot \left(-1 + \ln\left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}}\right) - r_0}{g_s - r_0}\right)\right)
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\dot{Z}_{17} &= \frac{\partial f_1}{\partial W_{max}} + \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial y_1}{\partial W_{max}} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial y_2}{\partial W_{max}} \\
\dot{Z}_{17} &= \frac{\alpha \cdot \Psi \cdot (W - W_{max} \cdot Z_{17})}{W_{max}^2} + \frac{1}{(1 + c_0 \cdot g_s)^2 \cdot W_{max}^2} \cdot \left(\beta \cdot \exp\left(\tfrac{17.2694 \cdot T}{237.3 + T}\right)\right. \\
&\quad \cdot (g_s \cdot (-100 + h_r + c_0 \cdot g_s \cdot (-100 + h_r)) \cdot (-0.0061078 \cdot W + 0.0061078 \cdot W_{max} \cdot Z_{17}) \\
&\quad \left. + (-0.61078 + 0.0061078 \cdot h_r) \cdot W \cdot W_{max} \cdot Z_{18})\right)
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
\dot{Z}_{18} &= \frac{\partial f_2}{\partial W_{max}} + \frac{\partial f_2}{\partial y_1} \cdot \frac{\partial y_1}{\partial W_{max}} + \frac{\partial f_2}{\partial y_2} \cdot \frac{\partial y_2}{\partial W_{max}} \\
\dot{Z}_{18} &= \frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0)}{W_{max} \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} - \frac{0.5 \cdot k \cdot N_2 \cdot (g_s - r_0) \cdot Z_{17}}{W \cdot \left(N_2 + (N_1 - 0.625 \cdot r_0) \cdot \sqrt{\frac{\beta \cdot W}{W_{max}}}\right)} \\
&\quad + k \cdot Z_{18} \cdot \left(-1 + \ln \left(\frac{1.6 \cdot \left(N_1 + \frac{N_2}{\sqrt{\frac{\beta \cdot W}{W_{max}}}} \right) - r_0}{g_s - r_0} \right) \right)
\end{aligned} \tag{42}
$$

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References