Statistical properties of velocity increments in two-dimensional turbulence

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Abstract The multiple-point probability density $f(v_1, r_1; v_2, r_2; \ldots v_N, r_N)$ of velocity increments v_i at different length scales r_i is investigated in a direct numerical simulation of two-dimensional turbulence. It has been shown for experimental data of three-dimensional turbulence, that this probability density can be represented by conditional probability densities in form of a Markov chain [1]. We have extended this analysis to the case of two-dimensional forced turbulence in the inverse cascade regime.

1 Introduction

Turbulence is commonly believed to exhibit universal statistical properties. This applies to the three- as well as to the two-dimensional case. In three dimensions the universal state is characterized by the direct energy cascade, which cascades energy from large scales down to smaller ones. Whereas in two-dimensional turbulence two cascades can be found: the enstrophy cascade and the inverse energy cascade. The latter one transports, in contrast to the three-dimensional case, energy injected at small scales towards larger scales. The main quantity of interest when describing those universal states of turbulence are the longitudinal velocity inrcrements $v(r,t)$ at scale r, defined by

$$v(r,t) = \frac{\boldsymbol{r}}{r} \cdot [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}(\boldsymbol{x},t)] \,,$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ is the velocity field at time t and location $\boldsymbol{x}$. Due to homogeneity and stationarity of the turbulence statistics the statistical properties of the longitudinal velocity increment do not depend on the reference point $\boldsymbol{x}$ and time t.

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A statistical description of the inverse cascade is formally conveyed in the N-point probability density function (pdf) $f(v_1, r_1; v_2, r_2; \dots v_N, r_N)$, where all r_i are within the inertial range. To simplify matters we choose in the following $r_{i+1} < r_i$. From this pdf the conditional probability density function $p(v_1, r_1|v_2, r_2; \dots v_N, r_N)$ may be obtained via

$$p(v_1, r_1|v_2, r_2; v_3, r_3; \dots v_N, r_N) = \frac{f(v_1, r_1; v_2, r_2; v_3, r_3; \dots v_N, r_N)}{f(v_2, r_2; v_3, r_3; \dots v_N, r_N)}.$$

The N-point pdf simplifies to a product of two-point conditionals pdfs, if the governing stochastic process is a Markov process, i.e. if the conditional pdfs fulfill the Markov property

$$p(v_1, r_1|v_2, r_2; \dots v_N, r_N) = p(v_1, r_1|v_2, r_2),$$

for all $N \geq 3$ and every set of length scales $r_1, \dots r_N$. Often one finds that this property only holds for distances $r_i - r_{i+1}$, $i = 1, \dots N-1$ that are larger than a certain length, which is frequently referred to as Markov–Einstein length l_{ME} (see e.g. [3]). As a consequence the multiple-point pdf $f(v_1, r_1; \dots v_N, r_N)$ can be evaluated on scales larger than l_{ME} via a Markov chain

$$f(v_1, r_1; \dots v_N, r_N) = p(v_1, r_1|v_2, r_2) \cdots p(v_{N-1}, r_{N-1}|v_N, r_N) f(v_N, r_N).$$

2 Numerical Treatment and Statistical Analysis

We have numerically solved the forced two-dimensional Navier–Stokes equation on a doubly periodic square domain of 2π side length and 1024^2 grid points. To obtain a statistically stationary flow we apply a small-scale forcing with a spatiotemporal correlation function $\langle f(x + r, t), f(x, t')\rangle \sim \delta(t - t') \exp(-r^2/2l_c^2)$, where l_c is the correlation length. The solution yields an inverse cascade with a well-defined inertial range and an energy spectrum $E(k) \approx k^{-5/3}$ (for a detailed description of the numerics see [2]). In the following L shall denote the integral scale and λ the Taylor scale of this flow.

We have evaluated the statistics of the longitudinal velocity increments from the numerical solution and checked the Markov property. A rigorous proof for the validity of this property would afford an investigation of $p(v_1, r_1|v_1, r_2; \dots v_N, r_N)$ for all $N \geq 3$ and every set of length scales $r_1, \dots r_N$. This is clearly not possible. The obtainable data allowed for an investigation of the conditional probability densities $p(v_1, r_1|v_2, r_2)$ and $p(v_1, r_1|v_2, r_2; v_3, r_3)$ for different length scales r_1, r_2 and r_3. One example for these pdfs is displayed in Fig 1.

Comparison of the two conditional pdfs shows that they coincide for all v_3 given the differences $r_1 - r_2$ and $r_2 - r_3$ are large enough. To quantify this result we have calculated the mean correlation of cuts through the conditional pdfs

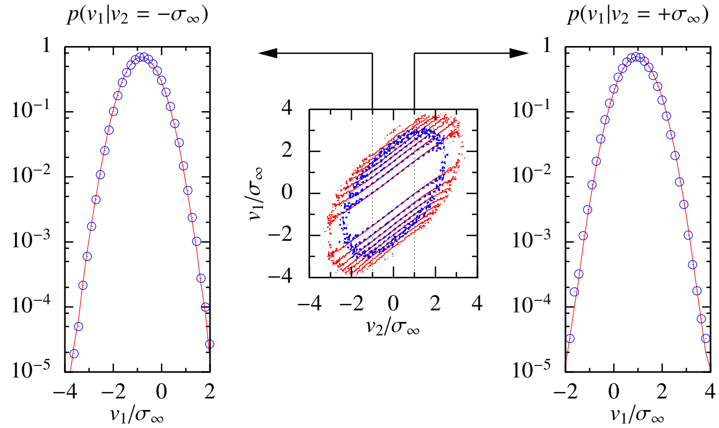

Fig. 1 The Graph in the middle shows the contour plots of the conditional pdfs $p(v_1, r_1|v_2, r_2)$ (solid lines) and $p(v_1, r_1|v_2, r_2; v_3 = 0, r_3)$ (dashed lines) for $r_1 = \lambda + L$, $r_2 = \lambda + L/2$, and $r_3 = \lambda$. At the sides, cuts through those pdfs at $v_2 = \pm\sigma_\infty$ respectiveley are shown. The two pdfs clearly coincide.

$$\mathrm{Corr}(r_1, \Delta r, v_3) = \left\langle \frac{\int p(v_1, r_1|v_2, r_2) p(v_1, r_1|v_2, r_2; v_3, r_3) \mathrm{d}v_1}{\sqrt{\int p(v_1, r_1|v_2, r_2)^2 \mathrm{d}v_1} \sqrt{\int p(v_1, r_1|v_2, r_2; v_3, r_3)^2 \mathrm{d}v_1}} \right\rangle_{v_2} ,$$

which proved to be a good measure for their accordance. Fig 2 shows that the coincidence occurs for $\Delta r = r_1 - r_2 = r_2 - r_3 > 0.4\lambda = l_{ME}$.

As a test for self-consistency of our results we have computed the one-point pdfs from a Markov Chain within the inertial range, i.e. we have calculated $f(v_i, r_i)$ via

$$\begin{aligned} f(v_i, r_i) &= \int \mathrm{d}v_{i+1} \cdots \mathrm{d}v_N\, f(v_i, r_i; v_{i+1}, r_{i+1}; \dots v_{N-1}, r_{N-1}; v_N, \lambda) \\ &= \int \mathrm{d}v_{i+1} \cdots \mathrm{d}v_N\, p(v_i, r_i|v_{i+1}, r_{i+1}) \cdots p(v_{N-1}, r_{N-1}|v_N, \lambda) f(v_N, \lambda), \qquad (1) \end{aligned}$$

for different r_i with the step size $r_{i+1} - r_i = l_{ME}$. Fig 3 compares the directly evaluated pdfs and the calculated ones. Their coincidence for all r_i within the inertial range affirms the validity of the Markov property.

3 Conclusion

Our results give evidence that the evolution of the pdfs of velocity increments within the inertial range can be described as a Markov process. This extends former statistical analyses of three-dimensional turbulence to the case of two-dimensional turbulence in the inverse cascade regime. For both cases a finite Markov–Einstein length within the range of the Taylor scale is found. These results suggest that the Markov

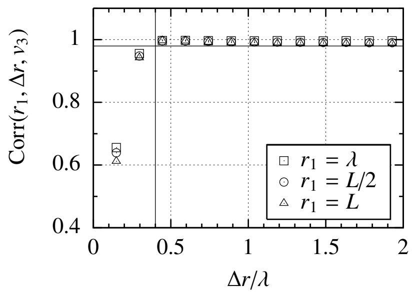

Fig. 2 Determination of the Markov–Einstein length. Shown is the mean correlation of $p(v_1, r_1|v_2, r_2)$ and $p(v_1, r_1|v_2, r_2; v_3 = 0, r_3)$, with $\Delta r = r_1 - r_2 = r_2 - r_3$. Accordance is supposed for Corr$(r_1, \Delta r, v_3) \geq 0.98$. For different v_3 similar results are obtained.

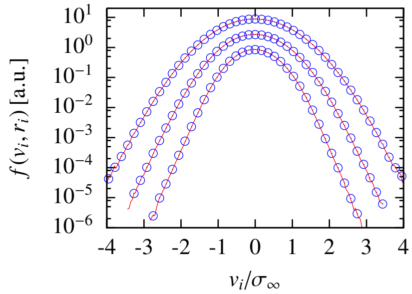

Fig. 3 Shown are the pdfs $f(v_i, r_i)$ for various r_i (from top to bottom: L, $L/2$, 2λ). Lines indicate directly evaluated pdfs, the values for the symbols were calculated by Eq (1). The pdfs have been shifted vertically for clarity of presentation.

property and the finite Markov–Einstein length might be universal properties of turbulent flows.

The existence of a Markov property demonstrates that the longitudinal velocity increment statistics at scale r_i can be determined from those at scale r_{i+1}. This process can be iterated and holds for the whole inertial range. It can be seen as a manifestation of the turbulent cascade.

As an application our results provide an approach to the modeling of increment statistics in turbulent flows in terms of Langevin and Fokker–Planck equations.

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