Mathify HSC

FURTHER FUNCTIONS

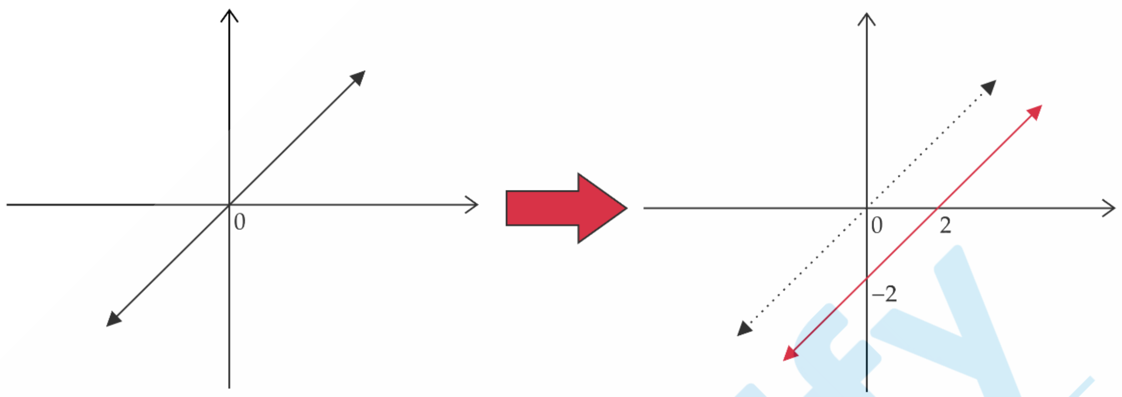
GRAPH SHIFTS AND TRANSLATIONS (II)

Contents include:

- Horizontal Shifts and Translations
- Vertical Shifts and Translations

- Horizontal Shifts & Translations

Let's say we are given the graph $f(x) = x$, and are asked to sketch this graph, but shifted 2 units to the right. Graphically, this would resemble:



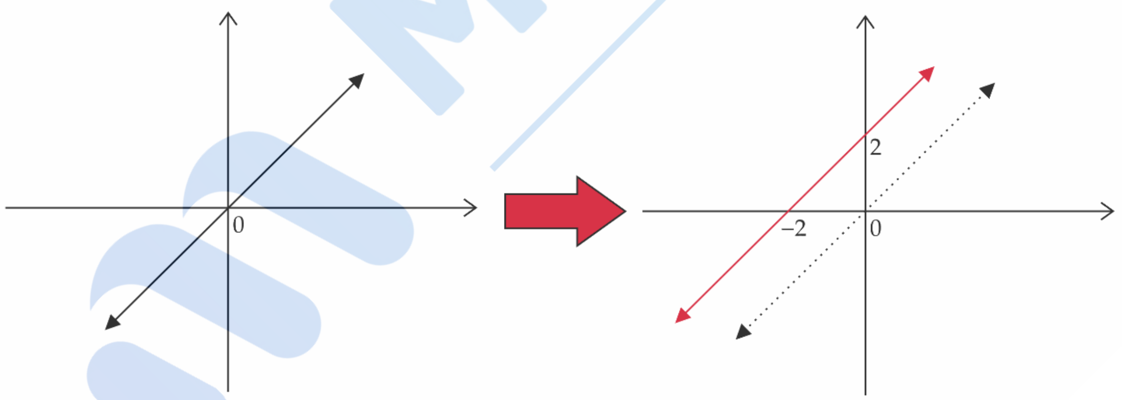
Looking at the new graph (the red line), we can see that its equation is $g(x) = x - 2$.

In other words, after **shifting 2 units right**:

$$y = x \;\rightarrow y = x - 2$$

Notice that we have replaced the "x" with "$x - 2$"

If we instead shifted $f(x) = x$, 2 units to the **left**, then graphically this would resemble:



Looking at the new graph now (the red line), we can see that its equation is $h(x) = x + 2$

In other words, after **shifting 2 units left**:

$$y = x \rightarrow y = x + 2$$

Notice that we have replaced the "x" with "$x + 2$"

Thus, in general for horizontal shifts/translations of a function $f(x)$:

- Shifting a units right means replacing every "x" with "$x - a$". In other words:

$$f(x) \to f(x-a)$$

- Shifting a units left means replacing every "x" with "$x + a$". In other words:

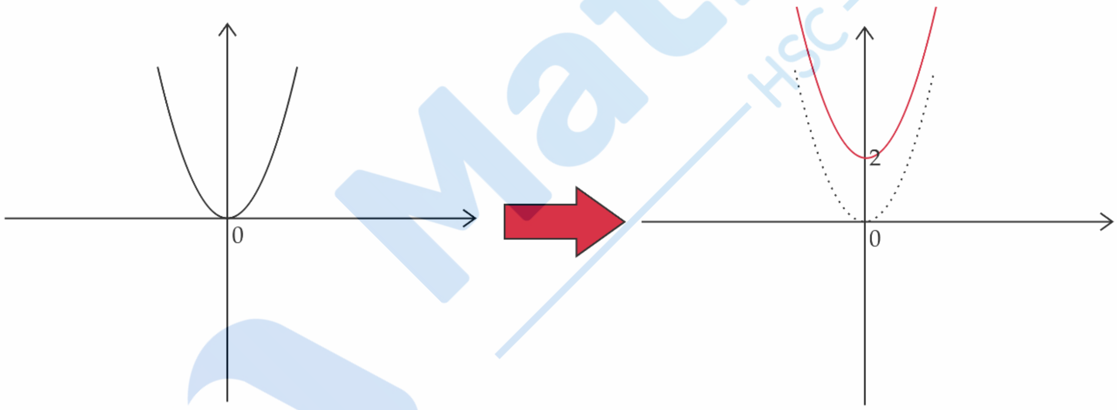
$$f(x) \to f(x+a)$$

Note: A shift of 'a' units to the **left** is the same as saying a shift of '$-a$' units to the **right**

For terminology, remember that "shifts" and "translation" mean the same thing!

- Vertical Shifts & Translations

Let's say we are now instead given the graph of $y = x^2$, and are asked to sketch this graph but shifted 2 units upwards. Graphically, this would resemble:

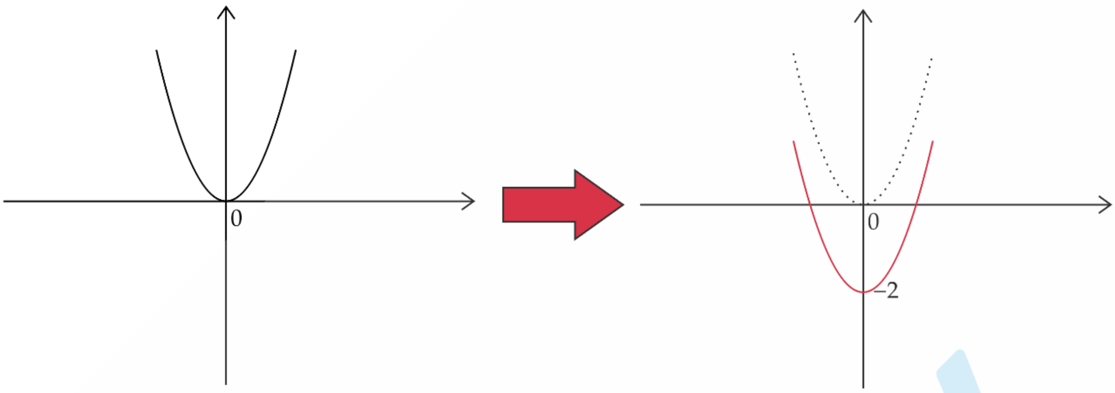


Looking at the new graph now (the red curve), we can see its equation is $y = x^2 + 2$, or in other words if we rearrange the equation:

$$y - 2 = x^2$$

Notice that we have replaced "y" with "$y - 2$"

If we instead shifted $f(x) = x^2$ 2 units **downwards**, graphically this would resemble:



Looking at the new graph now (the red curve), we can see its equation is $y = x^2 - 2$, or in other words if we rearrange the equation:

$$y + 2 = x^2$$

Notice that we have replaced "y" with "$y + 2$"

Thus, in general for vertical shifts/translations of a function $f(x)$:

- Shifting a units upwards means replacing every "y" with "$y - a$". In other words:

$$y \to y - a$$

- Shifting a units downwards means replacing every "y" with "$y + a$". In other words:

$$y \to y + a$$

Another way to remember vertical shifts is that if $y = f(x)$, then:

- Shifting upwards a units means:

$$y = f(x) + a$$

- Shifting downwards a units means:

$$y = f(x) - a$$

This is essentially the same as before, just rearranged!

Example 1: Write down the equation of the resulting graph when each transformation below is applied to the circle $(x-1)^2+(y+2)^2=1$

a) Shift left 3 units

To shift left 3 units, replace x with $x+3$:

$$\therefore (x+3-1)^2+(y-2)^2=1$$
$$(x+2)^2+(y-2)^2=1$$

b) Shift up 4 units

To shift up 4 units, replace y with $y-4$:

$$\therefore (x-1)^2+(y-4+2)^2=1$$
$$(x-1)^2+(y-2)^2=1$$

c) Shift down 2 units and right 1 unit

To shift down 2 units, replace y with $y+2$ and to shift right 1 units, replace x with $x-1$:

$$\therefore (x-1-1)^2+(y+2+2)^2=1$$
$$(x-2)^2+(y+4)^2=1$$

Graph Shifts and Translation Exercises

1. For each of the following parts, write down the new equation of the function after the given shift has been applied:

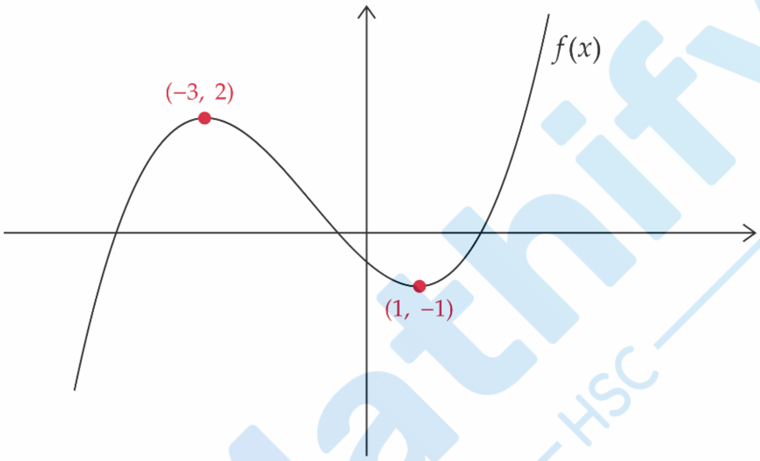
a) $y=x+5$, shift 4 units upwards
b) $y=-x+3$, shift 3 units to the left
c) $y=x^2-3x$, shift 2 units downwards
d) $y=x^2+x-4$, shift 2 units to the right
e) $y=2^x$, shift 5 units to the left
f) $x^2+y^2=4$, shift 1 unit upwards
g) $(x-2)^2+(y+4)^2=9$, shift 3 units to the left
h) $y=\sin x$, shift 5 units to the right
i) $y=\sqrt{x}$, shift 3 units downwards

2. Write down the new equation of each function after both given shifts have been applied:

a) $y=\cos(x+1)$, shift 3 units to the right, 2 units upwards
b) $y=x^3$, shift 1 unit to the left, 6 units upwards
c) $y=\frac{3}{x+5}$, shift 2 units to the left, 3 units downwards
d) $y=-\frac{4}{x^2+1}$, shift $\frac{1}{2}$ unit to the right, 5 units upwards
e) $(x+1)^2+(y-2)^2=16$, shift 2 units to the left, 1 unit downwards

3. We are given a function $f(x) = x^2$. Explain the necessary transformations required to change $f(x)$ into each of the following new equations:
a) $f(x) = x^2 + 3$
b) $f(x) = (x-2)^2$
c) $f(x) = (x+4)^2 - 4$
d) $f(x) = x^2 + 2x - 3$ (Hint: complete the square first)
e) $f(x) = x^2 + 6x + 15$ (Hint: complete the square first)

4. The graph for the function $f(x)$ is shown below:



Sketch the graph of each of the following on separate axes, labelling any important points:

a) $f(x+2)$
b) $f(x) - 1$

Graph Shifts and Translation Exercise Answers

1.
a)

Shifting 4 units upwards:

$$y = x + 5 + 4$$

$$\therefore y = x + 9$$

b)

Shifting 3 units to the left:

$$\begin{aligned} y &= -(x+3) + 3 \\ &= -x - 3 + 3 \\ &= -x \end{aligned}$$

c)

Shifting 2 units downwards:

$$y = x^2 - 3x - 2$$

d)

Shifting 2 units to the right:

$$\begin{aligned} y &= (x-2)^2 + (x-2) - 4 \\ &= x^2 - 4x + 4 + x - 2 - 4 \\ &= x^2 - 3x - 2 \end{aligned}$$

e)

Shifting 5 units to the left:

$$y = 2^{x+5}$$

f)

Shifting 1 unit upwards:

$$x^2 + (y-1)^2 = 4$$

g)

Shifting 3 units to the left:

$$(x + 3 - 2)^2 + (y+4)^2 = 9$$

$$(x+1)^2 + (y+4)^2 = 9$$

h)

Shifting 5 units to the right:

$$y = \sin(x-5)$$

i)

Shifting 3 units downwards:

$$y = \sqrt{x} - 3$$

2.
a)

First shifting 3 units to the right:

$$\begin{aligned} y &= \cos(x - 3 + 1) \\ &= \cos(x-2) \end{aligned}$$

Then shifting 2 units upwards:

$$y = \cos(x-2) + 2$$

b)

First shifting 1 unit to the left:

$$y = (x+1)^3$$

Then shifting 6 units upwards:

$$y = (x+1)^3 + 6$$

c)

First shifting 2 units to the left:

$$y = \frac{3}{(x+2)+5}$$
$$= \frac{3}{x+7}$$

Then shifting 3 units downwards:

$$y = \frac{3}{x+7} - 3$$

d)

First shifting $\frac{1}{2}$ unit to the right:

$$y = -\frac{4}{\left(x-\frac{1}{2}\right)^2 + 1}$$

Then shifting 5 units upwards:

$$y = -\frac{4}{\left(x-\frac{1}{2}\right)^2 + 1} + 5$$

e)

First shifting 2 units to the left:

$$(x+2+1)^2 + (y-2)^2 = 16$$
$$(x+3)^2 + (y-2)^2 = 16$$

Then shifting 5 units upwards:

$$(x+3)^2 + (y-5-2)^2 = 16$$
$$(x+3)^2 + (y-7)^2 = 16$$

3.

a)

Notice that since $x^2 \to x^2 + 3$, it means that $f(x)$ has been shifted up 3 units

b)

Since $x \to x - 2$, this means that $f(x)$ has been shifted right 2 units

c)

Since $x \to x + 4$, this means that $f(x)$ has been shifted left 4 units

Moreover, since $(x+4)^2 \to (x+4)^2 - 4$, this means that $f(x)$ has also been shifted down 4 units

d)

Completing the square for $f(x)$:

$$\begin{aligned} f(x) &= x^2 + 2x + 1 - 1 - 3 \\ &= (x+1)^2 - 3 \end{aligned}$$

Since $x \to x + 1$, this means that $f(x)$ has been shifted left 1 unit

Moreover, since $(x+1)^2 \to (x+1)^2 - 3$, this means that $f(x)$ has also been shifted down 3 units

e)

Completing the square for $f(x)$:

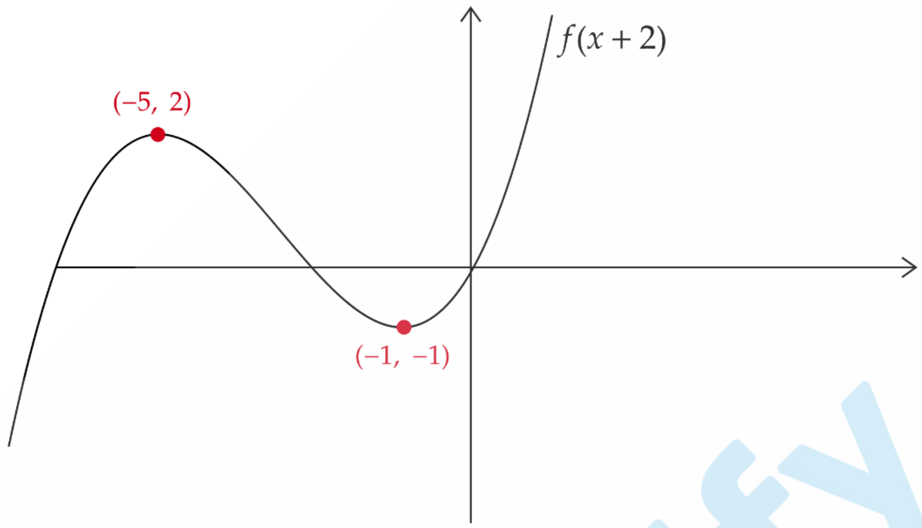
$$\begin{aligned} f(x) &= x^2 + 6x + 9 - 9 + 15 \\ &= (x+3)^2 + 6 \end{aligned}$$

Since $x \to x + 3$, this means that $f(x)$ has been shifted left 3 units

Moreover, since $(x+3)^2 \to (x+3)^2 + 6$, this means that $f(x)$ has also been shifted up 6 units

4.

a) $f(x+2)$ is the graph of the function shifted 2 units to the left:



b) Since $f(x) - 1$ is the graph of $f(x)$ shifted down 1 unit: