FURTHER FUNCTIONS

COMBINED TRANSFORMATIONS (IV)

Contents include:

- Does Order of Transformations Matter?
- The Universal Formula for Transformations

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- Does Order of Transformations Matter?

We can apply more than one transformation to a graph, i.e., dilations, translations, and reflections which we have covered previously. However, we must always be careful when applying multiple transformations because sometimes, the **order at which we apply them matters!**

Two transformations are said to **commute** if the order in which they are applied **does not matter**, meaning that if you do one transformation first and then the other transformation, the end result would be the same as if you did it vice versa instead.

The following is a table showing which transformations commute and which don't:

Commute	***Don't Commute***
Two translations	*A translation and dilation if both are either vertical or horizontal*
Two dilations	
A translation and dilation if one is vertical and the other is horizontal	

For example, if I shifted my graph up 1 units then right 1 unit, this would be the same as shifting right 1 unit then up 1 unit. This means the transformations **commute**

Instead, if I shifted 2 units up and stretched the graph vertically by a factor of 2, this would be different to first stretching the graph vertically by a factor of 2 and then shifting the graph up by 2 units. This means the transformations do **not commute**

Example 1:

a) A vertical dilation with factor $\frac{1}{2}$ and a translation down 3 units are applied to the function $y = x^2 + 4$. Find the equation of the resulting graph and sketch

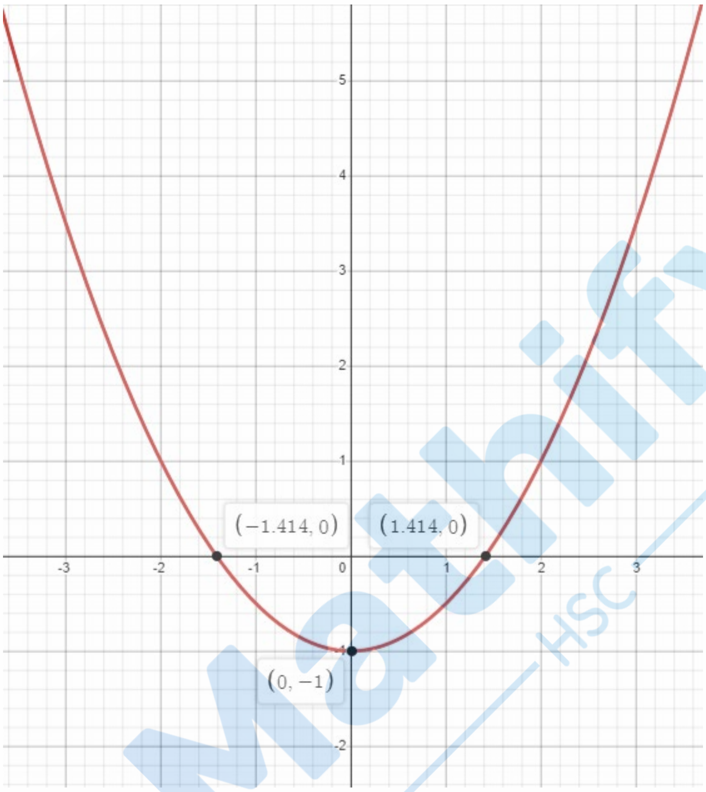
First, a vertical dilation means that y changes into $\frac{y}{\frac{1}{2}} = 2y$:

$$\therefore 2y = x^2 + 4$$
$$y = \frac{x^2}{2} + 2$$

Then, a translation down 3 units means that y changes into $y + 3$:

$$\therefore y + 3 = \frac{x^2}{2} + 2$$
$$y = \frac{x^2}{2} - 1$$

Sketching our final result:



b) Repeat the process but when the translation is applied first

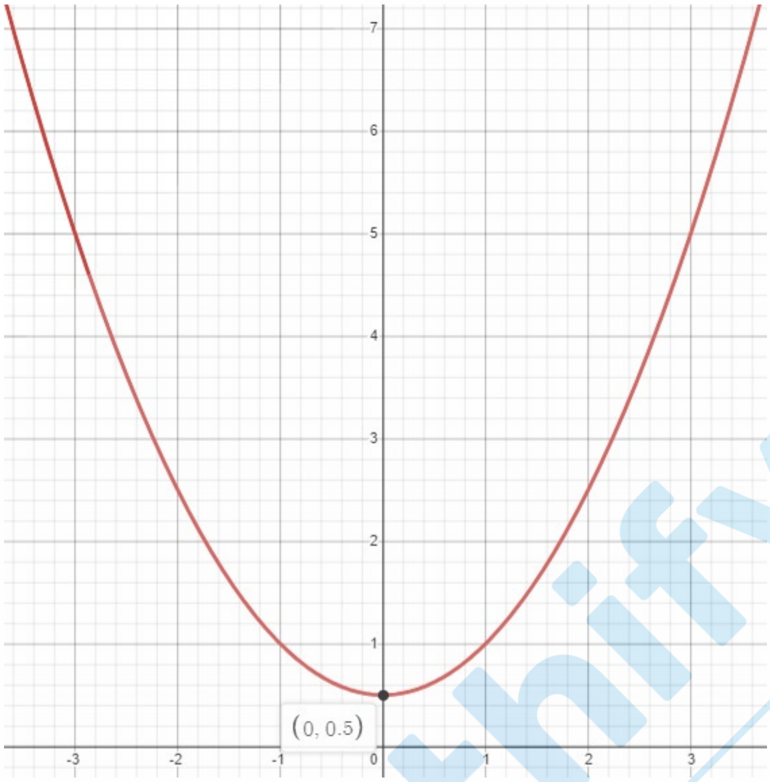
First, a translation down 3 units means that we change y into $y + 3$:

$$\therefore y + 3 = x^2 + 4$$
$$y = x^2 + 1$$

Then, a vertical dilation of factor $\frac{1}{2}$ means that we change y into $2y$:

$$2y = x^2 + 1$$
$$y = \frac{x^2}{2} + \frac{1}{2}$$

Sketching this:



- The Universal Formula for Transformations

The universal formula is:

$$y = kf\big(a(x+b)\big) + c$$

And is useful because it applies all four transformations to any function $y = f(x)$.

It is not a necessity to memorise, as it is more important to actually understand the methods of transformations rather than blindly following this formula. Moreover, full marks for working may not be given if the formula is used in some cases.

The following sequence of transformations transform the function $y = f(x)$ to $y = kf\big(a(x+b)\big) + c$:

1. Stretch horizontally with factor $\frac{1}{a}$
2. Shift left b units
3. Stretch vertically with factor k
4. Shift up c units

Example 2: Using the universal formula, find the transformations needed to change $y = x^2 + 4$ into $y = \frac{x^2}{2} - 1$

In this case:

$$f(x) = x^2 + 4$$

The universal formula is: $y = kf\big(a(x+b)\big) + c$

We want to try make the expression $\frac{x^2}{2} - 1$ in terms of $f(x)$ which is $x^2 + 4$. Therefore:

$y = \frac{x^2}{2} - 1$ can be written as $\frac{1}{2}(x^2 + 4) - 3$

This therefore tells us that a vertical dilation of $\frac{1}{2}$ and a shift downwards 3 units is needed.

This answer is consistent with example 1 part a).

<u>**Combinations of Transformations Exercises**</u>

1. Let $y = x^2 - 3x$.

a) Sketch the graph of this function showing intercepts and the vertex.

b) The parabola is then shifted up 2 units. Sketch the new graph

c) The previous graph is then vertically dilated by a factor of -2. Sketch the new graph

2. Write down the new equation for each function after the given transformations have been applied and draw a graph of it.

a) $y = x^2 - 1$: right 3 units, then dilate by factor $\frac{1}{2}$ horizontally

b) $y = 2^x$: down 2 units, then reflect in the $y - axis$

c) $y = \frac{1}{x}$: right 3 units then dilate by factor $\frac{1}{2}$ vertically

d) $x^2 + y^2 = 4$: up 3 units then dilate by factor $\frac{1}{2}$ vertically

e) $y = \sqrt{x}$: up 2 units then dilate by factor -1 horizontally

3. Determine the equation of the curve after the given transformations have been applied in the order stated:

a) $y = x^2 - 2x$: down 4 units, dilate horizontally by factor 2, left 1 unit

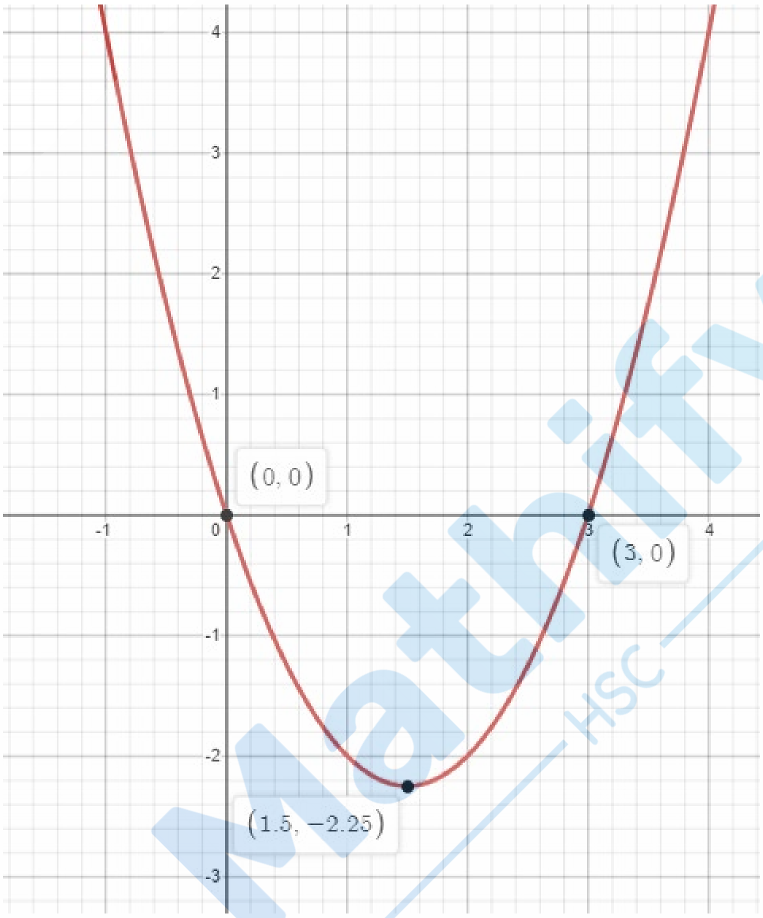
b) $y = 2^x + 1$: down 1 unit, right 1 unit, dilate vertically by -2

c) $y = \frac{1}{x}$: right 2 units, dilate vertically by 2, up 3 units

Combinations of Transformations Exercise Answers

1. $y = x^2 - 3x$

a) Sketching the graph:

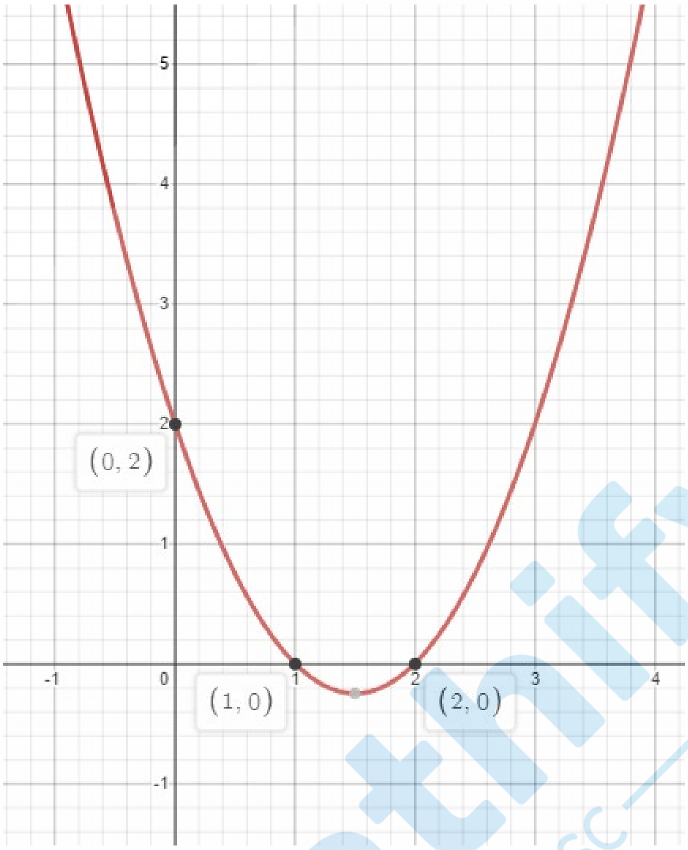


b) Shifting up 2 units means that y changes to $y - 2$

$$y - 2 = x^2 - 3x$$

$$\therefore y = x^2 - 3x + 2$$
$$= (x - 2)(x - 1)$$

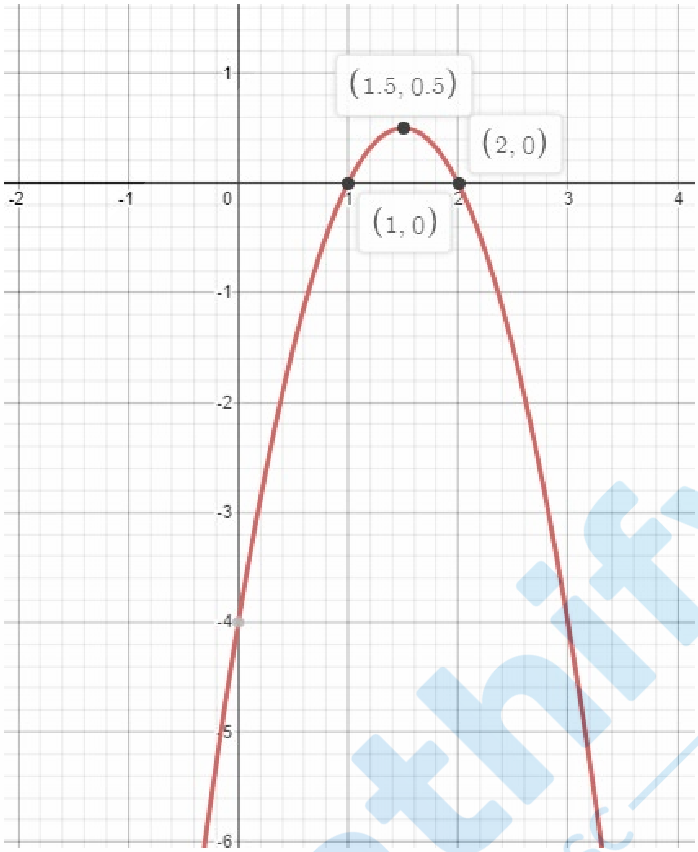
Sketching this graph:



c) Vertical dilation means that y is changed into $\frac{y}{-2}$

$$\therefore -\frac{y}{2} = x^2 - 3x + 2$$
$$y = -2(x^2 - 3x + 2)$$

Sketching:



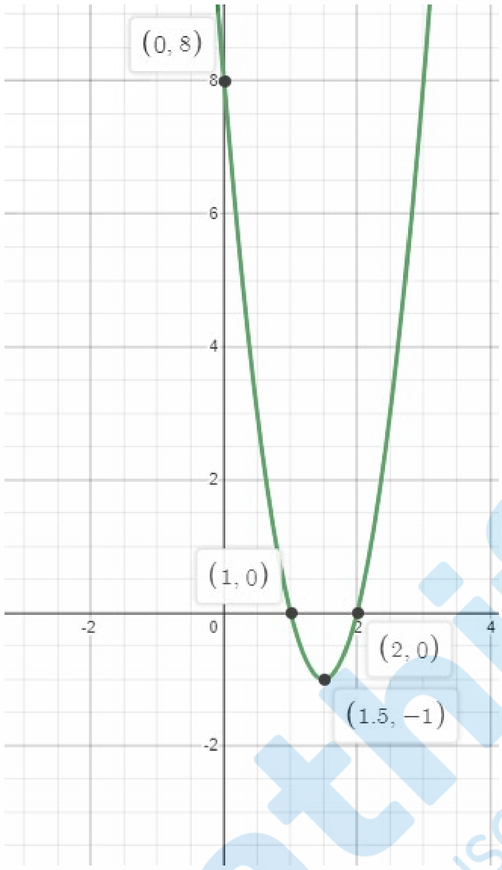
2.

a) Right 3 units means that $x \to x - 3$

$$y = (x-3)^2 - 1$$

Dilate horizontally by $\frac{1}{2}$ means that $x \to \frac{x}{\frac{1}{2}} = 2x$

$$\therefore y = (2x-3)^2 - 1$$

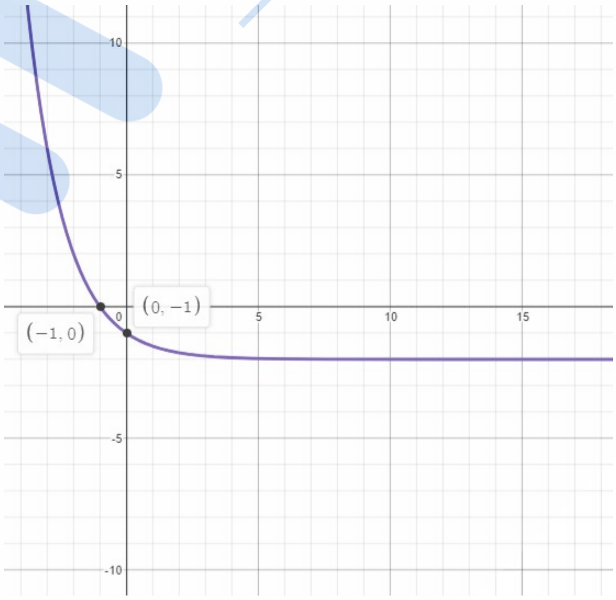


b) Down 2 units means that $y \to y + 2$

$$y = 2^x - 2$$

Reflect in the $y -$ axis means that $x \to -x$

$$\therefore y = 2^{-x} - 2$$

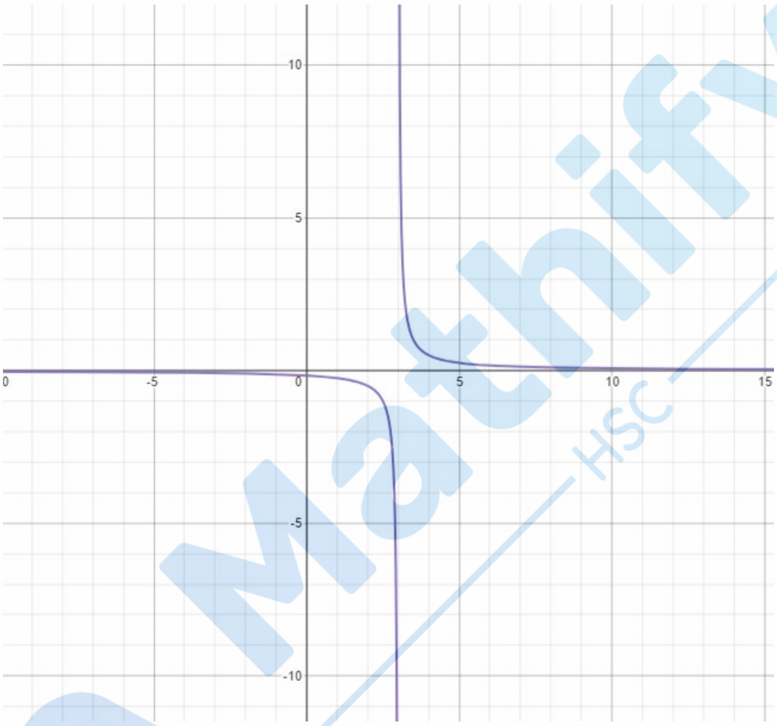


c) Right 3 units means that $x \to x - 3$

$$y = \frac{1}{x-3}$$

Dilate by factor $\frac{1}{2}$ vertically means that $y \to \frac{y}{\frac{1}{2}} = 2y$

$$2y = \frac{1}{x-3}$$
$$\therefore y = \frac{1}{2(x-3)}$$

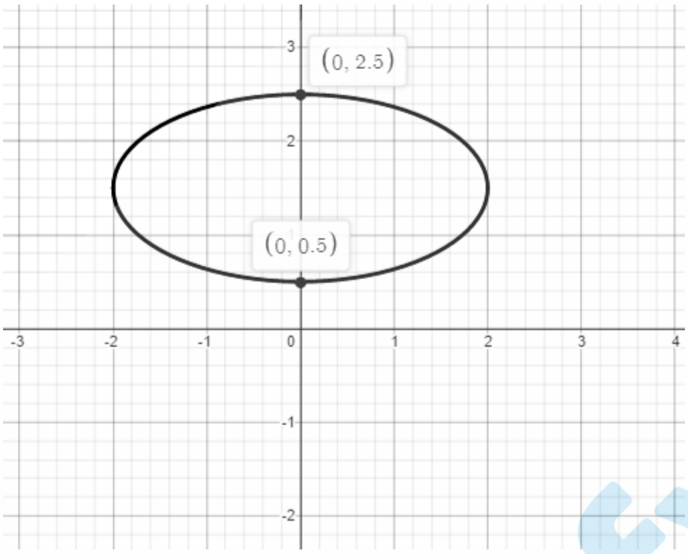


d) Up 3 units means that $y \to y - 3$

$$x^2 + (y-3)^2 = 4$$

Dilate by factor $\frac{1}{2}$ vertically means that $y \to \frac{y}{\frac{1}{2}} = 2y$

$$x^2 + (2y-3)^2 = 4$$

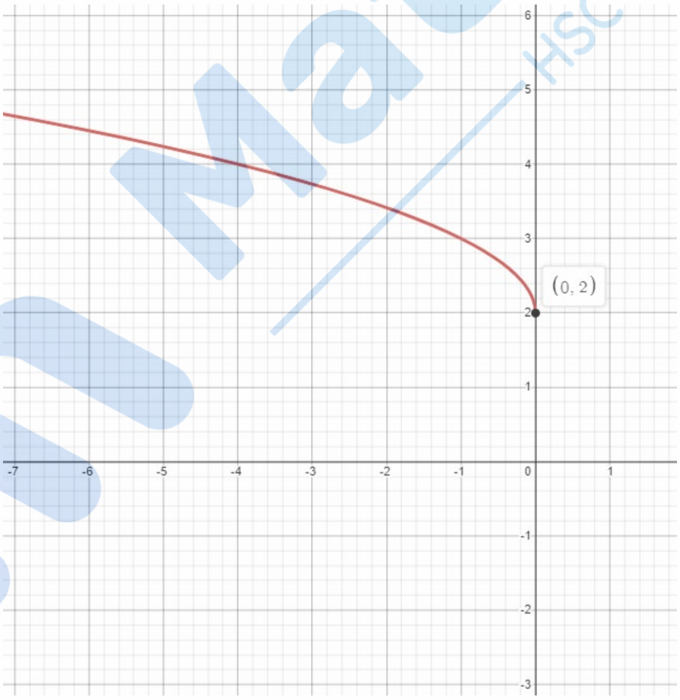


e) Up 2 units means that $y \to y - 2$

$$y = \sqrt{x} + 2$$

Dilate by factor -1 horizontally means that $x \to -x$

$$y = \sqrt{-x} + 2$$



3.

a) Down 4 units means that $y \to y + 4$

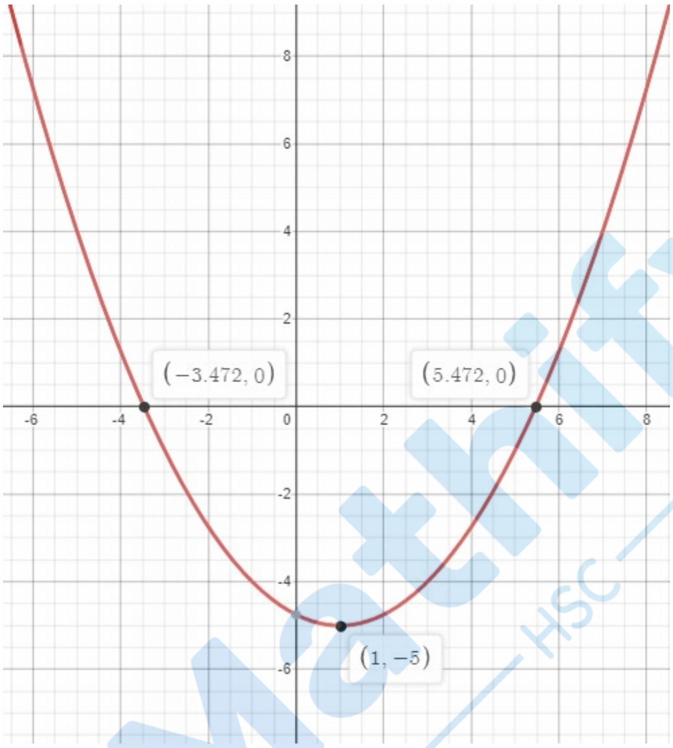
$$y = x^2 - 2x - 4$$

Dilate horizontally by factor 2 means that $x \to \frac{x}{2}$

$$y = \left(\frac{x}{2}\right)^2 - x - 4$$

Left 1 unit means that $x \rightarrow x+1$

$$\therefore y = \left(\frac{x+1}{2}\right)^2 - (x+1) - 4$$
$$= \left(\frac{x+1}{2}\right)^2 - x - 5$$



b) Down 1 unit means that $y \rightarrow y+1$

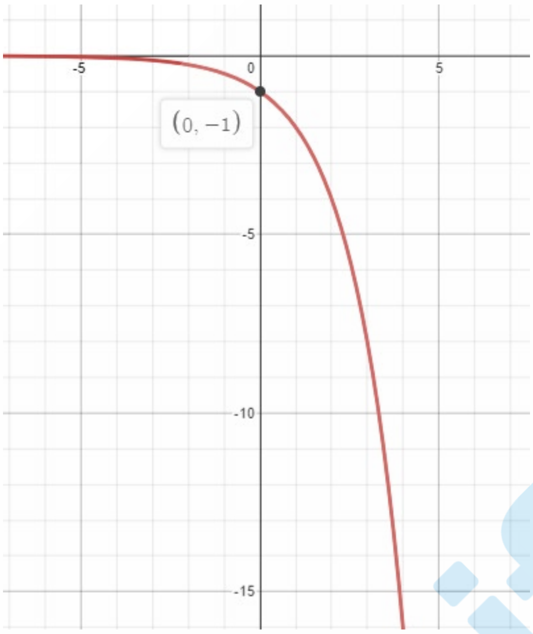
$$y = 2^x$$

Right 1 unit means that $x \rightarrow x-1$

$$y = 2^{x-1}$$

Dilate vertically by -2 means that $y \rightarrow \frac{y}{-2}$

$$\therefore \frac{y}{-2} = 2^{x-1}$$
$$y = -2 \times 2^{x-1}$$
$$= -2^x$$



c) Right 2 units means that $x \to x - 2$

$$y = \frac{1}{x-2}$$

Dilate vertically by 2 means that $y \to \frac{y}{2}$

$$y = \frac{2}{x-2}$$

Up 3 units means that $y \to y - 3$

$$\therefore y = \frac{2}{x-2} + 3$$