FUNCTIONS

POLYNOMIALS AND SKETCHING (X)

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- Polynomials Introduction

Polynomials are essentially expressions or functions that we have already unknowingly encountered, such as quadratics like $x^2 - 2x + 1$. More specifically:

A polynomial is any expression of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Where the coefficients $a_n, a_{n-1}, \dots, a_0$ are constants

The most important (and commonly forgotten) condition for an expression to be a polynomial is that **all the indices must be whole numbers!** They **cannot** be a fraction.

For example:

a) The function $y = x^4 + 4x^3 - 9x^2 - 2x + 5$ is a polynomial function!

As can be seen, this is because the function may be expressed in the form:

$$P(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

b) The function $y = x^3 + x^2 - x + \sqrt{x} - 4$ is **not** a polynomial function!

This is because of the "$\sqrt{x}$", where since $\sqrt{x} = x^{\frac{1}{2}}$, the index is not a whole number and thus the entire function does not classify as a polynomial function!

- Polynomials Terminology

There are a couple of definitions for polynomials that we have to remember:

- **Degree of a polynomial:** The degree of a polynomial is the value of the highest index
- **Leading term:** This is the term with the highest index
- **Leading coefficient:** This is the coefficient of the leading term

For example, looking at the following polynomial function:

Leading term $= -2x^5$

Degree $= 5$

$$y = -2x^5 + x^4 + 5x^3 - x^2 + 7x + 3$$

Leading coefficient $= -2$

- Monic Polynomials

Monic polynomials are defined as polynomials whose **leading coefficient** is 1

For example: $P(x) = x^3 - 4x^2 + 2x - 3$ would be a **monic** polynomial since the leading coefficient is 1

However, the expression $P(x) = -x^4 + x^2 - 6x + 9$ would **not be a monic polynomial!** This is because the leading coefficient is -1, not 1, so be careful! (Many often get this confused)

- Sketching Polynomials in Factored Form

Polynomials in factored form resemble:

$$P(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$$

In these situations, notice that the $x -$ intercepts of the polynomial would be $\alpha_1, \alpha_2, \dots, \alpha_n$.

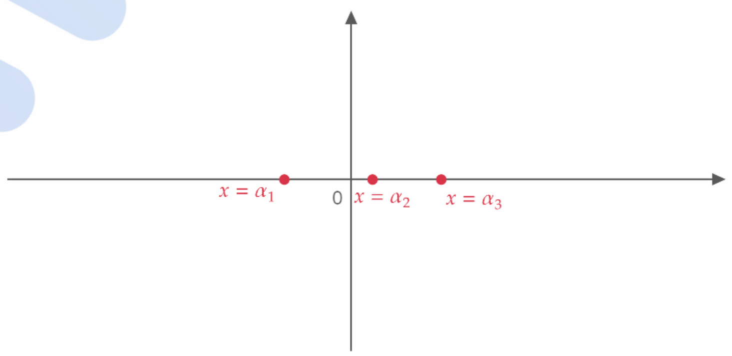
Hence, if we want to sketch a given polynomial that is in its factored form:

$$\boldsymbol{Step\ 1: Plot\ the\ x - intercepts}$$

As mentioned before, the $x -$ intercepts would occur at:

$$x = \alpha_1, x = \alpha_2, x = \alpha_3, etc.$$

This would look something like:

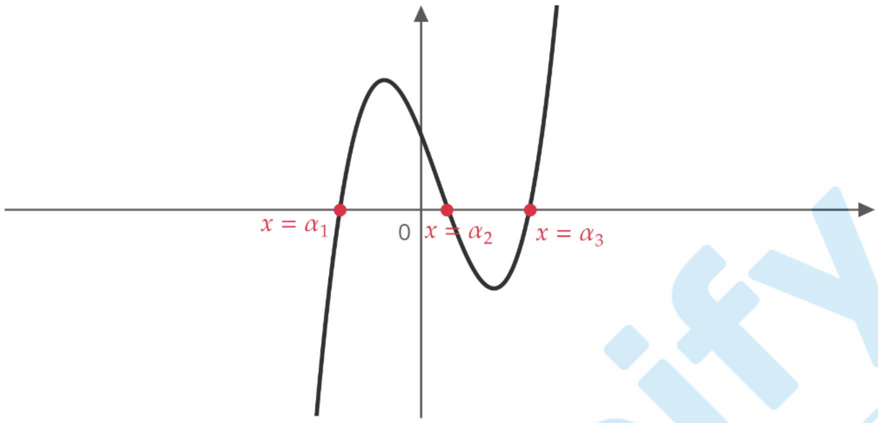


Since these are $x -$ intercepts, the graph will always intersect these points!

$$\boldsymbol{Step\ 2: Determine\ which\ side\ of\ the\ polynomial\ is\ positive/negative}$$

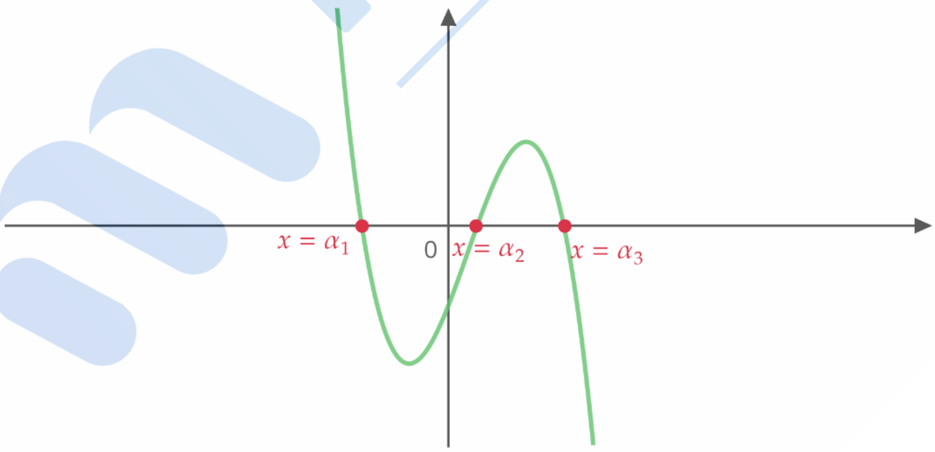
This is the harder step. Essentially, right now if we were to try sketch our polynomial or curve, there are two possibilities:

Possibility 1:



In this possibility, we see that the "left" side is negative (below the $x -$ axis) and the "right" side is positive (above the $x -$ axis).

Possibility 2:



In this possibility on the other hand, we see that the "left" side is now positive (above the $x -$ axis) and the "right" side is now negative (below the $x -$ axis)

- **How do we determine which possibility to use?**

$$\boldsymbol{Method\ 1: Substitute\ a\ Test\ Value}$$

We can essentially substitute in a "test value", any $x -$ value which lies to the right of the largest $x -$ intercept, i.e. $x > \alpha_3$, and see if its $y - value$ is negative or positive. This will determine which possibility the graph will look like, and thus you can finally draw the answer afterwards!

<u>Note:</u> The same test can be done with a $x - value$ which lies to the left of the smallest $x -$ intercept, i.e. $x < \alpha_1$. The process is the same!

$$\boldsymbol{Method\ 2: Use\ your\ y - intercept}!$$

Another popular method is finding the $y - intercept$, and from that value, determine which of the two possibilities will be your final graph!

Example 1: Sketch the polynomial $P(x) = (x + 2)(x - 1)(x - 4)$, making sure to plot all necessary intercepts

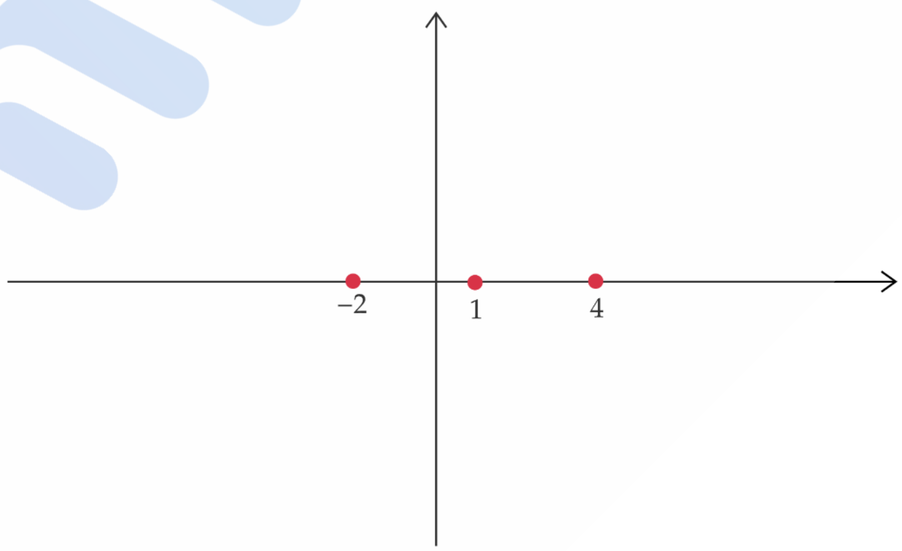
Solution:

$$Step\ 1: Plot\ the\ x - intercepts$$

From the equation, it can be seen that the $x - intercepts$ are:

$$x = 2, x = 1\ and\ x = 4$$

Hence, plotting this on our graph:

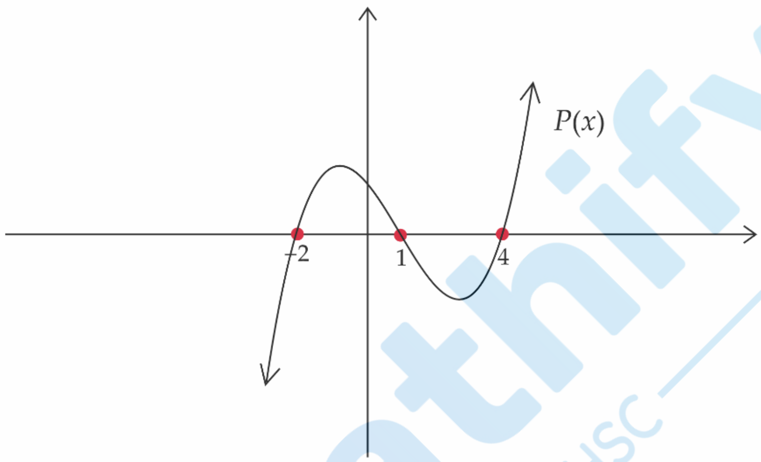


$$Step\ 2: Substitute\ a\ test\ value$$

Frequently, it is recommended that we test a x value after the right – most $x -$ intercept. In the scenario, let's test $x = 5$ and substitute it into our polynomial:

$$\begin{aligned} P(5) &= (5+2)(5-1)(5-4) \\ &= 7 \times 4 \times 1 \\ &= 28 \end{aligned}$$

Hence, we now know that the right side of my polynomial will be positive, so sketching $P(x)$ now:



Polynomials and Sketching Exercises

1. Sketch each of the following polynomials, labelling any $x -$ intercepts which may occur:

a) $P(x) = (x-1)(x+3)$
b) $P(x) = (x+2)(x-2)(x-4)$
c) $P(x) = -(x-3)(x-4)(x+1)$
d) $P(x) = (1-x)(x+1)(x-5)$
e) $P(x) = (2-x)(x-3)(x-5)$
f) $P(x) = (x-1)(x+1)(x-2)(x+2)$
g) $P(x) = -(x-3)(x+1)(x-2)(x+2)$
h) $P(x) = x(1-x)(x+2)(x+1)$

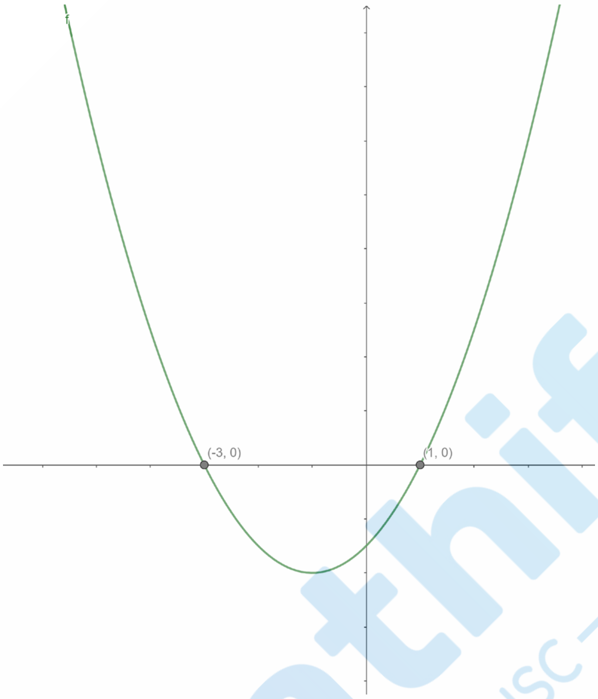
2. By first sketching a graph of the polynomial, solve each of the following inequalities:

a) $(x+1)(x-2)(x-5) \geq 0$
b) $(x+4)(x-2)(x-3) < 0$
c) $(x-1)(x+2)(3+x)(2-x) \leq 0$
d) $(x+2)\left(x-\frac{1}{2}\right)(x+5)(x-2) > 0$

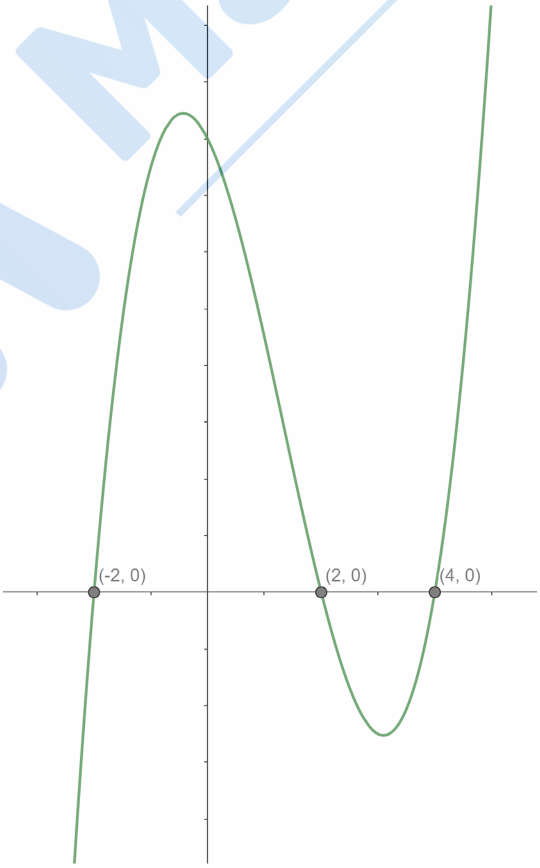
Polynomials and Sketching Exercise Answers

1.

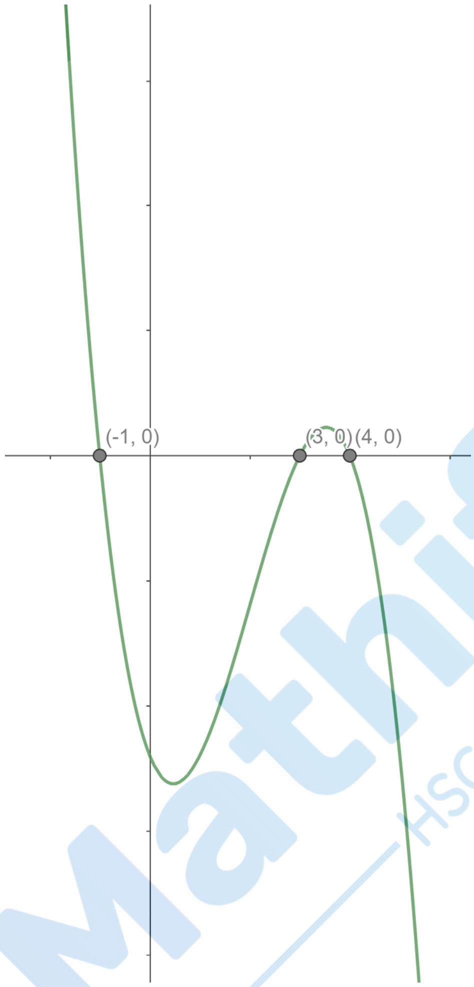
a)



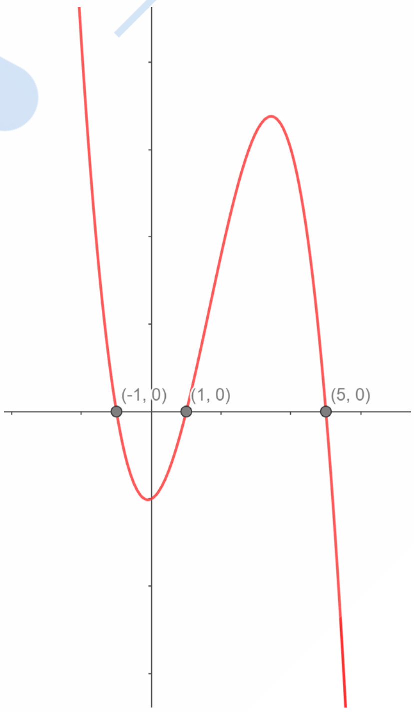
b)



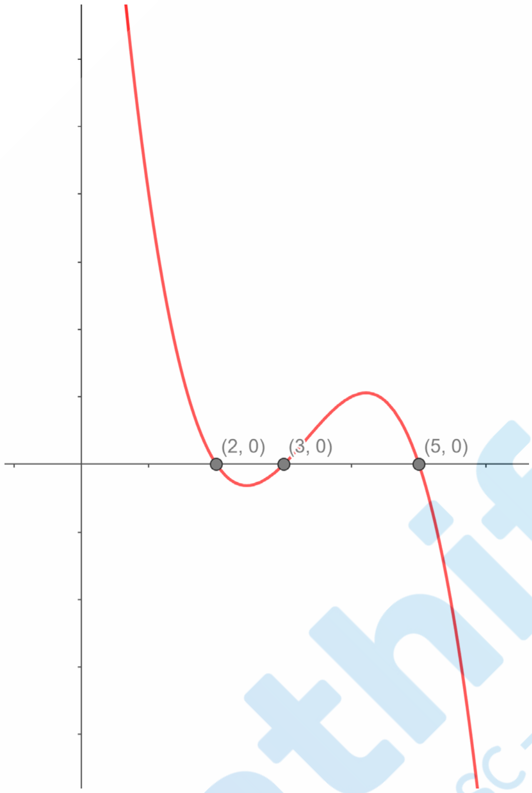
c)



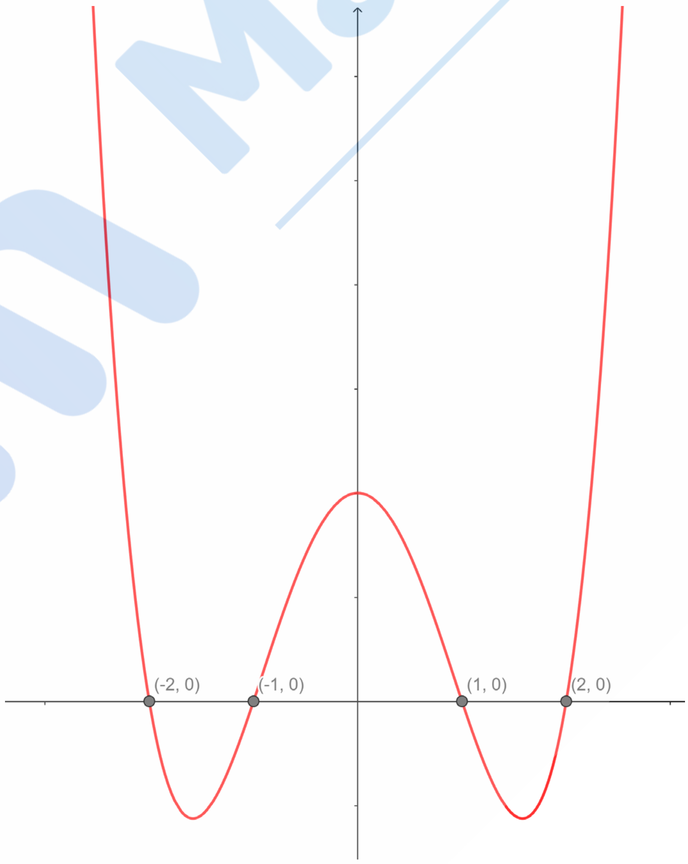
d)



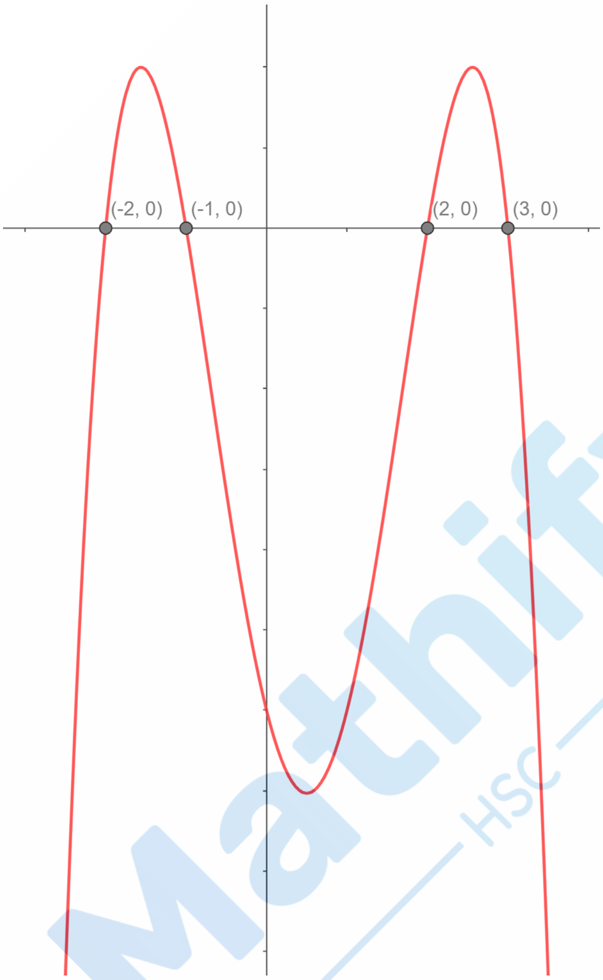
e)



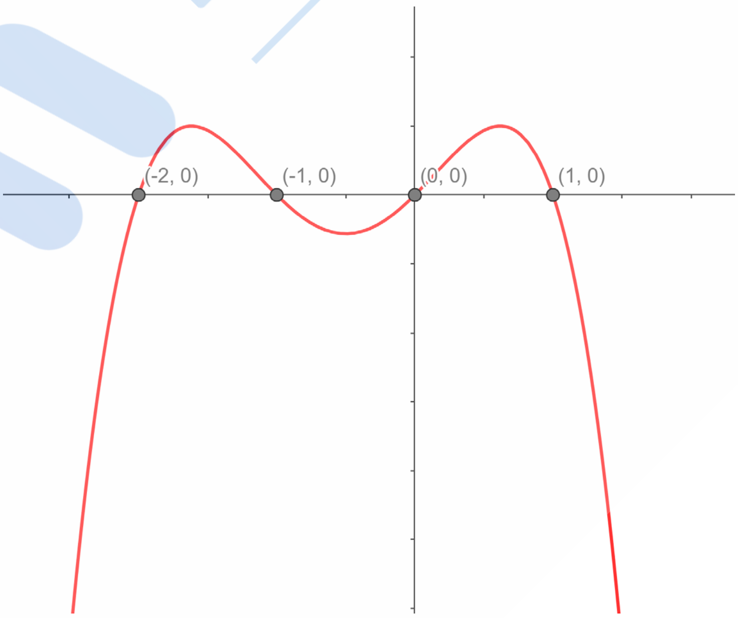
f)



g)

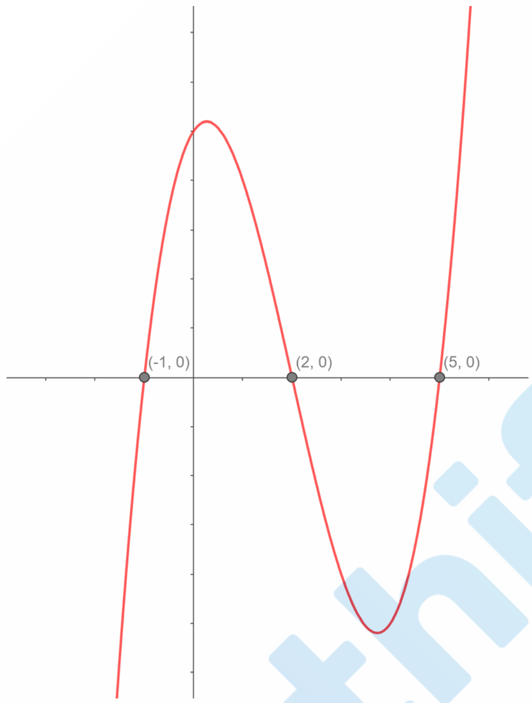


h)



2.

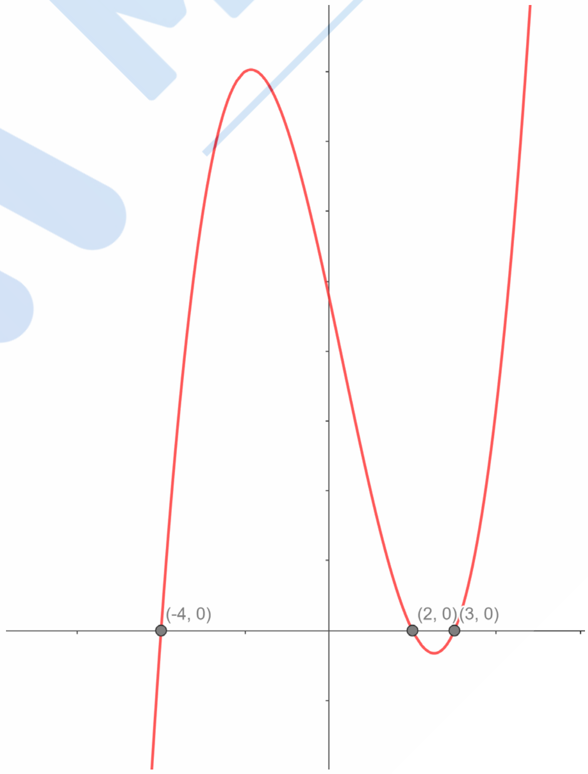
a) Sketching the polynomial $y = (x+1)(x-2)(x-5)$



Since we are looking for the interval where the polynomial is above or equal to the $x-$ axis:

$$\therefore -1 \leq x \leq 2 \; OR \; x \geq 5$$

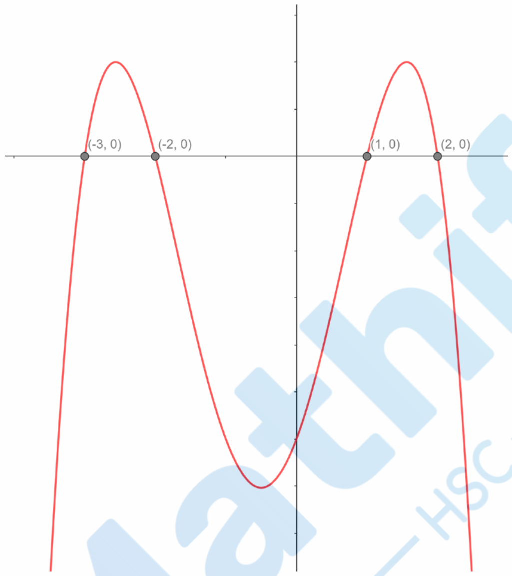
b) Sketching the polynomial $y = (x+4)(x-2)(x-3)$



Since we are looking for the interval where the polynomial is below the $x-$ axis:

$$x < -4, 2 < x < 3$$

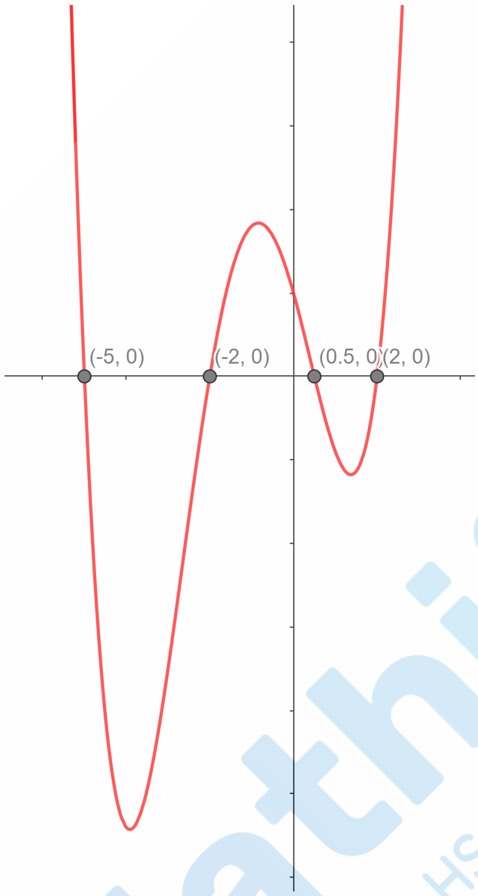
c) Sketching the polynomial $y = (x-1)(x+2)(3+x)(2-x)$



Since we are looking for the interval where the polynomial is above or equal to the $x-$ axis:

$$x \leq -3, -2 \leq x \leq 1, x \geq 2$$

d) Sketching the polynomial $y = (x+2)\left(x - \frac{1}{2}\right)(x+5)(x-2)$



Since we are looking for the interval where the polynomial is above the $x-$axis:

$$\therefore x < -5, \qquad -2 < x < 0.5, \qquad x > 2$$