DIFFERENTIATIAL CALCULUS

DERIVATIVE OF EXPONENTIALS (I)

Contents include:

- Differentiating Natural Exponentials
- Differentiating General Exponentials

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- Differentiating e^x

The natural exponential function is its own derivative, meaning that:

$$\frac{d}{dx}e^x = e^x$$

To find the standard form, we can apply our chain rule and substitution method:

Let $y = e^{f(x)}$

$Let\ u = f(x),$

$$so\ \frac{du}{dx} = f'(x)\ and\ \frac{dy}{du} = e^u$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du} \\ &= f'(x) \cdot e^u \\ &= f'(x)e^{f(x)}\end{aligned}$$

Hence, the standard form of differentiation for $y = e^{f(x)}$ is:

$$y' = f'(x)e^{f(x)}$$

Moreover, this means that if $y = e^{ax+b}$:

$$y' = ae^{ax+b}$$

Example 1: Find the derivative of $y = e^{4x-7}$

By the standard form $y' = ae^{ax+b}$:

$$\frac{d}{dx}(e^{4x-7}) = 4e^{4x-7}$$

Example 2: Find the derivative of $y = e^{\frac{1}{2}(9-x)}$

$$\begin{aligned} y &= e^{\frac{9}{2}-\frac{x}{2}} \\ &= e^{-\frac{x}{2}+\frac{9}{2}}\end{aligned}$$

By the standard form $y' = ae^{ax+b}$:

$$\therefore \frac{dy}{dx} = -\frac{1}{2}e^{-\frac{x}{2}+\frac{9}{2}}$$

Example 3: Find the derivative of $x^3e^{x^2}$

Applying our product rule $(uv)' = u'v + v'u$:

$$\frac{d}{dx}(x^3) = 3x^2$$
$$\frac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}\ using\ our\ standard\ form$$

$$\therefore \frac{d}{dx}\left(x^3e^{x^2}\right) = 3x^2e^{x^2} + x^3 \cdot 2xe^{x^2}$$
$$= 3x^2e^{x^2} + 2x^4e^{x^2}$$

- Differentiating a^x

We already know how to differentiate e^x:

$$(e^x)' = e^x$$

However, if the base is no longer e, and instead another real number, a, the derivative is:

$$(a^x)' = \ln a \times a^x$$

The proof for this requires our logarithmic laws, and is as follows:

$$a^x = e^{\log_e a^x}\ [logarithm\ law]$$
$$= e^{x \log_e a}\ [logarithm\ law]$$
$$= e^{x \ln a}$$

$$\therefore (a^x)' = \left(e^{x \ln a}\right)'$$
$$= (x \ln a)' \times e^{x \ln a}$$
$$= \ln a \times e^{x \ln a}$$
$$= \ln a \times a^x$$

The standard forms to remember when considering chain rule are:

$$\left(a^{f(x)}\right)' = f'(x) \times \ln a \times a^{f(x)}$$
$$(a^{px+q})' = p \times \ln a \times a^{px+q}$$

Example 5: Differentiate $y = 2^x$

$$y' = \ln 2 \times 2^x$$

Example 6: Differentiate $y = 5^{x^2-2x}$

Recalling the chain rule:

$$\left(a^{f(x)}\right)' = f'(x) \times \ln a \times a^{f(x)}$$

Hence, for this question:

$$\begin{aligned} \therefore y' &= (x^2 - 2x)' \times \ln 5 \times 5^{x^2-2x} \\ &= \ln 5\,(2x-2)5^{x^2-2x} \end{aligned}$$

Exponential Differentiation Exercises

1. Use the standard form $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$ to differentiate:

a) $y = 8e^{\frac{1}{4}x}$
b) $y = e^{-3x-6}$
c) $f(x) = 2e^{\frac{1}{3}x-3}$
d) $f(x) = 9e^{-3x-6}$
e) $y = \frac{e^{2x}}{2} - 2e^{8x}$
f) $f(x) = 3e^{5x-3} + \frac{1}{4}e^{(3x^2+3x)}$

2. Differentiate the following:

a) $f(x) = 8e^{-4x-3} - 7e^{5-6x}$
b) $f(x) = \frac{2}{3}e^{\frac{3}{2}x-4} - 5x + \frac{9}{16}e^{9x}$
c) $f(x) = 3x^3 + \frac{8}{x}e^{x^2} - 9e^{x^3-x^2+x-4}$
d) $y = 2\sqrt{e^x}$
e) $f(x) = \frac{4}{\sqrt[3]{e^x}}$

3. Differentiate the following expressions:

a) $y = x^2e^x$
b) $h = 3t^2e^{3t-4} + t$
c) $f(x) = \frac{x^3}{e^{2x-1}+3x}$
d) $f(x) = (x^2+3)^6e^{x^2+x}$
e) $y = \frac{(3x^2+2x-3)^4}{e^{3x}+x^2}$

f) $f(x) = e^{e^x}$

Exponential Differentiation Exercise Answers

1.

a) $y = 8e^{\frac{1}{4}x}$

$$\begin{aligned} y' &= 8 \times \frac{1}{4} e^{\frac{1}{4}x} \\ &= 2e^{\frac{1}{4}x} \end{aligned}$$

b) $y = e^{-3x-6}$

$$\begin{aligned} y' &= -3 \times e^{-3x-6} \\ &= -3e^{-3x-6} \end{aligned}$$

c) $f(x) = 2e^{\frac{1}{3}x-3}$

$$\begin{aligned} f'(x) &= 2 \times \frac{1}{3} e^{\frac{1}{3}x-3} \\ &= \frac{2}{3} e^{\frac{1}{3}x-3} \end{aligned}$$

d) $f(x) = 9e^{-3x-6}$

$$\begin{aligned} f'(x) &= 9 \times -3e^{-3x-6} \\ &= -27e^{-3x-6} \end{aligned}$$

e) $y = \frac{e^{2x}}{2} - 2e^{8x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} \times 2e^{2x} - 2 \times 8e^{8x} \\ &= e^{2x} - 16e^{8x} \end{aligned}$$

f) $f(x) = 3e^{5x-3} + \frac{1}{4}e^{3x^2+3x}$

For $\frac{1}{4}e^{3x^2+3x}$, let $u = 3x^2 + 3x$

$$\therefore \frac{du}{dx} = 6x + 3$$

$$\frac{d}{dx}\left(\frac{1}{4}e^{3x^2+3x}\right) = \frac{1}{4}(6x+3)e^{3x^2+3x}$$

$$\begin{aligned} \therefore f'(x) &= 3 \times 5e^{5x-3} + \frac{1}{4}(6x+3)e^{3x^2+3x} \\ &= 15e^{5x-3} + \frac{1}{4}(6x+3)e^{3x^2+3x} \end{aligned}$$

2.

a) $f(x) = 8e^{-4x-3} - 7e^{5-6x}$

$$\begin{aligned} f'(x) &= 8 \times -4e^{-4x-3} - 7 \times -6e^{5-6x} \\ &= -32e^{-4x-3} + 42e^{5-6x} \end{aligned}$$

b) $f(x) = \frac{2}{3}e^{\frac{3}{2}x-4} - 5x + \frac{9}{16}e^{9x}$

$$\begin{aligned} f'(x) &= \frac{2}{3} \times \frac{3}{2}e^{\frac{3}{2}x-4} - 5 + \frac{9}{16} \times 9e^{9x} \\ &= e^{\frac{3}{2}x-4} - 5 + \frac{81}{16}e^{9x} \end{aligned}$$

c) $f(x) = 3x^3 + \frac{8}{x}e^{x^2} - 9e^{x^3-x^2+x-4}$

$$\frac{d}{dx}(x^2) = 2x$$
$$\frac{d}{dx}(x^3 - x^2 + x - 4) = 3x^2 - 2x + 1$$

$$\begin{aligned} \therefore f'(x) &= 3 \times 3x^2 + \frac{8}{x} \times 2xe^{x^2} - 9 \times (3x^2 - 2x + 1)e^{x^3-x^2+x-4} \\ &= 9x^2 + 16e^{x^2} - 9(3x^2 - 2x + 1)e^{x^3-x^2+x-4} \end{aligned}$$

d) $y = 2\sqrt{e^x}$

$$y = 2e^{\frac{1}{2}x}$$
$$\begin{aligned} \therefore \frac{dy}{dx} &= 2 \times \frac{1}{2}e^{\frac{1}{2}x} \\ &= e^{\frac{1}{2}x} \end{aligned}$$

e) $f(x) = \frac{4}{\sqrt[3]{e^x}}$

$$f(x) = 4e^{-\frac{1}{3}x}$$
$$\begin{aligned} \therefore f'(x) &= 4 \times -\frac{1}{3}e^{-\frac{1}{3}x} \\ &= -\frac{4}{3}e^{-\frac{1}{3}x} \end{aligned}$$

3.

a) $y = x^2e^x$

Using the product rule:

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\therefore y' = 2xe^x + x^2e^x$$

b) $h = 3t^2e^{3t-4} + t$

Using the product rule:

$$\frac{d}{dx}(3t^2) = 6t$$
$$\frac{d}{dx}(e^{3t-4}) = 3e^{3t-4}$$

$$\begin{aligned}\therefore \frac{dh}{dt} &= 6t \times e^{3t-4} + 3t^2 \times 3e^{3t-4} + 1 \\ &= 6te^{3t-4} + 9t^2e^{3t-4} + 1\end{aligned}$$

c) $f(x) = \frac{x^3}{e^{2x-1}+3x}$

Using the quotient rule:

$$\frac{d}{dx}(x^3) = 3x^2$$
$$\frac{d}{dx}(e^{2x-1} + 3x) = 2e^{2x-1} + 3$$

$$\begin{aligned}\therefore f'(x) &= \frac{3x^2 \times (e^{2x-1} + 3x) - x^3 \times (2e^{2x-1} + 3)}{(e^{2x-1} + 3x)^2} \\ &= \frac{3x^2e^{2x-1} + 9x^3 - 2x^3e^{2x-1} + 3x^3}{(e^{2x-1} + 3x)^2} \\ &= \frac{3x^2e^{2x-1} - 2x^3e^{2x-1} + 12x^3}{(e^{2x-1} + 3x)^2}\end{aligned}$$

d) $f(x) = (x^2 + 3)^6e^{x^2+x}$

Using the product rule:

$$\begin{aligned}\frac{d}{dx}[(x^2 + 3)^6] &= 6(x^2 + 3)^5 \times 2x\ (chain\ rule) \\ &= 12x(x^2 + 3)^5\end{aligned}$$

$$\frac{d}{dx}\left(e^{x^2+x}\right) = (2x + 1)e^{x^2+x}$$

$$\begin{aligned}\therefore f'(x) &= 12x(x^2 + 3)^5 \times e^{x^2+x} + (x^2 + 3)^6 \times (2x + 1)e^{x^2+x} \\ &= 12x(x^2 + 3)^5e^{x^2+x} + (x^2 + 3)^6(2x + 1)e^{x^2+x}\end{aligned}$$

e) $y = \frac{(3x^2+2x-3)^4}{e^{3x}+x^2}$

Using the quotient rule:

$$\frac{d}{dx}[(3x^2+2x-3)^4] = 4(3x^2+2x-3)^3(6x+2)$$

$$\frac{d}{dx}(e^{3x}+x^2) = 3e^{3x}+2x$$

$$\therefore y' = \frac{4(3x^2+2x-3)^3(6x+2)(e^{3x}+x^2)-(3e^{3x}+2x)(3x^2+2x-3)^4}{(e^{3x}+x^2)^2}$$

f) $f(x) = e^{e^x}$

Let $u = e^x$

$$\therefore y = e^u$$

$$\frac{dy}{du} = e^u$$

$$\frac{du}{dx} = e^x$$

$$\begin{aligned}\therefore f'(x) = \frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= e^u \times e^x\\ &= e^{e^x}\times e^x\\ &= e^{e^x+x}\end{aligned}$$