DIFFERENTIATIAL CALCULUS

DERIVATIVE OF LOGARITHMS (II)

Contents include:

- Differentiating Natural Logarithms
- Differentiating General Logarithms

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- Differentiating Logarithms

The natural logarithm may be differentiated:

$$(\ln x)' = \frac{1}{x}$$

To find the standard form, we can apply our chain rule and substitution method:

Let $y = \ln\big(f(x)\big)$

$Let\ u = f(x),$

$so\,\frac{du}{dx} = f'(x)\ and\ y = \ln u$

$\frac{dy}{du} = \frac{1}{u}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du} \\ &= f'(x) \cdot \frac{1}{u} \\ &= \frac{f'(x)}{f(x)} \end{aligned}$$

Hence, the standard form of differentiation for $y = \ln\big(f(x)\big)$ is:

$$y' = \frac{f'(x)}{f(x)}$$

Moreover, this means that if $y = e^{ax+b}$:

$$y' = \frac{a}{ax+b}$$

Example 1: Differentiate $y = \ln(3x+2)$

Using the standard form of differentiation $y' = \frac{a}{ax+b}$:

$$y' = \frac{3}{3x+2}$$

Example 2: Differentiate $y = \ln(x^2 + x - 3)$

Using the standard form of differentiation $y' = \frac{f'(x)}{f(x)}$:

$$y' = \frac{(x^2 + x - 3)'}{x^2 + x - 3}$$
$$= \frac{2x + 1}{x^2 + x - 3}$$

- Differentiating Logarithms Using Log Laws

Recall the following logarithm laws:

$$\ln a + \ln b = \ln(ab)$$
$$\ln a - \ln b = \ln \frac{a}{b}$$
$$\ln a^n = n \ln a$$
$$a^x = b^{\log_b a^x}$$

For harder logarithm differentiation questions, we may be required to utilise these, such as in the following examples:

Example 3: Differentiate $y = \ln\left(\frac{3x-5}{x^3+2}\right)$

If we were to use the standard form, it would look something like:

$$y' = \left(\frac{3x - 5}{x^3 + 2}\right)' \times \frac{1}{\left(\frac{3x - 5}{x^3 + 2}\right)}$$

Which would be annoying because we would require using the quotient rule and so on…

Instead, we can split up the logarithm using our logarithm laws:

$$y = \ln\left(\frac{3x - 5}{x^3 + 2}\right) = \ln(3x - 5) - \ln(x^3 + 2)$$

Then, we can differentiate using our previous standard forms:

$$\therefore y' = \frac{3}{3x - 5} - \frac{3x^2}{x^3 + 2}$$

Example 4: Differentiate $y = \ln(x^3 + 4x^2 - 5)^{10}$

We can apply our logarithm law here so that:

$$y = \ln(x^3 + 4x^2 - 5)^{10} = 10 \ln(x^3 + 4x^2 - 5)$$

And then differentiate now:

$$\therefore y' = 10 \times \frac{(x^3 + 4x^2 - 5)'}{x^3 + 4x^2 - 5}$$
$$= 10 \times \frac{3x^2 + 8x}{x^3 + 4x^2 - 5}$$

$$= \frac{30x^2 + 80x}{x^3 + 4x^2 - 5}$$

- Differentiating $\log_a x$

We already know how to differentiate $\log_e x = \ln x$:

$$(\ln x)' = \frac{1}{x}$$

However, if the base of the logarithm is no longer e, and instead another real number, a, the derivative is then:

$$(\log_a x)' = \frac{1}{x \ln a}$$

The proof for this requires logarithmic laws once again, and is as follows:

$$\log_a x = \frac{\log_e x}{\log_e a} \; [\textit{change of base logarithm law}]$$
$$= \frac{\ln x}{\ln a}$$
$$\therefore (\log_a x)' = \left(\frac{\ln x}{\ln a}\right)'$$
$$= \frac{1}{\ln a}(\ln x)' \; [\textit{since a is a constant}]$$
$$= \frac{1}{\ln a} \times \frac{1}{x}$$
$$= \frac{1}{x \ln a}$$

The standard forms to remember when considering chain rule are:

$$(\log_a f(x))' = \frac{f'(x)}{(\ln a) f(x)}$$

$$(\log_a (px + q))' = \frac{p}{(\ln a)(px + q)}$$

Example 7: Differentiate $\log_{10} x$

$$(\log_{10} x)' = \frac{1}{x \ln 10}$$

Example 8: Differentiate $\log_5(x^3 - 2x)$

Recalling the standard form:

$$(\log_a f(x))' = \frac{f'(x)}{(\ln a)f(x)}$$

Hence, for this question:

$$\begin{aligned}(\log_5(x^3 - 2x))' &= \frac{(x^3 - 2x)'}{(\ln 5)(x^3 - 2x)} \\ &= \frac{3x^2 - 2}{(\ln 5)(x^3 - 2x)}\end{aligned}$$

Derivative of Logarithms Exercises

1. Use the standard form $\frac{d}{dx}(\ln(ax + b)) = \frac{a}{ax+b}$ to differentiate:

a) $y = \ln(5x - 3)$
b) $y = \ln(-x + 2)$
c) $y = 4\ln(-3x + 5)$
d) $y = -2\ln\left(\frac{7}{2}x + 4\right)$
e) $y = 3\ln(1 - x) - \ln(2x + 3)$

2. Differentiate the following expressions:

a) $f(x) = x^2 \ln(8x - 5)$
b) $f(x) = \frac{\ln(x^2-3x)}{x}$
c) $f(x) = \ln\left(\frac{5x+1}{8x^2-3}\right)$
d) $f(x) = \ln\left(\frac{(8x^3+3x)^5}{3x-2}\right)$

Derivative of Logarithms and General Exponential Exercise Answers

1.

a) $y = \ln(5x - 3)$

$$\therefore y' = \frac{5}{5x - 3}$$

b) $y = \ln(-x + 2)$

$$\begin{aligned}\therefore y' &= \frac{-1}{-x + 2} \\ &= -\frac{1}{2 - x}\end{aligned}$$

c) $y = 4\ln(-3x + 5)$

$$\therefore y' = 4 \times \frac{-3}{-3x+5}$$
$$= -\frac{12}{5-3x}$$

d) $y = -2\ln\left(\frac{7}{2}x+4\right)$

$$\therefore y' = -2 \times \frac{\frac{7}{2}}{\frac{7}{2}x+4}$$
$$= -\frac{7}{\frac{7}{2}x+4}$$
$$= -\frac{7}{\frac{7x+8}{2}}$$
$$= -\frac{14}{7x+8}$$

e) $y = 3\ln(1-x) - \ln(2x+3)$

$$\therefore y' = 3 \times \frac{-1}{1-x} - \frac{2}{2x+3}$$
$$= -\frac{3}{1-x} - \frac{2}{2x+3}$$

2.

a) Recalling the product rule:

$$(uv)' = u'v + v'u$$

$$\therefore f'(x) = (x^2)'\ln(8x-5) + x^2(\ln(8x-5))'$$
$$= 2x\ln(8x-5) + x^2 \cdot \frac{8}{8x-5}$$
$$= 2x\ln(8x-5) + \frac{8x^2}{8x-5}$$

b) Recalling the quotient rule:

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$\therefore f'(x) = \frac{(\ln(x^2-3x))'x - (x)'\ln(x^2-3x)}{x^2}$$
$$= \frac{\frac{2x-3}{x^2-3x} \cdot x - \ln(x^2-3x)}{x^2}$$
$$= \frac{\frac{2x-3}{x-3} - \ln(x^2-3x)}{x^2}$$

c) First using our logarithm laws:

$$f(x) = \ln\left(\frac{5x+1}{8x^2-3}\right) = \ln(5x+1) - \ln(8x^2-3)$$

Then differentiating using the standard forms:

$$\therefore f'(x) = \frac{5}{5x+1} - \frac{16x}{8x^2-3}$$

d) First using our logarithm laws:

$$f(x) = \ln\left(\frac{(8x^3+3x)^5}{3x-2}\right) = \ln((8x^3+3x)^5) - \ln(3x-2)$$
$$= 5\ln(8x^3+3x) - \ln(3x-2)$$

Then differentiating using the standard forms:

$$\therefore f'(x) = 5 \times \frac{24x^2+3}{8x^3+3x} - \frac{3}{3x-2}$$
$$= \frac{120x^2+15}{8x^3+3x} - \frac{3}{3x-2}$$