FURTHER FUNCTIONS

INEQUALITIES AND SOLVING BY GRAPHICAL METHODS (V)

Contents include:

- Inequalities Revision
- Basic Graphical Representations
- Solving Quadratic Inequalities
- Solving Inequalities Algebraically
- Solving Absolute Values in Inequalities
- Solving by Graphical Methods

www.mathifyhsc.com.au

- Inequalities Revision

Inequalities (sometimes called inequations) essentially describe the relationship between two expressions that are **not** equal to each other. Inequality symbols can include:

- $' < '$ which means less than
- $' \leq '$ which means less than or equal to
- $' > '$ which means more than
- $' \geq '$ which means more than or equal to

Inequality questions will thus ask students to find the interval of x for which a certain inequality (e.g. $f(x) > g(x)$) is true

- Basic Graphical Representations of $y \leq f(x)$ or $y \geq f(x)$

Graphically, inequalities are represented by shaded regions. If we want to sketch an inequality that's in the form:

$$y \leq f(x)\ OR\ y \geq f(x)$$

Where $f(x)$ is a function

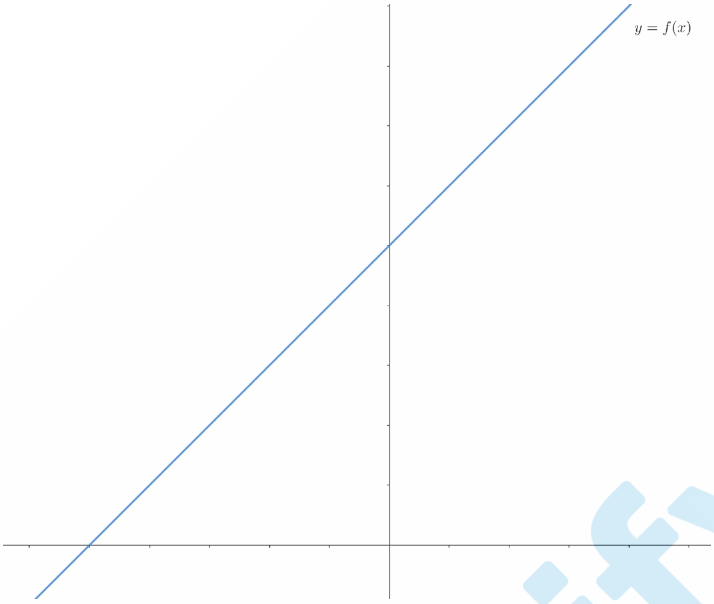
The following steps should be followed:

$$\boldsymbol{Step\ 1: Draw\ the\ curve\ representing\ f(x)}$$

If the inequality you are sketching is either $' < '$ or $' > '$, then the line for $f(x)$ should be **dotted!** This is because it **will** not be included as part of your inequality region

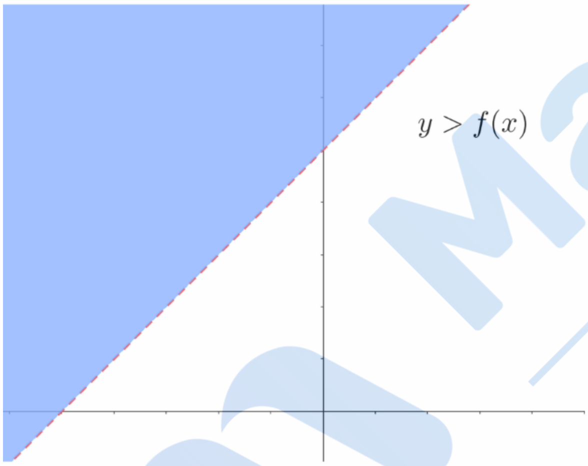
If the inequality you are sketching is either $' \leq '$ or $' \geq '$, then the line for $f(x)$ should be a normal solid line. This is because it **will** be included as part of your inequality region

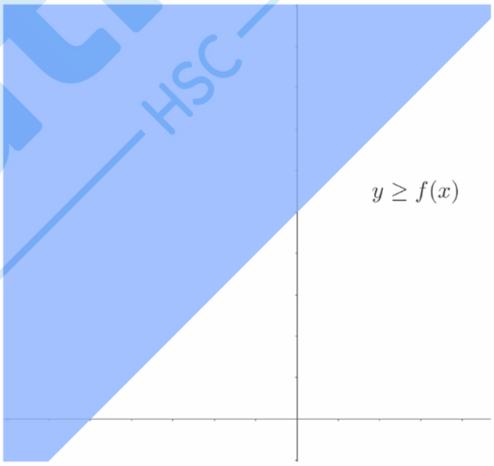
Assuming that $f(x)$ is a linear expression:



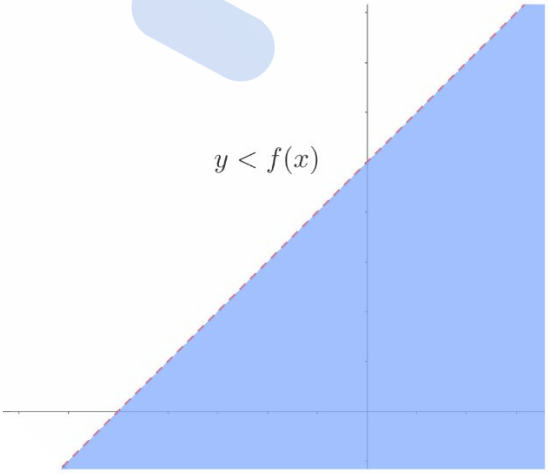
$\boldsymbol{Step\ 2}$: $\boldsymbol{Shade\ your\ region}$

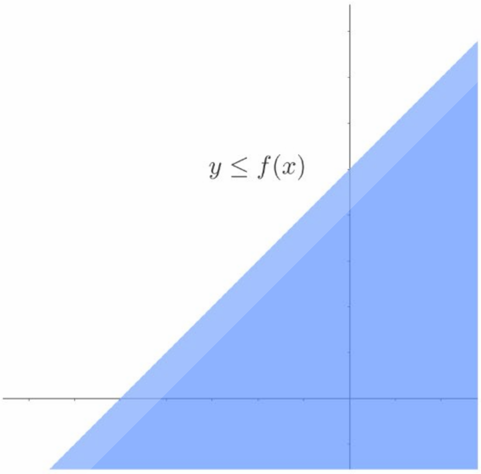
If the inequality symbols are $' > '$ OR $' \geq '$, then the region will be **above** the line:





If the inequality symbols are $' < '$ OR $' \leq '$, then the region will be **below** the line:





- Solving Quadratic Inequalities

To solve inequality questions, this generally involves the following steps:
$$\boldsymbol{Step\ 1: Move\ everything\ to\ one\ side}$$
This can be either LHS or RHS, it's up to you
$$\boldsymbol{Step\ 2: Simplify\ and\ factorise}$$
$$\boldsymbol{Step\ 3: Solve\ the\ inequality}$$
This last step may require you to draw a basic sketch of a parabola

Example 1: Solve the following inequality:

$$x + 6 \geq x^2$$

Solution:

$$Step\ 1: Move\ everything\ to\ RHS$$

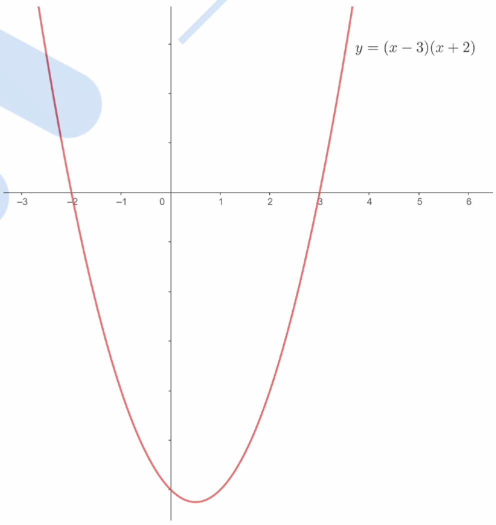
$$\therefore x^2 - x - 6 \leq 0$$

$$Step\ 2: Factorise\ the\ quadratic$$

$$(x - 3)(x + 2) \leq 0$$

$$Step\ 3: Solve\ the\ inequality$$

Drawing a basic sketch of the parabola $y = (x - 3)(x + 2)$:



We can see that the interval for which the parabola, $(x - 3)(x + 2) \leq 0$ is:

$$\therefore -2 \leq x \leq 3$$

- Solving Inequalities Algebraically

Continuing on from the last dot point, to solve these equations the following should **always** be remembered:

- Any addition or subtraction can be made provided that it is done to both sides of the inequality

- For multiplication and division, be careful since multiplying or dividing by a negative number requires you to flip the inequality.

  For example: We know that $2 < 4$, and if we multiplied both sides by $"-1"$ then it would have to turn into $-2 > -4$. Notice that the inequality sign has been flipped

- A denominator can be removed by multiplying both sides by its square. We **cannot** multiply both sides by the denominator itself since we cannot be certain if it is negative or not, so we don't know if the inequality sign should be flipped or not. Thus, since the square of anything is always positive, we multiply both sides by $denominator^2$ instead

- Always consider if **any restrictions** on the domain exist, especially in terms of the denominator and how it cannot equal 0

- A quick sketch of the graph could help as the final step to quickly solve the inequality. For example, to solve $(x-2)(x-5) < 0$ a quick sketch of the parabola would give the solution $2 < x < 5$. Note that whenever a quick sketch is made, details such as a sign table and y-intercepts etc. can be omitted.

Note: Please revise these as it is very common to make silly mistakes here!

Example 2: Solve $\frac{3}{x-4} \geq -1$

For this inequality, since there is a denominator, we multiply both sides by its square first.

$$\frac{3}{x-4} \times (x-4)^2 \geq -1 \times (x-4)^2$$
$$3(x-4) \geq -(x^2 - 8x + 16)$$
$$3x - 12 \geq -x^2 + 8x - 16$$
$$x^2 - 5x + 4 \geq 0$$
$$(x-4)(x-1) \geq 0$$

Then, considering the sketch of the parabola, the interval that satisfies this inequality is:

$$x \geq 4 \; or \; x \leq 1$$

However, remembering that we always need to look out for restrictions:

$$denominator \neq 0$$
$$x - 4 \neq 0$$
$$\therefore x \neq 4$$

Hence, the answer is actually:

$$x > 4 \; or \; x \leq 1$$

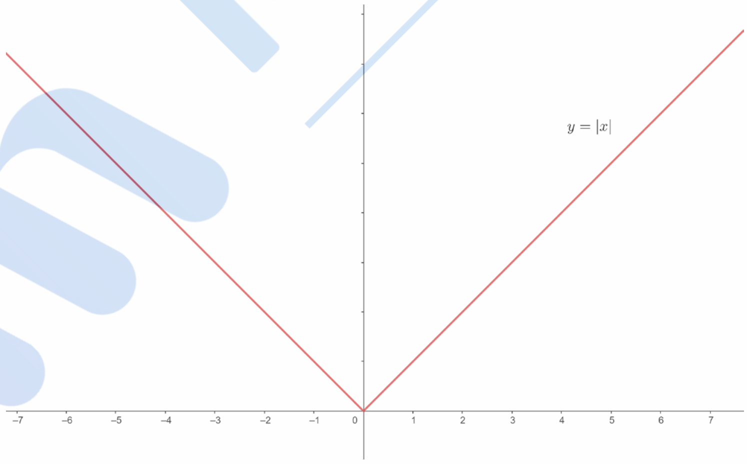
- Solving Absolute Values in Inequalities

Recall that:

Whenever an expression $f(x)$ or value a is absolute valued, it is made positive but the magnitude is still preserved. For example, $|-6| = 6$.

An alternative meaning of absolute value is that the absolute value $|x|$ of a number x is the distance from x to the origin on the number line.

Graphically, $y = |x|$ looks like:



When solving absolute value equations and inequations:

Let $a \geq 0$ be a real number

- Rewrite an equation $|f(x)| = a$ as $f(x) = -a$ or $f(x) = a$
- Rewrite an inequality $|f(x)| < a$ as $-a < f(x) < a$
- Rewrite an inequality $|f(x)| > a$ as $f(x) < -a$ or $f(x) > a$

Example 3:

a) Solve $|10 - x^2| = 6$

$$10 - x^2 = 6 \text{ or } 10 - x^2 = -6$$

$$x^2 = 4 \text{ or } x^2 = 16$$

$$\therefore x = 2, -2, 4 \text{ or } -4$$

b) Solve $|10 - x^2| < 6$

$$-6 < 10 - x^2 < 6$$
$$-16 < -x^2 < -4$$

Multiplying everything by -1 and remembering to flip the inequality:

$$4 < x^2 < 16$$

Remembering that if $x^2 = a^2$, then $x = a$ or $x = -a$:

Consider the positive square root first:

$$\sqrt{4} < x < \sqrt{16}$$
$$2 < x < 4$$

Considering the negative square root:

$$-2 > x > -4$$
$$-4 < x < -2$$

Therefore, the answer is $2 < x < 4$ or $-4 < x < -2$

c) Solve $|10 - x^2| > 6$

$$10 - x^2 > 6 \text{ or } 10 - x^2 < -6$$
$$x^2 < 4 \text{ or } x^2 > 16$$
$$\therefore -2 < x < 2 \text{ or } x < -4 \text{ or } x > 4$$

- Solving Equations Through Graphical Means

While it is often better and more efficient to solve an equation through algebraic methods, some questions may specifically request solving by graphical methods, so unfortunately in these situations we have no other choice.

If a question asks to solve the following equation:

$$f(x) = g(x)$$

The steps to solve through graphical methods is:

$$\boldsymbol{Step\ 1: Sketch\ f(x)}$$

$$\boldsymbol{Step\ 2: Sketch\ g(x) on\ the\ same\ axes}$$

$$\boldsymbol{Step\ 3: Find\ the\ x-value\ of\ the\ point\ of\ intersection(s)}$$

While this in itself isn't tricky, the tricky part sometimes is sketching $f(x)$ and $g(x)$ accurately. Make sure to make the graphs to scale in order to accurately determine the points of intersections!

Note: You can sometimes use algebraic methods to confirm that you have found the correct intersection points for a question! Just rub it out afterwards so the marker doesn't see it though…

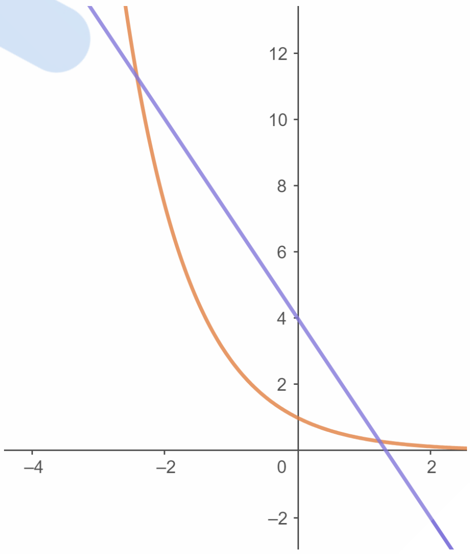
Example 1: Determine how many roots exist for the equation $e^{-x} + 3x - 4 = 0$

Solution:

This question is actually quite tricky as we cannot solve for the answer algebraically. Instead, we must rearrange the equation and solve graphically:

$$e^{-x} = 4 - 3x$$

Sketching on the same axes the graphs $y = e^{-x}$ and $y = 4 - 3x$:



We see that there are 2 intersection points. Hence, the equation $e^{-x} + 3x - 4 = 0$ has 2 solutions!

Inequality Exercises

1. Solve each of the following inequalities, giving answers in bracket interval notation:

a) $3 - 5x > 7$

b) $12 - 7x > -2x - 18$

c) $\frac{1}{6}(2 - x) - \frac{1}{3}(2 + x) \geq 2$

d) $3^x \geq 27$

e) $2^{-x} > 16$

f) $\log_2 x < 3$

2. Solve each of the following double inequalities:

a) $-5 < x - 3 \leq 4$

b) $-7 \leq 5x + 3 < 3$

c) $-4 < 1 - \frac{1}{3}x \leq 3$

d) $1 < 5^x \leq 125$

e) $-2 \leq \log_5 x \leq 4$

3. Solve the following equations and inequalities involving absolute values:

a) $|x - 4| = 1$

b) $|2x - 3| = 7$

c) $|4 - 2x| \leq 1$

d) $|3x + 4| > 2$

e) $1 < |x + 2| \leq 3$

f) $1 \leq |2x - 3| < 4$

4. Solve the following inequalities:

a) $\frac{4}{2-x} > 1$

b) $\frac{4}{3-2x} < 3$

c) $\frac{5}{4x-3} \geq -3$

d) $\frac{2x+1}{x-3} > 1$

5. Find the domain for which the hyperbola $y = \frac{2}{x-1}$ is above the line $y = \frac{1}{2}x - 2$

Inequality Exercise Answers

1.

a) $3 - 5x > 7$

$$5x < -4$$
$$\therefore x < -\frac{4}{5}$$

b) $12 - 7x > -2x - 18$

$$12 + 18 > -2x + 7x$$
$$5x < 30$$
$$\therefore x < 6$$

c) $\frac{1}{6}(2 - x) - \frac{1}{3}(2 + x) \geq 2$

Multiplying both sides by 6 to get rid of denominator:

$$(2 - x) - 2(2 + x) \geq 12$$
$$2 - x - 4 - 2x \geq 12$$
$$-3x - 3 \geq 12$$
$$-3x \geq 15$$

Dividing both sides by -3 and remembering to flip the inequality:

$$\therefore x \leq -5$$

d) $3^x \geq 27$

Logging both sides by base 3:

$$\log_3 3^x \geq \log_3 27$$
$$\therefore x \geq 3$$

e) $2^{-x} > 16$

Logging both sides by base 2:

$$\log_2 2^{-x} > \log_2 16$$
$$-x > 4$$
$$\therefore x < -4$$

f) $\log_2 x < 3$

Considering both sides as indices with base 2:

$$2^{\log_2 x} < 2^3$$
$$x < 8$$

Remembering to look for restrictions, since x is inside a log

$$\therefore x > 0$$

Answer is therefore $0 < x < 8$

2.

a) $-5 < x - 3 \leq 4$

Adding 3 to all sides:

$$\therefore -2 < x \leq 7$$

b) $-7 \leq 5x + 3 < 3$

Subtracting 3 to all sides:

$$-10 \leq 5x < 0$$

Dividing by 5 to all sides:

$$-2 \leq x < 0$$

c) $-4 < 1 - \frac{1}{3}x \leq 3$

$$-5 < -\frac{1}{3}x \leq 2$$

Multiplying by -3 to all sides:

$$15 > x \geq -6$$

$$\therefore -6 \leq x < 15$$

d) $1 < 5^x \leq 125$

Logging all sides with base 5:

$$\log_5 1 < \log_5 5^x < \log_5 125$$

$$\therefore 0 < x < 3$$

e) $-2 \leq \log_5 x \leq 4$

Considering all sides as indices with base 5:

$$5^{-2} \leq 5^{\log_5 x} \leq 5^4$$

$$\frac{1}{25} \leq x \leq 625$$

Answer is therefore $\frac{1}{25} \leq x \leq 625$

3.

a) $|x - 4| = 1$

$$x - 4 = 1 \; or \; x - 4 = -1$$
$$\therefore x = 5 \; or \; x = 3$$

b) $|2x-3|=7$

$$2x-3=7 \text{ or } 2x-3=-7$$
$$2x=10 \text{ or } 2x=-4$$
$$\therefore x=5 \text{ or } x=-2$$

c) $|4-2x|\leq 1$

$$-1\leq 4-2x\leq 1$$
$$-5\leq -2x\leq -3$$
$$\frac{5}{2}\geq x\geq \frac{3}{2}$$
$$\therefore \frac{3}{2}\leq x\leq \frac{5}{2}$$

d) $|3x+4|>2$

$$3x+4>2 \ OR \ 3x+4<-2$$
$$3x>-2 \ OR \ 3x<-6$$
$$x>-\frac{2}{3} \ OR \ x<-2$$

e) $1<|x+2|\leq 3$

$$1<x+2\leq 3 \ OR -3\leq x+2<-1$$
$$\therefore -1<x\leq 1 \ OR -5\leq x<-3$$

f) $1\leq |2x-3|<4$

$$1\leq 2x-3<4 \ OR -4<2x-3\leq -1$$
$$4\leq 2x<7 \ OR -1<2x\leq 2$$
$$2\leq x<\frac{7}{2} \ OR -\frac{1}{2}<x\leq 1$$

4.

a) $\frac{4}{2-x}>1$

$$\frac{4}{2-x}\times(2-x)^2>(2-x)^2$$
$$4(2-x)>4-4x+x^2$$
$$8-4x>4-4x+x^2$$
$$x^2<4$$
$$\therefore -2<x<2$$

Remembering the restriction that $denominator \neq 0$:

$$\therefore x\neq 2$$

Answer is therefore $-2<x<2 \ but \ x\neq 2$

b) $\frac{4}{3-2x}<3$

$$
\begin{aligned}
&\frac{4}{3-2x}\times(3-2x)^2 < 3(3-2x)^2\\
&4(3-2x) < 3(9-6x+4x^2)\\
&12-8x < 27-18x+12x^2\\
&12x^2-26x+15 > 0\\
&x = \frac{26\pm\sqrt{26^2-4\times15\times12}}{2\times12} = no\ solution
\end{aligned}
$$

Therefore, there are no roots for the quadratic and since the coefficient of x^2 is positive, it is concave up.

$$\therefore 12x^2-26x+15\ will\ be > 0\ for\ all\ x$$

However, there is a restriction since $denominator \neq 0$

$$
\begin{aligned}
&3-2x \neq 0\\
&\therefore x \neq \frac{3}{2}
\end{aligned}
$$

The answer is therefore $x \in \mathbb{R}\ but\ x \neq \frac{3}{2}$

c) $\frac{5}{4x-3} \geq -3$

$$
\begin{aligned}
&\frac{5}{4x-3}\times(4x-3)^2 \geq -3\times(4x-3)^2\\
&5(4x-3) \geq -3(16x^2-24x+9)\\
&20x-15 \geq -48x^2+72x-27\\
&-48x^2+52x-12 \leq 0\\
&x = \frac{-52\pm\sqrt{52^2-4\times12\times48}}{2\times-48}\\
&\quad = \frac{1}{3}\ or\ \frac{3}{4}
\end{aligned}
$$

Considering the concave down parabola, we therefore get:

$$x \leq \frac{1}{3}\ OR\ x \geq \frac{3}{4}$$

Remembering the restriction that $denominator \neq 0$:

$$
\begin{aligned}
&4x-3 \neq 0\\
&\therefore x \neq \frac{3}{4}
\end{aligned}
$$

Answer is therefore $x \leq \frac{1}{3}\ OR\ x > \frac{3}{4}$

d) $\frac{2x+1}{x-3} > 1$

$$
\begin{aligned}
&\frac{2x+1}{x-3}\times(x-3)^2 > (x-3)^2\\
&(2x+1)(x-3) > (x-3)^2
\end{aligned}
$$

$$2x^2 - 6x + x - 3 > x^2 - 6x + 9$$
$$x^2 + x - 12 > 0$$
$$(x + 4)(x - 3) > 0$$

Sketching a parabola to help, we therefore get:

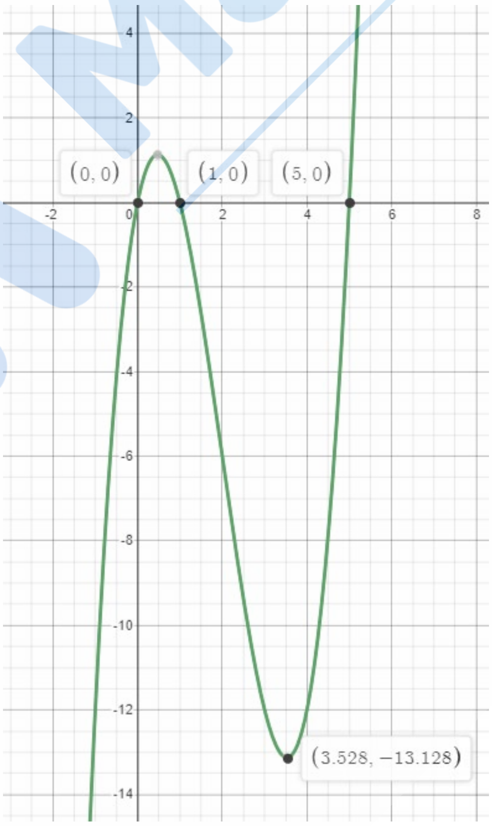
$$x < -4 \; OR \; x > 3$$

The restriction for this question is $x \neq 3$, but this doesn't affect our answer anyways here

5. The question is asking to solve for x the inequality $\frac{2}{x-1} > \frac{1}{2}x - 2$

$$\frac{2}{x-1} > \frac{x-4}{2}$$
$$\frac{2}{x-1} \times (x-1)^2 > \frac{x-4}{2} \times (x-1)^2$$
$$2 \times 2(x-1) > (x-4)(x-1)^2$$
$$4(x-1) > (x-4)(x^2 - 2x + 1)$$
$$4x - 4 > x^3 - 2x^2 + x - 4x^2 + 8x - 4$$
$$x^3 - 6x^2 + 5x < 0$$
$$x(x^2 - 6x + 5) < 0$$
$$x(x-5)(x-1) < 0$$

Sketching a basic sketch of this polynomial:



$$\therefore x < 0 \; OR \; 1 < x < 5$$

The restriction for this question is $x \neq 1$, but this doesn't affect our answer anyways here