TRIGONOMETRY

TRIGONOMETRIC EQUATIONS (VIII)

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- Trigonometric Equations

The bulk of exam trigonometry questions will deal with trigonometric equations, so it is critical to understand and practice this subtopic! Below is an example to introduce us

Let's say we want to solve the following trigonometric equation:

$$\sin\theta = \frac{1}{\sqrt{2}}, for\ 0° \le x \le 360°$$

If we are familiar with exact values, our immediate thought would be that:

$$\theta = 45°$$

However, recalling our previous work with trig quadrants and signs, we also know that for sine functions we need to consider the 2^{nd} quadrant angle $180° - 45° = 135°$, so:

$$\sin 135° = \sin 45° = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = 135°\ as\ well$$

Therefore, we actually have two answers for this equation

$$\theta = 45°, 135°$$

Do we need to consider the 3^{rd} quadrant? No, because $\sin\theta$ would then be negative

Do we need to consider the 4^{th} quadrant? No, because $\sin\theta$ would then be negative

Do we need to consider angles larger than one revolution? No, because our domain is defined as $0° \le x \le 360°$ for this question

Hence, those are our only two answers and we are done.

Thus, when solving trig equations, we follow these general steps:

$$\boldsymbol{Step\ 1: Find\ your\ related\ angle}$$

In the example before, this was when we found $\theta = 45°$.

Knowing your exact values for all the trig functions will help

$$\boldsymbol{Step\ 2: Find\ your\ solutions\ in\ other\ quadrants}$$

Remember the mnemonic: All Stations To Central

If it's 2^{nd} quadrant we think of $180° - \theta$, 3^{rd} quadrant we think of $180° + \theta$ and 4^{th} quadrant we think of $360° - \theta$

Here's a helpful summary if you're having trouble determining which quadrants to consider:

$Solution\ x\ will\ be\ in:$				
	$1st\ quadrant$	$2nd\ quadrant$	$3rd\ quadrant$	$4th\ quadrant$
$If\ \sin x > 0$	✓	✓		
$If\ \sin x < 0$			✓	✓

$Solution\ x\ will\ be\ in:$				
	$1st\ quadrant$	$2nd\ quadrant$	$3rd\ quadrant$	$4th\ quadrant$
$If\ cos\ x > 0$	✓			✓
$If\ cos\ x < 0$		✓	✓	

$Solution\ x\ will\ be\ in:$				
	$1st\ quadrant$	$2nd\ quadrant$	$3rd\ quadrant$	$4th\ quadrant$
$If\ tan\ x > 0$	✓		✓	
$If\ tan\ x < 0$		✓		✓

Note: If the question provides the domain of the question in radians, we must provide our answer in radians. If the question provides it in degrees, we must provide our answer in degrees.

- Paying Attention to the Domain

Typically, most trig equations will have the domain be defined as $0° \leq x \leq 360°$. However, this is not always guaranteed so we must **be careful of the domain** when solving trig equations.

Example 2: Solve the equation $\tan x = -\sqrt{3}$, for $0° \leq x \leq 720°$

Solution:

Since $\tan x < 0$, then this means x must be in the 2^{nd} and 4^{th} quadrant.

It is known that $\tan 60° = \sqrt{3}$

$$\therefore 60°\ is\ the\ related\ angle$$

Hence, the solutions for x will be:

$$180 - 60 = 120°\ OR\ 180 + 60 = 240°$$

However, since the domain is $0° \leq x \leq 720°$, we must consider the second revolution, meaning there are actually four solutions:

$$\therefore x = 120°, 240°, 120° + 360°, 240° + 360°$$
$$= 120°, 240°, 480°\ OR\ 600°$$

- Harder Trig Equations with Compound Angles

Sometimes, the trig function itself may not just have "x" by itself. These harder questions may look something like:

$$Solve \sin\left(x+\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}, for\ 0\le x\le 2\pi$$

$$Solve \cos(3x)=\frac{1}{2}, for\ 0\le x\le 2\pi$$

In these situations, we simply apply a substitution, where:

$$we\ let\ u=whatever\ is\ in\ the\ bracket$$

Using the examples above, this would look like:

$$Let\ u=x+\frac{\pi}{3}$$

$$Let\ u=3x$$

Thus, a summary of steps for solving compound equations are listed below:

$$\boldsymbol{Step\ 1: Make\ your\ u\ substitution}$$

As described before, we let $u=compound\ angle$

$$\boldsymbol{Step\ 2: Change\ your\ domain\ in\ accordance}$$

This is a critical but easily forgotten step! Since u is in terms of x, our restriction/domain for u must also match with the given domain for x. Using the examples before:

$$Since\ u=x+\frac{\pi}{3}, \qquad \therefore \frac{\pi}{3}\le u\le\frac{7\pi}{3}$$

$$Since\ u=3x, \qquad \therefore 0\le u\le 6\pi$$

$$\boldsymbol{Step\ 3: Solve\ the\ equation\ like\ normal\ for\ u}$$

$$\boldsymbol{Step\ 4: Convert\ back\ to\ x}$$

Example 3: Solve the equation $\cos(2x+\frac{\pi}{4})=\frac{1}{\sqrt{2}}$ for the domain $0\le x\le\pi$

Solution:

First, we let $u=2x+\frac{\pi}{4}$

Then, since our domain changes:

$$0 \leq 2x \leq 2\pi$$

$$0 \leq 2x + \frac{\pi}{4} \leq \frac{9\pi}{4}$$

$$\therefore 0 \leq u \leq \frac{9\pi}{4}$$

Now solving the equation $\cos u = \frac{1}{\sqrt{2}}$:

Since $\cos u > 0$, this means that u is in the 1st and 4th quadrant

$$\therefore u = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$
$$= \frac{\pi}{4}, \frac{7\pi}{4}$$

Moreover, since $0 \leq u \leq \frac{9\pi}{4}$ we should consider the second revolution, meaning that:

$$\therefore u = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{4} + 2\pi$$
$$= \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$$

Finally, converting back into x:

$$2x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$$
$$2x = 0, \frac{3\pi}{2}, 2\pi$$
$$\therefore x = 0, \frac{3\pi}{4}, \pi$$

- Harder Trig Equations Involving Multiple Trig Functions

Typically, if we see more than one trig function in an equation, it means that we should try rearranging our equation in a way so that:

$$\frac{\sin x}{\cos x} = \tan x$$

This way, it will simplify our equation into just one trig function for us to solve

Example 4: Solve the equation $\sin x - \sqrt{3}\cos x$ for the domain $0 \leq x \leq \pi$

Solution:

Rearranging our equation:

$$\sin x = \sqrt{3}\cos x$$
$$\frac{\sin x}{\cos x} = \sqrt{3}$$
$$\therefore \tan x = \sqrt{3}$$

Hence, for the domain $0 \leq x \leq \pi$:

$$x = \frac{\pi}{3}$$

- Trig Equations Reducible to Quadratics

For trig equations where we can see squared trig functions such as $\sin^2 x$, $\cos^2 x$ or $\tan^2 x$, we should:

$$Always\ see\ if\ factorisation\ is\ possible\ first$$

This would thus allow us to solve the equation easily

If that isn't possible, then we must consider reducing the problem to a quadratic. In other words:

$$let\ u = trig\ function$$

The steps to solve quadratic trig equations are summarised as follows:

$$\boldsymbol{Step\ 1: Substitute\ and\ let\ u = trig\ function}$$
$$\boldsymbol{Step\ 2: Solve\ the\ quadratic\ now\ in\ terms\ of\ u}$$
$$\boldsymbol{Step\ 3: Convert\ your\ u\ back\ into\ the\ trig\ function}$$
$$\boldsymbol{Step\ 4: Solve\ like\ normal}$$

Example 5: Solve the equation $\sin^2\theta - 2\sin\theta + 1 = 0$ for the domain $0 \leq x \leq 2\pi$

Solution:

Let $u = \sin\theta$

$$\therefore u^2 - 2u + 1 = 0$$
$$(u-1)^2 = 0$$
$$\therefore u = 1$$

Hence, we say that $\sin\theta = 1$

$$\therefore \theta = \frac{\pi}{2}$$

Trigonometric Equation Questions

1. Solve the following equations for $0° \leq \theta \leq 360°$
   a) $\sin^2 \theta - \sin \theta = 0$
   b) $4\cos^2 \theta + 3\sin \theta = 3$

For more questions, check out our online quizzes on the platform!

Trigonometric Equation Question Answers

1.
a) $\sin^2 \theta - \sin \theta = 0$

$$\sin \theta (\sin \theta - 1) = 0$$

$$\therefore \sin \theta = 0 \text{ or } \sin \theta = 1$$

If $\sin \theta = 0$:

$$\theta = 0°, 180° \text{ or } 360°$$

If $\sin \theta = 1$:

$$\theta = 90°$$

$$\therefore \theta = 0°, 90°, 180° \text{ or } 360°$$

b) $4\cos^2 \theta + 3\sin \theta = 3$

From the identity:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\therefore \cos^2 \theta = 1 - \sin^2 \theta$$

$$\therefore 4(1 - \sin^2 \theta) + 3\sin \theta = 3$$
$$4 - 4\sin^2 \theta + 3\sin \theta = 3$$
$$4\sin^2 \theta - 3\sin \theta - 1 = 0$$

Factorising like it's a quadratic:

$$(4\sin \theta + 1)(\sin \theta - 1) = 0$$

$$\therefore \sin \theta = -\frac{1}{4} \text{ or } \sin \theta = 1$$

If $\sin\theta = -\frac{1}{4}$:

$$\theta = 180° + \sin^{-1}\frac{1}{4}, 360° - \sin^{-1}\frac{1}{4}$$

If $\sin\theta = 1$:

$$\theta = 90°$$

$$\therefore \theta = 90°, 180° + \sin^{-1}\frac{1}{4}, 360° - \sin^{-1}\frac{1}{4}$$