FURTHER FUNCTIONS

VERTICAL AND HORIZONTAL ASYMPTOTES (I)

Contents include:

- Finding Vertical Asymptotes
- Finding Horizontal Asymptotes

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- Finding Vertical Asymptotes

Vertical asymptotes occur when the domain of a function is restricted at a certain value $x = a$.

This typically occurs because the denominator when $x = a$ is equal to 0.

$$\therefore \textit{Vertical asymptote occurs when denominator} \neq 0$$

Example 1: Find the vertical asymptotes for the function $f(x) = \frac{x^2}{(x-2)(x+4)}$

$$\because \textit{denominator} \neq 0$$
$$(x-2)(x+4) \neq 0$$
$$\therefore x \neq 2 \textit{ and } x \neq -4$$

Hence, the vertical asymptotes are $x = 2$ and $x = -4$

- Finding Horizontal Asymptotes

Finding horizontal asymptotes involves considering the behaviour of a function as x approaches ∞ and $-\infty$. In other words, we must use limits when finding horizontal asymptotes.

These questions typically involve *rational functions* where there is a polynomial on the numerator and denominator.

When investigating the behaviour of rational functions for large x:

1. Divide the numerator and denominator by the x with the highest power in the denominator

Now consider a couple of points:

- As $x \to \infty$ or as $x \to -\infty$ then $\frac{1}{x} \to 0$
- If $f(x)$ tends to a definite limit a as $x \to \infty$ or as $x \to -\infty$, then the horizontal line $y = a$ is a horizontal asymptote on the right or on the left

Example 2: Find the horizontal asymptotes for the function $f(x) = \frac{x-1}{x-4}$

$$\textit{Step } 1\textit{: Divide the top and bottom by } x$$

x is the highest term of the denominator, so that's why we divide by x

$$f(x) = \frac{1 - \frac{1}{x}}{1 - \frac{4}{x}}$$

$$Step\ 2: Consider\ x \to \infty$$

$$As\ x \to \infty, f(x) \to \frac{1-0}{1-0} = 1$$

$$Step\ 3: Consider\ x \to -\infty$$

$$As\ x \to -\infty, f(x) \to \frac{1+0}{1+0} = 1$$

Thus, we can conclude that the horizontal asymptote for this function is $y = 1$

Example 3: Find the horizontal asymptotes for the function $f(x) = \frac{x-1}{x^2-4}$

$$Step\ 1: Divide\ the\ top\ and\ bottom\ by\ x^2$$

$$f(x) = \frac{\frac{x}{x^2} - \frac{1}{x^2}}{1 - \frac{4}{x^2}}$$

$$Step\ 2: Consider\ x \to \infty$$

$$As\ x \to \infty, f(x) \to \frac{0-0}{1-0} = 0$$

$$Step\ 3: Consider\ x \to -\infty$$

$$As\ x \to -\infty, f(x) \to \frac{0-0}{1+0} = 0$$

Thus, we can conclude that the horizontal asymptote for this function is $y = 0$

Example 4: Find the horizontal asymptotes for the function $f(x) = \frac{x^2-1}{x-4}$

$$Step\ 1: Divide\ the\ top\ and\ bottom\ by\ x$$

$$f(x) = \frac{x - \frac{1}{x}}{1 - \frac{4}{x}}$$

$$Step\ 2: Consider\ x \to \infty$$

$$As\ x \to \infty, f(x) \to \frac{\infty - 0}{1-0} = \infty$$

Therefore, this means that as $x \to \infty$ the function approaches ∞

$$Step\ 3: Consider\ x \to -\infty$$

$$As\ x \to -\infty, f(x) \to \frac{-\infty + 0}{1+0} = -\infty$$

Therefore, this means that as $x \to -\infty$ the function approaches $-\infty$

Thus, we can conclude that there is no horizontal asymptote

Vertical and Horizontal Asymptote Exercises

1. Find the vertical asymptotes for each of the following functions:

a) $f(x) = \frac{1}{x-3}$

b) $f(x) = \frac{1}{x+2}$

c) $f(x) = \frac{x^2+x-6}{x-3}$

d) $f(x) = \frac{2}{(x-3)(x+5)}$

e) $f(x) = \frac{2(x+7)}{(x+1)(x-4)}$

f) $f(x) = \frac{2(x-7)}{\left(x-\frac{1}{2}\right)(x+4)(x-3)}$

2. Find the horizontal asymptotes for each of the following functions by first dividing through by the highest power of x in the denominator:

a) $f(x) = \frac{x+1}{x-3}$

b) $f(x) = \frac{x-5}{x+2}$

c) $f(x) = \frac{2x+2}{1-x}$

d) $f(x) = \frac{x-3}{x^2+2}$

e) $f(x) = \frac{x^2+2}{x^2+5x-4}$

f) $f(x) = \frac{2x^3}{x^3+x^2-2x}$

3. Find all asymptotes for each of the following functions:

a) $y = \frac{x^2-4x+4}{x^2+5x+4}$

b) $y = \frac{x-7}{x^2-x-6}$

c) $y = \frac{1+x^2}{1-4x^2}$

Vertical and Horizontal Asymptote Exercise Answers

1. Remember for these questions that the vertical asymptote occurs when the denominator of the function is equal to 0

a)

$$f(x) = \frac{1}{x-3}$$

$$\therefore x - 3 \neq 0$$

$$x \neq 3$$

Hence, vertical asymptote at $x = 3$

b)

$$f(x) = \frac{1}{x+2}$$

$$\therefore x + 2 \neq 0$$

$$x \neq -2$$

Hence, vertical asymptote at $x = -2$

c)

$$f(x) = \frac{x^2 + x - 6}{x - 3}$$

$$\therefore x - 3 \neq 0$$

$$x \neq 3$$

Hence, vertical asymptote at $x = 3$

d)

$$f(x) = \frac{2}{(x-3)(x+5)}$$

$$\therefore (x-3)(x+5) \neq 0$$

$$x \neq 3 \; and \; x \neq 5$$

Hence, vertical asymptote at $x = 3$ and $x = 5$

e)

$$f(x) = \frac{2(x+7)}{(x+1)(x-4)}$$

$$\therefore (x+1)(x-4) \neq 0$$

$$x \neq -1 \; and \; x \neq 4$$

Hence, vertical asymptote at $x = -1$ and $x = 4$

f)

$$f(x) = \frac{2(x-7)}{\left(x - \frac{1}{2}\right)(x+4)(x-3)}$$

$$\therefore \left(x - \frac{1}{2}\right)(x+4)(x-3) \neq 0$$

$$x \neq \frac{1}{2} \ and \ x \neq -4 \ and \ x \neq 3$$

Hence, vertical asymptote at $x = \frac{1}{2}$ and $x = 4$ and $x = 3$

2. Remember for these questions that $\frac{1}{\infty} = 0$ when we apply limits

a) For our first step, divide both top and bottom by x:

$$f(x) = \frac{1 + \frac{1}{x}}{1 - \frac{3}{x}}$$

Then, applying limit as $x \to \pm\infty$

$$\lim_{x \to \infty} \frac{1 + \frac{1}{x}}{1 - \frac{3}{x}} = 1$$

$$\lim_{x \to -\infty} \frac{1 + \frac{1}{x}}{1 - \frac{3}{x}} = 1$$

Hence, horizontal asymptote will occur at $x = 1$

b) For our first step, divide both top and bottom by x:

$$f(x) = \frac{1 - \frac{5}{x}}{1 + \frac{2}{x}}$$

Then applying limit as $x \to \pm\infty$:

$$\lim_{x \to \infty} \frac{1 - \frac{5}{x}}{1 + \frac{2}{x}} = 1$$

$$\lim_{x \to -\infty} \frac{1 - \frac{5}{x}}{1 + \frac{2}{x}} = 1$$

Hence, horizontal asymptote will occur at $x = 1$

c) For our first step, divide both top and bottom by x:

$$f(x) = \frac{2 + \frac{2}{x}}{\frac{1}{x} - 1}$$

Then applying limit as $x \to \pm\infty$:

$$\lim_{x \to \infty} \frac{2 + \frac{2}{x}}{\frac{1}{x} - 1} = \frac{2}{-1}$$
$$= -2$$

$$\lim_{x \to -\infty} \frac{2 + \frac{2}{x}}{\frac{1}{x} - 1} = \frac{2}{-1}$$
$$= -2$$

Hence, horizontal asymptote occurs at $x = -2$

d) For our first step, divide top and bottom by x^2:

$$f(x) = \frac{\frac{1}{x} - \frac{3}{x^2}}{1 + \frac{2}{x^2}}$$

Then applying limit as $x \to \pm\infty$:

$$\lim_{x \to \infty} \frac{\frac{1}{x} - \frac{3}{x^2}}{1 + \frac{2}{x^2}} = \frac{0 - 0}{1 + 0}$$
$$= 0$$

$$\lim_{x \to -\infty} \frac{\frac{1}{x} - \frac{3}{x^2}}{1 + \frac{2}{x^2}} = 0$$

Hence, horizontal asymptote occurs at $x = 0$

e) For our first step, divide top and bottom by x^2:

$$f(x) = \frac{1 + \frac{2}{x^2}}{1 + \frac{5}{x} - \frac{4}{x^2}}$$

Then applying limit as $x \to \pm\infty$:

$$\lim_{x\to\infty} \frac{1 + \frac{2}{x^2}}{1 + \frac{5}{x} - \frac{4}{x^2}} = \frac{1+0}{1+0-0}$$
$$= 1$$

$$\lim_{x\to-\infty} \frac{1 + \frac{2}{x^2}}{1 + \frac{5}{x} - \frac{4}{x^2}} = \frac{1+0}{1+0-0}$$
$$= 1$$

Hence, horizontal asymptote occurs at $x = 1$

f) For our first step, divide top and bottom by x^3

$$f(x) = \frac{2}{1 + \frac{1}{x} - \frac{2}{x^2}}$$

Then applying limit as $x \to \pm\infty$:

$$\lim_{x\to\infty} \frac{2}{1 + \frac{1}{x} - \frac{2}{x^2}} = \frac{2}{1+0-0}$$
$$= 2$$

$$\lim_{x\to-\infty} \frac{2}{1 + \frac{1}{x} - \frac{2}{x^2}} = 2$$

Hence, horizontal asymptote occurs at $x = 2$

3.
a) Before evaluating asymptotes, we must factorise both top and bottom:

$$y = \frac{(x-2)^2}{(x+4)(x+1)}$$

Then, since the denominator cannot equal to 0:

$$(x+4)(x+1) \neq 0$$

$$\therefore x \neq -4 \; and \; x \neq -1$$

Hence, vertical asymptotes occur at $x = -4$ and $x = -1$

Next, dividing top and bottom by x^2 we get:

$$y = \frac{1 - \frac{4}{x} + \frac{4}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}}$$

Then applying limit as $x \to \pm\infty$:

$$\lim_{x \to \infty} \frac{1 - \frac{4}{x} + \frac{4}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}} = 1$$

$$\lim_{x \to -\infty} \frac{1 - \frac{4}{x} + \frac{4}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}} = -1$$

Hence, horizontal asymptote occurs at $x = -1$

b) Factorising the top and bottom if possible:

$$y = \frac{x - 7}{(x - 3)(x + 2)}$$

Then, since the denominator cannot equal to 0:

$$(x - 3)(x + 2) \neq 0$$

Hence, vertical asymptotes occur at $x = 3$ and $x = -2$

Next, dividing the top and bottom by x^2 we get:

$$y = \frac{\frac{1}{x} - \frac{7}{x^2}}{1 - \frac{1}{x} - \frac{6}{x^2}}$$

Then applying limit as $x \to \pm\infty$:

$$\lim_{x \to \infty} \frac{\frac{1}{x} - \frac{7}{x^2}}{1 - \frac{1}{x} - \frac{6}{x^2}} = 0$$

$$\lim_{x \to -\infty} \frac{\frac{1}{x} - \frac{7}{x^2}}{1 - \frac{1}{x} - \frac{6}{x^2}} = 0$$

Hence, horizontal asymptote occurs at $x = 0$

c) Factorising the bottom using difference of squares:

$$y = \frac{1 + x^2}{(1 + 2x)(1 - 2x)}$$

Then, since the denominator cannot equal to 0:

$$(1 + 2x)(1 - 2x) \neq 0$$

$$\therefore x \neq -\frac{1}{2} \; and \; x \neq \frac{1}{2}$$

Hence, vertical asymptotes occur at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$

Next, dividing the top and bottom by x^2:

$$y = \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2} - 4}$$

Then applying limit as $x \to \pm\infty$:

$$\begin{aligned} \lim_{x \to \infty} \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2} - 4} &= \frac{0 + 1}{0 - 4} \\ &= -\frac{1}{4} \end{aligned}$$

$$\lim_{x \to \infty} \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2} - 4} = -\frac{1}{4}$$

Hence, horizontal asymptote occurs at $x = -\frac{1}{4}$