FUNCTIONS

ODD AND EVEN FUNCTIONS (VIII)

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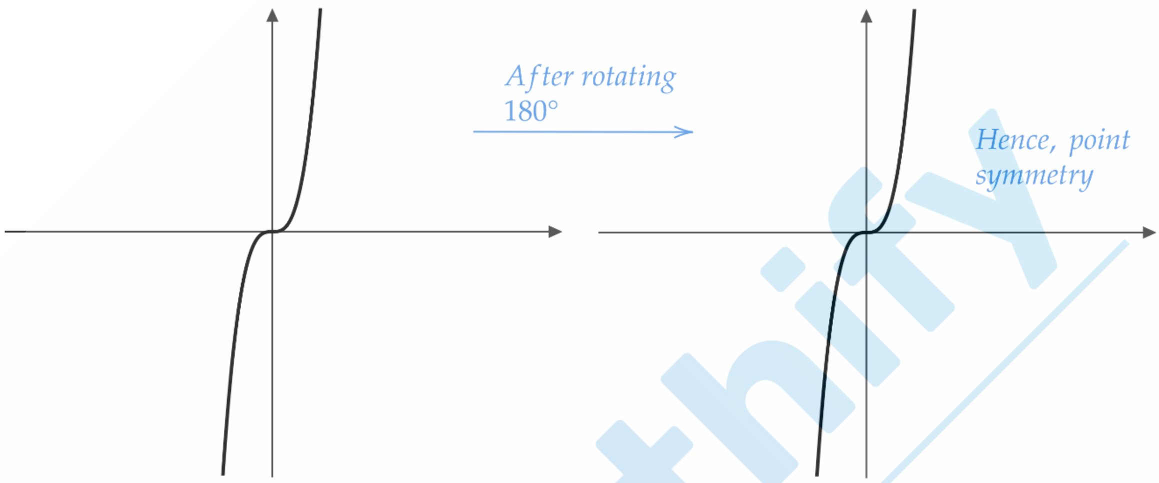
- Odd Functions

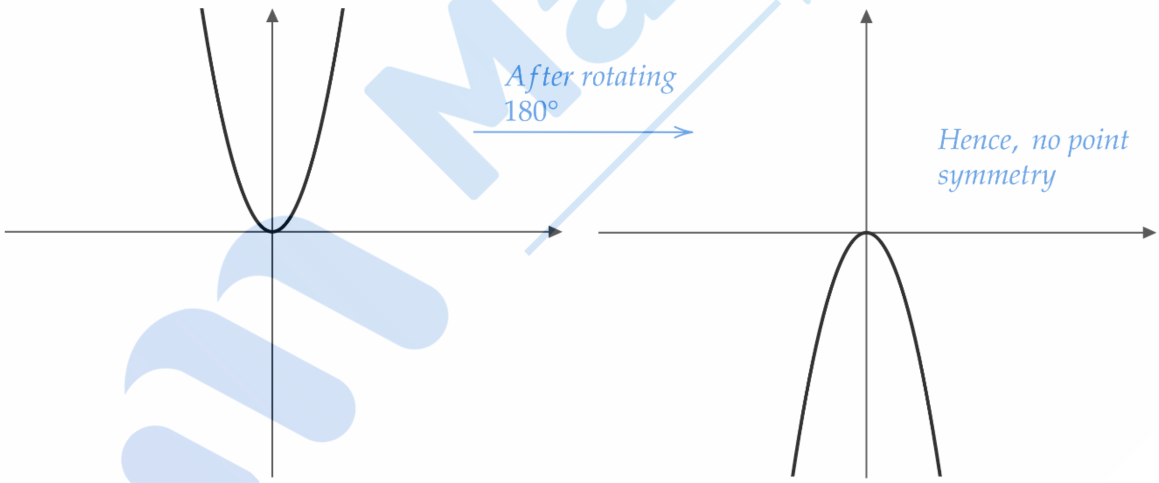
Odd functions are functions which are determined to have **point symmetry**. In other words, if you rotate the function 180° about the origin, it will look the same.

For example:

a) $y = x^3$ has point symmetry and is thus odd



b) $y = x^2$ does not have point symmetry and is thus not odd



However, having to check if a function is odd by sketching it and rotating is bothersome.

Instead, to determine if a given function $f(x)$ is odd algebraically, check:

$$If\ f(-x) = -f(x), then\ f(x)\ is\ odd$$

Note that in functions, when we say $f(-x)$, we are essentially replacing all the $x's$ that exist in our function with $'-x'$ instead!

Example 1: Determine if the function $f(x) = x^3 + x$ is odd or not

Solution:

First determining the equation of $f(-x)$:

$$\begin{aligned} f(-x) &= (-x)^3 + (-x) \\ &= -x^3 - x \end{aligned}$$

Now looking at $-f(x)$:

$$\begin{aligned} -f(x) &= -(x^3 + x) \\ &= -x^3 - x \end{aligned}$$

$$\therefore \textit{since } f(-x) = -f(x):$$

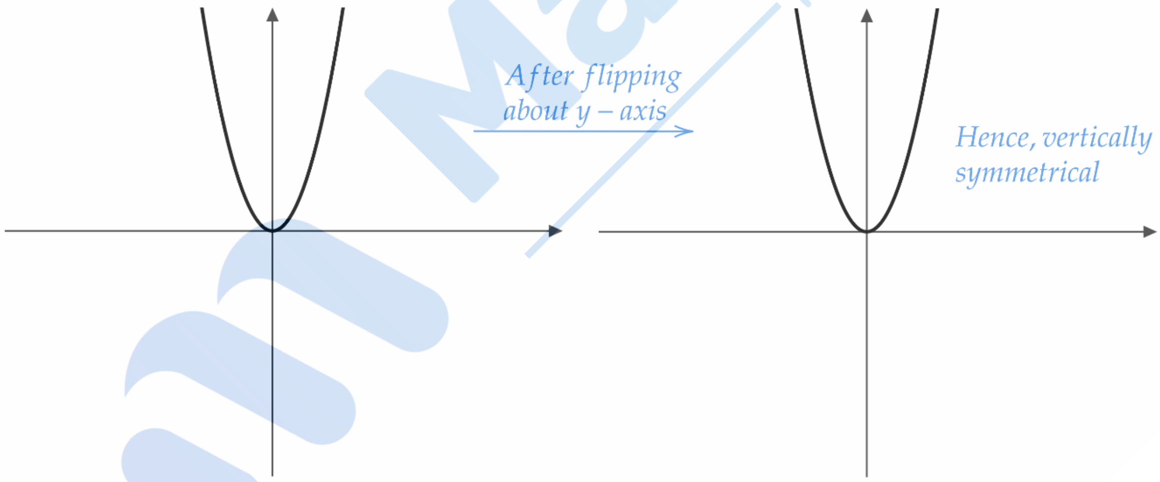
$$f(x) \textit{ is odd}$$

- Even Functions

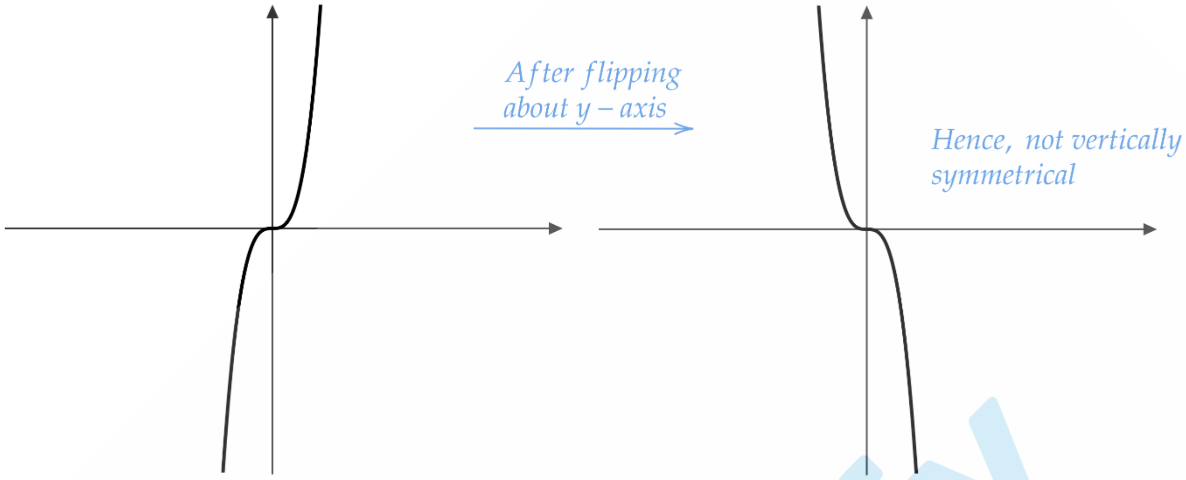
Even functions are functions which look the same when reflected about the $y-$ axis. In other words, they have vertical symmetry.

For example:

a) $y = x^2$ is vertically symmetrical and is thus even



b) $y = x^3$ is not vertically symmetrical and is thus not even



Once again though, having to test whether a function is even or not is quite bothersome

Instead, to determine if a function $f(x)$ is even algebraically, check:

$$If\ f(x) = f(-x), then\ f(x)\ is\ even$$

Note again that in functions, when we say $f(-x)$, we are essentially replacing all the $x's$ that exist in our function with $'-x'$ instead!

Example 2: Determine if the function $x^4 + 3x^2 - 5$ is even or not

Solution:

First determining the equation of $f(-x)$:

$$\begin{aligned} f(-x) &= (-x)^4 + 3(-x)^2 - 5 \\ &= x^4 + 3x^2 - 5 \\ &= f(x) \end{aligned}$$

Hence, since $f(-x) = f(x)$:

$$\therefore f(x)\ is\ even$$

- Determining Odd, Even or Neither

These questions essentially combine the methodologies of the previous two dot points, where we find expressions for $f(x), f(-x)$ and $-f(x)$ to see if any two are equal to one another. If none are equal, then the function $f(x)$ is neither odd nor even.

Example 3: Determine if $f(x) = 4x - 3x^2$ is odd, even or neither

Solution:

$$\begin{aligned} f(-x) &= 4(-x) - 3(-x)^2 \\ &= -4x - 3x^2 \end{aligned}$$

$$\begin{aligned} -f(x) &= -(4x - 3x^2) \\ &= -4x + 3x^2 \end{aligned}$$

Hence, since $f(x) \neq f(-x) \neq -f(x)$:

$$\therefore f(x) \; is \; neither \; odd \; nor \; even$$

Example 4: Determine if $f(x) = \frac{x}{x^2-1}$ is odd, even or neither

Solution:

$$\begin{aligned} f(-x) &= \frac{-x}{(-x)^2 - 1} \\ &= -\frac{x}{x^2 - 1} \end{aligned}$$

$$-f(x) = -\frac{x}{x^2 - 1}$$

Hence, since $f(-x) = -f(x)$:

$$\therefore f(x) \; is \; odd$$

Example 5: Determine if $f(x) = (x - 2)^2$ is odd, even or neither

Solution:

$$\begin{aligned} f(-x) &= (-x - 2)^2 \\ &= (-1)^2(x + 2)^2 \\ &= (x + 2)^2 \end{aligned}$$

$$-f(x) = -(x - 2)^2$$

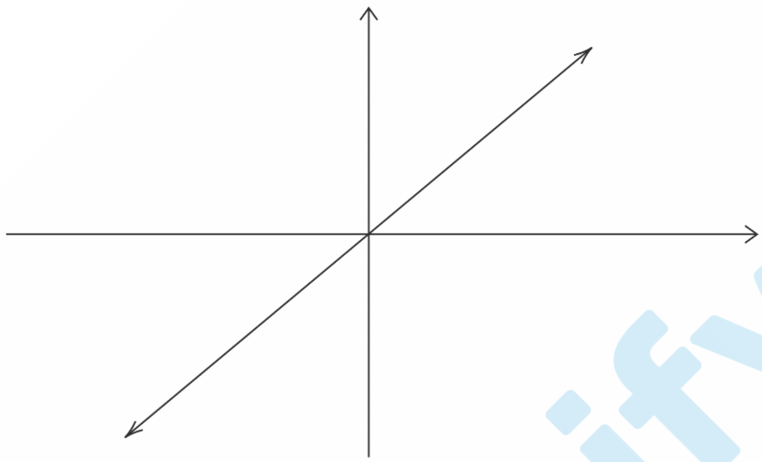
Hence, since $f(x) \neq f(-x) \neq -f(x)$:

$$\therefore f(x) \; is \; neither \; odd \; nor \; even$$

Odd and Even Function Exercises

1. Classify each of the following function, $y = f(x)$ as odd, even or neither

a)



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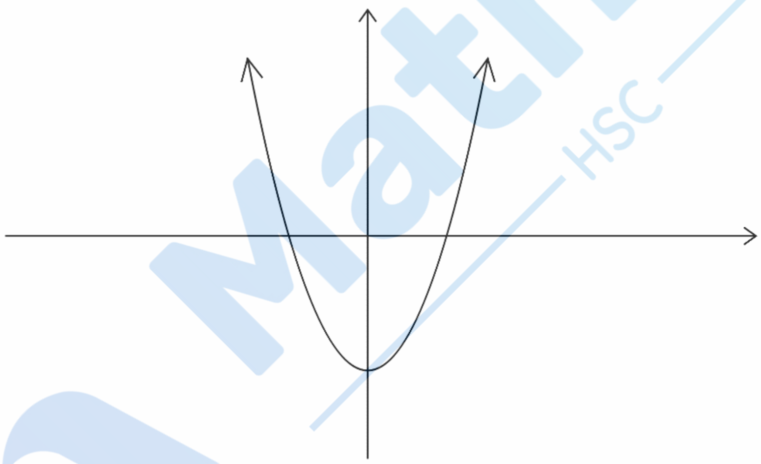
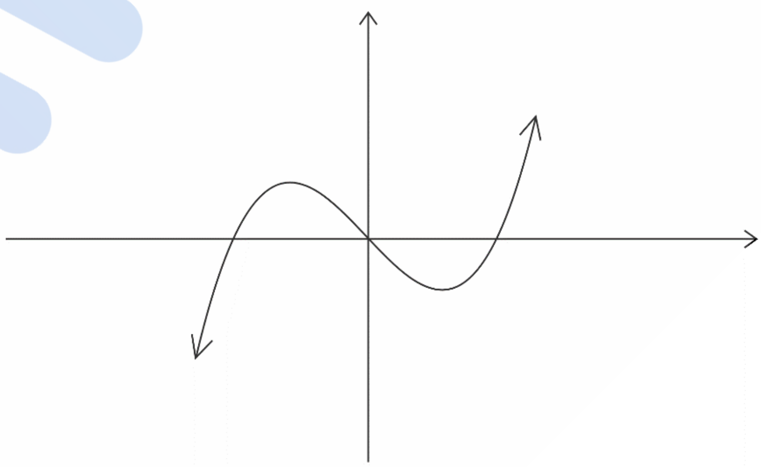
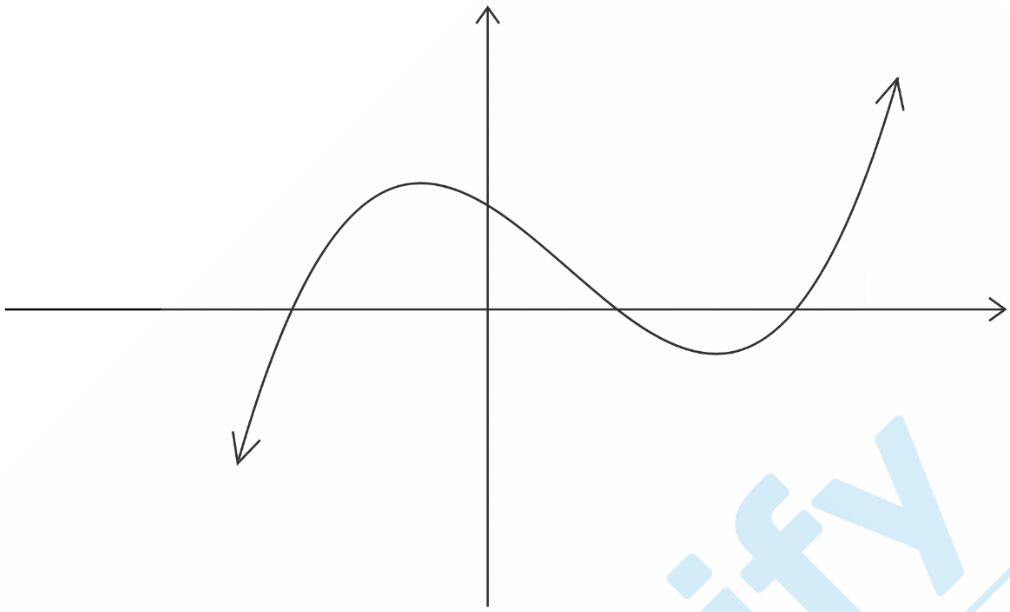


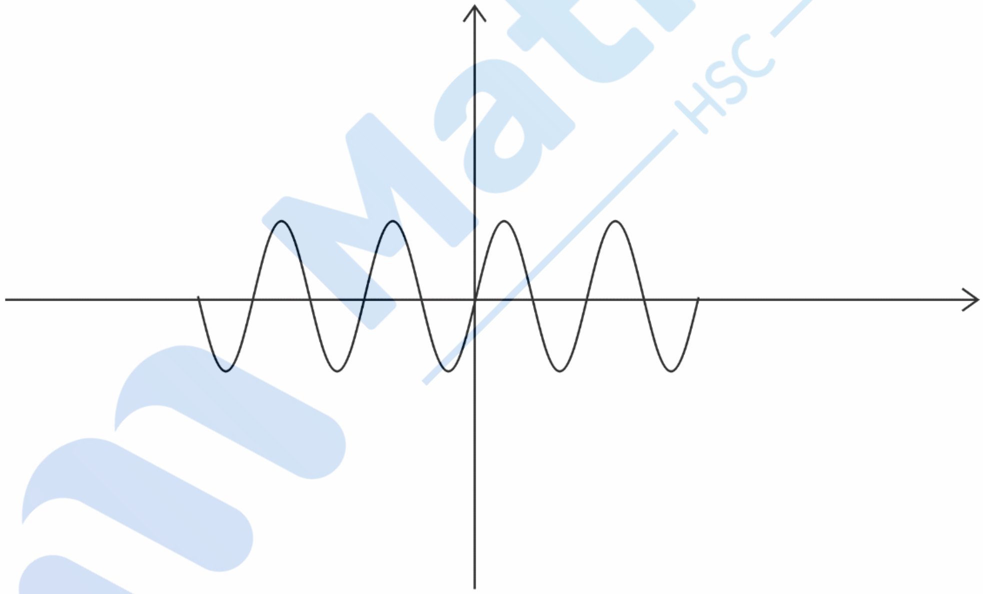
c)



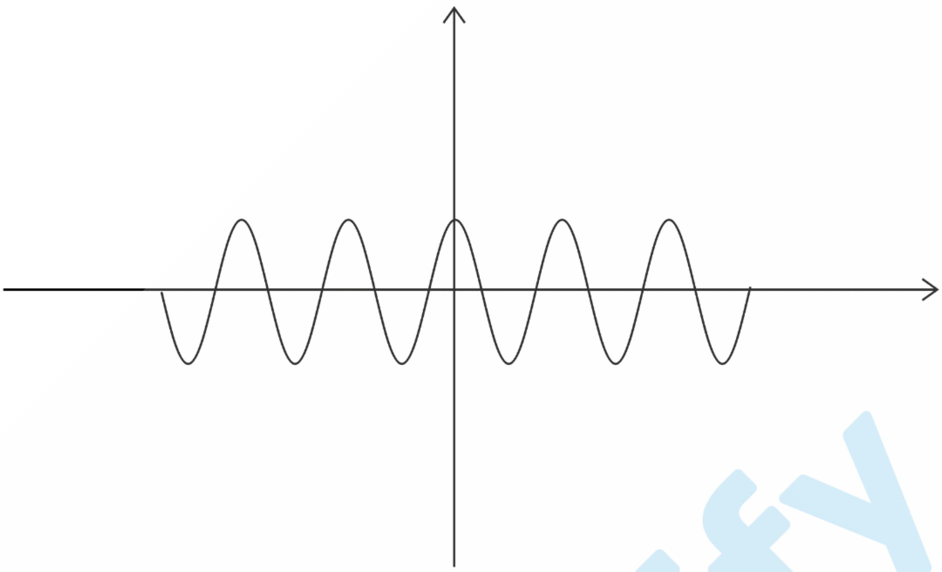
d)



e)



f)



2. Consider the function $f(x) = 2x^4 - x^2 + 7$:
a) Simplify the expression for $f(-x)$
b) Simplify the expression for $-f(x)$
c) Hence or otherwise, determine whether $f(x)$ is odd, even or neither

3. Consider the function $f(x) = 5x^3 - x$
a) Simplify the expression for $f(-x)$
b) Simplify the expression for $-f(x)$
c) Hence or otherwise, determine whether $f(x)$ is odd, even or neither

4. Consider the function $g(x) = -x^4 + 2x^2 - 3x + 5$
a) Simplify the expression for $g(-x)$
b) Simplify the expression for $-g(x)$
c) Hence or otherwise, determine whether $g(x)$ is odd, even or neither

5. Determine whether the following functions are odd, even or neither
a) $f(x) = e^x$
b) $f(x) = 2^x - 2^{-x}$
c) $f(x) = \sqrt{5 - x^2}$
d) $f(x) = \frac{1}{x^2+2}$
e) $f(x) = \frac{x}{x^3+3}$
f) $f(x) = 2^x + x^2$

6. Given that $f(x) = P(x) + Q(x)$, determine what symmetry $f(x0$ will have if:

a) Both $P(x)$ and $Q(x)$ are even
b) Both $P(x)$ and $Q(x)$ are odd
c) One of them is even and the other is odd

Odd and Even Function Exercise Answers

1.
a) Odd
b) Even
c) Odd
d) Neither
e) Odd
f) Even

2.
a) Considering $f(-x)$:

$$\begin{aligned} f(-x) &= 2(-x)^4 - (-x)^2 + 7 \\ &= 2x^4 - x^2 + 7 \end{aligned}$$

b) Considering $-f(x)$:

$$\begin{aligned} -f(x) &= -(2x^4 - x^2 + 7) \\ &= -2x^4 + x^2 - 7 \end{aligned}$$

c) Since it can be observed that $f(x) = f(-x)$:

$$\therefore f(x) \; is \; even$$

3.
a) Considering $f(-x)$:

$$\begin{aligned} f(-x) &= 5(-x)^3 - (-x) \\ &= -5x^3 + x \end{aligned}$$

b) Considering $-f(x)$:

$$\begin{aligned} -f(x) &= -(5x^3 - x) \\ &= -5x^3 + x \end{aligned}$$

c) Since it can be observed that $f(-x) = -f(x)$:

$$\therefore f(x) \; is \; odd$$

4.
a) Considering $g(-x)$:

$$g(-x) = -(-x)^4 + 2(-x)^2 - 3(-x) + 5$$
$$= -x^4 + 2x^2 + 3x + 5$$

b) Considering $-g(x)$:

$$-g(x) = -(-x^4 + 2x^2 - 3x + 5)$$
$$= x^4 - 2x^2 + 3x - 5$$

c) Since it can be observed that $g(x) \neq g(-x) \neq -g(x)$:

$$\therefore g(x) \textit{ is neither even nor odd}$$

5.

a) $f(x) = e^x$

$$f(-x) = e^{-x}$$

$$-f(x) = -e^x$$

Therefore since $f(x) \neq f(-x) \neq -f(x)$:

$$f(x) \textit{ is neither odd nor even}$$

b) $f(x) = 2^x - 2^{-x}$

$$f(-x) = 2^{-x} - 2^x$$

$$-f(x) = -(2^x - 2^{-x})$$
$$= -2^x + 2^{-x}$$

Therefore, since $-f(x) = f(-x)$:

$$f(x) \textit{ is odd}$$

c) $f(x) = \sqrt{5 - x^2}$

$$f(-x) = \sqrt{5 - (-x)^2}$$
$$= \sqrt{5 - x^2}$$

Therefore, since $f(x) = f(-x)$:

$$f(x) \textit{ is even}$$

d) $f(x) = \frac{1}{x^2+2}$

$$f(-x) = \frac{1}{(-x)^2 + 2}$$
$$= \frac{1}{x^2 + 2}$$

Therefore, since $f(x) = f(-x)$:

$$f(x) \textit{ is even}$$

e) $f(x) = \frac{x}{x^3+3}$

$$f(-x) = \frac{-x}{(-x)^3+3}$$
$$= \frac{-x}{-x^3+3}$$
$$-f(x) = -\frac{x}{x^3+3}$$

Therefore, since $f(x) \neq f(-x) \neq -f(x)$:

$$f(x) \text{ is neither odd nor even}$$

f) $f(x) = 2^x + x^2$

$$f(-x) = 2^{-x} + (-x)^2$$
$$= 2^{-x} + x^2$$
$$-f(x) = -(2^x + x^2)$$
$$= -2^x - x^2$$

Therefore, since $f(x) \neq f(-x) \neq -f(x)$:

$$f(x) \text{ is neither odd nor even}$$

6.
a)

If $P(x)$ and $Q(x)$ are both even:

$$\therefore P(x) = P(-x)$$
$$\therefore Q(x) = Q(-x)$$

Therefore, considering $f(-x)$:

$$f(-x) = P(-x) + Q(-x)$$
$$= P(x) + Q(x)$$
$$= f(x)$$
$$\therefore f(x) \text{ will be even}$$

b)

If $P(x)$ and $Q(x)$ are both odd:

$$\therefore P(-x) = -P(x)$$
$$\therefore Q(-x) = -Q(x)$$

Therefore, considering $f(-x)$:

$$f(-x) = P(-x) + Q(-x)$$

$$= -P(x) - Q(x)$$
$$= -\big(P(x) + Q(x)\big)$$
$$= -f(x)$$

$$\therefore f(x) \text{ will be odd}$$

c)

If one of $P(x)$ and $Q(x)$ are odd and the other is even:

$$\therefore P(-x) = P(x)$$

$$\therefore Q(-x) = -Q(x)$$

Therefore, considering $f(-x)$:

$$f(-x) = P(-x) + Q(-x)$$
$$= P(x) - Q(x)$$

Considering $-f(x)$:

$$-f(x) = -\big(P(x) + Q(x)\big)$$
$$= -P(x) - Q(x)$$

Hence, since $f(x) \neq f(-x) \neq -f(x)$:

$$\therefore f(x) \text{ will be neither}$$