TRIGONOMETRY

QUADRANTS & RELATED ANGLES (II)

Contents include:

- The Unit Circle
- Quadrants in the Cartesian Plane
- Related Acute Angles
- Signs of Trigonometric Functions
- Given one Trig Function, Find the Other
- Negative Angles
- Angles of Multiple Revolutions

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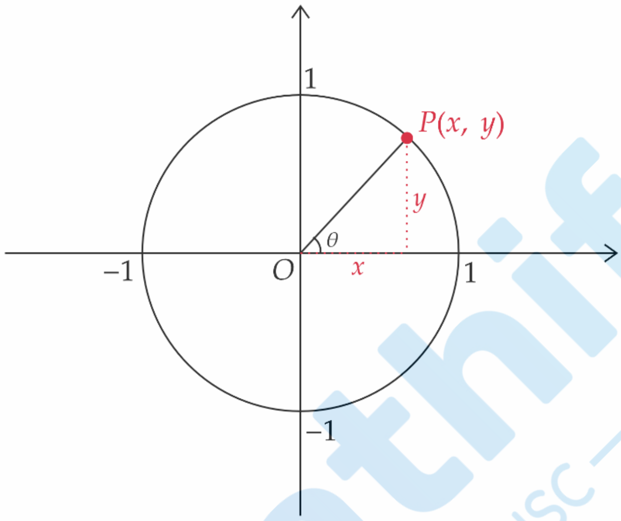
- The Unit Circle

Consider a circle of radius 1, whose centre is at the origin. The equation of this circle would be:

$$x^2 + y^2 = 1$$

Now take any point P on the circumference of the circle whose coordinates are (x, y) and consider the angle θ that interval OP makes with the positive $x-$ axis as shown below:



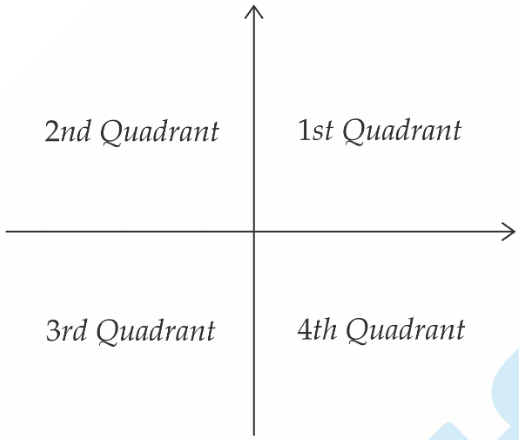
Since the hypotenuse (OP) of the triangle is equal to 1, we therefore can say:

$$\cos\theta = \frac{x}{1} = x$$

$$\sin\theta = \frac{y}{1} = y$$

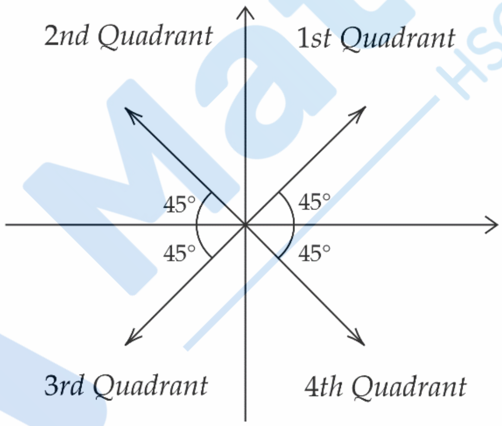
- Quadrants in the Cartesian Plane

The cartesian plane can be split up into 4 quadrants that are in **anti – clockwise order**:



- Related Acute Angles

The diagram below shows 4 rays that correspond to 4 angles:

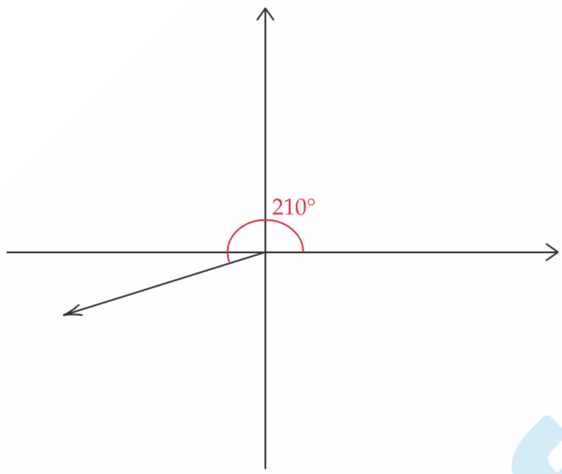


Notice that although the four rays are different, they make the same angle to the $x-$ axis.

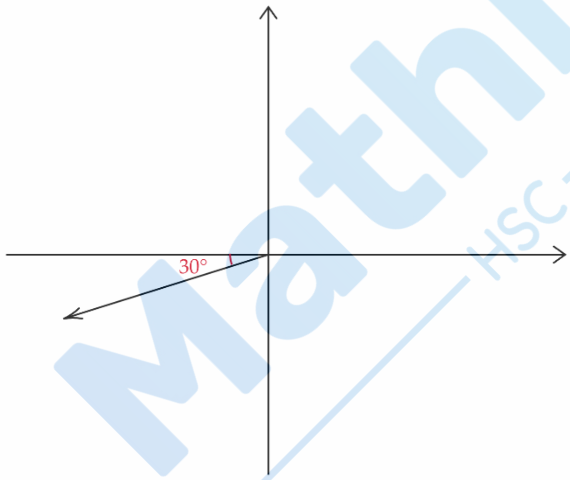
Thus, it is said that they have the same related angle, where:

$$related\ angle\ of\ \theta\ means\ the\ angle\ between\ a\ ray\ and\ the\ x-axis$$

For example, take the ray with an angle of $210°$ as shown:



This ray is also said to have a related angle of $30°$ because:



Thus, the actual angle of the ray is said to be $210°$, but its related angle is $30°$!

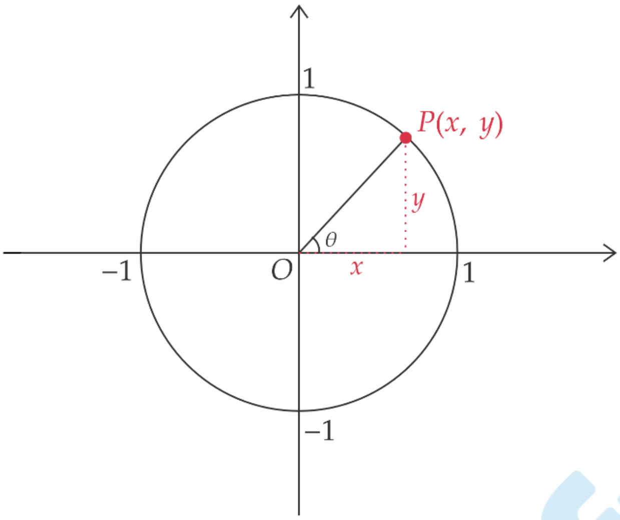
- Signs of Trigonometric Functions

Now comes one of the harder parts of Trigonometry. We will first explain the concept through using unit circles, but a summary will be included afterwards to memorise:

Considering the unit circle quadrants:

$$\textbf{\textit{First Quadrant:}}\ \mathbf{0° < \theta < 90°}$$

Consider the point P which lies on the unit circle in the first quadrant and has an angle of θ:



This is our most normal scenario, where:

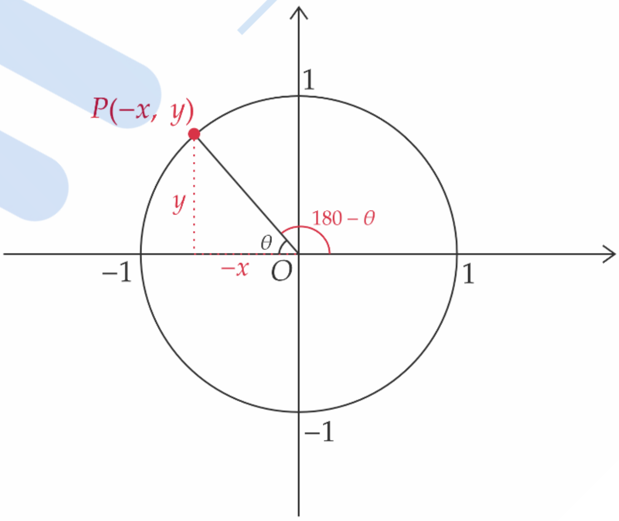
$$\cos\theta = x$$

$$\sin\theta = y$$

$$\tan\theta = \frac{y}{x}$$

$\boldsymbol{Second\ Quadrant}$: $\mathbf{90° < \theta < 180°}$

Now instead, consider the point P which is now in the second quadrant but still has a **related angle of $\boldsymbol{\theta}$**. This means that the actual angle of ray OP is $180° - \theta$, as shown below:



Notice now however, that the coordinate of P is $(-x, y)$

Hence:

$$\cos(180 - \theta) = -x = -\cos\theta$$
$$\sin(180 - \theta) = y = \sin\theta$$
$$\tan(180 - \theta) = \frac{y}{-x} = -\tan\theta$$

Example 1: Find the exact value of $\cos 150°$

Solution:

Notice that $150°$ may be written as:

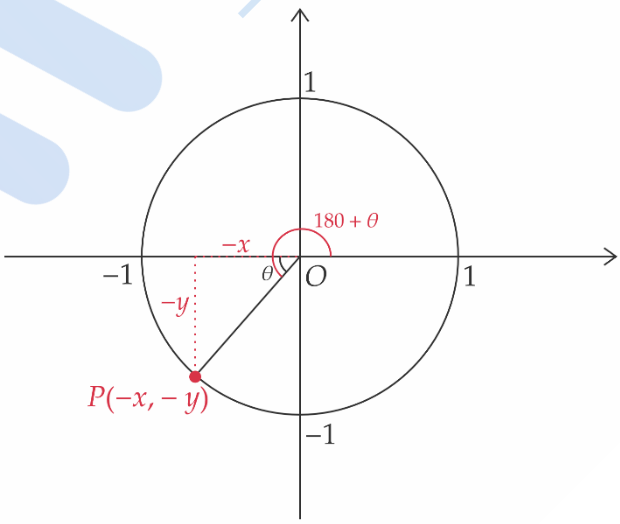
$$150° = 180° - 30°$$

Hence, since we know that $\cos(180 - \theta) = -\cos\theta$:

$$\therefore \cos 150° = -\cos 30°$$
$$= -\frac{\sqrt{3}}{2}$$

$\boldsymbol{Third\ Quadrant}$: $\mathbf{180° < \theta < 270°}$

Once again, consider the point P which is now in the third quadrant but still has a **related angle of $\boldsymbol{\theta}$**. This means that the actual angle of ray OP is $180° + \theta$, as shown below:



Notice now that the coordinate of P is $(-x, -y)$

Hence:

$$\cos(180+\theta) = -x = -\cos\theta$$

$$\sin(180+\theta) = -y = -\sin\theta$$

$$\tan(180+\theta) = \frac{-y}{-x} = \tan\theta$$

Example 2: Find the exact value of $\sin 210°$

Solution:

Notice that $210°$ may be written as:

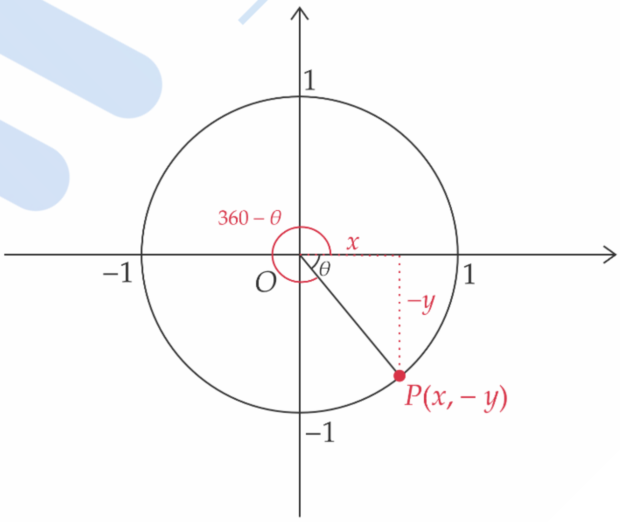
$$210 = 180 + 30$$

Hence, since we now know that $\sin(180+\theta) = -\sin\theta$:

$$\therefore \sin(180+30) = -\sin 30°$$
$$= -\frac{1}{2}$$

$\boldsymbol{Fourth\ Quadrant}: \mathbf{270° < \theta < 360°}$

Once again, consider the point P which is now in the fourth quadrant but still has a **related angle of $\boldsymbol{\theta}$**. This means that the actual angle of ray OP is $360° - \theta$, as shown below:



Hence:

$$\cos(360 - \theta) = x = \cos\theta$$

$$\sin(360 - \theta) = -y = -\sin\theta$$

$$\tan(360 - \theta) = \frac{-y}{x} = -\tan\theta$$

Example 3: Find the exact value of $\tan 300°$

Solution:

Notice that $300°$ may be written as:

$$300 = 360 - 60$$

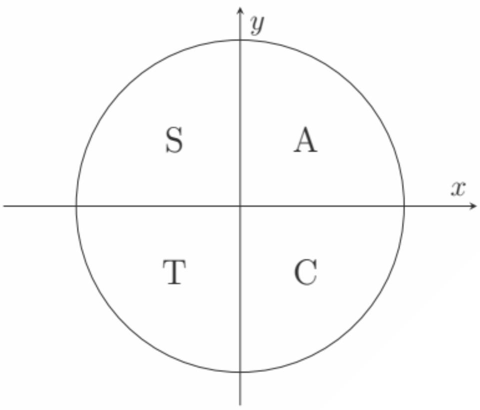
Hence, since we now know that $\tan(360 - \theta) = -\tan\theta$:

$$\therefore \tan(360 - 60) = -\sin 60°$$
$$= -\frac{\sqrt{3}}{2}$$

- Sign of the Trigonometric Ratios Summarised

The sign of cos, sin and tan for the four quadrants of the unit circle can be summarized as follows:

First Quadrant: *All are Positive* (A)
Second Quadrant: *Sin only is positive* (S)
Third Quadrant: *Tan only is positive* (T)
Fourth Quadrant: *Cos only is positive* (C)



A helpful mnemonic to remember this by is:

All Stations To Central

Moreover, if you are given a trig function with angle θ and want to determine which quadrant θ is in, remember that:

- If $0° < \theta < 90°$

$$\therefore \theta \text{ is in } 1st \text{ quadrant}$$

This is the normal case

- If $90° < \theta < 180°$

$$\therefore \theta \text{ is in } 2nd \text{ quadrant}$$

Hence, we should consider $180° - \theta$

- If $180° < \theta < 270°$

$$\therefore \theta \text{ is in } 3rd \text{ quadrant}$$

Hence, we should consider $180° + \theta$

- If $270° < \theta < 360°$

$$\therefore \theta \text{ is in } 4th \text{ quadrant}$$

Hence, we should consider $360° - \theta$

- Given one trigonometric function, find another

For these types of questions, we are given the value of one trigonometric function. **For example:**

$$\tan\theta = \frac{5}{11}$$

The question will then ask us to find the value of another trig function. In this case, let's say we are asked to find the exact value of $\sin\theta$.

Since we are dealing with exact values, we **do not use our calculator** to find the value of θ first then just substitute it back into the other trig function. Doing it this way will not get us an exact value, and thus is **wrong!**

Instead, we want to make use of our knowledge surrounding trigonometric ratios, quadrants and Pythagoras Theorem. The steps for these questions are highlighted in the next page:

$$\boldsymbol{Step\ 1: Sketch\ the\ right\ angle\ triangle}$$

With θ as an acute angle, label the sides of the triangle as appropriate with the trig ratio (SOHCAHTOA).

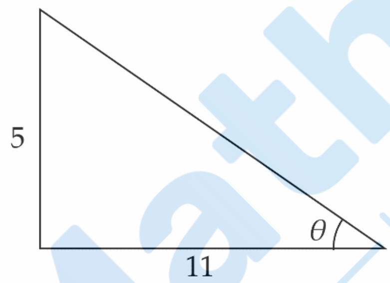
$$\boldsymbol{Step\ 2: Use\ Pythagoras\ theorem\ to\ find\ the\ length\ of\ the\ 3rd\ side}$$

$$\boldsymbol{Step\ 3: Find\ your\ trig\ ratio\ and\ answer}!$$

Thus, applying these steps to our example question, we know that:

$$\tan\theta = \frac{5}{11}$$

Thus, since $\tan\theta = \frac{opposite}{adjacent}$, drawing our right – angled triangle:



Applying Pythagoras theorem, we can determine the length of the hypotenuse:

$$\begin{aligned} hypotenuse &= \sqrt{5^2 + 11^2} \\ &= \sqrt{25 + 121} \\ &= \sqrt{146} \end{aligned}$$

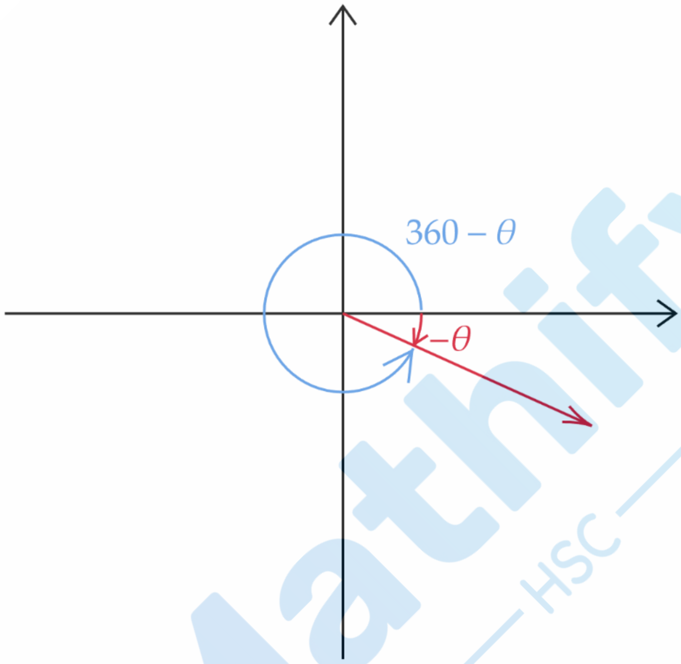
Hence, if we want the exact value of $\sin\theta$:

$$\begin{aligned} \therefore \sin\theta &= \frac{opposite}{hypotenuse} \\ &= \frac{5}{\sqrt{146}} \end{aligned}$$

- Negative Angles

If given a negative angle for a trig function, we can always convert it into an equivalent positive angle.

Letting the negative angle be $-\theta$, then the equivalent positive angle to it will be $360 - \theta$. This is shown in the diagram below:



Notice here that **negative angles** are illustrated as angles in a **clockwise** direction, whereas **positive angles** are illustrated in the **anti – clockwise** direction!

Note: You can either remember the formula as $360° - \theta$ OR $-\theta + 360°$.

Example 4: Evaluate the following in exact values:

a) $\sin(-45°)$
b) $\tan(-135°)$
c) $\cos(-225°)$

Solutions:

a) Following our above equation:

$$\begin{aligned}\sin(-45°) &= \sin(360° - 45°)\\ &= \sin(315°)\\ &= -\sin(45°)\ \ [4th\ quadrant]\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

b) Following our above equation:

$$\begin{aligned}\tan(-135°) &= \tan(360° - 135°)\\ &= \tan(225°)\\ &= \tan(180° + 45°)\\ &= \tan 45° \ \ [3rd\ quadrant]\\ &= 1\end{aligned}$$

c) Following our above equation:

$$\begin{aligned}\cos(-225°) &= \cos(360° - 225°)\\ &= \cos(135°)\\ &= \cos(180° - 45°)\\ &= -\cos 45° \ \ [2nd\ quadrant]\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

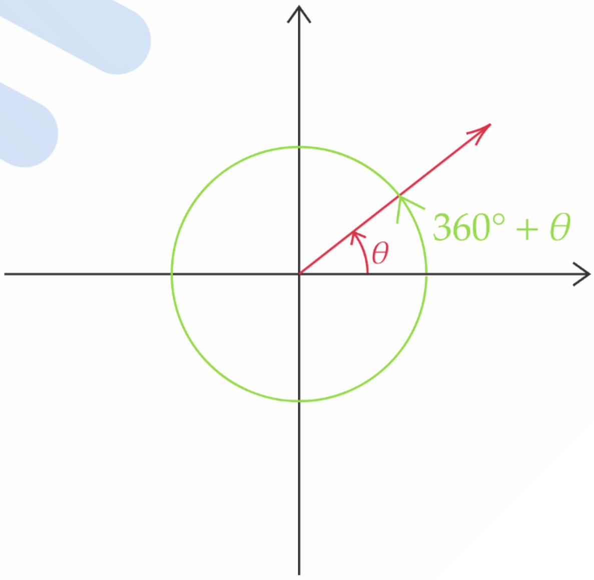
- Angles of Multiple Revolutions (bigger than 360°)

Put very simply, we should remember that for trig functions:

$$\theta = \theta + 360° = \theta + 720° = \theta + 1080° = \cdots$$

In other words, $\sin 30° = \sin 390° = \sin 750° = \frac{1}{2}$

This is because after 360°, the angle ends up back at the start and thus repeats itself every complete revolution. This concept is best understood through the diagram below:



Example 5: Evaluate the following in exact values:

a) $\sin 405°$
b) $\tan 945°$
c) $\cos -540°$

Solutions:

a) Subtracting $360°$ from the angle:

$$\begin{aligned}\sin 405° &= \sin(405° - 360°)\\ &= \sin 45°\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

b) Subtracting by $720°$ from the angle:

$$\begin{aligned}\tan 945° &= \tan(945° - 720°)\\ &= \tan 225°\\ &= \tan(180° + 45°)\\ &= \tan 45° \;\; [3rd\ quadrant]\\ &= 1\end{aligned}$$

c) This time, adding $360°$ to the angle:

$$\begin{aligned}\cos -540° &= \cos(-540 + 360°)\\ &= \cos -180°\\ &= \cos 180°\\ &= -1\end{aligned}$$

Quadrants and Related Angle Exercises

1. Write down the exact values of:

a) $\sin 120°$
b) $\cos 90°$
c) $\sec 135°$
d) $cosec\ 270°$
e) $\cot 240°$

2. Write down the exact values of:

a) $\sin 390°$
b) $\sec 405°$
c) $\tan 420°$
d) $\cos 495°$
e) $\cot 690°$

3. Write down the exact values of:

a) $\sin -60°$

b) $\cos -240°$

c) $\cot -210°$

d) $\sec -120°$

e) $cosec - 420°$

4. Find all the values of θ between $0°$ and $360°$ for which:

a) $\sin\theta = -\frac{\sqrt{3}}{2}$

b) $\cot\theta = \sqrt{3}$

c) $\sec\theta = -\sqrt{2}$

d) $2\cos\theta + 1 = 0$

e) $\sin\theta + \sqrt{3}\cos\theta = 0$

f) $cosec\,\theta = \sec\theta$

Quadrants and Related Angle Exercise Answers

1.

a) $\sin 120° = \sin(180° - 60°)$

$$= \sin 60° \ \ (because\ in\ 2nd\ quadrant)$$

b) $\cos 90° = 0$

c) $\sec 135° = \frac{1}{\cos 135°}$

$$= \frac{1}{\cos(180° - 45°)}$$

$$= \frac{1}{-\cos 45°} \ (because\ in\ 2nd\ quadrant)$$

$$= -\sqrt{2}$$

d) $cosec\ 270° = \frac{1}{\sin 270°}$

$$= \frac{1}{-1}$$

$$= -1$$

e) $\cot 240° = \frac{1}{\tan 240°}$

$$= \frac{1}{\tan(180° + 60°)}$$

$$= \frac{1}{\tan 60°}$$

$$= \frac{1}{\sqrt{3}}$$

2.

a) $\sin 390° = \sin(360° + 30°)$

$$= \sin 30° \ \ (\textit{since full revolution is } 360°)$$
$$= \frac{1}{2}$$

b) $\sec 405° = \frac{1}{\cos 405°}$

$$= \frac{1}{\cos(360° + 45°)} \ (\textit{since full revolution is } 360°)$$
$$= \frac{1}{\cos 45°}$$
$$= \sqrt{2}$$

c) $\tan 420° = \tan(360° + 60°)$

$$= \tan 60° \ \ (\textit{since full revolution is } 360°)$$
$$= \sqrt{3}$$

d) $3\cos 495° = \cos(360° + 135°)$

$$= \cos 135° \ \ (\textit{since full revolution is } 360°)$$
$$= \cos(180° - 45°)$$
$$= -\cos 45° \ \ (\textit{because in } 2\textit{nd quadrant})$$
$$= -\frac{1}{\sqrt{2}}$$

e) $\cot 690° = \frac{1}{\tan 690°}$

$$= \frac{1}{\tan(360° + 330°)} \ \ (\textit{since full revolution is } 360°)$$
$$= \frac{1}{\tan 330°}$$
$$= \frac{1}{\tan(360° - 30°)}$$
$$= -\tan 30° \ \ (\textit{because in } 4\textit{th quadrant})$$
$$= \frac{1}{\sqrt{3}}$$

3.

a) $\sin -60° = \sin 300° \ \ (\textit{add } 360° \textit{ to make a full revolution})$

$$= \sin(360° - 60°)$$

$$= -\sin 60° \ \ (\textit{because in } 4\textit{th quadrant})$$
$$= -\frac{\sqrt{3}}{2}$$

b) $\cos -240° = \cos 120°$ $(add\ 360°\ to\ make\ a\ full\ revolution)$

$$= -\cos 60° \ \ (because\ in\ 2nd\ quadrant)$$
$$= -\frac{1}{2}$$

c) $\cot -210° = \cot 150°$ $(add\ 360°\ to\ make\ a\ full\ revolution)$

$$= \frac{1}{\tan 150°}$$
$$= \frac{1}{-\tan 30°} \ (because\ in\ 2nd\ quadrant)$$
$$= -\sqrt{3}$$

d) $\sec -120° = \sec 240°$ $(add\ 360°\ to\ make\ a\ full\ revolution)$

$$= \frac{1}{\cos 240°}$$
$$= \frac{1}{-\cos 60°}$$
$$= -2$$

e) $cosec - 420° = cosec\ 300°$ $(add\ 720°\ to\ make\ two\ full\ revolutions)$

$$= \frac{1}{\sin 300\ °}$$
$$= \frac{1}{-\sin 60°}$$
$$= -\frac{2}{\sqrt{3}}$$

4. A

a) $\sin\theta = -\frac{\sqrt{3}}{2}$

Since $\sin\theta < 0$, then we know that θ must be in the 3rd or 4th quadrant

$$Normally, \sin 60° = \frac{\sqrt{3}}{2}$$

$$\therefore \theta = 180° + 60°\ OR\ 360° - 60°$$
$$= 240°\ OR\ 300°$$

b) $\cot\theta = \sqrt{3}$

$$Step\ 1{:}\ Convert\ to\ in\ terms\ of\ \tan\theta$$

$$\frac{1}{\tan\theta} = \sqrt{3}$$
$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$Step\ 2{:}\ Solve\ for\ \theta$$

Since $\tan\theta > 0$, then we know that θ must be in the 1st or 3rd quadrant

$$\therefore \theta = 30^\circ \ OR\ 180^\circ + 30^\circ$$
$$= 30^\circ \ OR\ 210^\circ$$

c) $\sec\theta = -\sqrt{2}$

$$Step\ 1: Convert\ to\ in\ terms\ of\ \cos\theta$$

$$\frac{1}{\cos\theta} = -\sqrt{2}$$
$$\cos\theta = -\frac{1}{\sqrt{2}}$$

$$Step\ 2: Solve\ for\ \theta$$

$$Normally, \cos 45^\circ = \frac{1}{\sqrt{2}}$$

Since $\cos\theta < 0$, then we know that θ must be in the 2nd or 3rd quadrant

$$\therefore \theta = 180^\circ - 45^\circ \ OR\ 180^\circ + 45^\circ$$
$$= 135^\circ \ OR\ 225^\circ$$

d) $2\cos\theta + 1 = 0$

$$Step\ 1: Rearrange\ our\ equation\ to\ make\ \cos\theta\ \ the\ subject$$

$$2\cos\theta = -1$$
$$\cos\theta = -\frac{1}{2}$$

$$Step\ 2: Solve\ for\ \theta$$

Since $\cos\theta < 0$, then we know that θ must be in the 2nd or 3rd quadrant

$$Normally, \cos 60^\circ = \frac{1}{2}$$

$$\therefore \theta = 180^\circ - 60^\circ \ OR\ 180^\circ + 60^\circ$$
$$= 120^\circ \ OR\ 240^\circ$$

e) $\sin\theta + \sqrt{3}\cos\theta = 0$

$$Step\ 1: Rearrange\ our\ equation\ to\ make\ a\ trig\ function\ the\ subject$$

Notice here how we have two trig functions, $\sin\theta\ \ and\ \cos\theta$. In order to solve for θ, we must consider reducing our expression to only have one trig function. Thus, we should consider forming a $\tan\theta$ since $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\sin\theta = -\sqrt{3}\cos\theta$$
$$\frac{\sin\theta}{\cos\theta} = -\sqrt{3}$$
$$\tan\theta = -\sqrt{3}$$

$$Step\ 2: Solve\ for\ \theta$$

Since $\tan\theta < 0$, then we know that θ must be in the 2nd and 4th quadrant

$$Normally, \tan 60° = \sqrt{3}$$

$$\therefore \theta = 180° - 60°\ OR\ 360° - 60°$$
$$= 120°\ OR\ 300°$$

f) $cosec\ \theta = \sec\theta$

$Step\ 1: Rearrange\ our\ equation$

Whenever we encounter $\sec\theta\ \ and\ cosec\ \theta$, we want to convert it to in terms of $\cos\theta\ \ or \sin\theta$

$$\frac{1}{\sin\theta} = \frac{1}{\cos\theta}$$
$$\frac{\sin\theta}{\cos\theta} = 1$$
$$\tan\theta = 1$$

$$Step\ 2: Solve\ for\ \theta$$

Since $\tan\theta > 0$, we know that θ must be in the 1st or 3rd

$$\therefore \theta = 45°\ OR\ 180° + 45°$$
$$= 45°\ OR\ 225°$$