TRIGONOMETRY

TRIGONOMETRIC IDENTITIES (VII)

Contents include:

- The Pythagorean Identities
- Complementary Angle Properties

- The Pythagorean Identities

The Pythagorean identities are the 3 most important identities you will encounter in trigonometry. In general, we must be able to recognize them, as well as recognize when we must apply them for questions. When you do apply them, make sure to state it as part of your working out!

The Pythagorean identities are as follows:

$$(1)\ \sin^2\theta + \cos^2\theta = 1$$

$$(2)\ 1 + \tan^2\theta = \sec^2\theta$$

$$(3)\ 1 + \cot^2\theta = cosec^2\theta$$

It is also important to note the different variations of these identities:

$$From\ (1), \cos^2\theta = 1 - \sin^2\theta$$
$$\sin^2\theta = 1 - \cos^2\theta$$

$$From\ (2), \tan^2\theta = \sec^2\theta - 1$$

$$From\ (3), \cot^2\theta = cosec^2\theta - 1$$

Example 1: Simplify $\frac{\sin^2\theta+\cos^2\theta}{\sin\theta}$

Solution:

Since $\sin^2\theta + \cos^2\theta = 1$:

$$\frac{\sin^2\theta + \cos^2\theta}{\sin\theta} = \frac{1}{\sin\theta}$$
$$= cosec\ \theta$$

Example 2: Simplify $\frac{\sin^2\theta+\cos^2\theta}{\sec^2\theta-1}$

Solution:

Since $\sin^2\theta + \cos^2\theta = 1$ and $\tan^2\theta = \sec^2\theta - 1$:

$$\therefore \frac{\sin^2\theta + \cos^2\theta}{\sec^2\theta - 1} = \frac{1}{\tan^2\theta}$$
$$= \cot^2\theta$$

- Complementary Angle Properties

The complementary angle properties essentially allow us to change trig functions under certain conditions.

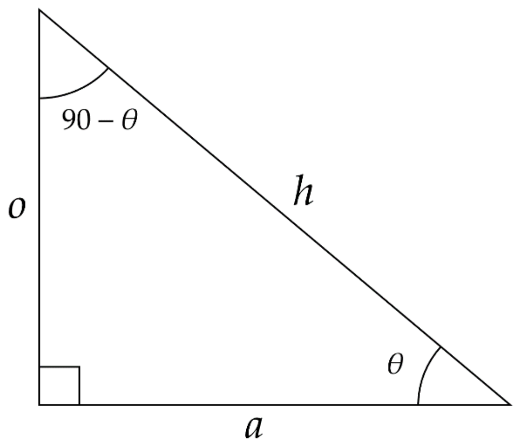
There are 3 properties in total which are summarised below:

$$\cos(90° - \theta) = \sin\theta$$

$$\sin(90° - \theta) = \cos\theta$$

$$\tan(90° - \theta) = \cot\theta$$

The proof for this can be investigated through a right – angled triangle:



It can be observed that:

$$\sin\theta = \frac{o}{h}$$

At the same time though:

$$\cos(90 - \theta) = \frac{o}{h}$$

$$\therefore \cos(90 - \theta) = \sin\theta$$

Similarly:

$$\sin(90 - \theta) = \cos\theta = \frac{a}{h}$$

$$\tan(90 - \theta) = \cot\theta = \frac{a}{0}$$

Hence, proven!

Example 3: Use the complementary identities to simplify $\frac{\cos(90°-\theta)}{\sin(90°-\theta)}$

Solution:

Since $\cos(90° - \theta) = \sin\theta$ and $\sin(90° - \theta) = \cos\theta$,

$$\begin{aligned}\therefore \frac{\cos(90° - \theta)}{\sin(90° - \theta)} &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta\end{aligned}$$

Trigonometric Identity Exercises

1. Prove the following identities:

a) $(1 + \tan^2\theta)\cos^2\theta = 1$

b) $\tan^2\gamma\cos^2\gamma + \cot^2\gamma\sin^2\gamma = 1$

c) $3\cos^2\alpha - 2 = 1 - 3\sin^2\alpha$

d) $\cot\theta\,(\sec^2\theta - 1) = \tan\theta$

e) $2\tan^2 A - 1 = 2\sec^2 A - 3$

2. Prove the following identities:

a) $cosec\ \emptyset - \sin\emptyset = \cos\emptyset\cot\emptyset$

b) $\tan\theta + \cot\theta = cosec\theta\sec\theta$

c) $\cot B\sec B = cosec\ B$

d) $\sec\emptyset - \cos\emptyset = \tan\emptyset\sin\emptyset$

3. Prove the following identities:

a) $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

b) $\sin\alpha + \cot\alpha\cos\alpha = cosec\ \alpha$

c) $\frac{1}{1+\sin A} + \frac{1}{1-\sin A} = 2\sec^2 A$

d) $\frac{1}{\sec\gamma-\tan\gamma} - \frac{1}{\sec\gamma+\tan\gamma} = 2\tan\gamma$

4. Prove that each expression is equal to a constant:

a) $\frac{\cos^2\beta}{1+\sin\beta} + \frac{\cos^2\beta}{1-\sin\beta}$

b) $\frac{\tan\delta+\cot\delta}{\sec\delta\, cosec\ \delta}$

c) $\tan\theta\,(1 - \cot^2\theta) + \cot\theta\,(1 - \tan^2\theta)$

d) $\frac{\tan\alpha+1}{\sec\alpha} - \frac{\cot\alpha+1}{cosec\ \alpha}$

Trigonometric Identity Exercise Answers

1. For these prove questions, the aim is to show that $LHS = RHS$

a) $(1+\tan^2\theta)\cos^2\theta = 1$

$$\begin{aligned} LHS &= (1+\tan^2\theta)\cos^2\theta \\ &= \cos^2\theta + \cos^2\theta\tan^2\theta \\ &= \cos^2\theta + \cos^2\theta \times \frac{\sin^2\theta}{\cos^2\theta} \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

b) $\tan^2\gamma\cos^2\gamma + \cot^2\gamma\sin^2\gamma = 1$

$$\begin{aligned} LHS &= \tan^2\gamma\cos^2\gamma + \cot^2\gamma\sin^2\gamma \\ &= \frac{\sin^2\gamma}{\cos^2\gamma} \times \cos^2\gamma + \frac{\cos^2\gamma}{\sin^2\gamma} \times \sin^2\gamma \\ &= \sin^2\gamma + \cos^2\gamma \\ &= 1 \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

c) $3\cos^2\alpha - 2 = 1 - 3\sin^2\alpha$

$$\begin{aligned} LHS &= 3\cos^2\alpha - 2 \\ &= 3(1-\sin^2\alpha) - 2 \\ &= 3 - 3\sin^2\alpha - 2 \\ &= 1 - 3\sin^2\alpha \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

d) $\cot\theta(\sec^2\theta - 1) = \tan\theta$

$$\begin{aligned} LHS &= \cot\theta(\sec^2\theta - 1) \\ &= \frac{1}{\tan\theta}(\tan^2\theta) \\ &= \tan\theta \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

e) $2\tan^2 A - 1 = 2\sec^2 A - 3$

$$\begin{aligned} LHS &= 2\tan^2 A - 1 \\ &= 2(\sec^2 A - 1) - 1 \\ &= 2\sec^2 A - 3 \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

2. For these prove questions, the aim is to show that $LHS = RHS$

a) $cosec\,\emptyset - \sin\emptyset = \cos\emptyset\cot\emptyset$

$$\begin{aligned} LHS &= cosec\,\emptyset - \sin\emptyset \\ &= \frac{1}{\sin\emptyset} - \sin\emptyset \\ &= \frac{1-\sin^2\emptyset}{\sin\emptyset} \\ &= \frac{\cos^2\emptyset}{\sin\emptyset} \\ &= \cos\emptyset\cot\emptyset \\ &= RHS \\ \therefore LHS &= RHS \end{aligned}$$

b) $\tan\theta + \cot\theta = cosec\,\theta\sec\theta$

$$\begin{aligned} LHS &= \tan\theta + \cot\theta \\ &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\ &= \frac{1}{\sin\theta\cos\theta} \\ &= cosec\,\theta\sec\theta \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

c) $\cot B\sec B = cosec\ B$

$$\begin{aligned} LHS &= \cot B\sec B \\ &= \frac{\cos B}{\sin B} \times \frac{1}{\cos B} \\ &= \frac{1}{\sin B} \\ &= cosec\ B \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

d) $\sec\emptyset - \cos\emptyset = \tan\emptyset\sin\emptyset$

$$\begin{aligned} LHS &= \sec\emptyset - \cos\emptyset \\ &= \frac{1}{\cos\emptyset} - \cos\emptyset \\ &= \frac{1-\cos^2\emptyset}{\cos\emptyset} \\ &= \frac{\sin^2\emptyset}{\cos\emptyset} \\ &= \tan\emptyset\sin\emptyset \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

3. For these prove questions, the aim is to show that $LHS = RHS$

a) $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

$$\begin{aligned} LHS &= \sin^4\theta - \cos^4\theta \\ &= (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta) \\ &= 1 \times (\sin^2\theta - \cos^2\theta) \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

b) $\sin\alpha + \cot\alpha\cos\alpha = cosec\ \alpha$

$$\begin{aligned} LHS &= \sin\alpha + \cot\alpha\cos\alpha \\ &= \sin\alpha + \frac{\cos\alpha}{\sin\alpha} \times \cos\alpha \\ &= \frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha} \\ &= \frac{1}{\sin\alpha} \\ &= cosec\ \alpha \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

c) $\frac{1}{1+\sin A} + \frac{1}{1-\sin A} = 2\sec^2 A$

$$\begin{aligned} LHS &= \frac{1}{1+\sin A} + \frac{1}{1-\sin A} \\ &= \frac{1-\sin A + 1 + \sin A}{(1+\sin A)(1-\sin A)} \\ &= \frac{2}{1-\sin^2 A} \\ &= \frac{2}{\cos^2 A} \\ &= 2\sec^2 A \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

d) $\frac{1}{\sec\gamma-\tan\gamma} - \frac{1}{\sec\gamma+\tan\gamma} = 2\tan\gamma$

$$\begin{aligned} LHS &= \frac{1}{\sec\gamma - \tan\gamma} - \frac{1}{\sec\gamma + \tan\gamma} \\ &= \frac{\sec\gamma + \tan\gamma - (\sec\gamma - \tan\gamma)}{(\sec\gamma - \tan\gamma)(\sec\gamma + \tan\gamma)} \\ &= \frac{2\tan\gamma}{\sec^2\gamma - \tan^2\gamma} \\ &= 2\tan\gamma \\ &= RHS \end{aligned}$$

$$\therefore LHS = RHS$$

4.

a) $\frac{\cos^2\beta}{1+\sin\beta}+\frac{\cos^2\beta}{1-\sin\beta}$

$$\begin{aligned}\frac{\cos^2\beta}{1+\sin\beta}+\frac{\cos^2\beta}{1-\sin\beta}&=\frac{\cos^2\beta\,(1-\sin\beta)+\cos^2\beta\,(1+\sin\beta)}{(1+\sin\beta)(1-\sin\beta)}\\&=\frac{\cos^2\beta-\sin\beta\cos^2\beta+\cos^2\beta+\sin\beta\cos^2\beta}{1-\sin^2\beta}\\&=\frac{2\cos^2\beta}{\cos^2\beta}\\&=2\end{aligned}$$

b) $\frac{\tan\delta+\cot\delta}{\sec\delta\, cosec\,\delta}$

$$\begin{aligned}\frac{\tan\delta+\cot\delta}{\sec\delta\; cosec\;\delta}&=\frac{\tan\delta+\cot\delta}{\frac{1}{\cos\delta}\times\frac{1}{\sin\delta}}\\&=\cos\delta\sin\delta\,(\tan\delta+\cot\delta)\\&=\cos\delta\sin\delta\left(\frac{\sin\delta}{\cos\delta}+\frac{\cos\delta}{\sin\delta}\right)\\&=\sin^2\delta+\cos^2\delta\\&=1\end{aligned}$$

c) $\tan\theta\,(1-\cot^2\theta)+\cot\theta\,(1-\tan^2\theta)$

$$\begin{aligned}&=\frac{1}{\cot\theta}(1-\cot^2\theta)+\frac{1}{\tan\theta}(1-\tan^2\theta)\\&=\frac{1}{\cot\theta}-\cot\theta+\frac{1}{\tan\theta}-\tan\theta\\&=\tan\theta-\cot\theta+\cot\theta-\tan\theta\\&=0\end{aligned}$$

d) $\frac{\tan\alpha+1}{\sec\alpha}-\frac{\cot\alpha+1}{cosec\,\alpha}$

$$\begin{aligned}\frac{\tan\alpha+1}{\sec\alpha}-\frac{\cot\alpha+1}{cosec\;\alpha}&=\frac{\tan\alpha+1}{\frac{1}{\cos\alpha}}-\frac{\cot\alpha+1}{\frac{1}{\sin\alpha}}\\&=\cos\alpha\,(\tan\alpha+1)-\sin\alpha\,(\cot\alpha+1)\\&=\cos\alpha\left(\frac{\sin\alpha}{\cos\alpha}+1\right)-\sin\alpha\left(\frac{\cos\alpha}{\sin\alpha}+1\right)\\&=\sin\alpha+\cos\alpha-\cos\alpha-\sin\alpha\\&=0\end{aligned}$$