Mathify HSC

EXPONENTIALS & LOGARITHMS

LOGARITHM LAWS (III)

Contents include:

- Adding Logarithms
- Subtracting Logarithms
- The Log of a Power
- Change of Base Law
- Exponential of a Logarithm

- Adding Logarithms

When adding logarithms with the same base, the expressions inside will multiply. This is shown below:

Must have the same base! *multiplied our expressions inside log!*

$$\underbrace{\log_a m + \log_a n}_{\text{Adding logarithms}} = \log_a mn$$

For example:

$$\begin{aligned}\log_2 8 + \log_2 16 &= \log_2 8 \times 16 \\ &= \log_2 128 \\ &= 7\end{aligned}$$

- Subtracting Logarithms

When subtracting logarithms with the same base, the expressions inside will multiply. This is shown below:

Must have the same base! *Divided our expressions inside log!*

$$\underbrace{\log_a m - \log_a n}_{\text{Subtracting logarithms}} = \log_a \frac{m}{n}$$

For example:

$$\begin{aligned}\log_3 81 - \log_3 27 &= \log_3 \frac{81}{27} \\ &= \log_3 3 \\ &= 1\end{aligned}$$

- The Log of a Power

When we log a power, we can bring the index of the power outside to the front of the logarithm. This is shown below:

Log of a power   *Can now take out n*

$$\log_a(m^n) = n(\log_a m)$$

For example:

$$\begin{aligned}\log_2(4^3) &= 3(\log_2 4)\\ &= 3\times 2\\ &= 6\end{aligned}$$

Moreover, as a consequence of this logarithmic law, we now also know that: $$\log_a \frac{1}{x} = -\log_a x$$

- Change of Base Law

To change the base of a logarithm, we remember to do the following:

New common base of "b"

Previous base of "a"

$$\log_a m = \frac{\log_b m}{\log_b a}$$

Base "a" goes on the bottom

Most of the time **we will be converting our base to either 10 or *e***, since the calculator HSC students use can only directly calculate $\log_{10} x$ or $\log_e x$, and not any other bases.

For example:

$$\log_2 8 = \frac{\log_{10} 8}{\log_{10} 2}$$
$$= 3$$

Note: The "log" button on your calculator is base 10, so means $\log_{10}$. The "ln" button is base e, so means $\log_e$

- Exponential of a Logarithm

When taking the exponential of a logarithm with the same base, it can be simplified such that:

Same base

$$a^{\log_a m} = m$$

For example:

$$e^{\log_e 5} = 5$$

<u>Logarithmic Law Exercises</u>

1. Use your knowledge of logarithmic laws to evaluate the following without a calculator:

a) $\log_6 12 + \log_6 3$
b) $\log_4 8 + \log_4 128$
c) $\log_5 100 - \log_5 4$
d) $\log_3 405 - \log_3 5$
e) $\log_8 \frac{1}{64} + \log_8 512$
f) $\log_{12} \frac{1}{144} + \log_{12} 12$
g) $\log_5 2 - \log_5 50$
h) $\log_6 \frac{1}{3} - \log_6 12$
i) $\log_{40} 4 + \log_{40} 5 + \log_{40} 2$
j) $\log_{15} 600 - \log_{15} 24 + \log_{15} 9$
k) $\log_{\frac{1}{32}} 5 - \log_{\frac{1}{32}} 10$

2. Given that $\log_a 2 = 1.55$, evaluate the following expressions using your knowledge of logarithmic laws (do not find the value of a)

a) $\log_a 8$
b) $\log_a 64$

c) $\log_a \frac{1}{4}$
d) $\log_a \sqrt{2}$
e) $\log_a \frac{1}{\sqrt[3]{2}}$
f) $\log_a 4 + \log_a 32$
g) $\log_a 8 - \log_a 4$

3. Simplify the following using the identity $a^{\log_a m} = m$:
a) $e^{\ln 6}$
b) $5^{\log_5 44.2}$
c) $10^{\log_{10} x^3}$
d) $9^{\log_9 4}$
e) $7^{-\log_7 2}$
f) $5^{2\log_5 3}$
g) $2^{\log_2 4 + \log_2 6}$
h) $7^{x + \log_7 x}$
i) $4^{x \log_4 x}$
j) $9^{\frac{\log_9 x}{x}}$

4. Use change of base laws to evaluate the following to the nearest 3 significant figures:
a) $\log_8 32$
b) $\log_9 45$
c) $\log_{\frac{3}{4}} 9.6$
d) $\log_{\frac{2}{3}} \frac{5}{7}$
e) $\log_4 \frac{11}{6}$
f) $\log_{0.06} 0.78$
g) $\log_{0.8} 1.45$

Logarithmic Law Exercise Answers

1.
a)

$$\begin{aligned}\log_6 12 + \log_6 3 &= \log_6 12 \times 3 \ \ [logarithmic\ law] \\ &= \log_6 36 \\ &= 2\end{aligned}$$

b)

$$\log_4 8 + \log_4 128 = \log_4 8 \times 128 \ \ [logarithmic\ law]$$

$$= \log_4 1024$$
$$= 5 \; [\because 4^5 = 1024]$$

c)

$$\log_5 100 - \log_5 4 = \log_5 \frac{100}{4} \; [logarithmic\ law]$$
$$= \log_5 25$$
$$= 2$$

d)

$$\log_3 405 - \log_3 5 = \log_3 \frac{405}{5} \; [logarithmic\ law]$$
$$= \log_3 81$$
$$= 4$$

e)

$$\log_8 \frac{1}{64} + \log_8 512 = \log_8 \left(\frac{1}{64} \times 512\right) \; [logarithmic\ law]$$
$$= \log_8 8$$
$$= 1$$

f)

$$\log_{12} \frac{1}{144} + \log_{12} 12 = \log_{12} \frac{1}{144} \times 12 \; [logarithmic\ law]$$
$$= \log_{12} \frac{1}{12}$$
$$= -1$$

g)

$$\log_5 2 - \log_5 50 = \log_5 \frac{2}{50} \; [logarithmic\ law]$$
$$= \log_5 \frac{1}{25}$$
$$= -2$$

h)

$$\log_6 \frac{1}{3} - \log_6 12 = \log_6 \frac{1}{3} \div 12 \; [logarithmic\ law]$$
$$= \log_6 \frac{1}{36}$$
$$= -2$$

i)

$$\log_{40} 4 + \log_{40} 5 + \log_{40} 2 = \log_{40}(4 \times 5 \times 2) \; [logarithmic\ law]$$
$$= \log_{40} 40$$
$$= 1$$

j)

$$\log_{15} 600 - \log_{15} 24 + \log_{15} 9 = \log_{15}\left(\frac{600}{24} \times 9\right)$$
$$= \log_{15} 225$$
$$= 2$$

k)

$$\log_{\frac{1}{32}} 5 - \log_{\frac{1}{32}} 10 = \log_{\frac{1}{32}} \frac{5}{10} \ [logarithmic\ law]$$
$$= \log_{\frac{1}{32}} \frac{1}{2}$$

This question is hard because the base of the logarithm is a fraction. However, since the base is also a power of 2, we can utilise change of base laws to help us here:

$$\log_{\frac{1}{32}} \frac{1}{2} = \frac{\log_2 \frac{1}{2}}{\log_2 \frac{1}{32}}$$
$$= \frac{\log_2 2^{-1}}{\log_2 2^{-5}}$$
$$= \frac{-1 \times \log_2 2}{-5 \times \log_2 2} \ [logarithmic\ law]$$
$$= \frac{1}{5}$$

2. Since we know the value of $\log_a 2$, for all these questions, we are going to try express numbers as a power of 2! This is so we can eventually end up with an expression in terms of $\log_a 2$

a)

$$\log_a 8 = \log_a 2^3$$
$$= 3\log_a 2 \ [logarithmic\ law]$$
$$= 3 \times 1.55 \ [given]$$
$$= 4.65$$

b)

$$\log_a 64 = \log_a 2^6$$
$$= 6\log_a 2 \ [logarithmic\ law]$$
$$= 6 \times 1.55 \ [given]$$
$$= 9.3$$

c)

$$\log_a \frac{1}{4} = \log_a 2^{-2}$$
$$= -2\log_a 2 \ [logarithmic\ law]$$

$$= -2 \times 1.55$$
$$= -3.1$$

d)

$$\begin{aligned}\log_a \sqrt{2} &= \log_a 2^{\frac{1}{2}} \\ &= \frac{1}{2}\log_a 2 \ \ [logarithmic\ law] \\ &= \frac{1}{2} \times 1.55 \\ &= 0.775\end{aligned}$$

e)

$$\begin{aligned}\log_a \frac{1}{\sqrt[3]{2}} &= \log_a 2^{-\frac{1}{3}} \\ &= -\frac{1}{3}\log_a 2 \ \ [logarithmic\ law] \\ &= -\frac{1}{3} \times 1.55\ [given] \\ &= -0.52\ (nearest\ 2\ d.p.)\end{aligned}$$

f)

$$\begin{aligned}\log_a 4 + \log_a 32 &= \log_a 2^2 + \log_a 2^5 \\ &= 2\log_a 2 + 5\log_a 2 \ \ [logarithmic\ law] \\ &= 2 \times 1.55 + 5 \times 1.55\ [given] \\ &= 10.85\end{aligned}$$

g)

$$\begin{aligned}\log_a 8 - \log_a 4 &= \log_a \frac{8}{4}\ [logarithmic\ law] \\ &= \log_a 2 \\ &= 1.55\end{aligned}$$

3.
a)

$$\begin{aligned}e^{\ln 6} &= e^{\log_e 6} \\ &= 6\ [logarithmic\ law]\end{aligned}$$

b)

$$5^{\log_5 44.2} = 44.2\ [logarithmic\ law]$$

c)

$$10^{\log_{10} x^3} = x^3\ [logarithmic\ law]$$

d)

$$9^{\log_9 4} = 4\ [logarithmic\ law]$$

e)

$$\begin{aligned} 7^{-\log_7 2} &= 7^{\log_7 2^{-1}}\ [logarithmic\ law] \\ &= 2^{-1}\ [logarithmic\ law] \\ &= \frac{1}{2} \end{aligned}$$

f)

$$\begin{aligned} 5^{2\log_5 3} &= 5^{\log_5 3^2}\ [logarithmic\ law] \\ &= 5^{\log_5 9} \\ &= 9\ [logarithmic\ law] \end{aligned}$$

g)

$$\begin{aligned} 2^{\log_2 4+\log_2 6} &= 2^{\log_2(4\times 6)}\ [logarithmic\ law] \\ &= 2^{\log_2 24} \\ &= 24\ [logarithmic\ law] \end{aligned}$$

h)

$$\begin{aligned} 7^{x+\log_7 x} &= 7^x \times 7^{\log_7 x} \\ &= x7^x\ [logarithmic\ law] \end{aligned}$$

i)

$$\begin{aligned} 4^{x\log_4 x} &= 4^{\log_4 x^x}\ [logarithmic\ law] \\ &= x^x\ [logarithmic\ law] \end{aligned}$$

j)

$$\begin{aligned} 9^{\frac{\log_9 x}{x}} &= \left(9^{\log_9 x}\right)^{\frac{1}{x}} \\ &= (x)^{\frac{1}{x}}\ [logarithmic\ law] \\ &= \sqrt[x]{x} \end{aligned}$$

4.
a)

Using the change of base law:

$$\log_8 32 = \frac{\ln 32}{\ln 8}$$

Using our calculator now to evaluate:

$$\frac{\ln 32}{\ln 8} \approx 1.67\ (nearest\ 3\ sig\ figs)$$

b)

Using the change of base law:

$$\log_9 45 = \frac{\ln 45}{\ln 9}$$

Using our calculator now to evaluate:

$$\frac{\ln 45}{\ln 9} \approx 1.73\ (nearest\ 3\ sig\ figs)$$

c)

Using the change of base law:

$$\log_{\frac{3}{4}} 9.6 = \frac{\ln 9.6}{\ln \frac{3}{4}}$$

Using our calculator now to evaluate:

$$\frac{\ln 9.6}{\ln \frac{3}{4}} \approx -7.86\ (nearest\ 3\ sig\ figs)$$

d)

Using the change of base law:

$$\log_{\frac{2}{3}} \frac{5}{7} = \frac{\ln \frac{5}{7}}{\ln \frac{2}{3}}$$

Using our calculator now to evaluate:

$$\frac{\ln \frac{5}{7}}{\ln \frac{2}{3}} \approx 0.830\ (nearest\ 3\ sig\ figs)$$

e)

Using the change of base law:

$$\log_4 \frac{11}{6} = \frac{\ln\left(\frac{11}{6}\right)}{\ln 4}$$

Using our calculator now to evaluate:

$$\frac{\ln\left(\frac{11}{6}\right)}{\ln 4} \approx 0.437\ (nearest\ 3\ sig\ figs)$$

f)

Using the change of base law:

$$\log_{0.06} 0.78 = \frac{\ln 0.78}{\ln 0.06}$$

Using our calculator now to evaluate:

$$\frac{\ln 0.78}{\ln 0.06} \approx 0.0883\ (nearest\ 3\ sig\ figs)$$

g)

Using the change of base law:

$$\log_{0.8} 1.45 = \frac{\ln 1.45}{\ln 0.8}$$

Using our calculator now to evaluate:

$$\frac{\ln 1.45}{\ln 0.8} \approx -1.67\ (nearest\ 3\ sig\ figs)$$