FURTHER FUNCTIONS

GRAPH STRETCHING AND DILATIONS (III)

Contents include:

- Stretching and Dilations Definition
- Vertical Dilations
- Horizontal Dilations
- Dilations With a Fractional Factor
- Dilations With a Negative Factor

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- Stretching and Dilations Definition

Dilations are another kind of transformation of curves and involve **stretching a curve either horizontally or vertically**. Put simply, dilation just means stretch (by a factor of a, any real number)

- Vertical Dilations

To stretch a graph of $f(x)$ in a vertical direction by a factor of a, replace y by $\frac{y}{a}$

- Alternatively, if the graph is a function, the new function rule is $y = af(x)$
- The axis of dilation for these transformations is the $x-$ axis

When plotting these vertical dilations, remember that they will change the $y-$ value of points on the graph, including the $\boldsymbol{y-intercept}$ by a factor of $\boldsymbol{a}$. However, the coordinates of the $x-intercepts$ do **not change.**

Example 1: Find the new equation of the graph $y = x(x-2)$ after it is vertically dilated by a factor of 3 and sketch both graphs

Solution:

Since the graph is vertically dilated by a factor of 3:

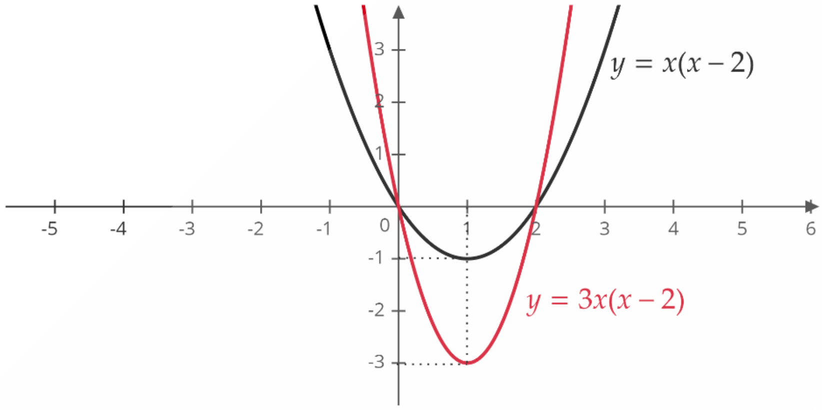
$$y \to \frac{y}{3}$$

$$\therefore \frac{y}{3} = x(x-2)$$

$$y = 3x(x-2)$$

Plotting the table of values for both graphs:

x	-2	-1	0	1	2	3	4
$x(x-2)$	8	3	0	-1	0	3	8
$3x(x-2)$	24	9	0	-3	0	9	24



- Horizontal Dilations

To stretch the graph of $f(x)$ in a horizontal direction by a factor of a, replace x by $\frac{x}{a}$

- Alternatively, if the graph is a function, the new function rule is $y = f\left(\frac{x}{a}\right)$
- The axis of dilation for these transformations is the $y -$ axis

When plotting these horizontal dilations, remember that they will change the $x -$ value of points on the graph, including the $x - intercept$ by a factor of a. However, the coordinates of the $y - intercepts$ do **not change.**

Example 2: Obtain the graph of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ from the graph of the circle $x^2 + y^2 = 1$

Solution:

We start off with the circle equation:

$$x^2 + y^2 = 1$$

Since we are considering changes to both 'x' and 'y', the solution may be rewritten as:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

This therefore tells us that the unit circle $x^2 + y^2 = 1$ has been stretched horizontally by a factor of 4 and vertically by a factor of 2.

- Dilations with a Fractional Factor

If the dilation factor is between 0 and 1, the graph is compressed instead of stretched

- Dilations with a Negative Factor

Put simply, a reflection is a dilation with factor -1

This means that any dilation with a negative factor is done the same way as if it was positive, except at the end it would also need to be reflected about the x or y axis.

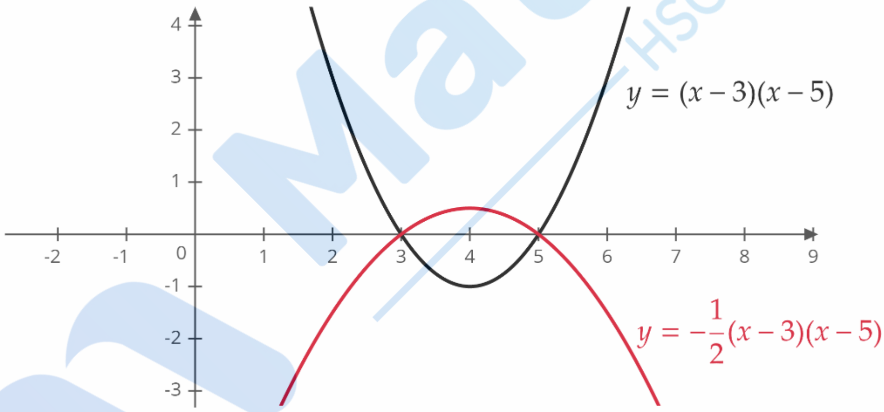
Example 3: Write down the new function when a vertical dilation of factor $-\frac{1}{2}$ is applied to the parabola $y = (x-3)(x-5)$ and sketch both curves on the same axes.

Solutions:

A vertical dilation means that we replace y with $\frac{y}{-\frac{1}{2}} = -2y$

$$\therefore -2y = (x-3)(x-5)$$
$$y = -\frac{1}{2}(x-3)(x-5)$$

Sketching both curves should resemble:



Dilation of graphs exercises

1. Write down the new equation for each equation after the given dilation has been applied:

a) $y = x^2 - 1$: Dilated vertically by -1

b) $x^2 + y^2 = 4$: Dilated vertically by $\frac{1}{3}$

c) $y = \sin\frac{x}{2}$: Dilated vertically by -4

d) $y = \frac{1}{x}$: Dilated horizontally by 2

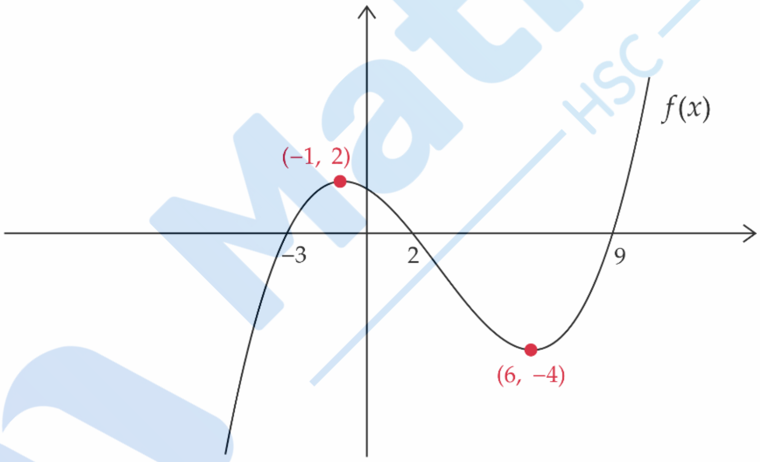
e) $y = \log_x 2$: Dilated horizontally by $-\frac{1}{4}$

f) $y = \sqrt{x^3 + 2x}$: Dilated horizontally by -3

2. In each of the following cases, identify how the graph of the second equation could be obtained from the first by dilation:
a) $y = x^2 - 2x$ and $y = 3x^2 - 6x$
b) $y = \frac{1}{x-4}$ and $y = \frac{1}{2x-4}$
c) $y = \cos x$ and $y = 3\cos\frac{x}{4}$

3. For each pair of curves, suggest 2 transformations by which the second equation may be obtained from the first
a) $y = 2^x$, $y = 2^{x+1}$
b) $y = \frac{1}{x}$, $y = \frac{k^2}{x}$
c) $y = 3^x$, $y = 3^{-x}$

4. The graph of the unknown function $f(x)$ has been drawn below:



Use dilations to sketch each of the following graphs:

a) $f\left(\frac{x}{2}\right)$
b) $2f(x)$
c) $\frac{1}{2}f(-2x)$

Dilations of graphs exercise answers

1.
a) Since it's a vertical dilation of factor -1, then $y \to \frac{y}{-1}$

$$\frac{y}{-1} = x^2 - 1$$

$$\therefore y = 1 - x^2$$

b) Since it's a vertical dilation of factor $\frac{1}{3}$, then $y \rightarrow \frac{y}{\frac{1}{3}}$

$$x^2 + \left(\frac{y}{\frac{1}{3}}\right)^2 = 4$$

$$x^2 + 9y^2 = 4$$

c) Since it's a vertical dilation of factor -4, then $y \rightarrow \frac{y}{-4}$

$$\frac{y}{-4} = \sin\frac{x}{2}$$

$$\therefore y = -4\sin\frac{x}{2}$$

d) Since it's a horizontal dilation of factor 2, then $x \rightarrow \frac{x}{2}$

$$y = \frac{1}{\frac{x}{2}}$$

$$\therefore y = \frac{2}{x}$$

e) Since it's a horizontal dilation of factor $-\frac{1}{4}$, then $x \rightarrow \frac{x}{-\frac{1}{4}}$

$$y = \log_{\frac{x}{-\frac{1}{4}}} 2$$

$$\therefore y = \log_{-4x} 2$$

f) Since it's a horizontal dilation of factor -3 , $x \rightarrow \frac{x}{-3}$

$$y = \sqrt{\left(-\frac{x}{3}\right)^3 + 2\left(-\frac{x}{3}\right)}$$

$$= \sqrt{-\frac{x^3}{27} - \frac{2x}{3}}$$

2.

a) $y = 3x^2 - 6x$

Rearranging the equation: $y = 3(x^2 - 2x)$

$$\therefore \frac{y}{3} = x^2 - 2x$$

Therefore, this shows us that a vertical dilation of factor 3 has occurred

b) $y = \frac{1}{2x-4}$

Rearranging the solution: $y = \frac{1}{2(x-2)}$

$$2y = \frac{1}{x-2}$$
$$\frac{y}{\frac{1}{2}} = \frac{1}{x-2}$$

Therefore, this shows us that a vertical dilation of factor $\frac{1}{2}$ has occurred

c) $y = 3\cos\left(\frac{x}{4}\right)$

Rearranging the equation:

$$\frac{y}{3} = \cos\left(\frac{x}{4}\right)$$

Therefore, this shows us that a vertical dilation of factor 3 has occurred, and a horizontal dilation of factor 4 has also occurred

3.

a) $y = 2^{x+1}$

Rearranging the solution: $y = 2 \times 2^x$

$$\frac{y}{2} = 2^x$$

Therefore, this shows us that it is a vertical dilation of 2

Alternatively:

x has been changed into $x + 1$. Therefore, this tells us that it is a horizontal shift of 1 unit to the left

b) $y = \frac{k^2}{x}$

Rearranging the solution: $\frac{y}{k^2} = \frac{1}{x}$

Therefore, this shows us that it is a vertical dilation of k^2

Alternatively:

Rearranging the solution: $y = \frac{1}{\frac{x}{k^2}}$

Therefore, this shows us that it is a horizontal dilation of k^2

c) $y = 3^{-x}$

Since we have changed x into $-x$, the transformation is reflecting about the y −axis.

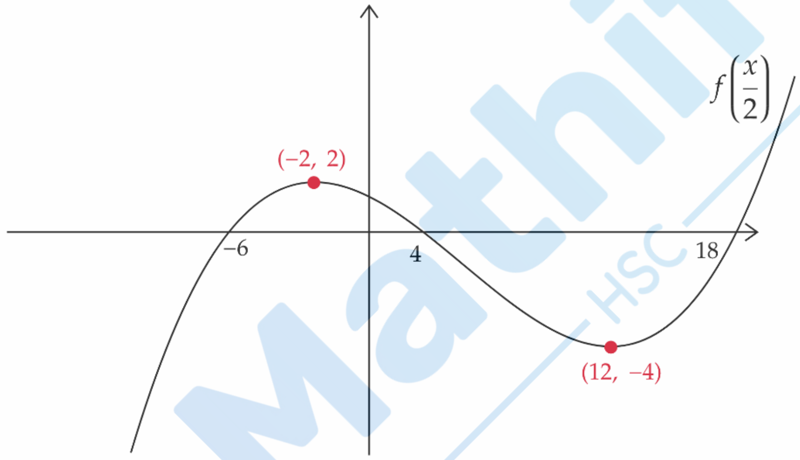
Alternatively:

This is the same as saying a horizontal dilation of -1

4.

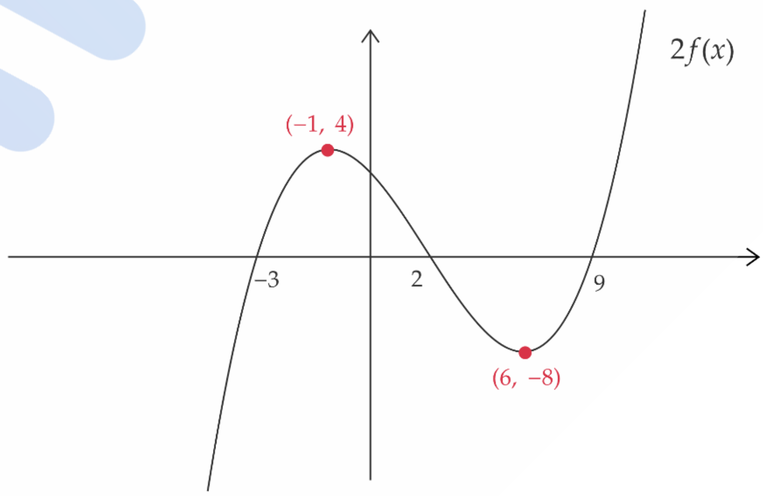
a) $f\left(\frac{x}{2}\right)$ is the graph of $f(x)$ except horizontally dilated by a factor of 2. Hence, we imagine that all x values are also multiplied by a factor of 2 (including $x-$ intercepts).

Sketching this now (not to scale):



b) $2f(x)$ is the graph of $f(x)$ vertically dilated by a factor of 2. Hence, we imagine that all y values are also multiplied by a factor of 2 (the $x-$ intercepts however will remain the same)

Sketching this now (not to scale):



c) $\frac{1}{2}f(-2x)$ is the graph of $f(x)$ except vertically dilated by a factor of $\frac{1}{2}$ and also horizontally dilated by a factor of $-\frac{1}{2}$. Hence, all $x -$ values are multiplied by a factor of $-\frac{1}{2}$ and all the $y -$ values are multiplied by a factor of $\frac{1}{2}$. Moreover, notice that since the $x -$ values are multiplied by a negative number, the graph will be reflected about the $y -$ axis as well.

Sketching this now (not to scale):