FUNCTIONS

LINEAR FUNCTIONS AND DIRECT VARIATION (IV)

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- Reviewing Linear Functions

Much of this chapter involves reviewing linear function concepts and equations so if you are already familiar with these, you may skip to the end for the direct variation subtopic.

Linear functions refer to any graphs of a **straight line**. There are two common ways to express linear functions:

- Gradient – Intercept form:

$$y = mx + b$$

In this form, it is known that:

$$m = gradient\ of\ function$$

$$b = y - intercept\ of\ function$$

- General form:

$$ax + by + c = 0$$

Example 1: Determine if the following equations are linear or non – linear

a) $y - 2 = -3x$

Since we can arrange the function into either the gradient – intercept form:

$$y = -3x + 2$$

Or the general form:

$$y + 3x - 2 = 0$$

Therefore, it can be said that the function is linear

b) $y = x^2 - 3x + 2$

This equation is non – linear, since we have an x^2 as part of the equation, and this cannot be expressed in gradient – intercept nor general form.

c) $3x - 2y + 5 = 0$

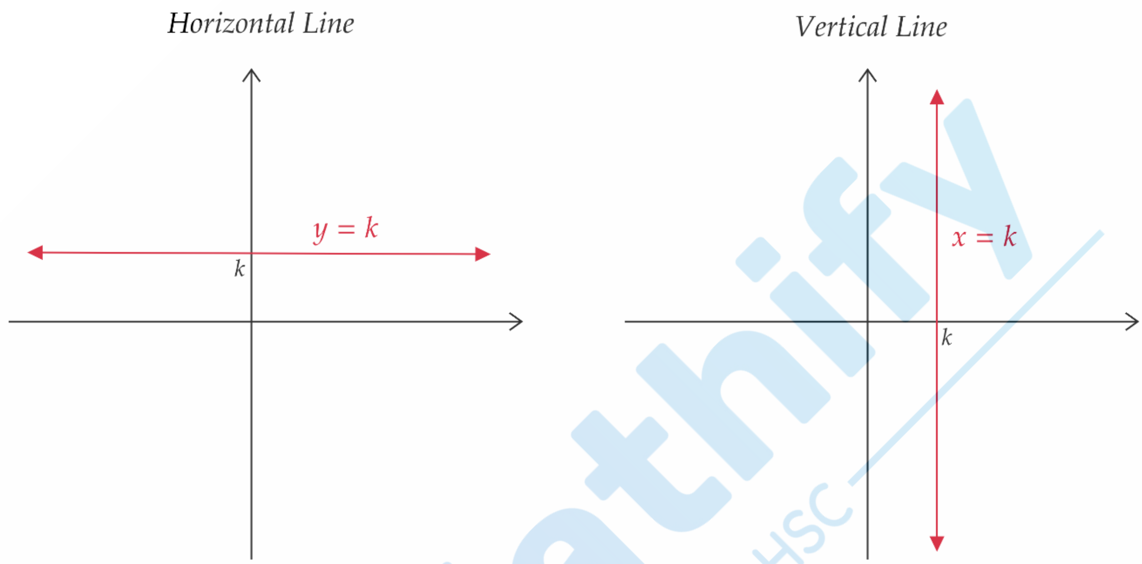
Since this equation is in general form, it can be said that the function is linear!

- Horizontal and Vertical Lines

A horizontal line is given by the equation $y = k$, where k is a constant

A vertical line is given by the equation $x = k$, where k is a constant

This is summarised in the diagram below:



- Gradient of a function

Put simply, the gradient of a function refers to **how steep it is**, with the common formula:

$$gradient = \frac{rise}{run} = \frac{change\ in\ y\ value}{change\ in\ x\ value}$$

Moreover, gradient is often denoted as the symbol m

To find the gradient of any line:

$$\boldsymbol{Step\ 1: Take\ any\ two\ points\ on\ the\ line}$$

Call these two points (x_1, y_1) and (x_2, y_2)

$$\boldsymbol{Step\ 2: Apply\ the\ gradient\ formula}$$

Since we know that $gradient = \frac{rise}{run}$:

$$\therefore gradient = m = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, if I want to find the gradient between two points $(1, 2)$ and $(3, 4)$:

$$\begin{aligned} m &= \frac{4-2}{3-1} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

- Parallel and Perpendicular Lines

Given two lines l_1 and l_2 with gradients m_1 and m_2 respectively:

If they are parallel:

$$m_1 = m_2$$

If they are perpendicular:

$$m_1 \times m_2 = -1$$

- Sketching Linear Functions

Typically, it is **easier to sketch** a linear function when you see it **in gradient – intercept form**, so we should always try to rearrange our question first if needed!

The steps to sketch are summarised below:

$$\boldsymbol{Step\ 1: Arrange\ the\ equation\ into\ gradient - intercept\ form}$$

$$\boldsymbol{Step\ 2: Plot\ the\ x - intercept\ and\ y - intercept}$$

The $x -$ intercept can be found by letting $y = 0$. Then:

$$\begin{aligned} 0 &= mx + b \\ mx &= -b \\ \therefore x &= -\frac{b}{m} \end{aligned}$$

The $y - intercept$ is simply the "b"

$$\boldsymbol{Step\ 3: Draw\ the\ line\ between\ the\ x\ and\ y\ intercept}$$

Make sure to extend this line past your intercepts though!

Example 2: Sketch the linear function $x + 2y - 8 = 0$

$$Step\ 1: Arranging\ into\ gradient - intercept\ form$$

$$x + 2y - 8 = 0$$

$$2y = -x - 8$$

$$\therefore y = -\frac{x}{2} - 4$$

$$Step\ 2: Plot\ the\ x\ and\ y\ intercept$$

When $y = 0$:

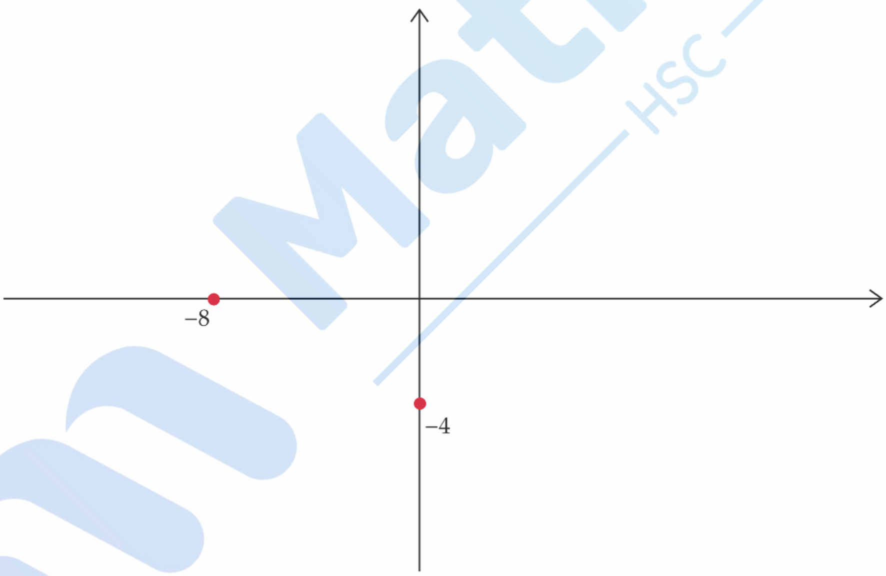
$$0 = -\frac{x}{2} - 4$$

$$\frac{x}{2} = -4$$

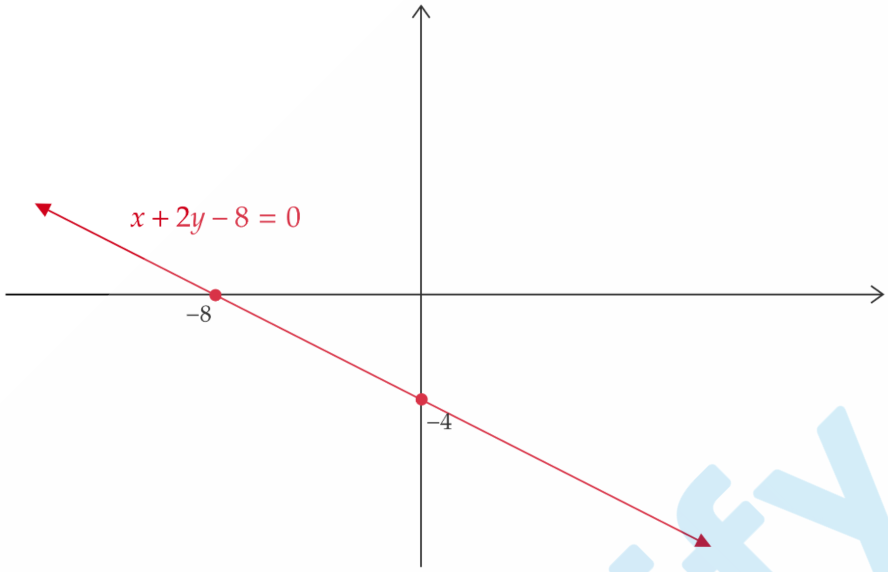
$$\therefore x = -8\ is\ x - intercept$$

The $y - intercept$ is -4

Hence, plotting this:



$$Step\ 3: Sketching\ the\ line$$



- Solving linear function equations $f(x) = k$ graphically

Let's say you are asked to solve the following equation:

$$mx + b = k$$

Before we begin, we must think about what this means:

We have two equations here, $y = mx + b$ and $y = k$. Hence, if we want to solve the equation $mx + b = k$, we essentially want to find when they intersect, as this is where their y coordinates will be equal!

The steps to solving $mx + b = k$ graphically would therefore be:

$$\boldsymbol{Step\ 1: Sketch\ the\ graph\ for\ y = mx + b}$$

$$\boldsymbol{Step\ 2: Sketch\ the\ graph\ for\ y = k\ on\ the\ same\ axes}$$

This should just be a horizontal line!

$$\boldsymbol{Step\ 3: Find\ where\ the\ two\ lines\ intersect}$$

The $x -$ coordinate of the intersection point will therefore be your solution!

Example 3: Solve the following equation through graphical means:

$$3x + 7 = 1$$

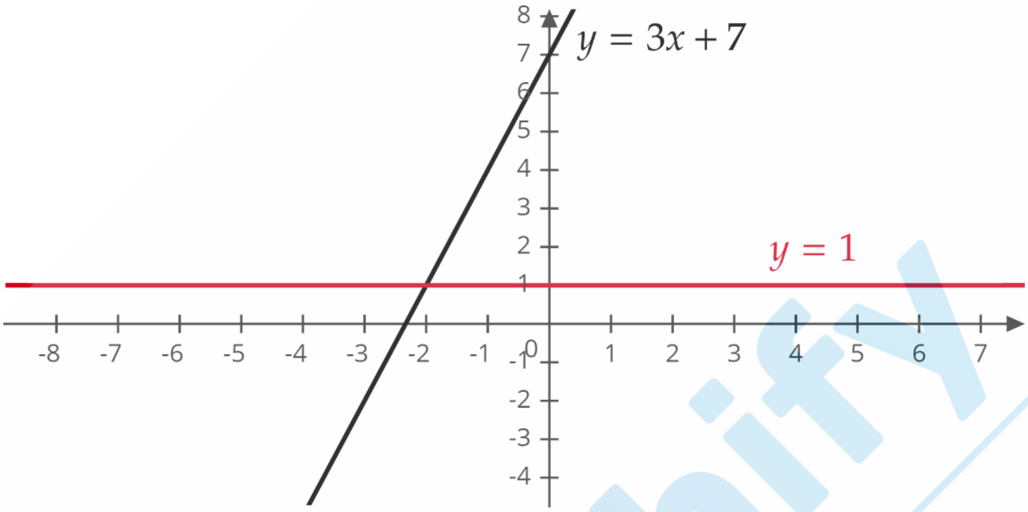
Solution:

$$Step\ 1\ \&\ 2: Sketch\ the\ two\ graphs\ on\ the\ same\ axes$$

For $y = 3x + 7$:

$$y - intercept\ occurs\ at\ y = 7$$

$$x - intercept\ occurs\ at\ x = -\frac{7}{3}$$



$$Step\ 3: Solve!$$

As can be seen, the two lines intersect when $x = -2$. Hence, the solution to our equation is $x = -2$

- Solving Linear Equations $f(x) = k$ Algebraically

It is much more efficient to solve equations algebraically and graphically, so it is recommended to do algebraic methods unless specified otherwise

The steps to solve $mx + b = k$ algebraically are summarised below:

$$\boldsymbol{Step\ 1: Move\ b\ to\ the\ other\ side}$$

$$mx + b = k$$

$$\therefore mx = k - b$$

$$\boldsymbol{Step\ 2: Make\ x\ the\ subject}$$

$$\therefore x = \frac{k - b}{m}$$

This will therefore be out solution!

Example 4:

Solve the equation $5x + 4 = -1$

Solution:

$$\begin{aligned} 5x &= -1 - 4 \\ &= -5 \end{aligned}$$

$$\therefore x = -1$$

Hence, the solution to the equation is $x = -1$

- Point – Gradient Formula

The point – gradient formula is **used to find the equation** of a straight line with gradient m that passes through a certain point (x_1, y_1). The line's equation is:

$$y - y_1 = m(x - x_1)$$

Example 5: Find the equation of the line that is parallel to $y = 2x - 3$ and passes through the point $(3, -4)$

Solution:

Since the line is parallel to $y = 2x - 3$, their gradients must be equal!

Hence, the gradient of our unknown line is $m = 2$. Therefore, using the point – gradient formula:

$$\begin{aligned} &y - -4 = 2(x - 3) \\ &y + 4 = 2x - 6 \\ &\therefore y = 2x - 10 \end{aligned}$$

- Equation of a line through two points

To find the equation of a line that is known to pass through two points, (x_1, y_1) and (x_2, y_2):

$$\boldsymbol{Step\ 1: Calculate\ the\ gradient}$$

This is done through using the gradient formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$\boldsymbol{Step\ 2: Use\ the\ point - gradient\ formula}$$

Now choose one of your two points (doesn't matter which one), and apply the point – gradient formula we learnt previously!

Example 6: Find the equation of a line that is known to pass through the points $(-2,-3)$ and $(4,2)$

Solution:

$$Step\ 1: Find\ gradient$$

$$\begin{aligned} m &= \frac{2--3}{4--2} \\ &= \frac{5}{6} \end{aligned}$$

$$Step\ 2: use\ the\ point - gradient\ formula$$

The line has gradient $m = \frac{5}{6}$ and passes through the point $(4,2)$ so:

$$\therefore y - 2 = \frac{5}{6}(x-4)$$

$$y - 2 = \frac{5}{6}x - \frac{20}{6}$$

$$y - 2 = \frac{5}{6}x - \frac{10}{3}$$

$$y = \frac{5}{6}x - \frac{10}{3} + 2$$

$$\therefore y = \frac{5}{6}x - \frac{4}{3}$$

- Direct Variations

Two variables are considered **directly proportional** to each other if it can be said that:

$$y = kx$$

Where k is a non – zero constant. It is commonly referred to as "*The constant of proportionality*".

It can be thought of that when two variables are directly proportional, as x increases, y will increase as well. Hence, the graph would be a line through the origin!

Note: The words "proportion" and "variation" are used interchangeably and essentially mean the same thing

Commonly, direct variation questions will ask students to **find the equation** relating two variables x and y. To do so, we use the following steps:

$$\boldsymbol{Step\ 1: Express\ that\ y = kx}$$

$$\boldsymbol{Step\ 2: Find\ the\ value\ of\ k}$$

We do this by substituting in known values into our equation for step 1, and rearrange to find k

$$\boldsymbol{Step\ 3: Write\ the\ final\ equation, with\ the\ correct\ k\ value}$$

Overall, these steps are pretty easy and are demonstrated in the example below:

Example 7: The total cost in dollars, C, for a batch of garden fertiliser varies directly with the mass (M) in kg. The gardener observes that a 20kg bag costs $15 to purchase. Find the equation relating C and M.

$$Step\ 1: Express\ the\ equation$$

Since C is directly proportional to M:

$$\therefore C = kM, where\ k\ is\ a\ constant$$

$$Step\ 2: Find\ the\ value\ of\ k$$

When $C = 15, M = 20$

$$\therefore 15 = 20k$$

$$k = \frac{15}{20} = \frac{3}{4}$$

$$Step\ 3: Final\ answer$$

$$\therefore C = \frac{3}{4}M$$

Linear Functions and Direct Variation Exercises

1. Solve the following equations:

a) $8x + 4 = 7$

b) $\frac{2}{3}x - 8 = 3$

c) $\frac{4}{5}x - 3 = 7\frac{1}{9}$

d) $\frac{x+5}{3} + \frac{x-1}{2} = 1$

e) $\frac{x+4}{3} - \frac{2x-3}{5} = -2$

f) $\frac{4x+1}{5} + \frac{2x-5}{8} - \frac{6x+10}{15} = \frac{2x}{3}$

g) $\frac{2}{3}(x-1)-\frac{1}{2}(3x+2)=0$

2. Sketch the following linear functions, showing any intercepts:

a) $y=-5x+3$

b) $y=\frac{2}{3}x-2$

c) $2y-3x+2=0$

d) $8y+x-4=0$

e) $\frac{4}{5}y-16x+5=0$

3. Given that a line is parallel to $3x+2y-9=0$ and passes through the point $(-1,5)$, find its equation

4. Given that a line is perpendicular to $6x-9y+2=0$ and passes through the point $(2,4)$, find its equation

5. Find the equation of the line that passes through the points $(4,9)$ and $(-2,-7)$

6. The amount of paint, V, used to cover Andy's house is directly proportional to the surface area, A, in square metres, to be covered. Andy is told from the store clerk that 8L of paint is needed to cover $75m^2$ of his house.

a) Find the equation relating A and V

b) If Andy's house is measured to have a surface area of $860m^2$, find how much paint is required

c) Unfortunately for Andy, he only has 70L of paint. What percentage of his house will he be able to paint? Give your answer to the nearest percentage

7. The cost, C, to maintain my Taxi business is directly proportional to the number of kilometres driven, D. It is known that if I drive 5000km, the cost is $1800.

a) Find the equation relating C and D

b) Given that I receive $4000 for every 7000km driven, calculate my profit after costs are accounted for

Linear Functions and Direct Variation Exercise Answers

1. For these questions, our end goal always is to eventually make x the subject

a)

$$8x + 4 = 7$$

$$8x = 3$$

$$\therefore x = \frac{3}{8}$$

b)

$$\frac{2}{3}x - 8 = 3$$

$$\frac{2}{3}x = 11$$

$$\therefore x = 11 \times \frac{3}{2}$$
$$= \frac{33}{2}$$

c)

$$\frac{4}{5}x - 3 = 7\frac{1}{9}$$

$$\frac{4}{5}x = 10\frac{1}{9}$$

$$\therefore x = \frac{5}{4} \times 10\frac{1}{9}$$
$$= \frac{5}{4} \times \frac{91}{9}$$
$$= \frac{455}{36}$$

d)

$$\frac{x+5}{3} + \frac{x-1}{2} = 1$$

$$\frac{x}{3} + \frac{5}{3} + \frac{x}{2} - \frac{1}{2} = 1$$

$$\frac{x}{3} + \frac{x}{2} = 1 - \frac{5}{3} + \frac{1}{2}$$

$$\frac{5}{6}x = -\frac{1}{6}$$

$$\therefore x = -\frac{1}{5}$$

e)

$$\frac{x+4}{3}-\frac{2x-3}{5}=-2$$

$$\frac{x}{3}+\frac{4}{3}-\frac{2x}{5}+\frac{3}{5}=-2$$

$$\frac{x}{3}-\frac{2x}{5}=-2-\frac{4}{3}-\frac{3}{5}$$

$$-\frac{x}{15}=-\frac{59}{15}$$

$$\therefore x=59$$

f)

$$\frac{4x+1}{5}+\frac{2x-5}{8}-\frac{6x+10}{15}=\frac{2x}{3}$$

Multiplying LHS and RHS by 120:

$$24(4x+1)+15(2x-5)-8(6x+10)=40(2x)$$

$$96x+24+30x-75-48x-80=80x$$

$$96x+30x-48x-80x=80+75-24$$

$$-2x=131$$

$$\therefore x=-\frac{131}{2}$$

g)

$$\frac{2}{3}(x-1)-\frac{1}{2}(3x+2)=0$$

$$\frac{2}{3}x-\frac{2}{3}-\frac{3}{2}x-1=0$$

$$\frac{2}{3}x-\frac{3}{2}x=1+\frac{2}{3}$$

$$-\frac{5}{6}x=\frac{5}{3}$$

$$\therefore x=-\frac{6}{5}\times\frac{5}{3}$$
$$=-2$$

2.
a)

$$y=-5x+3$$

Finding $y -$ intercept by letting $x = 0$:

$$y = 3$$

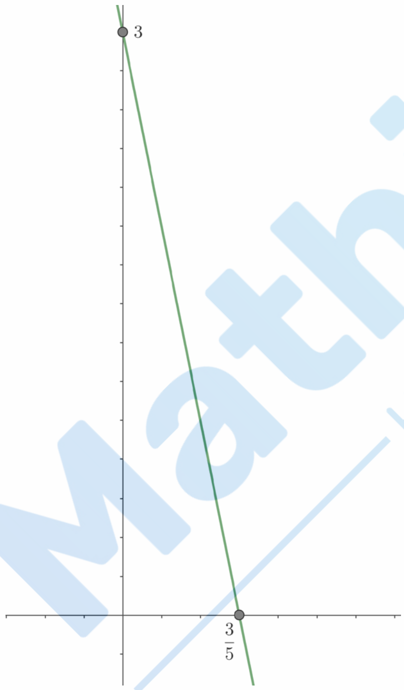
Finding $x -$ intercept by letting $y = 0$:

$$0 = -5x + 3$$

$$5x = 3$$

$$\therefore x = \frac{3}{5}$$

Sketching:



b)

$$y = \frac{2}{3}x - 2$$

Finding $y -$ intercept by letting $x = 0$:

$$y = -2$$

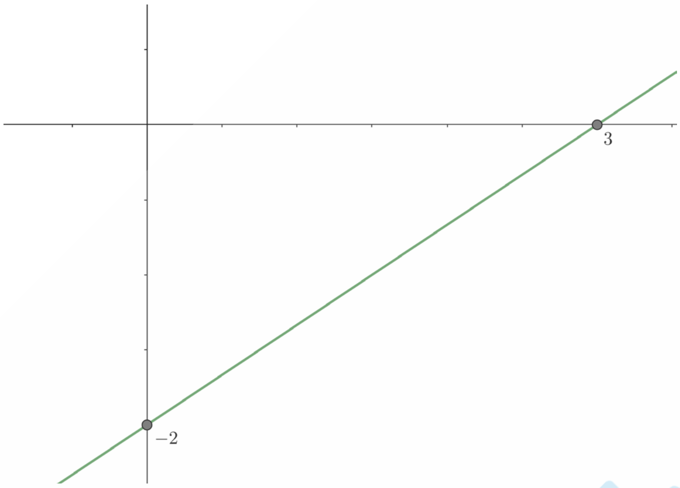
Finding $x -$ intercept by letting $y = 0$:

$$0 = \frac{2}{3}x - 2$$

$$\frac{2}{3}x = 2$$

$$\therefore x = 3$$

Sketching:



c)

$$2y - 3x + 2 = 0$$

Finding $y -$ intercept by letting $x = 0$:

$$2y + 2 = 0$$

$$\therefore y = -1$$

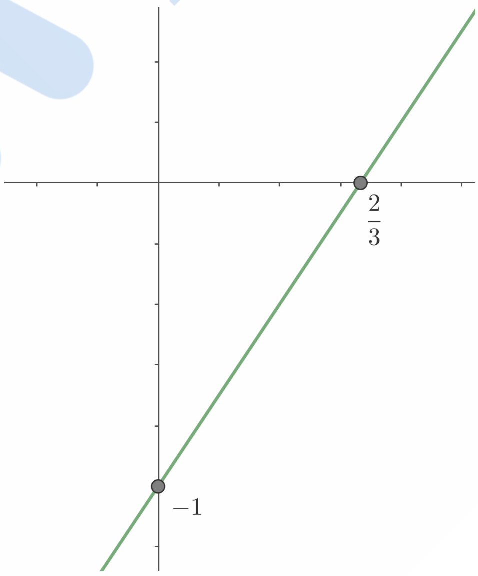
Finding $x -$ intercept by letting $y = 0$:

$$-3x + 2 = 0$$

$$3x = 2$$

$$\therefore x = \frac{2}{3}$$

Sketching:



d)

$$8y + x - 4 = 0$$

Finding $y -$ intercept by letting $x = 0$:

$$8y - 4 = 0$$

$$8y = 4$$

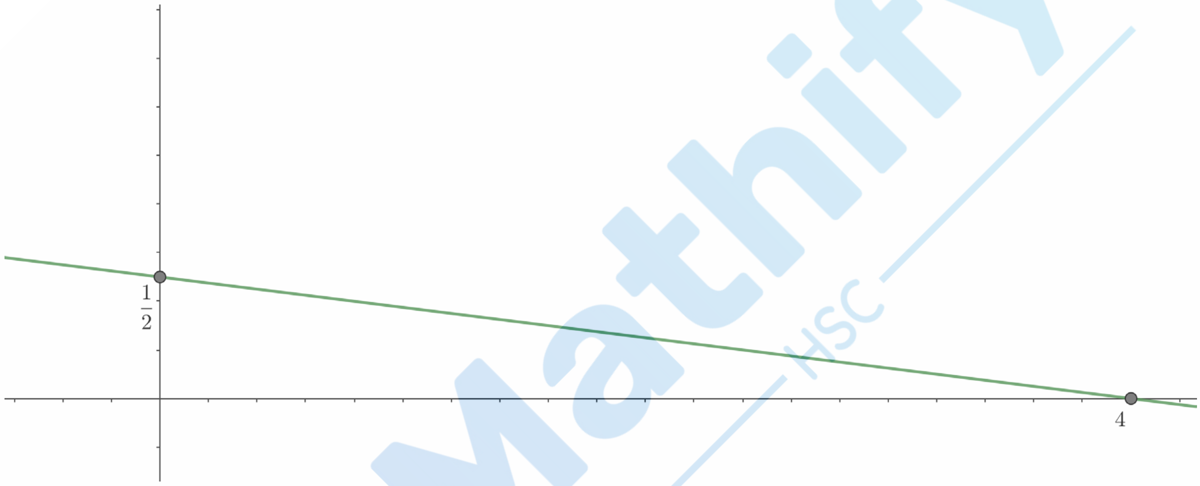
$$\therefore y = \frac{1}{2}$$

Finding $x -$ intercept by letting $y = 0$:

$$x - 4 = 0$$

$$\therefore x = 4$$

Sketching:



e)

$$\frac{4}{5}y - 16x + 5 = 0$$

Finding $y -$ intercept by letting $x = 0$:

$$\frac{4}{5}y + 5 = 0$$

$$\frac{4}{5}y = -5$$

$$\therefore y = -\frac{25}{4}$$

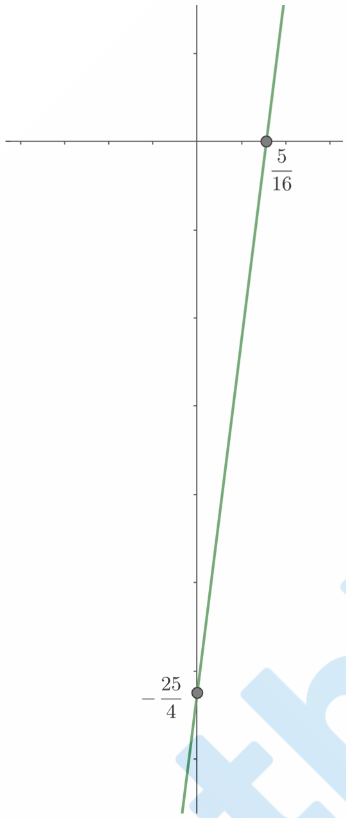
Finding $x -$ intercept by letting $y = 0$:

$$-16x + 5 = 0$$

$$16x = 5$$

$$\therefore x = \frac{5}{16}$$

Sketching:



3. Finding the gradient of the line $3x + 2y - 9 = 0$ by rearranging into gradient intercept form:

$$2y = -3x + 9$$

$$\therefore y = -\frac{3}{2}x + \frac{9}{2}$$

Since the line is parallel to $y = -\frac{3}{2}x + \frac{9}{2}$, this means that its gradient is $-\frac{3}{2}$

Hence, since the line also passes through $(-1, 5)$:

$$y - 5 = -\frac{3}{2}(x - -1)$$

$$y - 5 = -\frac{3}{2}(x + 1)$$

$$y - 5 = -\frac{3}{2}x - \frac{3}{2}$$

$$\therefore y = -\frac{3}{2}x - \frac{3}{2} + 5$$
$$= -\frac{3}{2}x + \frac{7}{2}$$

4. Finding the gradient of the line $6x - 9y + 2 = 0$ by rearranging into gradient – intercept form:

$$9y = 6x + 2$$
$$\therefore y = \frac{2}{3}x + \frac{2}{9}$$

Hence, the gradient of this line is $\frac{2}{3}$

Since the unknown line is perpendicular to the line, then its gradient will be:

$$m = -\frac{1}{\frac{2}{3}}$$
$$= -\frac{3}{2}$$

Hence, since the line passes through $(2, 4)$:

$$y - 4 = -\frac{3}{2}(x - 2)$$
$$y - 4 = -\frac{3}{2}x + 3$$
$$y = -\frac{3}{2}x + 7$$

5. Finding the gradient of the line through two points first:

$$gradient = m = \frac{9 - -7}{4 - -2}$$
$$= \frac{16}{6}$$
$$= \frac{8}{3}$$

Therefore, finding the equation:

$$y - 9 = \frac{8}{3}(x - 4)$$
$$y = \frac{8}{3}x - \frac{32}{3} + 9$$
$$\therefore y = \frac{8}{3}x - \frac{5}{3}$$

6.
a)

$$Step\ 1: Express\ the\ equation$$

Since V is directly proportional to A:

$$\therefore V = kA, where\ k\ is\ a\ constant$$

$$Step\ 2: Find\ the\ value\ of\ k$$

When $A = 75, V = 8$:

$$8 = 75k$$

$$\therefore k = \frac{8}{75}$$

$$Step\ 3: Final\ answer$$

$$\therefore V = \frac{8}{75}A$$

b)

Essentially, we need to find the value of V when $A = 860$:

$$\begin{aligned} V &= \frac{8}{75} \times 860 \\ &= \frac{1376}{15}L \end{aligned}$$

c)

If $V = 70$, finding the value of A:

$$70 = \frac{8}{75}A$$

$$\begin{aligned} \therefore A &= 70 \times \frac{75}{8} \\ &= \frac{2625}{4}\ m^2 \end{aligned}$$

Now finding this area as a percentage of the total:

$$\begin{aligned} Percentage\ of\ total\ area &= \frac{\frac{2625}{4}}{860} \times 100\% \\ &= \frac{656.25}{860} \times 100\% \\ &= 76\%\ (nearest\ percentage) \end{aligned}$$

7.

a)

$$Step\ 1: Express\ the\ Equation$$

Since C is directly proportional to D:

$$\therefore C = kD, where\ k\ is\ a\ constant$$

$$\boldsymbol{Step}\ 2: Find\ the\ value\ of\ k$$

When $C = 1800$, $D = 5000$:

$$1800 = 5000k$$

$$\therefore k = \frac{1800}{5000}$$
$$= \frac{9}{25}$$

$$Step\ 3: Final\ answer$$

$$\therefore C = \frac{9}{25}D$$

b) Calculating the costs for driving 7000km by letting $D = 7000$:

$$C = \frac{9}{25} \times 7000$$
$$= \$2520$$

Therefore, the profit will be:

$$Profit = \$4000 - \$2520$$
$$= \$1480$$