PROBABILITY

MULTI - STAGE EXPERIMENTS AND TREE DIAGRAMS (IV)

Contents include: Multi - stage experiments, tree diagrams and harder probability questions

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- Multi – stage experiments

Multi – stage experiments are essentially experiments with multiple stages or parts to it, where the action of the 1^{st} stage impacts the probability of the 2^{nd}, the action of the 2^{nd} impacts the probability of the 3^{rd}, and so on.

Probability tree diagrams are a helpful tool to solve these problems.

Probability tree diagrams involve: ○ "Sets" of branches equal to the number of stages ○ Writing down the probability of each branch ○ Calculating the probability of an event by multiplying each of its branch

Below are examples highlighting how this may be done with a 2 – stage experiment:

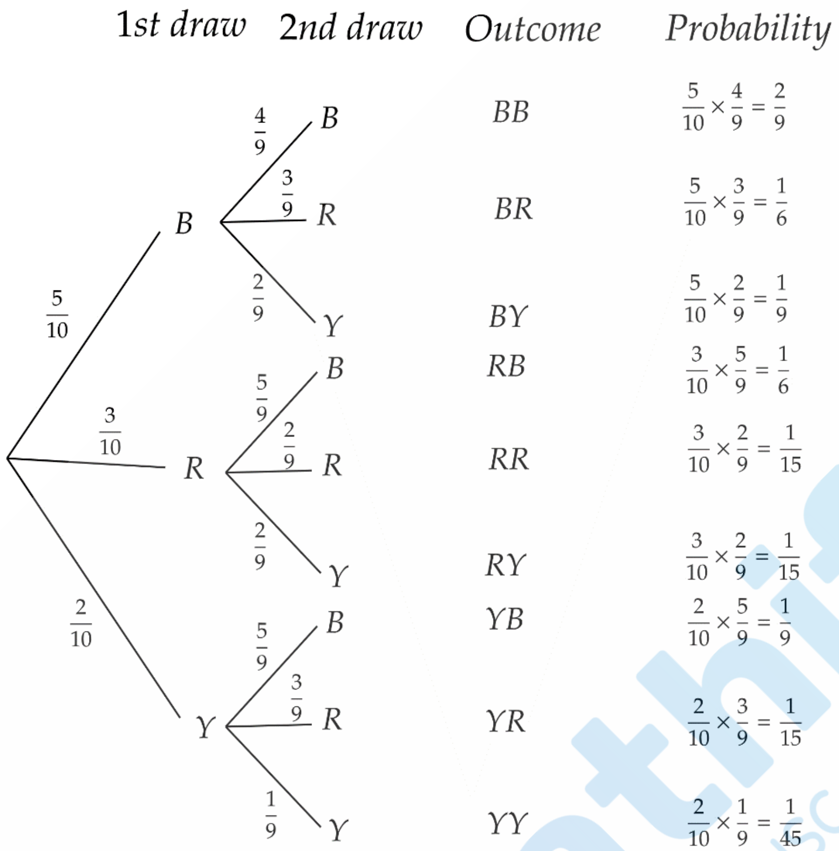
Example 1: There are 5 blue marbles, 3 red marbles and 2 yellow marbles in a bag. Two marbles are taken one by one out of the bag without replacement. Find the probability of drawing:

a) Two red marbles
b) A blue and then a red marble, in that order
c) A yellow and blue marble in any order
d) 2 marbles that are not red in colour

Remember for probability questions that: ○ **Without replacement** means once taken out, the marble is no longer put back in. The sample space is reduced ○ **With replacement/replaced** means that the marble is put back in. The sample space remains the same

Solutions:

Since 2 marbles are being taken out, this is a 2 – stage experiment with 2 sets of branches, just like in the tree diagram shown below:



a) Looking at the tree diagram:

$$\begin{aligned} P(Two\,reds) &= P(RR) \\ &= \frac{1}{15} \end{aligned}$$

b)

$$\begin{aligned} P(blue\ and\ then\ red) &= P(BR) \\ &= \frac{1}{6} \end{aligned}$$

c)

$$\begin{aligned} P(yellow\ and\ blue) &= P(YB) + P(BY) \\ &= \frac{1}{9} + \frac{1}{9} \\ &= \frac{2}{9} \end{aligned}$$

d)

$$\begin{aligned} P(no\ red\ marbles) &= P(YY) + P(BB) \\ &= \frac{1}{45} + \frac{2}{9} \\ &= \frac{11}{45} \end{aligned}$$

Example 2: There is an 80% chance that Lebron makes a free throw shot in basketball and a 55% chance that Simmons makes the free throw shot. Draw a probability tree diagram and find the chance that:

a) Lebron succeeds but Simmons fails
b) Simmons succeeds but Lebron fails
c) Only one of them make it
d) At least one fails

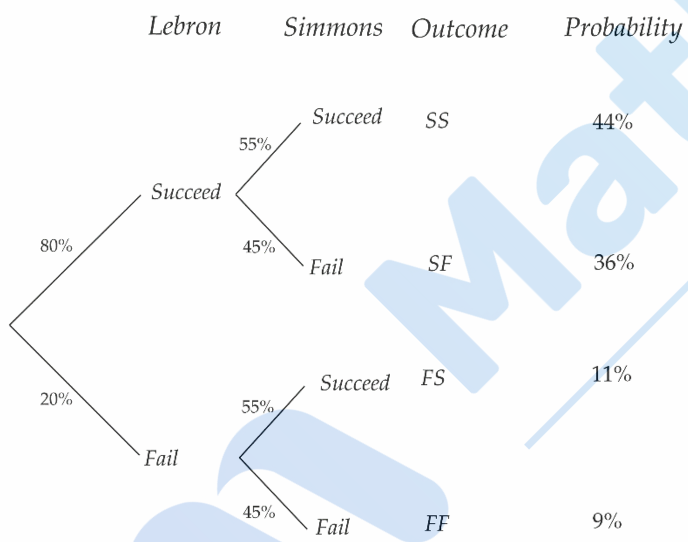
Solutions:

For this question, our probability tree diagram will have 2 stages:

The first stage and set of branches will have Lebron shooting a free throw, where he either "succeeds" or "fails"

The second stage and set of branches will have Simmons shooting a free throw, where he either "succeeds" or "fails"

Thus, drawing our probability tree diagram:



a)

If Lebron succeeds, and Simmons fails:

$$\begin{aligned} P(SF) &= 80\% \times 45\% \\ &= 36\% \end{aligned}$$

b)

If Simmons succeeds and Lebron fails:

$$\begin{aligned} P(FS) &= 20\% \times 55\% \\ &= 11\% \end{aligned}$$

c)

If only one of them makes it:

$$\begin{aligned} P(one\ of\ them\ makes\ it) &= P(SF) + P(FS) \\ &= 36\% + 11\% \\ &= 47\% \end{aligned}$$

d)

$$\begin{aligned} P(at\ least\ one\ fails) &= 1 - P(both\ fail) \\ &= 1 - P(FF) \\ &= 1 - 9\% \\ &= 91\% \end{aligned}$$

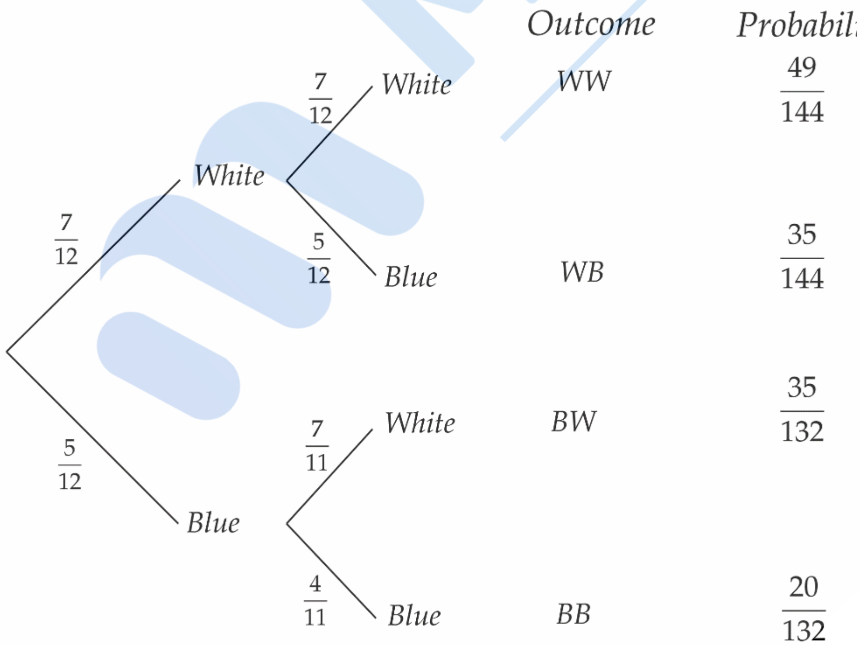
Example 3: A bag contains 7 white marbles and 5 blue marbles. Two marbles are drawn in succession, where if the marble drawn is white it is replaced, and if it is blue it is not replaced. Find the probability of drawing:

a) No blue marbles
b) 1 blue marble
c) 2 blue marbles

Solutions:

In this experiment, the 1st draw can impact the probability of the 2nd draw if a marble is not replaced due to the sample space decreasing.

Thus, a probability tree diagram should be drawn, with 2 sets of branches since 2 marbles are being drawn:



Thus, using the probability tree diagram, the answers are:

a)

$$P(no\ blue) = P(WW)$$
$$= \frac{49}{144}$$

b)

$$P(1\ blue) = P(WB) + P(BW)$$
$$= \frac{35}{144} + \frac{35}{132}$$
$$= \frac{805}{1584}$$

c)

$$P(2\ blues) = P(BB)$$
$$= \frac{20}{132}$$

- Harder Probability Questions

A couple of tips to remember when attempting harder probability questions:

- '**And**' means you multiply probabilities of events, '**Or**' means you add probabilities
- Utilise complimentary events when you see "**at least one**" or "**not**" in the question
- Some questions may require you to add probabilities
- Abbreviate events to single letters, eg. Heads may be represented as "H"

Note that for these questions, there is no requirement to draw a probability tree diagram unless specified by the question. Otherwise, you may choose to draw it or not depending on your confidence! It is recommended you do draw a tree diagram though if you are new to the topic.

The following is an example where a probability tree diagram is not used to solve our question:

Example 4: Giannis shoots 3 free throws. He has a 80% chance of it going in. Assuming each shot is independent, find the probability that:

a) He makes all 3
b) He misses all 3
c) He makes the last shot only
d) He makes only one shot
e) He makes only 2 shots

Hint: Use 'I' for if the shot goes in and 'O' for if the shot goes out

Solutions:

For these questions, remember that:

$$P(I) = \frac{8}{10}$$

$$P(O) = \frac{2}{10}$$

a)

$$P(III) = \frac{8}{10} \times \frac{8}{10} \times \frac{8}{10}$$
$$= \frac{64}{125}$$

b)

$$P(OOO) = \frac{2}{10} \times \frac{2}{10} \times \frac{2}{10}$$
$$= \frac{1}{125}$$

c)

$$P(OOI) = \frac{2}{10} \times \frac{2}{10} \times \frac{8}{10}$$
$$= \frac{4}{125}$$

d)

$$P(makes\ only\ 1) = P(IOO) + P(OIO) + P(OOI)$$

$$= \left(\frac{8}{10} \times \frac{2}{10} \times \frac{2}{10}\right) + \left(\frac{2}{10} \times \frac{8}{10} \times \frac{2}{10}\right) + \left(\frac{2}{10} \times \frac{2}{10} \times \frac{8}{10}\right)$$

$$= \frac{4}{125} + \frac{4}{125} + \frac{4}{125}$$
$$= \frac{12}{125}$$

e)

$$P(makes\ only\ 2) = P(OII) + P(IOI) + P(IIO)$$

$$= \left(\frac{2}{10} \times \frac{8}{10} \times \frac{8}{10}\right) + \left(\frac{8}{10} \times \frac{2}{10} \times \frac{8}{10}\right) + \left(\frac{8}{10} \times \frac{8}{10} \times \frac{2}{10}\right)$$
$$= \frac{16}{125} + \frac{16}{125} + \frac{16}{125}$$
$$= \frac{48}{125}$$

Probability Tree Diagram Exercises

1. Hamilton, Ricciardo and Verstappen take a driving test. The chances that they pass are $\frac{3}{4}, \frac{5}{8}$ and $\frac{4}{9}$ respectively. By drawing a probability tree diagram or otherwise:

a) Find the probability that only Hamilton passes
b) Find the probability that only one of them pass
c) Find the probability that both Ricciardo and Verstappen pass but not Hamilton

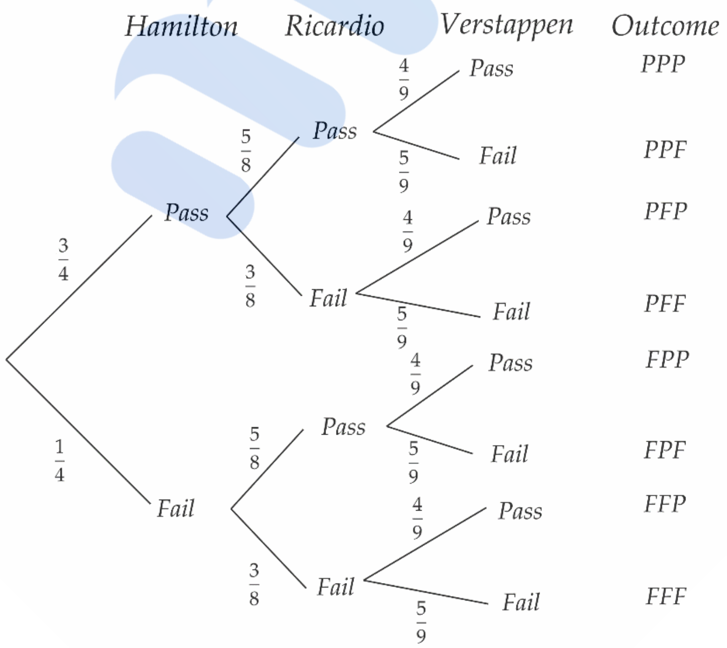
2. If a fair coin is flipped repeatedly, find the probability of:

a) Obtaining at least one tail in:
   i. Two flips
   ii. Five flips
   iii. Ten flips
b) Write down the probability of at least one tail in n flips
c) How many times would you need to flip a coin so that the probability of flipping at least one tail is greater than 0.9999?

3. A bag contains 3 green, 4 red and 3 blue marbles. Marbles are drawn at random, one by one without replacement, until 2 green marbles are drawn. Find the probability that exactly 3 draws will be required?

Multi – stage Experiment Exercise Answers

1. For this question, it is a 3-stage experiment, meaning that we have 3 sets of branches. Hence, sketching the tree diagram:



a)

$$\begin{aligned} P(\textit{only Hamilton passes}) &= P(PFF) \\ &= \frac{3}{4}\times\frac{3}{8}\times\frac{5}{9} \\ &= \frac{5}{32} \end{aligned}$$

b)

$$\begin{aligned} P(\textit{only one passes}) &= P(PFF)+P(FPF)+P(FFP) \\ &= \frac{5}{32}+\left(\frac{1}{4}\times\frac{5}{8}\times\frac{5}{9}\right)+\left(\frac{1}{4}\times\frac{3}{8}\times\frac{4}{9}\right) \\ &= \frac{5}{32}+\frac{25}{288}+\frac{1}{24} \\ &= \frac{41}{144} \end{aligned}$$

c)

$$\begin{aligned} P(\textit{both R and V pass}) &= P(FPP) \\ &= \frac{1}{4}\times\frac{5}{8}\times\frac{4}{9} \\ &= \frac{5}{72} \end{aligned}$$

2.

a) Considering the complementary:

i. In two flips:

$$\begin{aligned} P(\textit{at least one tail}) &= 1-P(\textit{no tails}) \\ &= 1-P(HH) \\ &= 1-\left(\frac{1}{2}\times\frac{1}{2}\right) \\ &= 1-\frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

ii. In five flips:

$$\begin{aligned} P(\textit{at least one tail}) &= 1-P(\textit{no tails}) \\ &= 1-P(HHHHH) \\ &= 1-\left(\frac{1}{2}\right)^5 \\ &= 1-\frac{1}{32} \\ &= \frac{31}{32} \end{aligned}$$

iii. In ten flips:

$$
\begin{aligned}
P(at\ least\ one\ tail) &= 1 - P(no\ tails) \\
&= 1 - P(10\ heads) \\
&= 1 - \left(\frac{1}{2}\right)^{10} \\
&= 1 - \frac{1}{1024} \\
&= \frac{1023}{1024}
\end{aligned}
$$

b)

Following our pattern from before and considering the complement for n trials:

$$
\begin{aligned}
P(at\ least\ one\ tail) &= 1 - P(no\ tails) \\
&= 1 - P(n\ heads) \\
&= 1 - \left(\frac{1}{2}\right)^{n} \\
&= 1 - \frac{1}{2^n} \\
&= \frac{2^n - 1}{2^n}
\end{aligned}
$$

c) Using our equation from the previous part:

$$P(at\ least\ one\ tail) = \frac{2^n - 1}{2^n}$$

Hence, if we want $P(at\ least\ one\ tail) \geq 0.9999$:

$$
\begin{aligned}
&\therefore \frac{2^n - 1}{2^n} \geq 0.9999 \\
&2^n - 1 \geq 0.9999(2^n) \\
&2^n - 0.9999(2^n) \geq 1 \\
&2^n(1 - 0.9999) \geq 1 \\
&2^n \geq \frac{1}{0.0001} \\
&2^n \geq 10000
\end{aligned}
$$

$$\therefore n \geq \log_2 10000$$

Using log laws to change base:

$$
\begin{aligned}
\therefore n &\geq \frac{\ln 10000}{\ln 2} \\
&\geq 13.29
\end{aligned}
$$

Hence, the minimum number of flips will be 14, we round up since 13 would not be enough for $P(at\ least\ one\ tail) \geq 0.9999$!

3. If exactly 3 draws are required, then we must consider all cases, where one of the first 2 marbles drawn is green and the 3rd marble drawn is green as well. Hence:

$$P(3\ draws\ required) = P(RGG) + P(BGG) + P(GRG) + P(GBG)$$

Where:

$$R = Red\ marble\ drawn$$
$$B = Blue\ marble\ drawn$$
$$G = Green\ marble\ drawn$$

Now calculating each individual probability:

$$P(RGG) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$$

$$P(BGG) = \frac{3}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{40}$$

$$P(GRG) = \frac{3}{10} \times \frac{4}{9} \times \frac{2}{8} = \frac{1}{30}$$

$$P(GBG) = \frac{3}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{40}$$

$$\therefore P(RGG) + P(BGG) + P(GRG) + P(GBG) = \frac{1}{30} + \frac{1}{40} + \frac{1}{30} + \frac{1}{40} = \frac{7}{60}$$

Note that the sample space is being reduced here because marbles are NOT replaced!

Hence, we can say that:

$$P(3\ draws\ required) = \frac{7}{60}$$