Mathify HSC

INTEGRATION

FURTHER WORK ON AREA (XI)

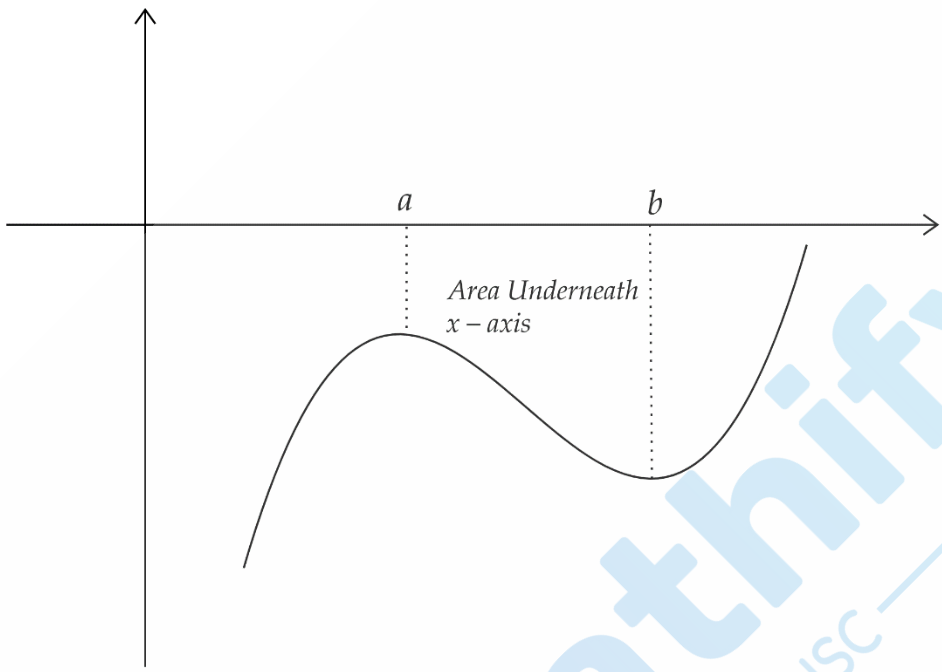
Contents include:

- Areas Underneath the Axis
- Area Between Two Curves
- Areas of Multiple Regions

www.mathifyhsc.com.au

- Areas Underneath the $x -$ axis

We need to be careful when evaluating areas underneath the $x -$ axis, such as in the diagram below:

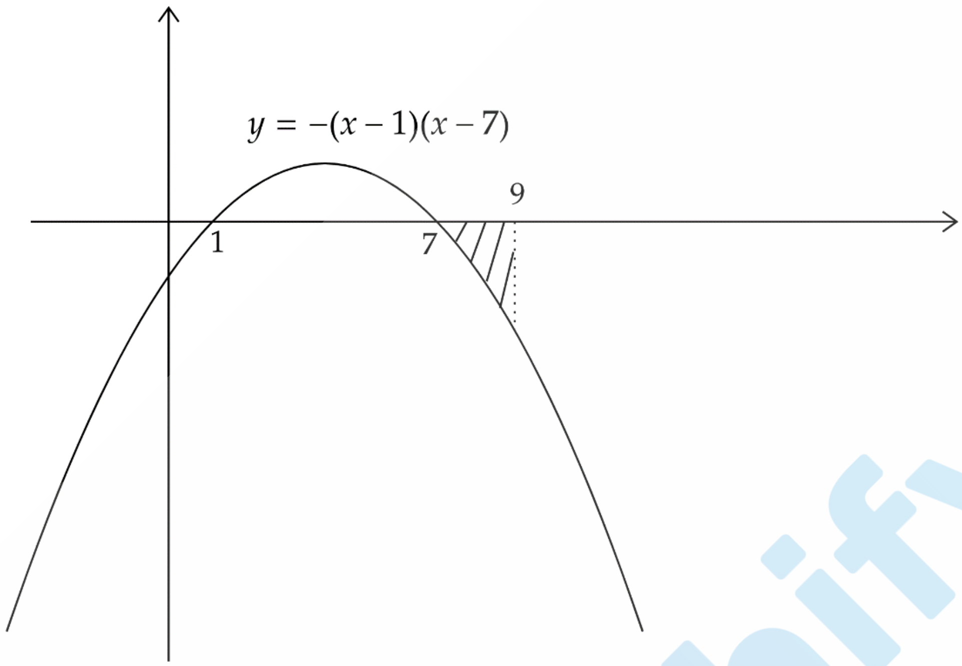


If we just use the definite integral to evaluate these regions the answer will be **negative**, which doesn't make sense since area can only be positive.

Hence, we must remember to absolute value our definite integral value!

$$\therefore \, Area \; below \; x - axis = \left| \int_a^b f(x)\,dx \right|$$

Example 1: Calculate the area of the region shown below:



Solution:

Since the area is below the $x-$axis, we must remember to use absolute values. In other words:

$$Area = \left|\int_7^9 -(x-1)(x-7)dx\right|$$

Expanding our brackets before integrating:

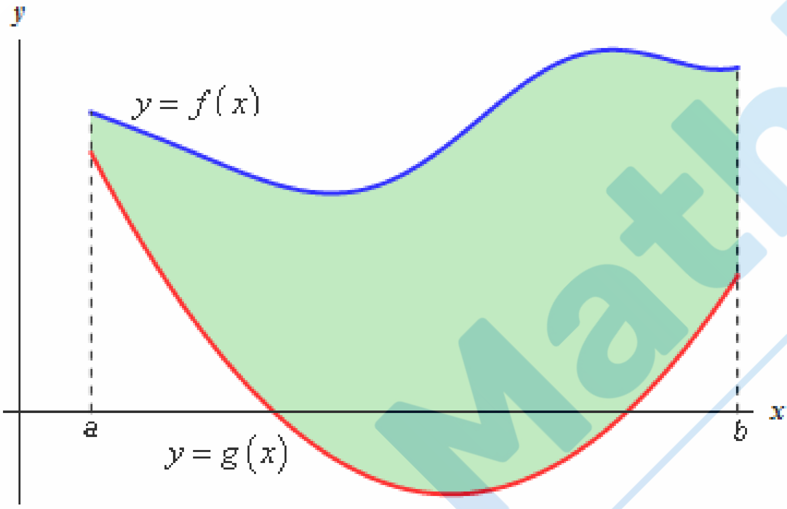
$$\begin{aligned}
\therefore Area &= \left|\int_7^9 -(x^2-8x+7)dx\right| \\
&= \left|\int_7^9 -x^2+8x-7dx\right| \\
&= \left|\left[-\frac{x^3}{3}+4x^2-7x\right]_7^9\right| \\
&= \left|\left(-\frac{9^3}{3}+4(9)^2-7(9)\right)-\left(-\frac{7^3}{3}+4(7)^2-7(7)\right)\right| \\
&= \left|(18)-\left(\frac{98}{3}\right)\right| \\
&= \left|-\frac{44}{3}\right| \\
&= \frac{44}{3}\ units^2
\end{aligned}$$

- Area between Two Curves

If $f(x)$ and $g(x)$ are two continuous functions where $f(x) > g(x)$ for the interval $a < x < b$, the area of the region bounded by the two curves is:

$$\begin{aligned} Area\ between\ 2\ curves &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b [f(x) - g(x)]\,dx \end{aligned}$$

This is illustrated in the diagram below:



<u>Note</u> that it doesn't matter whether or not $f(x)$ and $g(x)$ are above the $x-$ axis, **use the above formula no matter what** when calculating area between 2 curves. Don't overcomplicate it for yourself!

If the question does not give the bounds for an area, this means that you must find it yourself, where most of the time it will just be the intersection points between the two curves!!

Example 2: Calculate the area of the region enclosed by the graphs of $f(x) = x + 1$ and $g(x) = x^2 - x - 2$

Solution:

We currently are not sure what our lower and upper bounds are, so:

$$\textbf{\textit{Step 1: Find the lower and upper bounds for our integral}}$$

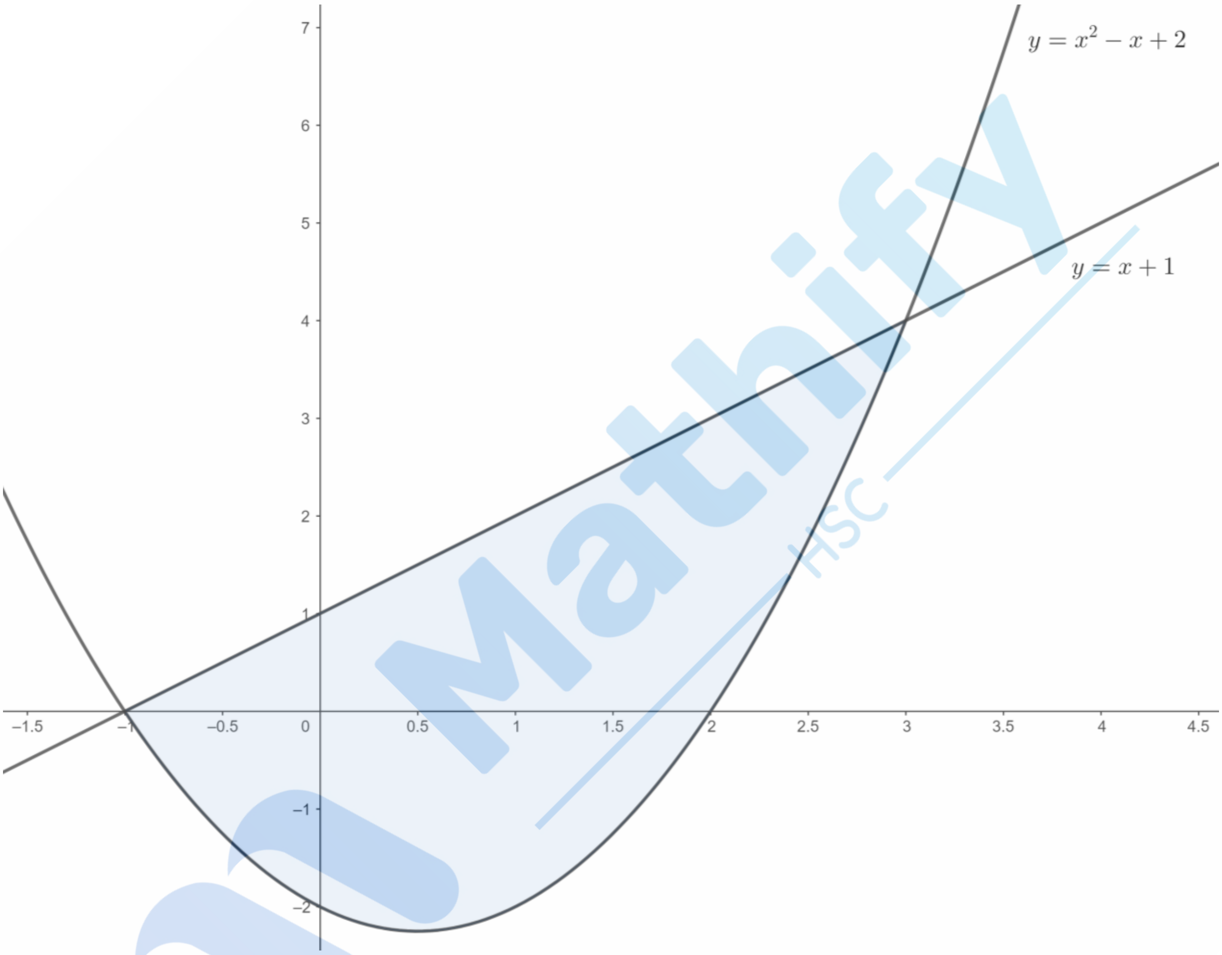
We can achieve this by finding the $x-$ values of the points of intersection between the 2 lines.

$$x^2 - x - 2 = x + 1$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$

$$\therefore x = 3 \text{ } or \text{ } x = -1 \text{ } are \text{ } where \text{ } the \text{ } graphs \text{ } intersect$$

Hence, lower bound is $x = -1$ and upper bound is $x = 3$

$$\boldsymbol{Step\ 2: Draw\ a\ graph\ with\ the\ given\ information}$$



$$\boldsymbol{Step\ 3: Derive\ the\ area\ expression}$$

$$A = \int_{-1}^{3} [(x+1) - (x^2 - x - 2)]\, dx$$
$$= \int_{-1}^{3} [-x^2 + 2x + 3]\, dx$$
$$= \left[-\frac{x^3}{3} + x^2 + 3x\right]_{-1}^{3}$$

$$\boldsymbol{Step\ 4: Substitute\ in\ bounds\ to\ determine\ the\ value\ of\ A}$$

$$\therefore A = \left(-\frac{3^3}{3} + 3^2 + 3(3)\right) - \left(-\frac{(-1)^3}{3} + (-1)^2 + 3(-1)\right)$$

$$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right)$$
$$= 9 - -\frac{5}{3}$$
$$= 10\frac{2}{3}\ units^2$$

As can be seen, even though the area was underneath the $x-$ axis, since we are finding area between two curves, there was no need for using absolute values!

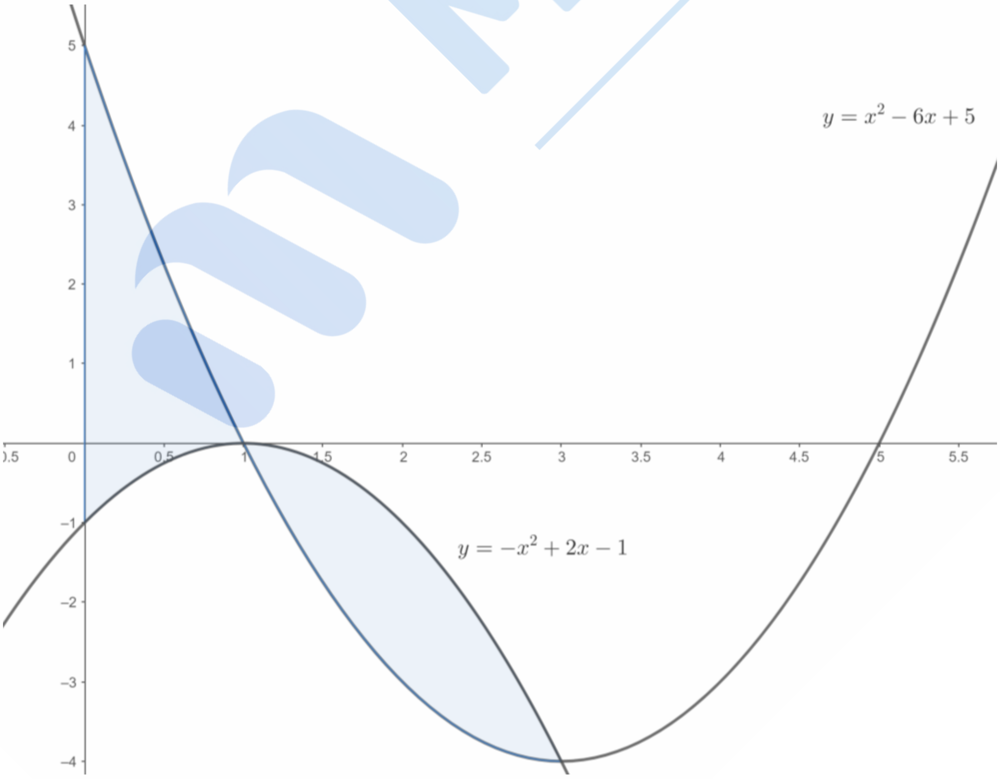
- Area of Multiple Regions

When dealing with more difficult area questions with multiple regions, intersections, or curves, it is recommended that we ***first draw a preliminary sketch***.

After completing our sketch, we can then if needed, split our area into regions before evaluating the definite integrals.

These area regions which we can label $A_1, A_2, A_3, \ldots$ are generally split by intersection points with the axes or between two curves.

Example 3: The diagram below shows the two curves $f(x) = -x^2 + 2x - 1$ and $g(x) = x^2 - 6x + 5$. Find the area of the shaded regions.



$$Step\ 1: Divide\ your\ total\ area\ into\ 2\ regions$$

Let A_1 be your left region and A_2 be your right region, where:

$$A = A_1 + A_2$$

$$Step\ 2: Derive\ the\ area\ expression$$

For A_1:

$g(x)$ is above $f(x)$, and the shaded area is between these two functions. Hence:

$$\begin{aligned} A_1 &= \int_0^1 g(x) - f(x)\,dx \\ &= \int_0^1 (x^2 - 6x + 5) - (-x^2 + 2x - 1)\,dx \\ &= \int_0^1 2x^2 - 8x + 6\ dx \end{aligned}$$

For A_2:

$f(x)$ is above $g(x)$, and the shaded area is between these two functions. Hence:

$$\begin{aligned} A_2 &= \int_1^3 f(x) - g(x)\,dx \\ &= \int_1^3 (-x^2 + 2x - 1) - (x^2 - 6x + 5)\,dx \\ &= \int_1^3 -2x^2 + 8x - 6\,dx \end{aligned}$$

$$Step\ 3: Evaluate\ the\ area\ expression$$

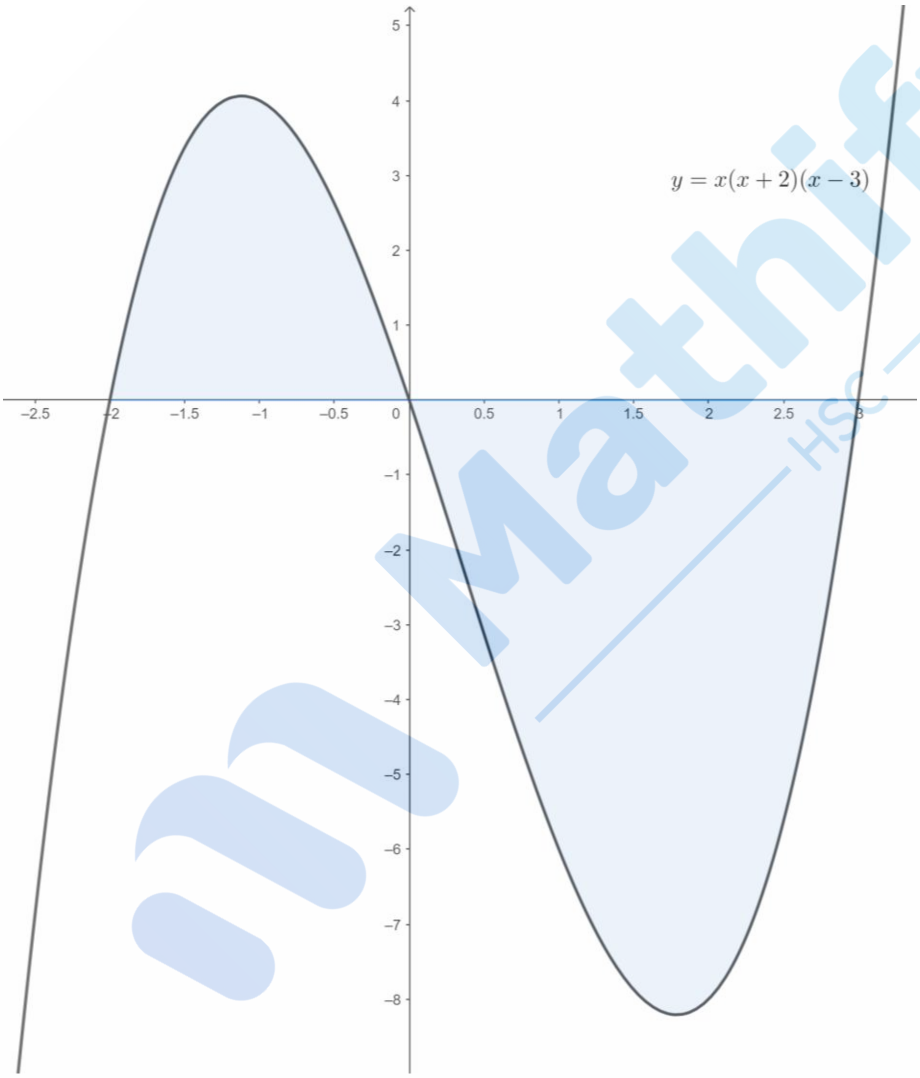
$$\begin{aligned} A_1 &= \left[\frac{2x^3}{3} - 4x^2 + 6x\right]_0^1 \\ &= \left(\frac{2}{3} - 4 + 6\right) - (0) \\ &= 2\frac{2}{3} \end{aligned}$$

$$\begin{aligned} A_2 &= \left[-\frac{2}{3}x^3 + 4x^2 - 6x\right]_1^3 \\ &= \left(-\frac{2}{3}(3)^3 + 4(3)^2 - 6(3)\right) - \left(-\frac{2}{3} + 4 - 6\right) \\ &= (-18 + 36 - 18) - \left(-2\frac{2}{3}\right) \\ &= 2\frac{2}{3} \end{aligned}$$

$$\begin{aligned}\therefore A &= A_1 + A_2 \\ &= 2\frac{2}{3} + 2\frac{2}{3} \\ &= 5\frac{1}{3}\ u^2\end{aligned}$$

Example 4: Calculate the total area bound between the curve $y = x(x+2)(x-3)$ and the $x -$ axis.

$$Step\ 1: Draw\ sketch\ and\ shade\ in\ the\ necessary\ area$$



$$Step\ 2: Derive\ the\ area\ expression$$

A comment mistake at this stage is to assume that:

$$A = \int_{-2}^{3} x(x+2)(x-3)\ dx$$

The **reason this is wrong** is because the green shaded area is *below* the $x -$ axis, which means that the definite integral when evaluated will be negative, which is an error because *area cannot be a negative*.

Instead, the correct way is to split our area expression into two regions:

$$A = \int_{-2}^{0} x(x+2)(x-3)\,dx + \left|\int_{0}^{3} x(x+2)(x-3)\,dx\right|$$

Notice the absolute value used for the green region which is underneath the curve. This is to ensure that the definite integral, which represents its area, ends up being positive.

$$\textit{Step } 3\textit{: Evaluate the area expression}$$

Expanding the brackets, $x(x+2)(x-3) = x^3 - x^2 - 6x$

$$A = \int_{-2}^{0} x^3 - x^2 - 6x\,dx + \left|\int_{0}^{3} x^3 - x^2 - 6x\,dx\right|$$

$$\begin{aligned}
\therefore A &= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{0} + \left|\left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{0}^{3}\right| \\
&= (0) - \left(\frac{(-2)^4}{4} - \frac{(-2)^3}{3} - 3(-2)^2\right) + \left|\left(\frac{3^4}{4} - \frac{3^3}{3} - 3(3)^2\right) - 0\right| \\
&= -\left(4 + \frac{8}{3} - 12\right) + \left|\frac{81}{4} - 9 - 27\right| \\
&= -\left(-\frac{16}{3}\right) + \left|-\frac{63}{4}\right| \\
&= \frac{16}{3} + \frac{63}{4} \\
&= \frac{253}{12}\,u^2
\end{aligned}$$

<u>Further Work on Area Exercises</u>

1. Calculate the area of the region bounded by the $x -$ axis and the curves with equation $y = \sqrt{x}$ and $y = 6 - x$

2. Calculate the area of the region bounded by the line $y = 2x$ and the parabola $y = x^2$

3. Find the area of the region enclosed by the line $y = 2x + 3$ and the parabola $y = x^2$

4. A straight line through the origin cuts the parabola $y = 4x - x^2$ at the point where $x = 3$.

a) Find the equation of this line
b) Calculate the area of the region bounded by the parabola and the straight line

5. In what ratio does the $x-$ axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$

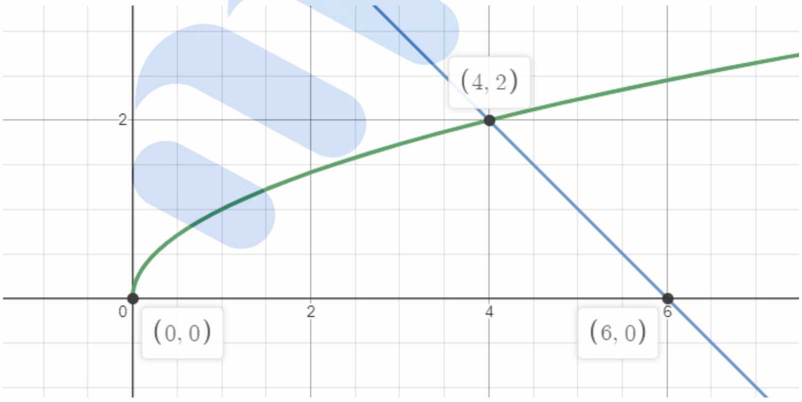
Further Work on Area Exercise Answers

1.

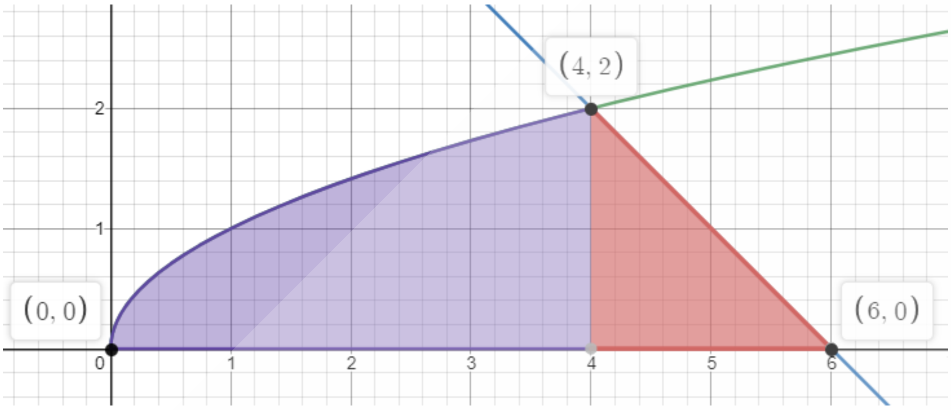
$$Step\ 1: Find\ the\ x-coordinate\ of\ the\ intersection\ point$$

$\sqrt{x} = 6 - x$
$x = (6 - x)^2$
$x = 36 - 12x + x^2$
$x^2 - 13x + 36 = 0$
$(x - 9)(x - 4) = 0$
$\therefore x = 4\ only\ (since\ \sqrt{x} \geq 0\ and\ so\ 6 - x\ must\ be\ \geq 0)$

$$Step\ 2: Draw\ a\ graph\ with\ the\ given\ information$$



Notice here how the two functions intersect, which means that our total area must be split into 2 parts as follows:



$$Step\ 3: Derive\ the\ expressions\ for\ area\ and\ evaluate$$

Let the purple area be A_1 and the red area be A_2, where $A = A_1 + A_2$:

$$\therefore A_1 = \int_0^4 \sqrt{x}\,dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$
$$= \frac{2}{3}(4)^{\frac{3}{2}} - 0$$
$$= \frac{16}{3}\ units^2$$

$$\therefore A_2 = \int_4^6 6 - x\,dx = \left[6x - \frac{x^2}{2}\right]_4^6$$
$$= \left[6(6) - \frac{6^2}{2}\right] - \left[6(4) - \frac{4^2}{2}\right]$$
$$= 18 - 16$$
$$= 2\ units^2$$

Thus, $A = \frac{16}{3} + 2 = 7\frac{1}{3}\ units^2$

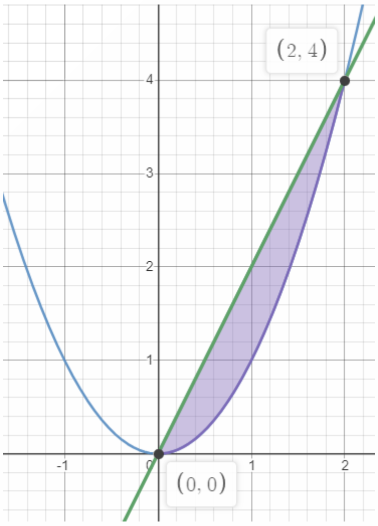
2.

$$Step\ 1: Find\ the\ x - coordinate\ of\ the\ intersection\ points$$

$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$

$$\therefore x = 0\ or\ x = 2$$

$$Step\ 2: Draw\ a\ diagram\ with\ the\ given\ information$$



The area that we want to find is the shaded region, where lower bound is $x = 0$ and upper bound is $x = 2$

$$Step\ 3: Derive\ the\ area\ expression\ and\ evaluate$$

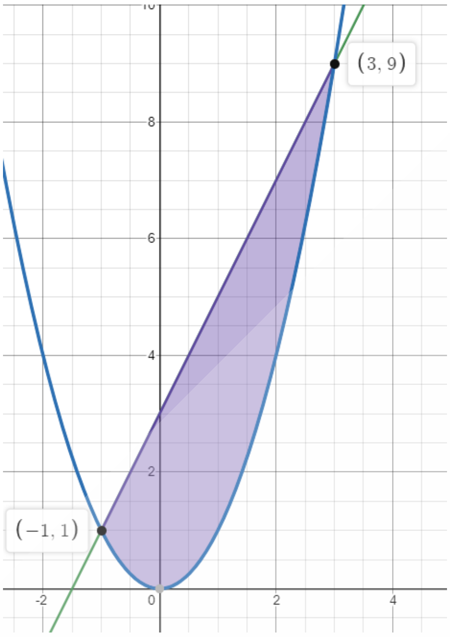
$$\therefore A = \int_0^2 2x - x^2\, dx$$
$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$
$$= \left[2^2 - \frac{2^3}{3}\right] - 0$$
$$= 1\frac{1}{3}\ units^2$$

3.

$$Step\ 1: Find\ the\ x - coordinate\ of\ the\ intersection\ points$$

$$2x + 3 = x^2$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$\therefore x = 3\ or\ x = -1$$

$$Step\ 2: Draw\ a\ diagram\ with\ the\ given\ information$$



The area that we want to find is the shaded region, where lower bound is $x = -1$ and the upper bound is $x = 3$

$$Step\ 3: Derive\ the\ area\ expression\ and\ evaluate$$

$$\therefore A = \int_{-1}^{3} (2x+3) - x^2\, dx$$

$$= \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3}$$

$$= \left[3^2 + 3(3) - \frac{3^3}{3}\right] - \left[(-1)^2 + 3(-1) - \frac{(-1)^3}{3}\right]$$

$$= 9 - -\frac{5}{3}$$

$$= \frac{32}{3}\ units^2$$

4.

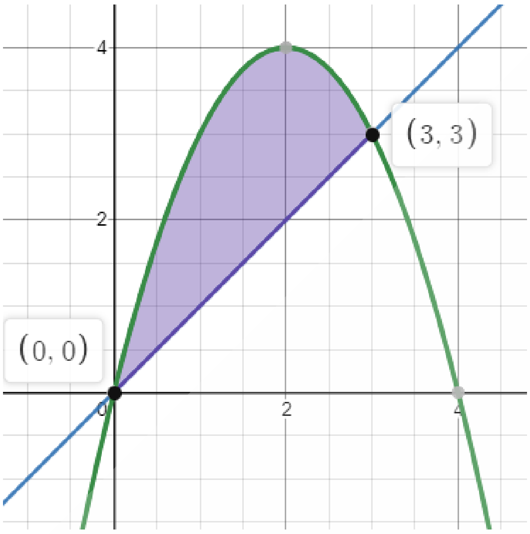
a) Since the straight line passes through the origin, we thus know that its y-intercept $= 0$ and so its equation is $y = mx$

$$At\ the\ point\ of\ intersection\ x = 3, so\ y = 4(3) - 3^2 = 3$$

$$\therefore m = \frac{y}{x} = \frac{3}{3} = 1$$

$$\therefore The\ equation\ of\ the\ line\ is\ y = x$$

b) $Step\ 1: Draw\ a\ diagram\ with\ the\ given\ information$



The area that we want to find is the shaded region, where lower bound is $x = 0$ and the upper bound is $x = 3$

$$Step\ 2: Derive\ the\ area\ expression\ and\ evaluate$$

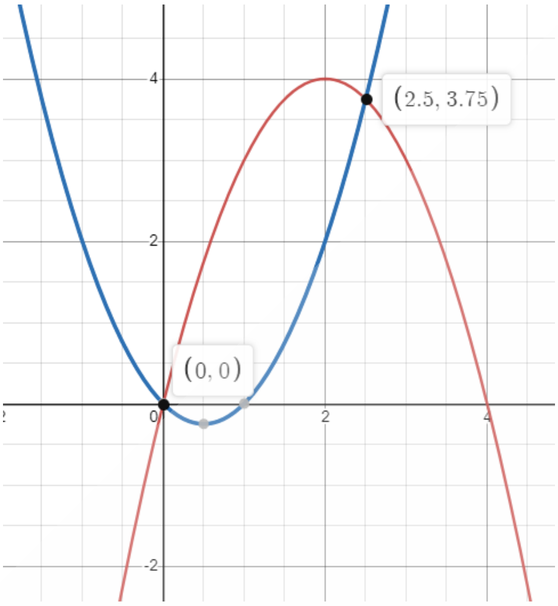
$$\begin{aligned}
\therefore A &= \int_0^3 4x - x^2\ dx \\
&= \left[2x^2 - \frac{x^3}{3}\right]_0^3 \\
&= \left[2(3)^2 - \frac{3^3}{3}\right] - 0 \\
&= 9\ units^2
\end{aligned}$$

5. $Step\ 1: Find\ the\ x - coordinate\ of\ the\ intersection\ points$

$$\begin{aligned}
&4x - x^2 = x^2 - x \\
&2x^2 - 5x = 0 \\
&x(2x - 5) = 0 \\
&\therefore x = 0\ or\ x = \frac{5}{2}
\end{aligned}$$

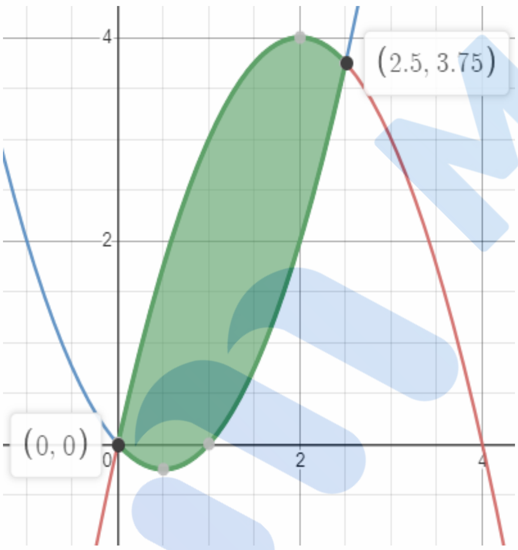
$$Step\ 2: Draw\ a\ diagram\ with\ the\ given\ information$$

In this case, drawing a diagram will also help us to understand the question better:



$Step\ 3$: *$Find\ the\ area\ of\ the\ region\ bound\ between\ the\ two\ curves$*

From the diagram above, the region's lower bound would be $x = 0$ and the upper bound would be $x = \frac{5}{2}$



$$\therefore A = \int_0^{\frac{5}{2}} [(4x - x^2) - (x^2 - x)]\ dx$$

$$= \int_0^{\frac{5}{2}} [5x - 2x^2]\ dx$$

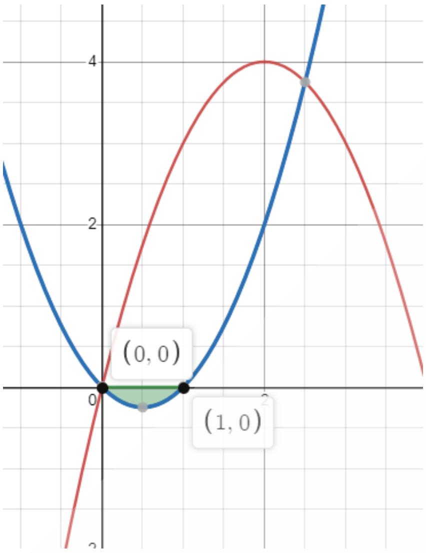
$$= \left[\frac{5x^2}{2} - \frac{2x^3}{3}\right]_0^{\frac{5}{2}}$$

$$= \left[\frac{5}{2}\left(\frac{5}{2}\right)^2 - \frac{2}{3}\left(\frac{5}{2}\right)^3\right] - 0$$

$$= \frac{125}{24}\ units^2$$

$Step\ 4$: *$Find\ the\ area\ bounded\ between\ the\ x-axis\ and\ y = x^2 - x$*

From the diagram above, the region's lower bound would be $x = 0$ and the upper bound would be $x = 1$

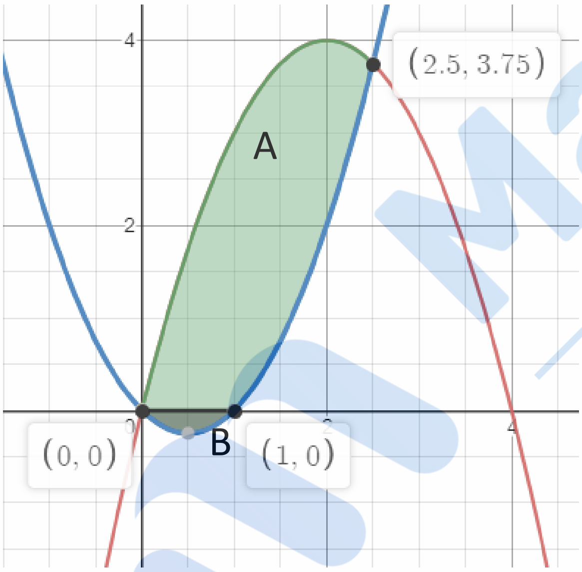


$$\therefore A = \int_0^1 x^2 - x \, dx$$
$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$
$$= \frac{1}{3} - \frac{1}{2}$$
$$= -\frac{1}{6}$$

However, note that area cannot equal to a negative value, so in these cases, since the region is underneath the x-axis, we absolute value our result to get the answer

$$\therefore A = \left|-\frac{1}{6}\right| = \frac{1}{6} \; units^2$$

$$Step\ 5: Find\ the\ ratio\ between\ the\ two\ areas$$



Since area of region A = $\frac{1}{6}$ $units^2$:

Area of region B =

$\frac{125}{24} - \frac{1}{6} = \frac{121}{24}$ $units^2$

Thus, the ratio between the areas A and B is:

$$A : B$$
$$\frac{1}{6} : \frac{121}{24}$$
$$4 : 121$$