Mathify HSC

DIFFERENTIATION

GEOMETRICAL APPLICATIONS: TANGENTS AND NORMAL (V)

Contents include:

- Finding Gradients
- Parallel and Perpendicular Lines
- Finding the Equation of Tangents and Normals
- Finding the Angle of Tangents

• Differentiating to find the Gradient

An important application of differentiation is using the derivative of a function to help find the gradient (which is the rate of change) at any point on the function.

Thus, given a function $f(x)$, to find the gradient of any point on $f(x)$:

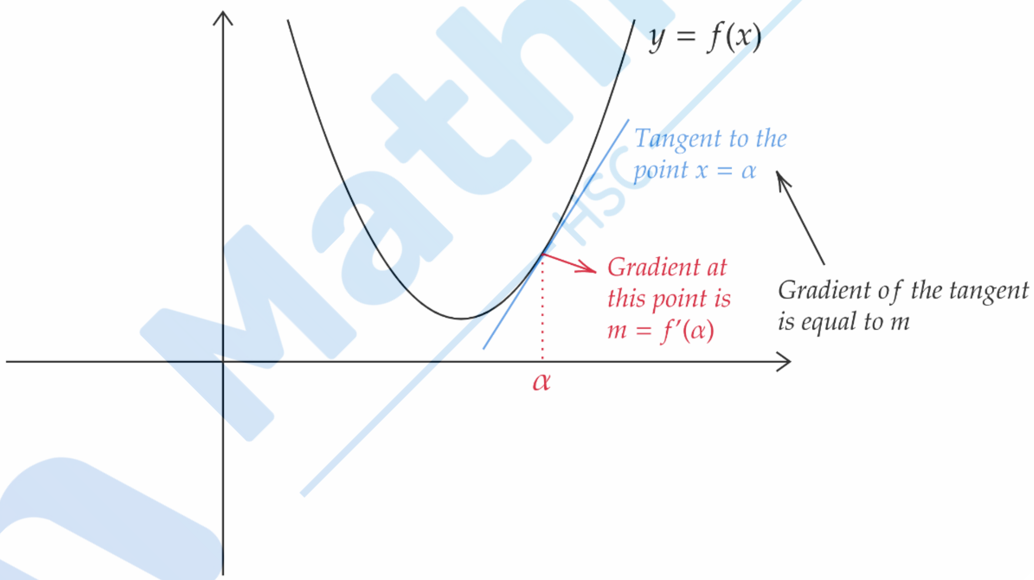
$$\boldsymbol{Step\ 1: Differentiate\ to\ find\ f'(x)}$$

$$\boldsymbol{Step\ 2: Substitute\ the\ x - coordinate\ into\ f'(x)}$$

The gradient, m, to a point with $x -$ coordinate α will therefore be:

$$m = f'(\alpha)$$

This concept is visually represented in the diagram below:



Example 1: Let $f(x) = x^2$, find the gradient of the tangent to the point (2, 4) on the function.

Solution:

$$Step\ 1: Differentiatiate\ f(x)$$

$$f'(x) = \frac{d}{dx}(x^2) = 2x$$

$$Step\ 2: Substitute\ your\ x\ value, 2, into\ f'(x)$$

$$f'(2) = 2 \times 2 = 4$$

$$\therefore m = 4$$

- Parallel and Perpendicular Lines

Recall that given two lines l_1 and l_2 with gradients m_1 and m_2 respectively:

If they are parallel:

$$m_1 = m_2$$

If they are perpendicular:

$$m_1 \times m_2 = -1$$

- Finding the Equation of the Tangent to a Point

Given that we already know the gradient, m, of a certain point $(\alpha, f(\alpha))$ on the function $f(x)$, the equation of the tangent to this point is:

$$y - y_1 = m(x - x_1)$$
$$y - \beta = m(x - f(\alpha))$$

This is known as the **point gradient formula**

After completing this step, we can then rearrange the equation to get our final answer:

$$y = mx + b$$

Where b would be the $y-$ intercept of the tangent

Example 2: Find the equation of the tangent to the curve $y = x^2 - 4x + 4$ at the point on the curve where $x = 3$

Solution:

$$\boldsymbol{Step\ 1: Determine\ the\ gradient\ of\ the\ tangent}$$

$$y' = 2x - 4$$

$$\therefore When\ x = 3, m = 2(3) - 4$$
$$= 2$$

$$\boldsymbol{Step\ 2: Hence\ determine\ the\ equation\ of\ the\ tangent}$$

$When\ x = 3, y = f(3) = 1$

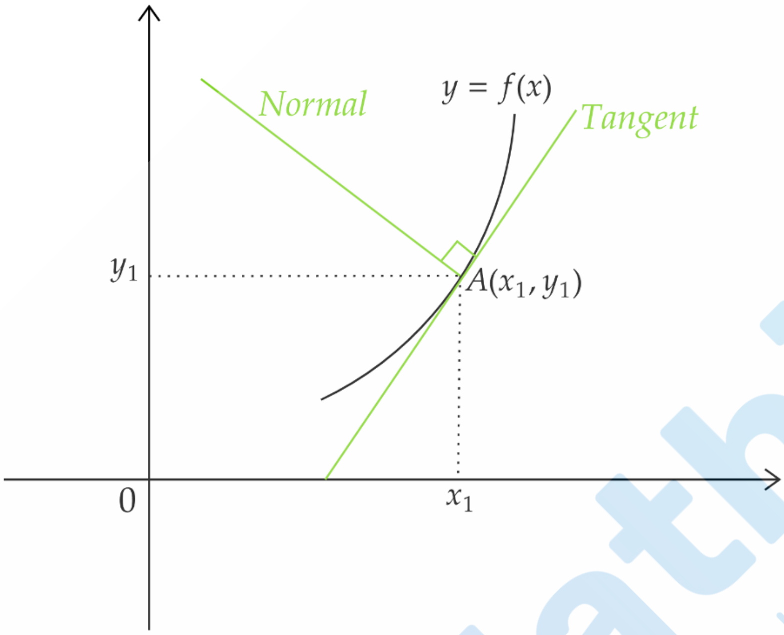
Thus, we now know that the tangent has a gradient of 2 and passes through the point (3, 1)

Now using the point gradient formula:

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 2(x - 3)$$
$$\therefore y = 2x - 5\ is\ the\ equation\ of\ our\ tangent$$

- Finding the Equation of the Normal to a point

The normal to a point is the line that is directly perpendicular to its tangent, as shown in the diagram below:



Therefore, for the gradient of the normal and tangent:

$$m_n \times m_t = -1$$

$$\therefore m_n = -\frac{1}{m_t}$$

Where:

m_n is the gradient of the normal

m_t is the gradient of the tangent

Hence, given that we already know the gradient, m, of a certain point $(\alpha, f(\alpha))$ on the function $f(x)$, the equation of the normal to this point is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - \alpha = -\frac{1}{m}\big(x - f(\alpha)\big)$$

Example 3: Find the equation of the normal to the curve $y = 3 - x - x^2$ where the curve crosses the $y -$ axis

Solution:

$$\textbf{\textit{Step 1: Determine the gradient of the tangent}}$$

Differentiating to find y:

$$y' = -1 - 2x$$

The point when the curve crosses the $y -$ axis is the $y -$ intercept, whose coordinate is (0, 3).

Hence, subbing in $x = 0$ into y' to find the gradient of the tangent, m_t:

$$\therefore m_t = -1 - 2(0) = -1$$

$$\textbf{\textit{Step 2: Determine the gradient of the normal}}$$

Hence, the gradient of the normal m_n is:

$$\begin{aligned} m_n &= -\frac{1}{m_t} \\ &= -\frac{1}{-1} \\ &= 1 \end{aligned}$$

$$\textbf{\textit{Step 3: Determine the equation of the normal}}$$

Using the point gradient formula:

$$y - y_1 = m_n(x - x_1)$$

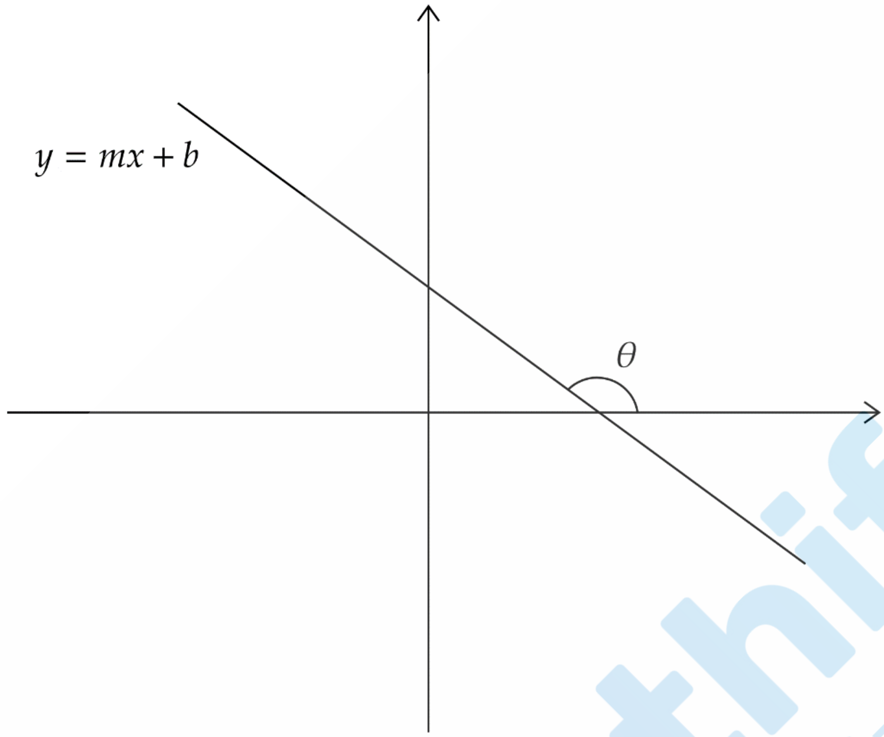
$$\begin{aligned} \therefore y - 3 &= 1(x - 0) \\ y - 3 &= x \\ y &= x + 3 \end{aligned}$$

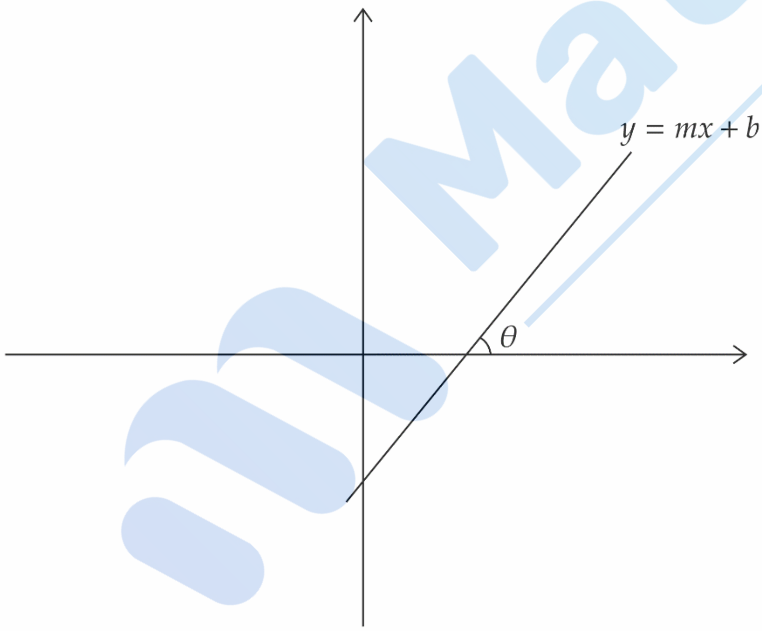
- Finding the angle a tangent makes to the $x -$ axis

If we consider a tangent that intersects through the $x -$ axis, let the angle it makes to the **positive $x -$ axis** be θ, and the gradient of the tangent be m.

It can therefore be said that: $$m = \tan\theta$$

This is represented through the following diagrams:





Example 4: Find the gradient at which the tangent to the function $f(x)$ makes an angle of $45°$ to the x-axis

Solution:

$$m = tan\ \theta$$
$$m = \tan(45°)$$
$$m = 1$$

Equations of the Tangent and Normal Exercises

1. Find the equation of the tangent and normal to the curve $y = \frac{1}{x}$ where $x = -2$

2. Prove that the parabolas $y = 2x^2 - 6x + 5$ and $y = x^2 - 2x + 1$ intersect, and find the equation of the common tangent

3. The normal to the curve $y = (x + 2)^2$ at the point $A(-3, 1)$ meets the curve again at B. Find:

a) The equation of the normal to the point A
b) The coordinates of B
c) The angle that the tangent through point B makes with the $x - axis$

4. Find the coordinates of the points on the curve $y = x^2(2x - 3)$ at which the tangents are parallel to the line $y - 12x = 1$

5. Find the gradient of the parabola $y = x^2 - x - 6$ at the points where it crosses the x-axis

6. For the graph of $f(x) = x^2 - 5x + 4$:

a) Find the gradients at the $x -$ intercept along with the angle of inclination
b) Find the point on the graph where the tangent has gradient -2

7. Find the coordinates of the points on the curve $y = x^2 - 7x + 12$ at which the tangent:

a) Makes an angle of $45°$ with the x-axis
b) Is parallel to the line with equation $3x + y - 4 = 0$
c) Is perpendicular to the line with equation $2y - x + 3 = 0$

8. Find the coordinates of the points on the parabola $y = x^2 - 2x - 8$ at which:

a) The gradient is zero
b) The tangent is parallel to the line $2x + y = 7$

Equation of the Tangent and Normal Exercise Answers

1.

$$y' = -x^{-2}$$

$$When\ x = -2, y = -\frac{1}{2}$$

$$\therefore gradient\ of\ our\ tangent\ is = -(-2)^{-2} = -2^{-2} = -\frac{1}{4}$$

Thus, $y - -\frac{1}{2} = -\frac{1}{4}(x - -2)$ $[using\ our\ point - gradient\ formula]$

$$y + \frac{1}{2} = -\frac{1}{4}(x + 2)$$

$$\therefore y = -\frac{1}{4}x - 1\ is\ the\ equation\ of\ our\ tangent$$

$$N \times m = -1,$$

$$\therefore N = -\frac{1}{m} = \frac{-1}{-\frac{1}{4}} = 4$$

$$Thus, y - -\frac{1}{2} = 4(x - -2)$$

$$\therefore y = 4x + 7\frac{1}{2}\ is\ the\ equation\ of\ our\ normal$$

2. $f(x) = 2x^2 - 6x + 5\ and\ g(x) = x^2 - 2x + 1$

$$\underline{Step\ 1: Find\ where\ f(x)\ and\ g(x)\ intersect}$$

$$f(x) = g(x)$$
$$2x^2 - 6x + 5 = x^2 - 2x + 1$$
$$x^2 - 4x + 4 = 0$$
$$(x - 2)^2 = 0$$

$$\therefore x = 2\ is\ the\ x - coordinate\ of\ the\ point\ of\ intersection$$

Now we also need to find the $y - coordinate$ so just sub $x = 2$ into $f(x)\ or\ g(x)$:

$g(2) = 4 - 4 + 1 = 1$

$$\therefore (2, 1)\ is\ the\ coordinates\ of\ the\ point\ of\ intersection$$

$$\underline{Step\ 2: Find\ the\ gradient\ of\ the\ tangent\ to\ the\ point}$$

To do this step, we may either sub $x = 2$ into $f'(x)\ or\ g'(x)$. It should produce the same answer regardless.

$$g'(x) = 2x - 2$$
$$m = g'(2) = 2$$

$$\underline{Step\ 3: Find\ the\ equation\ of\ the\ tangent}$$

$$\therefore We\ use\ the\ point - gradient\ formula:$$
$$y - y_1 = m(x - x_1)$$
$$y - 1 = 2(x - 2)$$

$$\therefore y = 2x - 3\ is\ the\ equation\ of\ the\ tangent$$

3.

a) $\underline{Step\ 1: Find\ the\ gradient\ of\ the\ tangent\ to\ the\ point\ A\ (-3,1)}$

$$y' = 2x + 4$$

$$\therefore m = 2(-3) + 4$$
$$= -2$$

$$\underline{Step\ 2: Find\ the\ gradient\ of\ the\ normal\ and\ its\ equation}$$

$$Since\ N \times m = -1\ and\ m = -2:$$
$$N = \frac{-1}{-2} = \frac{1}{2}$$

$$\therefore Using\ the\ point - gradient\ formula:$$
$$y - y_1 = m(x - x_1)$$
$$y - 1 = \frac{1}{2}(x - \ -3)$$

$$\therefore y = \frac{1}{2}x + \frac{5}{2}\ is\ the\ equation\ of\ our\ normal$$

b)

$$\underline{Step\ 1: Equate\ the\ normal\ and\ parabola\ curve\ formulas\ and\ solve\ for\ x}$$

$$\frac{1}{2}x + \frac{5}{2} = x^2 + 4x + 4$$
$$x^2 + 3\frac{1}{2}x + \frac{3}{2} = 0$$
$$2x^2 + 7x + 3 = 0$$
$$(2x + 1)(x + 3) = 0$$

$$\therefore x = -\frac{1}{2}\ and\ x = -3\ are\ points\ of\ intersections,$$
$$so\ x = -\frac{1}{2} must\ be\ the\ x - coordinate\ of\ B$$

$$\underline{Step\ 2: Find\ the\ y - coordinate\ of\ B}$$

Sub $x = -\frac{1}{2}$ into either the parabola or normal's equation. Result is the same

$$When\ x = -\frac{1}{2}, y = \frac{1}{2}\left(-\frac{1}{2}\right) + \frac{5}{2} = \frac{9}{4}$$

$$\therefore B\left(-\frac{1}{2},\frac{9}{4}\right)$$

c)

$$\underline{Step\ 1: Find\ the\ gradient\ of\ tangent\ to\ parabola\ at\ B}$$

$$y' = 2x + 4$$
$$m = 2\left(-\frac{1}{2}\right) + 4 = 3$$

$$\underline{Step\ 2: Find\ the\ angle, \theta, that\ it\ makes\ with\ the\ x - axis}$$

This is done by using the formula:

$$m = tan\theta$$

$$\therefore \theta = \tan^{-1} 3$$

4.

$y' = 6x^2 - 6x$

$$\underline{Step\ 1: Equate\ gradient\ of\ tangent\ to\ the\ line}$$

Since the line is given by equation: $y = 12x + 1$, the gradient is $m = 12$.

$$12 = 6x^2 - 6x$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$

$$\therefore x = 2\ and\ x = -1\ are\ the\ x - coordinates\ of\ where\ m = 12$$

$$\underline{Step\ 2: Find\ the\ y - coordinates\ of\ these\ points}$$

$When\ x = 2, y = 2(2)^3 - 3(2)^2 = 4$
$When\ x = -1, y = 2(-1)^3 - 3(-1)^2 = -5$

$$\therefore (2, 4)\ and\ (-1, -5)\ are\ points\ where\ the\ tangents\ are\ parallel$$

5.

$$\underline{Step\ 1: Differentiate\ y}$$

$$y' = 2x - 1$$

$$\underline{Step\ 2: Find\ the\ x - intercepts}$$

$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$\therefore x = 3\ or - 2$$

$$\underline{Step\ 3: Find\ the\ Gradients}$$

Substituting our $x -$ values into y' to find the gradients:

$$\therefore m = 2(3) - 1$$
$$= 5$$
or
$$\therefore m = 2(-2) - 1$$
$$= -5$$

6.

a)

$\underline{Step\ 1: Find\ f'(x)}$

$$f'(x) = 2x - 5$$

$\underline{Step\ 2: Find\ the\ x - intercepts}$

$$f(x) = x^2 - 5x + 4$$
$$= (x - 4)(x - 1)$$

$$\therefore x - intercepts\ when\ x = 4\ or\ x = 1$$

$\underline{Step\ 3: Find\ the\ gradients\ at\ these\ points}$

At $x = 4$:

$$f'(4) = 2(4) - 5$$
$$= 3$$

At $x = 1$:

$$f'(1) = 2(1) - 5$$
$$= -3$$

$\underline{Step\ 4: Find\ the\ angle\ of\ inclination\ at\ these\ points}$

At $x = 4$:

$$3 = \tan\theta$$
$$\therefore \theta = \tan^{-1} 3$$

At $x = 1$:

$$-3 = \tan\theta$$
$$\therefore \theta = \pi - \tan^{-1} 3$$

Note here that the reason $\theta \neq \tan^{-1}(-3)$ is because the angle of inclination is defined as the angle that a line makes with the **positive** $x - axis$. Hence, we say that θ is defined in the domain $0° \leq \theta \leq 180°$.

b)

Since we want to find the point where gradient is -2, we are going to equate $f'(x)$ to -2 and then solve for x:

$$-2 = 2x - 5$$
$$2x = 3$$
$$\therefore x = \frac{3}{2}$$

Now since the question is asking for a point, we also have to find the $y -$ coordinate when $x = \frac{3}{2}$:

$$y = \left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 4$$
$$= \frac{9}{4} - \frac{15}{2} + 4$$
$$= -\frac{5}{4}$$

Hence, the point is $\left(\frac{3}{2}, -\frac{5}{4}\right)$

7. a)

Differentiating to find y' expression:

$$y' = 2x - 7$$

$$m = \tan 45°$$
$$\therefore m = 1$$

Now equate our y' equation to 1 in order to find the x - coordinate of when this occurs

$$1 = 2x - 7$$
$$\therefore x = 4$$

$$\therefore \mathit{The\ coordinates\ of\ the\ point\ is}\ (4, 0)$$

b) Line with equation $3x + y - 4 = 0$ may be rewritten as:

$$y = -3x + 4$$

Thus, the gradient of this line is -3. Since we want the tangent to be parallel to this line, we let our tangent's gradient $= -3$.

$$y' = 2x - 7$$
$$= -3$$
$$2x = 4$$
$$x = 2$$

$$\therefore \mathit{The\ coordinates\ of\ the\ point\ is}\ (2,2)$$

c) Line with equation $2y - x + 3 = 0$ may be rewritten as:

$$y = \frac{x}{2} - \frac{3}{2}$$

Thus, the gradient of this line is $\frac{1}{2}$. Since we want the tangent to be perpendicular to this line, we let our tangent's gradient = -2 since $\frac{1}{2} \times -2 = -1$.

$$y' = 2x - 7 = -2$$
$$2x = 5$$
$$x = \frac{5}{2}$$

$$\therefore \textit{The coordinates of the point is } (\frac{5}{2}, \frac{3}{4})$$

8.

a) Differentiating to find y':

$$y' = 2x - 2$$
$$\therefore 2x - 2 = 0 \; (\textit{since we want gradient} = 0)$$
$$x = 1$$
$$\therefore \textit{The coordinates of the point is } (1, -9)$$

b) Line with equation $2x + y = 7$ may be rewritten as:

$$y = 7 - 2x$$

Thus, the gradient of this line is -2, and since we want the tangent to be parallel to this line, we aim to make our tangent's gradient = -2.

$$2x - 2 = -2$$
$$\therefore -2x = 0$$
$$\therefore x = 0$$

$$\therefore y = (0)^2 - 2(0) - 8$$
$$= -8$$

$$\therefore \textit{The coordinates of the point is } (0, -8)$$