DIFFERENTIATIAL CALCULUS

NATURAL GROWTH AND DECAY (IV)

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- Exponential Growth and Decay Questions

Exponential growth occurs when the rate of change of a quantity increases as the population itself increases. What this means is that the quantity will increase more and more with time. Some examples of this include population growth, growth of bacteria, and more recently, the number of cases of COVID-19. The same logic applies to exponential decay, except the value is decreasing more and more with time.

In each case of **exponential growth**, the following equation follows:

$$y = Ae^{kx}$$

Then

$$\begin{aligned}\frac{dy}{dx} &= kAe^{kx} \\ &= ky\end{aligned}$$

This states that the rate of change of one quantity, y, with respect to another quantity, x, is proportional to y.

For Example: The rate of growth of bacteria in a culture is proportional to the number of bacteria present at any instant. That is:

$$\frac{dN}{dt} = kN$$

In each case of **exponential decay**, the following equation follows:

$$y = Ae^{-kx}$$

Then

$$\begin{aligned}\frac{dy}{dx} &= -kAe^{-kx} \\ &= -ky\end{aligned}$$

This states that the rate of change of one quantity, y, with respect to another quantity, x, is proportional to y. In this case, the rate of change is negative since the quantity is decreasing

For example: The rate of decay of a radioactive element is proportional to the mass of that element present at any time. That is:

$$\frac{dM}{dt} = -kM$$

<u>Note:</u> The negative sign exists since it is decay that we are dealing with

- Solving Exponential Growth and Decay Questions

For most of these types of questions, we are given a bunch of initial conditions with the formula $y = Ae^{kx}$. These questions typically follow the same general steps:

$$\boldsymbol{Step\ 1: Find\ the\ value\ of\ A}$$

We do this by letting $x = 0$ and subbing in the appropriate value of y. Since $e = 0$, this means that $e^{kx} = e^0 = 1$.

$$\boldsymbol{Step\ 2: Find\ the\ value\ of\ k}$$

We do this by subbing x and y values for another given condition, along with our value of A which we should now know.

Example 1: A vessel containing water is being emptied and the volume $V(t)$ cubic metres remaining in the vessel after t minutes is given by:

$$V(t) = Ae^{-kt}$$

a) If $V(0) = 100$, find the value of A

When $t = 0, V = 100$

$$\begin{aligned} \therefore 100 &= Ae^{-k\times 0} \\ &= A \times 1 \\ &= A \end{aligned}$$

b) If $V(5) = 90$, find the value of k

When $t = 5, V = 90$

$$\begin{aligned} \therefore 90 &= 100e^{-5k} \\ e^{-5k} &= \frac{9}{10} \\ k &= -\frac{1}{5}\ln\left(\frac{9}{10}\right) \\ \therefore k &\approx 0.02 \end{aligned}$$

c) Find $V(20)$

$$V(20) = 100e^{20\times -\frac{1}{5}\ln\left(\frac{9}{10}\right)}$$
$$\therefore V(20) \approx 67\ (using\ calculator)$$

Exponential Growth and Decay Exercises

1. The charge, Q units, on the plate of a condenser t seconds after it starts to discharge is given by the formula

$$Q = Ae^{-kt}$$

a) If the original charge is 5000 units, find the value of A
b) If $\frac{dQ}{dt} = -2000$ when $Q = 1000$, find the value of k
c) Find the rate of discharge when $Q = 5000$

2. In a certain bacterial culture, the rate of increase is proportional to the number of bacteria present
a) If the number doubles in 3 hours, find the hourly growth rate
b) How many bacteria are there after 9 hours, if the original population is 10^4
c) After how many hours are there 4×10^4 bacteria?

3. The rate of increase in the number N of bacteria in a certain culture is given by $\frac{dN}{dt} = 0.15N$ where t is time in hours
a) If the original number of bacteria is 1000, express N as a function of t
b) After how many hours has the original number of bacteria doubled and what is the rate of increase at that time?

4. The rate of increase in the population $P(t)$ of a particular island is given by the equation $\frac{d}{dt}P(t) = kP(t)$ where t is time in years. In 1970 the population was 1000 and in 1980 it had decreased to 800
a) Find k, the annual growth rate
b) In how many years will the population be half that in 1970

Exponential Growth and Decay Exercise Answers

1.
a) $When\ t = 0, Q = 5000$

$$\therefore 5000 = Ae^{-k\times 0}$$
$$= A$$

b) $\frac{dQ}{dt} = -kAe^{-kt} = -k5000e^{-kt} = -kQ$
$When\ Q = 1000,$
$1000 = 5000e^{-kt}$
$So\ \frac{dQ}{dt} = -1000k$

$$\therefore -2000 = -1000k$$
$$k = 2$$

c) When $Q = 5000$,

$$\frac{dQ}{dt} = -kQ$$
$$= -5000 \times 2 \ (from\ b)$$
$$= -10000$$

$$Thus, the\ rate\ of\ discharge\ is\ 10000\ units/s$$

<u>Note:</u> Don't forget units for rates!

2. The rate of increase is proportional to N, the number of bacteria present

$$\therefore \frac{dN}{dt} = kN, where\ k\ is\ a\ constant$$

a) Since $\frac{dN}{dt} = kN$,

$$N = Ae^{kt}$$

When $t = 0$,

$$N = A$$

When $t = 3$,

$$N = Ae^{3k} = 2A\ (since\ it\ doubles)$$
$$\therefore e^{3k} = 2$$
$$k = \frac{1}{3}\ln(2)$$
$$= 0.23$$

b) Since the original population is $10^4 = 10000$,

$$Step\ 1: Find\ the\ value\ of\ A$$

When $t = 0$,

$$N = A = 10000$$
$$Step\ 2: Find\ N\ when\ t = 9$$

$$N = 10000e^{kt}, where\ k = 0.23\ from\ (a)$$

When $t = 9$

$$N = 10000e^{0.23\times 9}$$
$$= 79248.23$$

c) $N = 4 \times 10^4$

$$4 \times 10^4 = 10000e^{kt}, where\ k = 0.23\ from\ a)$$

$$e^{kt} = 4$$
$$t = \frac{1}{k} ln4$$
$$= 6.03\ hours$$

3.

a) $Step\ 1: Derive\ the\ expression\ for\ N\ in\ terms\ of\ t$

$$\frac{dN}{dt} = 0.15N$$
$$\therefore \int \frac{dN}{N} = \int 0.15\ dt$$
$$\ln|N| = 0.15t + C$$
$$N = e^{0.15t+C}$$
$$= Ae^{0.15t}$$

$$Step\ 2: Find\ what\ A\ is\ equal\ to$$

When $t = 0, N = 1000$

$$\therefore A = 1000$$

b) When original number of bacteria doubles, $N = 2000$

$$\therefore N = 2000 = 1000e^{0.15t}$$
$$e^{0.15t} = 2$$
$$t = \frac{1}{0.15} \ln(2)$$
$$= 4.62\ hours$$

$$At\ that\ time, the\ rate\ of\ increase \frac{dN}{dt} = 0.15 \times 2000 = 300\ bacteria/hr$$

4.

a) $Step\ 1: Derive\ the\ expression\ for\ P(t)\ in\ terms\ of\ t$

$$\frac{d\ P(t)}{dt} = kP(t)$$
$$\therefore \int \frac{dP(t)}{P(t)} = \int k\ dt$$
$$\ln|P(t)| = kt + C$$
$$P(t) = e^{kt+C}$$
$$= Ae^{kt}$$

$$Step\ 2: Determine\ the\ value\ of\ A$$

$When\ t = 0, i.e. the\ year\ is\ 1970, P(t) = 1000$

$$1000 = A$$

$$Step\ 3: Determine\ the\ value\ of\ k$$

$$When\ t = 10, i.e. the\ year\ is\ 1980, P(t) = 800$$

$$\therefore 800 = 1000e^{10k}$$
$$e^{10k} = \frac{8}{10}$$
$$k = \frac{1}{10}\ln\left(\frac{8}{10}\right)$$
$$= -0.0223$$

b) If population is half that in 1970, then $P(t) = 500$

$$500 = 1000e^{kt}, where\ k = -0.0223\ from\ a)$$

$$\therefore e^{kt} = \frac{1}{2}$$
$$t = \frac{1}{k}\ln\left(\frac{1}{2}\right)$$
$$\approx 31\ years$$