# INTEGRATION

## EXPONENTIALS (V)

Contents include:

- Integrating Natural Exponentials
- Standard Exponentials

- Integration of $e^x$

Recalling from differentiation that:

$$(e^x)' = e^x$$

Reversing this therefore gives:

$$\int e^x \, dx = e^x + C$$

Moreover, looking at the standard form of differentiation:

$$(e^{ax+b})' = ae^{ax+b}$$

Reversing this therefore gives the standard forms of exponential integrals:

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

This may also be **done using substitution**, by letting $u = ax + b$!

$$\int f'(x) e^{f(x)} dx = e^{f(x)} + C$$

This may also be **done using substitution,** by letting $u = f(x)$!

**Note** that it's good practice to use substitution in your working out, that way you don't have to remember or rely on using the formula sheet for all these different formulas when integrating!

**Example 1:** Find the integral $\int e^{3x+2} \, dx$

If we look at the standard form, $a = 3$ and $b = 2$:

$$\therefore \int e^{3x+2} dx = \frac{1}{3} e^{3x+2} + C$$

**Example 2:** Find the integral $\int (1 - x + e^x) \, dx$

$$\therefore \int (1 - x + e^x) \, dx = x - \frac{x^2}{2} + e^x + C$$

**Example 3:** Evaluate $\int_0^3 (2 + e^{-2x}) \, dx$

$$\therefore \int_0^3 (2+e^{-2x})\,dx = \left[2x - \frac{1}{2}e^{-2x}\right]_0^3$$
$$= \left(2\times 3 - \frac{1}{2}e^{-6}\right) - \left(0 - \frac{1}{2}e^{0}\right)$$
$$= 6 - \frac{1}{2}e^{-6} + \frac{1}{2}$$
$$= 6\frac{1}{2} - \frac{1}{2}e^{-6}$$

**Note:** Don't forget that $e^0 = 1$, a common mistake is saying that it's equal to 0

- Integration of $a^x$

Let's say instead of being asked to find $\int e^x dx$, we are instead tasked with integrating a different exponential such as $\int 2^x dx$. In this scenario, we must remember that:

$$\int a^x dx = \frac{1}{\ln a}a^x + C$$

Where $a$ is any constant

The proof for this involves a couple of logarithm laws as follows:

Since $a = e^{\ln a}$ [logarithm law]:

$$\therefore \int a^x\,dx = \int \left(e^{\ln a}\right)^x dx \ \ [logarithm\ law]$$
$$= \int e^{x\ln a}dx \ \ [logarithm\ law]$$
$$= \frac{1}{\ln a}e^{x\ln a} + C$$
$$= \frac{1}{\ln a}a^x + C \ [logarithm\ law]$$

Moreover, more standard forms that should be remembered include:

$$\int a^{cx+d}dx = \frac{a^{cx+d}}{c\ln a} + C$$

$$\int f'(x)\,a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + C$$

Both of these can once again also be found/proved using substitution!

**Example 4:** Evaluate $\int 4^x \, dx$

Recalling the standard integral:

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

$$\therefore \int 4^x \, dx = \frac{1}{\ln 4} 4^x + C$$

**Example 5:** Evaluate the integral $\int_0^2 3^{5x-4} dx$ to the nearest 2 decimal places

Solution:

Recalling the standard integral from the formula sheet:

$$\int a^{cx+d} dx = \frac{a^{cx+d}}{c \ln a} + C$$

$$\begin{aligned} \therefore \int_0^2 3^{5x-4} dx &= \left[\frac{3^{5x-4}}{5 \ln 3}\right]_0^2 \\ &= \left(\frac{3^{10-4}}{5 \ln 3}\right) - \left(\frac{3^{0-4}}{5 \ln 3}\right) \\ &= \frac{3^6}{5 \ln 3} - \frac{3^{-4}}{5 \ln 3} \\ &= 132.71 \end{aligned}$$

## Integral of Exponential Exercises

1. Find each indefinite integral

a) $\int 10e^{2x} \, dx$

b) $\int 12e^{7-2x} \, dx$

c) $\int \left(7 - \frac{1}{2} e^{1-3x}\right) dx$

d) $\int (3x - 4e^{4x+3}) \, dx$

e) $\int \left(4x^3 - 2 + 3e^{\frac{1}{3}x+2}\right) dx$

2. Evaluate the following definite integrals:

a) $\int_2^3 12e^{4x-5} \, dx$

b) $\int_1^2 6e^{8-3x} \, dx$

c) $\int_{-\frac{1}{2}}^{\frac{1}{2}} (3x - 5 + e^{3-2x}) \, dx$

d) $\int_{-1}^{3}(4x^3 + 9e^{6x})\,dx$

e) $\int_{-2}^{0}(12x - 7x^2 - e^{4x-3})\,dx$

3. Find each indefinite integral

a) $\int 5^{3x} dx$

b) $2\int 4^{6x+5} dx$

c) $\int 4 - 4^{1-x}\,dx$

d) $\int 3x^2 - 7^{2-3x}\,dx$

## Integral of Exponential Exercise Answers

1.

a) $\int 10e^{2x}\,dx$

$$\begin{aligned}\int 10e^{2x}\,dx &= \frac{10}{2}e^{2x} + C \\ &= 5e^{2x} + C\end{aligned}$$

b) $\int 12e^{7-2x}\,dx$

Let $u = 7 - 2x$

$$\frac{du}{dx} = -2, \qquad dx = \frac{du}{-2}$$

$$\begin{aligned}\therefore \int 12e^{7-2x}\,dx &= \int 12e^{u} \times \frac{du}{-2} \\ &= \frac{12}{-2}e^{u} + C \\ &= -6e^{7-2x} + C\end{aligned}$$

c) $\int 7 - \frac{1}{2}e^{1-3x}\,dx$

Let $u = 1 - 3x$

$$\frac{du}{dx} = -3, \qquad dx = \frac{du}{-3}$$

$$\therefore \int 7-\frac{1}{2}e^{1-3x}\,dx = \int 7\,dx - \int \frac{1}{2}e^{u} \times \frac{du}{-3}$$
$$= 7x + \frac{1}{6}e^{u} + C$$
$$= 7x + \frac{1}{6}e^{1-3x} + C$$

d) $\int (3x - 4e^{4x+3})\,dx$

Let $u = 4x + 3$

$$\frac{du}{dx} = 4, \qquad dx = \frac{du}{4}$$
$$\therefore \int (3x - 4e^{4x+3})\,dx = \int 3x\,dx - \int 4e^{u} \times \frac{du}{4}$$
$$= \frac{3x^2}{2} - e^{u} + C$$
$$= \frac{3x^2}{2} - e^{4x+3} + C$$

e) $\int \left(4x^3 - 2 + 3e^{\frac{1}{3}x+2}\right) dx$

Let $u = \frac{1}{3}x + 2$

$$\frac{du}{dx} = \frac{1}{3}, \qquad dx = 3du$$
$$\therefore \int \left(4x^3 - 2 + 3e^{\frac{1}{3}x+2}\right) dx = \int 4x^3 dx - \int 2\,dx + \int 3e^{u} \times 3du$$
$$= \frac{4x^4}{4} - 2x + 9e^{u} + C$$
$$= x^4 - 2x + 9e^{\frac{1}{3}x+2} + C$$

2.

a) $\int_2^3 12e^{4x-5}dx$

Let $u = 4x - 5$

$$\frac{du}{dx} = 4, \qquad dx = \frac{du}{4}$$

$$When\ x = 2, u = 3$$
$$When\ x = 3, u = 7$$

$$\int_2^3 12e^{4x-5}dx = \int_3^7 12e^{u} \times \frac{du}{4}$$
$$= \int_3^7 3e^{u}du$$
$$= [3e^{u}]_3^7$$
$$= 3e^7 - 3e^3$$

b) $\int_1^2 6e^{8-3x} dx$

Let $u = 8 - 3x$

$$\frac{du}{dx} = -3, \qquad dx = -\frac{du}{3}$$

$$When\ x = 1, u = 5$$
$$When\ x = 2, u = 2$$

$$\int_1^2 6e^{8-3x} dx = \int_5^2 6e^u \times -\frac{du}{3}$$
$$= \int_2^5 2e^u\, du$$
$$= [2e^u]_2^5$$
$$= 2e^5 - 2e^2$$

c) $\int_{-\frac{1}{2}}^{\frac{1}{2}} (3x - 5 + e^{3-2x}) dx$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (3x - 5 + e^{3-2x}) dx = \left[\frac{3x^2}{2} - 5x - \frac{1}{2}e^{3-2x}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$
$$= \left(\frac{3}{2} \times \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) - \frac{1}{2}e^{3-1}\right) - \left(\frac{3}{2} \times \left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) - \frac{1}{2}e^{3+1}\right)$$
$$= \frac{3}{8} - \frac{5}{2} - \frac{1}{2}e^2 - \left(\frac{3}{8} + \frac{5}{2} - \frac{1}{2}e^4\right)$$
$$= -5 - \frac{1}{2}e^2 + \frac{1}{2}e^4$$

d) $\int_{-1}^{3} (4x^3 + 9e^{6x})\, dx$

$$\int_{-1}^{3} (4x^3 + 9e^{6x})\, dx = \left[\frac{4x^4}{4} + \frac{9}{6}e^{6x}\right]_{-1}^{3}$$
$$= \left[x^4 + \frac{3}{2}e^{6x}\right]_{-1}^{3}$$
$$= \left[(3)^4 + \frac{3}{2}e^{18}\right] - \left[(-1)^4 + \frac{3}{2}e^{-6}\right]$$
$$= \left[81 + \frac{3}{2}e^{18}\right] - \left[1 + \frac{3}{2}e^{-6}\right]$$
$$= 80 + \frac{3}{2}e^{18} - \frac{3}{2}e^{-6}$$

e) $\int_{-2}^{0} (12x - 7x^2 - e^{4x-3})\, dx$

$$\int_{-2}^{0} (12x - 7x^2 - e^{4x-3})\, dx = \left[6x^2 - \frac{7}{3}x^3 - \frac{1}{4}e^{4x-3}\right]_{-2}^{0}$$

$$= \left(6 \times 0 - \frac{7}{3} \times (0)^3 - \frac{1}{4}e^{-3}\right) - \left(6 \times (-2)^2 - \frac{7}{3} \times (-2)^3 - \frac{1}{4}e^{-11}\right)$$
$$= -\frac{1}{4}e^{-3} - \left(24 + \frac{7}{3} \times 8 - \frac{1}{4}e^{-11}\right)$$
$$= -\frac{1}{4}e^{-3} - \frac{128}{3} + \frac{1}{4}e^{-11}$$

3.

a) $\int 5^{3x} dx$

Recalling the standard integral:

$$\int a^{cx+d}\, dx = \frac{a^{cx+d}}{c \ln a} + C$$
$$\therefore \int 5^{3x} dx = \frac{5^{3x}}{3 \ln 5} + C$$

b) $2 \int 4^{6x+5} dx$

Recalling the standard integral:

$$\int a^{cx+d}\, dx = \frac{a^{cx+d}}{c \ln a} + C$$
$$\therefore 2 \int 4^{6x+5} dx = 2 \times \frac{4^{6x+5}}{6 \ln 4} + C$$
$$= \frac{4^{6x+5}}{3 \ln 4} + C$$

c) $\int 4 - 4^{1-x}\, dx$

$$\int 4 - 4^{1-x}\, dx = \int 4\, dx - \int 4^{1-x} dx$$

Recalling the standard integral:

$$\int a^{cx+d}\, dx = \frac{a^{cx+d}}{c \ln a} + C$$
$$\therefore \int 4\, dx - \int 4^{1-x} dx = 4x - \frac{4^{1-x}}{-1 \cdot \ln 4} + C$$
$$= 4x + \frac{4^{1-x}}{\ln 4} + C$$

d) $\int 3x^2 - 7^{2-3x}\, dx$

$$\int 3x^2 - 7^{2-3x}\, dx = \int 3x^2 dx - \int 7^{2-3x} dx$$

Recalling the standard integral:

$$\int a^{cx+d}\,dx = \frac{a^{cx+d}}{c\ln a} + C$$

$$\therefore \int 3x^2 dx - \int 7^{2-3x} dx = 3\cdot\frac{x^3}{3} - \frac{7^{2-3x}}{-3\cdot\ln 7} + C$$

$$= x^3 + \frac{7^{2-3x}}{3\ln 7} + C$$