INTEGRATION

AREAS BOUNDED BY THE Y - AXIS (XII)

Contents include:

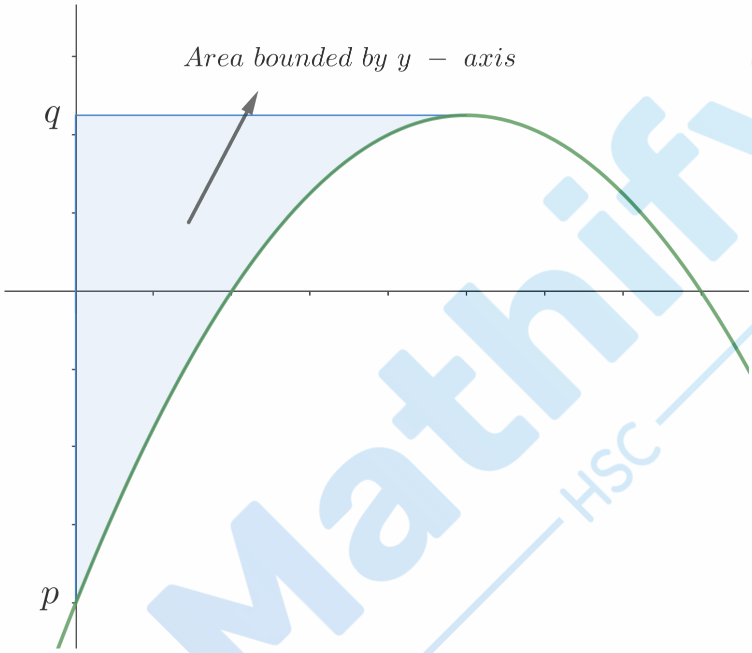
- Areas Bounded by the Y - Axis
- Splitting Areas

- Integration with respect to y

So far, we have been able to find the area between a function and the $x -$ axis through using a definite integral like so:

$$Area\ bounded\ by\ x - axis = \int_a^b f(x)dx$$

Let's say we are given a function $y = f(x)$, and are asked to find the area bounded between the graph and the $y - axis$ as shown:



To directly find the shaded area, we can learn to integrate with respect to y using the following steps:

$$\boldsymbol{Step\ 1: Rearrange\ your\ equation\ to\ make\ x\ the\ subject}$$

In other words, you should have an equation now like $x = f(y)$. Remember that this process is NOT the same as finding the inverse. We are purely making x the subject, nothing else.

$$\boldsymbol{Step\ 2: Form\ integral\ expression\ with\ appropriate\ bounds}$$

The integral expression should resemble:

$$Area\ bounded\ by\ y - axis = \int_p^q f(y)\,dy$$

Where p and q are the lower and upper bound values of y respectively

$$\boldsymbol{Step\ 3: Integrate!}$$

The integration step should be done like normal, except now with y as the variable

Here's an example to demonstrate:

Example 1: Find the area bounded between the curve $y = x^3$ and the $y -$ axis from $y = 1$ to $y = 8$

$$Step\ 1: Rearrange\ to\ make\ x\ the\ subject$$

$$x = \sqrt[3]{y}$$

$$Step\ 2: Form\ Integral\ expression$$

$$A = \int_1^8 \sqrt[3]{y}\, dy$$

$$Step\ 3: Evaluate\ the\ Integral$$

$$\begin{aligned}\int_1^8 \sqrt[3]{y}\, dy &= \int_1^8 y^{\frac{1}{3}}\, dy \\ &= \left[\frac{3}{4}y^{\frac{4}{3}}\right]_1^8 \\ &= \frac{3}{4} \times 8^{\frac{4}{3}} - \frac{3}{4} \times 1^{\frac{4}{3}} \\ &= \frac{3}{4} \times 16 - \frac{3}{4} \\ &= 11\frac{1}{4}\end{aligned}$$

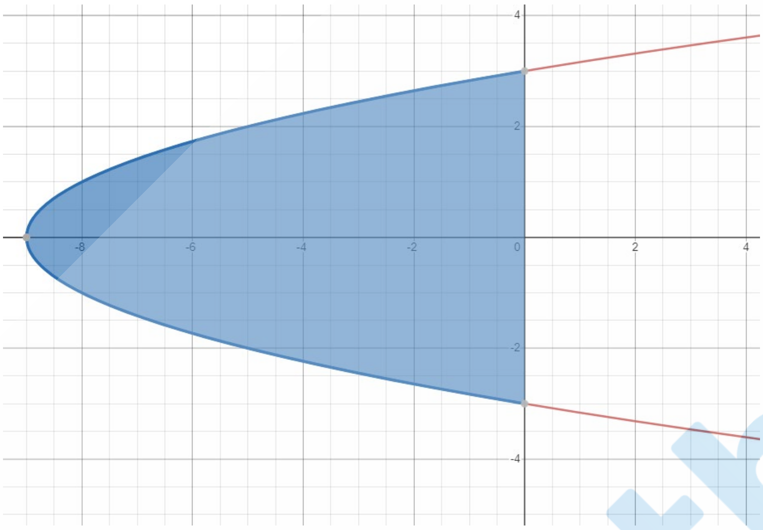
- Splitting Areas between the Left and Right Side of the $y -$ axis

Just like the process of finding areas bounded by the $x -$ axis, we sometimes have to split our shaded area into separate regions, and calculate each individual regions using a definite integral.

- Recall that for areas underneath the $x -$ axis, an absolute value must be used for the definite integral.
- Similarly, for areas to the left of the $y -$ axis, an absolute value must also be used for the definite integral representing it

Example 2: Find the area bounded by the curve $x = y^2 - 9$ and the $y -$ axis

$$Step\ 1: Sketch$$



$$Step\ 2: Find\ area\ integral\ and\ evaluate$$

$$\begin{aligned}
Area &= \left|\int_{-3}^{3} y^2 - 9\ dy\right| \\
&= \left|\left[\frac{y^3}{3} - 9y\right]_{-3}^{3}\right| \\
&= \left|\left[\frac{3^3}{3} - 9(3)\right] - \left[\frac{(-3)^3}{3} - 9(-3)\right]\right| \\
&= \left|\frac{27}{3} - 27 + \frac{27}{3} - 27\right| \\
&= \left|-\frac{108}{3}\right| \\
&= \frac{108}{3}\ units^2
\end{aligned}$$

<u>Areas Bounded by the $y -$ axis Exercises</u>

1. Find the exact area bounded by the curve $y = x^2 - 3$, the positive $y -$ axis and the line $y = 5$

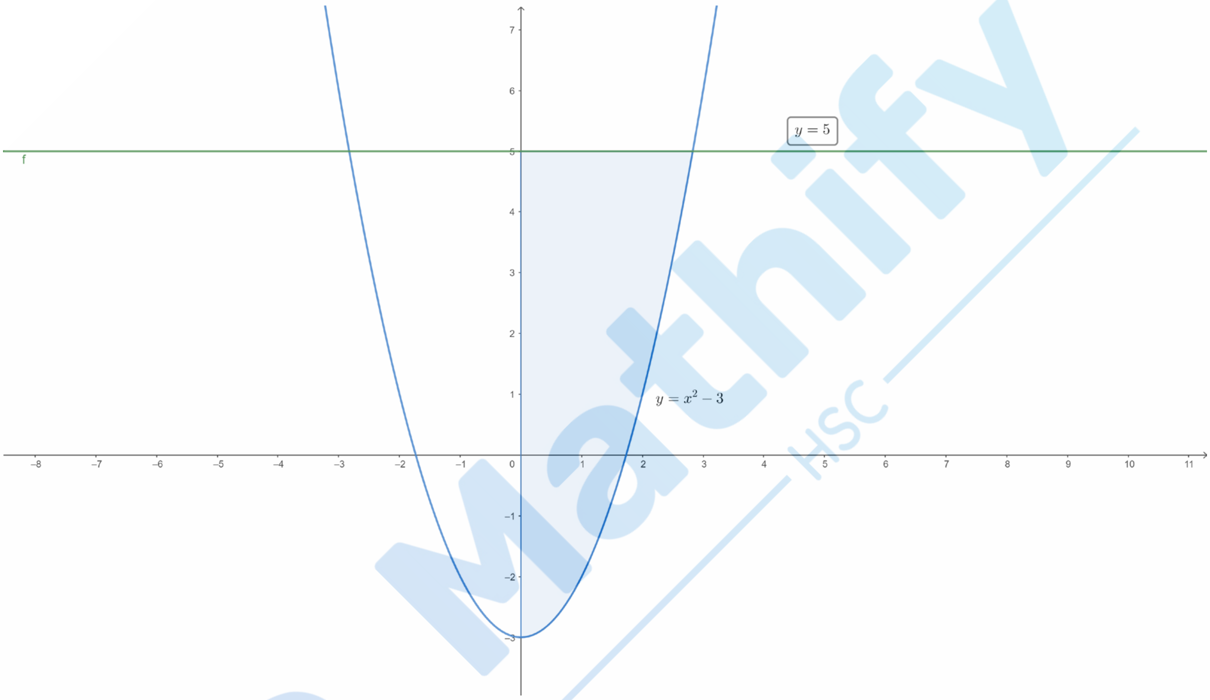
2. Find the area bounded by the curve $y = 4\sqrt{x}$, the line $y = 8$ and the $y -$ axis

Areas Bounded by the $y-$ axis Exercise Answers

1.

Solution:

Starting off with a sketch to visualise the area which must be found:



Notice that this area is bounded by the $y-$ axis from $y=-3$ to $y=5$, and thus it would be more efficient to evaluate the area by using dy.

First rearranging $y=x^2-3$ to make x the subject:

$$y = x^2 - 3$$
$$x^2 = y + 3$$
$$\therefore x = \sqrt{y+3} \, (\because x > 0)$$

The area will thus be represented by the definite integral:

$$\begin{aligned} Area &= \int_{-3}^{5} \sqrt{y+3}\, dy \\ &= \int_{-3}^{5} (y+3)^{\frac{1}{2}}\, dy \\ &= \left[\frac{(y+3)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-3}^{5} \end{aligned}$$

$$= \left[\frac{2}{3}(y+3)^{\frac{3}{2}}\right]_{-3}^{5}$$

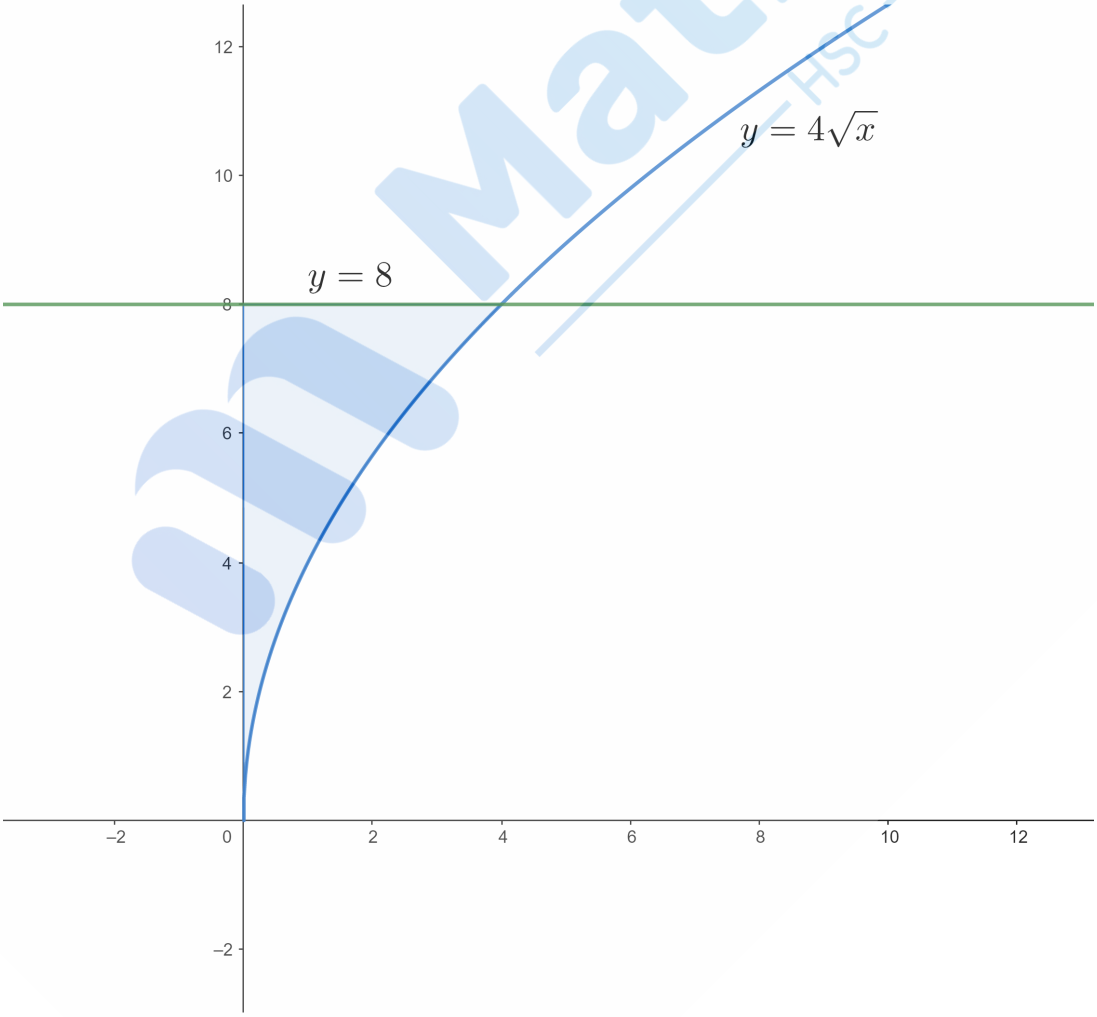
$$\begin{aligned} Area &= \left(\frac{2}{3}(5+3)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(-3+3)^{\frac{3}{2}}\right) \\ &= \frac{2}{3}(8)^{\frac{3}{2}} - 0 \\ &= \frac{2}{3}(2^3)^{\frac{3}{2}} \\ &= \frac{2}{3}(2)^{\frac{9}{2}}\, units^2 \end{aligned}$$

Hence, the area of the region is $\frac{2}{3} \times 2^{\frac{9}{2}}\, units^2$

2.

Solution:

First sketching a graph to best visualise the area we need to find:



Note that this area is bounded by the $y-$ axis, so we can find the area through using dy with lower bound $y=0$ and $y=8$:

First, rearranging $y=4\sqrt{x}$ to make x the subject:

$$y=4\sqrt{x}$$
$$\frac{y}{4}=\sqrt{x}$$
$$\therefore x=\frac{y^2}{16}$$

The area will thus be represented by the definite integral:

$$\begin{aligned} Area &= \int_0^8 \frac{y^2}{16}\,dy \\ &= \left[\frac{y^3}{48}\right]_0^8 \\ &= \left(\frac{8^3}{48}\right)-(0) \\ &= \frac{32}{3}\ units^2 \end{aligned}$$

Hence, the area is $\frac{32}{3}\ units^2$