INTEGRATION

DEFINITE INTEGRALS (IV)

Contents include:

- Calculating Definite Integrals
- Using Substitution for Definite Integrals

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- Definite Integrals

Previously, indefinite integrals were used to simply find the primitive expression.

A definite integral is essentially the same as an indefinite integral, except it involves an upper and lower bound which is used at the end of our working out to **find a numerical value**.

The definite integral is shown below:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Where:

'b' is the upper bound.

'a' is the lower bound.

Example 1: Evaluate $\int_1^6 12x^2\,dx$

$$\begin{aligned}\int_1^6 12x^2\,dx &= \left[12 \times \frac{x^3}{3} + C\right]_1^6 \\ &= [4(6)^3 + C] - [4(1)^3 + C] \\ &= 864 - 4 + C - C \\ &= 860\end{aligned}$$

__**Note:**__ it can be seen that the process of evaluating definite integrals is largely the same, except when we integrate we **no longer need to evaluate or write down C** since it cancels out anyways, as shown in the example above

Why do we use definite integrals? Don't worry – you'll find out about this later in booklet 8

Example 2: Evaluate $\int_1^4 2\sqrt{x} - \frac{3}{\sqrt{x}}\,dx$, showing all working

Solutions:

$$\textit{Step 1: Convert expression into indices form}$$

$$\int_1^4 2\sqrt{x} - \frac{3}{\sqrt{x}}\,dx = \int_1^4 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}\,dx$$

$$\textit{Step 2: Find the primitive function expression}$$

$$\int_1^4 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}\,dx = \left[2 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3 \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}}\right]_1^4$$

$$Step\ 3: Evaluate\ by\ substituting\ in\ the\ bounds$$

$$\therefore \left[\frac{4}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}}\right]_1^4 = \left(\frac{4}{3}(4)^{\frac{3}{2}} - 6(4)^{\frac{1}{2}}\right) - \left(\frac{4}{3}(1)^{\frac{3}{2}} - 6(1)^{\frac{1}{2}}\right)$$
$$= \left(\frac{4}{3} \times 8 - 6 \times 2\right) - \left(\frac{4}{3} - 6\right)$$
$$= -\frac{4}{3} - -\frac{14}{3}$$
$$= \frac{10}{3}$$

- Substitution Method in Definite Integrals

When using the substitution method in harder definite integral, we must remember that the bounds given to us are in terms of $x -$ values, so once we integrate with respect to our substitute $'u'$, we **must** then either:

- Change our lower and upper bounds to be in terms of u. (Recommended method for harder questions)

$$OR$$

- Convert back into terms of x before subbing in lower and upper bound (Used with the standard form of reverse chain rule, remember to do this!)

Here's an example of the first working out using substitution:

Example 3: Evaluate the integral $\int_0^1 4(4x+5)^3 dx$

$$\underline{Step\ 1: Let\ u = 4x + 5}$$

$$\frac{du}{dx} = 4, \therefore dx = \frac{du}{4}$$

$$\underline{Step\ 2: Change\ our\ bounds\ to\ be\ in\ terms\ of\ u}$$

This is the recommended step when integrating by substitution!

Since $u = 4x + 5$:

$$When\ x = 0, u = 4(0) + 5 = 5$$

$$When\ x = 1, u = 4(1) + 5 = 9$$

Hence, the new lower bound is 5 and the new upper bound is 9

$$\underline{Step\ 3: Substitute\ and\ integrate}$$

$$\int_0^1 4(4x+5)^3 dx = \int_5^9 4u^3 \times \frac{du}{4}$$

$$\begin{aligned}\int_5^9 u^3 du &= \left[\frac{u^4}{4}\right]_5^9 \\ &= \frac{9^4}{4} - \frac{5^4}{4} \\ &= 1484\end{aligned}$$

As can be seen, because we did **step 2**, we don't have to convert back into x in the end to get our answer!

Here's an example of the other method of working out:

Example 4: Evaluate the integral $\int_{-1}^{1} -5(2x+3)^4 dx$

Solution:

Recalling the reverse chain rule formula:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a \times (n+1)} + C$$

For this integral:

$$\begin{aligned}\therefore \int_{-1}^{1} -5(2x+3)^4 dx &= \left[-5 \times \frac{(2x+3)^5}{2 \times 5}\right]_{-1}^{1} \\ &= \left[-\frac{(2x+3)^5}{2}\right]_{-1}^{1} \\ &= \left(-\frac{(2(1)+3)^5}{2}\right) - \left(-\frac{(2(-1)+3)^5}{2}\right) \\ &= -\frac{5^5}{2} + \frac{1^5}{2} \\ &= -1562\end{aligned}$$

Definite Integral Exercises

1. Evaluate $\int_0^2 4x^3 - 2x\, dx$, showing all working

2. Evaluate $\int_1^5 \frac{x^3-3}{x^3} dx$, showing all working

3. Evaluate $\int_1^3 15(3x-1)^4 dx$, showing all working

4. Evaluate $\int_{-1}^2 2x(x^2+2)^3 dx$ by making a substitution, showing all working

5. Evaluate $\int_1^3 6x^2(x^3-1)^2 dx$ by making a substitution, showing all working

6. If $c\int_{-2}^2 (x-5)\, dx = 1$, evaluate c

<u>Definite Integral Exercise Solutions</u>

1.

$$\text{Step 1: Find the primitive function expression}$$

$$\int_0^2 4x^3 - 2x\, dx = \left[4 \times \frac{x^4}{4} - 2 \times \frac{x^2}{2}\right]_0^2$$
$$= [x^4 - x^2]_0^2$$

$$\text{Step 2: Evaluate by substituting in the bounds}$$

$$\therefore [2^4 - 2^2] - [0-0] = 12$$

2.

$$\text{Step 1: Simplify the expression}$$

Splitting the numerator:

$$\int_1^5 \frac{x^3-3}{x^3} dx = \int_1^5 1 - 3x^{-3}\, dx$$

$$\text{Step 2: Find the primitive function expression}$$

$$\int_1^5 1 - 3x^{-3}\ \ dx = \left[x - 3 \times \frac{x^{-2}}{-2}\right]_1^5$$
$$= [x + \frac{3}{2}x^{-2}]_1^5$$

$$\text{Step 3: Evaluate by substituting in the bounds}$$

$$\therefore \left[5 + \frac{3}{2}(5)^{-2}\right] - \left[1 + \frac{3}{2}(1)^{-2}\right] = \frac{253}{20} - \frac{5}{2}$$
$$= \frac{64}{25}$$

3. Recalling the standard form of reverse chain rule:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a \times (n+1)} + C$$

Applying this to our question:

$$\therefore \int_1^3 15(3x-1)^4 dx = \left[15 \times \frac{(3x-1)^5}{3 \times 5}\right]_1^3$$
$$= [(3x-1)^5]_1^3$$
$$= (3(3)-1)^5 - (3(1)-1)^5$$
$$= 8^5 - 2^5$$
$$= 32736$$

4. Making a substitution here of $u = x^2 + 2$:

$$\therefore \frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

When $x = -1, u = 3$

When $x = 2, u = 6$

Hence, substituting these into the original integral:

$$\therefore \int_{-1}^2 2x(x^2+2)^3 dx = \int_3^6 2x \cdot u^3 \times \frac{du}{2x}$$
$$= \int_3^6 u^3\, du$$
$$= \left[\frac{u^4}{4}\right]_3^6$$
$$= \frac{6^4}{4} - \frac{3^4}{4}$$
$$= \frac{1215}{4}$$

As you can see, the substitution method is great for harder questions, and really isn't a long process either!

5. Making a substitution here of $u = x^3 - 1$:

$$\therefore \frac{du}{dx} = 3x^2$$

$$dx = \frac{du}{3x^2}$$

When $x = 1, u = 0$

When $x = 3, u = 26$

Hence, substituting these into the original integral:

$$\therefore \int_1^3 6x^2(x^3-1)^2 dx = \int_0^{26} 6x^2 \cdot u^2 \cdot \frac{du}{3x^2}$$
$$= \int_0^{26} 2u^2\, du$$
$$= \left[2 \times \frac{u^3}{3}\right]_0^{26}$$
$$= \left(\frac{2(26)^3}{3}\right) - \left(\frac{2(0)^3}{3}\right)$$
$$= \frac{35152}{3}$$

6.

$$Step\ 1: Find\ the\ primitive\ function\ expression$$

$$c\int_{-2}^{2}(x-5)\,dx = c\left[\frac{x^2}{2} - 5x\right]_{-2}^{2}$$

$$Step\ 2: Evaluate\ by\ substituting\ bounds$$

$$\therefore c\left[\frac{2^2}{2} - 5(2)\right] - c\left[\frac{(-2)^2}{2} - 5(-2)\right]$$
$$= c[-8] - c[12]$$
$$= -20c$$

$$Step\ 3: Equate\ this\ value\ to\ our\ RHS$$

$$\therefore -20c = 1\ [given]$$
$$so\ c = -\frac{1}{20}$$