DIFFERENTIATIAL CALCULUS

GEOMETRICAL APPLICATIONS: FIRST DERIVATIVE (V)

Contents include:

- Revision of Stationary Points
- Finding the Nature of Stationary Points
- Sign Tables

- The First Derivative and Stationary Points

Recall from year 11 that:

When the gradient at any point is equal to 0, this is referred to as a **stationary point**. In other words:

$$\textit{Stationary points occur when } f'(x) = 0$$

Example 1: Find the coordinate of the stationary point for the function $f(x) = 2 - 3x - x^2$

Solution:

First differentiating $f(x)$:

$$f'(x) = -3 - 2x$$

Since stationary points occur when $f'(x) = 0$, solving for x:

$$0 = -3 - 2x$$
$$2x = -3$$
$$\therefore x = -\frac{3}{2}$$

When $x = -\frac{3}{2}$:

$$y = f\left(-\frac{3}{2}\right) = 2 - 3\left(-\frac{3}{2}\right) - \left(-\frac{3}{2}\right)^2$$
$$= 2 + \frac{9}{2} - \frac{9}{4}$$
$$= \frac{17}{4}$$

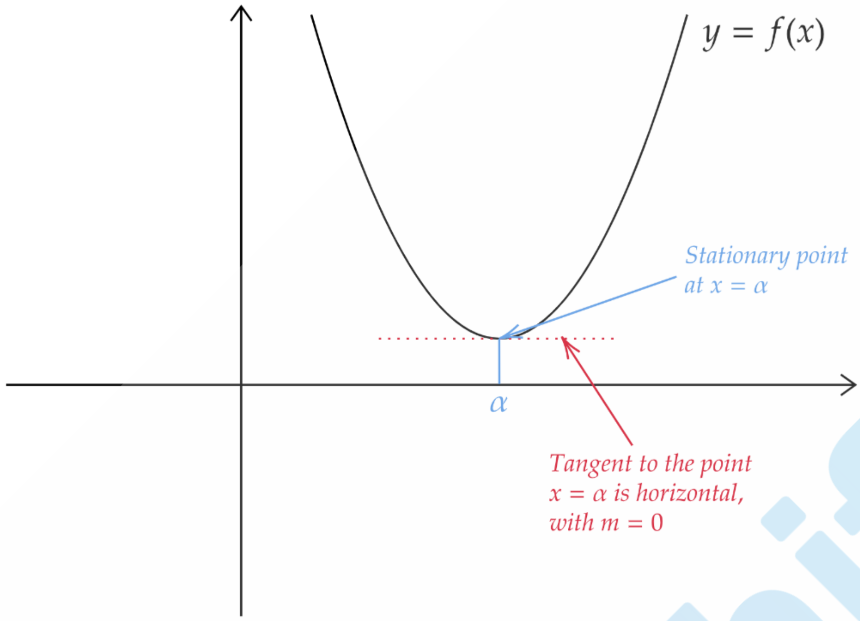
Hence, stationary point occurs at $\left(-\frac{3}{2}, \frac{17}{4}\right)$

- Types of Stationary Points

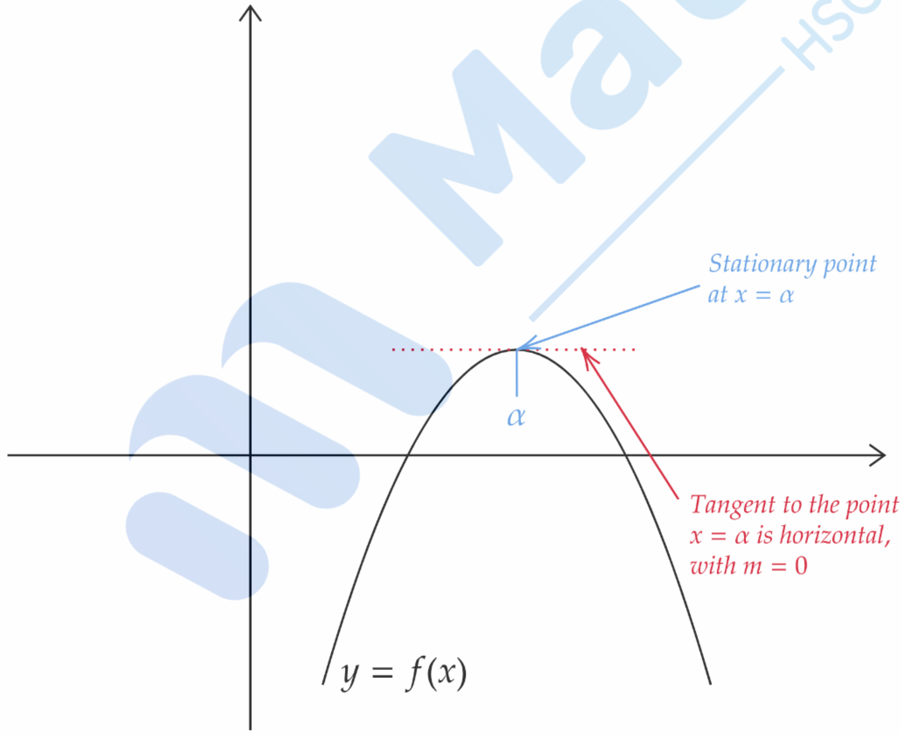
Recall from Year 11 that:

If we try to visualise a stationary point, there are two different ways we can do so:

Type 1:



Type 2:



Although both have $f'(x) = 0$, the shape of the curve is different. Thus, it can be said that stationary points may be split into two types:

Type 1: Minimum Turning Points

These occur when the left side of the point is increasing ($f'(x) > 0$) and the right side is decreasing ($f'(x) < 0$). In other words, it looks like a smiley face.

Type 2: Maximum Turning Points

These occur when the left side of the point is decreasing ($f'(x) < 0$) and the right side is increasing ($f'(x) > 0$). In other words, it looks like a sad face.

- Finding the Nature of Turning Points Using Sign Tables

If a question asks you to find the nature of a turning point, it is essentially asking you to find whether it is a minimum or maximum turning point.

To do so, we follow these steps:

$$\textbf{\textit{Step 1: Determine where the stationary points are}}$$

We do so by letting $f'(x) = 0$ and solving for x

$$\textbf{\textit{Step 2: Draw a sign table to determine their nature}}$$

A sign table is essentially a summarised form of whether the points to the left and right of your stationary points are increasing or decreasing, as this can help us determine if it's a minimum or maximum turning point.

To draw a sign table:

- Start off with 2 rows labelled x and $f'(x)$:

x					
$f'(x)$					

The number of columns depends on how many stationary points you have. If you have 1 stationary point, you need 3 columns, if you have 2 stationary points you need 5 columns, if you have 3 stationary points you need 7 columns, and so on…

- Write down the $x -$ coordinate of your stationary points (α *and* β) and their gradients, which should be 0

x		α		β	
$f'(x)$		0		0	

Make sure you leave an empty column to the left and right of each stationary point

- To the left of each stationary point, write down **any** $x - coordinate$ less than it (fractions included), and to the right of each stationary point write down **any** $x -$ coordinate more than it. (Try pick easy numbers if you can for your own sake!)

x	$\alpha - 1$	α	$number$ $between$ α and β	β	$\beta + 1$
$f'(x)$		0		0	

We have chosen $\alpha - 1$, a number between α and β, and $\beta + 1$ in this scenario, but remember it doesn't matter exactly what number you choose!

- Now calculate the gradient at each point you chose previously

x	$\alpha - 1$	α	$number$ $between$ α and β	β	$\beta + 1$
$f'(x)$	$f'(\alpha - 1)$	0	$f'(number)$	0	$f'(\beta + 1)$

Make sure you write down the proper value, so for example if $f'(\alpha - 1) = 2$, then I should be writing down '2' into the cell instead of $f'(\alpha - 1)$.

- Now depending on whether the point to the left and right of each stationary point is increasing or decreasing, we finally determine the nature of the stationary point

In this scenario, let's assume that $f'(\alpha - 1) > 0, f'(number) < 0$ and $f'(\beta + 1) > 0$ as an example. Hence:

x	$\alpha - 1$	α	$number$ $between$ α and β	β	$\beta + 1$
$f'(x)$	$f'(\alpha - 1)$	0	$f'(number)$	0	$f'(\beta + 1)$

$\therefore Maximum\ TP$

$\therefore Minimum\ TP$

Increasing since $f'(\alpha - 1) > 0$

Decreasing since $f'(number) < 0$

Increasing since $f'(\beta + 1) > 0$

$\textit{Step 3: Find the } y - \textit{coordinate of each point as well!}$

Many forget this step, but remember in your final answer you are giving a **coordinate** a lot of the time, so you must include the $y -$ coordinate as well!

In this case:

$$(\alpha, f(\alpha))\ is\ a\ maximum\ TP$$

$$\left(\beta, f(\beta)\right)\ is\ a\ minimum\ TP$$

Example 3: Determine the coordinates and nature (whether it's minimum or maximum) of the stationary point for the function $f(x) = x^2$

Solution:

$$\underline{Step\ 1: Determining\ the\ stationary\ point}$$
$$f'(x) = 2x$$

Now making $f'(x) = 0$ and solving to find the stationary point:

$$f'(x) = 2x = 0$$
$$\therefore x = 0\ is\ the\ x - coordinate\ of\ the\ stationary\ point$$

$$\underline{Step\ 2: Draw\ the\ sign\ table}$$

x	-1	0	1
$f'(x)$	-2	0	2

$$\therefore Minimum\ TP$$

$$\underline{Step\ 3: answer}!$$

When $x = 0, y = 0$

$$\therefore (0, 0)\ is\ a\ minimum\ turning\ point$$

<u>First Derivative Exercises</u>

1. For the graph of $f(x) = 2 - 3x - x^2$, find the values of x for which the function:
a) Increases when x increases
b) Decreases when x increases
c) Changes from increasing to decreasing

2. Consider the function $f(x) = 2x^3 - 15x^2 + 12$. Find the values of x for which:
a) $f'(x) = 0$
b) $f'(x) > 0$
c) $f'(x) < 0$

3. Consider the function $f(x) = x^3 - x^2 - x + 36$. Find the values of x for which:
a) $f'(x) = 0$
b) $f'(x) > 0$

c) $f'(x) < 0$

4. Locate the intercepts and turning points for the curve $y = x^3 - 6x^2$

5. Locate the intercepts and turning points for the curve $y = (2 - x)(1 + x^2)$

6. Determine the coordinates of any turning points and their nature in the function $y = 3x^2 - x^3$.

First Derivative Exercise Answers

1. $f(x) = 2 - 3x - x^2$

a) $f'(x) = -3 - 2x$

Since we need to find the interval/values for when the function $f(x)$ is increasing, we set our $f'(x)$ to be > 0.

i.e.
$$f'(x) = -3 - 2x > 0$$
$$2x < -3$$
$$x < -\frac{3}{2}$$

b) In this case, we now want to find when the function is decreasing, so we set our $f'(x)$ to be < 0.

$$f'(x) = -3 - 2x < 0$$
$$2x > -3$$
$$x > -\frac{3}{2}$$

c) Now the question is asking us to essentially find the x value of the maximum turning point. To do so we equate $f'(x)$ to 0

$$f'(x) = -3 - 2x = 0$$
$$x = -\frac{3}{2}$$

2. $f(x) = 2x^3 - 15x^2 + 12$

a) $f'(x) = 6x^2 - 30x$

$$6x^2 - 30x = 6x(x - 5) = 0$$
$$\therefore x = 0 \; or \; x = 5$$

b) $f'(x) = 6x(x - 5) > 0$

Drawing our parabola with x-intercepts of $x = 0 \; or \; x = 5$:

$$\therefore x < 0 \text{ or } x > 5 \text{ for } f'(x) > 0$$

c) $f'(x) = 6x(x-5) < 0$

Drawing our parabola with x-intercepts of $x = 0 \text{ or } x = 5$:

$$\therefore 0 < x < 5 \text{ for } f'(x) < 0$$

3. $f(x) = x^3 - x^2 - x + 36$

a) $f'(x) = 3x^2 - 2x - 1$

$$f'(x) = 3x^2 - 2x - 1 = 0$$
$$(3x+1)(x-1) = 0$$

$$\therefore x = 1 \text{ or } x = -\frac{1}{3}$$

b) $f'(x) = (3x+1)(x-1) > 0$

Drawing our parabola with x-intercepts of $x = 1 \text{ or } x = -\frac{1}{3}$:

$$\therefore x < -\frac{1}{3} \text{ or } x > 1$$

c) $f'(x) = (3x+1)(x-1) < 0$

Drawing our parabola with x-intercepts of $x = 1 \text{ or } x = -\frac{1}{3}$:

$$\therefore -\frac{1}{3} < x < 1$$

4. $y = x^3 - 6x^2$

$$\text{Step 1: Factorise to determine } x \text{ and } y \text{ intercepts}$$

$$y = x^2(x-6)$$

$$\therefore x - \text{intercepts occur at } x = 0 \text{ and } x = 6$$

$$y - \text{intercept occurs at } y = 0$$

$$\text{Step 2: Differentiate } y \text{ to hence find stationary points}$$
$$y' = 3x^2 - 12x$$
$$y' = 3x(x-4) = 0$$

$\therefore$ *stationary points occur when* $x = 0$ *or* $x = 4$
When $x = 0, y = 0$
When $x = 4, y = 4^3 - 6(4)^2 = -32$

$$\therefore \text{Stationary points at } (0,0) \text{ and } (4,-32)$$

$$Step\ 3: Construct\ a\ sign\ table\ to\ determine\ their\ nature$$

x	-1	0	1	4	5
y'	15	0	-9	0	15

$\therefore$ *From our sign table, we can determine that* $(0,0)$ *is a maximum turning point and that* $(4,-32)$ *is a minimum turning point.*

5. $y = (2-x)(1+x^2)$

Following the same steps as question 6:

$$x - intercept\ occurs\ at\ x = 2\ and\ y - intercept\ at\ y = 2$$

Before we differentiate y, expand it out first to make it easier:
$y = 2 + 2x^2 - x - x^3$

$$\therefore y' = 4x - 1 - 3x^2$$
$$= (-3x+1)(x-1)$$

Thus, we can determine that the stationary points occur at $x = 1, \frac{1}{3}$

When $x = 1, y = 2$
When $x = \frac{1}{3}, y = \frac{50}{27}$

$$\therefore Stationary\ points\ at\ (1,2)\ and\ (\frac{1}{3}, \frac{50}{27})$$

Constructing a sign table we get:

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	2
y'	-1	0	$\frac{1}{3}$	0	-5

$\therefore$ *From our sign table, we can determine that* $\left(\frac{1}{3}, \frac{50}{27}\right)$ *is a minimum turning point and that* $(1,2)$ *is a maxmimum turning point.*

6. $y = 3x^2 - x^3$

$y' = 6x - 3x^2$

$$6x - 3x^2 = 0$$
$$3x(2-x) = 0$$

Thus, we can determine that $x = 0$ *and* $x = 2$ *are stationary points*

$When\ x = 0, y = 0$
$When\ x = 2, y = 4$

$$\therefore Stationary\ points\ at\ (0,0)\ and\ (2,4)$$

Constructing our sign table we get:

x	-1	0	1	2	3
$f'(x)$	-9	0	3	0	-9

$$\therefore From\ our\ sign\ table, we\ can\ determine\ that\ (0,0)\ is\ a\ minimum\ turning\ point\ and\ that\ (2,4) is\ a\ maximum\ turning\ point$$