Mathify HSC

# EXPONENTIALS & LOGARITHMS

## LOGARITHMIC EQUATIONS (V)

Contents include:

- Taking Logarithms of Both Sides
- Logarithmic Equations Reducible to Quadratics

- Taking Logarithms of Both Sides

When we see exponentials with two completely different bases on both sides of the equation, we should think of taking the logarithm of both sides! When we do take logarithms, also make sure we are logging with base 10 or $e$ so that we can use our calculator later on.

Keep in mind that remembering your **log laws are essential** for these types of questions

Below is an example demonstrating how to log both sides to solve an equation:

**Example 1:** Solve $7^{3x} = 2 \times 7^{x+2}$ correct to the nearest 3 decimal places

Solution:

Taking the logarithm of both sides first

$$\log_{10} 7^{3x} = \log_{10}(2 \times 7^{x+2})$$
$$3x \times \log_{10} 7 = \log_{10} 2 + \log_{10} 7^{x+2} \ [logarithm\ laws]$$
$$3x \times \log_{10} 7 = \log_{10} 2 + (x+2)\log_{10} 7 \ [logarithm\ laws]$$

$$\therefore 3x \times \log_{10} 7 = \log_{10} 2 + x\log_{10} 7 + 2\log_{10} 7$$
$$2x\log_{10} 7 = \log_{10} 2 + 2\log_{10} 7$$

Finally, making $x$ the subject:

$$\therefore x = \frac{\log_{10} 2 + 2\log_{10} 7}{2\log_{10} 7}$$
$$\approx 1.178\ (nearest\ 3\ d.p.)$$

In summary, the following steps should be taken when logging both sides:

$\boldsymbol{Step\ 1: Log\ both\ sides\ with\ either\ base\ 10\ or\ base\ e}$

$\boldsymbol{Step\ 2: Apply\ logarithm\ laws\ with\ the\ goal\ of\ making\ x\ the\ subject}$

$\boldsymbol{Step\ 3: Once\ x\ is\ the\ subject, enter\ values\ into\ your\ calculator}$

- Logarithmic Equations Reducible to Quadratics

Similar to exponential equations, some logarithmic equations are also reducible to quadratics. Hence, in these scenarios we let $u = \log_a x$ and solve like any normal quadratic afterwards.

This is shown in the example below:

**Example 2:** Solve $\log_{10} x - \frac{3}{\log_{10} x} = 2$

Solution:

Since $\log_{10} x$ is on the denominator of a fraction, multiplying both sides by $\log_{10} x$:

$$(\log_{10} x)^2 - 3 = 2\log_{10} x$$

$$(\log_{10} x)^2 - 2\log_{10} x - 3 = 0$$

Then, letting $u = \log_{10} x$:

$$\therefore u^2 - 2u - 3 = 0$$

$$(u-3)(u+1) = 0$$

$$\therefore u = 3 \; OR -1$$

Converting $u$ back into $\log_{10} x$:

$$\log_{10} x = 3 \; OR \; \log_{10} x = -1$$

$$\therefore x = 10^3 \; OR \; x = 10^{-1}$$

$$x = 1000 \; or \, \frac{1}{10}$$

## Logarithmic Equation Exercises

1. By taking logarithms of base 10 or base $e$ to both sides, solve the following equations by giving your answer to the nearest 3 significant figures:

a) $3^{x-3} = 48$

b) $2^{x-5} = 5 \times 2^{2x}$

c) $7^{2x} = 5^{x-1}$

d) $4^{1-x} = 3 \times 5^{x+2}$

2. Solve the equation $\log_4(x+1) - \log_4 x = 2$

3. Solve $\log_e(2x+1) - \log_e 3 = 3$

4. Solve $\log_e x - \frac{5}{\log_e x} = 4$, giving your answers in exact form

## Logarithmic Equation Exercise Answers

1.

a)

$$3^{x-3} = 48$$

Logging both sides with base 10:

$$\log_{10} 3^{x-3} = \log_{10} 48$$

$$(x-3)\log_{10} 3 = \log_{10} 48 \ [logarithmic\ law]$$

$$\therefore x - 3 = \frac{\log_{10} 48}{\log_{10} 3}$$

$$x = \frac{\log_{10} 48}{\log_{10} 3} + 3$$

$$\approx 6.52\ (nearest\ 3\ sig\ figs)$$

b)

$$2^{x-5} = 5 \times 2^{2x}$$

Logging both sides with base 10:

$$\log_{10} 2^{x-5} = \log_{10}(5 \times 2^{2x})$$

$$= \log_{10} 5 + \log_{10} 2^{2x} \ [logarithmic\ law]$$

$$\therefore (x-5)\log_{10} 2 = \log_{10} 5 + 2x\log_{10} 2 \ [logarithmic\ law]$$

$$x\log_{10} 2 - 5\log_{10} 2 = \log_{10} 5 + 2x\log_{10} 2$$

$$-5\log_{10} 2 - \log_{10} 5 = x\log_{10} 2$$

$$\therefore x = \frac{-5\log_{10} 2 - \log_{10} 5}{\log_{10} 2}$$

$$\approx -7.32\ (nearest\ 3\ sig\ figs)$$

c)

$$7^{2x} = 5^{x-1}$$

Logging both sides with base $e$:

$$\ln(7^{2x}) = \ln(5^{x-1})$$

$$2x\ln 7 = (x-1)\ln 5 \ [logarithmic\ law]$$

$$2\ln 7 \times x = x\ln 5 - \ln 5$$

$$\ln 5 = x\ln 5 - 2\ln 7 \times x$$

$$\ln 5 = x(\ln 5 - 2\ln 7)$$

$$\therefore x = \frac{\ln 5}{\ln 5 - 2\ln 7}$$

$$\approx -0.705\ (nearest\ 3\ sig\ figs)$$

d)

$$4^{1-x} = 3 \times 5^{x+2}$$

Logging both sides with base $e$:

$$\ln(4^{1-x}) = \ln(3 \times 5^{x+2})$$

$$= \ln 3 + \ln(5^{x+2}) \ [logarithm\ law]$$

$$(1-x)\ln 4 = \ln 3 + (x+2)\ln 5 \ [logarithm\ law]$$

$$\ln 4 - x\ln 4 = \ln 3 + x\ln 5 + 2\ln 5$$

$$\ln 4 - \ln 3 - 2\ln 5 = x\ln 4 + x\ln 5$$

$$\ln 4 - \ln 3 - 2\ln 5 = x(\ln 4 + \ln 5)$$

$$\therefore x = \frac{\ln 4 - \ln 3 - 2\ln 5}{\ln 4 + \ln 5}$$

$$\approx -0.978\ (nearest\ 3\ sig\ figs)$$

2.

$$\log_4(x+1) - \log_4 x = 2$$

Using logarithmic law of subtraction:

$$\log_4\left(\frac{x+1}{x}\right) = 2$$

$$\therefore \frac{x+1}{x} = 4^2$$

$$x + 1 = 16x$$

$$15x = 1$$

$$\therefore x = \frac{1}{15}$$

3.

$$\log_e(2x+1) - \log_e 3 = 3$$

Using logarithmic law of subtraction:

$$\log_e\left(\frac{2x+1}{3}\right) = 3$$

$$\therefore \frac{2x+1}{3} = e^3$$

$$2x + 1 = 3e^3$$

$$2x = 3e^3 - 1$$

$$\therefore x = \frac{3e^3 - 1}{2}$$

4.

$$\log_e x - \frac{5}{\log_e x} = 4$$

Multiply both sides by $\log_e x$:

$$(\log_e x)^2 - 5 = 4\log_e x$$

$$(\log_e x)^2 - 4\log_e x - 5 = 0$$

Let $u = \log_e x$:

$$u^2 - 4u - 5 = 0$$

$$(u-5)(u+1) = 0$$

$$\therefore \log_e x = 5 \; OR \; \log_e x = -1$$

$$\therefore x = e^5 \; OR \; x = e^{-1}$$