Mathify HSC

DIFFERENTIATION

FIRST PRINCIPLES (II)

Contents include:

- Notation of Differentiation
- Differentiating Through First Principles

- The Notation of Differentiation

Differentiating a function is the process of **finding its instantaneous rate of change** based on the independent variable, usually x. This may not make much sense now, but will be discussed in greater depth later so don't worry about it too much for now!

> When asked to differentiate a function, $y = f(x)$, the result is what we call **the derivative**. The derivative may be expressed in numerous ways:
>
> $$The\ Derivative = f'(x) = \frac{dy}{dx} = y'$$

Note: The notation $\frac{\mathrm{dy}}{\mathrm{dx}}$ means to '*differentiate y with respect to* x'. The variable in the denominator is the independent variable (in this case 'x') and the variable in the numerator is the dependent variable (in this case 'y'). We always differentiate the numerator (dependent variable) with respect to the denominator (independent variable). E.g., If it was $\frac{dv}{dt}$ instead, we say that we are differentiating v with respect to t instead.

- Differentiating Through First Principles

> Differentiating through first principles is the method that we will learn first to differentiate a function $f(x)$. The formula we follow is:
>
> $$f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The general steps for completing exam questions are:

$$\boldsymbol{Step\ 1: Find\ the\ expanded\ form\ of\ f(x+h)}$$

This is done through replacing every "x" that you see with "$x + h$" instead

$$\boldsymbol{Step\ 2: Use\ the\ formula\ and\ simplify}$$

By the end of this step, you should be able to factorise out a h from the numerator, and cancel it with the bottom

$$\boldsymbol{Step\ 3: Apply\ your\ limit}$$

And then that's it!

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How we apply this is best learnt through examples:

Example 1: Differentiate $x^2 + 2x + 2$ from first principles

Recall from first principles that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We know that:

$$f(x) = x^2 + 2x + 2$$
$$f(x+h) = (x+h)^2 + 2(x+h) + 2$$

Hence, substituting these into the first principles equation:

$$\begin{aligned} \therefore f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) + 2 - (x^2 + 2x + 2)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h + 2 - x^2 - 2x - 2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \\ &= \lim_{h \to 0} 2x + h + 2 \\ &= 2x + 2 \end{aligned}$$

Example 2: Differentiate $\frac{1}{x}$ from first principles

Recall from first principles that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We know that:

$$f(x) = \frac{1}{x}$$
$$f(x+h) = \frac{1}{x+h}$$

Hence, substituting these into the first principles equation:

$$\begin{aligned} \therefore f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \to 0} \frac{\frac{x - x - h}{x(x+h)}}{h} \end{aligned}$$

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$$= \lim_{h \to 0} -\frac{h}{x^2h + xh^2}$$
$$= \lim_{h \to 0} -\frac{1}{x^2 + xh}$$
$$= -\frac{1}{x^2}$$

Example 3: Differentiate $f(x) = x^2$ using first principles

Recall from first principles that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We know that:

$$f(x) = x^2$$

$$f(x+h) = (x+h)^2$$

Hence, substituting these into the first principles equation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{h(2x+h)}{h}$$
$$= \lim_{h \to 0} (2x + h)$$
$$= 2x$$

Exercise 4: Find from first principles the derivative of $3 - 2x + 4x^2$

Recall from first principles that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We know that:

$$f(x) = 3 - 2x + 4x^2$$
$$f(x+h) = 3 - 2(x+h) + 4(x+h)^2$$
$$= 3 - 2x - 2h + 4(x^2 + 2xh + h^2)$$

Hence, substituting these into the first principles equation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\left(3 - 2x - 2h + 4(x^2 + 2xh + h^2)\right) - (3 - 2x + 4x^2)}{h}$$

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$$= \lim_{h \to 0} \frac{3 - 2x - 2h + 4x^2 + 8xh + 4h^2 - 3 + 2x - 4x^2}{h}$$
$$= \lim_{h \to 0} \frac{-2h + 8xh + 4h^2}{h}$$
$$= \lim_{h \to 0} -2 + 8x + 4h$$
$$= 8x - 2$$

Exercise 5: For the graph of the equation $f(x) = x^3 - 5x$, write down the value of:

a) $f'(x)$ using first principles
b) $f'(2)$
c) $f'(4)$

Solution:

a) Recall from first principles that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We know that:

$$f(x) = x^3 - 5x$$

$$\begin{aligned} f(x+h) &= (x+h)^3 - 5(x+h) \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h \end{aligned}$$

Hence, substituting these into the first principles equation:

$$\begin{aligned} \therefore f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h) - (x^3 - 5x)}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 5h}{h} \\ &= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 5 \\ &= 3x^2 - 5 \end{aligned}$$

b) from part a), $f'(x) = 3x^2 - 5$

$$\begin{aligned} \therefore f'(2) &= 3(2)^2 - 5 \\ &= 12 - 5 \\ &= 7 \end{aligned}$$

c) from part a), $f'(x) = 3x^2 - 5$

$$\begin{aligned} \therefore f'(4) &= 3(4)^2 - 5 \\ &= 48 - 5 \\ &= 43 \end{aligned}$$

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First Principles Exercises

1. Find from first principles the derivative of $(x-2)(x+1)$

2. Find from first principles the derivative of $3x(x+4)$

3. Find from first principles the derivative of x^3

4. For the function $f(x) = 2x^2 - 3x + 6$:

a) Find $f'(x)$ using first principles

b) Find the value of x for which $f'(x) = 0$

First Principle Exercise Answers

1. $f(x) = (x-2)(x+1) = x^2 - x - 2$

$$\begin{aligned}
f(x+h) &= (x+h)^2 - (x+h) - 2 \\
&= x^2 + 2xh + h^2 - x - h - 2 \\
\therefore f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - x - h - 2) - (x^2 - x - 2)}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} \\
&= \lim_{h\to 0} 2x + h - 1 \\
&= 2x - 1
\end{aligned}$$

2. $f(x) = 3x(x+4) = 3x^2 + 12x$

$$\begin{aligned}
f(x+h) &= 3(x+h)^2 + 12(x+h) \\
&= 3(x^2 + 2xh + h^2) + 12x + 12h \\
\therefore f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(3(x^2 + 2xh + h^2) + 12x + 12h) - (3x^2 + 12x)}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 12x + 12h - 3x^2 - 12x}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2 + 12h}{h} \\
&= \lim_{h\to 0} 6x + 3h + 12
\end{aligned}$$

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$$= 6x + 12$$

3. $f(x) = x^3$

$$\begin{aligned} f(x+h) &= (x+h)^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 \\ \therefore f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - (x^3)}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\ &= 3x^2 \end{aligned}$$

4. a) $f(x) = 2x^2 - 3x + 6$

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 3(x+h) + 6 \\ &= 2(x^2 + 2xh + h^2) - 3x - 3h + 6 \\ \therefore f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(2(x^2 + 2xh + h^2) - 3x - 3h + 6) - (2x^2 - 3x + 6)}{h} \\ &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 6 - 2x^2 + 3x - 6}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 - 3h}{h} \\ &= \lim_{h\to 0} 4x + 2h - 3 \\ &= 4x - 3 \end{aligned}$$

b) from a), $f'(x) = 4x - 3$

$$f'(x) = 4x - 3 = 0$$

$$\therefore x = \frac{3}{4}$$

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