INTEGRATION

THE REVERSE CHAIN RULE AND SUBSTITUTION (III)

Contents include:

- The Reverse Chain Rule
- Substitution Techniques in Integration

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- Reverse chain rule

Previously when we learnt **differentiation**, the chain rule had to be applied for harder composite functions (when more than one function is applied to x).

The same logic applies for integration, where we must make a substitution for u and determine what dx is equal to in terms of du before integrating.

The standard form for reverse chain rule is given by:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

The proof for this is done through substitution and is as follows:

$$\textbf{\textit{Step 1: Make your substitution}}$$

Let $u = ax + b$

$$\textbf{\textit{Step 2: Differentiate u and find an expression for dx}}$$

Thus:

$$\frac{du}{dx} = a, and\ dx = \frac{du}{a}$$

$$\textbf{\textit{Step 3: Substitute back into integral}}$$

Then we can substitute these new values into the integral

$$\int (ax+b)^n dx = \int u^n \times \frac{du}{a}$$

$$\textbf{\textit{Step 4: Integrate}}$$

Now we have successfully converted our expression to in terms of u, so integrating with respect to u we get:

$$\int u^n \times \frac{du}{a} = \frac{u^{n+1}}{a(n+1)} + C$$

$$\textbf{\textit{Step 5: Convert back into x}}$$

Substituting back our $u = ax + b$ we finally get:

$$\frac{u^{n+1}}{a(n+1)} + C = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

Example 1: Find the expression for $\int (5-3x)^2 dx$

It is recommended that when first learning integration, we do the entire process of reverse chain rule rather than just quoting the formula.

Let $u = 5 - 3x$

$\frac{du}{dx} = -3, so\ dx = -\frac{du}{3}$

$$\therefore \int (5-3x)^2 dx = \int u^2 \times -\frac{du}{3}$$
$$= -\frac{1}{3} \times \frac{u^3}{3} + C$$
$$= -\frac{1}{9}u^3 + C$$

Substituting back $u = 5 - 3x$, we therefore get:

$$-\frac{1}{9}u^3 + C = -\frac{1}{9}(5-3x)^3 + C$$

Example 2: Evaluate the integral $\int \frac{2}{(5x+3)^9} dx$

Solution:

A helpful tip is to **first convert fractions to negative indices** before integrating:

$$\therefore \int \frac{2}{(5x+3)^9} dx = \int 2(5x+3)^{-9} dx$$

If we would like to use the standard form for reverse chain rule, recall that:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

Therefore:

$$\int 2(5x+3)^{-9} dx = 2 \times \frac{(5x+3)^{-9+1}}{5(-9+1)} + C$$
$$= 2 \times \frac{(5x+3)^{-8}}{-40} + C$$
$$= -\frac{(5x+3)^{-8}}{20} + C$$
$$= -\frac{1}{20(5x+3)^8} + C$$

- Multiple attempts at substitution

Most of the time the question will provide you with information on what to substitute.

However, in harder questions they'll leave you to figure it out yourself what to substitute.

Some common expressions to let your substitution 'u' equal to include:

- The expression inside the bracket, especially one which has a power
- The denominator expression

Deciding on the most appropriate substitution though is sometimes a process of trial and error.

Example 3: Find the integral $\int \frac{12x+2}{(3x^2+x-2)^4} dx$

Solution:

We can let u in this case equal to either $12x + 2$ or $3x^2 + x - 2$. Following our previous advice before though:

Let $u = 3x^2 + x - 2$

$$\frac{du}{dx} = 6x + 1$$

$$\therefore dx = \frac{du}{6x + 1}$$

$$\begin{aligned}\int \frac{12x + 2}{(3x^2 + x - 2)^4} dx &= \int \frac{2(6x + 1)}{u^4} \times \frac{du}{6x + 1} \\ &= \int 2u^{-4} du \\ &= \frac{2u^{-3}}{-3} + C \\ &= -\frac{2}{3(3x^2 + x - 2)^3} + C\end{aligned}$$

Reverse Chain Rule Exercises

1. Using the standard form for reverse chain rule, find $\int \frac{1}{(x+1)^3} dx$

2. Find $\int 2(2x - 1)^{10} dx$, showing all working

3. Perform the integration $\int \frac{x^7 + x^4}{x^6} dx$

4. Find $\int \frac{8}{(4x+1)^5} dx$, showing all working

5. Perform the integration $\int x^2(5-3x)\, dx$

6. Perform the integration $\int \sqrt{x}(3\sqrt{x}-x)\, dx$

7. Find $\int \sqrt[3]{4x-1}\, dx$, showing all working

8. Find $\int \frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{x+2}} dx$, showing all working

Indefinite Integral Exercise Answers

1. It is highly recommended that for any questions, we convert our fraction into indices form to apply the formula easier.

$$Step\ 1: Convert\ into\ indices\ form$$

$$\int \frac{1}{(x+1)^3} dx = \int (x+1)^{-3} dx$$

$$Step\ 2: Apply\ the\ standard\ form$$

$$\int (x+1)^{-3} dx = \frac{(x+1)^{-2}}{-2} + C$$
$$= -\frac{1}{2(x+1)^2} + C$$

2.

$$Step\ 1: Apply\ our\ substitution\ for\ u$$

Let $u = 2x - 1$

$\frac{du}{dx} = 2, so\ dx = \frac{du}{2}$

$$\therefore \int 2(2x-1)^{10} dx = \int 2u^{10} \times \frac{du}{2}$$
$$= \int u^{10} du$$

$$Step\ 2: Integrate\ with\ respect\ to\ u$$

$$\int u^{10} du = \frac{u^{11}}{11} + C$$

$$Step\ 3: Convert\ our\ answer\ back\ to\ x$$

$$\therefore \frac{u^{11}}{11} + C = \frac{(2x-1)^{11}}{11} + C$$

3. It is recommended that **before integrating** our expression, we **simplify or expand it** beforehand.

$$Step\ 1: Simplify\ our\ expression$$

$$\int \frac{x^7 + x^4}{x^6} dx = \int x + \frac{1}{x^2}\, dx$$
$$= \int x + x^{-2}\, dx$$

$$Step\ 2: Integrate\ with\ respect\ to\ x$$

$$\int x + x^{-2}\, dx = \frac{x^2}{2} - x^{-1} + C$$
$$= \frac{x^2}{2} - \frac{1}{x} + C$$

4.

$$Step\ 1: Apply\ our\ substitution\ for\ u$$

Let $u = 4x + 1$

$\frac{du}{dx} = 4, so\ dx = \frac{du}{4}$

$$\therefore \int \frac{8}{(4x+1)^5} dx = 8\int \frac{1}{u^5} \times \frac{du}{4}$$
$$= 2\int u^{-5} du$$

$$Step\ 2: Integrate\ with\ respect\ to\ x$$

$$2\int u^{-5} du = 2 \times -\frac{u^{-4}}{4} + C$$
$$= -\frac{u^{-4}}{2} + C$$

$$Step\ 3: Convert\ our\ answer\ back\ to\ x$$

$$\therefore -\frac{u^{-4}}{2} + C = -\frac{(4x+1)^{-4}}{2} + C$$

5. Once again, it is recommended that before integrating our expression we should **expand our expression**

$$Step\ 1: Expand\ our\ expression$$

$$\int x^2(5-3x)\,dx = \int 5x^2 - 3x^3\,dx$$

$$Step\ 2: Integrate\ with\ respect\ to\ x$$

$$\therefore \int 5x^2 - 3x^3\,dx = \frac{5x^3}{3} - \frac{3x^4}{4} + C$$

6.

$$Step\ 1: Expand\ our\ expression$$

$$\begin{aligned}\int \sqrt{x}(3\sqrt{x}-x)\,dx &= \int 3x - x\sqrt{x}\,dx \\ &= \int 3x - x^{\frac{3}{2}}\,dx\end{aligned}$$

$$Step\ 2: Integrate\ with\ respect\ to\ x$$

$$\therefore \int 3x - x^{\frac{3}{2}}\,dx = \frac{3x^2}{2} - \frac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} + C$$

$$= \frac{3x^2}{2} - \frac{2x^{\frac{5}{2}}}{5} + C$$

7. $\int \sqrt[3]{4x-1}\,dx = \int (4x-1)^{\frac{1}{3}}\,dx$

$$Step\ 1: Apply\ our\ substitution\ for\ u$$

Let $u = 4x - 1$

$\frac{du}{dx} = 4$, so $dx = \frac{du}{4}$

$$\begin{aligned}\therefore \int (4x-1)^{\frac{1}{3}}\,dx &= \int u^{\frac{1}{3}} \times \frac{du}{4} \\ &= \frac{1}{4}\int u^{\frac{1}{3}}\,du\end{aligned}$$

$$Step\ 2: Integrate\ with\ respect\ to\ u$$

$$\begin{aligned}\frac{1}{4}\int u^{\frac{1}{3}}\,du &= \frac{1}{4} \times \frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C \\ &= \frac{1}{4} \times \frac{3}{4}u^{\frac{4}{3}} + C \\ &= \frac{3}{16}u^{\frac{4}{3}} + C\end{aligned}$$

$$Step\ 3: Convert\ our\ answer\ back\ to\ x$$

$$\therefore \frac{3}{16}u^{\frac{4}{3}} + C = \frac{3}{16}(4x-1)^{\frac{4}{3}} + C$$

8. Notice how in this question reverse chain rule does not need to be considered as if we let $u = x + 1, \frac{du}{dx} = 1$ and thus $du = dx$

$$Step\ 1: Convert\ to\ indices\ form$$

$$\int \frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{x+2}}\, dx = \int (x+1)^{-\frac{1}{2}} + (x+2)^{-\frac{1}{2}}\, dx$$

$$Step\ 2: Apply\ the\ standard\ form$$

$$\int (x+1)^{-\frac{1}{2}} + (x+2)^{-\frac{1}{2}}\, dx = \frac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}} + \frac{(x+2)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= 2(x+1)^{\frac{1}{2}} + 2(x+2)^{\frac{1}{2}} + C$$

$$\therefore = 2\sqrt{x+1} + 2\sqrt{x+2} + C$$