EXPONENTIALS & LOGARITHMS

INDEX LAWS (I)

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- Exponentials Revision

For an expression a^n, it is referred to as a "Power" or "Exponential" and its English translation is "a to the power of n".

Moreover:

This is called the "base"

This is called the "index" or the "exponent"

$$a^n$$

Indices are essentially a quicker way to represent multiplications of the same number many times. In other words:

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{a \text{ is multiplied } n \text{ times}}$$

For example:

$$6^4 = 6 \times 6 \times 6 \times 6$$

$$9^3 = 9 \times 9 \times 9$$

The one exception to this case is that:

$$a^0 = 1$$

No matter the value of a, any number to the power of 0 will equal to 1!

Note:

- Negative Indices

Negative indices are essentially **reciprocals** of your usual positive index.

What this means is that:

$$a^{-n} = \frac{1}{a^n}$$

Where n is a positive constant

For example:

$$7^{-3} = \frac{1}{7^3}$$

- Multiplying Exponentials

When you **multiply exponentials** with the same base, the **indices must add!**

Hence:

bases must be the same!

add the indices

$$\underbrace{a^m \times a^n}_{\textit{Multiplying powers}} = a^{m+n}$$

For example:

$$2^7 \times 2^3 = 2^{10}$$

- Dividing Exponentials

When you **divide exponentials** with the same base, the **indices must subtract!**

Hence:

bases must be the same!

Subtract the indices

$$\underbrace{a^m \div a^n}_{\textit{Dividing powers}} = a^{m-n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

For example:

$$\frac{5^8}{5^3} = 5^5$$

- Power of a Power

When we **power a power**, the indices must multiply!

Hence:

Multiply the indices

$$\underbrace{\left(a^{m}\right)^{n}}_{\textit{Powering a power}} = a^{mn}$$

For example:

$$(11^4)^5 = 11^{20}$$

- Fractional Indices

Fractional indices essentially represent roots of a base, where:

$$a^{\frac{1}{m}} = \sqrt[m]{a}$$

"a root m"

For example:

$$\begin{aligned} 27^{\frac{1}{3}} &= \sqrt[3]{27} \\ &= 3 \end{aligned}$$

Moreover, the fractional index doesn't just have to be a reciprocal. Thus, we must remember that:

$$a^{\frac{n}{m}} = \sqrt[m]{a^n}$$

"a^n root m"

For example:

$$\begin{aligned} 8^{\frac{2}{3}} &= \sqrt[3]{8^2} \\ &= \sqrt[3]{64} = 4 \end{aligned}$$

- Combined Index Laws

When given complicated indices to solve or simplify, we should complete the following steps:

$$\boldsymbol{Step\ 1: Remove\ the\ negative\ index\ by\ taking\ the\ reciprocal}$$

$$\boldsymbol{Step\ 2: If\ the\ index\ is\ a\ fraction, try\ remove\ the\ denominator\ by\ rooting}$$

$$\boldsymbol{Step\ 3: Finally, take\ the\ power\ of\ the\ index}$$

Example 1: Simplify $\left(\frac{16}{9}\right)^{-\frac{5}{2}}$

Solution:

First, removing the negative:

$$\left(\frac{16}{9}\right)^{-\frac{5}{2}} = \left(\frac{9}{16}\right)^{\frac{5}{2}}$$

Then, removing the denominator of the index:

$$\left(\frac{9}{16}\right)^{\frac{5}{2}} = \left(\sqrt{\frac{9}{16}}\right)^{5}$$

$$= \left(\frac{3}{4}\right)^{5}$$

Finally, taking the power and getting our answer:

$$\left(\frac{3}{4}\right)^{5} = \frac{243}{1024}$$

- The Natural Exponential Function

A common base for an index is the **natural number** e, so it is important to remember that e is a constant number (it is irrational) and not just another variable. In other words:

$$e^x = (2.7182\ \ldots)^x$$

Exponentials and Index Law Exercises

1. Simplify the following expressions:

a) 4^3

b) $\left(\frac{3}{5}\right)^3$

c) 7^{-2}

d) $\left(\frac{2}{7}\right)^{-2}$

e) $64^{\frac{1}{3}}$

f) $\left(\frac{1}{36}\right)^{\frac{1}{2}}$

g) $(109248)^0$

2. Simplify, leaving your answer in index form:

a) $x^2 \times x^5$

b) $3^{-2} \times 3^5$

c) $a^9 \div a^3$

d) $6^{-2} \div 6^{-3}$

e) $y^4 \times y^3 \times y^2$

f) $5^3 \times 2^5 \div 5^2 \div 2^3$

g) $(x^3)^4$

h) $(t^2)^5 \div t^7$

i) $(a^3)^{-2} \times a^3 \div a^4$

3. Simplify the following powers, converting any decimals to fractions as your first step:

a) $\left(\frac{25}{49}\right)^{\frac{1}{2}}$

b) $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

c) $(0.25)^{\frac{7}{2}}$

d) $(0.09)^{\frac{3}{2}}$

e) $(0.04)^{\frac{3}{2}}$

f) $\left(\frac{1}{16}\right)^{-\frac{3}{4}}$

g) $\left(\frac{125}{8}\right)^{-\frac{2}{3}}$

4. Expand the brackets in the following:

a) $\left(\frac{x}{y}\right)^4$

b) $\left(\frac{3a}{4}\right)^3$

c) $\left(\frac{7x}{2y}\right)^{-2}$
d) $(abc)^3$
e) $\left(\frac{5t}{2pq}\right)^{-4}$

5. Rewrite the following using indices instead of surds:
a) $\sqrt{2x}$
b) $9\sqrt[5]{x}$
c) $12\sqrt{x^3}$
d) $36\sqrt[5]{x^4}$
e) $5\sqrt[7]{x^2}$

6. Simplify each of the following expressions, giving the answer without negative indices
a) $x^{-5}y^3 \times x^6y^{-2}$
b) $(3x^2y^2)^3 \times (xy^2)^2$
c) $(3a^{-2}y^3)^{-2} \times (3ay^{-2})^3$
d) $(2x^2y^4)^2 \times (xy)^{-2}$
e) $(5ab^2)^3 \div (5a^{-1}b^3)^2$
f) $(8x^2y^3)^3 \div (2x^{-1}y^2)^3$

7. Use the index laws to simplify the following expressions, leaving your answers in surd form instead of fractional indices:
a) $\left(x^{-\frac{2}{3}}\right)^9$
b) $x^4 \times x^{-\frac{1}{2}}$
c) $(9s^{-4}q^6)^{\frac{3}{2}}$
d) $(8x^{-3}y^5)^{\frac{1}{2}}$

Exponentials and Index Law Exercise Answers

1.
a) $4^3 = 64$
b) $\left(\frac{3}{5}\right)^3 = \frac{3^3}{5^3}$

$$\therefore \frac{3^3}{5^3} = \frac{27}{125}$$

c) $7^{-2} = \frac{1}{7^2}$

$$\therefore \frac{1}{7^2} = \frac{1}{49}$$

d) $\left(\frac{2}{7}\right)^{-2} = \frac{1}{\left(\frac{2}{7}\right)^2}$

$$\therefore \frac{1}{\left(\frac{2}{7}\right)^2} = \left(\frac{7}{2}\right)^2$$
$$= \frac{49}{4}$$

e) $64^{\frac{1}{3}} = \sqrt[3]{64}$

$$\therefore \sqrt[3]{64} = 4$$

f) $\left(\frac{1}{36}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{36}}$

$$\therefore \sqrt{\frac{1}{36}} = \frac{1}{\sqrt{36}}$$
$$= \frac{1}{6}$$

2.

a) x^7

b) 3^3

c) a^6

d) $6^{-2--3} = 6^1$

e) $y^{4+3+2} = y^9$

f) $5^3 \times 2^5 \div 5^2 \div 2^3 = 5^3 \div 5^2 \times 2^5 \div 2^3$

$$\therefore 5^3 \div 5^2 \times 2^5 \div 2^3 = 5 \times 2^2$$
$$= 20$$

g) x^{12}

h) $t^{10} \div t^7 = t^3$

i) $a^{-6} \times a^3 \div a^4 = a^{-6+3-4}$

$$a^{-6+3-4} = a^{-7}$$
$$= \frac{1}{a^7}$$

3.

a) $\left(\frac{25}{49}\right)^{\frac{1}{2}} = \sqrt{\frac{25}{49}}$

$$\therefore \sqrt{\frac{25}{49}} = \frac{5}{7}$$

b) $\left(\frac{8}{27}\right)^{-\frac{2}{3}} = \left(\frac{27}{8}\right)^{\frac{2}{3}}$

$$\begin{aligned}\therefore \left(\frac{27}{8}\right)^{\frac{2}{3}} &= \left(\frac{3^3}{2^3}\right)^{\frac{2}{3}} \\ &= \frac{3^{3\times\frac{2}{3}}}{2^{3\times\frac{2}{3}}} \\ &= \frac{3^2}{2^2} \\ &= \frac{9}{4}\end{aligned}$$

c) $(0.25)^{\frac{7}{2}}$

$$\begin{aligned}\left(\frac{1}{4}\right)^{\frac{7}{2}} &= \left(\frac{1}{2^2}\right)^{\frac{7}{2}} \\ &= \frac{1}{2^{2\times\frac{7}{2}}} \\ &= \frac{1}{2^7} \\ &= \frac{1}{128}\end{aligned}$$

d) $(0.09)^{\frac{3}{2}}$

$$\begin{aligned}\left(\frac{9}{100}\right)^{\frac{3}{2}} &= \left(\frac{3^2}{10^2}\right)^{\frac{3}{2}} \\ &= \left(\frac{3^{2\times\frac{3}{2}}}{10^{2\times\frac{3}{2}}}\right) \\ &= \frac{3^3}{10^3} \\ &= \frac{27}{1000}\end{aligned}$$

e) $(0.04)^{\frac{3}{2}}$

$$\begin{aligned}\left(\frac{1}{25}\right)^{\frac{3}{2}} &= \left(\frac{1}{5^2}\right)^{\frac{3}{2}} \\ &= \left(\frac{1}{5^{2\times\frac{3}{2}}}\right)\end{aligned}$$

$$= \left(\frac{1}{5^3}\right)$$
$$= \frac{1}{125}$$

f) $\left(\frac{1}{16}\right)^{-\frac{3}{4}}$

$$\left(\frac{1}{16}\right)^{-\frac{3}{4}} = \left(\frac{1}{2^4}\right)^{-\frac{3}{4}}$$
$$= \frac{1}{2^{4\times-\frac{3}{4}}}$$
$$= \frac{1}{2^{-3}}$$
$$= \frac{1}{\frac{1}{2^3}}$$
$$= 2^3$$
$$= 8$$

g) $\left(\frac{125}{8}\right)^{-\frac{2}{3}}$

$$\left(\frac{125}{8}\right)^{-\frac{2}{3}} = \left(\frac{8}{125}\right)^{\frac{2}{3}}$$
$$= \left(\frac{2^3}{5^3}\right)^{\frac{2}{3}}$$
$$= \frac{2^{3\times\frac{2}{3}}}{5^{3\times\frac{2}{3}}}$$
$$= \frac{2^2}{5^2}$$
$$= \frac{4}{25}$$

4.
a)

$$\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$$

b) A

$$\left(\frac{3a}{4}\right)^3 = \frac{3^3a^3}{4^3}$$
$$= \frac{27a^3}{64}$$

c) $\left(\frac{7x}{2y}\right)^{-2}$

$$\begin{aligned}\left(\frac{7x}{2y}\right)^{-2} &= \left(\frac{2y}{7x}\right)^{2}\\ &= \frac{2^2y^2}{7^2x^2}\\ &= \frac{4y^2}{49x^2}\end{aligned}$$

d) $(abc)^3 = a^3b^3c^3$

e)

$$\begin{aligned}\left(\frac{5t}{2pq}\right)^{-4} &= \left(\frac{2pq}{5t}\right)^{4}\\ &= \frac{2^4p^4q^4}{5^4t^4}\\ &= \frac{16p^4q^4}{625t^4}\end{aligned}$$

5.

a) $\sqrt{2x} = 2^{\frac{1}{2}}x^{\frac{1}{2}}$

b) $9\sqrt[5]{x} = 9x^{\frac{1}{5}}$

c) $12\sqrt{x^3} = 12x^{\frac{3}{2}}$

d) $36\sqrt[5]{x^4} = 36x^{\frac{4}{5}}$

e) $5\sqrt[7]{x^2} = 5x^{\frac{2}{7}}$

6.

f)

$$\begin{aligned}x^{-5}y^3 \times x^6y^{-2} &= x^{-5+6}y^{3-2}\\ &= xy\end{aligned}$$

g)

$$\begin{aligned}(3x^2y^2)^3 \times (xy^2)^2 &= 3^3x^6y^6 \times x^2y^4\\ &= 27x^{6+2}y^{6+4}\\ &= 27x^8y^{10}\end{aligned}$$

h)

$$\begin{aligned}(3a^{-2}y^3)^{-2} \times (3ay^{-2})^3 &= 3^{-2}a^4y^{-6} \times 3^3a^3y^{-6}\\ &= 3^{-2+3}a^{4+3}y^{-6-6}\\ &= 3a^7y^{-12}\\ &= \frac{3a^7}{y^{12}}\end{aligned}$$

i)

$$\begin{aligned}(2x^2y^4)^2 \times (xy)^{-2} &= 2^2x^4y^8 \times x^{-2}y^{-2}\\ &= 4x^{4-2}y^{8-2}\\ &= 4x^2y^6\end{aligned}$$

j)

$$\begin{aligned}(5ab^2)^3 \div (5a^{-1}b^3)^2 &= 5^3a^3b^6 \div 5^2a^{-2}b^6\\ &= 5^{3-2}a^{3--2}b^{6-6}\\ &= 5a^5\end{aligned}$$

k)

$$\begin{aligned}(8x^2y^3)^3 \div (2x^{-1}y^2)^3 &= 8^3x^6y^9 \div 2^3x^{-3}y^6\\ &= 8^3x^6y^9 \div 8x^{-3}y^6\\ &= 8^{3-1}x^{6--3}y^{9-6}\\ &= 8^2x^9y^3\\ &= 64x^9y^3\end{aligned}$$

7.
a)

$$\begin{aligned}\left(x^{-\frac{2}{3}}\right)^9 &= x^{-\frac{2}{3}\times 9}\\ &= x^{-6}\\ &= \frac{1}{x^6}\end{aligned}$$

b)

$$\begin{aligned}x^4 \times x^{-\frac{1}{2}} &= x^{4-\frac{1}{2}}\\ &= x^{3\frac{1}{2}}\\ &= x^3\sqrt{x}\end{aligned}$$

c)

$$\begin{aligned}(9s^{-4}q^6)^{\frac{3}{2}} &= 9^{\frac{3}{2}}s^{-6}q^9\\ &= \frac{27q^9}{s^6}\end{aligned}$$

d)

$$(8x^{-3}y^5)^{\frac{1}{2}} = 8^{\frac{1}{2}}x^{-\frac{3}{2}}y^{\frac{5}{2}}$$

$$= \frac{\sqrt{8}y^{\frac{5}{2}}}{x^{\frac{3}{2}}}$$

$$= \frac{2\sqrt{2}y^2\sqrt{y}}{x\sqrt{x}}$$