Mathify HSC

INTEGRATION

TRIG INTEGRALS (VII)

Contents include:

- Standard Trigonometric Integrals
- Using Trig Identities in Integration
- Integral of tan x

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- The Standard Integrals

When we consider the anti-derivatives of the trigonometric functions:

First, $\frac{d}{dx}\sin x = \cos x$

Reversing this we get:

$$\int \cos x \, dx = \sin x$$

Example 1: Evaluate $\int_{\frac{\pi}{3}}^{\pi} \cos x \, dx$

Solution:

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\pi} \cos x \, dx &= [\sin x]_{\frac{\pi}{3}}^{\pi} \\ &= (\sin \pi) - \left(\sin \frac{\pi}{3}\right) \\ &= 0 - \frac{\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Secondly, $\frac{d}{dx}\cos x = -\sin x$

Reversing this we get:

$$\int \sin x \, dx = -\cos x + C$$

Example 2: Evaluate the integral $\int_0^{\pi} \sin x \, dx$

Solution:

$$\begin{aligned}\int_0^{\pi} \sin x \, dx &= [-\cos x]_0^{\pi} \\ &= (-\cos \pi) - (-\cos 0) \\ &= --1 + 1 \\ &= 2\end{aligned}$$

Thirdly, $\frac{d}{dx}\tan x = \sec^2 x$

Reversing this we get:

$$\int \sec^2 x\, dx = \tan x + C$$

Example 3: Evaluate the integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x\, dx$

Solution:

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x\, dx &= [\tan x]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\ &= \left(\tan\frac{\pi}{4}\right) - \left(\tan\frac{\pi}{6}\right) \\ &= 1 - \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}-1}{\sqrt{3}}\end{aligned}$$

Remember that for trigonometric integrals, the reverse chain rule and substitution still applies! Often if we need to make a substitution, it will be whatever expression is inside the trig function.

Thus, the standard forms include:

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$
$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$
$$\int \sec^2(ax+b)\ dx = \frac{1}{a}\tan(ax+b) + C$$

Example 4: Evaluate the integral $\int_{\pi}^{2\pi} \sin\frac{1}{4}x\, dx$

Solution:

Recall that:

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int_{\pi}^{2\pi} \sin\frac{1}{4}x\, dx = \left[-4\cos\frac{1}{4}x\right]_{\pi}^{2\pi}$$

$$= -4\cos\frac{1}{4} \times 2\pi - -4\cos\frac{1}{4} \times \pi$$
$$= -4\cos\frac{\pi}{2} - -4\cos\frac{\pi}{4}$$
$$= 0 + 4 \times \frac{1}{\sqrt{2}}$$
$$= \frac{4}{\sqrt{2}}$$

- Using Trigonometric Identities in Integration

Recalling our Pythagorean identities for trigonometry:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = cosec^2 x$$

In some of the harder trigonometric integration questions, students will be required to apply these identities prior to integrating anything. Thus, if a trigonometric integral seems impossible/difficult, remember to try use your identities!

Example 5: Find $\int \tan^2 x + 1\, dx$

Solution:

If we try to integrating $\int \tan^2 x \; dx$, we are NOT allowed to say:

$$\int \tan^2 x \; dx = \frac{\tan^3 x}{3}$$

The reason this is WRONG is because we are integrating with respect to x, NOT $\tan x$. We would need to undergo the substitution method if we wanted to do it correctly.

Thus, since this integral doesn't seem possible through normal methods, we must consider trig identities, where:

Since $1 + \tan^2 x = \sec^2 x$:

$$\therefore \int \tan^2 x + 1\, dx = \int \sec^2 x \; dx$$
$$= \tan x + C$$

- Finding the integral of $\tan x$

Recall that:

$$\tan x = \frac{\sin x}{\cos x}$$

Thus, if we want to find the integral of $\tan x$, we must use substitution:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$ ($\because$ it is the denominator)

$$\frac{du}{dx} = -\sin x$$

$$\therefore dx = -\frac{du}{\sin x}$$

Hence:

$$\begin{aligned} \int \frac{\sin x}{\cos x} dx &= \int \frac{\sin x}{u} \times -\frac{du}{\sin x} \\ &= \int -\frac{1}{u} du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \end{aligned}$$

Therefore, it can be said that:

$$\int \tan x \, dx = -\ln|\cos x| + C$$

Integration of Trigonometric Function Exercises

1. Evaluate the following integrals

a) $\int_0^{\pi} (3\sin x - 3\sin 2x) \, dx$

b) $\int_0^{\frac{\pi}{2}} (\sec^2 \frac{1}{2}x + \cos x) \, dx$

2. The derivative of a certain function is $f'(x) = \frac{1}{2}\cos\frac{1}{2}x$, and the graph of the function has y-intercept (0, 0). Find the original function $f(x)$ and then find $f\left(\frac{\pi}{2}\right)$

3. Given that the function $f'(x) = 3\sin 3x$ and $f(\pi) = 1$:

a) Find the function $f(x)$

b) Find $f\left(\frac{\pi}{3}\right)$

4.
a) Use the chain rule to differentiate $\cos^5 x$
b) Hence find $\int_0^{\pi} \sin x \cos^4 x \; dx$

5. Find the primitive function of $f(x) = \cos x \sin^3 x$

6. A curve passing through the origin has gradient function $y' = \cos x - 4\sin 4x$, provide an expression for y

7. Evaluate the integral:

$$\int_0^{\frac{\pi}{4}} \tan^2 x \; dx$$

8. Find $\int \tan(5x)\, dx$

Integration of Trigonometric Function Exercise Answers

1. Using the standard form of the integral:
a)

$$\begin{aligned}\int_0^{\pi} 3\sin x - 3\sin 2x\, dx &= \left[-3\cos x + \frac{3}{2}\cos 2x\right]_0^{\pi} \\ &= \left[-3\cos\pi + \frac{3}{2}\cos 2\times\pi\right] - \left[-3\cos 0 + \frac{3}{2}\cos 2\times 0\right] \\ &= \left[-3\times -1 + \frac{3}{2}\times 1\right] - \left[-3\times 1 + \frac{3}{2}\times 1\right] \\ &= \frac{9}{2} - -\frac{3}{2} \\ &= 6\end{aligned}$$

b)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \left(\sec^2\frac{1}{2}x + \cos x\right) dx &= \left[2\tan\frac{1}{2}x + \sin x\right]_0^{\frac{\pi}{2}} \\ &= \left[2\tan\frac{1}{2}\times\frac{\pi}{2} + \sin\frac{\pi}{2}\right] - \left[2\tan\frac{1}{2}\times 0 + \sin 0\right] \\ &= \left[2\tan\frac{\pi}{4} + \sin\frac{\pi}{2}\right] - [2\tan 0 + \sin 0] \\ &= 2 + 1 - 0 \\ &= 3\end{aligned}$$

2.

$$Step\ 1: Find\ the\ indefinite\ integral\ f(x)$$

$$f(x) = \int \frac{1}{2}\cos\frac{1}{2}x\ dx = \sin\frac{1}{2}x + C$$

$$Step\ 2: Substitute\ in\ known\ point\ (0,0)\ to\ determine\ C$$

$$\therefore 0 = \sin\frac{1}{2} \times 0 + C$$
$$C = 0$$

$$\therefore f(x) = \sin\frac{1}{2}x$$

$$Step\ 3: Find\ the\ value\ of\ f\left(\frac{\pi}{2}\right)$$

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \sin\frac{1}{2} \times \frac{\pi}{2} \\ &= \sin\frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

3.

a) $Step\ 1: Find\ the\ indefinite\ integral\ f(x)$

$$f(x) = \int 3\sin 3x\, dx = -\cos 3x + C$$

$$Step\ 2: Substitute\ in\ known\ point\ (\pi, 1)\ to\ determine\ C$$

$$\begin{aligned} 1 &= -\cos 3 \times \pi + C \\ 1 &= -\cos\pi + C \\ 1 &= --1 + C \\ \therefore C &= 0 \end{aligned}$$

$$f(x) = -\cos 3x$$

b)

$$\begin{aligned} f\left(\frac{\pi}{3}\right) &= -\cos\left(3 \times \frac{\pi}{3}\right) \\ &= -\cos\pi \\ &= --1 \\ &= 1 \end{aligned}$$

4. Notice here how a substitution must be made

a) $Let\ u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$\therefore \frac{d}{dx}(\cos^5 x) = \frac{d}{du}(u^5) \times \frac{du}{dx}$$
$$= 5u^4 \times -\sin x$$
$$= -5\sin x \cos^4 x$$

b) From a), we have determined that:

$$\frac{d}{dx}(\cos^5 x) = -5\sin x \cos^4 x$$

Thus, considering the anti-derivative, we therefore can determine that:

$$\int \frac{d}{dx}(\cos^5 x)\ dx = \int -5\sin x \cos^4 x\ \ dx$$
$$\therefore -5\int \sin x \cos^4 x\ \ dx = \cos^5 x + C_1$$
$$\int \sin x \cos^4 x\ dx = -\frac{1}{5}\cos^5 x + C$$

Note: $C \neq C_1$, but both are still constants, so it doesn't really matter

Thus, when we consider our definite integral with bounds:

$$\int_0^\pi \sin x \cos^4 x\ \ dx = \left[-\frac{1}{5}\cos^5 x\right]_0^\pi$$
$$= \left[-\frac{1}{5}\cos^5 \pi\right] - \left[-\frac{1}{5}\cos^5 0\right]$$
$$= \left[-\frac{1}{5} \times (-1)^5\right] - \left[-\frac{1}{5} \times (1)^5\right]$$
$$= \frac{1}{5} + \frac{1}{5}$$
$$= \frac{2}{5}$$

5. A trig substitution has to be made for $f(x) = \int \cos x \sin^3 x\ \ dx$

$Let\ u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$$\therefore \int \cos \sin^3 x\ \ dx = \int \cos x\, u^3 \times \frac{du}{\cos x}$$
$$= \int u^3\, du$$
$$= \frac{u^4}{4} + C$$
$$= \frac{\sin^4 x}{4} + C$$

6. $y' = \cos x - 4\sin 4x$

$$\begin{aligned} \therefore y &= \int \cos x - 4\sin 4x \, dx \\ &= \sin x + \cos 4x + C \end{aligned}$$

7. First applying our trig identity:

Since $1 + \tan^2 x = \sec^2 x$:

$$\therefore \tan^2 x = \sec^2 x - 1$$

Hence:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \tan^2 x \; dx &= \int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx \\ &= [\tan x - x]_0^{\frac{\pi}{4}} \\ &= \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0) \\ &= 1 - \frac{\pi}{4} \end{aligned}$$

8.

Since its is known that:

$$\tan x = -\ln|\cos x| + C$$

Then therefore, by the reverse chain rule:

$$\tan 5x = -\frac{1}{5}\ln|\cos 5x| + C$$