INTEGRATION

RATES OF CHANGE AND MOTION (XIII)

Contents include:

- Integrating in Motion
- Maximum Height of an Upwards Particle

- Motion

Recall back in Differentiation (VIII) that:

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Where:

x stands for displacement

t stands for time

v stands for velocity

a stands for acceleration

Now considering the reverse for integration, we can say that:

$$x = \int v\, dt$$

$$v = \int a\, dt$$

In other words:

$$displacement\ is\ the\ integral\ of\ velocity$$

$$velocity\ is\ the\ integral\ of\ acceleration$$

When completing these questions you must be careful with **direction** and know which way is negative or positive! Generally:

$$upwards\ or\ towards\ the\ right = positive$$

$$downwards\ or\ towards\ the\ left = negative$$

Example 1: The velocity $v\ m/s$ of a body moving in a straight line is given by $v = 3t^2 - 2t - 1$. The body initially has a displacement of 1m from the origin O.

a) Find the displacement at any time t
b) Find the distance travelled in the first 2 seconds

Solution:

a) Since we are asked to find displacement from velocity:

$$x = \int 3t^2 - 3t - 1\,dt$$
$$= 3 \times \frac{t^3}{3} - 3 \times \frac{t^2}{2} - t + C$$
$$= t^3 - \frac{3t^2}{2} - t + C$$

When $t = 0$, $x = 1$. Hence:

$$1 = 0^3 - \frac{3(0)^2}{2} - 0 + C$$
$$\therefore C = 1$$

Hence, the displacement equation is:

$$x = t^3 - \frac{3t^2}{2} - t + 1$$

b)

We must be careful whenever the question asks for distance!!!

We can't just sub in $t = 2$ into the displacement equation in these situations because **displacement is not the same as distance.** We must check for when $v = 0$ and see if the particle turns around at all before $t = 2$.

Since $v = 3t^2 - 2t - 1$, factorising:

$$v = (3t + 1)(t - 1)$$

Hence, when $t = 1$, $v = 0$

Now we must plot our object's movement as time progresses:

When $t = 0$:

$$x = 1$$

When $t = 1$:

$$x = 1 - \frac{3}{2} - 1 + 1$$

$$= -\frac{1}{2}$$

When $t = 2$:

$$x = 2^3 - \frac{3}{2}(2)^2 - 2 + 1$$
$$= 8 - \frac{3}{2} \times 4 - 2 + 1$$
$$= 8 - 6 - 2 + 1$$
$$= 1$$

Between $t = 0$ and $t = 1$, we can see that the distance we travelled is $1 - -\frac{1}{2} = \frac{3}{2}m$

Between $t = 1$ and $t = 2$, we can see that the distance we travelled is $1 - -\frac{1}{2} = \frac{3}{2}m$

Hence, the total distance travelled is:

$$\therefore Total\ distance = \frac{3}{2} + \frac{3}{2} = 3m$$

- Maximum height of an upwards particle

Imagine if we throw a ball up into the air. Due to gravity, it's upward velocity will decreases slowly, all the way until it reaches 0, before it starts falling down again back into your hand.

When did the highest point of the ball occur? The highest point occurs when $v = 0$

Note that for these questions, a common value for acceleration due to gravity is:

$$a = -10\ m/s^2\ OR\ a = -9.8\ m/s^2$$

Example 2: A ball is projected vertically upwards from the top of a building $30m$ high with an upwards velocity of $25\ m/s$. Assume that acceleration due to gravity is $10\ m/s^2$. Find:

a) The time taken to reach the highest point
b) How long it will take the ball to reach the ground
c) The speed with which the ball strikes the ground

Solutions:

a) Since we are looking for when the highest point occurs, we need to first find the velocity time equation:

$$a = -10$$

$$\therefore v = \int -10\, dt$$
$$= -10t + C$$

When $t = 0, v = 25$

$$\therefore 25 = 0 + C$$
$$C = 25$$

Hence:

$$v = -10t + 25$$

Now finding when $v = 0$:

$$0 = -10t + 25$$
$$10t = 25$$
$$\therefore t = 2.5\ seconds$$

b) To find the time taken for the ball to reach the ground again, AKA $x = 0$, we must first find the displacement equation:

$$v = -10t + 25$$
$$\therefore x = \int -10t + 25\, dt$$
$$= -10 \times \frac{t^2}{2} + 25t + C$$
$$= -5t^2 + 25t + C$$

When $t = 0$, $x = 30$ (since we start off on a 30m tall building):

$$\therefore 30 = 0 + 0 + C$$
$$C = 30$$
$$\therefore x = -5t^2 + 25t + 30$$

Hence, factorising x:

$$x = 5(-t^2 + 5t + 6)$$
$$= 5(-t + 6)(t + 1)$$

Now letting $x = 0$ and solving for t:

$$\therefore t = 6\ ONLY\ (\because t \geq 0)$$

Hence, the ball reaches the ground after 6 seconds

c) Since we know the ball hits the ground after 6 seconds, the speed at which it hits the ground is:

$$\therefore v = -10(6) + 25$$

$$= -60 + 25$$
$$= -35$$

Hence, the speed at which the ball strikes the ground is $35\ m/s$. Notice that we don't include the negative sign here since speed is a scalar and no direction is needed.