DIFFERENTIATIAL CALCULUS

OPTIMISATION (VIII)

Contents include:

- Optimisation Questions

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- Optimisation Problems

Optimisation refers to the process of finding the most efficient usage of resources.

For example, if I'm given some planks of wood and I'm asked to find what is the largest crate I can build using these planks. As you can probably guess, optimisation questions essentially use everything we've learnt so far in differentiation and are very tricky!

Moreover, the types of questions asked vary greatly, but generally involves similar steps:

$$\textbf{Step 1}: \textit{Finding the formula for the quantity that needs to be maximised or minimised, ensuring that there is only one independent variable}$$

$$\textbf{Step 2}: \textit{Differentiating the formula to find the stationary point by letting } \frac{dy}{dx} = 0$$

$$\textbf{Step 3}: \textit{Finding the nature of the stationary points}$$

$$\textbf{Step 4}: \textit{Answering the question while including units}$$

Practice is your best friend, especially for these types of problems! It is purely an application of differential calculus theory, so there is nothing extra to learn for this booklet!

Example 1: A piece of wire of length 24cm is bent in the shape of a rectangle. Find the maximum area of the rectangle

A rectangle with a set perimeter, in this case 24cm, may have multiple different areas. Thus, the length and width of the rectangle are variable.

Since $Perimeter = 2 \times (width + length) = 24$,

$$\therefore width + length = 12$$

Letting $length = x, this\ therefore\ means\ that\ width = 12 - x$ and:

$$\begin{aligned} Area &= width \times length \\ &= x(12 - x) \end{aligned}$$

$$\begin{aligned} \therefore A(x) &= 12x - x^2 \\ A'(x) &= 12 - 2x \end{aligned}$$

Maximum area will occur when $A'(x) = 0$.

$$\begin{aligned} 12 - 2x &= 0 \\ x &= 6 \end{aligned}$$

$When\ x < 6, A'(x) > 0$
$When\ x > 6, A'(x) < 0$

$$\therefore Maximum\ area\ when\ x = 6$$

Hence, the dimensions of the rectangle will be 6×6, meaning that it is a square

Optimisation Exercises

1. Find the maximum area of a rectangular plot of ground that can be enclosed by 160m of fencing

2. A rectangular field is to be fenced around 3 sides with 300m of fencing. Find the dimensions of such a field if the area to be enclosed is as great as possible

3. The area of a rectangle is $400cm^2$. If the length of one of its sides is xcm, express the length of the other side and hence the perimeter in terms of x. Find the value of x which makes the perimeter a minimum

4. A company manufactures items at \$2 per item and sells them for $\$x$ per item. If the number sold is $\frac{800}{x^2}$ per month, find the value of x for which the company could expect to maximize its monthly profit.

5. The cost of running a truck at an average speed of vkm/hr is $64 + \frac{v^2}{100}$ dollars per hour. Calculate the average speed for which the total cost of running a truck from Sydney to Katoomba, a distance of 100km, will be a minimum.

Optimisation Exercise Answers

1. Since $Perimeter = 2 \times (width + length) = 160$,

$$\therefore width + length = 80$$

Letting $length = x, this\ therefore\ means\ that\ width = 80 - x$ and:

$$\begin{aligned} Area &= width\ \times length \\ &= x(80 - x) \end{aligned}$$

$$\begin{aligned} \therefore A(x) &= 80x - x^2 \\ A'(x) &= 80 - 2x \end{aligned}$$

Maximum area will occur when $A'(x) = 0$.

$$\begin{aligned} 80 - 2x &= 0 \\ x &= 40 \end{aligned}$$

$When\ x < 40, A'(x) > 0$
$When\ x > 40, A'(x) < 0$

$$\therefore Maximum\ area\ when\ x = 40$$

Thus, the maximum area is $A(40) = 40 \times 40 = 1600m^2$

2. Since the perimeter of 3 sides is 300m, if the length is x, then the width is $300 - 2x$.

$$\therefore Area = A(x) = x(300 - 2x)$$
$$= 300x - 2x^2$$

$$A'(x) = 300 - 4x$$

Maximum area will occur when $A'(x) = 0$.

$$300 - 4x = 0$$

$$x = 75$$

$When\ x < 75, A'(x) > 0$

$When\ x > 75, A'(x) < 0$

$$\therefore Maximum\ area\ when\ x = 75$$

Hence, $width = 300 - 2 \times 75 = 150$

Dimensions of our rectangle will therefore be 75×150

3. Let the other side of the rectangle by y

$$\therefore xy = 400$$
$$y = \frac{400}{x}$$

The expression for the perimeter of our rectangle is thus:

$$P = 2\left(x + \frac{400}{x}\right)$$
$$P = 2x + 800x^{-1}$$

$$P'(x) = 2 - 800x^{-2}$$

Minimum perimeter will occur when $P'(x) = 0$.

$$2 - \frac{800}{x^2} = 0$$
$$\frac{2x^2 - 800}{x^2} = 0$$

$$\therefore 2x^2 - 800 = 0$$
$$x^2 = 400$$
$$x = 20\ (x > 0)$$

$When\ x < 20, P'(x) < 0$
$When\ x > 20, P'(x) > 0$

$$\therefore Minimum\ perimeter\ when\ x = 20$$

4. We should first derive an expression for profit, $P(x)$

Essentially, $P(x) = Total\ money\ earnt - Costs$

In this case, for each unit sold we earn $x - 2$ dollars, with $\frac{800}{x^2}$ units in total

$$\begin{aligned}\therefore P(x) &= \frac{800}{x^2}(x-2)\\ &= \frac{800}{x} - \frac{1600}{x^2}\\ &= 800x^{-1} - 1600x^{-2}\end{aligned}$$

$$P'(x) = -800x^{-2} + 3200x^{-3}$$

Maximum profit will occur when $P'(x) = 0$

Thus, $-\frac{800}{x^2} + \frac{3200}{x^3} = 0$

$\frac{3200}{x^3} = \frac{800}{x^2}$

$3200x^2 = 800x^3$

$800x^3 - 3200x^2 = 0$

$800x^2(x-4) = 0$

$$\therefore x = 4\ (Since\ x > 0)$$

$When\ x < 4, P'(x) > 0$

$When\ x > 4, P'(x) < 0$

$$\therefore Maximum\ Profit\ will\ occur\ when\ x = 4$$

5. The total cost of the journey each hour may be given by:

$$Cost\ per\ hour = 64 + \frac{v^2}{100}$$

The total number of hours $= \frac{distance}{speed} = \frac{100}{v}$

Thus, the total cost will be given by:

$$\begin{aligned}c(v) &= \frac{100}{v}\left(64 + \frac{v^2}{100}\right)\\ &= \frac{6400}{v} + v\end{aligned}$$

$$c'(v) = -6400v^{-2} + 1$$

Minimum cost will occur when $c'(v) = 0$

$$1 - \frac{6400}{v^2} = 0$$
$$1 = \frac{6400}{v^2}$$
$$v^2 = 6400$$
$$\therefore v = 80 \ (v > 0)$$

$$When\ v < 80, c'(v) < 0$$
$$When\ v > 80, c'(v) > 0$$

$$\therefore v = 80\ is\ average\ speed\ for\ minimum\ cost$$