DIFFERENTIATION

PHYSICAL APPLICATIONS: MOTION (VIII)

Contents include:

- Linear Motion
- Scalar and Vector Quantities
- Displacement, Velocity and Acceleration

- Linear Motion and Kinematics

The concept of the first derivative being used to measure the rate of something may be applied to the study of objects in motion, known as **kinematics**.

In kinematics and thus in this course, we do not consider the force or forces causing the motion, but are instead interested in certain quantities like distance, velocity, speed, acceleration and displacement, the definitions of which we will cover in the dot point below.

- Scalar and Vector Quantities

In maths it is important to distinguish between 2 types of quantities or values:

1. Vector Quantities

Vector quantities have **both magnitude and direction**. This means that it is important to consider which direction is forward or "positive" since sometimes our value will be positive. Some examples of vector quantities include:

- **Displacement:** The difference in position between the end point and start point, independent of the path taken
- **Velocity:** Speed with direction associated. Given by the formula:

$$Velocity = \frac{displacement}{time}$$

- **Acceleration:** How fast something's velocity changes. Formula may be given by:

$$Acceleration = \frac{Final\ velocity - Initial\ velocity}{time\ taken}$$

2. Scalar Quantities

Scalar quantities have magnitude but **no direction** associated with it. Some examples include:

- **Distance:** How far something has travelled
- **Speed:** How fast something is moving, i.e. $\frac{distance}{time}$

Note: A common mistake is assuming $distance = displacement$. They are NOT the same. For example, if I run a lap of the oval, my displacement would be $0m$ as I have arrived back at the starting position, but my distance travelled would be $400m$.

- Velocity as a derivative of displacement

Since velocity is defined as the rate of change of position over time, we may express it as a derivative of displacement x with respect to time.

$$velocity = v(t) = x'(t) = \frac{dx}{dt} = \dot{x}$$

- Acceleration as a derivative of velocity

Acceleration is defined as the rate of change of velocity, and thus we may express it as a derivative of velocity v with respect to time.

$$acceleration = a(t) = v'(t) = \frac{dv}{dt}$$

Moreover, since we know that $v(t) = \frac{dx}{dt}$, we can determine that acceleration is also the second derivative of displacement x

$$a(t) = \frac{d^2x}{dx^2} = f''(t) = \ddot{x}$$

When completing these questions you must be careful with **direction** and know which way is negative or positive! Generally:

$$upwards\ or\ towards\ the\ right = positive$$

$$downwards\ or\ towards\ the\ left = negative$$

Example 1: A particle moves in a straight line such that its position x m from a fixed-point O on the line at time t s $(t \geq 0)$ is given by $x = t^3 - 12t + 16$. Find

a) Its initial position, velocity and acceleration
b) The time when its velocity is zero

Solutions:

a)

Initial position will occur when $t = 0$,

$$\therefore Initial\ position = 0^3 - 12 \times 0 + 16 = 16m$$

Finding the velocity equation:

$$v = \frac{dx}{dt} = 3t^2 - 12$$

Now find initial velocity by letting $t = 0$,

$$\therefore Initial\ velocity = 3 \times (0)^2 - 12 = -12m/s$$

Finding the acceleration equation:

$$a = \frac{dv}{dt} = 6t$$

Now find initial acceleration by letting $t = 0$,

$$\therefore Initial\ velocity = 6 \times 0 = 0\ m/s^2$$

Note: Always remember to put units in your answers

b)

$$\underline{Step\ 1: Equate\ our\ velocity - time\ equation\ to\ 0}$$

From a), $v = 3t^2 - 12$

$$\therefore 3t^2 - 12 = 0$$

$$\underline{Step\ 2: Solve\ for\ t}$$

$$3t^2 = 12$$
$$t^2 = 4$$
$$t = 2\ seconds\ (t > 0)$$

__**Motion Exercises**__

1. A cheetah runs in a straight line such that its position x m from the origin O on the line at time t s $(t \geq 0)$ is given by $x = t^3 - 12t + 15$. Find:

a) The cheetah's initial position, velocity and acceleration
b) The time when its velocity is zero, along with its position and acceleration at that time

2. A particle is moving in a straight line xm from the origin at time t, where:

$$x = 2t^3 - 20t^2 + 24t$$

a) Find the velocity and acceleration at any time t
b) Find the initial velocity and acceleration
c) When is the velocity zero?
d) At what time is the acceleration zero?
e) Find the interval of time for which the velocity is negative

3. The position s m at time t s of a particle moving in a straight line is given by:

$$s = 2t^3 - 9t^2 + 12t + 6$$

Find:

a) When its acceleration is zero and its velocity then

b) When its velocity is zero and its acceleration then

4. The distance d m at time t s ($t \geq 0$) of a particle moving in a straight line is given by $d = t^2 - 5t + 6$

a) What is the value of t when velocity is zero?
b) What is the acceleration when velocity is zero?
c) What is the distance travelled in the first 4 seconds?
d) At what time is the velocity 11m/s?

Motion Exercise Answers

1.

a) $x = t^3 - 12t + 15$

When $t = 0$:

$$x = 0^3 - 12 \times 0 + 15$$
$$= 15m$$

Therefore, initial position is $15m$

$$v = \frac{dx}{dt} = 3t^2 - 12$$

When $t = 0$:

$$v = 3 \times 0^2 - 12$$
$$= -12m/s$$

Therefore, initial velocity is $-12m/s$

$$a = \frac{dv}{dt} = 6t$$

When $t = 0$:

$$a = 0m/s^2$$

Therefore, initial acceleration is $0m/s^2$

b) The velocity – time equation is $v = 3t^2 - 12$

Letting $v = 0$:

$$0 = 3t^2 - 12$$
$$3t^2 = 12$$
$$t^2 = 4$$

$$\therefore t = 2\ ONLY\ (\because t > 0)$$

When $t = 2$:

$$x = t^3 - 12t + 15$$

$$= 2^3 - 12 \times 2 + 15$$
$$= -1m$$

$$\begin{aligned} a &= 6t \\ &= 6 \times 2 \\ &= 12m/s^2 \end{aligned}$$

2.

a) $x = 2t^3 - 20t^2 + 24t$

$$\begin{aligned} v = \frac{dx}{dt} &= 3 \times 2t^2 - 20 \times 2t + 24 \\ &= 6t^2 - 40t + 24 \end{aligned}$$

$$\begin{aligned} a = \frac{dv}{dt} &= 6 \times 2t - 40 \\ &= 12t - 40 \end{aligned}$$

b) When $t = 0$:

$$\begin{aligned} v &= 6 \times 0^2 - 40 \times 0 + 24 \\ &= 24m/s \end{aligned}$$

$$\begin{aligned} a &= 12 \times 0 - 40 \\ &= -40m/s^2 \end{aligned}$$

c) $v = 6t^2 - 40t + 24$

Let $v = 0$:

$$0 = 6t^2 - 40t + 24$$
$$3t^2 - 20t + 12 = 0$$
$$(3t - 2)(t - 6) = 0$$

$$\therefore t = \frac{2}{3} \text{ or } 6 \text{ seconds}$$

d) $a = 12t - 40$

When $a = 0$

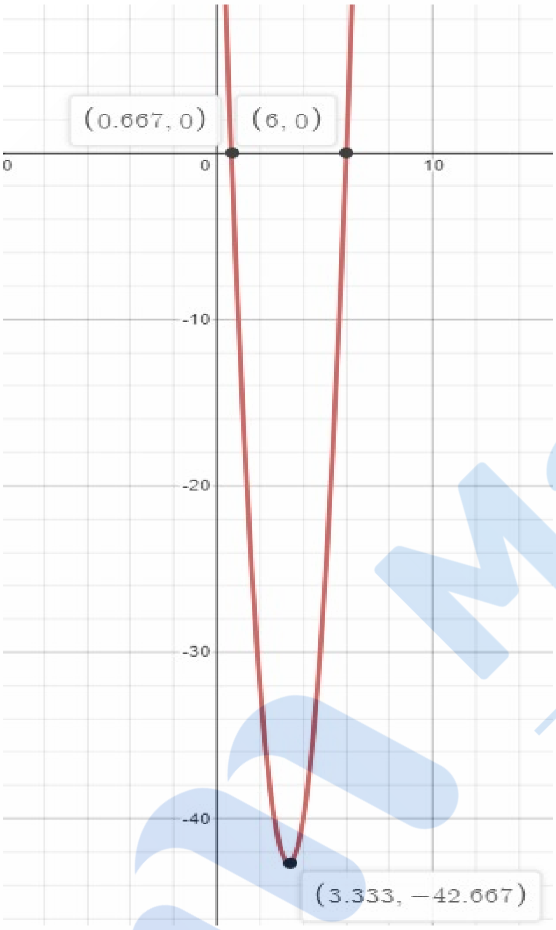
$$0 = 12t - 40$$
$$12t = 40$$
$$t = \frac{40}{12}$$

$$\therefore t = \frac{10}{3} \text{seconds}$$

e) $v = 6t^2 - 40t + 24$

$$v = 6t^2 - 40t + 24 = 2(3t - 2)(t - 6)$$

Since it's a concave up parabola (shown in the sketch), the time interval for which velocity is negative is $\frac{2}{3} < v < 6$



3.

a) $s = 2t^3 - 9t^2 + 12t + 6$

$$v = \frac{ds}{dt} = 3 \times 2t^2 - 9 \times 2t + 12$$
$$= 6t^2 - 18t + 12$$

$$a = \frac{dv}{dt} = 6 \times 2t - 18$$
$$= 12t - 18$$

When $a = 0$:

$$0 = 12t - 18$$
$$12t = 18$$
$$\therefore t = \frac{18}{12} = \frac{3}{2}\ s$$

When $t = \frac{3}{2}$:

$$v = 6\left(\frac{3}{2}\right)^2 - 18\left(\frac{3}{2}\right) + 12$$
$$= 6 \times \frac{9}{4} - 18 \times \frac{3}{2} + 12$$
$$= \frac{54}{4} - \frac{54}{2} + 12$$
$$= -\frac{3}{2}\ m/s$$

b) When $v = 0$:

$$0 = 6t^2 - 18t + 12$$
$$t^2 - 3t + 2 = 0$$
$$(t-2)(t-1) = 0$$
$$\therefore t = 1\ or\ t = 2\ s$$

When $t = 1$:

$$a = 12(1) - 18$$
$$= -6\ m/s^2$$

When $t = 2$:

$$a = 12(2) - 18$$
$$= 6\ m/s^2$$

4.
a) $d = t^2 - 5t + 6$

$$v = d' = 2t - 5$$

When $v = 0$

$$0 = 2t - 5$$
$$2t = 5$$
$$\therefore t = \frac{5}{2}$$

b) $v = 2t - 5$

$$a = \frac{dv}{dt} = 2$$

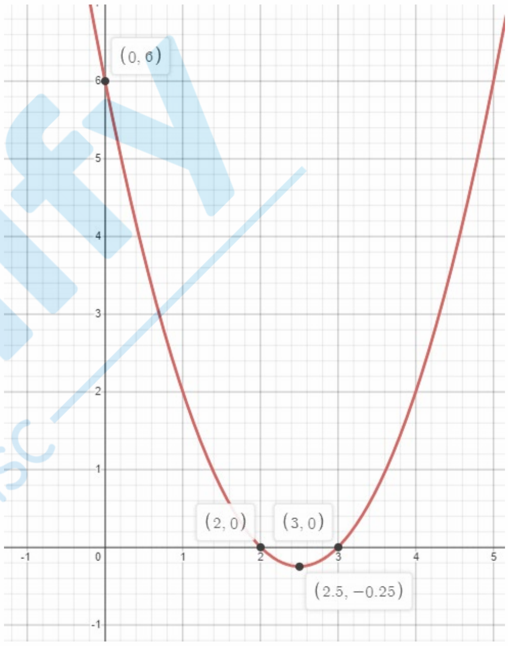
Therefore, acceleration is constantly $2\ m/s^2$

c) When $t = 4$:

$$
\begin{aligned}
d &= 4^2 - 5(4) + 6 \\
&= 16 - 20 + 6 \\
&= 2
\end{aligned}
$$

However, note that the question is asking for distance travelled in 4 seconds, NOT the displacement after 4 seconds. Hence, drawing a graph first would be recommended:

The vertex is an important point, and occurs when $t = \frac{5}{2}\ s$, and $d = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 6 = -\frac{1}{4}$. Moreover, when $t = 0$, $d = 6m$, meaning that the particle starts 6m from the origin.



Hence:

Between $t = 0$ and $t = 2.5s$, the particle travels $6 + 0.25 = 6.25m$

Between $t = 0$ and $t = 4s$, the particle travels $2 + 0.25 = 2.25m$

$$\therefore\ total\ distance = 6.25 + 2.25 = 8.5m$$

d) When $v = 11$:

$$
\begin{aligned}
11 &= 2t - 5 \\
2t &= 16
\end{aligned}
$$

$$\therefore t = 8\ s$$