Mathify HSC

DIFFERENTIATION

LIMITS (I)

Contents include:

- Limits
- Limit Properties

www.mathifyhsc.com.au

- What are limits?

A limit in mathematics tells us that our x value of a function $f(x)$ approaches some value a. This concept of a limit is very important in calculus, and the reason why we say 'x *approaches some number* a' in some cases rather than saying $x = a$ is because $x = a$ may not be defined for the function $f(x)$ sometimes.

> The notation used for the limit of $f(x)$ as x approaches a is $\lim_{x \to a} f(x)$, where '*lim*' is the abbreviation for limit.
>
> $x \to a$ means 'x approaches the value of a'

Example 1: For the function $f(x) = x + 3$, find $\lim_{x \to 2}(x + 3)$

The following table shows $f(x)$ for values of x in the neighbourhood of 2

x	1.95	1.99	1.995	$\to 2 \leftarrow$	2.005	2.01	2.05
$f(x)$	4.95	4.99	4.995	$\to 5 \leftarrow$	5.005	5.01	5.05

This table shows that as x approaches 2 from either above or below 2, our function $f(x)$ approaches 5. We can make $f(x)$ as close as possible to 5 by making our x as close as possible to 2. Hence, we may write:

$$\lim_{x \to 2}(x + 3) = 5$$

Thus, we notice that in this case, $\lim_{x \to 2}(x + 3) \to f(2) = 5$

Example 2: For the function $f(x) = \frac{x^2-9}{x-3}$, find $\lim_{x \to 3} \frac{x^2-9}{x-3}$.

If we follow the previous example's steps and immediately let $x = 3$ we get:

$$f(3) = \frac{3^2 - 9}{3 - 3} = \frac{0}{0}$$

Notice that this function is not defined since we can never divide a number by zero. Hence, *this method is invalid.*

The first step to solve these types of limit questions is to factorise our equation as much as possible BEFORE we apply our limit. Hence, we may write:

$$\begin{aligned} \frac{x^2 - 9}{x - 3} &= \frac{(x + 3)(x - 3)}{x - 3} \\ &= x + 3 \end{aligned}$$

Then we can apply the limit:

$$\therefore \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x + 3)$$

$$= 6$$

Thus, we notice that in this case, $\lim_{x \to 3} f(x) \neq f(3)$.

Limit Properties

The limit properties are required to be known, but their proofs will be omitted as it is not expected that students know them.

- **Property 1: Limit of a sum = the sum of the limits**

$$\lim_{x \to a}\big(f(x) + g(x)\big) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

Example 3: Evaluate $\lim_{x \to 2}(x^2 + 3x)$

Applying our property:

$$\begin{aligned} \lim_{x \to 2}(x^2 + 3x) &= \lim_{x \to 2}(x^2) + \lim_{x \to 2}(3x) \\ &= 4 + 6 \\ &= 10 \end{aligned}$$

- **Property 2: Limit of a difference = the difference of the limits**

$$\lim_{x \to a}\big(f(x) - g(x)\big) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

Example 4: Evaluate $\lim_{x \to 4}(x^2 - 5x)$

Applying our property:

$$\begin{aligned} \lim_{x \to 4}(x^2 - 5x) &= \lim_{x \to 4}(x^2) - \lim_{x \to 4}(5x) \\ &= 4^2 - 5(4) \\ &= 16 - 20 \\ &= -4 \end{aligned}$$

- **Property 3: Limit of a product = the product of the limits**

$$\lim_{x \to a}\big(f(x) \cdot g(x)\big) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Example 5: Evaluate $\lim_{x \to 1} 3x(x^2 - 2)$

Applying our properties:

$$\begin{aligned} \lim_{x \to 1} 3x(x^2 - 2) &= \lim_{x \to 1}(3x) \times \lim_{x \to 1}(x^2 - 2) \\ &= 3(1) \times (1^2 - 2) \\ &= 3 \times -1 \\ &= -3 \end{aligned}$$

- **Property 4: Limit of a quotient = the quotient of the limits**

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

Example 6: Evaluate $\lim\limits_{x\to 2}\frac{x^2+1}{x+3}$

Applying our properties:

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+1}{x+3}&=\frac{\lim\limits_{x\to 2}(x^2+1)}{\lim\limits_{x\to 2}(x+3)}\\&=\frac{2^2+1}{2+3}\\&=\frac{5}{5}\\&=1\end{aligned}$$

<u>Limit Exercises</u>

1. $\lim\limits_{x\to 4} 4x$
2. $\lim\limits_{x\to -3}(9-2x^2)$
3. $\lim\limits_{x\to 3}(x+3)(x+5)$
4. $\lim\limits_{x\to -4}\frac{(x+5)(x+4)}{(x+4)}$
5. $\lim\limits_{x\to 1}\frac{x-1}{x^2+x-2}$
6. $\lim\limits_{t\to 5}\frac{t-5}{2t^2-9t-5}$
7. $\lim\limits_{x\to -2}\frac{x+2}{x^2-4}$
8. $\lim\limits_{x\to 0}\frac{(1+x)^4-2}{1+x}$

9. $\lim\limits_{x\to 3} f(x)$ where $f(x)=\begin{cases}2x^2 \ \ when\ x\geq 3\\ -2x+4 \ \ when\ x<3\end{cases}$

10. Evaluate $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ where:

a) $f(x)=x^2-2$
b) $f(x)=x(x+3)$

Limit Exercise Answers

1. $\lim_{x \to 4} 4x = 4 \times 4 = 16$
2.
$$\begin{aligned}\lim_{x \to -3} (9 - 2x^2) &= \lim_{x \to -3} 9 - \lim_{x \to -3} 2x^2 \\ &= 9 - 2(-3)^2 \\ &= 9 - 2 \times 9 \\ &= -9\end{aligned}$$
3.
$$\begin{aligned}\lim_{x \to 3} (x+3)(x+5) &= \lim_{x \to 3} (x+3) \cdot \lim_{x \to 3} (x+5) \\ &= 6 \times 8 \\ &= 48\end{aligned}$$
4. Simplify the fraction first before applying the limit
$$\begin{aligned}\lim_{x \to -4} \frac{(x+5)(x+4)}{(x+4)} &= \lim_{x \to -4} (x+5) \\ &= -4 + 5 \\ &= 1\end{aligned}$$
5. Factorise and simplify the fraction first before applying the limit
$$\begin{aligned}\lim_{x \to 1} \frac{x-1}{x^2+x-2} &= \lim_{x \to 1} \frac{x-1}{(x+2)(x-1)} \\ &= \lim_{x \to 1} \frac{1}{x+2} \\ &= \frac{1}{3}\end{aligned}$$
6. Factorise and simplify the fraction first before applying the limit
$$\begin{aligned}\lim_{t \to 5} \frac{t-5}{2t^2-9t-5} &= \lim_{t \to 5} \frac{t-5}{(2t+1)(t-5)} \\ &= \lim_{t \to 5} \frac{1}{2t+1} \\ &= \frac{1}{11}\end{aligned}$$
7. Factorise and simplify the fraction first before applying the limit
$$\begin{aligned}\lim_{x \to -2} \frac{x+2}{x^2-4} &= \lim_{x \to -2} \frac{x+2}{(x+2)(x-2)} \\ &= \lim_{x \to -2} \frac{1}{x-2} \\ &= -\frac{1}{4}\end{aligned}$$
8.
$$\begin{aligned}\lim_{x \to 0} \frac{(1+x)^4 - 2}{1+x} &= \frac{(1+0)^4 - 2}{1+0} \\ &= \frac{1-2}{1} \\ &= -1\end{aligned}$$
9. When approaching from the left side of the piecewise function:

$$\lim_{x \to 3} -2x + 4 = -2(3) + 4 = -2$$

When approaching from the right side of the piecewise function:

$$\lim_{x \to 3} 2x^2 = 2 \times 3^2 = 18$$

As can be seen, different result are obtained when the limit is approached from opposite sides. Therefore, in this case we say that this limit cannot be found.

10.

a)

$$\begin{aligned} f(x) &= x^2 - 2 \\ \therefore f(x+h) &= (x+h)^2 - 2 \\ &= x^2 + 2xh + h^2 - 2 \\ \therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2 - (x^2 - 2)}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} 2x + h \\ &= 2x \end{aligned}$$

b)

$$\begin{aligned} f(x) &= x(x+3) \\ &= x^2 + 3x \\ \therefore f(x+h) &= (x+h)(x+h+3) \\ &= x^2 + xh + 3x + hx + h^2 + 3h \\ &= x^2 + 2xh + h^2 + 3x + 3h \end{aligned}$$

$$\begin{aligned} \therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - (x^2 + 3x)}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} \\ &= \lim_{h \to 0} 2x + h + 3 \\ &= 2x + 3 \end{aligned}$$