DIFFERENTIATIAL CALCULUS

GEOMETRICAL APPLICATIONS: SECOND DERIVATIVE (VI)

Contents include:

- Second Derivative
- Concavity of Curves
- Points of Inflections
- Horizontal Points of Inflections

- What is the Second Derivative?

The differentiation of the first derivative $f'(x)$ produces the second derivative, which can be expressed as:

$$\boxed{\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=f''(x)=y''}$$

Example 1: Find the second derivative of $y=x^3-x^2-x+1$

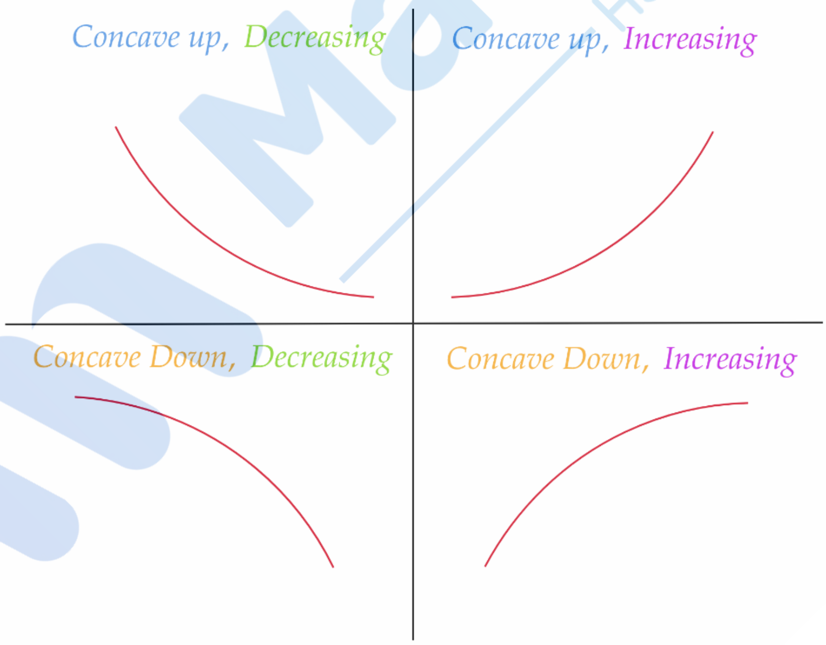
Solution:

$$y'=3x^2-2x-1$$
$$y''=6x-2$$

$$\therefore \textit{The second derivative } y''=6x-2$$

- Concavity

The second derivative is used to help determine the concavity of a shape, which may look like:



When $f''(x)<0$:
This means that the graph is concave down.
When $f''(x)>0$:
This means that the graph is concave up

Also, since a maximum turning point is concave down and a minimum turning point is concave up, we can also say that:

- A maximum turning point occurs at $x = a$ when $f'(a) = 0$ **and** $f''(a) < 0$

This is because a maximum turning point (sad face) is concave down

- A minimum turning point occurs at $x = a$ when $f'(a) = 0$ **and** $f''(a) > 0$

This is because a minimum turning point (smiley face) is concave up

This information is helpful for us, because instead of having to draw a sign table, we can now utilise the second derivative to determine the nature of stationary points

Example 2: Determine the nature of the stationary points for $f(x) = x^3 - 3x + 6$

Solution:

First differentiating $f(x)$:

$$f'(x) = 3x^2 - 3$$

Letting $f'(x) = 0$ and solving for x to find the stationary points:

$$\begin{aligned} 0 &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x + 1)(x - 1) \end{aligned}$$

$$\therefore x = 1 \ and \ x = -1 \ are \ stationary \ points$$

Now finding $f''(x)$:

$$f''(x) = 6x$$

Substituting $x = 1$ into $f''(x)$:

$$f''(1) = 6$$

$$\therefore Since \ f''(1) > 0, x = 1 \ is \ a \ minimum \ turning \ point$$

Substituting $x = -1$ into $f''(x)$:

$$f''(-1) = -6$$

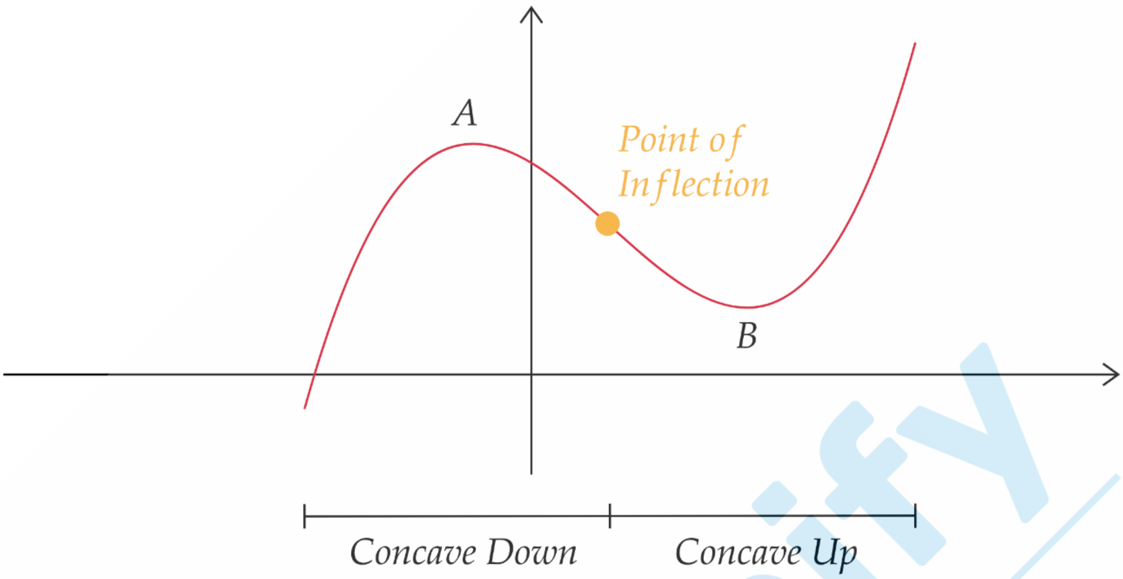
$$\therefore Since \ f''(-1) < 0, x = -1 \ is \ a \ maximum \ turning \ point$$

- Point of Inflection

A point of inflection (POI) is defined as:

The point where the concavity of a function $f(x)$ changes

The image below is a visual representation of a point of inflection:



- The point of inflection (POI) is also the point at which the **most negative or largest positive gradient occurs** between two stationary points (points A and B in the above diagram)

To find a point of inflection:

$$\textbf{\textit{Step 1: Let }} f''(x) = 0 \textbf{\textit{ and solve for }} x$$

This is achieved through differentiating twice to get $f''(x)$

$$\textbf{\textit{Step 2: Draw a sign table for }} f''(x)$$

Although rare, there are some rare cases where a point has $f''(x) = 0$ but is not a point of inflection. Hence, we always must draw a sign table to make sure this is not the case. The following is an example of what this might look like (it's basically the same as drawing our previous sign tables, just that we have $f''(x)$ instead of $f'(x)$)!

x	0	1	2
$f''(x)$	-4	0	3

Concave down since $f''(x) < 0$

Concave up since $f''(x) > 0$

Hence, $x = 1$ is a point of inflection since there **is a change in concavity**

Example 3: Find the point of inflection of $y = -x^3 + 3x^2 - 3x$

Solution:

Finding the second derivative:

$$y' = -3x^2 + 6x - 3$$

$$y'' = -6x + 6$$

Then letting $y'' = 0$ and solving for x:

$$0 = -6x + 6$$
$$6x = 6$$
$$\therefore x = 1$$

Then drawing a sign table to confirm it is a POI:

x	0	1	2
$f''(x)$	6	0	-6

Concave up　　　　　　　　　　Concave down

When $x = 1$:

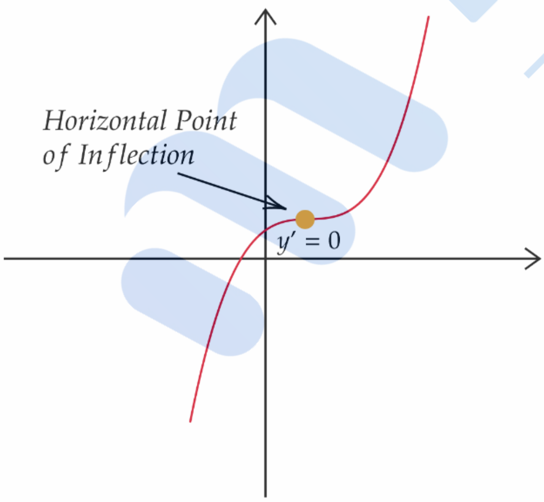
$$y = -1 + 3 - 3$$
$$= -1$$

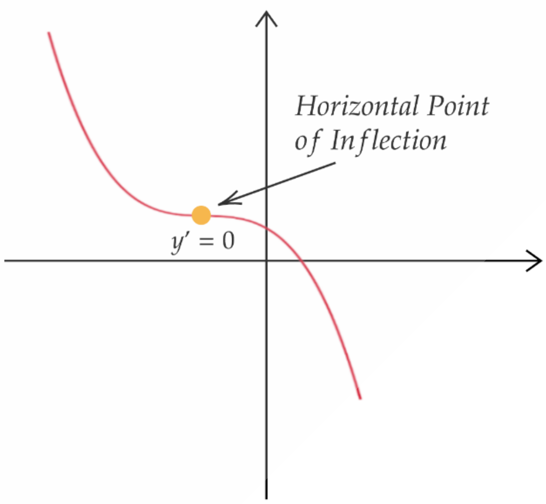
Hence, $(1, -1)$ is a point of inflection

- Horizontal Points of Inflection

A horizontal point of inflection is a special case where the point is both a stationary point and also a point of inflection. Hence:

$$f'(x) = 0 \; and \; f''(x) = 0$$





<u>Second Derivative Exercises</u>

1. For the function $f(x) = x^3 - 6x^2 + 2$, find:

i) The maxima and minima points using the first and second derivative.

ii) The point of inflection.

2. Find the greatest and least values of the function given by $f(x) = x^2 + 5x + 4$ in the domain $-3 \leq x \leq 1$

Second Derivative Exercise Answers

1. $f(x) = x^3 - 6x^2 + 2$

i) $f'(x) = 3x^2 - 12x$

$f'' = 6x - 12$

$$Step\ 1: Find\ the\ stationary\ points$$

$f'(x) = 3x^2 - 12x = 3x(x-4) = 0$

$$x = 0\ and\ x = 4\ are\ the\ x - coordinates\ of\ the\ stationary\ points$$

$When\ x = 0, y = f(0) = 2$
$When\ x = 4, y = f(4) = -30$

$$\therefore Stationary\ points\ at\ (0,2) and\ (4,-30)$$

$$Step\ 2: Use\ the\ second\ derivative\ to\ determine\ \min\ or\ \max$$

$Substitute\ x = 0\ into\ f''(x)$
$f''(0) = -12, which\ is < 0$

$$\therefore Since\ f''(x) < 0, it\ is\ concave\ down\ and\ is\ thus\ a\ \max point$$

$Substitute\ x = 4\ into\ f''(x)$
$f''(4) = 12, which\ is > 0$

$$\therefore Since\ f''(x) > 0, it\ is\ concave\ up\ and\ is\ thus\ a\ \min point$$

ii) To find the point of inflection, equate $f''(x)$ to 0

$$f''(x) = 6x - 12 = 0$$
$$x = 2\ is\ the\ x - coordinate\ of\ my\ POI$$

Now sketching the sign table:

x	1	2	3
$f''(x)$	-6	0	6

Hence, a change in concavity has occured

$When\ x = 2, y = f(2) = -14$

$$\therefore Point\ of\ inflection\ occurs\ at\ (2,-14)$$

2. $f(x) = x^2 + 5x + 4$

$$f'(x) = 2x + 5$$

$Finding\ the\ stationary\ point\ by\ making\ f'(x) = 0$:

$$f'(x) = 2x + 5 = 0$$

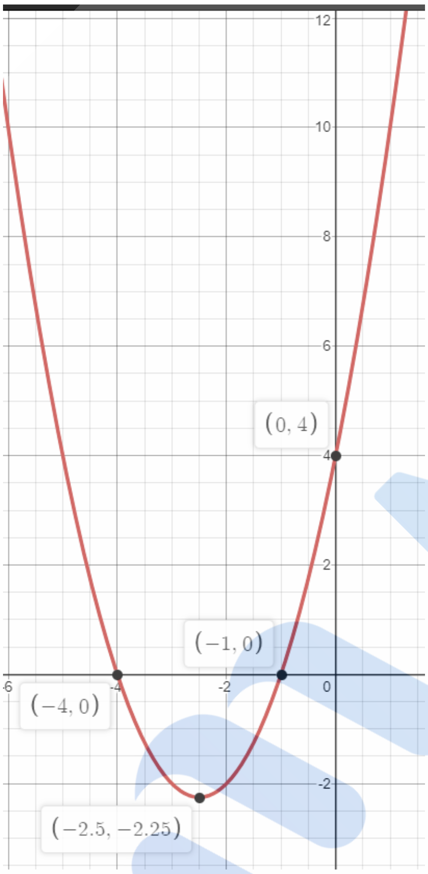
$$x = -\frac{5}{2} \textit{ is the } x - \textit{coordinate of the stationary point}$$

$$When\ x = -\frac{5}{2}, y = f\left(-\frac{5}{2}\right) = -\frac{9}{4}$$

Since the coefficient of x^2 in our quadratic equation is > 0, we also know that the parabola is concave up. This therefore means that:

$$\left(-\frac{5}{2}, -\frac{9}{4}\right) \textit{ is the minimum turning point}$$

The maximum value in this domain can be found by drawing the graph of $f(x)$



$$\therefore \textit{Maximum value occurs when } x = 1, \quad f(1) = 10$$

$$\therefore \textit{In the domain of } -3 \le x \le 1:$$

$$\textit{Maximum value} = 10$$

$$\textit{Minimum value} = -\frac{9}{4}$$

Note: a quicker alternative method to determining the 'vertex' or turning point of a parabola with equation $ax^2 + bx + c$ is by using the formula:

$$x = -\frac{b}{2a}$$

$$\textit{Applied to this question: } x = -\frac{5}{2(1)} = -\frac{5}{2}$$