Mathify HSC

INTEGRATION

ESTIMATING AREA WITH TRAPEZOIDAL RULE (IX)

Contents include:

- The Trapezoidal Rule
- Determining Overestimation or Underestimation of Area

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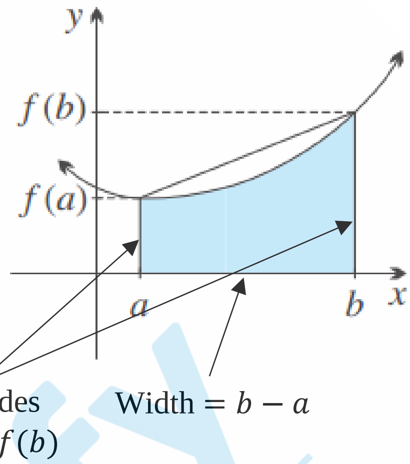
- The Trapezoidal Rule

When estimating the area underneath the curve, an effective way of doing so is connecting an interval from one end of the curve to the other (as shown to the right). This thus forms a trapezium, where its area is given as:



$$Trapezium\ area = \frac{sum\ of\ the\ two\ parallel\ sides}{2} \times width$$

$$= \frac{f(a)+f(b)}{2} \times (b-a)$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right)$$

Notice here how the area under the curve (shaded in blue) is approximately equal to the area of the trapezium, hence:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left(f(a)+f(b)\right)$$

- Subdividing the Interval and Improving Accuracy

In the above example, we estimated our area using one trapezium. Using the trapezoidal rule however, we can **use more than one trapezium** to estimate the area.

Although this is more of a hassle, the more subdivisions/trapeziums we use to estimate the area, the more accurate the estimation is.

The formula to follow if we want n – number of trapeziums or subdivisions to estimate an area is:

Let $f(x)$ be a continuous function in the domain $[a, b]$, therefore:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(f(a)+f(b)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})\right)$$

Where $x_r = a + r \times \frac{b-a}{n}$, for $r = 1, 2, 3, \ldots, n-1$

It is recommended when using this formula to **first construct a table of values** for our function values $a, x_1, x_2, \ldots, x_{n-1}, b$ and then $f(a), f(x_1), f(x_2), \ldots, f(x_{n-1}), f(b)$ so that we can cleanly plug these values into our formula afterwards. Our function values must be evenly distributed! (take a look at the examples below)

Note: In questions, '***function values***' **and** '***subdivisions***' **do not** mean the same thing. If there are n number of subdivisions, it means that there are $n+1$ function values as part of our formula

An easier way to remember the textbook trapezoidal rule formula above is:

$$\int_a^b f(x)\,dx \approx \left(\frac{width\ of\ each\ subinterval}{2}\right) \times \left(1st + last + 2(values\ in\ between)\right)$$

These refer to $f(x)$ values!

Example 1: Use the trapezoidal rule formula with the five given function values to approximate $\int_{-6}^{6} f(x)\,dx$

$\boldsymbol{x}$	-6	-3	0	3	6
$\boldsymbol{f(x)}$	-11	-5	3	7	14

Solution:

Notice here how we have 5 function values, and thus only 4 subdivisions

$$\therefore n = 4$$

Using the trapezoidal rule formula outlined above:

$$\begin{aligned}\int_{-6}^{6} f(x)\,dx &\approx \frac{6 - -6}{2(4)}\Big(f(-6) + f(6) + 2\big(f(-3) + f(0) + f(3)\big)\Big)\\ &= \frac{12}{8}\big(-11 + 14 + 2(-5 + 3 + 7)\big)\\ &= \frac{3}{2}\big(3 + 2(5)\big)\\ &= \frac{3}{2} \times 13\\ &= \frac{39}{2}\end{aligned}$$

Example 2: Approximate $\int_1^5 x^2\,dx$ using:

a) 1 subdivision
b) 2 subdivisions
c) 4 subdivisions

Solutions:

a) If there is only 1 subdivision, $n = 1$ and there are 2 function values. Hence, drawing our table:

x	1	5
$f(x)$	1	25

Using the trapezoidal rule formula:

$$\therefore \int_1^5 x^2\,dx \approx \frac{5-1}{2(1)}\big(f(1)+f(5)\big)$$
$$= \frac{4}{2}(1+25)$$
$$= 52$$

b) If there are 2 subdivision, $n = 2$ and there are 3 function values. Hence, drawing our table:

x	1	3	5
$f(x)$	1	9	25

Using the trapezoidal rule formula:

$$\therefore \int_1^5 x^2\,dx \approx \frac{5-1}{2(2)}\big(f(1)+f(5)+2f(3)\big)$$
$$= \frac{4}{4}(1+25+2\times 9)$$
$$= 1(26+18)$$
$$= 44$$

c) If there are 4 subdivisions, $n = 4$ and there are 5 function values. Hence, drawing our table:

x	1	2	3	4	5
$f(x)$	1	4	9	16	25

Using the trapezoidal rule formula:

$$\therefore \int_1^5 x^2\,dx \approx \frac{5-1}{2(4)}\Big(f(1)+f(5)+2\big(f(2)+f(3)+f(4)\big)\Big)$$
$$= \frac{4}{8}\big(1+25+2(4+9+16)\big)$$
$$= \frac{1}{2}\big(26+2(29)\big)$$
$$= \frac{1}{2}(26+58)$$
$$= \frac{1}{2}\times 84$$
$$= 42$$

Example 3: Approximate $\int_2^7 x^2 - 2x + 1\,dx$ using:

a) 2 function values

b) 6 function values

Solutions:

a) If there are 2 function values, then there is only 1 subinterval meaning that $n = 1$. Hence, drawing our table of values:

x	2	7
$f(x)$	1	36

Using the trapezoidal rule formula:

$$\therefore \int_2^7 x^2 - 2x + 1\, dx \approx \frac{7-2}{2(1)}(1+36)$$
$$= \frac{5}{2} \times 37$$
$$= \frac{185}{2}$$

b) If there are 6 function values, then there are 5 subintervals meaning that $n = 5$. Hence, drawing our table of values:

x	2	3	4	5	6	7
$f(x)$	1	4	9	16	25	36

Using the trapezoidal rule formula:

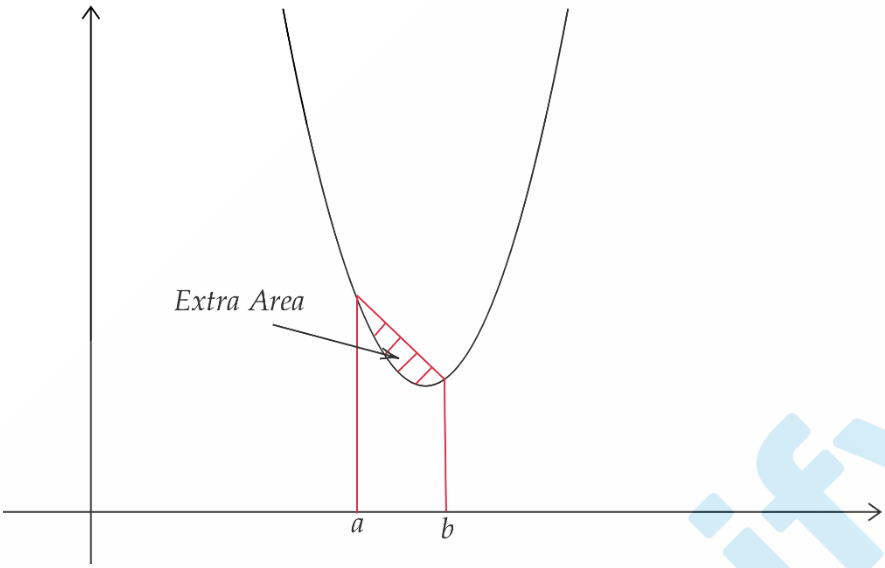
$$\therefore \int_2^7 x^2 - 2x + 1\, dx \approx \frac{7-2}{2(5)}\big(1 + 36 + 2(4 + 9 + 16 + 25)\big)$$
$$= \frac{5}{10}\big(1 + 36 + 2(54)\big)$$
$$= \frac{1}{2}(37 + 108)$$
$$= \frac{1}{2} \times 145$$
$$= \frac{145}{2}$$

- Concavity of Curves

Depending on the concavity of the curve, the trapezoidal rule either **overestimates or underestimates** the area of the curve:

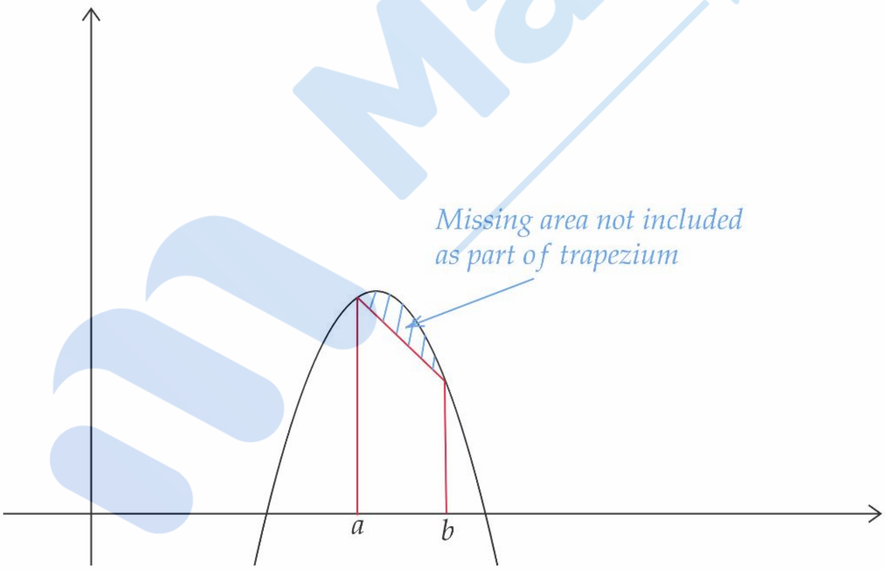
- If the curve is concave up, i.e., $\frac{d^2y}{dx^2} > 0$, the trapezoidal rule overestimates the area

We can visualise this through the following diagram:



- If the curve is concave down, i.e., $\frac{d^2y}{dx^2} < 0$, the trapezoidal rule underestimates the area

We can also visualise this through the following diagram:



Example 4:

a) Use the trapezoidal rule with 3 subintervals, along with appropriate log laws, to approximate $\int_2^8 \ln x \, dx$

b) Determine if your answer in part a) is greater than or less than $\int_2^8 \ln x \, dx$. Provide reasoning for your answer

Solutions:

a) If there are 3 subintervals, $n = 3$ and there are 4 function values. Hence, drawing our table of values:

x	2	4	6	8
$f(x)$	$\ln 2$	$\ln 4$	$\ln 6$	$\ln 8$

Using the trapezoidal rule formula:

$$\therefore \int_2^8 \ln x \, dx \approx \frac{8-2}{2(3)}\left(\ln 2 + \ln 8 + 2(\ln 4 + \ln 6)\right)$$
$$= \frac{6}{6}(\ln 2 + \ln 8 + 2\ln 4 + 2\ln 6)$$
$$= \ln 2 + \ln 8 + \ln 16 + \ln 36 \ [logarithm\ laws]$$
$$= \ln(2 \times 8 \times 16 \times 36)$$
$$= \ln 9216$$

b) First determining the concavity of $f(x) = \ln x$:

$$f'(x) = \frac{1}{x}$$
$$= x^{-1}$$
$$\therefore f''(x) = -x^{-2}$$
$$= -\frac{1}{x^2}$$

Hence, since $x^2 > 0$, $f''(x) < 0$ for all x

This therefore means that the curve $f(x) = \ln x$ is always concave down since $f''(x) < 0$

When the curve is concave down, the trapezoidal rule underestimates area

$$\therefore answer\ in\ part\ a < \int_2^8 \ln x \, dx$$

Example 5:

a) Evaluate the definite integral $\int_{-4}^{11} \sqrt{x+5}\, dx$

b) Approximate $\int_{-4}^{11} \sqrt{x+5}\, dx$ using 4 function values

c) Hence, prove the inequality $5 + 2\sqrt{6} + 2\sqrt{11} < \frac{84}{5}$

Solutions:

a) First converting to indices form:

$$\int_{-4}^{11} \sqrt{x+5}\, dx = \int_{-4}^{11} (x+5)^{\frac{1}{2}}\, dx$$

Then integrating:

$$\therefore \int_{-4}^{11} (x+5)^{\frac{1}{2}}\,dx = \left[\frac{2}{3}(x+5)^{\frac{3}{2}}\right]_{-4}^{11}$$
$$= \frac{2}{3}(11+5)^{\frac{3}{2}} - \frac{2}{3}(-4+5)^{\frac{3}{2}}$$
$$= \frac{2}{3}(16)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}$$
$$= \frac{2}{3} \times 64 - \frac{2}{3}$$
$$= 42$$

b) With 4 function values, this means that there are 3 subintervals so $n = 3$. Hence, drawing our table of values with $f(x) = \sqrt{x+5}$

x	-4	1	6	11
$f(x)$	1	$\sqrt{6}$	$\sqrt{11}$	4

Using the trapezoidal rule formula:

$$\therefore \int_{-4}^{11} \sqrt{x+5}\,dx \approx \frac{11 - -4}{2(3)}\left(1 + 4 + 2\left(\sqrt{6} + \sqrt{11}\right)\right)$$
$$= \frac{15}{6}\left(5 + 2\sqrt{6} + 2\sqrt{11}\right)$$
$$= \frac{5}{2}\left(5 + 2\sqrt{6} + 2\sqrt{11}\right)$$

c) When we see an inequality question with trapezoidal rule involved, immediately think of the concavity!

In this scenario, considering the concavity of $f(x) = \sqrt{x+5}$:

$$f'(x) = \frac{1}{2}(x+5)^{-\frac{1}{2}}$$

$$f''(x) = \frac{1}{2} \times -\frac{1}{2}(x+5)^{-\frac{3}{2}}$$
$$= -\frac{1}{4}(x+5)^{-\frac{3}{2}}$$
$$= -\frac{1}{4(x+5)^{\frac{3}{2}}}$$

$f(x)$ is defined within the domain $x \geq -5$, and in this domain, since $(x+5)^{\frac{3}{2}} > 0$, then:

$$-\frac{1}{4(x+5)^{\frac{3}{2}}} = f''(x) < 0$$

Hence, $f(x)$ is always concave down

Thus, since $f(x)$ is concave down, the trapezoidal rule understimates the area

$$\therefore \textit{Trapezoidal area} < \textit{definite integral}$$

$$\frac{5}{2}\left(5 + 2\sqrt{6} + 2\sqrt{11}\right) < 42$$

$$5 + 2\sqrt{6} + 2\sqrt{11} < 42 \times \frac{2}{5}$$

$$\therefore 5 + 2\sqrt{6} + 2\sqrt{11} < \frac{84}{5}$$