INTEGRATION

INTEGRATING TO LOGARITHMS (VI)

Contents include:

- Integrating to Logarithms

- Integration of $\int \frac{1}{x} dx$

When asked to find the indefinite integral $\int \frac{1}{x} dx$, if we integrate it using the standard formula $\int x^n dx = \frac{x^{n+1}}{n+1}$ then:

$$\int \frac{1}{x} dx = \int x^{-1} dx$$
$$= \frac{x^{-1+1}}{-1+1}$$
$$= \frac{x^0}{0}$$

This is undefined since we cannot divide by 0, so **we cannot do it this way!**

Instead, we do complete the integral a different way and it is important to remember that:

$$\int \frac{1}{x} dx = \ln|x| + C$$

Note: Remember to put those absolute values since it is forgotten very often!

Moreover, more standard forms that should be remembered include:

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + C$$
$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

We can use the substitution method to prove these formulas, and just like for the previous booklet, it is recommended that we use substitution when practicing so that we don't have to rely on memorisation/the formula sheet too much!

Example 4: Find the integral $\int \frac{5}{x} dx$

$$\int \frac{5}{x} dx = 5 \ln|x| + C$$

Example 5: Find the integral $\int \frac{7}{4x+3} dx$

$$Let\ u = 4x + 3$$

$$\therefore \frac{du}{dx} = 4$$
$$dx = \frac{du}{4}$$
$$\therefore \int \frac{7}{4x+3}\,dx = \int \frac{7}{u} \times \frac{du}{4}$$
$$= \frac{7}{4}\ln|u| + C$$
$$= \frac{7}{4}\ln|4x+3| + C$$

Integrating to Logarithm Exercises

1. Find each indefinite integral:

a) $\int \frac{4}{5x-1}\,dx$

b) $\int \frac{1}{7-2x}\,dx$

c) $\int \frac{2x}{x^2+2}\,dx$

d) $\int \frac{x}{9-x^2}\,dx$

2. Evaluate the following definite integrals:

a) $\int_0^{11} \frac{5}{2x-11}\,dx$

b) $\int_{-1}^{1} \frac{3}{7-3x}\,dx$

c) $\int_1^2 \frac{2x^2+x-4}{x}\,dx$

d) $\int_2^4 \frac{3x^2-2x}{x^2}\,dx$

3. Find the primitives of the following functions:

a) $e^x(e^x+1)$

b) $(e^x-1)^2$

c) $\frac{x^4-x+2}{x^2}$

d) $\frac{e^{2x}+1}{e^x}$

e) $\frac{1-8x}{9x}$

f) $\frac{2e^x-e^{2x}}{e^{3x}}$

g) $\frac{1}{\sqrt[3]{e^x}}$

4. Find $f(x)$ and then find $f(2)$ given that:

a) $f'(x) = 1 + \frac{2}{x}$ and $f(1) = 1$

b) $f'(x) = 2x + \frac{1}{3x}$ and $f(1) = 2$

c) $f'(x) = 3e^x + \frac{5}{2x-1}$ and $f(0) = 0$

d) $f'(x) = e^{2x} + 2x + \frac{1}{3x+1}$ and $f(0) = 2$

5.

a) Differentiate $y = e^{3x^2+4x+1}$

b) Hence evaluate $\int_{-1}^{0} (3x+2)\, e^{3x^2+4x+1} dx$

6.

a) Differentiate $y = xe^x$

b) Hence evaluate $\int_0^2 xe^x$

Integration of Natural Numbers and Logs Exercise Answers

1.

a) $\int \frac{4}{5x-1} dx$

Let $u = 5x - 1$

$$\frac{du}{dx} = 5, \qquad dx = \frac{du}{5}$$

$$\int \frac{4}{5x-1} dx = \int \frac{4}{u} \times \frac{du}{5}$$
$$= \frac{4}{5}\ln|u| + C$$
$$= \frac{4}{5}\ln|5x-1| + C$$

b) $\int \frac{1}{7-2x} dx$

Let $u = 7 - 2x$

$$\frac{du}{dx} = -2, \qquad dx = -\frac{du}{2}$$

$$\int \frac{1}{7-2x} dx = \int \frac{1}{u} \times -\frac{du}{2}$$
$$= -\frac{1}{2}\ln|u| + C$$
$$= -\frac{1}{2}\ln|7-2x| + C$$

c) $\int \frac{2x}{x^2+2} dx$

Let $u = x^2 + 2$

$$\frac{du}{dx} = 2x, \qquad dx = \frac{du}{2x}$$

$$\begin{aligned}\int \frac{2x}{x^2+2}\,dx &= \int \frac{2x}{u} \times \frac{du}{2x} \\ &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|x^2+2| + C\end{aligned}$$

d) $\int \frac{x}{9-x^2}\,dx$

Let $u = 9 - x^2$

$$\frac{du}{dx} = -2x, \qquad dx = -\frac{du}{2x}$$

$$\begin{aligned}\int \frac{x}{9-x^2}\,dx &= \int \frac{x}{u} \times -\frac{du}{2x} \\ &= \int -\frac{du}{2u} \\ &= -\frac{1}{2}\ln|u| + C \\ &= -\frac{1}{2}\ln|9-x^2| + C\end{aligned}$$

2.

a) $\int_0^{11} \frac{5}{2x-11}\,dx$

Recalling that $\int \frac{1}{f(x)}\,dx = \frac{1}{f'(x)}\ln|f(x)|$

$$\begin{aligned}\int_0^{11} \frac{5}{2x-11}\,dx &= \left[\frac{5}{2}\ln|2x-11|\right]_0^{11} \\ &= \frac{5}{2}\ln|22-11| - \frac{5}{2}\ln|-11| \\ &= \frac{5}{2}\ln 11 - \frac{5}{2}\ln 11 \\ &= 0\end{aligned}$$

b) $\int_{-1}^{1} \frac{3}{7-3x}\,dx$

Recalling that $\int \frac{1}{f(x)}\,dx = \frac{1}{f'(x)}\ln|f(x)|$

$$\int_{-1}^{1} \frac{3}{7-3x}\,dx = \left[\frac{3}{-3}\ln|7-3x|\right]_{-1}^{1}$$
$$= (-\ln 4) - (-\ln 10)$$
$$= \ln 10 - \ln 4$$
$$= \ln\frac{5}{2}$$

c) $\int_1^2 \frac{2x^2+x-4}{x}\,dx$

First splitting the numerator:

$$\int_1^2 \frac{2x^2+x-4}{x}\,dx = \int_1^2 \left(2x+1-\frac{4}{x}\right)dx$$
$$= [x^2 + x - 4\ln|x|]_1^2$$
$$= (4+2-4\ln 2) - (1+1-4\ln 1)$$
$$= 4 - 4\ln 2$$

d) $\int_2^4 \frac{3x^2-2x}{x^2}\,dx$

First splitting the numerator:

$$\int_2^4 \frac{3x^2-2x}{x^2}\,dx = \int_2^4 3 - \frac{2}{x}\,dx$$
$$= [3x - 2\ln|x|]_2^4$$
$$= (3\times 4 - 2\ln 4) - (3\times 2 - 2\ln 2)$$
$$= 6 + 2\ln 2 - 2\ln 4$$

Note that $\ln 4 = \ln 2^2 = 2\ln 2$

$$\therefore = 6 + 2\ln 2 - 2\times 2\ln 2$$
$$= 6 - 2\ln 2$$

3.

a) $\int e^x(e^x+1)\,dx$

$$Step\ 1: Expand\ brackets$$

$$\int e^x(e^x+1)\,dx = \int e^{2x} + e^x\,dx$$

$$Step\ 2: Integrate$$

$$\int e^{2x} + e^x\,dx = \frac{1}{2}e^{2x} + e^x + C$$

b) $\int (e^x-1)^2\,dx$

$$Step\ 1: Expand\ brackets$$

$$\int (e^x - 1)^2\, dx = \int (e^{2x} - 2e^x + 1)\, dx$$

$$Step\ 2: Integrate$$

$$\int (e^{2x} - 2e^x + 1)\, dx = \frac{1}{2}e^{2x} - 2e^x + x + C$$

c) $\int \frac{x^4 - x + 2}{x^2} dx$

First splitting the numerator and expressing it in indices form:

$$\begin{aligned}\int \frac{x^4 - x + 2}{x^2} dx &= \int x^2 - \frac{1}{x} + 2x^{-2}\, dx \\ &= \frac{x^3}{3} - \ln|x| + 2 \times \frac{x^{-1}}{-1} + C \\ &= \frac{x^3}{3} - \ln|x| - \frac{2}{x} + C\end{aligned}$$

d) $\int \frac{e^{2x} + 1}{e^x} dx$

First splitting the numerator and expressing it in indices form:

$$\begin{aligned}\int \frac{e^{2x} + 1}{e^x} dx &= \int e^x + e^{-x}\, dx \\ &= e^x - e^x + C\end{aligned}$$

e) $\int \frac{1 - 8x}{9x} dx$

First splitting the numerator:

$$\begin{aligned}\int \frac{1 - 8x}{9x} dx &= \int \frac{1}{9x} - \frac{8}{9} dx \\ &= \frac{1}{9}\ln|x| - \frac{8}{9}x + C\end{aligned}$$

f) $\int \frac{2e^x - e^{2x}}{e^{3x}} dx$

First splitting the numerator and expressing it in indices form:

$$\begin{aligned}\int \frac{2e^x - e^{2x}}{e^{3x}} dx &= \int 2e^{-2x} - e^{-x}\, dx \\ &= \frac{2}{-2}e^{-2x} - \frac{e^{-x}}{-1} + C \\ &= -e^{-2x} + e^{-x} + C\end{aligned}$$

g) $\int \frac{1}{\sqrt[3]{e^x}} dx$

First expressing the function in indices form:

$$\int \frac{1}{\sqrt[3]{e^x}} dx = \int e^{-\frac{1}{3}x} dx$$

$$= -3e^{-\frac{1}{3}x} + C$$

4.

a) $f'(x) = 1 + \frac{2}{x}$

$$f(x) = \int 1 + \frac{2}{x}\,dx$$
$$= x + 2\ln x + C$$

$$f(1) = 1 + 2\ln 1 + C = 1$$
$$C = 0$$

$$\therefore f(x) = x + 2\ln x$$

b) $f'(x) = 2x + \frac{1}{3x}$

$$f(x) = \int 2x + \frac{1}{3x}\,dx$$
$$= x^2 + \frac{1}{3}\ln|x| + C$$

$$f(1) = 1 + \frac{1}{3}\ln 1 + C = 2$$
$$C = 1$$

$$\therefore f(x) = x^2 + \frac{1}{3}\ln|x| + 1$$

c) $f'(x) = 3e^x + \frac{5}{2x-1}$

$$f(x) = \int 3e^x + \frac{5}{2x-1}\,dx$$
$$= \int 3e^x\,dx + \int \frac{5}{2x-1}\,dx$$

Remembering that $\int \frac{1}{f(x)}\,dx = \frac{1}{f'(x)}\ln|f(x)| + C$

Let $u = 2x - 1$

$$\frac{du}{dx} = 2, \qquad dx = \frac{du}{2}$$

$$\therefore\ f(x) = 3e^x + \frac{5}{2}\ln|2x-1| + C$$

$$f(0) = 3e^0 + \frac{5}{2}\ln|-1| + C = 0$$
$$3 + C = 0$$
$$C = -3$$

$$\therefore f(x) = 3e^x + \frac{5}{2}\ln|2x-1| - 3$$

d) $f'(x) = e^{2x} + 2x + \frac{1}{3x+1}$

$$f(x) = \int \left(e^{2x} + 2x + \frac{1}{3x+1}\right) dx$$
$$= \frac{1}{2}e^{2x} + x^2 + \frac{1}{3}\ln|3x+1| + C$$

$$f(0) = \frac{1}{2}e^0 + \frac{1}{3}\ln 1 + C = 2$$
$$\frac{1}{2} + C = 2$$
$$C = \frac{3}{2}$$

$$\therefore f(x) = \frac{1}{2}e^{2x} + x^2 + \frac{1}{3}\ln|3x+1| + \frac{3}{2}$$

5.

a) $y = e^{3x^2+4x+1}$

Let $u = 3x^2 + 4x + 1$

$$\frac{du}{dx} = 6x + 4$$
$$y = e^u$$
$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \ [chain\ rule]$$
$$\frac{dy}{dx} = e^u \times (6x+4)$$
$$\frac{dy}{dx} = (6x+4)e^{3x^2+4x+1}$$

b) From part a) we get that:

$$\frac{dy}{dx} = 2(3x+2)e^{3x^2+4x+1}$$

Therefore, looking at the anti-derivative, where $\int \frac{dy}{dx} dx = y + C$:

$$\int 2(3x+2)e^{3x^2+4x+1} dx = \int \frac{dy}{dx}\, dx$$
$$= y + C$$
$$= e^{3x^2+4x+1} + C$$

Dividing both sides by 2, we can therefore say that:

$$\therefore \int (3x+2)e^{3x^2+4x+1}dx = \frac{1}{2}e^{3x^2+4x+1} + C$$

Hence, now considering definite bounds:

$$\int_{-1}^{0} (3x+2)e^{3x^2+4x+1}\,dx = \frac{1}{2}\left[e^{3x^2+4x+1}\right]_{-1}^{0}$$
$$= \frac{1}{2}(\mathrm{e}^1) - \frac{1}{2}\mathrm{e}^{3(-1)^2+4(-1)+1}$$
$$= \frac{1}{2}\mathrm{e} - \frac{1}{2}\mathrm{e}^{3-4+1}$$
$$= \frac{1}{2}\mathrm{e} - \frac{1}{2}\mathrm{e}^{0}$$
$$= \frac{1}{2}\mathrm{e} - \frac{1}{2}$$

6.

a) $y = xe^x$

Using the product rule to differentiate:

$$(uv)' = u'v + v'u$$
$$\therefore \frac{dy}{dx} = e^x + xe^x$$

b) From a) we get that:

$$(xe^x)' = e^x + xe^x$$
$$\therefore xe^x = e^x - (xe^x)'$$

Therefore, integrating both sides:

$$\therefore \int_0^2 xe^x dx = \int_0^2 (e^x - (xe^x)')\,dx$$
$$= \int_0^2 e^x\,dx - \int_0^2 (xe^x)'dx$$

Note here that $\int f'(x)dx = f(x)$:

$$\therefore \int_0^2 e^x\,dx - \int_0^2 (xe^x)'dx = [e^x]_0^2 - [xe^x]_0^2$$
$$= \mathrm{e}^2 - \mathrm{e}^0 - (2\mathrm{e}^2 - 0)$$
$$= \mathrm{e}^2 - 1 - 2\mathrm{e}^2$$
$$= -\mathrm{e}^2 - 1$$