PROBABILITY

CONDITIONAL PROBABILITY (VII)

Contents include: Conditional probability

www.mathifyhsc.com.au

- Conditional Probability

Probably the most tested subtopic for Probability, so it is very important you understand this section!

Conditional probability deals with questions where either the **sample space is reduced**, or event space is reduced when a piece of information is given.

Take for example, a die being rolled:

The probability that the number will be a 2 is:

$$P(number\ is\ 2) = \frac{1}{6}$$

Now if we are given the information that the dice's number is even, what is the probability of the number being a 2?

With new information now, our sample space has been reduced from 6 to 3. Therefore:

$$P(number\ is\ 2\ given\ that\ is\ even) = \frac{1}{3}$$

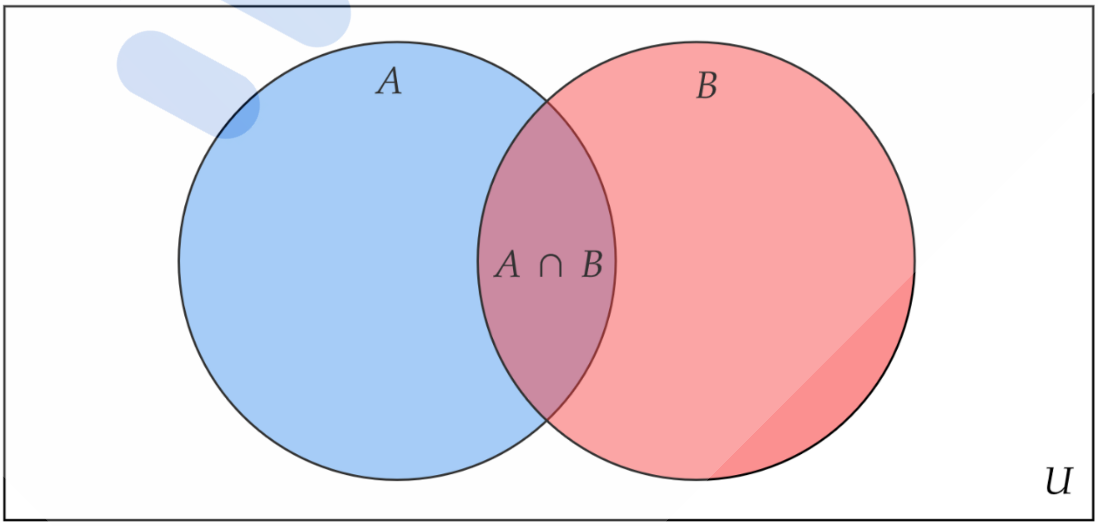
The **notation** that we use in maths for when we are **given** new information is:

$$Probability\ of\ A\ given\ B = P(A\ given\ B) = P(A|B)$$

The conditional probability formula is given as:

$$P(A|B) = \frac{|reduced\ event\ space|}{|reduce\ event\ space|}\ and\ P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The reasoning behind this formula can be best demonstrated through a Venn diagram:



If we first try to find $P(A)$, we can think of it as:

$$P(A) = \frac{event\ space}{sample\ space} = \frac{Blue\ circle}{U}$$

Hence, we can say that the **event space** is the blue circle and the **sample space** is U

Now if we are instead asked to find $P(A)$ **given that** it is a part of B, i.e., $P(A|B)$:

With the new information given that it is in B, two things now occur:

1. New sample space

The sample space changes from U to red circle B

2. New event space

Since we now know that the event space is in the red circle, the only part of A which is a part of the red circle is the part that overlaps, i.e. $A \cap B$

Hence, the new probability is:

$$\begin{aligned} P(A|B) &= \frac{new\ event\ space}{new\ sample\ space} \\ &= \frac{Purple\ Overlap}{Red\ circle} \\ &= \frac{P(A \cap B)}{P(B)} \end{aligned}$$

Example 1: Use the conditional probability formula to answer the following:

a) Find $P(A|B)$ if $P(A \cap B) = 0.1$ and $P(B) = 0.6$
b) Find $P(X|Y)$ if $P(X \cap Y) = 0.25$ and $P(Y) = 0.75$
c) Find $P(A \cap B)$ if $P(A|B) = 0.2$ and $P(B) = 0.7$
d) Find $P(B)$ if $P(A|B) = 0.3$ and $P(A \cap B) = 0.15$

Solutions:

a) Recalling the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Substituting in given values:

$$\begin{aligned} \therefore P(A|B) &= \frac{0.1}{0.6} \\ &= \frac{1}{6} \end{aligned}$$

b) Recalling the conditional probability formula:

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$$

Substituting in given values:

$$\therefore P(X|Y) = \frac{0.25}{0.75}$$
$$= \frac{1}{3}$$

c) Recalling the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Substituting in given values:

$$\therefore 0.2 = \frac{P(A \cap B)}{0.7}$$

$$\therefore P(A \cap B) = 0.2 \times 0.7$$
$$= 0.14$$

d) Recalling the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Substituting in given values:

$$\therefore 0.3 = \frac{0.15}{P(B)}$$

$$\therefore P(B) = \frac{0.15}{0.3}$$
$$= \frac{1}{2}$$

Exercise 2: A bag of marbles contains 3 red, 1 green and 2 blue marbles. Steph selects one marble and then, without replacing it, selects another. Find the probability that:

a) Both marbles are blue
b) At least one of the marbles is blue
c) Given that the first marble is blue, the other is also blue

Solution:

a) Since there are 6 marbles in total, 2 of which are blue:

$$P(BB) = \frac{2}{6} \times \frac{1}{6}$$

$$= \frac{1}{18}$$

b) Considering the complement:

$$P(at\ least\ one\ blue) = 1 - P(none\ blue)$$

Now finding $P(none\ blue)$:

$$P(none\ blue) = \frac{4}{6} \times \frac{3}{5}$$
$$= \frac{12}{30}$$
$$= \frac{2}{5}$$

$$\therefore P(at\ least\ one\ blue) = 1 - \frac{2}{5}$$
$$= \frac{3}{5}$$

c) If we are given that the first blue, then this means in the bag there are 3 red, 1 green and 1 blue marbles.

Therefore, the chance of drawing a blue marble for my second pick is:

$$P(blue\ again) = \frac{1}{5}$$

Exercise 3: Two dice are rolled. A 4 appears on at least one of the dice and the sum of the two die is recorded. Find the probability that the sum is greater than 7.

Solution:

Since we are given that one of the two dice has a 4, then in order for the sum to be greater than 7, the other die must have a number of 4 or above. Therefore:

$$P(sum\ greater\ than\ 7) = P(number \geq 4)$$
$$= \frac{3}{6}$$
$$= \frac{1}{2}$$

Example 4: In a cohort, 45% has black hair, 20% have brown eyes and 10% have brown eyes and black hair. A student is chosen at random

a) Find the probability that they have brown eyes, given that they have black hair
b) Find the probability that they have black hair, given that they have brown eyes

Solution:

a) Recalling the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Therefore, for this question:

$$P(brown\ eyes\ given\ black\ hair) = \frac{P(brown\ eyes\ and\ black\ hair)}{P(black\ hair)}$$
$$= \frac{0.1}{0.45}$$
$$= \frac{2}{9}$$

b) Recalling the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Therefore, for this question:

$$P(black\ hair\ given\ brown\ eyes) = \frac{P(black\ hair\ and\ brown\ eyes)}{P(brown\ eyes)}$$
$$= \frac{0.1}{0.2}$$
$$= \frac{1}{2}$$

Exercise 5: Jaylen picks two notes at random from three \$5, four \$10 and one \$20 notes. Given that at least one of the notes he picked is a \$5, find the probability that the total of his two notes is \$15 or more.

Solution:

If we are given that one of the notes Jaylen picked is a \$5, then this means in order for the total to be \$15 or greater, the second notes must be either a \$10 or \$20 note.

Since a \$5 note has been picked already, there are 7 notes in total:

$$\therefore P(Sum \geq 15) = P(\$10\ or\ \$20)$$
$$= P(\$10) + P(\$20)$$
$$= \frac{4}{7} + \frac{1}{7}$$
$$= \frac{5}{7}$$