EXPONENTIALS & LOGARITHMS

EXPONENTIAL EQUATIONS (IV)

Contents include:

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- Solving Basic Index Equations

When solving basic index equations for x, we must utilise our knowledge of logarithms. We remember that:

$$To\ solve\ a^x = b$$

Where a, b are constants

$$\begin{aligned} \therefore x &= \log_a b \\ &= \frac{\log_{10} b}{\log_{10} a}\ [Change\ of\ base] \end{aligned}$$

Example 1: Solve $3^x = 8$ for x to the nearest 2 decimal places

Solution:

First converting to logarithm

$$\therefore x = \log_3 8$$

Then changing our base

$$x = \frac{\log 8}{\log 3}$$

Finally entering this into our calculator:

$$\therefore x \approx 1.89\ (nearest\ 2\ d.p.)$$

- Exponential Equations Reducible to Quadratics

Recall that quadratic equations are in the form:

$$ax^2 + bx + c = 0$$

Sometimes, we may be required to solve exponential equations which resemble this form and thus can be solved using quadratic methods such as factorisation or the quadratic formula. We let u equal to whatever exponent is involved in the quadratic equation.

Below is an example demonstrating this:

Example 2: Solve the equation $(2^x)^2 - 5(2^x) + 6 = 0$

First, we let $u = 2^x$. Hence:

$$\therefore u^2 - 5u + 6 = 0$$

As can now be seen, the equation is a quadratic, so factorising like normal:

$$(u-3)(u-2)=0$$

$$\therefore u = 3\ OR\ 2$$

Since $u = 2^x$:

$$\therefore 2^x = 3\ OR\ 2^x = 2$$

$$x = \log_2 3\ \ OR\ x = \log_2 2 = 1$$

Hence, our two solutions are $x = \log_2 3$ and $x = 1$

However, not all reducible equations are as easy as the previous example to spot. As a general rule of thumb, if you see an **equation with two or more exponentials** that have bases which are squares of each other, think of **reducing to quadratics!**

Below is an example of a harder and more hidden exponential equation:

Example 3: Solve the equation $16^x + 4^x - 6 = 0$

From the equation, we see that:

$$16^x + 4^x - 6 = (4^x)^2 + 4^x - 6$$

Letting $u = 4^x$ now:

$$\therefore u^2 + u - 6 = 0$$

Factorising:

$$(u+3)(u-2) = 0$$

$$\therefore u = -3\ OR\ 2$$

Since $u = 4^x$ though, and $4^x > 0$, this means that $u \neq -3$. Hence:

$$u = 4^x = 2\ ONLY\ (\because u > 0)$$

Rearranging to logarithm:

$$\begin{aligned} x &= \log_4 2 \\ &= \frac{\log 2}{\log 4}\ [Change\ of\ base] \\ &= \frac{1}{2} \end{aligned}$$

Hence, the solution is $x = \frac{1}{2}\ ONLY$

Remember that for any index, $a^x > 0$ so we must remove all negative solutions!

- Solving Exponential Inequations

If it is given that:

$$a^x > a^k$$

Where a and k are constants. Then therefore:

$$x > k$$

Similarly, if $a^x < a^k$ then $x < k$

Hence, whenever we are given an exponential inequality in the form of $a^x > b$, then we should **try to express the constant $'b'$ as an index with base a**.

- This can sometimes be an easy process, such as in the following example:

Example 4: Solve the inequation $2^x > 16$

Solution:

Changing 16 to be a power with base 2:

$$2^x > 2^4$$

$$\therefore x > 4$$

- It can however also be quite difficult, and requires us to use the logarithmic law:

$$b = a^{\log_a b}$$

Example 5: Solve the inequation $3^x > 20$

Solution:

Changing 20 to be a power with base 3:

$$20 = 3^{\log_3 20}$$

$$\therefore 3^x > 3^{\log_3 20}$$

Hence, now it can be said that:

$$x > \log_3 20$$

Using change of base law:

$$x > \frac{\ln 20}{\ln 3}$$

$$\therefore x > 2.73 \; (nearest\ 3\ sig\ figs)$$

Alternatively, we can solve this inequality by taking the logarithm of both sides with base a. Hence, this alternative solution will resemble:

Logging both sides with base 3:

$$3^x > 20$$

$$\therefore \log_3 3^x > \log_3 20$$

Now applying logarithmic law:

$$x \log_3 3 > \log_3 20$$

$$\therefore x > \log_3 20$$

It's your choice which method to use when solving these inequalities, so go with whatever seems easier!

Exponential Equation Exercises

1. Use the change of base formula to solve the following equations for x, giving your answer to the nearest 3 significant figures when necessary

a) $3^x = 15$
b) $5^x = 44$
c) $12^x = 160$
d) $0.8^x = 0.24$
e) $26^x = 4$
f) $0.88^x = 0.03$

2. Solve the following inequations, giving exact solutions and without using a calculator:

a) $2^x < 64$
b) $3^x > 81$
c) $5^x \geq 25$
d) $4^x < \frac{1}{16}$
e) $7^x \geq \sqrt{7}$
f) $10^x \leq 0.001$
g) $2^x \leq 1$

3. Solve the following inequations, giving solutions to the nearest 3 significant figures when necessary

a) $2^x > 14$

b) $6^x \leq 250$
c) $3^x \geq \frac{3}{5}$
d) $5^x < 0.68$
e) $0.78^x > 2$
f) $\left(\frac{1}{2}\right)^x > 52$
g) $\left(\frac{3}{4}\right)^x \leq 4.5$

4. Use the substitution $u = 2^x$ to solve the equation $4^x - 6 = 2^x$, leaving your answer in exact form

5. Use the substitution $u = 5^x$ to solve the equation $25^x - 20 = 3 \times 5^x$, leaving your answer to the nearest 3 significant figures

Exponential Equations Exercise Answers

1.
a)

$$3^x = 15$$

$$\therefore x = \log_3 15$$

Using change of base law:

$$x = \frac{\ln 15}{\ln 3}$$
$$\approx 2.46\ (nearest\ 3\ sig\ figs)$$

b)

$$5^x = 44$$

$$\therefore x = \log_5 44$$

Using change of base law:

$$x = \frac{\ln 44}{\ln 5}$$
$$\approx 2.35\ (nearest\ 3\ sig\ figs)$$

c)

$$12^x = 160$$

$$\therefore x = \log_{12} 160$$

Using change of base law:

$$x = \frac{\ln 160}{\ln 12}$$
$$\approx 2.04 \ (nearest\ 3\ sig\ figs)$$

d)

$$0.8^x = 0.24$$

$$\therefore x = \log_{0.8} 0.24$$

Using change of base law:

$$x = \frac{\ln 0.24}{\ln 0.8}$$
$$\approx 6.40 \ (nearest\ 3\ sig\ figs)$$

e)

$$26^x = 4$$

$$\therefore x = \log_{26} 4$$

Using change of base law:

$$x = \frac{\ln 4}{\ln 26}$$
$$\approx 0.425 \ (nearest\ 3\ sig\ figs)$$

f)

$$0.88^x = 0.03$$

$$\therefore x = \log_{0.88} 0.03$$

Using change of base law:

$$x = \frac{\ln 0.03}{\ln 0.88}$$
$$\approx 27.4 \ (nearest\ 3\ sig\ figs)$$

2.
a)

$$2^x < 64$$

Converting 64 to an index with base 2:

$$64 = 2^6$$

$$\therefore 2^x < 2^6$$

$$x < 6$$

b)

$$3^x > 81$$

Converting 81 to an index with base 3:

$$81 = 3^4$$

$$\therefore 3^x > 3^4$$

$$x > 4$$

c)

$$5^x \geq 25$$

Converting 25 to an index with base 5:

$$25 = 5^2$$

$$\therefore 5^x \geq 5^2$$

$$x \geq 2$$

d)

$$4^x < \frac{1}{16}$$

Converting $\frac{1}{16}$ to an index with base 4:

$$\frac{1}{16} = 4^{-2}$$

$$\therefore 4^x < 4^{-2}$$

$$x < -2$$

e)

$$7^x \geq \sqrt{7}$$

Converting $\sqrt{7}$ to an index with base 7:

$$\sqrt{7} = 7^{\frac{1}{2}}$$

$$\therefore 7^x \geq 7^{\frac{1}{2}}$$

$$x \geq \frac{1}{2}$$

f)

$$10^x \leq 0.001$$

Converting 0.001 to an index with base 10:

$$0.001 = 10^{-3}$$

$$\therefore 10^x \leq 10^{-3}$$

$$x \leq -3$$

g)

$$2^x \leq 1$$

Converting 1 to an index with base 2:

$$1 = 2^0$$

$$\therefore 2^x \leq 2^0$$

$$x \leq 0$$

3. Remember that there are 2 methods which can be used to solve this set of questions, so either way is fine!

a)

$$2^x > 14$$

Expressing 14 as a power of 2:

$$14 = 2^{\log_2 14}$$

$$\therefore 2^x > 2^{\log_2 14}$$

Hence:

$$x > \log_2 14$$

Using change of base law:

$$x > \frac{\ln 14}{\ln 2}$$

$$\therefore x > 3.81\ (nearest\ 3\ sig\ figs)$$

b)

$$6^x \leq 250$$

Expressing 250 as a power of 6:

$$250 = 6^{\log_6 250}$$

$$\therefore 6^x \leq 6^{\log_6 250}$$

Hence:

$$x \leq \log_6 250$$

Using change of base law:

$$x \leq \frac{\ln 250}{\ln 6}$$

$$\therefore x \leq 3.08\ (nearest\ 3\ sig\ figs)$$

c)

$$3^x \geq \frac{3}{5}$$

Expressing $\frac{3}{5}$ as a power of 3:

$$\frac{3}{5} = 3^{\log_3 \frac{3}{5}}$$

$$\therefore 3^x \geq 3^{\log_3 \frac{3}{5}}$$

Hence:

$$x \geq \log_3 \frac{3}{5}$$

Using change of base law:

$$x \geq \frac{\ln \frac{3}{5}}{\ln 3}$$

$$\therefore x \geq -0.465\ (nearest\ 3\ sig\ figs)$$

d)

$$5^x < 0.68$$

Expressing 0.68 as a power of 5:

$$0.68 = 5^{\log_5 0.68}$$

$$\therefore 5^x < 5^{\log_5 0.68}$$

Hence:

$$x < \log_5 0.68$$

Using change of base law:

$$x < \frac{\ln 0.68}{\ln 5}$$

$$\therefore x < -0.240\ (nearest\ 3\ sig\ fig)$$

e)

$$0.78^x > 2$$

Logging both sides with base 0.78:

$$\log_{0.78} 0.78^x > \log_{0.78} 2$$

$$\therefore x \log_{0.78} 0.78 > \log_{0.78} 2$$

$$x > \log_{0.78} 2$$

Using change of base law:

$$x > \frac{\ln 2}{\ln 0.78}$$

$$\therefore x > -2.79 \ (nearest\ 3\ sig\ figs)$$

f)

$$\left(\frac{1}{2}\right)^x > 52$$

Logging both sides with base $\frac{1}{2}$:

$$\log_{\frac{1}{2}} \left(\frac{1}{2}\right)^x > \log_{\frac{1}{2}} 52$$

$$\therefore x \log_{\frac{1}{2}} \frac{1}{2} > \log_{\frac{1}{2}} 52$$

$$x > \log_{\frac{1}{2}} 52$$

Using change of base law:

$$x > \frac{\ln 52}{\ln \frac{1}{2}}$$

$$\therefore x > -5.70 \ (nearest\ 3\ sig\ figs)$$

g)

$$\left(\frac{3}{4}\right)^x \leq 4.5$$

Logging both sides with base $\frac{3}{4}$:

$$\log_{\frac{3}{4}} \left(\frac{3}{4}\right)^x \leq \log_{\frac{3}{4}} 4.5$$

$$\therefore x \log_{\frac{3}{4}} \frac{3}{4} \leq \log_{\frac{3}{4}} 4.5$$

$$x \leq \log_{\frac{3}{4}} 4.5$$

Using change of base law:

$$x \leq \frac{\ln 4.5}{\ln \left(\frac{3}{4}\right)}$$

$$\therefore x \leq -5.23$$

4.

$$4^x - 6 = 2^x$$

Let $u = 2^x$. Since $4^x = (2^x)^2$ we can reduce the equation to quadratics:

$$u^2 - 6 = u$$

$$u^2 - u - 6 = 0$$

$$(u-3)(u+2) = 0$$

$$\therefore u = 3 \; OR \; -2$$

Since $u = 2^x$ however, this means that $u > 0$ [because indices must be positive]

$$\therefore 2^x = 3 \; ONLY \; (\because 2^x > 0)$$

Applying logarithm:

$$\therefore x = \log_2 3$$

5.

$$25^x - 20 = 3 \times 5^x$$

Let $u = 5^x$. Since $25^x = (5^x)^2$ we can reduce the equation to quadratics:

$$u^2 - 20 = 3u$$

$$\therefore u^2 - 3u - 20 = 0$$

Using the quadratic formula:

$$\begin{aligned} u &= \frac{3 \pm \sqrt{(-3)^2 - 4 \times -20}}{2} \\ &= \frac{3 \pm \sqrt{9+80}}{2} \\ &= \frac{3 \pm \sqrt{89}}{2} \end{aligned}$$

Since $u = 5^x$ however, this means that $u > 0$ [because indices must be positive]

$$\therefore 5^x = \frac{3+\sqrt{89}}{2} \; ONLY \; (\because 5^x > 0)$$

Applying logarithm:

$$\therefore x = \log_5 \left(\frac{3+\sqrt{89}}{2}\right)$$

Since the question wants the answer to the nearest 3 significant figures, using change of base law:

$$\begin{aligned} x &= \frac{\ln\left(\dfrac{3+\sqrt{89}}{2}\right)}{\ln 5} \\ &\approx 1.14\ (nearest\ 3\ sig\ figs) \end{aligned}$$