DIFFERENTIATION

FURTHER DIFFERENTIATION (IV)

Contents include:

- The Chain Rule
- Substitution Method
- Product Rule
- Quotient Rule

- Chain Rule and Substitution

Normally, when we write out $\frac{dy}{dx}$, this means that we are differentiating y with respect to x. In most cases, this is easy enough as we tend to just follow the $x^n = nx^{n-1}$ rule and work from there.

However, this method does not always apply when we deal with *composite functions*, $f\big(g(x)\big)$. In these situations, we may **need to change the variable** which we differentiate with respect to by making a substitution.

Example 1: Consider the function $y = (x^2 + 2)^2$. If we were to use the $x^n = nx^{n-1}$ rule directly on $f(x)$, we would get:

$$y' = 2(x^2 + 2) = 2x^2 + 4$$

However, **we know that this answer is wrong** since if we expand $f(x)$ first before differentiating, we get:

$$y = x^4 + 4x^2 + 4$$
$$y' = 4x^3 + 8x$$

Hence, for functions like these, we use a process called the "**Chain Rule**" where we essentially **substitute** a new variable to help simplify the function.

$$\boldsymbol{Step\ 1: Make\ a\ substitution}$$

Usually, this is **whatever is in the bracket**, so if we let $u = x^2 + 2$, then:

$$y = u^2$$

$$\boldsymbol{Step\ 2: Find\ \frac{du}{dx}}$$

Since $u = x^2 + 2$:

$$\therefore \frac{du}{dx} = 2x$$

$$\boldsymbol{Step\ 3: Find\ \frac{dy}{dx}}$$

Since $y = u^2$

$$\therefore \frac{dy}{du} = 2u$$

$$\boldsymbol{Step\ 4: Consider\ chain\ rule}$$

The chain rule essentially states that:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Thus, in this example:

$$\begin{aligned}\frac{dy}{dx}&=2u\times 2x\\&=4xu\end{aligned}$$

$$\boldsymbol{Step\ 5: Convert\ u\ back\ into\ in\ terms\ of\ x}$$

Since we said that $u=x^2+2$:

$$\begin{aligned}\therefore\frac{dy}{dx}&=4xu\\&=4x(x^2+2)\\&=4x^3+8x\end{aligned}$$

Which is our answer!

Thus, in summary:

To find the derivative of composite functions, consider the chain rule:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

A good thing to remember is that if $y=(ax+b)^n$, then by the chain rule:

$$y'=an(ax+b)^{n-1}$$

Moreover, if $y=\big(f(x)\big)^n$

$$y'=f'(x)\cdot n\big(f(x)\big)^{n-1}$$

Note: The substitution method is a very powerful method that students should use for difficult questions. For more straightforward problems, you can directly use the chain rule formula like in the following example

Example 2: Differentiate $y=(x^2+2x+5)^6$ using substitution

$$Step\ 1: Substitute\ expression\ inside\ the\ bracket\ for\ u$$

Let $u = x^2 + 2x + 5$

$$\therefore y = u^6$$

$$Step\ 2: Find\ the\ derivative\ expressions$$

$$\frac{du}{dx} = 2x + 2$$

$$\frac{dy}{du} = 6u^5$$

$$Step\ 3: Apply\ Chain\ Rule$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 6u^5 \times (2x + 2) \\ &= 6(2x + 2)(x^2 + 2x + 5)^5\end{aligned}$$

Example 3: Differentiate $y = (x^2 + 4)^{-2}$

Considering chain rule, where if $y = \left(f(x)\right)^n$:

$$y' = f'(x) \cdot n\left(f(x)\right)^{n-1}$$

Hence:

$$\begin{aligned}y' &= (x^2 + 4)' \cdot -2(x^2 + 4)^{-3} \\ &= 2x \cdot -2(x^2 + 4)^{-3} \\ &= -4x(x^2 + 4)^{-3}\end{aligned}$$

- Product Rule

When we are asked to differentiate the product of two functions, in simpler cases we can just expand our expression first then differentiate.

For Example:

$$\begin{aligned}If\ f(x) &= (x + 1)(x + 2), then: \\ f'(x) &= \frac{d[(x + 1)(x + 2)]}{dx} \\ &= \frac{d(x^2 + 3x + 2)}{dx} \\ &= 2x + 3\end{aligned}$$

However, sometimes expanding our expression is not so easy, like for the following-example:

$$f(x) = (x^3 + 3x + 2)(x^2 - 2)$$

When we encounter these sorts of problems, we must thus utilise the product rule:

If $g(x)$ and $h(x)$ are two differentiable functions, then:

$$\frac{d}{dx}[g(x)\cdot h(x)] = g'(x)\cdot h(x) + g(x)\cdot h'(x)$$

An easier way to remember is that:

$$(uv)' = u'v + v'u$$

Example 4: Taking our previous function, find $f'(x)$.

$$f(x) = (x^3 + 3x + 2)(x^2 - 2)$$

letting $u = x^3 + 3x + 2$ and $v = x^2 - 2$,

$$\begin{aligned} u' &= (x^3 + 3x + 2)' \\ &= 3x^2 + 3 \end{aligned}$$

$$\begin{aligned} v' &= (x^2 - 2)' \\ &= 2x \end{aligned}$$

∴ Using our product rule:

$$\begin{aligned} f'(x) &= u'v + v'u \\ &= (3x^2 + 3)(x^2 - 2) + (x^3 + 3x + 2)(2x) \\ &= 3x^4 - 6x^2 + 3x^2 - 6 + 2x^4 + 6x^2 + 4x \\ &= 5x^4 + 3x^2 + 4x - 6 \end{aligned}$$

- Quotient Rule

Similar to the product rule, sometimes when asked to differentiate the quotient of two functions we may be able to first simplify our expression first before differentiating.

For Example:

If $f(x) = \dfrac{x^4 + x^3 + x}{x}$, then:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x^3 + x^2 + 1] \\ &= 3x^2 + 2x \end{aligned}$$

However, sometimes we can't simply our expression enough to differentiate it like usual, like in the following example:

$$If\ f(x) = \frac{2x + 1}{4x - 3}, x \neq \frac{3}{4}$$

When we encounter these sorts of problems, we must utilise the quotient rule:

$$If\ f(x) = \frac{g(x)}{h(x)}, h(x) \neq 0, then:$$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

An easier way to remember is that:

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

Example 5: Taking our previous function, find $f'(x)$.

$$f(x) = \frac{2x+1}{4x-3}$$

letting $g(x) = 2x + 1$ and $h(x) = 4x - 3$,

$$g'(x) = (2x+1)' = 2$$

$$h'(x) = (4x-3)' = 4$$

∴ Using our quotient rule:

$$\begin{aligned} f'(x) &= \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2} \\ &= \frac{2(4x-3) - 4(2x+1)}{(4x-3)^2} \\ &= \frac{8x - 6 - 8x - 4}{16x^2 - 24x + 9} \\ &= -\frac{10}{16x^2 - 24x + 9} \end{aligned}$$

Differentiation Worksheet

1. Differentiate $f(x) = (3x^3 - x^2 + 2x)^4$
2. Differentiate $\sqrt[3]{x^2 - 4}$
3. Using the product rule, differentiate $(5x + 4)(x^2 - 2x)$
4. Using the product rule, differentiate $(3x - 1)(3x^2 + 1)$

5. Differentiate $(x^2+x+3)^5$

6. Differentiate $\sqrt{x^3+3x}$

7. Use the quotient rule to differentiate $\frac{x}{x^2-5x+6}$

8. Use the quotient rule to differentiate $\frac{2x+5}{x+2}$

9. Use the quotient rule to differentiate $\frac{x^2+3x+4}{2x-1}$

10. Differentiate $(x-1)^6(x+2)$

11. Differentiate $(x-3)\sqrt{x+3}$

Differentiation Worksheet Answers

1. $f(x)=(3x^3-x^2+2x)^4$

$$Step\ 1: Substitute\ expression\ inside\ the\ bracket\ for\ u$$

Let $u=3x^3-x^2+2x$

$$\therefore y=u^4$$

$$Step\ 2: Find\ the\ derivative\ expressions$$

$$\begin{aligned}\frac{du}{dx}&=3\times 3x^2-2x+2\\&=9x^2-2x+2\end{aligned}$$

$$\frac{dy}{du}=4u^3$$

$$Step\ 3: Apply\ Chain\ Rule$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4u^3 \times (9x^2 - 2x + 2) \\ &= 4(9x^2 - 2x + 2)(3x^3 - x^2 + 2x)^3\end{aligned}$$

2. $f(x) = \sqrt[3]{x^2 - 4}$

For ~~these~~ questions with a root, it's easier to write it in indices form first:

$$i.e. f(x) = y = (x^2 - 4)^{\frac{1}{3}}$$

Now we can apply our chain rule to the function in the following steps:

Step 1: Substitute your expression inside the bracket as u

$$\begin{aligned}u &= x^2 - 4 \\ \therefore \frac{du}{dx} &= 2x\end{aligned}$$

Step 2: Now using your substitution, find $\frac{dy}{du}$

$$\begin{aligned}y &= u^{\frac{1}{3}} \\ \therefore \frac{dy}{du} &= \frac{1}{3}u^{-\frac{2}{3}} \\ &= \frac{1}{3}(x^2 - 4)^{-\frac{2}{3}}\end{aligned}$$

Step 3: Now apply chain rule to find $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{3}(x^2 - 4)^{-\frac{2}{3}} \times 2x \\ &= \frac{2x}{3\sqrt[3]{(x^2 - 4)^2}}\end{aligned}$$

3. $f(x) = (5x + 4)(x^2 - 2x)$

$$\begin{aligned}f'(x) &= (5)(x^2 - 2x) + (5x + 4)(2x - 2) \\ &= 5x^2 - 10x + (10x^2 - 10x + 8x - 8) \\ &= 15x^2 - 12x - 8\end{aligned}$$

4. $f(x) = (3x - 1)(3x^2 + 1)$

$$\begin{aligned}f'(x) &= 3(3x^2 + 1) + (3x - 1)(6x) \\ &= 9x^2 + 3 + 18x^2 - 6x \\ &= 27x^2 - 6x + 3\end{aligned}$$

5. $f(x) = (x^2 + x + 3)^5$

Step 1: Substitute your expression inside the bracket as u

$$u = x^2 + x + 3$$
$$\therefore \frac{du}{dx} = 2x + 1$$

Step 2: Now using your substitution, find $\frac{dy}{du}$

$$y = u^5$$
$$\therefore \frac{dy}{du} = 5u^4$$
$$= 5(x^2 + x + 3)^4$$

Step 3: Now apply chain rule to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 5(x^2 + x + 3)^4(2x + 1)$$

6. $f(x) = y = \sqrt{x^3 + 3x} = (x^3 + 3x)^{\frac{1}{2}}$

Step 1: Substitute your expression inside the bracket as u

$$u = x^3 + 3x$$
$$\therefore \frac{du}{dx} = 3x^2 + 3$$

Step 2: Now using your substitution, find ~~out~~ $\frac{dy}{du}$

$$y = u^{\frac{1}{2}}$$
$$\therefore \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$
$$= \frac{1}{2}(x^3 + 3x)^{-\frac{1}{2}}$$

Step 3: Now apply chain rule to find $\frac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{2}(x^3 + 3x)^{-\frac{1}{2}} \times (3x^2 + 3) \\ &= \frac{3x^2 + 3}{2\sqrt{x^3 + 3x}}\end{aligned}$$

7. $f(x) = \frac{x}{x^2 - 5x + 6}$

$$\begin{aligned}f'(x) &= \frac{(1)(x^2 - 5x + 6) - x(2x - 5)}{(x^2 - 5x + 6)^2} \\ &= \frac{x^2 - 5x + 6 - 2x^2 + 5x}{(x^2 - 5x + 6)^2} \\ &= \frac{-x^2 + 6}{(x^2 - 5x + 6)^2}\end{aligned}$$

8. $f(x) = \frac{2x+5}{x+2}$

$$\begin{aligned} f'(x) &= \frac{2(x+2) - (2x+5)(1)}{(x+2)^2} \\ &= \frac{2x+4-2x-5}{(x+2)^2} \\ &= -\frac{1}{(x+2)^2} \end{aligned}$$

9. $f(x) = \frac{x^2+3x+4}{2x-1}$

$$\begin{aligned} f'(x) &= \frac{(2x+3)(2x-1) - (x^2+3x+4)(2)}{(2x-1)^2} \\ &= \frac{4x^2-2x+6x-3-2x^2-6x-8}{(2x-1)^2} \\ &= \frac{2x^2-2x-11}{(2x-1)^2} \end{aligned}$$

10. $f(x) = (x-1)^6(x+2)$

$$\frac{d}{dx}(x-1)^6 = 6(x-1)^5$$

$$\begin{aligned} f'(x) &= 6(x-1)^5(x+2) + (x-1)^6(1) \\ &= (x-1)^5[6(x+2) + (x-1)] \\ &= (x-1)^5(7x+11) \end{aligned}$$

11. $f(x) = (x-3)\sqrt{x+3} = (x-3)(x+3)^{\frac{1}{2}}$

$$\frac{d}{dx}(x+3)^{\frac{1}{2}} = \frac{1}{2}(x+3)^{-\frac{1}{2}}$$

$$\begin{aligned} f'(x) &= (1)\sqrt{x+3} + \frac{1}{2}(x+3)^{-\frac{1}{2}}(x-3) \\ &= \sqrt{x+3} + \frac{x-3}{2\sqrt{x+3}} \\ &= \frac{2(x+3)+x-3}{2\sqrt{x+3}} \\ &= \frac{3x+3}{2\sqrt{x+3}} \end{aligned}$$