INTEGRATION

APPLICATIONS OF DEFINITE INTEGRALS TO AREA (VIII)

Contents include:

- Area Underneath the Curve and Definite Integrals

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- Area Underneath the Curve

The region underneath any given function $f(x)$ bounded by the x-axis, $x = a$ and $x = b$ may be found through an evaluation of the definite integral:

$$Area\ under\ curve = \int_a^b f(x)\,dx$$

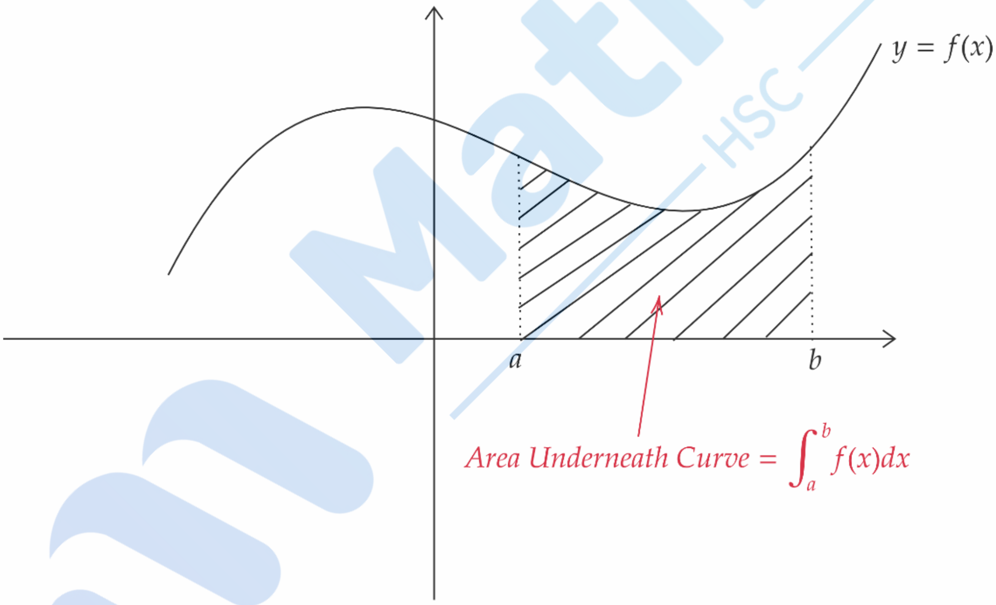
Where:

a is the lower bound

b is the upper bound

Don't forget to add $units^2$ to your area answer in the end!

This is illustrated in the following diagram:



Important to note:

- For regions above the $x -$ axis:

The result of the definite integral expression will be **positive**

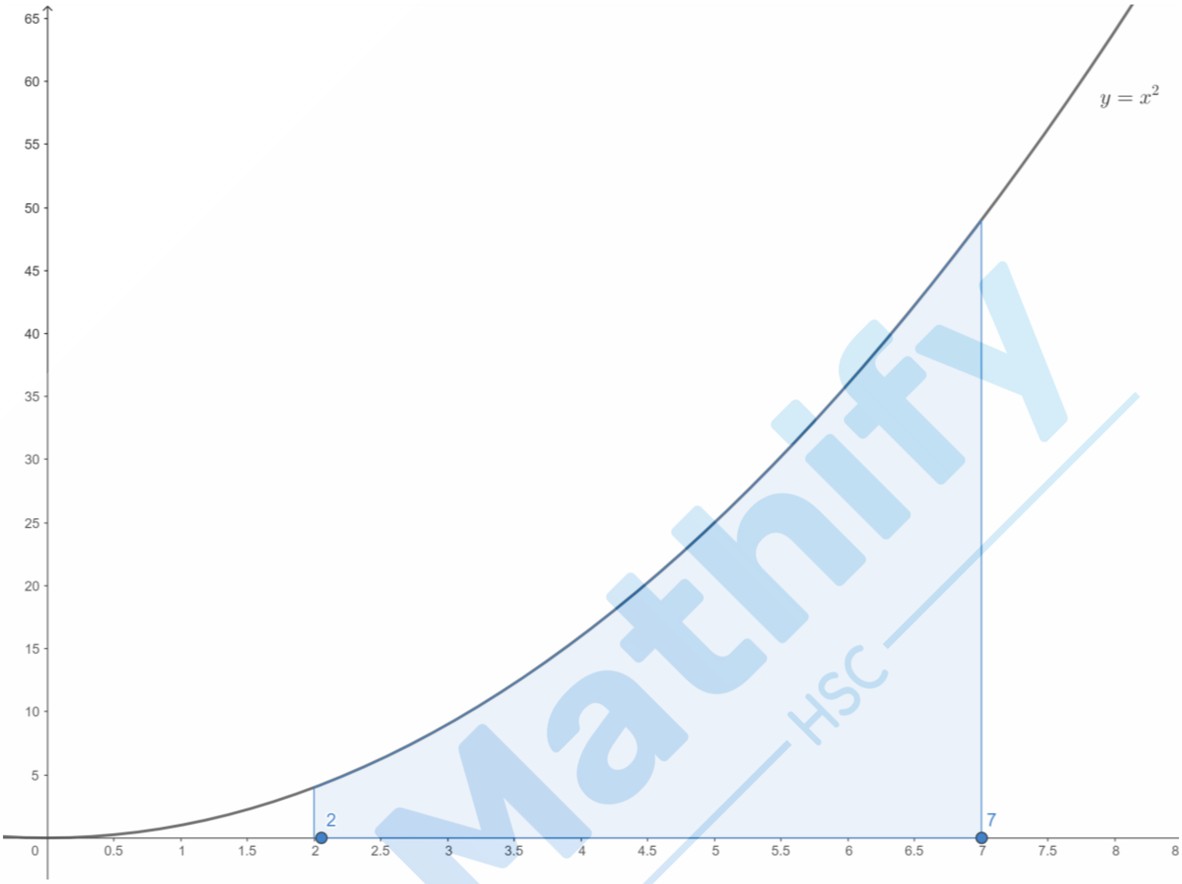
- For regions underneath the $x -$ axis:

The result of the definite integral expression will be **negative**

Example 1: Find the area underneath the graph $y = x^2$ bounded by the $x -$ axis for the given domain of $2 \leq x \leq 7$

Solution:

For our first step, you can choose to draw a basic sketch of the question before finding the area, but this is optional.



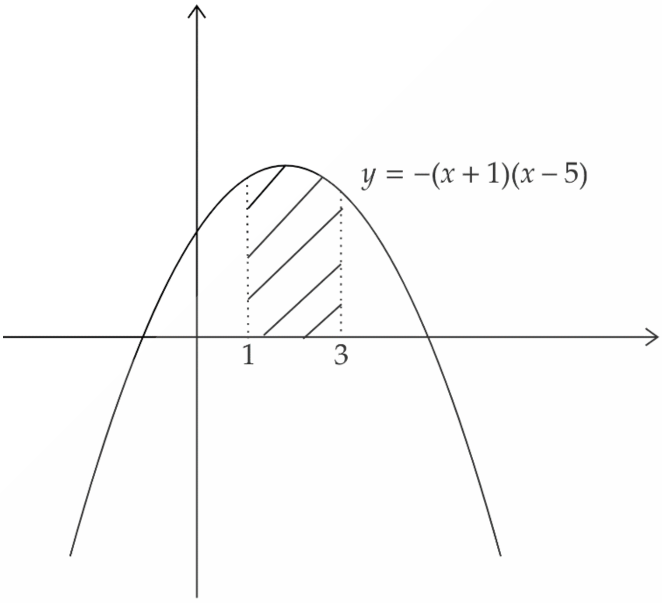
Since we are evaluating area underneath the curve, we use the definite integral here with lower bound $x = 2$ and upper bound $x = 7$

$$\therefore Area = \int_2^7 x^2 \, dx = \left[\frac{x^3}{3}\right]_2^7$$

$$A = \frac{7^3}{3} - \frac{2^3}{3}$$

$$= \frac{335}{3} units^2$$

Example 2: Calculate the total area of the following region:



Solution:

The shaded region may be expressed by the definite integral expression:

$$
\begin{aligned}
A &= \int_1^3 -(x+1)(x-5)dx \\
&= \int_1^3 -(x^2-5x+x-5)dx \\
&= \int_1^3 -(x^2-4x-5)dx \\
&= \int_1^3 -x^2+4x+5\,dx \\
&= \left[-\frac{x^3}{3}+4\cdot\frac{x^2}{2}+5x\right]_1^3 \\
&= \left[-\frac{x^3}{3}+2x^2+5x\right]_1^3 \\
&= \left(-\frac{3^3}{3}+2(3)^2+5(3)\right)-\left(-\frac{1^3}{3}+2(1)^2+5(1)\right) \\
&= (-9+18+15)-\left(-\frac{1}{3}+2+5\right) \\
&= 24-\left(\frac{20}{3}\right) \\
&= \frac{52}{3}\ units^2
\end{aligned}
$$

Definite Integral Exercises

1. Find the area between the function $y = -(x-1)(x-4)$ and the $x - axis$

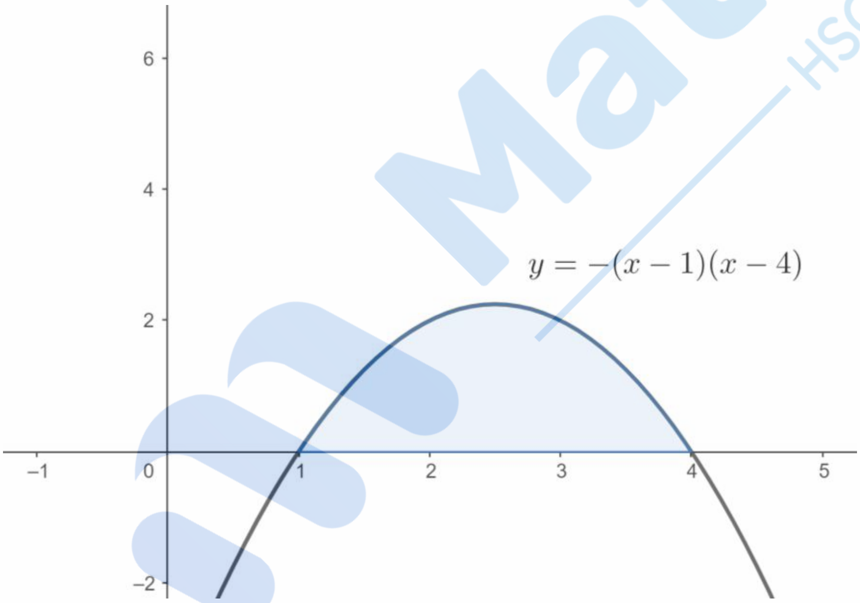
2. Calculate the area bounded by the curve $y = x^2(3-x)$ and the $x - axis$

3. Calculate the area of the region bounded by the graph of $f(x) = x^2 - 4x + 4$, the $x - axis$ and the lines $x = 1 \; and \; x = 4$.

4. Find the positive number, k, such that the area of the region bounded by the graph of $f(x) = kx(2-x)^2$ and the $x - axis$ is equal to 1 unit.

Definite Integral Exercise Answers

1. $Step\ 1: Draw\ a\ diagram\ of\ \ y = -(x-1)(x-4)$



Thus, from the diagram we can determine that the bounds of our region are $x = 1 \; and \; x = 4$

$$Step\ 2: Expand\ the\ expression\ then\ find\ the\ primitive$$

$$\begin{aligned} y &= -(x^2 - 4x - x + 4) \\ &= -x^2 + 5x - 4 \end{aligned}$$

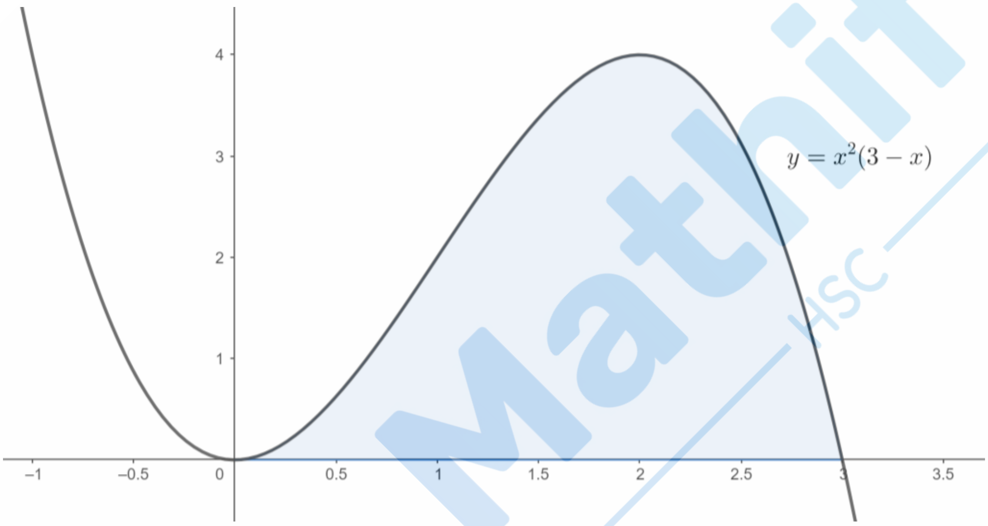
$$\therefore \int_1^4 -x^2 + 5x - 4\, dx = \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x \right]_1^4$$

$$Step\ 3: Evaluate\ area\ by\ substituting\ in\ the\ bounds$$

$$\therefore A = \left[-\frac{4^3}{3} + \frac{5(4)^2}{2} - 4(4)\right] - \left[-\frac{1}{3} + \frac{5(1)^2}{2} - 4(1)\right]$$
$$= \left[-\frac{64}{3} + \frac{80}{2} - 16\right] - \left[-\frac{1}{3} + \frac{5}{2} - 4\right]$$
$$= \frac{8}{3} - -\frac{11}{6}$$
$$= \frac{9}{2} units^2$$

2.

$$Step\ 1: Draw\ a\ diagram\ of\ y = x^2(3 - x)$$



Thus, from the diagram we can determine that the bounds of our region are $x = 0\ and\ x = 3$

$$Step\ 2: Expand\ the\ expression\ then\ find\ the\ primitive$$

$$y = x^2(3 - x)$$
$$= 3x^2 - x^3$$

$$\therefore \int_0^3 3x^2 - x^3\, dx = \left[3 \times \frac{x^3}{3} - \frac{x^4}{4}\right]_0^3$$
$$= \left[x^3 - \frac{x^4}{4}\right]_0^3$$

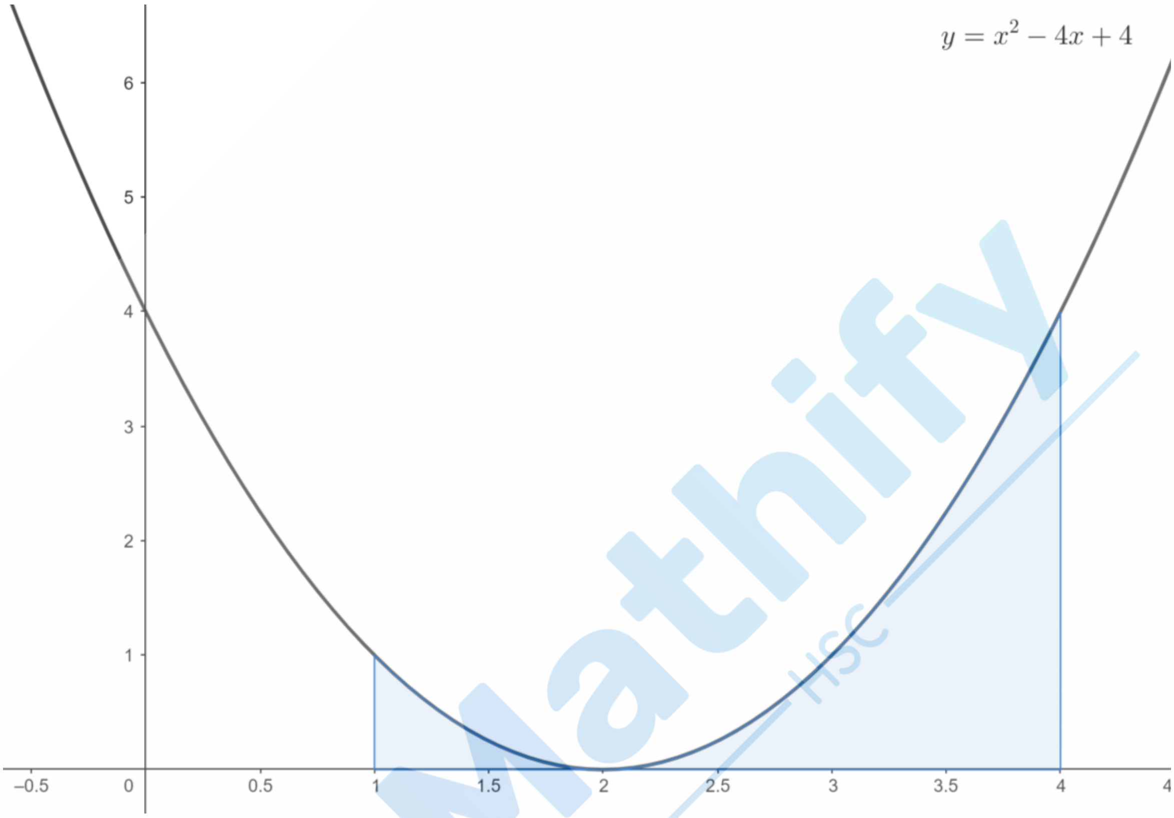
$$Step\ 3: Evaluate\ area\ by\ substituting\ the\ bounds$$

$$\therefore A = \left[3^3 - \frac{3^4}{4}\right] - [0 - 0]$$
$$= \frac{27}{4} units^2$$

3.

$$Step\ 1: Draw\ a\ diagram\ of\ f(x) = x^2 - 4x + 4$$

Notice here how the bounds are $x = 1$ and $x = 4$, therefore the regions we are concerned with finding are:



$$Step\ 2: Find\ the\ primitive\ function\ expression$$

$$\int_1^4 x^2 - 4x + 4\, dx = \left[\frac{x^3}{3} - 4 \times \frac{x^2}{2} + 4x\right]_1^4$$

$$= \left[\frac{x^3}{3} - 2x^2 + 4x\right]_1^4$$

$$Step\ 3: Evaluate\ area\ by\ substituting\ bounds$$

$$\therefore A = \left[\frac{4^3}{3} - 2(4)^2 + 4(4)\right] - \left[\frac{1^3}{3} - 2(1)^2 + 4\right]$$

$$= \left[\frac{64}{3} - 32 + 16\right] - \left[\frac{1}{3} - 2 + 4\right]$$

$$= \frac{16}{3} - \frac{7}{3}$$

$$= 3\ units^2$$

4.

$$Step\ 1: Find\ the\ primitive\ function\ expression$$

Since $f(x) = kx(2-x)^2$, we can determine that the x-intercepts of the function are $x = 2\ and\ x = 0$. Thus, these values will be the lower and upper bounds of our region.

$$f(x) = kx(2-x)^2 = kx(4-4x+x^2)$$
$$= 4kx - 4kx^2 + kx^3$$

$$\therefore Area = \int_0^2 4kx - 4kx^2 + kx^3\,dx$$

$$\int_0^2 4kx - 4kx^2 + kx^3\,dx = \left[4k \times \frac{x^2}{2} - 4k \times \frac{x^3}{3} + k \times \frac{x^4}{4}\right]_0^2$$
$$= \left[2kx^2 - \frac{4}{3}kx^3 + \frac{kx^4}{4}\right]_0^2$$

$$Step\ 2: Evaluate\ area\ by\ substituting\ bounds$$

$$\therefore A = \left[2k(2)^2 - \frac{4}{3}k(2)^3 + \frac{k}{4}(2)^4\right] - 0$$
$$= 8k - \frac{32}{3}k + 4k$$
$$= \frac{4}{3}k$$

$$Step\ 3: Equate\ the\ area\ value\ to\ 1$$

$$\therefore \frac{4}{3}k = 1$$
$$k = \frac{3}{4}$$