FUNCTIONS

COMPOSITE FUNCTIONS AND PIECEWISE NOTATION (IX)

Contents include:

- Composite Functions
- Piecewise Functions and Notation

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- Composite Functions

Composite functions are essentially **functions in a function**. In other words:

$$f(g(x)) \ is \ a \ composite \ function$$

Composite functions may also be expressed as:

$$f \circ g(x) = f(g(x))$$

For all x for which $g(x)$ and $g(f(x))$ are defined

Note: For a composite function, think of replacing every "x" that we see with your function "$g(x)$" instead! Then, you may expand your brackets and simplify where possible

Example 1: If $f(x) = x^2 + 2$ and $g(x) = x + 1$, then derive the expression for $f \circ g(x)$:

Solution:

Considering composite functions and replacing every "x" with "$g(x)$" in $f(x)$:

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= (g(x))^2 + 2 \\ &= (x+1)^2 + 2 \\ &= x^2 + 2x + 1 + 2 \\ &= x^2 + 2x + 3 \end{aligned}$$

Example 2: If $f(x) = x + 3$ and $g(x) = (x + 1)^2$, find:

a) $g \circ f(5)$

First finding the expression for $g \circ f(x)$:

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= (f(x) + 1)^2 \\ &= (x + 3 + 1)^2 \\ &= (x + 4)^2 \end{aligned}$$

Now letting $x = 5$:

$$\begin{aligned} \therefore g \circ f(5) &= (5 + 4)^2 \\ &= 9^2 \\ &= 81 \end{aligned}$$

b) $f \circ g(6)$

First finding the expression for $f \circ g(x)$:

$$f \circ g(x) = f(g(x))$$

$$= g(x) + 3$$
$$= (x+1)^2 + 3$$

Now letting $x = 6$:

$$\therefore f \circ g(6) = (6+1)^3 + 3$$
$$= 7^3 + 3$$
$$= 343 + 3$$
$$= 346$$

- Piecewise Functions and Notation

Put simply, a piecewise function is a function that is made up of multiple, smaller sub – functions that change depending on what our value of x is.

For example, a piecewise function is shown below:

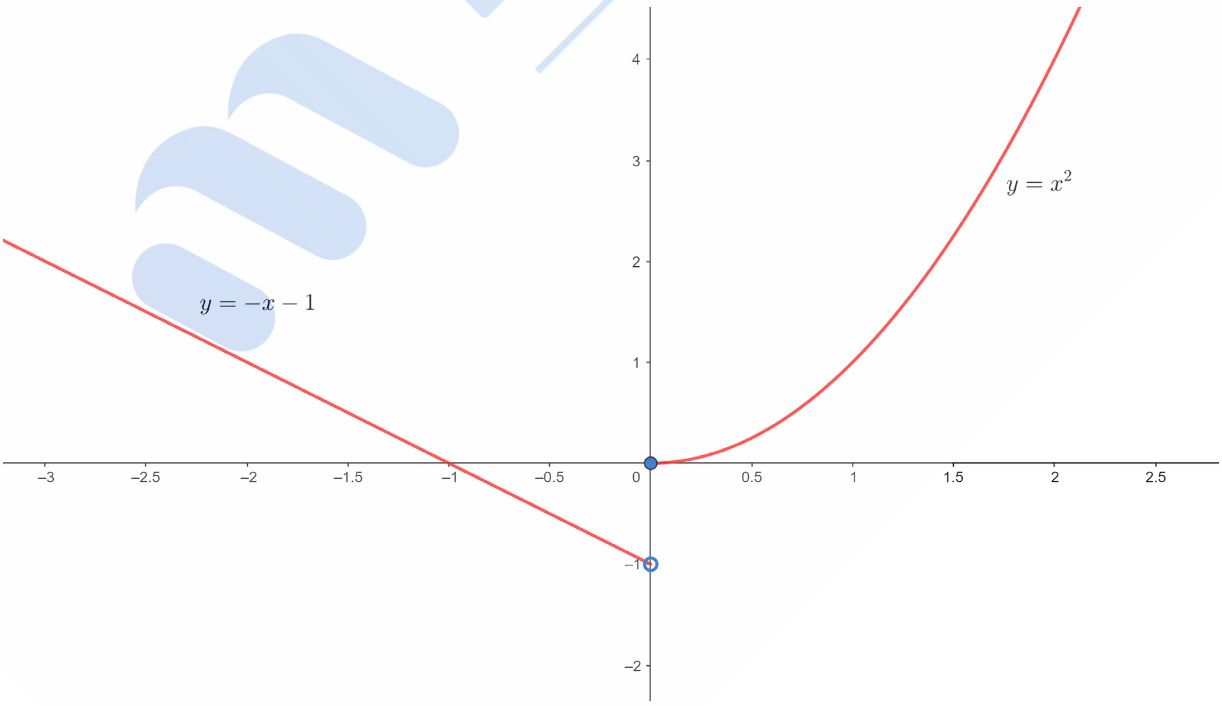
$$f(x) = \begin{cases} x^2 & for\ x \geq 0 \\ -x-1 & for\ x < 0 \end{cases}$$

What this essentially means is that when $x < 0$, the graph of the function is $y = -x - 1$. Then, when $x \geq 0$, the function resembles $y = x^2$ instead.

Note: Be careful of the difference between " $<$ "," $>$ " and " $\leq$ "," $\geq$ "!

- If the interval is either " $<$ " or " $>$ ", then the endpoint is an open circle
- If the interval is either " $\leq$ " or " $\geq$ ", then the endpoint is a closed circle

Hence, sketching $f(x)$ keeping this in mind:



Composite Function Exercises

1. Given that $f(x) = 3x + 5$ and $g(x) = \frac{1}{2}x + 2$, find the value of each of the following:

a) $f\big(g(3)\big)$
b) $f\big(g(-2)\big)$
c) $g\big(f(4)\big)$
d) $g\big(f(-1)\big)$

2. Given that $f(x) = 2x^2 + 5$ and $g(x) = 2x - 1$, find the value of each of the following:

a) $f\big(g(1)\big)$
b) $f\big(g(-3)\big)$
c) $g\left(f\left(\frac{1}{2}\right)\right)$
d) $g\big(f(-2)\big)$

3. Given that $h(x) = f\big(g(x)\big)$, find the expression for $h(x)$ if it is known that:

a) $f(x) = \frac{1}{x+1}$ and $g(x) = 3x + 2$
b) $f(x) = -x^3 + 1$ and $g(x) = 3x$
c) $f(x) = e^x$ and $g(x) = 3x - 4$
d) $f(x) = x^2 + 5$ and $g(x) = x + h$, where h is a constant
e) $f(x) = \sqrt{x^2 + 5x}$ and $g(x) = \frac{1}{x^2}$

Composite Function Exercise Answers

1.
a)

$$\begin{aligned} f\big(g(x)\big) &= 3\big(g(x)\big) + 5 \\ &= 3\left(\frac{1}{2}x + 2\right) + 5 \\ &= \frac{3}{2}x + 11 \end{aligned}$$

$$\begin{aligned} \therefore f\big(g(3)\big) &= \frac{3}{2}(3) + 11 \\ &= \frac{31}{2} \end{aligned}$$

b)

Since $f\big(g(x)\big) = \frac{3}{2}x + 11$

$$f\big(g(-2)\big) = \frac{3}{2}(-2) + 11$$
$$= 8$$

c)

$$g\big(f(x)\big) = \frac{1}{2}f(x) + 2$$
$$= \frac{1}{2}(3x + 5) + 2$$
$$= \frac{3}{2}x + \frac{5}{2} + 2$$
$$= \frac{3}{2}x + \frac{9}{2}$$

$$\therefore g\big(f(4)\big) = \frac{3}{2}(4) + \frac{9}{2}$$
$$= 6 + \frac{9}{2}$$
$$= \frac{21}{2}$$

d)

Since $g\big(f(x)\big) = \frac{3}{2}x + 11$

$$\therefore g\big(f(-1)\big) = \frac{3}{2}(-1) + \frac{9}{2}$$
$$= -\frac{3}{2} + \frac{9}{2}$$
$$= 3$$

2.
a)

$$f\big(g(x)\big) = 2\big(g(x)\big)^2 + 5$$
$$= 2(2x - 1)^2 + 5$$
$$= 2(4x^2 - 4x + 1) + 5$$
$$= 8x^2 - 8x + 2 + 5$$
$$= 8x^2 - 8x + 7$$

$$\therefore f\big(g(1)\big) = 8(1)^2 - 8(1) + 7$$
$$= 8 - 8 + 7$$
$$= 7$$

b)

Since $f\big(g(x)\big) = 8x^2 - 8x + 7$

$$\therefore f\big(g(-3)\big) = 8(-3)^2 - 8(-3) + 7$$

$$= 8(9) + 24 + 7$$
$$= 103$$

c)

$$\begin{aligned} g(f(x)) &= 2(f(x)) - 1 \\ &= 2(2x^2 + 5) - 1 \\ &= 4x^2 + 10 - 1 \\ &= 4x^2 + 9 \end{aligned}$$

$$\begin{aligned} \therefore g\left(f\left(\frac{1}{2}\right)\right) &= 4\left(\frac{1}{2}\right)^2 + 9 \\ &= 4\left(\frac{1}{4}\right) + 9 \\ &= 10 \end{aligned}$$

d)

Since $g(f(x)) = 4x^2 + 9$:

$$\begin{aligned} \therefore g(f(-2)) &= 4(-2)^2 + 9 \\ &= 4(4) + 9 \\ &= 25 \end{aligned}$$

3.

a)

$$\begin{aligned} f(g(x)) &= \frac{1}{g(x) + 1} \\ &= \frac{1}{(3x + 2) + 1} \\ &= \frac{1}{3x + 3} \end{aligned}$$

b)

$$\begin{aligned} f(g(x)) &= -(g(x))^3 + 1 \\ &= -(3x)^3 + 1 \\ &= -27x^3 + 1 \end{aligned}$$

c)

$$\begin{aligned} f(g(x)) &= e^{g(x)} \\ &= e^{3x-4} \end{aligned}$$

d)

$$\begin{aligned} f(g(x)) &= (g(x))^2 + 5 \\ &= (x + h)^2 + 5 \end{aligned}$$

$$= x^2 + 2xh + h^2 + 5$$

e)

$$f\left(g(x)\right) = \sqrt{\left(g(x)\right)^2 + 5g(x)}$$

$$= \sqrt{\left(\frac{1}{x^2}\right)^2 + 5\left(\frac{1}{x^2}\right)}$$

$$= \sqrt{\frac{1}{x^4} + \frac{5}{x^2}}$$

$$= \sqrt{\frac{1}{x^4} + \frac{5x^2}{x^4}}$$

$$= \frac{\sqrt{1 + 5x^2}}{x^2}$$