DIFFERENTIATION

PHYSICAL APPLICATIONS: RATES OF CHANGE (VII)

Contents include:

- Gradient as a Rate Measurer
- Average vs Instantaneous Rate of Change
- Rate Notation

- Gradient as a rate measurer

The gradient of a function acts as a **rate measurer**, essentially telling us how quickly a quantity may increase or decrease at a certain point in time.

It may or may not be at a constant rate and may either show a value is increasing (for positive gradients) or decreasing (for negative gradients). Hence, we say that derivative functions are **related to an instantaneous rate of change** of a certain value such as volume (V) or quantity (Q).

In other words:

$$The\ instantaneous\ rate\ of\ change = f'(x)$$

For example: The rate of change of a volume, V is given as:

$$rate\ of\ change\ of\ volume = V'$$

- Average vs Instantaneous rate of change

What does it mean by "Instantaneous"?

Instantaneous basically means at a **specific point** in time. This is different to "average", which instead refers to a **period or interval** of time such as for example, average rate of change over a 1-minute period. For most questions in calculus, we are dealing with the instantaneous rate unless specified otherwise!

The formula for average rate of change if you are ever asked for it is:

$$average\ rate\ of\ change = \frac{change\ in\ quantity}{change\ in\ time}$$

Remember:

$$Instantaneous\ rate\ of\ change \neq Average\ rate\ of\ change$$

- The importance of $\frac{dy}{dx}$ notation and their relationship to rates

When we are given a rate, we can typically split it into two parts:

1. What is changing, this is your "y" or dependent variable
2. What we are changing with respect to, this is your "x" or independent variable

Whenever we are given a rate, we can and always should translate it into derivative $\frac{dy}{dx}$ notation, where (1) is the variable on the top and (2) is the variable on the bottom.

For example, when we talk about speed, it is (1) the rate of change of **distance (x)** with (2) respect to time. Hence, for speed, it can be represented as $\frac{dx}{dt}$ in derivative notation.

<u>**Note:**</u> Often we can use the units of our given rate as a hint to what its derivative notation is. Looking at speed, its units is typically km/hr or m/s. In both cases, a measurement of distance (x) is on top and a measurement of time (t) is on the bottom. Hence, its notation will be $\frac{dx}{dt}$.

Example 1: Convert the following sentences into derivative notation:

a) A block of ice melts so that its volume decreases at a constant rate of 30 cm^3/hr

$$\frac{dV}{dt} = -30\ cm^3/hr$$

Notice here how a negative sign was used since the volume is **decreasing**

b) The rate of change of a circle's area with respect to its radius

$$rate = \frac{dA}{dr}$$

c) The rate at which a cube's side length (r) is increasing is $2cm/min$

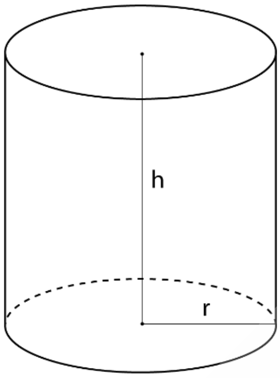
$$\frac{dr}{dt} = 2\ cm/min$$

Example 2: The volume V of an object may increase or decrease with time and $\frac{dV}{dt}$ is the rate of change of volume.

A right circular cylinder of volume V has height h and radius of base r. Find:

a) The rate of change of volume with respect to height, if the radius of the base is constant

The question requires us to find the expression for $\frac{dV}{dh}$



Recall that the volume of a cylinder was $V = \pi r^2 h$

$$\therefore \text{Since } r \text{ is a constant}, \frac{dV}{dh} = \pi r^2$$

b) The rate of change of volume with respect to the radius of the base, if the height is constant

The question requires us to find the expression for $\frac{dV}{dr}$

$$\therefore \text{Since } h \text{ is a constant}, \frac{dV}{dr} = 2\pi rh$$

Example 3: The quantity output of a machine, Q, may increase with time and $\frac{dQ}{dt}$ is the rate that the machine is working.

A toy machine manufactures toys at a variable rate given by:

$$Q = t^2 + 2t + 1$$

Where Q is the number of toys manufactured in time t minutes

a) At what rate is the machine working initially?

$$\frac{dQ}{dt} = 2t + 2$$

Since $t = 0$ initially,

$$\therefore \text{Rate } \frac{dQ}{dt} = 2 \text{ initially}$$

b) After 10 minutes?

After 10 minutes, $t = 10$

$$\therefore \text{Rate } \frac{dQ}{dt} = 22 \text{ when } t = 10$$

c) The total number of toys manufactured in the first 10 minutes?

To find the total number of toys Q, sub in $t = 10$:

$$\begin{aligned} Q &= (10)^2 + 2(10) + 1 \\ &= 121 \end{aligned}$$

Example 4: A hole is being dug by a team who remove V cubic metres of soil in t minutes, where:

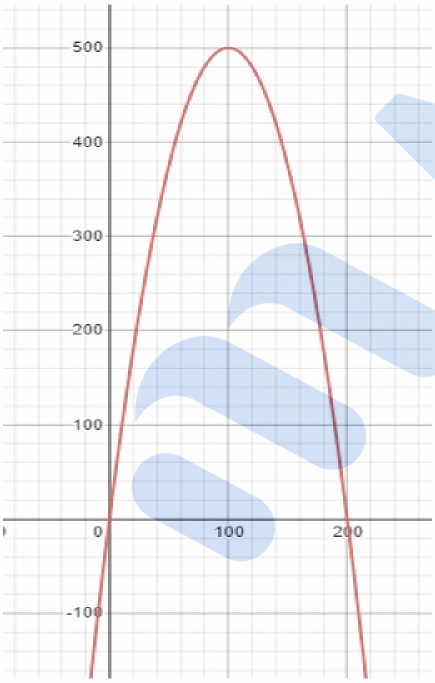
$$V = 10t - \frac{t^2}{20}$$

a) State the values of t during which soil is being removed
b) At what rate is the soil being removed at the end of 40 minutes?
c) What is their initial rate of work when $t = 0$?
d) At what time are they removing soil at the rate of $5m^3$ per minute?

Solutions:

a)

Amount of soil removed must be greater than or equal to 0 and V must always be increasing. Hence, drawing the graph of $V = 10t - \frac{t^2}{20}$:



Since we want $V > 0$ and always increasing, the domain for t must therefore be:

$$0 \leq t \leq 100$$

As seen in the graph.

b) The rate of soil removal is:

$$\frac{dV}{dt} = 10 - \frac{2t}{20}$$
$$= 10 - \frac{t}{10}$$

At the end of 40 minutes, $t = 40$:

$$\therefore \frac{dV}{dt} = 10 - \frac{40}{10}$$
$$= 6\ m^3/min$$

c) When $t = 0$:

$$\frac{dV}{dt} = 10 - \frac{0}{10}$$
$$= 10\ m^3/min$$

d) Find the value of t for which $\frac{dV}{dt} = 5$

$$\frac{dV}{dt} = 10 - \frac{t}{10}$$

$$\therefore 10 - \frac{t}{10} = 5$$
$$\frac{t}{10} = 5$$
$$t = 50\ minutes$$

Example 5: A water tank is being emptied and the quantity of water, Q litres, remaining in the tank at any time, t minutes, after it starts to empty is given by:

$$Q(t) = 1000(20 - t)^2, t \geq 0$$

a) At what rate is the tank being emptied at any time t?
b) How long does it take to empty the tank?
c) At what time is the water flowing out at the rate of 20,000 litres per minute?
d) What is the average rate at which the water flows out in the first 5 minutes?

Solutions:

a)

The question here wants us to find the rate at which volume is changing, in other words, $\frac{dQ}{dt}$.

Hence, since $Q = 1000(20 - t)^2$:

$$\frac{dQ}{dt} = 1000 \times 2(20 - t) \times -1$$
$$= -2000(20 - t)$$

Since $\frac{dQ}{dt} < 0$, this means that volume is decreasing and the tank is therefore being emptied at a rate of $2000(20 - t)$ litres per minute.

b)

To find the length of time taken to empty the taken, find the value of t when $Q = 0$.

$$0 = 1000(20 - t)^2$$
$$\therefore t = 20$$

c)

Equate our previous $\frac{dQ}{dt}$ expression to -20000 ($-$ since water is flowing out):

$$\frac{dQ}{dt} = -2000(20 - t) = -20000$$

$$20 - t = 10$$
$$\therefore t = 10$$

d) We are concerned with average rate here, the equation of which is:

$$average\ rate = \frac{total\ volume}{total\ time}$$

When $t = 0$:

$$\begin{aligned} Q(0) &= 1000(20 - 0)^2 \\ &= 400000 \end{aligned}$$

After 5 minutes, $t = 5$:

$$\begin{aligned} Q(5) &= 1000(20 - 5)^2 \\ &= 1000(15)^2 \\ &= 225000 \end{aligned}$$

$$\begin{aligned} \therefore average\ rate &= \frac{400000 - 225000}{5} \\ &= \frac{175000}{5} \\ &= 35000\ litres/min \end{aligned}$$