DIFFERENTIATIAL CALCULUS

TRIG DERIVATIVES (III)

Contents include:

- Standard Forms of Trig Differentiation

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- Standard forms of Trig Differentiation

There are 3 main derivatives to remember when differentiating trig:

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$

The chain rule still applies for trigonometric functions. Often if we need to find a substitution to make, it will be whatever expression is inside the trig function. Thus, in general:

$$\frac{d}{dx}\sin(f(x)) = f'(x)\cos(f(x))$$
$$\frac{d}{dx}\cos(f(x)) = -f'(x)\sin(f(x))$$
$$\frac{d}{dx}\tan(f(x)) = f'(x)\sec^2(f(x))$$

Note: The chain rule, product and quotient rules which were covered in previous lessons also still apply for trigonometric functions as normal.

Example 1: Differentiate $\sin(2x-1)$ with respect to x

Solution:

Since $(2x-1)' = 2$, using the standard derivative:

$$[\sin(2x-1)]' = 2\cos(2x-1)$$

Example 2: Differentiate $\cos x^2$ with respect to x

Solution:

We may complete this question using substitution and chain rule:

$$Let\ u = x^2$$
$$\frac{du}{dx} = 2x$$
$$\frac{d}{du}(\cos u) = -\sin u$$
$$\therefore \frac{d}{dx}(\cos x^2) = \frac{d}{du}(\cos u) \times \frac{du}{dx}$$
$$= -2x\sin u$$

$$= -2x\sin x^2$$

Alternatively, we can use the standard derivative $\frac{d}{dx}\cos(f(x)) = -f'(x)\sin(f(x))$:

$$\frac{d}{dx}\cos x^2 = -2x\sin x^2$$

Example 3: Differentiate $x^2\tan x^2$ with respect to x

Solution:

Recall the product rule:

$$(uv)' = u'v + v'u$$

Hence:

$$\textit{Step 1: Find the derivative of } \tan x^2$$

$$\text{Let } u = x^2$$
$$\frac{du}{dx} = 2x$$
$$\frac{d}{du}(\tan u) = \sec^2 u$$
$$\therefore \frac{d}{dx}(\tan x^2) = \frac{d}{du}(\tan u) \times \frac{du}{dx}$$
$$\frac{d}{dx}(\tan x^2) = 2x\sec^2 u$$
$$= 2x\sec^2 x^2$$

$$\textit{Step 2: Apply our product rule}$$

$$\frac{d}{dx}(x^2\tan x^2) = (x^2)'\tan x^2 + x^2(\tan x^2)'$$
$$= 2x\tan x^2 + x^2 \times 2x\sec^2 x^2$$
$$= 2x\tan x^2 + 2x^3\sec^2 x^2$$

<u>Differentiation of Trigonometric Function Exercises</u>

1. Differentiate the following functions with respect to x

a) $y = 3\sin x$
b) $y = \frac{7}{2}\cos x$
c) $y = 2\tan x$
d) $y = -\sin 2x$
e) $y = 6\cos\frac{x}{3}$
f) $y = -\tan 2x$
g) $y = 6\sin\frac{x+1}{2}$
h) $y = 20\cos\frac{2x+1}{5}$
i) $y = \cos^2 x$

2. Find the first, second and third derivatives of:

a) $y = 7\sin(6 - 3x)$

b) $y = 4\sin x + \cos 5x$

3. If $f(x) = \sin\left(\frac{1}{5}x + \frac{\pi}{3}\right)$, find $f'(x)$ and $f''(x)$ then find:

a) $f'(0)$

b) $f'\left(\frac{5\pi}{2}\right)$

4. Differentiate $y = 3x^3 \cos 2x^2$ with respect to x

5. Differentiate $y = \frac{x^2}{3+\tan x}$ with respect to x

6. Differentiate $y = \frac{x^3}{2+x\sin x}$ with respect to x

Differentiation of Trigonometric Function Exercise Answers

1.

a) Using the standard form of differentiation,

$$y' = 3\cos x$$

b) Using the standard form of differentiation,

$$y' = -\frac{7}{2}\sin x$$

c) Using the standard form of differentiation,

$$y' = 2\sec^2 x$$

d) Using the standard form of differentiation,

$$y' = -2\cos 2x$$

e) Using the standard form of differentiation,

$$y' = -2\sin\frac{x}{3}$$

f) Using the standard form of differentiation,

$$y' = -2\sec^2 2x$$

g) Using the standard form of differentiation,

$$y' = 3\cos\frac{x+1}{2}$$

h) Using the standard form of differentiation,

$$\left(\frac{2x+1}{5}\right)' = \frac{2}{5}$$

$$\therefore y' = 20 \times \frac{2}{5} - \sin\frac{2x+1}{5}$$
$$= -8\sin\frac{2x+1}{5}$$

i) $y = \cos^2 x$

$Let\ u = \cos x$

$\frac{du}{dx} = -\sin x$

$$\therefore y = u^2$$
$$\frac{dy}{dx} = 2u \times -\sin x \ \ (chain\ rule)$$
$$= -2\cos x \sin x$$

2.

a) $(6-3x)' = -3$

$$\therefore y' = -3 \times 7\cos(6-3x)$$
$$= -21\cos(6-3x)$$

To find the second derivative, differentiate again:

$$y'' = -3 \times -21 \times -\sin(6-3x)$$
$$= -63\sin(6-3x)$$

Remember that $(\cos x)' = -\sin x$

To find the third derivative, differentiate again:

$$y''' = -3 \times -63\cos(6-3x)$$
$$= 189\cos(6-3x)$$

b) $y = 4\sin x + \cos 5x$

$$\therefore y' = 4\cos x - 5\sin 5x$$

To find the second derivative, differentiate again:

$$y'' = -4\sin x - 25\cos 5x$$

To find the third derivative, differentiate again:

$$y''' = -4\cos x + 125\sin 5x$$

3. $f(x) = \sin\left(\frac{1}{5}x + \frac{\pi}{3}\right)$

$$f'(x) = \frac{1}{5}\cos\left(\frac{1}{5}x + \frac{\pi}{3}\right)$$

$$f''(x) = -\frac{1}{25}\sin\left(\frac{1}{5}x + \frac{\pi}{3}\right)$$

a) $f'(0) = \frac{1}{5}\cos\left(\frac{\pi}{3}\right)$

Remembering our exact values:

$$\therefore f'(0) = \frac{1}{5} \times \frac{1}{2}$$
$$= \frac{1}{10}$$

b) $f'\left(\frac{5\pi}{2}\right) = \frac{1}{5}\cos\left(\frac{1}{5} \times \frac{5\pi}{2} + \frac{\pi}{3}\right)$

$$\therefore f'(0) = \frac{1}{5}\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$$
$$= \frac{1}{5}\cos\left(\frac{5\pi}{6}\right)$$

Remembering our exact values:

$$= \frac{1}{5} \times -\frac{\sqrt{3}}{2}$$
$$= \frac{-\sqrt{3}}{10}$$

4. $y = 3x^3 \, cos\, 2x^2$

$$Step\ 1: Find\ out\ the\ derivative\ of\ \cos 2x^2$$

$$(\cos 2x^2)' = -4x\sin 2x^2$$

$$Step\ 2: Apply\ the\ product\ rule$$

$$\begin{aligned} y' &= (3x^3)\cos 2x^2 + 3x^3(\cos 2x^2)' \\ &= 9x^2\cos 2x^2 + 3x^3 \times -4x\sin 2x^2 \\ &= 9x^2\cos 2x^2 - 12x^4\sin 2x^2 \end{aligned}$$

5. $y = \frac{x^2}{3+\tan x}$

$$Applying\ the\ quotient\ rule:$$

$$\begin{aligned} y' &= \frac{(x^2)'(3+\tan x) - x^2(3+\tan x)'}{(3+\tan x)^2} \\ &= \frac{2x(3+\tan x) - x^2(3+\sec^2 x)}{(3+\tan x)^2} \end{aligned}$$

$$= \frac{6x - 3x^2 + 2x\tan x - x^2\sec^2 x}{(3 + \tan x)^2}$$

6. $y = \frac{x^3}{2+xsinx}$

$$Step\ 1{:}\, Find\ the\ derivative\ of\ the\ denominator, 2 + xsinx$$

Notice that we have to use the product rule

$$(2 + x\sin x)' = x'sinx + x(\sin x)'$$
$$= sinx + x\cos x$$

$$Step\ 2{:}\, Use\ the\ quotient\ rule\ to\ differentiate\ y$$

$$y' = \frac{(x^3)'(2 + x\sin x) - x^3(2 + x\sin x)'}{(2 + x\sin x)^2}$$
$$= \frac{3x^2(2 + x\sin x) - x^3(\sin x + x\cos x)}{(2 + x\sin x)^2}$$
$$= \frac{6x^2 + 3x^3\sin x - x^3\sin x - x^4\cos x}{(2 + x\sin x)^2}$$
$$= \frac{6x^2 + 2x^3\sin x - x^4\cos x}{(2 + x\sin x)^2}$$